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Cohomological aspects of non-Kähler manifolds

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Introduction

By a remarkable result by W. L. Chow, [Cho49, Theorem V] (see also [Ser56]), *projective manifolds* (that is, compact complex submanifolds of $\mathbb{CP}^n := \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \left(\mathbb{C} \setminus \{0\}\right)$, for $n \in \mathbb{N}$) are in fact *algebraic* (that is, they can be described as the zero set of finitely many homogeneous holomorphic polynomials). One is hence interested in relaxing the projective condition, looking for special properties on compact manifolds sharing a weaker structure than projective manifolds. For example, a large amount of developed analytic techniques allows to prove strong cohomological properties for compact *Kähler manifolds* (that is, compact complex manifolds endowed with a Kähler metric, namely, a Hermitian metric admitting a local potential function), [SvD30, Käh33], see also [Wei58], which are, in a certain sense, the "analytic-versus-algebraic", [Cho49, Theorem V], or the "$\mathbb{R}$-versus-$\mathbb{Q}$", [Kod54, Theorem 4], version of projective manifolds. Kähler manifolds are in fact endowed with three different structures, interacting each other: a *complex* structure, a *symplectic* structure, and a *metric* structure; it is the strong linking between them that allows to develop many analytic tools and hence to derive the very special properties of Kähler manifolds. In order to better understand any of such properties, it is natural to ask what of these three structures is actually involved and required. Therefore, one is led to study complex, symplectic, and metric contribution separately, possibly weakening either the interactions between them, or one of these structures. For example, by relaxing the metric condition, one could ask what properties of a compact complex manifold can be deduced by the existence of special Hermitian metrics defined by conditions similar to, but weaker than, the defining condition of the Kähler metrics (for example, metrics being *balanced* in the sense of M. L. Michelsohn [Mic82], *pluriclosed* [Bis89], *astheno-Kähler* [JY93, JY94], *Gauduchon* [Gau77], *strongly-Gauduchon* [Pop09]); by relaxing the complex structure, one is led to study properties of *almost-complex* manifolds, possibly endowed with compatible symplectic structures.

In particular, we are concerned with studying cohomological properties of compact (almost-)complex manifolds, and of manifolds endowed with special structures, e.g., symplectic structures, **D**-complex structures in the sense of F. R. Harvey and H. B. Lawson, exhaustion functions satisfying positivity conditions. Part of the original results have been published or will appear in [AT11, AT12a, Ang11, AT12b, AR12, ATZ12, AC12]; some other results have been collected in a preprint, see [AT12c]; some more results have not yet been submitted for publication.

We recall that a *complex* manifold X is endowed with a natural *almost-complex* structure, that is, an endomorphism $J \in \mathrm{End}(TX)$ of the tangent bundle of X such that $J^2 = -\operatorname{id}_{TX}$, which actually satisfies a further integrability condition, [NN57, Theorem 1.1]. By considering the decomposition into eigen-spaces, just the datum of the almost-complex structure yields a splitting of the complexified tangent bundle, namely,

$$TX \otimes \mathbb{C} \;=\; T^{1,0}X \oplus T^{0,1}X \;,$$

and hence it induces also a splitting of the bundle of complex differential forms, namely,

$$\wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C} \;=\; \bigoplus_{p+q=\bullet} \wedge^{p,q} X \;.$$

Furthermore, on a complex manifold, the integrability condition of such an almost-complex structure yields a further structure on $\wedge^{\bullet,\bullet}X$, namely, a structure of double complex $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$, where ∂ and $\overline{\partial}$ are the components of the $\mathbb{C}$-linear extension of the exterior differential d.

Hence, on a complex manifold X, one can consider both the de Rham cohomology

$$H^\bullet_{dR}(X;\mathbb{C}) \;:=\; \frac{\ker \mathrm{d}}{\operatorname{im} \mathrm{d}}$$

and the Dolbeault cohomology

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) \;:=\; \frac{\ker \overline{\partial}}{\operatorname{im} \overline{\partial}} \;;$$

whenever X is compact, the Hodge theory assures that they have finite-dimension as $\mathbb{C}$-vector spaces. On a compact complex manifold, in general, no natural map between $H^{\bullet,\bullet}_{\overline{\partial}}(X)$ and $H^{\bullet}_{dR}(X;\mathbb{C})$ exists; on the other hand, the structure of double complex of $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ gives rise to a spectral sequence

$$E_1^{\bullet,\bullet} \;\simeq\; H^{\bullet,\bullet}_{\overline{\partial}}(X) \;\Rightarrow\; H^{\bullet}_{dR}(X;\mathbb{C}) \;,$$

from which one gets the Frölicher inequality, [Frö55, Theorem 2]: for every $k \in \mathbb{N}$,

$$\dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C}) \;\leq\; \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) \;.$$

On a complex manifold, a "bridge" between the Dolbeault and the de Rham cohomology is provided, in a sense, by the *Bott-Chern cohomology*,

$$H^{\bullet,\bullet}_{BC}(X) \;:=\; \frac{\ker\partial \cap \ker\overline{\partial}}{\mathrm{im}\,\partial\overline{\partial}} \;,$$

and the *Aeppli cohomology*,

$$H^{\bullet,\bullet}_{A}(X) \;:=\; \frac{\ker\partial\overline{\partial}}{\mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial}} \;.$$

In fact, the identity induces the maps of (bi-)graded $\mathbb{C}$-vector spaces

$$\begin{tikzcd} & H^{\bullet,\bullet}_{BC}(X) \arrow[dl] \arrow[d] \arrow[dr] & \\ H^{\bullet,\bullet}_{\partial}(X) \arrow[dr] & H^{\bullet}_{dR}(X;\mathbb{C}) \arrow[d] & H^{\bullet,\bullet}_{\overline{\partial}}(X) \arrow[dl] \\ & H^{\bullet,\bullet}_{A}(X) & \end{tikzcd}$$

which, in general, are neither injective nor surjective.

We recall that, whenever X is compact, the Hodge theory can be performed also for Bott-Chern and Aeppli cohomologies, [Sch07, §2], yielding their finite-dimensionality; more precisely, one has that, on a compact complex manifold X of complex dimension n endowed with a Hermitian metric,

$$H^{\bullet,\bullet}_{BC}(X) \;\simeq\; \tilde{\Delta}_{BC} \qquad \text{and} \qquad H^{\bullet,\bullet}_{A}(X) \;\simeq\; \tilde{\Delta}_{A} \;,$$

where $\tilde{\Delta}_{BC}$ and $\tilde{\Delta}_{A}$ are 4^{th} order self-adjoint elliptic differential operators; furthermore, the Hodge-$*$-operator associated to any Hermitian metric on X induces an isomorphism $H^{p,q}_{BC}(X) \simeq H^{n-q,n-p}_{A}(X)$, for every $p,q \in \mathbb{N}$.

By the definitions, the map $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$ is injective if and only if every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact: a compact complex manifold fulfilling this property is said to satisfy the *$\partial\overline{\partial}$-Lemma*; see [DGMS75] by P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan, where consequences of the validity of the $\partial\overline{\partial}$-Lemma on the real homotopy type of a compact complex manifold are investigated. When the $\partial\overline{\partial}$-Lemma holds, it turns out that actually all the above maps are isomorphisms, [DGMS75, Lemma 5.15, Remark 5.16, 5.21]: in particular, one gets a decomposition

$$H^{\bullet}_{dR}(X;\mathbb{C}) \;\simeq\; \bigoplus H^{\bullet,\bullet}_{\overline{\partial}}(X) \qquad \text{such that} \qquad H^{\bullet_1,\bullet_2}_{\overline{\partial}}(X) \;\simeq\; \overline{H^{\bullet_2,\bullet_1}_{\overline{\partial}}(X)} \;.$$

A very remarkable property of compact Kähler manifolds is that they satisfy the $\partial\overline{\partial}$-Lemma, [DGMS75, Lemma 5.11]: this follows from the Kähler identities, which can be proven as a consequence of the fact that the Kähler metrics osculate to order 2 the standard Hermitian metric of $\mathbb{C}^n$ at every point. Therefore, the above decomposition holds true, in particular, for compact Kähler manifolds, [Wei58, Théorème IV.3].

In particular, if X is a compact complex manifold satisfying the $\partial\overline{\partial}$-Lemma, then, for every $k \in \mathbb{N}$,

$$\dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C}) \;=\; \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{BC}(X) \;.$$

In the first chapter, we study cohomological properties of compact complex manifolds, studying in particular the Bott-Chern and Aeppli cohomologies, and their relation with the $\partial\overline{\partial}$-Lemma.

In fact, the first result we prove states a Frölicher-type inequality for the Bott-Chern and Aeppli cohomologies, which provides also a characterization of the compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma just in terms of the dimensions of the Bott-Chern cohomology groups, [AT12b, Theorem A, Theorem B]; a key tool in the proof of the Frölicher-type inequality relies on exact sequences by J. Varouchas, [Var86]. More precisely, we state the following result.

Theorem (see Theorem 1.22 and Theorem 1.25). *Let X be a compact complex manifold. Then, for every $k \in \mathbb{N}$, the following inequality holds:*

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right) \;\geq\; 2\, \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C}) \;.$$

The equality

$$\dim_{\mathbb{C}} H^{k}_{BC}(X) + \dim_{\mathbb{C}} H^{k}_{A}(X) \;=\; 2\, \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C})$$

holds for every $k \in \mathbb{N}$ if and only if X satisfies the $\partial\overline{\partial}$-Lemma.

Note that the equality $\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) = \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C})$ for every $k \in \mathbb{N}$ (which is equivalent to the degeneration of the Hodge and Frölicher spectral sequence at the first step, $E_1 \simeq E_\infty$) is not sufficient to let X satisfy the $\partial\overline{\partial}$-Lemma: in some sense, the above result states that the Bott-Chern cohomology, together with its dual, the Aeppli cohomology, encodes "more informations" on the double complex $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ than just the Dolbeault cohomology.

As a straightforward consequence of the previous theorem, we obtain another proof, see [AT12b, Corollary 2.7], of the following result, see [Voi02, Proposition 9.21], [Wu06, Theorem 5.12], [Tom08, §B].

Corollary (see Corollary 1.28). *Satisfying the $\partial\overline{\partial}$-Lemma is a stable property under small deformations of the complex structure, that is, if $\left\{X_t\right\}_{t\in B}$ is a complex-analytic family of compact complex manifolds and X_{t_0} satisfies the $\partial\overline{\partial}$-Lemma for some $t_0 \in B$, then X_t satisfies the $\partial\overline{\partial}$-Lemma for every t in an open neighbourhood of t_0 in B.*

A class of manifolds that turns out to be particularly interesting in non-Kähler geometry, as a fruitful source of examples, is provided by the class of *nilmanifolds*, and, more in general, of *solvmanifolds*, namely, compact quotients of connected simply-connected nilpotent, respectively solvable, Lie groups by co-compact discrete subgroups. In fact, on the one hand, non-tori nilmanifolds admit no Kähler structure, [BG88, Theorem A], [Has89, Theorem 1, Corollary], and, on the other hand, focusing on left-invariant geometric structures on solvmanifolds, one can often reduce their study at the level of the associated Lie algebra; this turns out to hold true, in particular, for the de Rham cohomology of completely-solvable solvmanifolds, [Nom54, Hat60], and for the Dolbeault cohomology of nilmanifolds endowed with certain left-invariant complex structures, [Sak76, CFGU00, CF01, Rol09a, Rol11a], see, e.g., [Con06, Rol11a].

More precisely, on a nilmanifold $X = \Gamma \backslash G$, the inclusion of the subcomplex composed of the G-left-invariant forms on X (which is isomorphic to the complex $(\wedge^\bullet\mathfrak{g}^*,\, \mathrm{d})$, where $\mathfrak{g}$ is the associated Lie algebra) turns out to be a quasi-isomorphism, [Nom54, Theorem 1], that is,

$$i\colon H^{\bullet}_{dR}\left(\mathfrak{g};\mathbb{R}\right) \;:=\; H^{\bullet}\left(\wedge^\bullet\mathfrak{g}^*,\, \mathrm{d}\right) \;\stackrel{\simeq}{\to}\; H^{\bullet}_{dR}(X;\mathbb{R}) \;;$$

a similar result holds true also for completely-solvable solvmanifolds, [Hat60, Corollary 4.2], and for the Dolbeault cohomology of nilmanifolds endowed with left-invariant complex structures belonging to certain classes, [Sak76, Theorem 1], [CFGU00, Main Theorem], [CF01, Theorem 2, Remark 4], [Rol09a, Theorem 1.10], [Rol11a, Corollary 3.10].

As a matter of notation, denote by $H^{\bullet,\bullet}_{\sharp}\left(\mathfrak{g}_{\mathbb{C}}\right)$, for $\sharp \in \left\{\overline{\partial},\, \partial,\, BC,\, A\right\}$, the cohomology of the corresponding subcomplex of G-left-invariant forms on a solvmanifold $X = \Gamma \backslash G$, with Lie algebra $\mathfrak{g}$, endowed with a G-left-invariant complex structure. The following result states a Nomizu-type theorem also for the Bott-Chern and Aeppli cohomologies, [Ang11, Theorem 3.7, Theorem 3.8, Theorem 3.9].

Theorem (see Theorem 1.37, Theorem 1.39, Remark 1.41, and Theorem 1.42). *Let $X = \Gamma \backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Suppose that the inclusions of the subcomplexes of G-left-invariant forms on X into the corresponding complexes of differential forms on X yield the isomorphisms*

$$i\colon H^{\bullet}_{dR}(\mathfrak{g};\mathbb{C}) \stackrel{\simeq}{\to} H^{\bullet}_{dR}(X;\mathbb{C}) \qquad \text{and} \qquad i\colon H^{\bullet,\bullet}_{\overline{\partial}}\left(\mathfrak{g}_{\mathbb{C}}\right) \stackrel{\simeq}{\to} H^{\bullet,\bullet}_{\overline{\partial}}(X) \;;$$

in particular, this holds true if one of the following conditions holds:

- *X is holomorphically parallelizable;*

- *J is an Abelian complex structure;*
- *J is a nilpotent complex structure;*
- *J is a rational complex structure;*
- *$\mathfrak{g}$ admits a torus-bundle series compatible with J and with the rational structure induced by Γ;*
- *$\dim_{\mathbb{R}} \mathfrak{g} = 6$ and $\mathfrak{g}$ is not isomorphic to $\mathfrak{h}_7 := \left(0^3,\, 12,\, 13,\, 23\right)$.*

Then also

$$i\colon H^{\bullet,\bullet}_{BC}\left(\mathfrak{g}_{\mathbb{C}}\right) \stackrel{\simeq}{\to} H^{\bullet,\bullet}_{BC}(X) \qquad \text{and} \qquad i\colon H^{\bullet,\bullet}_{A}\left(\mathfrak{g}_{\mathbb{C}}\right) \stackrel{\simeq}{\to} H^{\bullet,\bullet}_{A}(X)$$

are isomorphisms.

Furthermore, if $\mathcal{C}\left(\mathfrak{g}\right)$ denotes the set of G-left-invariant complex structures on X, then the set

$$\mathcal{U} \;:=\; \left\{J \in \mathcal{C}\left(\mathfrak{g}\right) \;:\; i\colon H^{\bullet,\bullet}_{\sharp_J}\left(\mathfrak{g}_{\mathbb{C}}\right) \;\stackrel{\simeq}{\hookrightarrow}\; H^{\bullet,\bullet}_{\sharp_J}\left(X\right)\right\}$$

is open in $\mathcal{C}\left(\mathfrak{g}\right)$, for $\sharp \in \{\partial,\, \overline{\partial},\, BC,\, A\}$.

The above result allows to explicitly compute the Bott-Chern cohomology for the *Iwasawa manifold*

$$\mathbb{I}_3 \;:=\; \mathbb{H}\left(3;\mathbb{Z}\left[\mathrm{i}\right]\right)\backslash\, \mathbb{H}(3;\mathbb{C})$$

and for its small deformations, where

$$\mathbb{H}(3;\mathbb{C}) \;:=\; \left\{\begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3;\mathbb{C}) \;:\; z^1,\, z^2,\, z^3 \in \mathbb{C}\right\} \quad \text{and} \quad \mathbb{H}\left(3;\mathbb{Z}\left[\mathrm{i}\right]\right) \;:=\; \mathbb{H}(3;\mathbb{C}) \cap \mathrm{GL}\left(3;\mathbb{Z}\left[\mathrm{i}\right]\right) \;.$$

The Iwasawa manifold is one of the simplest example of compact non-Kähler complex manifold: as an example of a complex-parallelizable manifold, it has been studied by I. Nakamura, [Nak75], who computed its Kuranishi space and classified the small deformations of $\mathbb{I}_3$ by means of the dimensions of their Dolbeault cohomology groups.

In §1.4.4, [Ang11, §5.3], we explicitly compute the Bott-Chern cohomology of the small deformations of the Iwasawa manifold, showing that it makes possible to give a finer classification of the small deformations $\left\{X_{\mathbf{t}}\right\}_{\mathbf{t}\in\Delta(\mathbf{0},\varepsilon)\subset\mathbb{C}^6}$ of $\mathbb{I}_3$ than the Dolbeault cohomology: more precisely, classes *(ii)* and *(iii)* in I. Nakamura's classification [Nak75, §3] are further subdivided into subclasses *(ii.a)* and *(ii.b)*, respectively *(iii.a)* and *(iii.b)*, according to the value of $\dim_{\mathbb{C}} H^{2,2}_{BC}\left(X_{\mathbf{t}}\right)$.

Another class that could provide several interesting examples is given by complex orbifolds of the type $\tilde{X} = X \big/ G$, where X is a complex manifold and G is a finite group of biholomorphisms of X. Orbifolds of such a global-quotient-type have been considered and studied, e.g., by D. D. Joyce in constructing examples of compact 7-dimensional manifolds with holonomy G_2, [Joy96b] and [Joy00, Chapters 11-12], and examples of compact 8-dimensional manifolds with holonomy Spin(7), [Joy96a, Joy99] and [Joy00, Chapters 13-14].

One can define the space of differential forms $\wedge^{\bullet,\bullet}\tilde{X}$ on a complex orbifold of the type $\tilde{X} = X\big/G$ as the space of G-invariant differential forms on X; hence, one can define the de Rham, Dolbeault, Bott-Chern, and Aeppli cohomologies also for $\tilde{X}$. Analogously, one can define the space of currents $\mathcal{D}^{\bullet,\bullet}\tilde{X}$ on $\tilde{X}$ as the space of G-invariant currents on X, as well as a Hermitian metric on $\tilde{X}$ as a G-invariant Hermitian metric on X.

As a first tool to investigate the Bott-Chern and Aeppli cohomologies of compact complex orbifolds of global-quotient-type, we obtain the following result.

Theorem (see Theorem 1.55). *Let $\tilde{X} = X\big/G$ be a compact complex orbifold of complex dimension n, where X is a complex manifold and G is a finite group of biholomorphisms of X. Then, for any $p, q \in \mathbb{N}$, there are canonical isomorphisms*

$$H^{p,q}_{BC}(\tilde{X}) \;\simeq\; \frac{\ker\left(\partial\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p,q+1}\tilde{X}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)} \;.$$

Furthermore, given a Hermitian metric on X, there are canonical isomorphisms

$$H^{\bullet,\bullet}_{BC}(\tilde{X}) \;\simeq\; \ker\tilde{\Delta}_{BC} \qquad \text{and} \qquad H^{\bullet,\bullet}_{A}(\tilde{X}) \;\simeq\; \ker\tilde{\Delta}_{A} \;.$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet_1,\bullet_2}_{BC}(\tilde{X}) \;\simeq\; H^{n-\bullet_2,n-\bullet_1}_{A}(\tilde{X}) \;.$$

In the second chapter, we do not require the integrability of the almost-complex structure, and we study cohomological properties of almost-complex manifolds, that is, differentiable manifolds endowed with a (possibly non-integrable) almost-complex structure. In this case, the Dolbeault cohomology is not defined. However, following T.-J. Li and W. Zhang [LZ09], one can consider, for every $p, q \in \mathbb{N}$, the subgroups

$$H_J^{(p,q),(q,p)}(X;\mathbb{R}) := \left\{[\alpha] \in H_{dR}^{p+q}(X;\mathbb{R}) \;:\; \alpha \in \left(\wedge^{p,q}X \oplus \wedge^{q,p}X\right) \cap \wedge^{p+q}X\right\} \subseteq H_{dR}^{p+q}(X;\mathbb{R}) ,$$

and their complex counterpart

$$H_J^{(p,q)}(X;\mathbb{C}) := \left\{[\alpha] \in H_{dR}^{p+q}(X;\mathbb{C}) \;:\; \alpha \in \wedge^{p,q}X\right\} \subseteq H_{dR}^{p+q}(X;\mathbb{C}) .$$

If X is a compact Kähler manifold, then $H_J^{(p,q)}(X;\mathbb{C}) \simeq H_{\overline{\partial}}^{p,q}(X)$ for every $p, q \in \mathbb{N}$, [DLZ10, Lemma 2.15, Theorem 2.16]; therefore these subgroups can be considered, in a sense, as a generalization of the Dolbeault cohomology groups to the non-Kähler, or to the non-integrable, case.

Two remarks need to be pointed out. Firstly, note that, in general, neither the equality in

$$\sum_{\substack{p+q=k \\ p\leq q}} H_J^{(p,q),(q,p)}(X;\mathbb{R}) \subseteq H_{dR}^{p+q}(X;\mathbb{R}) , \qquad \text{or} \qquad \sum_{p+q=k} H_J^{(p,q)}(X;\mathbb{C}) \subseteq H_{dR}^{p+q}(X;\mathbb{C}) ,$$

holds, nor the sum is direct, nor there are relations between the equality holding and the sum being direct, see, e.g., Proposition 2.12. Hence, one may be interested in studying compact almost-complex manifolds for which one of the above properties holds, at least for a fixed $k \in \mathbb{N}$, see [LZ09, DLZ10, DLZ11, FT10, AT11, AT12a, Zha11, ATZ12, DZ11, TWZZ11, HMT11, LT12, DLZ12]. A remarkable result by T. Drăghici, T.-J. Li, and W. Zhang, [DLZ10, Theorem 2.3], states that every almost-complex structure J on a compact 4-dimensional manifold X^4 satisfies the cohomological decomposition

$$H_{dR}^2\left(X^4;\mathbb{R}\right) = H_J^{(2,0),(0,2)}\left(X^4;\mathbb{R}\right) \oplus H_J^{(1,1)}\left(X^4;\mathbb{R}\right) .$$

Secondly, note that $J\lfloor_{\wedge^2X}$ satisfies $\left(J\lfloor_{\wedge^2X}\right)^2 = \mathrm{id}_{\wedge^2X}$, therefore the above subgroups of $H_{dR}^2(X;\mathbb{R})$ can be interpreted as the subgroup represented by J-invariant forms,

$$H_J^+(X) := H_J^{(1,1)}(X;\mathbb{R}) = \left\{[\alpha] \in H_{dR}^2(X;\mathbb{R}) \;:\; J\alpha = \alpha\right\} ,$$

and the subgroup represented by J-anti-invariant forms,

$$H_J^-(X) := H_J^{(2,0),(0,2)}(X;\mathbb{R}) = \left\{[\alpha] \in H_{dR}^2(X;\mathbb{R}) \;:\; J\alpha = -\alpha\right\} .$$

Note also that, if g is any Hermitian metric on X whose associated $(1,1)$-form $\omega := g(J\cdot, \cdot\cdot) \in \wedge^{1,1}X \cap \wedge^2X$ is d-closed (namely, g is an *almost-Kähler* metric on X), then $[\omega] \in H_J^+(X)$.

In fact, T.-J. Li and W. Zhang's interest in studying such subgroups and $\mathcal{C}^\infty$*-pure-and-full* almost-complex structures (that is, almost-complex structures for which the decomposition

$$H_{dR}^2(X;\mathbb{R}) = H_J^+(X) \oplus H_J^-(X)$$

holds, [LZ09, Definition 2.2, Definition 2.3, Lemma 2.2]) arises in investigating the symplectic cones of an almost-complex manifolds, that is, the *J-tamed cone*

$$\mathcal{K}_J^t := \left\{[\omega] \in H_{dR}^2(X;\mathbb{R}) \;:\; \omega_x\left(v_x, J_xv_x\right) > 0 \text{ for every } v_x \in T_xX \setminus \{0\} \text{ and for every } x \in X\right\}$$

and the *J-compatible cone*

$$\mathcal{K}_J^c := \left\{[\omega] \in H_{dR}^2(X;\mathbb{R}) \;:\; \omega_x\left(v_x, J_xv_x\right) > 0 \text{ for every } v_x \in T_xX \setminus \{0\} \text{ and for every } x \in X, \text{ and } J\omega = \omega\right\} .$$

Indeed, they proved in [LZ09, Theorem 1.1] that, given a $\mathcal{C}^\infty$-pure-and-full almost-Kähler structure on a compact manifold X, the J-anti-invariant subgroup $H_J^-(X)$ of $H_{dR}^2(X;\mathbb{R})$ measures the quantitative difference between the J-tamed cone and the J-compatible cone, namely,

$$\mathcal{K}_J^t = \mathcal{K}_J^c \oplus H_J^-(X) .$$

A natural question concerns the qualitative comparison between the tamed cone and the compatible cone: more precisely, one could ask whether, whenever an almost-complex structure J admits a J-tamed symplectic form, there exists also a J-compatible symplectic form. This turns out to be false, in general, for non-integrable almost-complex structures in dimension greater than 4, [MT00, Tom02]; on the other hand, it is not known whether, for almost-complex structures on compact 4-dimensional manifolds, as asked by S. K. Donaldson, [Don06, Question 2], or for complex structures on compact manifolds of complex dimension greater than or equal to 3, as asked by T.-J. Li and W. Zhang, [LZ09, page 678], and by J. Streets and G. Tian, [ST10, Question 1.7], it holds that $\mathcal{K}_J^c$ is non-empty if and only if $\mathcal{K}_J^t$ is non-empty. We prove the following result, stating that no counterexample can be found among 6-dimensional non-tori nilmanifolds endowed with left-invariant complex structures, [AT11, Theorem 3.3]; note that the same holds true, more in general, for higher dimensional nilmanifolds, as proven by N. Enrietti, A. Fino, and L. Vezzoni, [EFV12, Theorem 1.3].

Theorem (see Theorem 2.67). *Let $X = \Gamma \backslash G$ be a 6-dimensional nilmanifold endowed with a G-left-invariant complex structure J. If X is not a torus, then there is no J-tamed symplectic structure on X.*

One can study further cones in cohomology, which are related to special metrics, other than Kähler metrics; a key tool is provided by the theory of cone structures on differentiable manifolds developed by D. P. Sullivan, [Sul76]. In order to compare, in particular, the cone associated to *balanced metrics* (that is, Hermitian metrics whose associated $(1,1)$-form is co-closed, [Mic82, Definition 1.4, Theorem 1.6]) and the cone associated to *strongly-Gauduchon metrics* (that is, Hermitian metrics whose associated $(1,1)$-form ω satisfies the condition that $\partial\left(\omega^{\dim_{\mathbb{C}} X-1}\right)$ is $\overline{\partial}$-exact [Pop09, Definition 3.1]), we give the following result, [AT12a, Theorem 2.9], which is the semi-Kähler counterpart of [LZ09, Theorem 1.1]. (We refer to §2.4.3 for the definitions of the cones $\mathcal{K}b_J^t$ and $\mathcal{K}b_J^c$ on a manifold X endowed with an almost-complex structure J.)

Theorem (see Theorem 2.74). *Let X be a compact $2n$-dimensional manifold endowed with an almost-complex structure J. Assume that $\mathcal{K}b_J^c \neq \varnothing$ (that is, there exists a semi-Kähler structure on X) and that $0 \notin \mathcal{K}b_J^t$. Then*

$$\mathcal{K}b_J^t \cap H_J^{(n-1,n-1)}(X;\mathbb{R}) \;=\; \mathcal{K}b_J^c$$

and

$$\mathcal{K}b_J^c + H_J^{(n,n-2),(n-2,n)}(X;\mathbb{R}) \;\subseteq\; \mathcal{K}b_J^t \;.$$

Moreover, if the equality $H_{dR}^{2n-2}(X;\mathbb{R}) = H_J^{(n,n-2),(n-2,n)}(X;\mathbb{R}) + H_J^{(n-1,n-1)}(X;\mathbb{R})$ holds, then

$$\mathcal{K}b_J^c + H_J^{(n,n-2),(n-2,n)}(X;\mathbb{R}) \;=\; \mathcal{K}b_J^t \;.$$

In order to better understand cohomological properties of compact almost-complex manifolds, and in view of the Hodge decomposition theorem for compact Kähler manifolds, it could be interesting to investigate the subgroups $H_J^{(p,q),(q,p)}(X;\mathbb{R})$ for almost-complex manifolds endowed with special structures. For example, we prove the following result, [ATZ12, Proposition 4.1], providing a strong difference between the Kähler case and the almost-Kähler case.

Proposition (see Proposition 2.42). *The differentiable manifold X underlying the Iwasawa manifold $\mathbb{I}_3 := \mathbb{H}(3;\mathbb{Z}[\mathrm{i}]) \backslash \mathbb{H}(3;\mathbb{C})$ admits a non-$\mathcal{C}^\infty$-pure-and-full almost-Kähler structure.*

A further study on almost-Kähler structures $(J,\,\omega,\,g)$ on a compact $2n$-dimensional manifold X yields the following result, [ATZ12, Theorem 2.3], which relates $\mathcal{C}^\infty$-pure-and-fullness with the *Lefschetz-type property on 2-forms* firstly considered by W. Zhang, that is, the property that the Lefschetz operator

$$\omega^{n-2} \wedge \cdot \colon \; \wedge^2 X \to \wedge^{2n-2} X$$

takes g-harmonic 2-forms to g-harmonic $(2n-2)$-forms.

Theorem (see Theorem 2.35). *Let X be a compact manifold endowed with an almost-Kähler structure $(J,\,\omega,\,g)$. Suppose that there exists a basis of $H_{dR}^2(X;\mathbb{R})$ represented by g-harmonic 2-forms which are of pure type with respect to J. Then the Lefschetz-type property on 2-forms holds on X.*

As a tool to study explicit examples, we provide a Nomizu-type theorem for the subgroups $H_J^{(p,q),(q,p)}(X;\mathbb{R})$ of a completely-solvable solvmanifold $X = \Gamma \backslash G$ endowed with a G-left-invariant almost-complex structure J, [ATZ12, Theorem 5.4], see Proposition 2.19, and Corollary 2.20.

A remarkable result by K. Kodaira and D. C. Spencer states that the Kähler property on compact complex manifolds is stable under small deformations of the complex structure, [KS60, Theorem 15]: more precisely, it states that, given a compact complex manifold admitting a Kähler structure, every small deformation still admits a Kähler structure; it can be proven as a consequence of the semi-continuity properties for the dimensions of the cohomology groups of a compact Kähler manifold. Hence, a natural question in non-Kähler geometry is to investigate the (in)stability of weaker properties than being Kähler. As a first result in this direction, L. Alessandrini and G. Bassanelli proved that, given a compact complex manifold, the property of admitting a *balanced metric* (that is, a Hermitian metric whose associated $(1,1)$-form is co-closed) is not stable under small deformations of the complex structure, [AB90, Proposition 4.1]; on the other hand, they proved that the class of balanced manifolds is stable under modifications, [AB96, Corollary 5.7]. Another result in this context is the stability of the property of satisfying the $\partial\overline{\partial}$-Lemma under small deformations of the complex structure, as already recalled, see Corollary 1.28.

Therefore, it is natural to investigate stability properties for the cohomological decomposition by means of the subgroups $H_J^{(p,q),(q,p)}(X;\mathbb{R})$ on (almost-)complex manifolds $(X,\, J)$. More precisely, we consider the Iwasawa manifold $\mathbb{I}_3 := \mathbb{H}\left(3;\mathbb{Z}\left[\mathrm{i}\right]\right)\backslash\mathbb{H}(3;\mathbb{C})$, showing that the subgroups $H_J^{(p,q),(q,p)}(X;\mathbb{R})$ provide a cohomological decomposition for $\mathbb{I}_3$ but not for some of its small deformations, Theorem 2.49. We prove the following result, [AT11, Theorem 3.2].

Theorem (see Theorem 2.48). *The properties of being $\mathcal{C}^\infty$-pure-and-full is not stable under small deformations of the complex structure.*

More in general, one could try to study directions along which the curves of almost-complex structures on a differentiable manifold preserve the property of being $\mathcal{C}^\infty$-pure-and-full. Using a procedure by J. Lee, [Lee04, §1], to construct curves of almost-complex structures through an almost-complex structure J, by means of J-anti-invariant real 2-forms, we provide the following result, [AT11, Theorem 4.1].

Theorem (see Theorem 2.53). *There exists a compact manifold $N^6(c)$ endowed with an almost-complex structure J and a J-Hermitian metric g such that:*

(i) J is $\mathcal{C}^\infty$-pure-and-full;

(ii) each J-anti-invariant g-harmonic form gives rise to a curve $\{J_t\}_{t\in(-\varepsilon,\varepsilon)}$ of $\mathcal{C}^\infty$-pure-and-full almost-complex structures on $N^6(c)$ (where $\varepsilon>0$ is small enough);

(iii) furthermore, the function

$$(-\varepsilon,\varepsilon) \;\ni\; t\mapsto \dim_{\mathbb{R}} H_{J_t}^{(2,0),(0,2)}\left(N^6(c);\mathbb{R}\right) \;\in\; \mathbb{N}$$

is upper-semi-continuous at 0.

Another problem in deformation theory is the study of semi-continuity properties for the dimensions of the subgroups $H_J^+(X)$ and $H_J^-(X)$. As a consequence of the Hodge theory for compact 4-dimensional manifolds, T. Drăghici, T.-J. Li, and W. Zhang proved in [DLZ11, Theorem 2.6] that, given a curve $\{J_t\}_{t\in I\subseteq\mathbb{R}}$ of ($\mathcal{C}^\infty$-pure-and-full) almost-complex structures on a compact 4-dimensional manifold X, the functions

$$I \;\ni\; t\mapsto \dim_{\mathbb{R}} H_{J_t}^-(X) \;\in\; \mathbb{N} \qquad \text{and} \qquad I \;\ni\; t\mapsto \dim_{\mathbb{R}} H_{J_t}^+(X) \;\in\; \mathbb{N}$$

are, respectively, upper-semi-continuous and lower-semi-continuous.

In higher dimension this fails to be true, as we show in explicit examples. We provide hence the following result, [AT12a, Proposition 4.1, Proposition 4.3].

Proposition (see Proposition 2.55 and Proposition 2.56). *In dimension higher than* 4*, there exist compact manifolds X endowed with families $\{J_t\}_{t\in I}$ of almost-complex structures such that either the function $I\ni t\mapsto \dim_{\mathbb{R}} H_{J_t}^-(X)\in\mathbb{N}$ is not upper-semi-continuous, or the function $I\ni t\mapsto \dim_{\mathbb{R}} H_{J_t}^+(X)\in\mathbb{N}$ is not lower-semi-continuous.*

Motivated by such counterexamples, we study a stronger semi-continuity property on almost-complex manifolds (namely, that, for every d-closed J-invariant real 2-form α, there exists a d-closed J_t-invariant real 2-form $\eta_t=\alpha+\mathrm{o}(1)$, depending real-analytically in t, for $t\in(-\varepsilon,\,\varepsilon)$ with $\varepsilon>0$ small enough): we give a formal characterization of the curves of almost-complex structures satisfying such a property, see Proposition 2.57, and we provide also a counterexample to such a stronger semi-continuity property, see Proposition 2.60.

In the third chapter, motivated by the problem to study cohomological obstructions induced by special structures on differentiable manifolds, we investigate cohomological properties of symplectic manifolds, **D**-complex manifolds, and strictly p-convex domains.

We recall that compact Kähler manifolds have special cohomological properties not just in the complex framework, but also from the symplectic viewpoint. More precisely, another important result, other than the Hodge decomposition theorem, [Wei58, Théorème IV.3], is the Lefschetz decomposition theorem, [Wei58, Théorème IV.5], which states a decomposition in terms of *primitive* subgroups of the cohomology, namely,

$$H^\bullet_{dR}(X;\mathbb{C}) \;=\; \bigoplus_{r\in\mathbb{N}} L^r\left(\ker\left(\Lambda\colon H^{\bullet-2r}_{dR}(X;\mathbb{C})\to H^{\bullet-2r-2}_{dR}(X;\mathbb{C})\right)\right) \;,$$

where Λ is the adjoint operator of the Lefschetz operator $L := \omega \wedge \cdot \colon \wedge^\bullet X \to \wedge^{\bullet+2} X$ with respect to the pairing induced by ω. Hence, after having investigated cohomological properties of almost-complex manifolds, we turn our attention to cohomological properties of symplectic manifolds.

In particular, in §3.1, we provide a symplectic counterpart of T.-J. Li and W. Zhang's cohomological theory for almost-complex-manifolds, studying compact symplectic manifolds $(X,\, \omega)$ for which the Lefschetz decomposition on differential forms,

$$\wedge^\bullet X \;=\; \bigoplus_{r\in\mathbb{N}} L^r\, \mathrm{P}\wedge^{\bullet-2r} X \;,$$

(where $\mathrm{P}\wedge^\bullet X := \ker \Lambda$ is the space of *primitive* forms,) gives rise to a decomposition of the de Rham cohomology by means of the subgroups

$$H^{(r,s)}_{\omega}(X;\mathbb{R}) \;:=\; \left\{ \left[L^r\, \beta^{(s)} \right] \in H^{2r+s}_{dR}(X;\mathbb{R}) \;:\; \beta^{(s)} \in \mathrm{P}\wedge^s X \right\} \;\subseteq\; H^{2r+s}_{dR}(X;\mathbb{R}) \;.$$

In particular, we provide the following result, [AT12c, Theorem 2.6], which gives a symplectic counterpart to T. Drăghici, T.-J. Li, and W. Zhang's decomposition theorem [DLZ10, Theorem 2.3] in the almost-complex setting (in fact, without the restriction to dimension 4).

Theorem (see Theorem 3.14). *Let X be a compact manifold endowed with a symplectic structure ω. Then*

$$H^2_{dR}(X;\mathbb{R}) \quad = \quad H^{(1,0)}_{\omega}(X;\mathbb{R}) \oplus H^{(0,2)}_{\omega}(X;\mathbb{R}) \;.$$

A Nomizu-type theorem for the subgroups $H^{(r,s)}_{\omega}(X;\mathbb{R})$ of a completely-solvable solvmanifold $X = \Gamma\backslash G$ endowed with a G-left-invariant symplectic structure ω is provided, see Proposition 3.18, giving an useful tool in order to investigate explicit examples.

In a sense, **D**-complex Geometry provides a "hyperbolic analogue" of Complex Geometry. An *almost-***D***-complex* structure is, by definition, the datum of an endomorphism $K \in \mathrm{End}(TX)$ of the tangent bundle of a differentiable manifold X such that $K^2 = \mathrm{id}_{TX}$ and with the additional property that the eigen-bundles T^+X and T^-X have the same rank; a natural notion of *integrability* can be defined by requiring that the two distributions T^+X and T^-X are involutive. Many connections between **D**-complex Geometry and other problems both in Mathematics and Physics (in particular, concerning product structures, bi-Lagrangian geometry, and optimal transport theory) have been investigated in the last years: see, e.g., [HL83, AMT09, CMMS04, CMMS05, CM09, CFAG96, KMW10, ABDMO05, AS05, Kra10, Ros12a, Ros12b] and the references therein for further details on **D**-complex structures and motivations for their study.

We study cohomological decomposition for compact manifolds X endowed with (almost-)**D**-complex structures K. Note that the elliptic theory in the complex setting has not a **D**-complex counterpart: for example, a **D**-complex counterpart of the Dolbeault cohomology is possibly infinite-dimensional, even if the manifold is compact. This fact makes natural to consider the **D**-complex counterpart $H^{(p,q)}_{K}(X;\mathbb{R})$ of T.-J. Li and W. Zhang's subgroups $H^{(p,q),(q,p)}_{J}(X;\mathbb{R})$ as a possible substitute of the **D**-Dolbeault cohomology, and hence to study the subgroups

$$H^{2\,+}_{K}(X;\mathbb{R}) \;:=\; \left\{ [\alpha] \in H^2_{dR}(X;\mathbb{R}) \;:\; K\alpha = \alpha \right\} \;\subseteq\; H^2_{dR}(X;\mathbb{R})$$

and

$$H^{2\,-}_{K}(X;\mathbb{R}) \;:=\; \left\{ [\alpha] \in H^2_{dR}(X;\mathbb{R}) \;:\; K\alpha = -\alpha \right\} \;\subseteq\; H^2_{dR}(X;\mathbb{R}) \;.$$

Nevertheless, several important differences arise between the complex and the **D**-complex cases. For example, after having stated and proved a Nomizu-type result for the subgroups $H^{(p,q)}_{K}(X;\mathbb{R})$ of a completely-solvable solvmanifold $X = \Gamma\backslash G$ endowed with a G-left-invariant **D**-complex structure K, we are able to prove the following result, [AR12, Proposition 3.3], which turns out to be very different from the complex case, see [LZ09, Proposition 2.1], or [DLZ10, Lemma 2.15, Theorem 2.16]. (Recall that a **D**-*Kähler* structure on a **D**-complex manifold is the datum of an anti-invariant symplectic form with respect to the **D**-complex structure.)

Proposition (see Proposition 3.34). *Admitting a* **D***-Kähler structure is not a sufficient condition for either the sum*

$$H^{2\,+}_{K}(X;\mathbb{R}) + H^{2\,-}_{K}(X;\mathbb{R}) \;\subseteq\; H^2_{dR}(X;\mathbb{R})$$

being direct, or the equality holding.

A partial **D**-complex counterpart of T. Drăghici, T.-J. Li, and W. Zhang's decomposition theorem [DLZ10, Theorem 2.3] is provided by the following result, [AR12, Theorem 3.17].

Theorem (see Theorem 3.47). *Every left-invariant* $\mathbf{D}$*-complex structure on a* 4*-dimensional nilmanifold satisfies the cohomological decomposition*

$$H^2_{dR}(X;\mathbb{R}) \;=\; H^{2\,+}_{K}(X;\mathbb{R}) \oplus H^{2\,-}_{K}(X;\mathbb{R}) \;.$$

Note that the hypothesis in Theorem 3.47 can not be weakened, as Example 3.32 and Example 3.33, Example 3.49, and Example 3.35 show.

Concerning deformations of the $\mathbf{D}$-complex structure, we provide another strong difference with the complex case: in contrast with the stability theorem of K. Kodaira and D. C. Spencer, [KS60, Theorem 15], we prove the following result in the $\mathbf{D}$-complex context, [AR12, Theorem 4.2].

Theorem (see Theorem 3.50). *The property of being* $\mathbf{D}$*-Kähler is not stable under small deformations of the* $\mathbf{D}$*-complex structure.*

Analogously to Theorem 2.48 for almost-complex structures, we provide also the following instability result, [AR12, Proposition 4.3].

Proposition (see Proposition 3.51). *The properties of either the sum*

$$H^{2\,+}_{K}(X;\mathbb{R}) + H^{2\,-}_{K}(X;\mathbb{R}) \;\subseteq\; H^2_{dR}(X;\mathbb{R})$$

being direct, or the equality holding are not stable under small deformations of the $\mathbf{D}$*-complex structure.*

Finally, we prove that, even in the $\mathbf{D}$-complex case, no general result on semi-continuity holds for the dimensions of the K-(anti-)invariant subgroups of the de Rham cohomology, [AR12, Proposition 4.6].

Proposition (see Proposition 3.54). *Let* X *be a compact manifold and let* $\{K_t\}_{t\in I\subseteq\mathbb{R}}$ *be a curve of* $\mathbf{D}$*-complex structures on* X*. Then, in general, the functions*

$$I \ni t \mapsto \dim_{\mathbb{R}} H^{2\,+}_{K_t}(X;\mathbb{R}) \in \mathbb{N} \qquad \text{and} \qquad I \ni t \mapsto \dim_{\mathbb{R}} H^{2\,-}_{K_t}(X;\mathbb{R}) \in \mathbb{N}$$

are not upper-semi-continuous or lower-semi-continuous.

Finally, motivated by A. Andreotti and H. Grauert's vanishing result for the higher Dolbeault cohomology groups of a *q-complete* domain in $\mathbb{C}^n$ (that is, a domain in $\mathbb{C}^n$ admitting a smooth proper exhaustion function whose Levi form has at least $n-q+1$ positive eigen-values), we turn our interest to study cohomological properties of Riemannian manifolds endowed with exhaustion functions whose Hessian satisfies positivity conditions.

In particular, a first case to be considered is the case of *strictly* p*-convex* domains in $\mathbb{R}^n$ in the sense of F. R. Harvey and H. B. Lawson, [HL12, HL11], that is, domains in $\mathbb{R}^n$ admitting a smooth proper exhaustion function u such that, at every point, every sum of p different eigenvalues of the Hessian of u is positive. Adapting the L^2-techniques developed by L. Hörmander, [Hör65], and used also by A. Andreotti and E. Vesentini, [AV65a, AV65b], (and which could be hopefully applied in a wider context,) we give a different proof of a vanishing result following from J.-P. Sha's theorem [Sha86, Theorem 1], and from H. Wu's theorem [Wu87, Theorem 1], for the de Rham cohomology of strictly p-convex domains in $\mathbb{R}^n$ in the sense of F. R. Harvey and H. B. Lawson; more precisely, the following result holds, [AC12, Theorem 3.1], see [Sha86, Theorem 1], [Wu87, Theorem 1], [HL11, Proposition 5.7].

Theorem (see Theorem 3.67 and Theorem 3.68). *Let* X *be a strictly* p*-convex domain in* $\mathbb{R}^n$*, and fix* $k \in \mathbb{N}$ *such that* $k \geq p$*. Then, every* d*-closed* k*-form is* d*-exact, that is,*

$$H^k_{dR}(X;\mathbb{R}) \;=\; \{0\}$$

for every $k \geq p$*.*

The plan of the thesis is as follows.

In Chapter 0, which contains no original material, we collect the basic notions concerning almost-complex, complex, and symplectic structures, we recall the main results on Hodge theory for Kähler manifolds, and we summarize the classical results on deformations of complex structures, on currents and de Rham homology, and on solvmanifolds.

In Chapter 1, we study cohomological properties of compact complex manifolds, and in particular the Bott-Chern cohomology, [AT12b, Ang11]. By using exact sequences introduced by J. Varouchas, [Var86], we prove a Frölicher-type inequality for the Bott-Chern cohomology, Theorem 1.22, which also provides a characterization of the validity of the $\partial\overline{\partial}$-Lemma in terms of the dimensions of the Bott-Chern cohomology groups, Theorem 1.25. We then prove a Nomizu-type result for the Bott-Chern cohomology, showing that, for certain classes of complex structures on nilmanifolds, the Bott-Chern cohomology is completely determined by the associated Lie algebra endowed with the induced linear complex structure, Theorem 1.37, Theorem 1.39, and Theorem 1.42. As an application, in §1.4, we explicitly study the Bott-Chern and Aeppli cohomologies of the Iwasawa manifold and of its small deformations. Finally, we study the Bott-Chern cohomology of complex orbifolds of the type X/G, where X is a compact complex manifold and G a finite group of biholomorphisms of X, Theorem 1.55.

In Chapter 2, we study cohomological properties of almost-complex manifolds, [AT11, AT12a, ATZ12]. Firstly, in §2.1, we recall the notion of $\mathcal{C}^\infty$-pure-and-full almost-complex structure, which has been introduced by T.-J. Li and W. Zhang in [LZ09] in order to investigate the relations between the compatible and the tamed symplectic cones on a compact almost-complex manifold and with the aim to throw light on a question by S. K. Donaldson, [Don06, Question 2]. In particular, we are interested in studying when certain subgroups, related to the almost-complex structure, let a splitting of the de Rham cohomology of an almost-complex manifold, and their relations with cones of metric structures. In §2.2, we focus on $\mathcal{C}^\infty$-pure-and-fullness on several classes of (almost-)complex manifolds, e.g., solvmanifolds endowed with left-invariant almost-complex structures, semi-Kähler manifolds, almost-Kähler manifolds. In §2.3, we study the behaviour of $\mathcal{C}^\infty$-pure-and-fullness under small deformations of the complex structure and along curves of almost-complex structures, investigating properties of stability, Theorem 2.48, Theorem 2.53, and of semi-continuity for the dimensions of the invariant and anti-invariant subgroups of the de Rham cohomology with respect to the almost-complex structure, Proposition 2.55, Proposition 2.56, Proposition 2.57, Proposition 2.60. In §2.4, we consider the cone of semi-Kähler structures on a compact almost-complex manifold and, in particular, by adapting the results by D. P. Sullivan on cone structures, [Sul76], we compare the cones of balanced metrics and of strongly-Gauduchon metrics on a compact complex manifold, Theorem 2.74.

In Chapter 3, we study the cohomological properties of (differentiable) manifolds endowed with special structures, other than (almost-)complex structures, [AT12c, AR12, AC12]. More precisely, in Section 3.1, we investigate the cohomology of symplectic manifolds; in Section 3.2, we study cohomological decompositions on **D**-complex manifolds in the sense of F. R. Harvey and H. B. Lawson; finally, in Section 3.3, we consider domains in $\mathbb{R}^n$ endowed with a smooth proper strictly p-convex exhaustion function, and, using L^2-techniques, we give another proof of a consequence of J.-P. Sha's theorem [Sha86, Theorem 1], and of H. Wu's theorem [Wu87, Theorem 1], on the vanishing of the higher degree de Rham cohomology groups.

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Chapter 0

Preliminaries on (almost-)complex manifolds

In this preliminary chapter (which contains no original material), we summarize the basic notions and the classical results concerning (almost-)complex and symplectic structures. In particular, we start by setting some definitions and notation concerning (almost-)complex structures, §0.1, and symplectic structures, §0.2; then we recall the main results in the Hodge theory for Kähler manifolds, §0.3, and in the Kodaira, Spencer, Nirenberg, and Kuranishi theory of deformations of complex structures, §0.4; furthermore, we summarize the basic definitions and some useful facts about currents and de Rham homology, §0.5, and about solvmanifolds, §0.6, in order to set the notation for the following chapters. (As a matter of notation, unless otherwise stated, by "manifold" we mean "connected differentiable manifold", and by "compact manifold" we mean "closed manifold".)

0.1 Almost-complex structures and integrability

The tangent bundle of a complex manifold X is naturally endowed with an endomorphism $J \in \mathrm{End}(TX)$ such that $J^2 = -\,\mathrm{id}_{TX}$, satisfying a further integrability property. It is hence natural to study differentiable manifolds endowed with such an endomorphism, the so-called *almost-complex* manifolds. It turns out that the vanishing of the *Nijenhuis tensor* Nij_J characterizes the almost-complex structures J on X naturally induced by a structure of complex manifold, [NN57, Theorem 1.1].

In this section, we recall the notions of almost-complex structure, complex manifold, and Dolbeault cohomology, and some of their properties.

0.1.1 Almost-complex structures

Let X be a (differentiable) manifold endowed with an *almost-complex structure* J, namely, an endomorphism $J \in \mathrm{End}(TX)$ such that $J^2 = -\,\mathrm{id}_{TX}$.

Extending J by $\mathbb{C}$-linearity to $TX \otimes \mathbb{C}$, we get the decomposition

$$TX \otimes \mathbb{C} \;=\; T^{1,0}X \oplus T^{0,1}X \;,$$

where $T^{1,0}X$ (respectively, $T^{0,1}X$) is the sub-bundle of $TX \otimes \mathbb{C}$ given by the i-eigen-spaces (respectively, the $(-\,\mathrm{i})$-eigen-spaces) of $J \in \mathrm{End}\,(TX \otimes \mathbb{C})$: that is, for every $x \in X$,

$$\left(T^{1,0}X\right)_x \;=\; \left\{v_x - \mathrm{i}\; J_x v_x \;:\; v_x \in T_xX\right\} \;, \qquad \left(T^{0,1}X\right)_x \;=\; \left\{v_x + \mathrm{i}\; J_x v_x \;:\; v_x \in T_xX\right\} \;.$$

Considering the dual of J, again denoted by $J \in \mathrm{End}\,(T^*X)$, we get analogously a decomposition at the level of the cotangent bundle:

$$T^*X \otimes \mathbb{C} \;=\; \left(T^{1,0}X\right)^* \oplus \left(T^{0,1}X\right)^* \;,$$

where $\left(T^{1,0}X\right)^*$ (respectively, $\left(T^{0,1}X\right)^*$) is the sub-bundle of $T^*X \otimes \mathbb{C}$ given by the i-eigen-spaces (respectively, the $(-\,\mathrm{i})$-eigen-spaces) of the $\mathbb{C}$-linear extension $J \in \mathrm{End}\,(T^*X \otimes \mathbb{C})$. Extending the endomorphism J to the bundle $\wedge^\bullet\,(T^*X) \otimes \mathbb{C}$ of complex-valued differential forms, we get, for every $k \in \mathbb{N}$, the bundle decomposition

$$\wedge^k\,(T^*X) \otimes \mathbb{C} \;=\; \bigoplus_{p+q=k} \wedge^p \left(T^{1,0}X\right)^* \otimes \wedge^q \left(T^{0,1}X\right)^* \;.$$

As a matter of notation, we will denote by $\mathcal{C}^\infty\,(X; F)$ the space of smooth sections of a vector bundle F over X, and, for every $k \in \mathbb{N}$ and $p, q \in \mathbb{N}$, we will denote by $\wedge^k X := \mathcal{C}^\infty\,\left(X;\, \wedge^k\,(T^*X)\right)$ the space of smooth sections of $\wedge^k\,(T^*X)$ over X and by $\wedge^{p,q}X :=: \wedge^{p,q}_J X := \mathcal{C}^\infty \left(X;\, \wedge^p \left(T^{1,0}X\right)^* \otimes \wedge^q \left(T^{0,1}X\right)^*\right)$ the space of smooth sections of $\wedge^p \left(T^{1,0}X\right)^* \otimes \wedge^q \left(T^{0,1}X\right)^*$ over X.

Since $\mathrm{d}\left(\wedge^0 X \otimes_{\mathbb{R}} \mathbb{C}\right) \subseteq \wedge^{1,0}X \oplus \wedge^{0,1}X$ and $\mathrm{d}\left(\wedge^1 X \otimes_{\mathbb{R}} \mathbb{C}\right) \subseteq \wedge^{2,0}X \oplus \wedge^{1,1}X \oplus \wedge^{0,2}X$, since every differential form is locally a finite sum of decomposable differential forms, and by the Leibniz rule, the $\mathbb{C}$-linear extension of the exterior differential, $\mathrm{d}\colon \wedge^\bullet X \otimes \mathbb{C} \to \wedge^{\bullet+1} X \otimes \mathbb{C}$, splits into four components:

$$\mathrm{d} \;=\; A + \partial + \overline{\partial} + \bar{A}$$

where

$$A\colon \wedge^{\bullet,\bullet}X \to \wedge^{\bullet+2,\bullet-1}X \;, \qquad \partial\colon \wedge^{\bullet,\bullet}X \to \wedge^{\bullet+1,\bullet}X \;, \qquad \overline{\partial}\colon \wedge^{\bullet,\bullet}X \to \wedge^{\bullet,\bullet+1}X \;, \qquad \bar{A}\colon \wedge^{\bullet,\bullet}X \to \wedge^{\bullet-1,\bullet+2}X \;;$$

in terms of these components, the condition $\mathrm{d}^2 = 0$ is written as

$$\left\{\begin{array}{rcl} A^2 &=& 0 \\ A\,\partial + \partial\,A &=& 0 \\ A\,\overline{\partial} + \partial^2 + \overline{\partial}\,A &=& 0 \\ A\,\bar{A} + \partial\,\overline{\partial} + \overline{\partial}\,\partial + A\,\bar{A} &=& 0 \\ \partial\,\bar{A} + \overline{\partial}^2 + \bar{A}\,\partial &=& 0 \\ \bar{A}\,\overline{\partial} + \overline{\partial}\,\bar{A} &=& 0 \\ \bar{A}^2 &=& 0 \end{array}\right. .$$

0.1.2 Complex structures, and Dolbeault cohomology

If X is a complex manifold, then there is a natural almost-complex structure on X: locally, in a holomorphic coordinate chart $\left(U,\, \left\{z^\alpha =: x^{2\alpha-1} + \mathrm{i}\, x^{2\alpha}\right\}_{\alpha\in\{1,\ldots,\dim_{\mathbb{C}} X\}}\right)$, with $\left(U,\, \{x^\alpha\}_{\alpha\in\{1,\ldots,2\dim_{\mathbb{C}} X\}}\right)$ a (differential) coordinate chart, one defines, for every $\alpha \in \{1,\ldots,\dim_{\mathbb{C}} X\}$,

$$J\left(\frac{\partial}{\partial x^{2\alpha-1}}\right) \stackrel{\text{loc}}{:=} \frac{\partial}{\partial x^{2\alpha}} \;, \qquad J\left(\frac{\partial}{\partial x^{2\alpha}}\right) \stackrel{\text{loc}}{:=} -\frac{\partial}{\partial x^{2\alpha-1}} \;;$$

note that this local definition does not depend on the coordinate chart, by the Cauchy and Riemann equations.

Conversely, an almost-complex structure on a manifold X is called *integrable* if it is the natural almost-complex structure induced by a structure of complex manifold on X. The following theorem by A. Newlander and L. Nirenberg characterizes the integrable almost-complex structures on a manifold X in terms of the *Nijenhuis tensor* Nij_J, defined as

$$\mathrm{Nij}_J(\cdot,\,\cdot\cdot) \;:=\; [\cdot,\,\cdot\cdot] + J\,[J\,\cdot,\,\cdot\cdot] + J\,[\cdot,\,J\,\cdot\cdot] - [J\cdot,\,J\,\cdot\cdot] \;.$$

Theorem 0.1 ([NN57, Theorem 1.1])**.** *Let X be a manifold. An almost-complex structure J on X is integrable if and only if* $\mathrm{Nij}_J = 0$.

By a straightforward computation, the integrability of an almost-complex structure J turns out to be equivalent to the vanishing of the components A and $\bar{A}$ of the exterior differential, equivalently, to $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ being a double complex of $\mathcal{C}^\infty(X;\mathbb{C})$-modules (see, e.g., [Wel08, §2.6], [Mor07, Proposition 8.2]).

Therefore, for a complex manifold X, one can consider, for every $p \in \mathbb{N}$, the differential complex $\left(\wedge^{p,\bullet}X,\, \overline{\partial}\right)$ and its cohomology, defining the *Dolbeault cohomology*, as the bi-graded $\mathbb{C}$-vector space

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) \;:=\; \frac{\ker\overline{\partial}}{\operatorname{im}\overline{\partial}} \;.$$

For every $p, q \in \mathbb{N}$, denote by $\mathcal{A}^{p,q}_X$ the (fine) sheaf of germs of (p,q)-forms on X. For every $p \in \mathbb{N}$, denote by Ω^p_X the sheaf of germs of *holomorphic p-forms* on X, that is, the kernel sheaf of the map $\overline{\partial}\colon \mathcal{A}^{p,0}_X \to \mathcal{A}^{p,1}_X$. By the Dolbeault and Grothendieck Lemma, see, e.g., [Dem12, I.3.29], one has that

$$0 \to \Omega^p_X \to \mathcal{A}^{p,\bullet}_X$$

is a fine resolution of Ω^p_X; hence, one gets the following result.

Theorem 0.2 (Dolbeault theorem, [Dol53])**.** *Let X be a complex manifold. For every $p,\, q \in \mathbb{N}$,*

$$H^{p,q}_{\overline{\partial}}(X) \;\simeq\; \check{H}^q\left(X;\Omega^p\right) \;.$$

This gives a sheaf-theoretic interpretation of the Dolbeault cohomology. On the other hand, also an analytic interpretation can be provided.

Suppose X is a compact complex manifold of complex dimension n, and fix g a Hermitian metric on X and vol the induced volume form on X (recall that every complex manifold is orientable, see, e.g., [GH94, pages 17–18]); denote by $\omega := g(J\cdot, \cdot\cdot) \in \wedge^{1,1}X \cap \wedge^2 X$ the associated $(1,1)$-form to g. Recall that g induces a Hermitian inner product $\langle \cdot, \cdot\cdot \rangle$ on the space $\wedge^{\bullet,\bullet}X$ of global differential forms on X, and that the *Hodge-$*$-operator* associated to g is the $\mathbb{C}$-linear map

$$*\lfloor_{\wedge^{p,q}X} \colon \wedge^{p,q} X \to \wedge^{n-q,n-p} X$$

defined requiring that, for every $\alpha, \beta \in \wedge^{p,q}X$,

$$\alpha \wedge *\bar{\beta} \;=\; \langle \alpha,\, \beta \rangle \text{ vol } .$$

Define

$$\overline{\partial}^* \;:=\; - * \partial * \colon \wedge^{\bullet,\bullet} X \to \wedge^{\bullet,\bullet-1} X \;;$$

the operator $\overline{\partial}^* \colon \wedge^{\bullet,\bullet} X \to \wedge^{\bullet,\bullet-1}X$ is the adjoint of $\overline{\partial} \colon \wedge^{\bullet,\bullet} X \to \wedge^{\bullet,\bullet+1}X$ with respect to $\langle \cdot, \cdot\cdot \rangle$. Define

$$\overline{\square} \;:=\; \left[\overline{\partial}, \overline{\partial}^*\right] \;:=\; \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} \colon \wedge^{\bullet,\bullet} X \to \wedge^{\bullet,\bullet} X \;;$$

$\overline{\square}$ being a 2nd order self-adjoint elliptic differential operator, (see, e.g., [Kod05, Theorem 3.16]), one gets the following result.

Theorem 0.3 (Hodge theorem, [Hod89])**.** *Let X be a compact complex manifold endowed with a Hermitian metric. There is an orthogonal decomposition*

$$\wedge^{\bullet,\bullet}X \;=\; \ker\overline{\square} \overset{\perp}{\oplus} \overline{\partial}\wedge^{\bullet,\bullet-1} X \overset{\perp}{\oplus} \overline{\partial}^* \wedge^{\bullet,\bullet+1} X \;,$$

and hence an isomorphism

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) \;\simeq\; \ker\overline{\square} \;.$$

In particular, $\dim_{\mathbb{C}} H^{\bullet,\bullet}_{\overline{\partial}}(X) < +\infty$*.*

Note that, for any $p, q \in \mathbb{N}$, the Hodge-$*$-operator $* \colon \wedge^{p,q} X \to \wedge^{n-q,n-p}X$ sends a $\overline{\square}$-harmonic (p,q)-form ψ (that is, $\psi \in \wedge^{p,q}X$ is such that $\overline{\square}\psi = 0$) to a $\square$-harmonic $(n-q, n-p)$-form $*\psi$, where $\square := [\partial, \partial^*] := \partial\partial^* + \partial^*\partial \in \operatorname{End}(\wedge^{\bullet,\bullet}X)$ is the conjugate operator to $\overline{\square}$, and hence, by conjugating, one gets a $\overline{\square}$-harmonic $(n-p, n-q)$-form $\overline{*\psi}$. Hence, one gets the following result.

Theorem 0.4 (Serre duality, [Ser55, Théorème 4])**.** *Let X be a compact complex manifold of complex dimension n, endowed with a Hermitian metric. For every p, $q \in \mathbb{N}$, the Hodge-$*$-operator induces an isomorphism*

$$* \colon H^{p,q}_{\overline{\partial}}(X) \xrightarrow{\simeq} \overline{H^{n-p,n-q}_{\overline{\partial}}(X)} \;.$$

Since a $\overline{\partial}$-closed form is not necessarily d-closed, Dolbeault cohomology classes do not define, in general, de Rham cohomology classes, that is, in general, on a compact complex manifold, there is no natural map between the Dolbeault cohomology and the de Rham cohomology (as we will see, in the special case of compact Kähler manifolds, or more in general of compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma, the de Rham cohomology actually can be decomposed by means of the Dolbeault cohomology groups, [Wei58, Théorème IV.3], [DGMS75, Lemma 5.15, Remark 5.16, 5.21]). Nevertheless, the Frölicher inequality provides a relation between the dimension of the Dolbeault cohomology and the dimension of the de Rham cohomology; it follows by considering the Hodge and Frölicher spectral sequence, which we recall here.

The structure of double complex of $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ gives rise to two natural filtrations of $\wedge^\bullet X \otimes \mathbb{C}$, namely, (for $p, q \in N$ and for $k \in \mathbb{N}$,)

$$'F^p\left(\wedge^k X \otimes \mathbb{C}\right) \;:=\; \bigoplus_{\substack{r+s=k\\ r\geq p}} \wedge^{r,s}X \qquad \text{and} \qquad ''F^q\left(\wedge^k X \otimes \mathbb{C}\right) \;:=\; \bigoplus_{\substack{r+s=k\\ s\geq q}} \wedge^{r,s}X \;;$$

these filtrations induce two spectral sequences (see, e.g., [McC01, §2.4], [GH94, §3.5]),

$$\left\{\left(E_r^{\bullet,\bullet},\, \mathrm{d}_r\right) \;=:\; \left('E_r^{\bullet,\bullet},\, '\,\mathrm{d}_r\right)\right\}_{r\in\mathbb{N}} \qquad \text{and, respectively,} \qquad \left\{\left(''E_r^{\bullet,\bullet},\, ''\,\mathrm{d}_r\right)\right\}_{r\in\mathbb{N}} \;,$$

called *Hodge and Frölicher spectral sequences* (or *Hodge to de Rham spectral sequences*): one has

$$'E_1^{\bullet,\bullet} \simeq H^{\bullet,\bullet}_{\overline{\partial}}(X) \Rightarrow H^\bullet_{dR}(X;\mathbb{C}) \quad \text{and} \quad ''E_1^{\bullet,\bullet} \simeq H^{\bullet,\bullet}_{\partial}(X) \Rightarrow H^\bullet_{dR}(X;\mathbb{C}) .$$

An explicit description of $\{(E_r,\, \mathrm{d}_r)\}_{r\in\mathbb{N}}$ is given in [CFUG97]: for any $p,\, q \in \mathbb{N}$ and $r \in \mathbb{N}$, its terms are

$$E_r^{p,q} \simeq \frac{\mathcal{X}_r^{p,q}}{\mathcal{Y}_r^{p,q}} ,$$

where, for $r = 1$,

$$\mathcal{X}_1^{p,q} := \left\{\alpha \in \wedge^{p,q}X \;:\; \overline{\partial}\alpha = 0\right\} , \qquad \mathcal{Y}_1^{p,q} := \overline{\partial}\wedge^{p,q-1}X ,$$

and, for $r \geq 2$,

$$\begin{aligned} \mathcal{X}_r^{p,q} &:= \left\{\alpha^{p,q} \in \wedge^{p,q}X \;:\; \overline{\partial}\alpha^{p,q} = 0 \text{ and, for any } i \in \{1,\ldots,r-1\}, \text{ there exists } \alpha^{p+i,q-i} \in \wedge^{p+i,q-i}X \right. \\ &\qquad \left. \text{such that } \partial\alpha^{p+i-1,q-i+1} + \overline{\partial}\alpha^{p+i,q-i} = 0\right\} , \\ \mathcal{Y}_r^{p,q} &:= \left\{\partial\beta^{p-1,q} + \overline{\partial}\beta^{p,q-1} \in \wedge^{p,q}X \;:\; \text{for any } i \in \{2,\ldots,r-1\}, \text{ there exists } \beta^{p-i,q+i-1} \in \wedge^{p-i,q+i-1}X \right. \\ &\qquad \left. \text{such that } \partial\beta^{p-i,q+i-1} + \overline{\partial}\beta^{p-i+1,q+i-2} = 0 \text{ and } \overline{\partial}\beta^{p-r+1,q+r-2} = 0\right\} , \end{aligned}$$

see [CFUG97, Theorem 1], and, for any $r \geq 1$, the map $\mathrm{d}_r \colon E_r^{\bullet,\bullet} \to E_r^{\bullet+r,\bullet-r+1}$ is given by

$$\mathrm{d}_r \colon \{[\alpha^{p,q}] \in E_r^{p,q}\}_{p,q\in\mathbb{N}} \mapsto \left\{\left[\partial\alpha^{p+r-1,q-r+1}\right] \in E_r^{p+r,q-r+1}\right\}_{p,q\in\mathbb{N}} ,$$

see [CFUG97, Theorem 3].

As a consequence of $'E_1^{\bullet,\bullet} \simeq H^{\bullet,\bullet}_{\overline{\partial}}(X) \Rightarrow H^\bullet_{dR}(X;\mathbb{C})$, one gets the following inequality by A. Frölicher.

Theorem 0.5 (Frölicher inequality, [Frö55, Theorem 2])**.** *Let X be a compact complex manifold. Then, for every $k \in \mathbb{N}$,*

$$\dim_{\mathbb{C}} H^k_{dR}(X;\mathbb{C}) \leq \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) .$$

As a matter of notation, for $k \in \mathbb{N}$ and $p, q \in \mathbb{N}$, we will denote by $b_k := \dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R})$, respectively $h^{p,q}_{\overline{\partial}} := \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X)$, the k^{th} *Betti number*, respectively the $(p,q)^{th}$ *Hodge number* of X.

In the next chapter, we will provide a Frölicher-type inequality also for the Bott-Chern cohomology, Theorem 1.22, showing that it allows to characterize the compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma just in terms of the dimensions of the Bott-Chern cohomology and of the de Rham cohomology, Theorem 1.25.

Remark 0.6. Other than the Dolbeault cohomology, other cohomologies can be defined for a complex manifold X; more precisely, since, for every $p, q \in \mathbb{N}$,

$$\wedge^{p-1,q-1}X \stackrel{\partial\overline{\partial}}{\to} \wedge^{p,q}X \stackrel{\partial+\overline{\partial}}{\to} \wedge^{p+1,q}X \oplus \wedge^{p,q+1}X \qquad \text{and} \qquad \wedge^{p-1,q}X \oplus \wedge^{p,q-1}X \stackrel{(\partial,\overline{\partial})}{\to} \wedge^{p,q}X \stackrel{\partial\overline{\partial}}{\to} \wedge^{p+1,q+1}X$$

are complexes, one can define the *Bott-Chern cohomology* $H^{\bullet,\bullet}_{BC}(X)$ and the *Aeppli cohomology* $H^{\bullet,\bullet}_{A}(X)$ of X as

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\ker\partial \cap \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}} \qquad \text{and} \qquad H^{\bullet,\bullet}_{A}(X) := \frac{\ker\partial\overline{\partial}}{\operatorname{im}\partial + \operatorname{im}\overline{\partial}} ;$$

we refer to §1.1 for further details.

0.2 Symplectic structures

In this section, we recall some definitions and results concerning symplectic manifolds, that is, differentiable manifolds endowed with a non-degenerate d-closed 2-form. An interesting class of examples of symplectic manifolds is provided by the Kähler manifolds. Moreover, given a differentiable manifold X, its cotangent bundle T^*X is endowed with a natural symplectic structure (see, e.g., [CdS01, §2]): in fact, Symplectic Geometry has applications and motivations in the study of Hamiltonian Mechanics, see, e.g., [CdS01, Part VII].

Let X be a compact $2n$-dimensional manifold endowed with a *symplectic form*, namely, a non-degenerate d-closed 2-form $\omega \in \wedge^2 X$.

The main difference between Symplectic Geometry and Riemannian Geometry is provided by G. Darboux's theorem.

Theorem 0.7 (Darboux theorem, [Dar82])**.** *Let X be a $2n$-dimensional manifold endowed with a symplectic form ω. Then, for every $x \in X$, there exists a coordinate chart $\left(U,\, \left\{x^j\right\}_{j\in\{1,\ldots,2n\}}\right)$, with $x \in U$, such that*

$$\omega \;\stackrel{loc}{=}\; \sum_{j=1}^{n} \mathrm{d}\, x^{2j-1} \wedge \mathrm{d}\, x^{2j} \;.$$

By exploiting the parallelism with Riemannian Geometry, one can try to develop a Hodge theory also for compact symplectic manifolds, [Bry88]. The first tool that can be introduced is an analogue of the Hodge-$*$-operator.

Note that every symplectic manifold is orientable, $\frac{\omega^n}{n!}$ giving a canonical orientation.

Denote by $I\colon TX \to T^*X$ the natural isomorphism of vector bundles induced by ω, namely, $I(v)(\cdot) := \omega(v, \cdot) \in \mathrm{Hom}\,(T_xX; \mathbb{R})$, for every $v \in T_xX$ and $x \in X$. Then, for every $k \in \mathbb{N}$, the form ω gives rise to a bi-$\mathcal{C}^\infty(X;\mathbb{R})$-linear form on $\wedge^k X$ denoted by $\left(\omega^{-1}\right)^k$, which is skew-symmetric, respectively symmetric, according that k is odd, respectively even, and defined on the simple elements $\alpha^1 \wedge \ldots \wedge \alpha^k$, $\beta^1 \wedge \ldots \wedge \beta^k \in \wedge^k X$ as

$$\left(\omega^{-1}\right)^k \left(\alpha^1 \wedge \ldots \wedge \alpha^k, \beta^1 \wedge \ldots \wedge \beta^k\right) \;:=\; \det \left(\omega^{-1}\left(\alpha^\ell, \beta^m\right)\right)_{\ell,m\in\{1,\ldots,k\}} \;,$$

where $\omega^{-1}\left(\alpha^\ell, \beta^m\right) \;:=\; \omega\left(I^{-1}\left(\alpha^\ell\right), I^{-1}\left(\beta^m\right)\right)$ for every $\ell, m \in \{1, \ldots, k\}$. In a Darboux coordinate chart $\left(U,\, \left\{x^j\right\}_{j\in\{1,\ldots,2n\}}\right)$, the canonical *Poisson bi-vector* $\Pi := \omega^{-1} \in \wedge^2 TX$ associated to ω is written as $\omega^{-1} \stackrel{\mathrm{loc}}{=} \sum_{j=1}^{n} \frac{\partial}{\partial x^{2j-1}} \wedge \frac{\partial}{\partial x^{2j}}$.

The *symplectic-$\star$-operator*

$$\star_\omega \colon \wedge^\bullet X \to \wedge^{2n-\bullet} X \;,$$

introduced by J.-L. Brylinski, [Bry88, §2], is defined requiring that, for every $k \in \mathbb{N}$, and for every α, $\beta \in \wedge^k X$,

$$\alpha \wedge \star_\omega \beta \;=\; \left(\omega^{-1}\right)^k (\alpha, \beta)\, \frac{\omega^n}{n!} \;.$$

As for (almost-)complex manifolds, on a symplectic manifold X one has a decomposition of differential forms in symplectic-type components, the so-called Lefschetz decomposition; it is a consequence of a $\mathfrak{sl}(2;\mathbb{R})$-representation on $\wedge^2 X$ by means of operators related to the symplectic structure.

More precisely, define the operators L, Λ, $H \in \mathrm{End}^\bullet\left(\wedge^\bullet X\right)$ as

$$\begin{aligned} L\colon \wedge^\bullet X \to \wedge^{\bullet+2} X \;, &\qquad \alpha \mapsto \omega \wedge \alpha \;, \\ \Lambda\colon \wedge^\bullet X \to \wedge^{\bullet-2} X \;, &\qquad \alpha \mapsto -\iota_\Pi \alpha \;, \\ H\colon \wedge^\bullet X \to \wedge^{\bullet} X \;, &\qquad \alpha \mapsto \sum_k (n-k)\, \pi_{\wedge^k X} \alpha \end{aligned}$$

(where $\iota_\xi \colon \wedge^\bullet X \to \wedge^{\bullet-2} X$ denotes the interior product with $\xi \in \wedge^2 (TX)$, and, for $k \in \mathbb{N}$, the map $\pi_{\wedge^k X} \colon \wedge^\bullet X \to \wedge^k X$ denotes the natural projection onto $\wedge^k X$). Note that, using the symplectic-$\star$-operator $\star_\omega$, one can write, [Yan96, Lemma 1.5],

$$\Lambda \;=\; - \star_\omega\, L \star_\omega \;.$$

The following result holds.

Theorem 0.8 ([Yan96, Corollary 1.6])**.** *Let X be a manifold endowed with a symplectic structure. Then*

$$[L,\, H] \;=\; 2\,L \;, \qquad [\Lambda,\, H] \;=\; -2\,\Lambda \;, \qquad [L,\, \Lambda] \;=\; H \;,$$

and hence

$$\mathfrak{sl}(2;\mathbb{R}) \;\simeq\; \langle L,\, \Lambda,\, H\rangle \to \mathrm{End}^\bullet\left(\wedge^\bullet X\right)$$

gives a $\mathfrak{sl}(2;\mathbb{R})$-representation on $\wedge^\bullet X$.

(See, e.g., [Hum78, §7] for general results concerning $\mathfrak{sl}(2;\mathbb{R})$-representations.)

The above $\mathfrak{sl}(2;\mathbb{R})$-representation, having finite H-spectrum, induces a decomposition of the space of the differential forms.

Theorem 0.9 ([Yan96, Corollary 2.6]). *Let X be a manifold endowed with a symplectic structure. Then one has the* Lefschetz decomposition *on differential forms,*

$$\wedge^\bullet X \;=\; \bigoplus_{r\in\mathbb{N}} L^r \, \mathrm{P}\wedge^{\bullet-2r} X \;,$$

where

$$\mathrm{P}\wedge^\bullet X \;:=\; \ker \Lambda$$

is the space of primitive forms.

Note (see, e.g., [Huy05, Proposition 1.2.30(v)]) that, for every $k\in\mathbb{N}$,

$$\mathrm{P}\wedge^k X \;=\; \ker L^{n-k+1}\lfloor_{\wedge^k X} \;.$$

In general, see, e.g., [TY12b, pages 7–8], the Lefschetz decomposition of $A^{(k)} \in \wedge^k X$ reads as

$$A^{(k)} \;=\; \sum_{r\geq \max\{k-n,\,0\}} \frac{1}{r!}\, L^r\, B^{(k-2r)}$$

where, for $r \geq \max\{k-n,\,0\}$,

$$B^{(k-2r)} \;:=\; \left(\sum_{\ell\in\mathbb{N}} a_{r,\ell,(n,k)}\, \frac{1}{\ell!}\, L^\ell\, \Lambda^{r+\ell}\right) A^{(k)} \;\in\; \mathrm{P}\wedge^{k-2r}X$$

and, for $r \geq \max\{k-n,\,0\}$ and $\ell\in\mathbb{N}$,

$$a_{r,\ell,(n,k)} \;:=\; (-1)^\ell \cdot (n-k+2r+1)^2 \cdot \prod_{i=0}^{r} \frac{1}{n-k+2r+1-i} \cdot \prod_{j=0}^{\ell} \frac{1}{n-k+2r+1+j} \in \mathbb{Q} \,.$$

We recall that

$$L\lfloor_{\bigoplus_{k=-1}^{n-2} \wedge^{n-k-2}X} \colon \bigoplus_{k=-1}^{n-2} \wedge^{n-k-2}X \to \wedge^{n-k}X$$

is injective, [Yan96, Corollary 2.8], and that, for every $k\in\mathbb{N}$,

$$L^k\colon\; \wedge^{n-k}\, X \to \wedge^{n+k} X$$

is an isomorphism, [Yan96, Corollary 2.7].

Since $[L,\,\mathrm{d}]=0$, for any $k\in\mathbb{N}$, the map $L^k\colon \wedge^{n-k}X \to \wedge^{n+k}X$ induces a map $L^k\colon H^{n-k}_{dR}(X;\mathbb{R}) \to H^{n+k}_{dR}(X;\mathbb{R})$ in cohomology. One says that X satisfies the *Hard Lefschetz Condition*, shortly HLC, if

$$\text{for every } k\in\mathbb{N}\;, \qquad L^k\colon H^{n-k}_{dR}(X;\mathbb{R}) \stackrel{\simeq}{\to} H^{n+k}_{dR}(X;\mathbb{R}) \;. \tag{HLC}$$

By continuing in the parallelism between Riemannian Geometry and Symplectic Geometry, one can introduce the d^Λ operator with respect to a symplectic structure ω as

$$\mathrm{d}^\Lambda\lfloor_{\wedge^k X} \;:=\; (-1)^{k+1} \star_\omega \mathrm{d} \star_\omega$$

for any $k\in\mathbb{N}$, and interpret it as the symplectic counterpart of the Riemannian d^* operator with respect to a Riemannian metric. In light of this, J.-L. Brylinski proposed in [Bry88] a Hodge theory for compact symplectic manifolds, conjecturing that, on a compact manifold endowed with a symplectic structure ω, every de Rham cohomology class admits a (possibly non-unique) *ω-symplectically-harmonic representative*, namely, a d-closed d^Λ-closed representative, [Bry88, Conjecture 2.2.7]. (Note that $\mathrm{d}\,\mathrm{d}^\Lambda + \mathrm{d}^\Lambda\,\mathrm{d} = 0$, [Bry88, Theorem 1.3.1], [Kos85, page 265], provides a strong difference in the parallelism between Symplectic Geometry and Riemannian geometry; in particular, it follows that a ω-symplectically-harmonic representative, whenever it exists, is not unique.)

For an almost-Kähler structure $(J,\,\omega,\,g)$ on a compact manifold X (that is, $\omega\in\wedge^2X$ is a symplectic form on X, $J\in\mathrm{End}\,(TX)$ is an almost-complex structure on X, and g is a J-Hermitian metric on X such that ω is the associated $(1,1)$-form to g), the symplectic-$\star$-operator $\star_\omega$ and the Hodge-$*$-operator $*_g$ are related by

$$\star_\omega \;=\; J *_g \;,$$

and hence

$$\mathrm{d}^{\Lambda} \;=\; -\left(\mathrm{d}^{c}\right)^{*_g}$$

where $\mathrm{d}^c := J^{-1}\, \mathrm{d}\, J$ and $\left(\mathrm{d}^c\right)^{*_g} \lfloor_{\wedge^k X} := (-1)^{k+1} *_g \mathrm{d} *_g$ for every $k \in \mathbb{N}$ (note that, when J is integrable, then $\mathrm{d}^c = -\,\mathrm{i}\left(\partial - \overline{\partial}\right)$). Moreover, on a compact manifold X endowed with a Kähler structure $(J,\, \omega,\, g)$, by the Hodge decomposition theorem, [Wei58, Théorème IV.3], the pure-type components with respect to J of the harmonic representatives of the de Rham cohomology classes are themselves harmonic. Hence, it follows that Brylinski's conjecture holds true for compact Kähler manifolds, [Bry88, Corollary 2.4.3].

O. Mathieu in [Mat95], and D. Yan in [Yan96], provided counterexamples to Brylinski's conjecture, characterizing the compact symplectic manifolds satisfying Brylinski's conjecture in terms of the validity of the Hard Lefschetz Condition. Furthermore, S. A. Merkulov in [Mer98], see also [Cav05], and V. Guillemin in [Gui01], proved that the Hard Lefschetz Condition on compact symplectic manifolds is equivalent to satisfying the $\mathrm{d}\,\mathrm{d}^{\Lambda}$-Lemma, namely, to every d-exact d^{Λ}-closed form being $\mathrm{d}\,\mathrm{d}^{\Lambda}$-exact. Summarizing, we recall the following result.

Theorem 0.10 ([Mat95, Corollary 2], [Yan96, Theorem 0.1], [Mer98, Proposition 1.4], [Gui01], [Cav05, Theorem 5.4])**.** *Let X be a compact manifold endowed with a symplectic structure ω. The following conditions are equivalent:*

(i) every de Rham cohomology class admits a representative being both d*-closed and* d^{Λ}*-closed (i.e., Brylinski's conjecture [Bry88, Conjecture 2.2.7] holds true on X);*

(ii) X satisfies the Hard Lefschetz Condition;

(iii) X satisfies the $\mathrm{d}\,\mathrm{d}^{\Lambda}$*-Lemma.*

Note that, by the Lefschetz decomposition theorem, [Wei58, Théorème IV.5] (see §0.3), compact Kähler manifolds satisfy the Hard Lefschetz Condition.

Remark 0.11. The Complex Generalized Geometry, introduced by N. J. Hitchin in [Hit03] and developed, among others, by M. Gualtieri, [Gua04, Gua11], and G. R. Cavalcanti, [Cav05], see also [Hit10, Cav07], allows to frame symplectic structures and complex structures in the same context (in a sense, this add more significance to the term "symplectic", which was invented by H. Weyl, [Wey97, §VI], substituting the Greek root in the term "complex" with the corresponding Latin root). In such a framework, the d^{Λ} operator associated to a symplectic structure should be interpreted as the symplectic counterpart of the operator $\mathrm{d}^c := -\,\mathrm{i}\,\left(\partial - \overline{\partial}\right)$ associated to a complex structure, [Cav05].

0.3 Kähler structures and cohomological decomposition

Note that, given a manifold X endowed with a symplectic form ω, there is always a (possibly non-integrable) almost-complex structure J on X such that $g := \omega(\cdot,\, J\cdot\cdot)$ is a Hermitian metric on X with ω as the associated $(1,1)$-form, see, e.g., [CdS01, Corollary 12.7] (in fact, the set of such almost-complex structures is contractible, see, e.g., [AL94, Corollary II.1.1.7], [CdS01, Proposition 13.1]; see also [Gro85, Corollary 2.3.C$'_2$], which proves that the space of almost-complex structures on X tamed by a given 2-form on X is contractible). Instead, the datum of an integrable almost-complex structure with the above property yields a Kähler structure on X. The notion of Kähler manifold has been studied for the first time by J. A. Schouten and D. van Dantzig [SvD30], see also [Sch29], and by E. Kähler [Käh33], and the terminology has been fixed by A. Weil [Wei58].

Kähler structures can be defined in different ways, according to the point of view which is stressed, §0.3.1. The presence of three different structures (complex, symplectic, and Riemannian) allows to make use of the tools available for any of them; in addition, the relations between such structures make available further tools, which yield many interesting results on Hodge theory, §0.3.2. Finally, we will study a cohomological property of compact Kähler manifolds, namely, the $\partial\overline{\partial}$-Lemma, §0.3.3: other than being a very useful tool in Kähler Geometry (compare, e.g., its role in S.-T. Yau's proof [Yau77, Yau78] of E. Calabi's conjecture [Cal57]), it provides obstructions to the existence of Kähler structures on differentiable manifolds, by means of the notion of formality introduced by D. P. Sullivan, [Sul77, §12].

0.3.1 Kähler metrics

Let X be a compact complex manifold of complex dimension n, and denote by J its natural integrable almost-complex structure.

A *Kähler metric* on X is a Hermitian metric g such that the associated $(1,1)$-form $\omega := g(J\cdot,\, \cdot\cdot)$ is d-closed (that is, ω is a symplectic form on X).

Remark 0.12. Let X be a complex manifold endowed with a Kähler metric g, and denote the associated $(1,1)$-form to g by ω. By the Poincaré lemma, see, e.g., [Dem12, I.1.22, Theorem I.2.24], and the Dolbeault and Grothendieck lemma, see, e.g., [Dem12, I.3.29], the property that $\mathrm{d}\,\omega = 0$ is equivalent to ask that, for every $x \in X$, there exist an open neighbourhood U in X with $x \in U$ and a smooth function $u \in \mathcal{C}^\infty(U;\mathbb{R})$ such that $\omega \stackrel{\text{loc}}{=} \mathrm{i}\,\partial\overline{\partial} u$ in U, that is, the metric has a local potential, [Käh33] (see, e.g., [Mor07, Proposition 8.8]).

Remark 0.13. For every $n \in \mathbb{N}$, the complex projective space $\mathbb{CP}^n$ admits a Kähler metric, the so-called *Fubini and Study metric*, [Fub04, Stu05], which is induced by the fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n$; more precisely, by using the homogeneous coordinates $[z_0 : \cdots : z_n]$, one has that the associated $(1,1)$-form ω_{FS} to the Fubini and Study metric is

$$\omega_{\mathrm{FS}} \;=\; \frac{\mathrm{i}}{2\pi}\,\partial\overline{\partial}\log\left(\sum_{\ell=0}^{n}\left|z^\ell\right|^2\right) .$$

It follows that complex projective manifolds provide examples of Kähler manifolds. Conversely, by the Kodaira embedding theorem [Kod54, Theorem 4], if X is a compact complex manifold endowed with a Kähler metric ω such that $[\omega] \in H^2_{dR}(X;\mathbb{R}) \cap \mathrm{im}\left(H^2(X;\mathbb{Z}) \to H^2_{dR}(X;\mathbb{R})\right)$, then there exists a complex-analytic embedding of X into a complex projective space $\mathbb{CP}^N$ for some $N \in \mathbb{N}$. In a sense, this suggest that projective manifolds are to Kähler manifolds as $\mathbb{Q}$ is to $\mathbb{R}$. Hence, it is natural to ask if every compact Kähler manifold is a deformation of a projective manifold (which is known as the *Kodaira problem*). Since Riemann surfaces are projective, this is trivially true in complex dimension 1. Furthermore, K. Kodaira proved in [Kod63, Theorem 16.1] that every compact Kähler surface is a deformation of an algebraic surface, as conjectured by W. Hodge; another proof, which does not make use of the classification of elliptic surfaces, has been given by N. Buchdahl, [Buc08, Theorem]. In higher dimension, a negative answer to the Kodaira problem has been given by C. Voisin, who constructed examples of compact Kähler manifolds, of any complex dimension greater than or equal to 4, which do not have the homotopy type of a complex projective manifold, [Voi04, Theorem 2] (indeed, recall that, by Ehresmann's theorem, if two compact complex manifolds can be obtained by deformation, then they are homeomorphic, and hence they have the same homotopy type). The examples in [Voi04] being, by construction, bimeromorphic to manifolds that can be deformed to projective manifolds, one could ask (as done by N. Buchdahl, F. Campana, S.-T. Yau) whether, in higher dimension, a birational version of the Kodaira problem may hold true; in [Voi06, Theorem 3], C. Voisin provided a negative answer to the birational version of the Kodaira problem, proving that, in any even complex dimension greater that or equal to 10, there exist compact Kähler manifolds X such that, for any compact Kähler manifold X' bimeromorphic to X, X' does not have the homotopy type of a projective complex manifold.

In the definition of a Kähler manifold, three different structures are involved: a complex structure, a symplectic structure, and a metric structure. Therefore, changing the point of view allows to give several equivalent definitions of Kähler structure (see, e.g., [Bal06, Theorem 4.17]): we review here two of these characterizations.

Firstly, it is straightforward to prove that a Hermitian metric g on a compact complex manifold X is a Kähler metric if and only if, for every point $x \in X$, there exists a holomorphic coordinate chart $\left(U,\, \left\{z^j\right\}_{j\in\{1,\ldots,n\}}\right)$, with $x \in U$, such that

$$g \;=\; \sum_{\alpha,\beta=1}^{n}\left(\delta_{\alpha\beta} + \mathrm{o}\left(|z|\right)\right)\mathrm{d}\,z^\alpha \odot \mathrm{d}\,\bar{z}^\beta \qquad \text{at } x ,$$

that is, g *osculates to order* 2 the standard Hermitian metric of $\mathbb{C}^n$ (see, e.g., [GH94, pages 107–108], [Huy05, Proposition 1.3.12], [Mor07, Theorem 11.6]).

As regards the second characterization, we recall that, on a compact complex manifold X endowed with a Hermitian metric g, there is a unique connection ∇^C such that

(i) $\nabla^C g = 0$,

(ii) $\nabla^C J = 0$, and

(iii) $\pi_{\wedge^{0,1}X}\nabla^C \lfloor_{\mathcal{C}^\infty(X;\mathbb{C})} = \overline{\partial}\lfloor_{\mathcal{C}^\infty(X;\mathbb{C})}$;

such a connection is called the *Chern connection* of X (see, e.g., [Huy05, Proposition 4.2.14], [Bal06, Theorem 3.18], [Mor07, Theorem 10.3]). Let g be a Hermitian metric on a compact complex manifold X, and set $\omega := g(J\cdot, \cdot\cdot)$ its associated $(1,1)$-form, where J is the natural integrable almost-complex structure on X; consider the Levi Civita connection ∇^{LC}. One can prove that, for every $x, y, z \in \mathcal{C}^\infty(X;TX)$,

$$\mathrm{d}\,\omega(x,y,z) \;=\; g\left(\left(\nabla^{LC}_x J\right)y,\, z\right) + g\left(\left(\nabla^{LC}_y J\right)z,\, x\right) + g\left(\left(\nabla^{LC}_z J\right)x,\, y\right) ,$$

and

$$2\,g\left(\left(\nabla^{LC}_{x} J\right) y,\, z\right) \;=\; \mathrm{d}\,\omega\,(x,\, y,\, z) - \mathrm{d}\,\omega\,(x, Jy,\, Jz) - g\left(\mathrm{Nij}_J\,(y,\, Jz)\,,\, x\right)\;;$$

(see, e.g., [Bal06, Theorem 4.16], [Tia00, Proposition 1.5]); in particular, it follows that g is a Kähler metric if and only if $\nabla^{LC} J = 0$ if and only if the Chern connection is the Levi Civita connection (see, e.g., [Bal06, Theorem 4.17], [Mor07, Proposition 11.8]).

0.3.2 Hodge theory for Kähler manifolds

The complex, symplectic, and metric structures being related on a Kähler manifold, one gets the following identities concerning the corresponding operators (see, e.g., [Huy05, Proposition 3.1.12]); see also [Hod35, Hod89]. (In [Dem86, Theorem 1.1, Theorem 2.12], commutation relations on arbitrary Hermitian manifolds are provided; see also [Gri66], [Dem12, §VI.6.2].)

Theorem 0.14 (Kähler identities, [Wei58, Théorème II.1, Théorème II.2, Corollaire II.1])**.** *Let X be a compact Kähler manifold. Consider the differential operators ∂ and $\overline{\partial}$ associated to the complex structure, the symplectic operators L and Λ associated to the symplectic structure, and the Hodge-$*$-operator associated to the Hermitian metric. Then, these operators are related as follows:*

(i) $\left[\overline{\partial},\, L\right] = \left[\partial,\, L\right] = 0$ *and* $\left[\Lambda,\, \overline{\partial}^*\right] = \left[\Lambda,\, \partial^*\right] = 0$*;*

(ii) $\left[\overline{\partial}^*,\, L\right] = \mathrm{i}\,\partial$ *and* $\left[\partial^*,\, L\right] = -\,\mathrm{i}\,\overline{\partial}$*, and* $\left[\Lambda,\, \overline{\partial}\right] = -\,\mathrm{i}\,\partial^*$ *and* $\left[\Lambda,\, \partial\right] = \mathrm{i}\,\overline{\partial}^*$*.*

Therefore, considering the 2^{nd} order self-adjoint elliptic differential operators $\Box := \left[\partial,\, \partial^*\right]$, $\overline{\Box} := \left[\overline{\partial},\, \overline{\partial}^*\right]$*, and* $\Delta := \left[\mathrm{d},\, \mathrm{d}^*\right]$*, one gets that*

(iii) $\Box = \overline{\Box} = \frac{1}{2}\Delta$*, and Δ commutes with* $*$, ∂, $\overline{\partial}$, ∂^*, $\overline{\partial}^*$, L, Λ*.*

The previous identities can be proven either using the $\mathfrak{sl}\,(2;\mathbb{C})$ representation $\langle L,\, \Lambda,\, H\rangle \to \mathrm{End}^\bullet\left(\wedge^\bullet X \otimes \mathbb{C}\right)$, or reducing to prove the corresponding identities on $\mathbb{C}^n$ with the standard Kähler structure (which are known as Y. Akizuki and S. Nakano's identities, [AN54, §3]) and hence using that every Kähler metric osculates to order 2 the standard Hermitian metric on $\mathbb{C}^n$.

As a consequence, one gets the following theorems, stating a decomposition of the de Rham cohomology of a Kähler manifold related to the complex, respectively symplectic, structure (see, e.g., [Huy05, Corollary 3.2.12], respectively [Huy05, Proposition 3.2.13]).

Theorem 0.15 (Hodge decomposition theorem, [Wei58, Théorème IV.3])**.** *Let X be a compact complex manifold endowed with a Kähler structure. Then there exist a decomposition*

$$H^\bullet_{dR}(X;\mathbb{C}) \;\simeq\; \bigoplus_{p+q=\bullet} H^{p,q}_{\overline{\partial}}(X)\;,$$

and, for every $p, q \in \mathbb{N}$, an isomorphism

$$H^{p,q}_{\overline{\partial}}(X) \;\simeq\; \overline{H^{q,p}_{\overline{\partial}}(X)}\;.$$

Theorem 0.16 (Lefschetz decomposition theorem, [Wei58, Théorème IV.5])**.** *Let X be a compact complex manifold, of complex dimension n, endowed with a Kähler structure. Then there exist a decomposition*

$$H^\bullet_{dR}(X;\mathbb{C}) \;=\; \bigoplus_{r\in\mathbb{N}} L^r\left(\ker\left(\Lambda\colon H^{\bullet-2r}_{dR}(X;\mathbb{C}) \to H^{\bullet-2r-2}_{dR}(X;\mathbb{C})\right)\right)\;,$$

and, for every $k \in \mathbb{N}$, an isomorphism

$$L^k\colon H^{n-k}_{dR}(X;\mathbb{C}) \stackrel{\simeq}{\to} H^{n+k}_{dR}(X;\mathbb{C})\;.$$

0.3.3 $\partial\overline{\partial}$-Lemma and formality for compact Kähler manifolds

The Hodge decomposition theorem and the Lefschetz decomposition theorem provide obstructions to the existence of Kähler structures on a compact complex manifold. In this section, we study another property of compact Kähler manifolds, namely, formality, which provides an obstruction to the existence of a Kähler structure on a compact (differentiable) manifold. Such a property turns out to be a consequence of the validity of the $\partial\overline{\partial}$-Lemma on compact complex manifolds.

Firstly, we need to recall some general notions regarding homotopy theory of differential algebras; we will then summarize some results concerning the homotopy type of Kähler manifolds: by the classical result by P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan, [DGMS75, Main Theorem], the real homotopy type of a Kähler manifold X is a formal consequence of its cohomology ring $H^{\bullet}_{dR}(X;\mathbb{R})$.

We recall that a *differential graded algebra* (shortly, *dga*) over a field $\mathbb{K}$ is a graded $\mathbb{K}$-algebra $A^{\bullet}$ (where the structure of $\mathbb{K}$-algebra is induced by an inclusion $\mathbb{K}\subseteq A^0$) being graded-commutative (that is, for every $x\in A^{\deg x}$ and $y\in A^{\deg y}$, it holds $x\cdot y=(-1)^{\deg x\cdot\deg y}\,y\cdot x$) and endowed with a differential $\mathrm{d}\colon A^{\bullet}\to A^{\bullet+1}$ satisfying the graded-Leibniz rule (that is, for every $x\in A^{\deg x}$ and $y\in A^{\deg y}$, it holds $\mathrm{d}\,(x\cdot y)=\mathrm{d}\,x\cdot y+(-1)^{\deg x}\,x\cdot\mathrm{d}\,y$). A *morphism of differential graded algebras* $F\colon\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)\to\left(B^{\bullet},\,\mathrm{d}_{B^{\bullet}}\right)$ is a morphism $A^{\bullet}\to B^{\bullet}$ of $\mathbb{K}$-algebras such that $F\circ\mathrm{d}_{A^{\bullet}}=\mathrm{d}_{B^{\bullet}}\circ F$.

Given a dga $\left(A^{\bullet},\,\mathrm{d}\right)$ over $\mathbb{K}$, the cohomology $H^{\bullet}\left(A^{\bullet},\,\mathrm{d}\right):=\frac{\ker\mathrm{d}}{\mathrm{im}\,\mathrm{d}}$ endowed with the zero differential has a natural structure of dga over $\mathbb{K}$; furthermore, every morphism $F\colon\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)\to\left(B^{\bullet},\,\mathrm{d}_{B^{\bullet}}\right)$ of dgas induces a morphism $F^{*}\colon\left(H^{\bullet}\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right),\,0\right)\to\left(H^{\bullet}\left(B^{\bullet},\,\mathrm{d}_{B^{\bullet}}\right),\,0\right)$ of dgas in cohomology; a morphism $F\colon\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)\to\left(B^{\bullet},\,\mathrm{d}_{B^{\bullet}}\right)$ of dgas is called a *quasi-isomorphism* (shortly, *qis*) if the corresponding morphism $F^{*}\colon\left(H^{\bullet}\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right),\,0\right)\to\left(H^{\bullet}\left(B^{\bullet},\,\mathrm{d}_{B^{\bullet}}\right),\,0\right)$ is an isomorphism.

The de Rham complex $\left(\wedge^{\bullet}X,\,\mathrm{d}\right)$ of a compact (differentiable) manifold X has a structure of dga over $\mathbb{R}$, whose cohomology is the dga $\left(H^{\bullet}_{dR}(X;\mathbb{R}),\,0\right)$.

Given a dga $\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)$ over $\mathbb{K}$, the differential $\mathrm{d}_{A^{\bullet}}$ is called *decomposable* if

$$\mathrm{d}_{A^{\bullet}}\left(A^{\bullet}\right)\subseteq\left(\bigoplus_{k\in\mathbb{N}\setminus\{0\}}A^k\right)\cdot\left(\bigoplus_{k\in\mathbb{N}\setminus\{0\}}A^k\right)\,.$$

Given a dga $\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)$ over $\mathbb{K}$, an *elementary extension* of $\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)$ is a dga $\left(B^{\bullet},\,\mathrm{d}_{B^{\bullet}}\right)$ over $\mathbb{K}$ such that

(i) $B^{\bullet}=A^{\bullet}\otimes_{\mathbb{K}}\wedge^{\bullet}V_k$ for V_k a finite-dimensional $\mathbb{K}$-vector space and $k>0$, where $\wedge^{\bullet}V_k$ is the free graded $\mathbb{K}$-algebra generated by V_k, the elements of V_k having degree k, and

(ii) $\mathrm{d}_{B^{\bullet}}\lfloor_{A^{\bullet}}=\mathrm{d}_{A^{\bullet}}$ and $\mathrm{d}_{B^{\bullet}}\left(V_k\right)\subseteq A^{\bullet}$.

A dga $\left(M^{\bullet},\,\mathrm{d}_{M^{\bullet}}\right)$ over $\mathbb{K}$ is called *minimal* if it can be written as an increasing union of sub-dga,

$$(\mathbb{K},\,0)\;=\;\left(M^{\bullet}_0,\,\mathrm{d}_{M^{\bullet}_0}\right)\;\subset\;\left(M^{\bullet}_1,\,\mathrm{d}_{M^{\bullet}_1}\right)\;\subset\;\left(M^{\bullet}_2,\,\mathrm{d}_{M^{\bullet}_2}\right)\;\subseteq\;\cdots\;,\qquad\left(M^{\bullet},\,\mathrm{d}_{M^{\bullet}}\right)\;=\;\bigcup_{j\in\mathbb{N}}\left(M^{\bullet}_j,\,\mathrm{d}_{M^{\bullet}_j}\right)\;,$$

such that

(i) for any $j\in\mathbb{N}$, the dga $\left(M^{\bullet}_{j+1},\,\mathrm{d}_{M^{\bullet}_{j+1}}\right)$ is an elementary extension of the dga $\left(M^{\bullet}_j,\,\mathrm{d}_{M^{\bullet}_j}\right)$, and

(ii) $\mathrm{d}_{M^{\bullet}}$ is decomposable.

A *minimal model* for a dga $\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)$ over $\mathbb{K}$ is the datum of a minimal dga $\left(M^{\bullet},\,\mathrm{d}_{M^{\bullet}}\right)$ over $\mathbb{K}$ and a quasi-isomorphism $\rho\colon\left(M^{\bullet},\,\mathrm{d}_{M^{\bullet}}\right)\stackrel{\text{qis}}{\to}\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)$ of dgas.

Two dgas $\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)$ and $\left(B^{\bullet},\,\mathrm{d}_{B^{\bullet}}\right)$ over $\mathbb{K}$ are *equivalent* if there exist an integer $n\in\mathbb{N}\setminus\{0\}$, a family $\left\{\left(C^{\bullet}_j,\,\mathrm{d}_{C^{\bullet}_j}\right)\right\}_{j\in\{0,\ldots,2n\}}$ of dgas over $\mathbb{K}$ with $\left(C^{\bullet}_0,\,\mathrm{d}_{C^{\bullet}_0}\right)=\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)$ and $\left(C^{\bullet}_{2n},\,\mathrm{d}_{C^{\bullet}_{2n}}\right)=\left(B^{\bullet},\,\mathrm{d}_{B^{\bullet}}\right)$, and a family

$$\left\{\left(C^{\bullet}_{2j+1},\,\mathrm{d}_{C^{\bullet}_{2j+1}}\right)\stackrel{\text{qis}}{\to}\left(C^{\bullet}_{2j},\,\mathrm{d}_{C^{\bullet}_{2j}}\right),\;\left(C^{\bullet}_{2j+1},\,\mathrm{d}_{C^{\bullet}_{2j+1}}\right)\stackrel{\text{qis}}{\to}\left(C^{\bullet}_{2j+2},\,\mathrm{d}_{C^{\bullet}_{2j+2}}\right)\right\}_{j\in\{0,\ldots,n-1\}}$$

of quasi-isomorphisms. A dga $\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)$ over $\mathbb{K}$ is called *formal* if it is equivalent to a dga $\left(B^{\bullet},\,0\right)$ over $\mathbb{K}$ with zero differential, that is, if it is equivalent to $\left(H^{\bullet}\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right),\,0\right)$.

A compact manifold X is called *formal* if its de Rham complex $\left(\wedge^{\bullet}X,\,\mathrm{d}\right)$ is a formal dga over $\mathbb{R}$.

Let $\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)$ be a dga over $\mathbb{K}$. Given

$$[\alpha_{12}]\in H^{\deg\alpha_{12}}\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)\,,\qquad[\alpha_{23}]\in H^{\deg\alpha_{23}}\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)\,,\qquad\text{and}\qquad[\alpha_{34}]\in H^{\deg\alpha_{34}}\left(A^{\bullet},\,\mathrm{d}_{A^{\bullet}}\right)$$

such that

$$[\alpha_{12}]\cdot[\alpha_{23}]\;=\;0\qquad\text{and}\qquad[\alpha_{23}]\cdot[\alpha_{34}]\;=\;0\,,$$

let $\alpha_{13}\in A^{\deg\alpha_{12}+\deg\alpha_{23}-1}$ and $\alpha_{24}\in A^{\deg\alpha_{23}+\deg\alpha_{34}-1}$ be such that

$$(-1)^{\deg\alpha_{12}}\,\alpha_{12}\cdot\alpha_{23}\;=\;\mathrm{d}_{A^{\bullet}}\alpha_{13}\qquad\text{and}\qquad(-1)^{\deg\alpha_{23}}\,\alpha_{23}\cdot\alpha_{34}\;=\;\mathrm{d}_{A^{\bullet}}\alpha_{24}\,;$$

one can then define the *triple Massey product* $\langle [\alpha_{12}], [\alpha_{23}], [\alpha_{34}] \rangle$ as

$$\langle [\alpha_{12}], [\alpha_{23}], [\alpha_{34}] \rangle := \left[(-1)^{\deg \alpha_{12}} \alpha_{12} \cdot \alpha_{24} + (-1)^{\deg \alpha_{13}} \alpha_{13} \cdot \alpha_{34} \right]$$

$$\in \frac{H^{\deg \alpha_{12} + \deg \alpha_{23} + \deg \alpha_{34} - 1} (A^\bullet, \mathrm{d}_{A^\bullet})}{H^{\deg \alpha_{12}} (A^\bullet, \mathrm{d}_{A^\bullet}) \cdot H^{\deg \alpha_{23} + \deg \alpha_{34} - 1} (A^\bullet, \mathrm{d}_{A^\bullet}) + H^{\deg \alpha_{34}} (A^\bullet, \mathrm{d}_{A^\bullet}) \cdot H^{\deg \alpha_{12} + \deg \alpha_{23} - 1} (A^\bullet, \mathrm{d}_{A^\bullet})} .$$

One can define the higher order Massey product by induction. Fixed $m \in \mathbb{N}$ such that $m \geq 4$, and given

$$[\alpha_{12}] \in H^{\deg \alpha_{12}} (A^\bullet, \mathrm{d}_{A^\bullet}) , \qquad \ldots , \qquad [\alpha_{m,m+1}] \in H^{\deg \alpha_{m,m+1}} (A^\bullet, \mathrm{d}_{A^\bullet})$$

such that all the Massey products of order lower than or equal to $m-1$ vanish, let $\{\alpha_{rs}\}_{1 \leq r < s \leq m+1} \subseteq A^\bullet$ be such that

$$\sum_{h < \ell < k} (-1)^{\deg \alpha_{h\ell}} \alpha_{h\ell} \cdot \alpha_{\ell k} = \mathrm{d}\, \alpha_{hk} ,$$

for any $h, k \in \{1, \ldots, m+1\}$ with $k - h < m$. Then define the m^{th} *order Massey product* as

$$\langle [\alpha_{12}], \ldots, [\alpha_{m,m+1}] \rangle := \left[\sum_{1 < \ell < m+1} (-1)^{\deg \alpha_{1k}} \alpha_{1k} \cdot \alpha_{k,m+1} \right]$$

belonging to a quotient of $H^\bullet (A^\bullet, \mathrm{d}_{A^\bullet})$.

As a direct consequence of the definitions, the Massey products (of any order) on a formal dga are zero.

Now, let X be a compact manifold endowed with a Kähler structure.

The Kähler identities allow to prove the following result, known as $\partial\overline{\partial}$*-Lemma* (see, e.g., [Huy05, Corollary 3.2.10]), which, in a sense, summarizes many of the cohomological properties of compact Kähler manifolds.

Theorem 0.17 ($\partial\overline{\partial}$-Lemma for compact Kähler manifolds, [DGMS75, Lemma 5.11])**.** *Let X be a compact Kähler manifold. Then every ∂-closed, $\overline{\partial}$-closed, d-exact form is also $\partial\overline{\partial}$-exact.*

Using the differential operator $\mathrm{d}^c := J^{-1}\, \mathrm{d}\, J = -\mathrm{i} \left(\partial - \overline{\partial} \right)$ (where J is the integrable almost-complex structure naturally associated to the structure of complex manifold on X), and noting that $\ker \partial \cap \ker \overline{\partial} = \ker \mathrm{d} \cap \ker \mathrm{d}^c$ and $\operatorname{im} \partial\overline{\partial} = \operatorname{im} \mathrm{d}\,\mathrm{d}^c$, the following equivalent formulation can be provided.

Theorem 0.18 ($\mathrm{d}\,\mathrm{d}^c$-Lemma for compact Kähler manifolds, [DGMS75, Lemma 5.11])**.** *Let X be a compact Kähler manifold. Then every d-closed, d^c-closed, d-exact form is also $\mathrm{d}\,\mathrm{d}^c$-exact.*

Actually, the $\partial\overline{\partial}$-Lemma holds true for a larger class of compact complex manifolds than the compact Kähler manifolds: indeed, it holds, for examples, for any compact complex manifold that can be blown up to a Kähler manifold, [DGMS75, Theorem 5.22], e.g., for compact complex manifolds in class $\mathcal{C}$ of Fujiki, or for Moĭšezon manifolds; we refer to §1.1.3 for further results concerning the $\partial\overline{\partial}$-Lemma for compact complex manifolds.

If X is a compact Kähler manifold (or, more in general, any compact complex manifold for which the $\partial\overline{\partial}$-Lemma, equivalently the $\mathrm{d}\,\mathrm{d}^c$-Lemma, holds), then one has the following quasi-isomorphisms of dgas:

$$\begin{tikzcd} & \left(\ker \mathrm{d}^c,\, \mathrm{d}\lfloor_{\ker \mathrm{d}^c}\right) \arrow[dl, "\text{qis}"'] \arrow[dr, "\text{qis}"] & \\ \left(\wedge^\bullet X,\, \mathrm{d}\right) & & \left(\frac{\ker \mathrm{d}^c}{\operatorname{im} \mathrm{d}^c},\, 0\right) \end{tikzcd} ;$$

in particular, the dga $(\wedge^\bullet X, \mathrm{d})$ is equivalent to a dga with zero differential, and hence it is formal. This proves the following result by P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan.

Theorem 0.19 ([DGMS75, Main Theorem])**.** *Let X be a compact complex manifold for which the $\partial\overline{\partial}$-Lemma holds (e.g., a compact Kähler manifold, or a manifold in class $\mathcal{C}$ of Fujiki). Then the differentiable manifold underlying X is formal (that is, the differential graded algebra $(\wedge^\bullet X, \mathrm{d})$ is formal).*

In particular, all Massey products (of any order) on a compact complex manifold satisfying the $\partial\overline{\partial}$-Lemma are zero, [DGMS75, Corollary 1]. This provide an obstruction to the existence of Kähler structures on compact differentiable manifolds.

0.4 Deformations of complex structures

A natural way to construct new complex structures on a manifold is by "deforming" a given complex structure. Natural questions arise naturally from this construction, concerning, for example, what properties (e.g., the existence of some special metric) remain still valid after such a small deformation.

We recall in this section the basic notions and the classical results concerning the K. Kodaira, D. C. Spencer, L. Nirenberg, and M. Kuranishi theory of deformations of complex manifolds, [KS58, KS60, KNS58, Kur62], referring to [Huy05], see also, e.g., [Kod05, MK06].

Let B be a complex (respectively, differentiable) manifold. A family $\{X_t\}_{t\in B}$ of compact complex manifolds is said to be a *complex-analytic* (respectively, *differentiable*) *family of compact complex manifolds* if there exist a complex (respectively, differentiable) manifold $\mathcal{X}$ and a surjective holomorphic (respectively, smooth) map $\pi\colon \mathcal{X} \to B$ such that *(i)* $\pi^{-1}(t) = X_t$ for any $t \in B$, and *(ii)* π is a proper holomorphic (respectively, smooth) submersion. A compact complex manifold X is said to be a *deformation* of a compact complex manifold Y if there exist a complex-analytic family $\{X_t\}_{t\in B}$ of compact complex manifolds, and $b_0,\, b_1 \in B$ such that $X_{b_0} = X_s$ and $X_{b_1} = X_t$.

A complex-analytic (respectively, differentiable) family $\mathcal{X} \stackrel{\pi}{\to} B$ of compact complex manifolds is said to be *trivial* if $\mathcal{X}$ is bi-holomorphic (respectively, diffeomorphic) to $B \times X_b \stackrel{\pi_B}{\to} B$ for some $b \in B$ (where $\pi_B\colon B\times X_b \to B$ denotes the natural projection onto B); it is said to be *locally trivial* if, for any $b \in B$, there exists an open neighbourhood U of b in B such that $\pi^{-1}(U) \stackrel{\pi\lfloor_{\pi^{-1}(U)}}{\to} U$ is trivial. The following theorem by C. Ehresmann states the local triviality of a differentiable family of compact complex manifolds (see, e.g., [Kod05, Theorem 2.3, Theorem 2.5], [MK06, Theorem 1.4.1]).

Theorem 0.20 (Ehresmann theorem, [Ehr47])**.** *Let $\{X_t\}_{t\in B}$ be a differentiable family of compact complex manifolds. For any $s,\, t \in B$, the manifolds X_s and X_t are diffeomorphic.*

As a consequence of Ehresmann's theorem, a complex-analytic family $\{X_t\}_{t\in B}$ of compact complex manifolds with B contractible can be viewed as a family of complex structures on a compact differentiable manifold.

We recall some other useful definitions, see, e.g., [Huy05, §6.2]. Let $\pi\colon \mathcal{X} \to B$ be a complex-analytic family of compact complex manifolds, deformations of $X := \pi^{-1}(0)$. We recall that, given $f\colon (B',\, 0') \to (B,\, 0)$ a morphism of germs with a distinguished point, the pull-back $f^*\mathcal{X} := \mathcal{X} \times_B B'$ gives a complex-analytic family of deformations of X. The complex-analytic family $\pi\colon \mathcal{X} \to B$ of deformations of X is called *complete* if, for any complex-analytic family $\pi'\colon \mathcal{X}' \to B'$ of deformations of X, there exists a morphism $f\colon B' \to B$ of germs with a distinguished point such that $\mathcal{X}' = f^*\mathcal{X}$. The complex-analytic family $\pi\colon \mathcal{X} \to B$ of deformations of X is called *universal* if, for any complex-analytic family $\pi'\colon \mathcal{X}' \to B'$ of deformations of X, there exists a unique morphism $f\colon B' \to B$ of germs with a distinguished point such that $\mathcal{X}' = f^*\mathcal{X}$. The complex-analytic family $\pi\colon \mathcal{X} \to B$ of deformations of X is called *versal* if, for any complex-analytic family $\pi'\colon \mathcal{X}' \to B'$ of deformations of X, there exists a morphism $f\colon B' \to B$ of germs with a distinguished point such that $\mathcal{X}' = f^*\mathcal{X}$ and such that $\mathrm{d}\, f\colon T_{0'}B' \to T_0 S$ is uniquely determined.

The theory of complex-analytic deformations of compact complex manifolds has been introduced by K. Kodaira and D. C. Spencer, [KS58, KS60], and developed also by L. Nirenberg, [KNS58], and M. Kuranishi, [Kur62, Kur65], see also [Kod05, MK06]. In recalling the main results of this theory, we follow the approach in [Huy05], based on the construction of a differential graded Lie algebra structure on $\mathcal{C}^\infty\left(X;\, T^{1,0}X \otimes \wedge^{0,\bullet}X\right)$, see also [Man04].

Let X be a compact manifold endowed with an integrable almost-complex structure J. Every section $s \in \mathcal{C}^\infty\left(X; T^{1,0}_J X \otimes \wedge^{0,1}_J X\right)$ near to the zero section determines an almost-complex structure J', defined in such a way that $\wedge^{1,0}_{J'} X$ is the graph of $-s\colon\, \wedge^{1,0}_J X \to \wedge^{0,1}_J X$; it turns out that J' is integrable if and only if the *Maurer and Cartan equation*

$$\overline{\partial} s + \frac{1}{2}\,[s, s] \;=\; 0 \tag{MC}$$

holds (see, e.g., [Huy05, Lemma 6.1.2]), where

- $[\cdot, \cdot\cdot] : \mathcal{C}^\infty\left(X;\, T^{1,0}_J X \otimes \wedge^{0,p}_J X\right) \times \mathcal{C}^\infty\left(X;\, T^{1,0}_J X \otimes \wedge^{0,q}_J X\right) \to \mathcal{C}^\infty\left(X;\, T^{1,0}_J X \otimes \wedge^{0,p+q}_J X\right)$ is defined as

  $$\left[X \otimes \bar\alpha,\, Y \otimes \bar\beta\right] \;:=\; X \otimes \left(\bar\beta \wedge \mathcal{L}_Y \bar\alpha\right) + Y \otimes \left(\bar\alpha \wedge \mathcal{L}_X \bar\beta\right) + [X,\, Y] \otimes \left(\bar\alpha \wedge \bar\beta\right) \;,$$

  where $\mathcal{L}_W \varphi := \iota_W\, \mathrm{d}\,\varphi + \mathrm{d}\,(\iota_W \varphi)$ is the Lie derivative of φ along W; locally, in a chart with holomorphic coordinates $\left\{z^j\right\}_j$, one has

  $$\left[w \otimes \mathrm{d}\,\bar z^{\ell_1} \wedge \cdots \wedge \mathrm{d}\,\bar z^{\ell_p},\, w' \otimes \mathrm{d}\,\bar z^{m_1} \wedge \cdots \wedge \mathrm{d}\,\bar z^{m_q} \wedge\right] \;\stackrel{\mathrm{loc}}{=}\; [w,\, w'] \otimes \mathrm{d}\,\bar z^{\ell_1} \wedge \cdots \wedge \mathrm{d}\,\bar z^{\ell_p} \wedge \mathrm{d}\,\bar z^{m_1} \wedge \cdots \wedge \mathrm{d}\,\bar z^{m_q} \;;$$

- $\overline{\partial}\colon \mathcal{C}^\infty\left(X;\, T^{1,0}_J X \otimes \wedge^{0,p}_J X\right) \to \mathcal{C}^\infty\left(X;\, T^{1,0}_J X \otimes \wedge^{0,p+1}_J X\right)$ is defined as

$$\overline{\partial}\varphi\left(\bar{Z},\, \bar{W}\right) \;:=\; \left[\bar{Z},\, \varphi\left(\bar{W}\right)\right]^{1,0} - \left[\bar{W},\, \varphi\left(\bar{Z}\right)\right]^{1,0} - \varphi\left(\left[\bar{Z},\, \bar{W}\right]\right) \;,$$

where $X^{1,0} := X - \mathrm{i}\, J\, X$ is the $(1,0)$-component of X; locally, in a chart with holomorphic coordinates $\left\{z^j\right\}_j$, one has

$$\overline{\partial}\left(\frac{\partial}{\partial z^\ell} \otimes \alpha\right) \;\stackrel{\text{loc}}{=}\; \frac{\partial}{\partial z^\ell} \otimes \overline{\partial}\alpha \;.$$

Hence, to study complex-analytic families of infinitesimal deformations of a compact complex manifold X, it suffices to study complex-analytic families $\left\{s(\mathbf{t})\right\}_{\mathbf{t}\in\Delta(0,\varepsilon)\subset\mathbb{C}^m} \subseteq \mathcal{C}^\infty\left(X;\, T^{1,0}X \otimes \wedge^{0,1}X\right)$ (where $\varepsilon > 0$ is small enough) with $s(0) = 0$. Consider the power series expansion in $\mathbf{t}$ of $s(\mathbf{t})$,

$$s(\mathbf{t}) \;=:\; \sum_{k\in\mathbb{N}} s_k(\mathbf{t}) \;,$$

where $s_k(\mathbf{t}) \in \mathcal{C}^\infty\left(X;\, T^{1,0}X \otimes \wedge^{0,1}X\right)$ is homogeneous of degree k in $\mathbf{t}$, and $s_0(\mathbf{t}) = 0$. Then the Maurer and Cartan equation (MC) can be rewritten, for every $\mathbf{t} \in \Delta(0,\varepsilon)$, as the system

$$\left\{\begin{array}{rcl} \overline{\partial}s_1(\mathbf{t}) & = & 0 \\ \overline{\partial}s_k(\mathbf{t}) & = & -\sum_{1\leq j\leq k-1}\left[s_j(\mathbf{t}),\, s_{k-j}(\mathbf{t})\right] \quad \text{for} \quad k \geq 2 \end{array}\right. \;;$$

in particular, $s_1(\mathbf{t})$ defines a class in $H^{0,1}\left(X;\, \Theta_X\right)$, where Θ_X denotes the sheaf of the germs of holomorphic vector fields on X; up to the action of $\mathrm{Diff}(X)$, one has that $s_1(\mathbf{t})$ is uniquely determined by its class in $H^{0,1}\left(X;\, \Theta_X\right)$ (see, e.g., [Huy05, Lemma 6.14]).

Fix now a Hermitian metric g on X. Consider the decomposition

$$T^{1,0}X \otimes \wedge^{0,1}X \;=\; \left(T^{1,0}X \otimes \ker\overline{\square}\lfloor_{\wedge^{0,1}X}\right) \oplus \left(T^{1,0}X \otimes \overline{\partial}\wedge^{0,0}X\right) \oplus \left(T^{1,0}X \otimes \overline{\partial}^{*}\wedge^{0,2}X\right) \;,$$

and the corresponding projections

$$H_{\overline{\partial}}\colon T^{1,0}X \otimes \wedge^{0,1}X \to T^{1,0}X \otimes \ker\overline{\square}\lfloor_{\wedge^{0,1}X} \;, \qquad P_{\overline{\partial}}\colon T^{1,0}X \otimes \wedge^{0,1}X \to T^{1,0}X \otimes \overline{\partial}\wedge^{0,0}X \;.$$

In order that $s(\mathbf{t})$ satisfies (MC), for every $\mathbf{t} \in \Delta(0,\varepsilon)$, one should have

$$\overline{\partial}s_k(\mathbf{t}) \;=\; -P_{\overline{\partial}}\left(\sum_{1\leq j\leq k-1}\left[s_j(\mathbf{t}),\, s_{k-j}(\mathbf{t})\right]\right) \;.$$

Hence, one gets

$$\overline{\partial}s(\mathbf{t}) + \left[s(\mathbf{t}),\, s(\mathbf{t})\right] \;=\; H_{\overline{\partial}}\left(\left[s(\mathbf{t}),\, s(\mathbf{t})\right]\right) \;.$$

Therefore, define the map

$$\mathrm{obs}\colon H^{0,1}\left(X;\, \Theta_X\right) \to H^{0,2}\left(X;\, \Theta_X\right)$$

as follows. Let $\left\{X_j \otimes \bar{\omega}^k\right\}_{\substack{j\in\{1,\ldots,n\}\\ k\in\{1,\ldots,m\}}}$ be a basis of $H^{0,1}\left(X;\, \Theta_X\right)$. Given $\mu =: \sum_{\substack{j\in\{1,\ldots,n\}\\ k\in\{1,\ldots,m\}}} t^j_k\, X_j \otimes \bar{\omega}^k$, denote $\mathbf{t} =: \left(t^j_k\right)_{\substack{j\in\{1,\ldots,n\}\\ k\in\{1,\ldots,m\}}}$, and define $s_1(\mathbf{t}) := \mu$ and $s_k(\mathbf{t})$ such that $\overline{\partial}s_k(\mathbf{t}) := -P_{\overline{\partial}}\left(\sum_{1\leq j\leq k-1}\left[s_j(\mathbf{t}),\, s_{k-j}(\mathbf{t})\right]\right)$ for $k \geq 2$; hence, define the formal power series $s(\mathbf{t}) := \sum_{k\in\mathbb{N}} s_k(\mathbf{t})$. Define

$$\mathrm{obs}\left(\mu\right) \;:=\; H_{\overline{\partial}}\left(\left[s(\mathbf{t}),\, s(\mathbf{t})\right]\right) \;.$$

Hence, one has then that $\left\{s(\mathbf{t})\right\}_{\mathbf{t}\in\Delta(0,\varepsilon)\subset\mathbb{C}^m} \subseteq \mathcal{C}^\infty\left(X; T^{1,0}_J X \otimes \wedge^{0,1}_J X\right)$ (where $\varepsilon > 0$ is small enough) defines an infinitesimal family of compact complex manifolds if $\mathrm{obs}\left(s_1\left(\mathbf{t}\right)\right) = 0$ for every $\mathbf{t} \in \Delta(0,\varepsilon)$ (indeed, for $\varepsilon > 0$ small enough, the formal power series converges, see, e.g., [Kod05, §5.3], [MK06, §2.3]).

One gets the following result by M. Kuranishi.

Theorem 0.21 ([Kur62, Theorem 2])**.** *Let X be a compact complex manifold. Then X admits a versal complex-analytic family of deformations.*

Fixed a Hermitian metric on X, such a family of deformations, which is called the *Kuranishi space* $\mathrm{Kur}(X)$ of X, is parametrized by

$$\mathrm{Kur}(X) \;=\; \left\{\mu \in H^{0,1}\left(X;\, \Theta_X\right) \;:\; \|\mu\| \ll 1,\; \mathrm{obs}(\mu) = 0\right\} \;.$$

Remark 0.22. A compact complex manifold X is called *non-obstructed* if $\mathrm{Kur}(X)$ is non-singular. In particular, if $H^{0,2}\left(X;\, \Theta_X\right) = \{0\}$, then X is non-obstructed. There are other interesting cases in which the Kuranishi space turns out to be non-singular: as announced by F. A. Bogomolov, [Bog78], and proven by G. Tian, [Tia87], and, independently, by A. N. Todorov, [Tod89, Theorem 1], this happens for *Calabi-Yau manifolds* (that is, compact complex manifolds X of complex dimension n endowed with a Kähler structure $(J,\, \omega,\, g)$ and with a nowhere vanishing $\epsilon \in \wedge^{n,0}X$ such that *(i)* $\nabla^{LC}\epsilon = 0$,where ∇^{LC} denotes the Levi Civita connection associated to g, and *(ii)* $\epsilon \wedge \bar\epsilon = (-1)^{\frac{n(n+1)}{2}}\, \mathrm{i}^n\, \frac{\omega^n}{n!}$). In [dBT12], P. de Bartolomeis and A. Tomassini introduced the notion of *quantum inner state manifold*, [dBT12, Definition 2.2], as a possible generalization of Calabi-Yau manifolds, proving that, under a suitable hypothesis, the moduli space of quantum inner state deformations of a compact Calabi-Yau manifold is totally unobstructed, [dBT12, Theorem 3.6]. On the other hand, in [Rol11b], S. Rollenske studied the Kuranishi space of holomorphically parallelizable nilmanifolds, proving that it is cut out by polynomial equations of degree at most equal to the step of nilpotency of the nilmanifold, [Rol11b, Theorem 4.5], and it is smooth if and only if the associated Lie algebra is a free 2-step nilpotent Lie algebra, [Rol11b, Corollary 4.9].

It could be interesting to study what properties are, in a sense, compatible with the construction of small deformations of the complex structure. In such a context, a property $\mathcal{P}$ concerning compact complex manifolds is called *open under (holomorphic) deformations of the complex structure* (or *stable under small deformations of the complex structure*) if, for every complex-analytic family $\{X_t\}_{t\in B}$ of compact complex manifolds, and for every $b_0 \in B$, if X_{b_0} has the property $\mathcal{P}$, then X_b has the property $\mathcal{P}$ for every b in an open neighbourhood of b_0; it is called *closed under (holomorphic) deformations of the complex structure* if, for every complex-analytic family $\{X_t\}_{t\in B}$ of compact complex manifolds, and for every converging sequence $\{b_k\}_{k\in\mathbb{N}} \subset B$ with $b_\infty := \lim_{k\to+\infty} b_k \in B$, if X_{b_k} has the property $\mathcal{P}$ for every $k \in \mathbb{N}$, then X_{b_∞} has the property $\mathcal{P}$.

We recall here the following classical result by K. Kodaira and D. C. Spencer, stating that admitting a Kähler metric is a stable property under deformations of the complex structure.

Theorem 0.23 ([KS60, Theorem 15])**.** *Let $\{X_t\}_{t\in B}$ be a differentiable family of compact complex manifolds. If X_t admits a Kähler metric for some $t \in B$, then X_s admits a Kähler metric for every s in an open neighbourhood of t in B. Moreover, given any Kähler metric ω on X_t, one can choose an open neighbourhood U of t in B and a Kähler metric ω_s on X_s for any $s \in U$ such that ω_s depends differentiably in s and $\omega_t = \omega$.*

Remark 0.24. In [Hir62], it is proven that admitting a Kähler structure is not a closed property under deformations of the complex structure: in fact, H. Hironaka provided an explicit example of a complex-analytic family of compact complex manifolds of complex dimension 3 such that *(i)* one of the complex manifold is non-Kähler (indeed, it carries a positive 1-cycle algebraically equivalent to zero), and *(ii)* the others are Kähler and, in fact, bi-regularly embedded in a projective space (and hence projective, [Moı66, Theorem 11]), [Hir62, Theorem]. (Note that, in complex dimension 2, the Kähler property is also closed under small deformations of the complex structure, since a compact complex surface is Kähler if and only if its 1st Betti number is even, by [Kod64, Miy74, Siu83], or [Lam99, Corollaire 5.7], or [Buc99, Theorem 11].) It is not known whether the limit of compact Kähler manifolds admits some special structure; J.-P. Demailly and M. Păun conjectured that, given a complex-analytic family $\{X_t\}_{t\in S}$ of compact complex manifolds such that one of the fibers, X_{t_0}, is endowed with a Kähler structure, then there exists a countable union $S' \subsetneq S$ of analytic subsets in the base such that X_t admits a Kähler structure for $t \in S \setminus S'$, [DP04, Conjecture 5.1]; they also guessed that a "natural expectation" is that the remaining fibres, X_t for $t \in S'$, are in class $\mathcal{C}$ of Fujiki, [DP04, page 1272]. In [Pop09, Pop10], D. Popovici studied limits of projective, respectively Moĭšezon manifolds under holomorphic deformations of complex structures, stating, in particular, (by means of a class of Hermitian metrics called strongly-Gauduchon metrics,) that the limit of Moĭšezon manifolds is still Moĭšezon. C. LeBrun and Y. S. Poon [LP92], and F. Campana [Cam91] showed that being in class $\mathcal{C}$ of Fujiki is not a stable property under small deformations of the complex structures, [LP92, Theorem 1], [Cam91, Corollary 3.13], studying twistor spaces. It is conjectured that being in class $\mathcal{C}$ of Fujiki is a closed property under deformations of the complex structure, see, e.g., [Pop11, Standard Conjecture 1.17].

We refer to [Pop11] for a review on the behaviour under holomorphic deformations of properties concerning, e.g., the existence of various types of Hermitian metrics on compact complex manifolds. See also Corollary 1.28, Theorem 2.48 for some results concerning stability or instability of special properties of complex manifolds, and Theorem 2.47, Theorem 3.50 for other instability results for almost-complex or **D**-complex manifolds.

0.5 Currents and de Rham homology

In this section, we recall the basic notions and results concerning currents on (differentiable) manifolds and de Rham homology: they turn out to be a useful tool to study the geometry of complex manifolds (as an example, we recall F. R. Harvey and H. B. Lawson's intrinsic characterization of Kähler manifolds by means of currents, [HL83, Proposition 12, Theorem 14], or M. L. Michelsohn's intrinsic characterization of balanced manifolds by means of currents [Mic82, Theorem 4.7], see also Theorem 2.73, or J. P. Demailly and M. Păun's characterization of compact complex manifolds in class $\mathcal{C}$ of Fujiki by means of Kähler currents [DP04, Theorem 3.4]). We refer, e.g., to [dR84, Chapter 3], [Dem12, §I.2], and [Fed69] (see also [Ale98, Ale10]) for further details.

Let X be a m-dimensional oriented differentiable manifold.

For every compact set $L \subseteq X$ and for every $s \in \mathbb{N}$, define the semi-norm ρ^s_L on $\wedge^\bullet X$ as follows: chosen $\left(U,\, \left\{x^j\right\}_{j\in\{1,\ldots,m\}}\right)$ a coordinate chart with $U \supset L$, and given

$$\varphi \stackrel{\text{loc}}{=:} \sum_{\substack{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,m\}\\ i_1<\cdots<i_k}} \varphi_I \, \mathrm{d}\, x^{i_1} \wedge \cdots \wedge \mathrm{d}\, x^{i_k} \;\in\; \wedge^\bullet X \;,$$

set

$$\rho^s_L(\varphi) \;:=\; \sup_L \; \sup_{\substack{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,m\}\\ i_1<\cdots<i_k}} \; \sup_{\substack{(\alpha_1,\ldots,\alpha_m)\in\mathbb{N}^m\\ \alpha_1+\cdots+\alpha_m\leq s}} \left| \frac{\partial^{\alpha_1+\cdots+\alpha_m}\varphi_I}{\left(\partial x^1\right)^{\alpha_1}\cdots\left(\partial x^m\right)^{\alpha_m}} \right| \;\in\; \mathbb{R} \,.$$

Consider $\wedge^\bullet X$ endowed with the topology induced by the family of semi-norms ρ^s_L, varying L among the compact sets in X, and $s \in \mathbb{N}$: the manifold X being second-countable, $\wedge^\bullet X$ has a structure of a Fréchet space. Let $\wedge^\bullet_{\mathrm{c}} X$ be the topological subspace of $\wedge^\bullet X$ consisting of differential forms with compact support in X.

For any $k \in \mathbb{N}$, the space of *currents* of dimension k (or degree $m-k$), denoted by

$$\mathcal{D}_k X \;:=:\; \mathcal{D}^{m-k} X \;,$$

is defined as the topological dual space of $\wedge^k_{\mathrm{c}} X$; the space $\mathcal{D}_\bullet X$ is endowed with the weak-$*$ topology.

Two basic examples of currents are the following.

- If Z is a (possibly non-closed) k-dimensional oriented compact submanifold of X, then

$$[Z] \;:=\; \int_Z \cdot \;\in\; \mathcal{D}_k X$$

  is a current of dimension k.

- If $\varphi \in \wedge^k X$, then

$$T_\varphi \;:=\; \int_X \varphi \wedge \cdot \;\in\; \mathcal{D}^k X$$

  is a current of degree k.

The exterior differential $\mathrm{d}\colon \wedge^\bullet X \to \wedge^{\bullet+1} X$ induces a differential on $\mathcal{D}_\bullet X$ by duality:

$$\mathrm{d}\colon \mathcal{D}_\bullet X \to \mathcal{D}_{\bullet-1} X$$

is defined, for every $T \in \mathcal{D}^k X$, as

$$\mathrm{d}\, T \;:=\; (-1)^{k+1}\, T\,(\mathrm{d}\,\cdot) \;.$$

In particular, if Z is a k-dimensional oriented closed submanifold of X, then $\mathrm{d}\,[Z] = (-1)^{m-k+1}\,[\mathrm{b}\, Z]$, where b is the boundary operator; if $\varphi \in \wedge^k X$, then $\mathrm{d}\, T_\varphi = T_{\mathrm{d}\,\varphi}$.

By definition, the *de Rham homology* $H^{dR}_\bullet(X;\mathbb{R})$ of X is the homology of the differential complex $(\mathcal{D}_\bullet X,\, \mathrm{d})$. By means of a regularization process, [dR84, Theorem 12], (see also [Dem12, §2.D.3, §2.D.4],) one can prove, [dR84, Theorem 14], that

$$H^\bullet_{dR}(X;\mathbb{R}) \;\simeq\; H^{dR}_{2n-\bullet}(X;\mathbb{R}) \;.$$

Since, for every $k \in \mathbb{N}$, the sheaf $\mathcal{A}^k_X$ is a sheaf of $\mathcal{C}^\infty_X$-module over a paracompact space (where $\mathcal{C}^\infty_X$ denotes the sheaf of germs of smooth functions over X), and by the Poincaré lemma for forms, see, e.g., [Dem12, I.1.22], one has that

$$0 \to \mathbb{R}_X \to \left(\mathcal{A}^\bullet_X,\, \mathrm{d}\right)$$

is a fine (and hence acyclic, see, e.g., [Dem12, Corollary IV.4.19]) resolution of the constant sheaf $\underline{\mathbb{R}}_X$, and hence

$$\check{H}^\bullet\left(X;\underline{\mathbb{R}}_X\right) \simeq \frac{\ker\left(\mathrm{d}\colon \wedge^\bullet X \to \wedge^{\bullet+1} X\right)}{\mathrm{im}\left(\mathrm{d}\colon \wedge^{\bullet-1} X \to \wedge^\bullet X\right)} =: H^\bullet_{dR}\left(X;\mathbb{R}\right) ,$$

see, e.g., [Dem12, IV.6.4].

Analogously, the regularization process [dR84, Theorem 12] allows to prove the analogue of the Poincaré lemma for currents, see, e.g., [Dem12, Theorem I.2.24], and hence, the sheaf $\mathcal{D}^k_X$ being fine for every $k \in \mathbb{N}$ since it is a sheaf of $\mathcal{C}^\infty_X$-module over a paracompact space, one has that

$$0 \to \underline{\mathbb{R}}_X \to \left(\mathcal{D}^\bullet_X,\, \mathrm{d}\right)$$

is a fine (and hence acyclic, see, e.g., [Dem12, Corollary IV.4.19]) resolution of the constant sheaf $\underline{\mathbb{R}}_X$ over X, and hence

$$\check{H}^\bullet\left(X;\underline{\mathbb{R}}_X\right) \simeq \frac{\ker\left(\mathrm{d}\colon \mathcal{D}^\bullet X \to \mathcal{D}^{\bullet+1} X\right)}{\mathrm{im}\left(\mathrm{d}\colon \mathcal{D}^{\bullet-1} X \to \mathcal{D}^\bullet X\right)} =: H^{dR}_{2n-\bullet}\left(X;\mathbb{R}\right) ,$$

see, e.g., [Dem12, IV.6.4].

If X is compact, then it follows that the map $T_\cdot\colon \wedge^\bullet X \to \mathcal{D}^\bullet X$ is injective and a quasi-isomorphism of differential complexes: indeed, fixed a Riemannian metric g on X, if α is a Δ-harmonic form (i.e., a d-closed d^*-closed form), then $T_\alpha(*\alpha) = \|\alpha\|^2$.

Suppose now that X is a $2n$-dimensional manifold endowed with an almost-complex structure $J \in \mathrm{End}(TX)$. Considering the induced endomorphisms $J \in \mathrm{End}\left(\wedge^\bullet X\right)$ and $J \in \mathrm{End}\left(\wedge^\bullet_c X\right)$, one can define $J \in \mathrm{End}\left(\mathcal{D}^\bullet X\right)$ by duality. In the same way as $J \in \mathrm{End}\left(\wedge^\bullet X\right)$ defines a bi-graduation on $\wedge^\bullet X \otimes \mathbb{C}$, one has that $J \in \mathrm{End}\left(\mathcal{D}^\bullet X\right)$ defines the splitting

$$\mathcal{D}_\bullet X \otimes \mathbb{C} = \bigoplus_{p,q\in\mathbb{N}} \mathcal{D}_{p,q} X ;$$

note that $\mathcal{D}_{p,q} X =: \mathcal{D}^{n-p,n-q} X$ is the topological dual of $\wedge^{p,q} X \cap \left(\wedge^\bullet_c X \otimes_{\mathbb{R}} \mathbb{C}\right)$, for every $p, q \in \mathbb{N}$.

0.6 Solvmanifolds

Nilmanifolds and solvmanifolds provide an important class of examples in non-Kähler geometry. Indeed, on the one hand, in studying their properties, one often can reduce to study left-invariant objects on them, which is the same to study linear objects on the corresponding Lie algebra (this allows, for example, to reduce the study of the de Rham cohomology of a nilmanifold to the study of the cohomology of a complex of finite-dimensional vector spaces, [Nom54, Theorem 1]); on the other hand, they do not admit too strong structures, e.g., they do not admit any Kähler structure.

In this section, we recall the main definitions and results concerning the theory of nilmanifolds and solvmanifolds, setting also the notation for the following chapters.

A *nilmanifold*, respectively *solvmanifold*, $X = \Gamma\backslash G$ is a compact quotient of a connected simply-connected nilpotent, respectively solvable, Lie group G by a co-compact discrete subgroup Γ. A solvmanifold $X = \Gamma\backslash G$ is called *completely-solvable* if, for any $g \in G$, all the eigenvalues of $\mathrm{Ad}g \in \mathrm{End}(\mathfrak{g})$ are real, equivalently, if, for any $X \in \mathfrak{g}$, all the eigenvalues of $\mathrm{ad}X \in \mathrm{End}(\mathfrak{g})$ are real.

Given a $2n$-dimensional solvmanifold $X = \Gamma\backslash G$, consider $(\mathfrak{g}, [\cdot, \cdot\cdot])$ the Lie algebra naturally associated to the Lie group G; given a basis $\{e_1, \ldots, e_{2n}\}$ of $\mathfrak{g}$, the Lie algebra structure of $\mathfrak{g}$ is characterized by the *structure constants* $\left\{c^k_{\ell m}\right\}_{\ell,m,k\in\{1,\ldots,2n\}} \subset \mathbb{R}$: namely, for any $k \in \{1, \ldots, 2n\}$,

$$\mathrm{d}_{\mathfrak{g}}\, e^k =: \sum_{\ell,m} c^k_{\ell m}\, e^\ell \wedge e^m ,$$

where $\left\{e^1, \ldots, e^{2n}\right\}$ is the dual basis of $\mathfrak{g}^*$ of $\{e_1, \ldots, e_{2n}\}$ and $\mathrm{d}_{\mathfrak{g}}\colon \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$ is defined by

$$\mathfrak{g}^* \ni \alpha \mapsto \mathrm{d}_{\mathfrak{g}}\,\alpha(\cdot, \cdot\cdot) := -\alpha\left([\cdot, \cdot\cdot]\right) \in \wedge^2\mathfrak{g}^* .$$

To shorten the notation, as in [Sal01], we will refer to a given solvmanifold $X = \Gamma\backslash G$ writing the structure equations of its Lie algebra: for example, writing

$$X := \left(0^4,\ 12,\ 13\right) , \qquad \left(\text{or } \mathfrak{g} := \left(0^4,\ 12,\ 13\right) ,\right)$$

we mean that $X = \Gamma\backslash G$ and there exists a basis of the Lie algebra $\mathfrak{g}$ naturally associated to G, let us say $\{e_1, \ldots, e_6\}$, whose dual will be denoted by $\left\{e^1, \ldots, e^6\right\}$, such that the structure equations with respect to such basis are

$$\begin{cases} \mathrm{d}\, e^1 \;=\; \mathrm{d}\, e^2 \;=\; \mathrm{d}\, e^3 \;=\; \mathrm{d}\, e^4 \;=\; 0 \\ \mathrm{d}\, e^5 \;=\; e^1 \wedge e^2 \;=:\; e^{12} \\ \mathrm{d}\, e^6 \;=\; e^1 \wedge e^3 \;=:\; e^{13} \end{cases} ,$$

where we also shorten $e^{AB} := e^A \wedge e^B$.

The following theorem by A. I. Mal'tsev characterizes the nilpotent Lie algebras $\mathfrak{g}$ for which the naturally associated connected simply-connected Lie group admits a co-compact discrete subgroup, and hence such that there exists a nilmanifold with $\mathfrak{g}$ as Lie algebra.

Theorem 0.25 ([Mal49, Theorem 7])**.** *In order that a simply-connected connected nilpotent Lie group contain a discrete co-compact Lie group it is necessary and sufficient that the Lie algebra of this group have rational constant structures with respect to an appropriate basis.*

Dealing with *G-left-invariant* objects on X, we mean objects induced by objects on G being invariant under the left-action of G on itself given by left-translations. By means of left-translations, G-left-invariant objects will be identified with objects on the Lie algebra $\mathfrak{g}$.

For example, a G-left-invariant complex structure $J \in \mathrm{End}(TX)$ on X is uniquely determined by a linear complex structure $J \in \mathrm{End}(\mathfrak{g})$ on $\mathfrak{g}$ satisfying the integrability condition $\mathrm{Nij}_J = 0$, [NN57, Theorem 1.1], where

$$\mathrm{Nij}_J(\cdot, \cdot\cdot) \;:=\; [\cdot, \cdot\cdot] + J\,[J\cdot, \cdot\cdot] + J\,[\cdot, J\cdot\cdot] - [J\cdot, J\cdot\cdot] \;\in\; \wedge^2\mathfrak{g}^* \otimes_{\mathbb{R}} \mathfrak{g}\;;$$

we will denote the set of G-left-invariant complex structures on X by

$$\mathcal{C}\left(\mathfrak{g}\right) := \left\{J \in \mathrm{End}\left(\mathfrak{g}\right) \;:\; J^2 = -\,\mathrm{id}_{\mathfrak{g}} \quad \text{and} \quad \mathrm{Nij}_J = 0\right\} \;.$$

By the Leibniz rule, the map $\mathrm{d}_{\mathfrak{g}}\colon \wedge^1 \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$ induces a differential operator $\mathrm{d}\colon \wedge^\bullet \mathfrak{g}^* \to \wedge^{\bullet+1}\mathfrak{g}^*$ giving a graded differential algebra $(\wedge^\bullet\mathfrak{g}^*,\, \mathrm{d})$, and hence a differential complex $(\wedge^\bullet\mathfrak{g}^*,\, \mathrm{d})$; we will denote by $H^\bullet_{dR}\left(\mathfrak{g};\mathbb{R}\right) := H^\bullet\left(\wedge^\bullet\mathfrak{g}^*,\, \mathrm{d}\right)$ the cohomology of such a differential complex.

In general, on a solvmanifold, the inclusion $(\wedge^\bullet\mathfrak{g}^*,\, \mathrm{d}) \hookrightarrow (\wedge^\bullet X,\, \mathrm{d})$ induces an injective map in cohomology, $i\colon H^\bullet_{dR}\left(\mathfrak{g};\mathbb{R}\right) \hookrightarrow H^\bullet_{dR}(X;\mathbb{R})$ (compare [CF01, Lemma 9] and Lemma 1.36, for the Dolbeault, respectively Bott-Chern, cohomology), which is not always an isomorphism, as the example in [dBT06, Corollary 4.2, Remark 4.3] shows. On the other hand, the following theorem by K. Nomizu says that the de Rham cohomology of a nilmanifold can be computed as the cohomology of the subcomplex of left-invariant forms (some results in this direction have been provided also by Y. Matsushima in [Mat51, Theorem 5, Theorem 6]).

Theorem 0.26 ([Nom54, Theorem 1])**.** *Let $X = \Gamma\backslash G$ be a nilmanifold and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. The complex $(\wedge^\bullet\mathfrak{g}^*, \mathrm{d})$ is a minimal model for $(\wedge^\bullet X,\, \mathrm{d})$. In particular, the map $(\wedge^\bullet\mathfrak{g}^*,\, \mathrm{d}) \to (\wedge^\bullet X,\, \mathrm{d})$ of differential complexes is a quasi-isomorphism:*

$$i\colon H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) \stackrel{\simeq}{\to} H^\bullet_{dR}(X;\mathbb{R})\;.$$

The proof rests on an inductive argument, which can be performed since every nilmanifold can be seen as a principal torus-bundle over a lower dimensional nilmanifold, see [Mal49, Lemma 4], [Mat51, Theorem 3].

A similar result holds also in the case of completely-solvable solvmanifolds, as proven by A. Hattori, as a consequence of the Mostow structure theorem, [Mos54, Mos57, Theorem 2].

Theorem 0.27 ([Hat60, Corollary 4.2])**.** *Let $X = \Gamma\backslash G$ be a completely-solvable solvmanifold and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. The map $(\wedge^\bullet\mathfrak{g}^*,\, \mathrm{d}) \to (\wedge^\bullet X,\, \mathrm{d})$ of differential complexes is a quasi-isomorphism:*

$$i\colon H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) \stackrel{\simeq}{\to} H^\bullet_{dR}(X;\mathbb{R})\;.$$

(For some results concerning the de Rham cohomology of (non-necessarily completely-solvable) solvmanifolds, see [Gua07, CF11].)

In some cases, we will see that the study of (properties of) geometric structures on a solvmanifold is reduced to the study of the corresponding (properties of) geometric structures on the associated Lie algebra (see, e.g., Theorem 2.67, Proposition 2.19, Proposition 3.18, Proposition 3.30, Theorem 2.47). To this aim, we need the following lemma by J. Milnor. (Recall that a Lie group G, with associated Lie algebra $\mathfrak{g}$, is called *unimodular* if, for all $X \in \mathfrak{g}$, it holds $\mathrm{tr}\,\mathrm{ad}\, X = 0$.)

Lemma 0.28 ([Mil76, Lemma 6.2]). *Any connected Lie group that admits a discrete subgroup with compact quotient is unimodular and in particular admits a bi-invariant volume form* η.

We will also need the following trick by F. A. Belgun (see also [FG04, Theorem 2.1]).

Lemma 0.29 (F. A. Belgun's symmetrization trick, [Bel00, Theorem 7]). *Let* $X = \Gamma \backslash G$ *be a solvmanifold, and denote the Lie algebra naturally associated to* G *by* $\mathfrak{g}$. *Let* η *be a* G*-bi-invariant volume form on* G *such that* $\int_X \eta = 1$, *whose existence follows from J. Milnor's lemma [Mil76, Lemma 6.2]. Up to identifying* G*-left-invariant forms on* X *and linear forms over* $\mathfrak{g}^*$ *through left-translations, define the* F. A. Belgun's symmetrization map

$$\mu\colon \wedge^\bullet X \to \wedge^\bullet \mathfrak{g}^* , \qquad \mu(\alpha) := \int_X \alpha\lfloor_m \eta(m) .$$

One has that

$$\mu\lfloor_{\wedge^\bullet \mathfrak{g}^*} = \mathrm{id}\lfloor_{\wedge^\bullet \mathfrak{g}^*} ,$$

and that

$$\mathrm{d} \circ \mu = \mu \circ \mathrm{d} .$$

In particular, the symmetrization map μ induces a map $\mu\colon (\wedge^\bullet X, \mathrm{d}) \to (\wedge^\bullet \mathfrak{g}^*, \mathrm{d})$ of differential complexes, and hence a map $\mu\colon H^\bullet_{dR}(X;\mathbb{R}) \to H^\bullet_{dR}(\mathfrak{g};\mathbb{R})$ in cohomology. Since $\mu\lfloor_{\wedge^\bullet \mathfrak{g}^*} = \mathrm{id}\lfloor_{\wedge^\bullet \mathfrak{g}^*}$, if the inclusion $(\wedge^\bullet \mathfrak{g}^*, \mathrm{d}) \hookrightarrow (\wedge^\bullet X, \mathrm{d})$ is a quasi-isomorphism (for example, if X is a nilmanifold, by [Nom54, Theorem 1], or a completely-solvable solvmanifold, by [Hat60, Corollary 4.2]), then the map $\mu\colon (\wedge^\bullet X, \mathrm{d}) \to (\wedge^\bullet \mathfrak{g}^*, \mathrm{d})$ turns out to be a quasi-isomorphism.

K. Nomizu's theorem [Nom54, Theorem 1], A. Hattori's theorem [Hat60, Corollary 4.2], and F. A. Belgun's theorem [Bel00, Theorem 7] suggest that nilmanifolds, and, more in general, solvmanifolds, may provide a very useful and interesting class of examples in non-Kähler geometry. On the other hand, another reason for this statement is given by the following results by Ch. Benson and C. S. Gordon, and by K. Hasegawa.

Theorem 0.30 ([BG88, Theorem A]). *Let* X *be a nilmanifold endowed with a symplectic structure* ω *such that the Hard Lefschetz Condition holds. Then* X *is diffeomorphic to a torus.*

(Actually, one can prove that any $2n$-dimensional nilmanifold X endowed with a symplectic structure ω such that the map $[\omega]^{n-1} : H^1_{dR}(X;\mathbb{R}) \to H^{2n-1}_{dR}(X;\mathbb{R})$ is an isomorphism is diffeomorphic to a torus, [LO94], see, e.g., [FOT08, Theorem 4.98]. A minimal model proof of Ch. Benson and C. S. Gordon's theorem [BG88, Theorem A] is due to G. Lupton and J. Oprea, [LO94, Theorem 3.5].)

Theorem 0.31 ([Has89, Theorem 1, Corollary]). *Let* X *be a nilmanifold. If* X *is formal, then* X *is diffeomorphic to a torus.*

In particular, since compact Kähler manifolds satisfy the Hard Lefschetz Condition, [Wei58, Théorème IV.5], and are formal, [DGMS75, Main Theorem], it follows that a nilmanifold admits a Kähler structure if and only if it is diffeomorphic to a torus (compare also [Han57, Theorem II, Footnote 1]). More in general, compact completely-solvable Kähler solvmanifolds are tori, as proven by A. Tralle and J. Kedra in [TK97, Theorem 1], solving a conjecture by Ch. Benson and C. S. Gordon, [BG90, page 972]. In fact, the following result by K. Hasegawa gives a complete characterization of Kähler solvmanifolds.

Theorem 0.32 ([Has06, Main Theorem]). *Let* X *be a compact homogeneous space of solvable Lie group, that is, a compact differentiable manifold on which a connected solvable Lie group acts transitively. Then* X *admits a Kähler structure if and only if it is a finite quotient of a complex torus which has a structure of a complex torus-bundle over a complex torus. In particular, a completely-solvable solvmanifold has a Kähler structure if and only if it is a complex torus.*

Chapter 1

Cohomology of complex manifolds

In this chapter, we study cohomological properties of compact complex manifolds. In particular, we are concerned with studying the *Bott-Chern cohomology*, which, in a sense, constitutes a bridge between the de Rham cohomology and the Dolbeault cohomology of a complex manifold.

In §1.1, we recall some definitions and results on the *Bott-Chern* and *Aeppli* cohomologies, see, e.g., [Sch07], and on the *$\partial\overline{\partial}$-Lemma*, referring to [DGMS75]. In §1.2, we provide a Frölicher-type inequality for the Bott-Chern cohomology, Theorem 1.22, which also allows to characterize the validity of the *$\partial\overline{\partial}$-Lemma* in terms of the dimensions of the Bott-Chern cohomology groups, Theorem 1.25; the proof of such inequality is based on two exact sequences, firstly considered by J. Varouchas in [Var86]. In §1.3, we show that, for certain classes of complex structures on *nilmanifolds* (that is, compact quotients of connected simply-connected nilpotent Lie groups by co-compact discrete subgroups), the Bott-Chern cohomology is completely determined by the associated Lie algebra endowed with the induced linear complex structure, Theorem 1.39, giving a sort of Nomizu-type result for the Bott-Chern cohomology. This will allow us to explicitly study the Bott-Chern and Aeppli cohomologies of the *Iwasawa manifold* and of its small deformations, in §1.4. In §1.5, we investigate the Bott-Chern cohomology of complex *orbifolds* of the type X/G, where X is a compact complex manifold and G a finite group of biholomorphisms of X, Theorem 1.55.

Some of the original results of this chapter have been obtained in [Ang11], and jointly with A. Tomassini in [AT12b]; §1.5 contains some original results that have not yet been submitted for publication.

1.1 Cohomologies of complex manifolds

The *Bott-Chern cohomology* and the *Aeppli cohomology* provide important invariants for the study of the geometry of compact (especially, non-Kähler) complex manifolds. These cohomology groups have been introduced by R. Bott and S. S. Chern in [BC65], and by A. Aeppli in [Aep65], and hence studied by many authors, e.g., B. Bigolin [Big69, Big70] (both from the sheaf-theoretic and from the analytic viewpoints), A. Andreotti and F. Norguet [AN71] (to study cycles of algebraic manifolds), J. Varouchas [Var86] (to study the cohomological properties of a certain class of compact complex manifolds), M. Abate [Aba88] (to study annular bundles), L. Alessandrini and G. Bassanelli [AB96] (to investigate the properties of balanced metrics), S. Ofman [Ofm85a, Ofm85b, Ofm88] (in view of applications to integration on analytic cycles), S. Boucksom [Bou04] (in order to extend divisorial Zariski decompositions to compact complex manifolds), J.-P. Demailly [Dem12] (as a tool in Complex Geometry), M. Schweitzer [Sch07] (in the context of cohomology theories), L. Lussardi [Lus10] (in the non-compact Kähler case), R. Kooistra [Koo11] (in the framework of cohomology theories), J.-M. Bismut [Bis11b, Bis11a] (in the context of Chern characters), L.-S. Tseng and S.-T. Yau [TY11] (in the framework of Generalized Geometry and type II String Theory).

In this preliminary section, we recall the basic notions and classical results concerning cohomologies of complex manifolds. More precisely, we recall the definitions of the Bott-Chern and Aeppli cohomologies, and some results on Hodge theory, referring to [Sch07]; then, we recall the notion of $\partial\overline{\partial}$-Lemma, referring to [DGMS75].

1.1.1 The Bott-Chern cohomology

Let X be a complex manifold. The *Bott-Chern cohomology* of X is the bi-graded algebra

$$H^{\bullet,\bullet}_{BC}(X) \;:=\; \frac{\ker\partial\cap\ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}} \;.$$

Unlike in the case of the Dolbeault cohomology groups, for every $p,q\in\mathbb{N}$, the conjugation induces an isomorphism

$$H^{p,q}_{BC}(X) \;\simeq\; H^{q,p}_{BC}(X) \;.$$

Furthermore, since $\ker \partial \cap \ker \overline{\partial} \subseteq \ker \mathrm{d}$ and $\mathrm{im}\, \partial\overline{\partial} \subseteq \mathrm{im}\, \mathrm{d}$, one has the natural map of graded $\mathbb{C}$-vector spaces

$$\bigoplus_{p+q=\bullet} H^{p,q}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C}) \;,$$

and, since $\ker \partial \cap \ker \overline{\partial} \subseteq \ker \overline{\partial}$ and $\mathrm{im}\, \partial\overline{\partial} \subseteq \mathrm{im}\, \overline{\partial}$, one has the natural map of bi-graded $\mathbb{C}$-vector spaces

$$H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet,\bullet}_{\overline{\partial}}(X) \;.$$

In general, even for compact complex manifolds, these maps are neither injective nor surjective: see, e.g., the examples in [Sch07, §1.c] or in §1.4.4. A case of special interest is when X is a compact complex manifold satisfying the $\partial\overline{\partial}$-Lemma, namely, the property that every ∂-closed $\overline{\partial}$-closed d-exact form is also $\partial\overline{\partial}$-exact, [DGMS75], that is, the natural map $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$ is injective (we recall that compact Kähler manifolds and, more in general, manifolds in class $\mathcal{C}$ of Fujiki, [Fuj78], that is, compact complex manifolds admitting a proper modification from a Kähler manifold, satisfy the $\partial\overline{\partial}$-Lemma, [DGMS75, Lemma 5.11, Corollary 5.23]; we refer to §1.1.3 for further details). In fact, we recall the following result.

Theorem 1.1 ([DGMS75, Lemma 5.15, Remark 5.16, 5.21])**.** *Let X be a compact complex manifold. If X satisfies the $\partial\overline{\partial}$-Lemma, then the natural maps*

$$\bigoplus_{p+q=\bullet} H^{p,q}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C}) \qquad \textit{and} \qquad H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$$

induced by the identity are isomorphisms.

As for the de Rham and the Dolbeault cohomologies, a Hodge theory can be developed also for the Bott-Chern cohomology for compact complex manifolds: we recall here some results, referring to [Sch07, §2] (see also [Big69, §5], and [Lus10]).

Suppose that X is a compact complex manifold. Fix a Hermitian metric on X, and define the differential operator

$$\tilde{\Delta}_{BC} \;:=\; \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\overline{\partial}^*\partial\right)\left(\overline{\partial}^*\partial\right)^* + \left(\overline{\partial}^*\partial\right)^*\left(\overline{\partial}^*\partial\right) + \overline{\partial}^*\overline{\partial} + \partial^*\partial \;,$$

see [KS60, Proposition 5] (where it is used to prove the stability of the Kähler property under small deformations of the complex structure), and also [Sch07, §2.b], [Big69, §5.1]. One has the following result.

Theorem 1.2 ([KS60, Proposition 5], see also [Sch07, §2.b])**.** *Let X be a compact complex manifold endowed with a Hermitian metric. The operator $\tilde{\Delta}_{BC}$ is a 4^{th} order self-adjoint elliptic differential operator, and*

$$\ker \tilde{\Delta}_{BC} \;=\; \ker \partial \cap \ker \overline{\partial} \cap \ker \overline{\partial}^* \partial^* \;.$$

Therefore, as a consequence of the general theory of self-adjoint elliptic differential operators, see, e.g., [Kod05, page 450], the following result holds.

Theorem 1.3 ([Sch07, Théorème 2.2], [Sch07, Corollaire 2.3])**.** *Let X be a compact complex manifold, endowed with a Hermitian metric. Then there exist an orthogonal decomposition*

$$\wedge^{\bullet,\bullet}X \;=\; \ker \tilde{\Delta}_{BC} \,\oplus\, \mathrm{im}\, \partial\overline{\partial} \,\oplus\, \left(\mathrm{im}\, \partial^* \,+\, \mathrm{im}\, \overline{\partial}^*\right)$$

and an isomorphism

$$H^{\bullet,\bullet}_{BC}(X) \;\simeq\; \ker \tilde{\Delta}_{BC} \;.$$

In particular, the Bott-Chern cohomology groups of X are finite-dimensional $\mathbb{C}$-vector spaces.

Another consequence of general results in spectral theory, see, e.g., [KS60, Theorem 4], [Kod05, Theorem 7.3], is the semi-continuity property for the dimensions of the Bott-Chern cohomology.

Theorem 1.4 ([Sch07, Lemme 3.2])**.** *Let $\{X_t\}_{t\in B}$ a complex-analytic family of compact complex manifolds. Then, for every $p, q \in \mathbb{N}$, the function*

$$B \ni t \mapsto \dim_{\mathbb{C}} H^{p,q}_{BC}\left(X_t\right) \in \mathbb{N}$$

is upper-semi-continuous.

By using the Kähler identities (in particular, the fact that $\overline{\square} = \square$ and that $\partial^*\overline{\partial} + \overline{\partial}\partial^* = 0 = \overline{\partial}^*\partial + \partial\overline{\partial}^*$), one can prove that, on a compact Kähler manifold,

$$\tilde{\Delta}_{BC} \;=\; \overline{\square}^2 + \partial^*\partial + \overline{\partial}^*\overline{\partial} \;,$$

[KS60, Proposition 6], [Sch07, Proposition 2.4], and hence $\ker \tilde{\Delta}_{BC} = \ker \overline{\square} = \ker \Delta$; in particular, it follows that, on a compact Kähler manifold, the de Rham cohomology, the Dolbeault cohomology, and the Bott-Chern cohomology are isomorphic (actually, since the $\partial\overline{\partial}$-Lemma holds on every compact Kähler manifold, one gets an isomorphism that does not depend on the choice of the Hermitian metric).

1.1.2 The Aeppli cohomology

Let X be a complex manifold. Dualizing the definition of the Bott-Chern cohomology, one can define another cohomology on X, the *Aeppli cohomology*: it is the bi-graded $H^{\bullet,\bullet}_{BC}(X)$-module

$$H^{\bullet,\bullet}_{A}(X) \;:=\; \frac{\ker \partial\overline{\partial}}{\operatorname{im}\partial \,+\, \operatorname{im}\overline{\partial}} \;.$$

As for the Bott-Chern cohomology, the conjugation induces, for every $p,q \in \mathbb{N}$, the isomorphism

$$H^{p,q}_{A}(X) \;\simeq\; H^{q,p}_{A}(X) \;.$$

Furthermore, since $\ker \mathrm{d} \subseteq \ker \partial\overline{\partial}$ and $\operatorname{im}\mathrm{d} \subseteq \operatorname{im}\partial + \operatorname{im}\overline{\partial}$, one has the natural map of graded $\mathbb{C}$-vector spaces

$$H^{\bullet}_{dR}(X;\mathbb{C}) \to \bigoplus_{p+q=\bullet} H^{p,q}_{A}(X) \;,$$

and, since $\ker \overline{\partial} \subseteq \ker \partial\overline{\partial}$ and $\operatorname{im}\overline{\partial} \subseteq \operatorname{im}\partial + \operatorname{im}\overline{\partial}$, one has the natural map of bi-graded $\mathbb{C}$-vector spaces

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) \to H^{\bullet,\bullet}_{A}(X) \;;$$

as we have noted for the Bott-Chern cohomology, such maps are, in general, neither injective nor surjective, but they are isomorphisms whenever X is compact and satisfies the $\partial\overline{\partial}$-Lemma, [DGMS75, Lemma 5.15, Remark 5.16, 5.21], and hence, in particular, if X is a compact complex manifold admitting a Kähler structure, [DGMS75, Lemma 5.11], or if X is a compact complex manifold in class $\mathcal{C}$ of Fujiki, [DGMS75, Corollary 5.23].

Remark 1.5. On a compact Kähler manifold X, the associated $(1,1)$-form ω of the Kähler metric defines a non-zero class in $H^2_{dR}(X;\mathbb{R})$. For general Hermitian manifolds, special classes of metrics are often defined in terms of closedness of powers of ω, so they define classes in the Bott-Chern or Aeppli cohomology groups (e.g., a Hermitian metric on a complex manifold of complex dimension n is said *balanced* if $\mathrm{d}\,\omega^{n-1} = 0$ [Mic82], *pluriclosed* if $\partial\overline{\partial}\omega = 0$ [Bis89], *astheno-Kähler* if $\partial\overline{\partial}\omega^{n-2} = 0$ [JY93, JY94], *Gauduchon* if $\partial\overline{\partial}\omega^{n-1} = 0$ [Gau77]). (Note that, they define possibly the zero class in the Bott-Chern or Aeppli cohomologies: for the balanced case, see [FLY12, Corollary 1.3], where it is shown that, for $k \geq 2$, the complex structures on $\sharp_{j=1}^{k}\left(\mathbb{S}^3 \times \mathbb{S}^3\right)$ constructed from the conifold transitions admit balanced metrics.)

We refer to [Sch07, §2.c] for the following results, concerning Hodge theory for the Aeppli cohomology on compact complex manifolds.

Suppose that X is a compact complex manifold. Once fixed a Hermitian metric on X, one defines the differential operator

$$\tilde{\Delta}_A \;:=\; \partial\partial^* + \overline{\partial}\overline{\partial}^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\overline{\partial}\partial^*\right)^*\left(\overline{\partial}\partial^*\right) + \left(\overline{\partial}\partial^*\right)\left(\overline{\partial}\partial^*\right)^* \;,$$

which turns out to be a 4^{th} order self-adjoint elliptic differential operator such that

$$\ker \tilde{\Delta}_A \;=\; \ker \partial\overline{\partial} \cap \ker \partial^* \cap \ker \overline{\partial}^* \;.$$

Hence one has an orthogonal decomposition

$$\wedge^{\bullet,\bullet}X \;=\; \ker \tilde{\Delta}_A \,\oplus\, \left(\operatorname{im}\partial \,+\, \operatorname{im}\overline{\partial}\right) \,\oplus\, \operatorname{im}\left(\partial\overline{\partial}\right)^*$$

from which one gets an isomorphism

$$H^{\bullet,\bullet}_{A}(X) \;\simeq\; \ker \tilde{\Delta}_A \;;$$

in particular, this proves that the Aeppli cohomology groups of a compact complex manifold are finite-dimensional $\mathbb{C}$-vector spaces.

Furthermore, as for the Bott-Chern cohomology, if $\{X_t\}_{t\in B}$ is a complex-analytic family of compact complex manifolds, with B a complex manifold, then, for every $p,q \in \mathbb{N}$, the function $B \ni t \mapsto \dim_{\mathbb{C}} H^{p,q}_{A}(X_t) \in \mathbb{N}$ is upper-semi-continuous.

Once again, whenever X is a compact Kähler manifold, by using the Kähler identities, one has

$$\tilde{\Delta}_A \;=\; \overline{\Box}^2 + \partial\partial^* + \overline{\partial}\overline{\partial}^* \;;$$

indeed, recall that $\overline{\square} = \square$ and that $\partial^*\bar\partial = \mathrm{i}\left[\Lambda, \bar\partial\right]\bar\partial = -\mathrm{i}\,\bar\partial\,\Lambda\,\bar\partial = -\mathrm{i}\,\bar\partial\left[\Lambda, \bar\partial\right] = -\bar\partial\partial^*$, and hence $\bar\partial^*\partial = -\partial\bar\partial^*$; therefore

$$
\begin{aligned}
\overline{\square}^2 \;=\; \overline{\square}\square \;&=\; \bar\partial\bar\partial^*\partial\partial^* + \bar\partial\bar\partial^*\partial^*\partial + \bar\partial^*\bar\partial\partial\partial^* + \bar\partial^*\bar\partial\partial^*\partial \\
&=\; -\bar\partial\partial\bar\partial^*\partial^* - \bar\partial\partial^*\bar\partial^*\partial - \bar\partial^*\partial\bar\partial\partial^* - \bar\partial^*\partial^*\bar\partial\partial \\
&=\; \partial\bar\partial\bar\partial^*\partial^* + \bar\partial\partial^*\partial\bar\partial^* + \partial\bar\partial^*\bar\partial\partial^* + \bar\partial^*\partial^*\partial\bar\partial \\
&=\; \tilde\Delta_A - \partial\partial^* - \bar\partial\bar\partial^* \,.
\end{aligned}
$$

In particular, it follows that, on a compact Kähler manifold, $\ker\tilde\Delta_A = \ker\overline{\square} = \ker\Delta$, and hence the de Rham cohomology, the Dolbeault cohomology, and the Aeppli cohomology are isomorphic (actually, since the $\partial\bar\partial$-Lemma holds on every compact Kähler manifold, one gets an isomorphism that does not depend on the choice of the Hermitian metric).

In fact, since $\ker\tilde\Delta_{BC} = \ker\partial \cap \ker\bar\partial \cap \ker\bar\partial^*\partial^*$ and $\ker\tilde\Delta_A = \ker\partial\bar\partial \cap \ker\partial^* \cap \ker\bar\partial^*$, one has the following isomorphism between the Bott-Chern cohomology and the Aeppli cohomology.

Theorem 1.6 ([Sch07, §2.c]). *Let X be a compact complex manifold of complex dimension n. For any $p, q \in \mathbb{N}$, the Hodge-$*$-operator associated to a Hermitian metric on X induces an isomorphism,*

$$
*\colon H^{p,q}_{BC}(X) \simeq H^{n-q,n-p}_{A}(X) \,,
$$

between the Bott-Chern and the Aeppli cohomologies.

Remark 1.7. We refer to [Dem12, §VI.12], [Sch07, §4], [Koo11, §3.2, §3.5] for a sheaf cohomology interpretation of the Bott-Chern and Aeppli cohomologies (see also Remark 1.56).

1.1.3 The $\partial\bar\partial$-Lemma

Let X be a compact complex manifold, and consider its complex de Rham $H^\bullet_{dR}(X;\mathbb{C})$, Dolbeault $H^{\bullet,\bullet}_{\bar\partial}(X)$, conjugate Dolbeault $H^{\bullet,\bullet}_{\partial}(X)$, Bott-Chern $H^{\bullet,\bullet}_{BC}(X)$, and Aeppli $H^{\bullet,\bullet}_{A}(X)$ cohomologies.

The identity map induces the following natural maps of (bi-)graded $\mathbb{C}$-vector spaces:

$$
\begin{tikzcd}
 & H^{\bullet,\bullet}_{BC}(X) \arrow[dl] \arrow[d] \arrow[dr] & \\
H^{\bullet,\bullet}_{\partial}(X) \arrow[dr] & H^{\bullet}_{dR}(X;\mathbb{C}) \arrow[d] & H^{\bullet,\bullet}_{\bar\partial}(X) \arrow[dl] \\
 & H^{\bullet,\bullet}_{A}(X) &
\end{tikzcd}
$$

In general, these maps are neither injective nor surjective: see, e.g., the examples in [Sch07, §1.c] or in §1.4.4.

By [DGMS75, Lemma 5.15, Proposition 5.17], it turns out that, if one of the above map is an isomorphism, then all the maps are isomorphisms, [DGMS75, Remark 5.16]; this is encoded in the notion of *$\partial\bar\partial$-Lemma*, which can be introduced in the more general setting of bounded double complexes of vector spaces. We start by recalling the following general result by P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan, [DGMS75].

Proposition 1.8 ([DGMS75, Lemma 5.15]). *Let $\left(K^{\bullet,\bullet}, \mathrm{d}', \mathrm{d}''\right)$ be a bounded double complex of vector spaces (or, more in general, of objects of any Abelian category), and let $\left(K^\bullet, \mathrm{d}\right)$ be the associated simple complex, where $\mathrm{d} := \mathrm{d}' + \mathrm{d}''$. For each $h \in \mathbb{N}$, the following conditions are equivalent:*

$(a)_h$ $\ker\mathrm{d}' \cap \ker\mathrm{d}'' \cap \mathrm{im}\,\mathrm{d} = \mathrm{im}\,\mathrm{d}'\,\mathrm{d}''$ *in K^h;*

$(b)_h$ $\ker\mathrm{d}'' \cap \mathrm{im}\,\mathrm{d}' = \mathrm{im}\,\mathrm{d}'\,\mathrm{d}''$ *and* $\ker\mathrm{d}' \cap \mathrm{im}\,\mathrm{d}'' = \mathrm{im}\,\mathrm{d}'\,\mathrm{d}''$ *in K^h;*

$(c)_h$ $\ker\mathrm{d}' \cap \ker\mathrm{d}'' \cap \left(\mathrm{im}\,\mathrm{d}' + \mathrm{im}\,\mathrm{d}''\right) = \mathrm{im}\,\mathrm{d}'\,\mathrm{d}''$ *in K^h;*

$(a^*)_{h-1}$ $\mathrm{im}\,\mathrm{d}' + \mathrm{im}\,\mathrm{d}'' + \ker\mathrm{d} = \ker\mathrm{d}'\,\mathrm{d}''$ *in K^{h-1};*

$(b^*)_{h-1}$ $\mathrm{im}\,\mathrm{d}'' + \ker\mathrm{d}' = \ker\mathrm{d}'\,\mathrm{d}''$ *and* $\mathrm{im}\,\mathrm{d}' + \ker\mathrm{d}'' = \ker\mathrm{d}'\,\mathrm{d}''$ *in K^{h-1};*

$(c^*)_{h-1}$ $\operatorname{im} \mathrm{d}' + \operatorname{im} \mathrm{d}'' + \left(\ker \mathrm{d}' \cap \ker \mathrm{d}''\right) = \ker \mathrm{d}' \mathrm{d}''$ *in* K^{h-1}.

The above equivalent conditions define the validity of the $\mathrm{d}' \mathrm{d}''$-Lemma for a double complex.

Definition 1.9 ([DGMS75]). Let $\left(K^{\bullet,\bullet},\, \mathrm{d}',\, \mathrm{d}''\right)$ be a bounded double complex of vector spaces (or, more in general, of objects of any Abelian category), and let $(K^\bullet,\, \mathrm{d})$ be the associated simple complex, where $\mathrm{d} := \mathrm{d}' + \mathrm{d}''$. One says that $\left(K^{\bullet,\bullet},\, \mathrm{d}',\, \mathrm{d}''\right)$ *satisfies the* $\mathrm{d}' \mathrm{d}''$*-Lemma* if, for every $h \in \mathbb{N}$, the equivalent conditions in [DGMS75, Lemma 5.15] hold.

The following result by P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan, [DGMS75], gives a characterization for the validity of the $\mathrm{d}' \mathrm{d}''$-Lemma.

Theorem 1.10 ([DGMS75, Proposition 5.17]). *Let* $\left(K^{\bullet,\bullet},\, \mathrm{d}',\, \mathrm{d}''\right)$ *be a bounded double complex of vector spaces, and let* $(K^\bullet,\, \mathrm{d})$ *be the associated simple complex, where* $\mathrm{d} := \mathrm{d}' + \mathrm{d}$. *The following conditions are equivalent:*

(i) $\left(K^{\bullet,\bullet},\, \mathrm{d}',\, \mathrm{d}''\right)$ *satisfies the* $\mathrm{d}' \mathrm{d}''$*-Lemma;*

(ii) $K^{\bullet,\bullet}$ *is a sum of double complexes of the following two types:*

(dots) *complexes which have only a single component, with* $\mathrm{d}' = 0$ *and* $\mathrm{d}'' = 0$*;*

(squares) *complexes which are a square of isomorphisms,*

$$\begin{tikzcd} K^{p-1,q} \arrow[r, "\mathrm{d}'"] & K^{p,q} \\ K^{p-1,q-1} \arrow[u, "\mathrm{d}''", "\simeq"'] \arrow[r, "\mathrm{d}'"] & K^{p,q-1} \arrow[u, "\mathrm{d}''", "\simeq"'] \end{tikzcd}$$

(iii) the spectral sequence defined by the filtration associated to either degree (denoted by $'F$ *or* $''F$*) degenerates at* E_1 *(namely,* $E_1 = E_\infty$*) and, for every* $h \in \mathbb{N}$*, the two induced filtrations are* h*-opposite on* $H^h_{dR}(X;\mathbb{C})$*, i.e.,* $'F^p \oplus {}''F^q \xrightarrow{\simeq} H^h_{dR}(X;\mathbb{C})$ *for* $p+q-1 = h$.

In particular, we are interested in dealing with compact complex manifolds X, where one considers the double complex $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$.

Definition 1.11 ([DGMS75]). A compact complex manifold X is said to *satisfy the* $\partial\overline{\partial}$*-Lemma* if $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ satisfies the $\partial\overline{\partial}$-Lemma, namely, if

$$\ker \partial \cap \ker \overline{\partial} \cap \operatorname{im} \mathrm{d} \;=\; \operatorname{im} \partial\overline{\partial} \;,$$

that is, in other words, if the natural map $H^{\bullet,\bullet}_{BC}(X) \to H^\bullet_{dR}(X;\mathbb{C})$ of graded $\mathbb{C}$-vector spaces induced by the identity is injective.

Remark 1.12. Let X be a compact complex manifold. By considering the differential operator

$$\mathrm{d}^c \;:=\; -\mathrm{i}\, \left(\partial - \overline{\partial}\right) \;,$$

one can say that X *satisfies the* $\mathrm{d}\,\mathrm{d}^c$, by definition, if

$$\operatorname{im} \mathrm{d} \cap \ker \mathrm{d}^c \;=\; \operatorname{im} \mathrm{d}\,\mathrm{d}^c \;.$$

Since $\mathrm{d}\,\mathrm{d}^c = 2\,\mathrm{i}\,\partial\overline{\partial}$, and $\partial = \frac{1}{2}\left(\mathrm{d} + \mathrm{i}\,\mathrm{d}^c\right)$ and $\overline{\partial} = \frac{1}{2}\left(\mathrm{d} - \mathrm{i}\,\mathrm{d}^c\right)$, one has

$$\ker \mathrm{d} \cap \ker \mathrm{d}^c \;=\; \ker \partial \cap \ker \overline{\partial} \qquad \text{and} \qquad \operatorname{im} \mathrm{d}\,\mathrm{d}^c = \operatorname{im} \partial\overline{\partial} \;;$$

and hence X satisfies the $\mathrm{d}\,\mathrm{d}^c$-Lemma if and only if X satisfies the $\partial\overline{\partial}$-Lemma.

For compact complex manifolds, P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan's characterization [DGMS75, Proposition 5.17] is rewritten as follows.

Theorem 1.13 ([DGMS75, 5.21]). *A compact complex manifold* X *satisfies the* $\partial\overline{\partial}$*-Lemma if and only if (i) the Hodge and Frölicher spectral sequence degenerates at the first step (that is,* $E_1 \simeq E_\infty$*), and (ii) the natural filtration on* $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ *induces, for every* $k \in \mathbb{N}$*, a Hodge structure of weight* k *on* $H^k_{dR}(X;\mathbb{C})$ *(that is,* $H^k_{dR}(X;\mathbb{C}) = \bigoplus_{p+q=k} F^p H^k_{dR}(X;\mathbb{C}) \cap \bar{F}^q H^k_{dR}(X;\mathbb{C})$*, where* $F^\bullet H^\bullet_{dR}(X;\mathbb{C})$ *is the filtration induced by* $F^\bullet \wedge^{\bullet_1,\bullet_2} X := \bigoplus_{p\geq\bullet,\, q} \wedge^{p,q} X$ *on* $H^\bullet_{dR}(X;\mathbb{C})$ *and* $\bar{F}^\bullet H^\bullet_{dR}(X;\mathbb{C})$ *is the conjugated filtration to* $F^\bullet H^\bullet_{dR}(X;\mathbb{C})$*).*

Another characterization for the validity of the $\partial\overline{\partial}$-Lemma, in terms of the dimensions of the Bott-Chern cohomology, will be given in Theorem 1.25.

Actually, as already mentioned, if a compact complex manifold satisfies the $\partial\overline{\partial}$-Lemma, then all the natural maps between cohomologies induced by the identity turn out to be isomorphisms.

Theorem 1.14 ([DGMS75, Lemma 5.15, Remark 5.16, 5.21])**.** *A compact complex manifold X satisfies the $\partial\overline{\partial}$-Lemma if and only if all the natural maps*

$$\begin{tikzcd} & H^{\bullet,\bullet}_{BC}(X) \arrow[dl] \arrow[d] \arrow[dr] & \\ H^{\bullet,\bullet}_{\partial}(X) \arrow[dr] & H^{\bullet}_{dR}(X;\mathbb{C}) \arrow[d] & H^{\bullet,\bullet}_{\overline{\partial}}(X) \arrow[dl] \\ & H^{\bullet,\bullet}_{A}(X) & \end{tikzcd}$$

induced by the identity are isomorphisms.

We recall that if X is a compact complex manifold endowed with a Kähler structure, then X satisfies the $\partial\overline{\partial}$-Lemma, [DGMS75, Lemma 5.11]. Moreover, one has the following result.

Theorem 1.15 ([DGMS75, Theorem 5.22])**.** *Let X and Y be compact complex manifolds of the same dimension, and let $f\colon X \to Y$ be a holomorphic birational map. If X satisfies the $\partial\overline{\partial}$-Lemma, then also Y satisfies the $\partial\overline{\partial}$-Lemma.*

Indeed, one has that, if X and Y are complex manifolds of the same dimension, and $\pi\colon X \to Y$ is a proper surjective holomorphic map, then the maps

$$\pi^*\colon H^{\bullet}_{dR}(Y;\mathbb{C}) \to H^{\bullet}_{dR}(X;\mathbb{C}) \qquad \text{and} \qquad \pi^*\colon H^{\bullet,\bullet}_{\overline{\partial}}(Y) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$$

induced by $\pi\colon X \to Y$ are injective, see, e.g., [Wel74, Theorem 3.1]; then one can use the characterization in [DGMS75, 5.21].

In particular, it follows that *Moĭšezon manifolds* (that is, compact complex manifolds X such that the degree of transcendence over $\mathbb{C}$ of the field of meromorphic functions over X is equal to the complex dimension of X, [Moı66], equivalently, compact complex manifolds admitting a proper modification from a projective manifold, [Moı66, Theorem 1]), and, more in general, manifolds *in class $\mathcal{C}$ of Fujiki* (that is, compact complex manifolds admitting a proper modification from a Kähler manifold, [Fuj78]) satisfy the $\partial\overline{\partial}$-Lemma. (We recall that a proper holomorphic map $f\colon X \to Y$ from the complex manifold X to the complex manifold Y is called a *modification* if there exists a nowhere dense closed analytic subset $B \subset Y$ such that $f\lfloor_{X\setminus f^{-1}(B)}\colon X \setminus f^{-1}(B) \to Y \setminus B$ is a biholomorphism.)

Corollary 1.16 ([DGMS75, Lemma 5.11, Corollary 5.23])**.** *The $\partial\overline{\partial}$-Lemma holds for compact Kähler manifolds, for Moĭšezon manifolds, and for manifolds in class $\mathcal{C}$ of Fujiki.*

Remark 1.17. In [Hir62], H. Hironaka provided an example of a non-Kähler Moĭšezon manifold of complex dimension 3 with arbitrary small deformations being projective (in fact, as stated by D. Popovici, the limit of projective manifolds under holomorphic deformations is Moĭšezon, [Pop09, Theorem 1.1], and, more in general, the limit of Moĭšezon manifolds under holomorphic deformations is Moĭšezon, [Pop10, Theorem 1.1]); in particular, H. Hironaka's manifold provides an example of a non-Kähler manifold satisfying the $\partial\overline{\partial}$-Lemma. Studying twistor spaces, C. LeBrun and Y. S. Poon, and F. Campana, showed that being in class $\mathcal{C}$ of Fujiki is not a stable property under small deformations of the complex structures, [LP92, Theorem 1], [Cam91, Corollary 3.13]; since the property of satisfying the $\partial\overline{\partial}$-Lemma is stable under small deformations of the complex structure, Corollary 1.28, or [Voi02, Proposition 9.21], or [Wu06, Theorem 5.12], or [Tom08, §B], C. LeBrun and Y. S. Poon's, and F. Campana's, result yields examples of compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma and not belonging to class $\mathcal{C}$ of Fujiki.

Finally, we recall the following obstructions to the existence of complex structures satisfying the $\partial\overline{\partial}$-Lemma on a compact (differentiable) manifold.

Theorem 1.18 ([DGMS75, Main Theorem, Corollary 1])**.** *Let X be a compact manifold. If X admits a complex structure such that the $\partial\overline{\partial}$-Lemma holds, then the differential graded algebra $(\wedge^{\bullet}X, \mathrm{d})$ is formal. In particular, all the Massey products of any order are zero.*

Indeed, if X satisfies the $\partial\overline{\partial}$-Lemma, equivalently, the $\mathrm{d}\,\mathrm{d}^c$-Lemma, then the inclusion $\ker \mathrm{d}^c \to \wedge^\bullet X$ and the projection $\ker \mathrm{d}^c \to \frac{\ker \mathrm{d}^c}{\mathrm{im}\, \mathrm{d}^c}$ induce the quasi-isomorphisms

$$\begin{tikzcd}[column sep=small] & (\ker \mathrm{d}^c,\, \mathrm{d}) \arrow[dl, "\text{qis}"'] \arrow[dr, "\text{qis}"] & \\ (\wedge^\bullet X,\, \mathrm{d}) & & \left(\frac{\ker \mathrm{d}^c}{\mathrm{im}\, \mathrm{d}^c},\, 0\right) \end{tikzcd}$$

of differential graded algebras, proving that $(\wedge^\bullet X,\, \mathrm{d})$ is equivalent to $\left(\frac{\ker \mathrm{d}^c}{\mathrm{im}\, \mathrm{d}^c},\, 0\right)$, and hence formal.

1.2 Cohomological properties of compact complex manifolds and the $\partial\overline{\partial}$-Lemma

In this section, we study some cohomological properties of compact complex manifolds, especially in relation with the $\partial\overline{\partial}$-Lemma. More precisely, we prove a Frölicher-type inequality for the Bott-Chern cohomology, Theorem 1.22, and we characterize the validity of the $\partial\overline{\partial}$-Lemma in terms of the dimensions of the Bott-Chern cohomology groups, Theorem 1.25. This has been the matter of a joint work with A. Tomassini, [AT12b].

Let X be a compact complex manifold of complex dimension n.

As a matter of notation, for every $p, q \in \mathbb{N}$, for every $k \in \mathbb{N}$, and for $\sharp \in \left\{\overline{\partial},\, \partial,\, BC,\, A\right\}$, we will denote

$$h^{p,q}_{\sharp} \;:=\; \dim_{\mathbb{C}} H^{p,q}_{\sharp}(X) \;<\; +\infty \qquad \text{and} \qquad h^{k}_{\sharp} \;:=\; \sum_{p+q=k} h^{p,q}_{\sharp} \;<\; +\infty \;,$$

while recall that the Betti numbers are denoted by

$$b_k \;:=\; \dim_{\mathbb{C}} H^k_{dR}(X;\mathbb{C}) \;<\; +\infty \;.$$

Recall that, for every $p, q \in \mathbb{N}$, the conjugation induces the isomorphisms $H^{p,q}_{BC}(X) \stackrel{\simeq}{\to} H^{q,p}_{BC}(X)$, $H^{p,q}_{A}(X) \stackrel{\simeq}{\to} H^{q,p}_{A}(X)$, and $H^{p,q}_{\overline{\partial}}(X) \stackrel{\simeq}{\to} H^{q,p}_{\partial}(X)$, and the Hodge-$*$-operator associated to any given Hermitian metric induces the isomorphisms $H^{p,q}_{BC}(X) \stackrel{\simeq}{\to} H^{n-q,n-p}_{A}(X)$ and $H^{p,q}_{\overline{\partial}}(X) \stackrel{\simeq}{\to} H^{n-q,n-p}_{\partial}(X)$; hence, for every $p, q \in \mathbb{N}$, one has the equalities

$$h^{p,q}_{BC} \;=\; h^{q,p}_{BC} \;=\; h^{n-p,n-q}_{A} \;=\; h^{n-q,n-p}_{A} \qquad \text{and} \qquad h^{p,q}_{\overline{\partial}} \;=\; h^{q,p}_{\partial} \;=\; h^{n-p,n-q}_{\overline{\partial}} \;=\; h^{n-q,n-p}_{\partial} \;,$$

and therefore, for every $k \in \mathbb{N}$, one has the equalities

$$h^{k}_{BC} \;=\; h^{2n-k}_{A} \qquad \text{and} \qquad h^{k}_{\overline{\partial}} \;=\; h^{k}_{\partial} \;=\; h^{2n-k}_{\overline{\partial}} \;=\; h^{2n-k}_{\partial} \;;$$

Finally, recall that the Hodge-$*$-operator (of any given Riemannian metric and volume form on X) yields, for every $k \in \mathbb{N}$, the isomorphism $H^k_{dR}(X;\mathbb{R}) \stackrel{\simeq}{\to} H^{2n-k}_{dR}(X;\mathbb{R})$, and hence the equality

$$b_k \;=\; b_{2n-k} \;.$$

1.2.1 J. Varouchas' exact sequences

In order to prove a Frölicher-type inequality for the Bott-Chern and Aeppli cohomologies and to give therefore a characterization of compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma in terms of the dimensions of their Bott-Chern cohomology groups, we need to recall two exact sequences from [Var86].

Following J. Varouchas, one defines the (finite-dimensional) bi-graded $\mathbb{C}$-vector spaces

$$A^{\bullet,\bullet} \;:=\; \frac{\mathrm{im}\,\overline{\partial} \cap \mathrm{im}\,\partial}{\mathrm{im}\,\partial\overline{\partial}} \;, \qquad B^{\bullet,\bullet} \;:=\; \frac{\ker\overline{\partial} \cap \mathrm{im}\,\partial}{\mathrm{im}\,\partial\overline{\partial}} \;, \qquad C^{\bullet,\bullet} \;:=\; \frac{\ker\partial\overline{\partial}}{\ker\overline{\partial} + \mathrm{im}\,\partial}$$

and

$$D^{\bullet,\bullet} \;:=\; \frac{\mathrm{im}\,\overline{\partial} \cap \ker\partial}{\mathrm{im}\,\partial\overline{\partial}} \;, \qquad E^{\bullet,\bullet} \;:=\; \frac{\ker\partial\overline{\partial}}{\ker\partial + \mathrm{im}\,\overline{\partial}} \;, \qquad F^{\bullet,\bullet} \;:=\; \frac{\ker\partial\overline{\partial}}{\ker\overline{\partial} + \ker\partial} \;.$$

For every $p, q \in \mathbb{N}$ and $k \in \mathbb{N}$, we will denote their dimensions by

$$a^{p,q} := \dim_{\mathbb{C}} A^{p,q} , \qquad b^{p,q} := \dim_{\mathbb{C}} B^{p,q} , \qquad c^{p,q} := \dim_{\mathbb{C}} C^{p,q} ,$$
$$d^{p,q} := \dim_{\mathbb{C}} D^{p,q} , \qquad e^{p,q} := \dim_{\mathbb{C}} E^{p,q} , \qquad f^{p,q} := \dim_{\mathbb{C}} F^{p,q} ,$$

and

$$a^{k} := \sum_{p+q=k} a^{p,q} , \qquad b^{k} := \sum_{p+q=k} b^{p,q} , \qquad c^{k} := \sum_{p+q=k} c^{p,q} ,$$
$$d^{k} := \sum_{p+q=k} d^{p,q} , \qquad e^{k} := \sum_{p+q=k} e^{p,q} , \qquad f^{k} := \sum_{p+q=k} f^{p,q} .$$

The previous vector spaces give the following exact sequences, by J. Varouchas.

Theorem 1.19 ([Var86, §3.1]). *The sequences*

$$0 \to A^{\bullet,\bullet} \to B^{\bullet,\bullet} \to H^{\bullet,\bullet}_{\overline{\partial}}(X) \to H^{\bullet,\bullet}_{A}(X) \to C^{\bullet,\bullet} \to 0 \tag{1.2.1}$$

and

$$0 \to D^{\bullet,\bullet} \to H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet,\bullet}_{\overline{\partial}}(X) \to E^{\bullet,\bullet} \to F^{\bullet,\bullet} \to 0 \tag{1.2.2}$$

are exact sequences of finite-dimensional bi-graded $\mathbb{C}$-vector spaces.

Proof. We first prove the exactness of (1.2.1). Since $\operatorname{im}\overline{\partial} \subseteq \ker\overline{\partial}$, the map $A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ is injective. The kernel of the map $B^{\bullet,\bullet} \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$ is $\frac{\ker\overline{\partial}\cap\operatorname{im}\partial\cap\operatorname{im}\overline{\partial}}{\operatorname{im}\partial\overline{\partial}} = \frac{\operatorname{im}\overline{\partial}\cap\operatorname{im}\partial}{\operatorname{im}\partial\overline{\partial}}$, that is, the image of the map $A^{\bullet,\bullet} \to B^{\bullet,\bullet}$. The kernel of the map $H^{\bullet,\bullet}_{\overline{\partial}}(X) \to H^{\bullet,\bullet}_{A}(X)$ is $\frac{\ker\overline{\partial}\cap\operatorname{im}\partial}{\operatorname{im}\overline{\partial}}$, that is, the image of the map $B^{\bullet,\bullet} \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$. The kernel of the map $H^{\bullet,\bullet}_{A}(X) \to C^{\bullet,\bullet}$ is $\frac{\ker\overline{\partial}\cap\ker\partial\overline{\partial}}{\operatorname{im}\partial+\operatorname{im}\overline{\partial}} = \frac{\ker\overline{\partial}}{\operatorname{im}\partial+\operatorname{im}\overline{\partial}}$, that is, the image of the map $H^{\bullet,\bullet}_{\overline{\partial}}(X) \to H^{\bullet,\bullet}_{A}(X)$. Finally, since $\operatorname{im}\partial + \operatorname{im}\overline{\partial} \subseteq \ker\overline{\partial} + \operatorname{im}\partial$, the map $H^{\bullet,\bullet}_{A}(X) \to C^{\bullet,\bullet}$ is surjective. In particular, since $H^{\bullet,\bullet}_{A}(X) \to C^{\bullet,\bullet}$ is surjective, then $C^{\bullet,\bullet}$ has finite dimension; since the identity induces an injective map $B^{\bullet,\bullet} \to H^{\bullet,\bullet}_{BC}(X)$, then $B^{\bullet,\bullet}$ has finite dimension; hence, since $A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ is injective, then also $A^{\bullet,\bullet}$ has finite dimension.

We prove now the exactness of (1.2.2). Since $\operatorname{im}\overline{\partial} \subseteq \ker\overline{\partial}$, the map $D^{\bullet,\bullet} \to H^{\bullet,\bullet}_{BC}(X)$ is injective. The kernel of the map $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$ is $\frac{\ker\partial\cap\ker\overline{\partial}\cap\operatorname{im}\overline{\partial}}{\operatorname{im}\partial\overline{\partial}} = \frac{\operatorname{im}\overline{\partial}\cap\ker\partial}{\operatorname{im}\partial\overline{\partial}}$, that is, the image of the map $D^{\bullet,\bullet} \to H^{\bullet,\bullet}_{BC}(X)$. The kernel of the map $H^{\bullet,\bullet}_{\overline{\partial}}(X) \to E^{\bullet,\bullet}$ is $\frac{\ker\overline{\partial}\cap\left(\ker\partial+\operatorname{im}\overline{\partial}\right)}{\operatorname{im}\overline{\partial}} = \frac{\ker\overline{\partial}\cap\ker\partial}{\operatorname{im}\overline{\partial}}$, that is, the image of the map $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$. The kernel of the map $E^{\bullet,\bullet} \to F^{\bullet,\bullet}$ is $\frac{\ker\partial\overline{\partial}\cap\left(\ker\overline{\partial}+\ker\partial\right)}{\ker\partial+\operatorname{im}\overline{\partial}} = \frac{\ker\partial\overline{\partial}\cap\ker\overline{\partial}}{\ker\partial+\operatorname{im}\overline{\partial}}$, that is, the image of the map $H^{\bullet,\bullet}_{\overline{\partial}}(X) \to E^{\bullet,\bullet}$. Finally, since $\ker\partial + \operatorname{im}\overline{\partial} \subseteq \ker\overline{\partial} + \ker\partial$, the map $E^{\bullet,\bullet} \to F^{\bullet,\bullet}$ is surjective. In particular, since $D^{\bullet,\bullet} \to H^{\bullet,\bullet}_{BC}(X)$ is injective, then $D^{\bullet,\bullet}$ has finite dimension; since the identity induces a surjective map $H^{\bullet,\bullet}_{A}(X) \to E^{\bullet,\bullet}$, then $E^{\bullet,\bullet}$ has finite dimension; hence, since $E^{\bullet,\bullet} \to F^{\bullet,\bullet}$ is surjective, then also $F^{\bullet,\bullet}$ has finite dimension.

□

Note, [Var86, §3.1], that the conjugation yields, for every $p, q \in \mathbb{N}$, the equalities

$$a^{p,q} = a^{q,p} , \qquad f^{p,q} = f^{q,p} , \qquad d^{p,q} = b^{q,p} , \qquad e^{p,q} = c^{q,p} , \tag{1.2.3}$$

and the isomorphisms $\overline{\partial} \colon C^{\bullet,\bullet} \stackrel{\simeq}{\to} D^{\bullet,\bullet+1}$ and $\partial \colon E^{\bullet,\bullet} \stackrel{\simeq}{\to} B^{\bullet+1,\bullet}$ yield the equalities

$$c^{p,q} = d^{p,q+1} , \qquad e^{p,q} = b^{p+1,q} ;$$

hence, for every $k \in \mathbb{N}$, one gets the equalities

$$d^{k} = b^{k} , \qquad e^{k} = c^{k} , \qquad \text{and} \qquad c^{k} = d^{k+1} , \qquad e^{k} = b^{k+1} .$$

Remark 1.20. Following the same argument used in [Sch07] to prove the duality between Bott-Chern and Aeppli cohomology groups, we can prove the duality between $A^{\bullet,\bullet}$ and $F^{\bullet,\bullet}$, and, similarly, between $C^{\bullet,\bullet}$ and $\overline{D^{\bullet,\bullet}}$.

Indeed, note that the pairing

$$A^{\bullet,\bullet} \times F^{\bullet,\bullet} \to \mathbb{C} , \qquad ([\alpha] , [\beta]) \mapsto \int_X \alpha \wedge \overline{\beta} ,$$

is non-degenerate: choose a Hermitian metric g on X; if $[\alpha] \in A^{\bullet,\bullet} \subseteq H^{\bullet,\bullet}_{BC}(X)$, then there exists a $\tilde{\Delta}_{BC}$-harmonic representative $\tilde{\alpha}$ in $[\alpha] \in A^{\bullet,\bullet}$, by [Sch07, Corollaire 2.3], that is, $\partial\tilde{\alpha} = \overline{\partial}\tilde{\alpha} = \partial\overline{\partial} * \tilde{\alpha} = 0$; hence, $[*\tilde{\alpha}] \in F^{\bullet,\bullet}$, and $([\tilde{\alpha}]\,,\,[*\tilde{\alpha}]) = \int_X \tilde{\alpha} \wedge \overline{*\tilde{\alpha}}$ is zero if and only if $\tilde{\alpha}$ is zero if and only if $[\alpha] \in A^{\bullet,\bullet}$ is zero.

Analogously, the pairing

$$C^{\bullet,\bullet} \times \overline{D^{\bullet,\bullet}} \to \mathbb{C}\;, \qquad ([\alpha]\,,\,[\beta]) \mapsto \int_X \alpha \wedge \overline{\beta}\;,$$

is non-degenerate: indeed, choose a Hermitian metric g on X; if $[\alpha] \in \overline{D^{\bullet,\bullet}} \subseteq \overline{H^{\bullet,\bullet}_{BC}(X)}$, then there exists a $\tilde{\Delta}_{BC}$-harmonic representative $\tilde{\alpha}$ in $[\alpha] \in \overline{D^{\bullet,\bullet}}$, by [Sch07, Corollaire 2.3], that is, $\partial\tilde{\alpha} = \overline{\partial}\tilde{\alpha} = \partial\overline{\partial} * \tilde{\alpha} = 0$; hence, $[*\tilde{\alpha}] \in C^{\bullet,\bullet}$, and $([\tilde{\alpha}]\,,\,[*\tilde{\alpha}]) = \int_X \tilde{\alpha} \wedge \overline{*\tilde{\alpha}}$ is zero if and only if $\tilde{\alpha}$ is zero if and only if $[\alpha] \in \overline{D^{\bullet,\bullet}}$ is zero.

1.2.2 A Frölicher-type inequality for the Bott-Chern cohomology

We can now state and prove a Frölicher-type inequality for the Bott-Chern and Aeppli cohomologies, Theorem 1.22.

Firstly, we recall that, on a compact complex manifold X, the *Frölicher inequality* [Frö55, Theorem 2] relates the Hodge numbers and the Betti numbers.

Theorem 1.21 ([Frö55, Theorem 2]). *Let X be a compact complex manifold. Then, for every $k \in \mathbb{N}$, the following inequality holds:*

$$\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) \;\geq\; \dim_{\mathbb{C}} H^k_{dR}(X;\mathbb{C})\;.$$

The equality $\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) = \dim_{\mathbb{C}} H^k_{dR}(X;\mathbb{C})$ holds for every $k \in \mathbb{N}$ if and only if the Hodge and Frölicher spectral sequence $\{(E_r,\, \mathrm{d}_r)\}_{r\in\mathbb{N}}$ degenerates at the first step.

It is in general not true that h^k_{BC} (respectively, h^k_A) is higher than the k^{th} Betti number of X for every $k \in \mathbb{N}$: an example is provided by the small deformations of the Iwasawa manifold $\mathbb{I}_3 := \mathbb{H}\left(3;\mathbb{Z}\left[\mathrm{i}\right]\right)\backslash\,\mathbb{H}(3;\mathbb{C})$ (see §1.4.1). In the following table, we summarize the dimensions of the Bott-Chern and Aeppli cohomology groups for $\mathbb{I}_3$ (which have been computed in [Sch07, Proposition 1.2]) and for the small deformations of $\mathbb{I}_3$ (see §1.4.4). We recall that the small deformations of the Iwasawa manifold, according to I. Nakamura's classification, [Nak75, §3], are divided into three classes, *(i)*, *(ii)*, and *(iii)*, in terms of their Hodge numbers; it turns out that the Bott-Chern cohomology yields a finer classification of the Kuranishi space of $\mathbb{I}_3$, allowing a further subdivision of class *(ii)*, respectively class *(iii)*, into subclasses *(ii.a)*, *(ii.b)*, respectively *(iii.a)*, *(iii.b)*, see §1.4.1.

classes	$\mathbf{h^1_{\overline{\partial}}}$	$\mathbf{h^1_{BC}}$	$\mathbf{h^1_{A}}$	$\mathbf{h^2_{\overline{\partial}}}$	$\mathbf{h^2_{BC}}$	$\mathbf{h^2_{A}}$	$\mathbf{h^3_{\overline{\partial}}}$	$\mathbf{h^3_{BC}}$	$\mathbf{h^3_{A}}$	$\mathbf{h^4_{\overline{\partial}}}$	$\mathbf{h^4_{BC}}$	$\mathbf{h^4_{A}}$	$\mathbf{h^5_{\overline{\partial}}}$	$\mathbf{h^5_{BC}}$	$\mathbf{h^5_{A}}$
(i)	5	4	6	11	10	12	14	14	14	11	12	10	5	6	4
(ii.a)	4	4	6	9	8	11	12	14	14	9	11	8	4	6	4
(ii.b)	4	4	6	9	8	10	12	14	14	9	10	8	4	6	4
(iii.a)	4	4	6	8	6	11	10	14	14	8	11	6	4	6	4
(iii.b)	4	4	6	8	6	10	10	14	14	8	10	6	4	6	4
		$\mathbf{b_1 = 4}$			$\mathbf{b_2 = 8}$			$\mathbf{b_3 = 10}$			$\mathbf{b_4 = 8}$			$\mathbf{b_5 = 4}$	

The following result, [AT12b, Theorem A], gives a Frölicher-type inequality for the Bott-Chern cohomology. (We recall that, on a compact complex manifold X of complex dimension n, for any $p,q \in \mathbb{N}$, one has the equality $\dim_{\mathbb{C}} H^{p,q}_{BC}(X) = \dim_{\mathbb{C}} H^{n-q,n-p}_{A}(X)$, and, for any $k \in \mathbb{N}$, the equality $\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{BC}(X) = \sum_{r+s=2n-k} \dim_{\mathbb{C}} H^{r,s}_{A}(X)$, [Sch07, §2.c].)

Theorem 1.22. *Let X be a compact complex manifold. Then, for every $p,q \in \mathbb{N}$, the following inequality holds:*

$$\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X) \;\geq\; \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) + \dim_{\mathbb{C}} H^{p,q}_{\partial}(X)\;. \tag{1.2.4}$$

In particular, for every $k \in \mathbb{N}$, the following inequality holds:

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right) \;\geq\; 2\,\dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C})\;. \tag{1.2.5}$$

Proof. Fix $p, q \in \mathbb{N}$. The exact sequences (1.2.1), respectively (1.2.2), yield the equality

$$h^{p,q}_A \;=\; h^{p,q}_{\overline{\partial}} + c^{p,q} + a^{p,q} - b^{p,q} \;,$$

respectively

$$h^{p,q}_{BC} \;=\; h^{p,q}_{\overline{\partial}} + d^{p,q} + f^{p,q} - e^{p,q} \;;$$

using also the symmetries $h^{p,q}_A = h^{q,p}_A$ and $h^{p,q}_{\overline{\partial}} = h^{q,p}_{\partial}$, and the equalities (1.2.3), we get

$$\begin{aligned} h^{p,q}_{BC} + h^{p,q}_A &= h^{p,q}_{BC} + h^{q,p}_A \\ &= h^{p,q}_{\overline{\partial}} + h^{q,p}_{\overline{\partial}} + f^{p,q} + a^{q,p} + d^{p,q} - b^{q,p} - e^{p,q} + c^{q,p} \\ &= h^{p,q}_{\overline{\partial}} + h^{p,q}_{\partial} + f^{p,q} + a^{p,q} \\ &\geq h^{p,q}_{\overline{\partial}} + h^{p,q}_{\partial} \;, \end{aligned}$$

which proves (1.2.4).

Now, fix $k \in \mathbb{N}$; summing over $(p,q) \in \mathbb{N} \times \mathbb{N}$ such that $p+q=k$, we get

$$\begin{aligned} h^k_{BC} + h^k_A &= \sum_{p+q=k} \left(h^{p,q}_{BC} + h^{p,q}_A\right) \\ &\geq \sum_{p+q=k} \left(h^{p,q}_{\overline{\partial}} + h^{p,q}_{\partial}\right) \;=\; h^k_{\overline{\partial}} + h^k_{\partial} \\ &\geq 2\, b_k \;, \end{aligned}$$

from which we get (1.2.5). □

Remark 1.23. Note that small deformations of the Iwasawa manifold show that both the inequalities (1.2.4) and (1.2.5) can be strict.

For example, for small deformations of $\mathbb{I}_3$ in class *(i)*, one has,

$$h^1_{BC} + h^1_A \;=\; 10 \;>\; 8 \;=\; 2 \cdot b_1 \;, \qquad h^2_{BC} + h^2_A \;=\; 22 \;>\; 16 \;=\; 2 \cdot b_2 \;, \qquad h^3_{BC} + h^3_A \;=\; 28 \;>\; 20 \;=\; 2 \cdot b_3 \;,$$

showing that (1.2.5) is strict for every $k \in \{1,\, 2,\, 3,\, 4,\, 5\}$.

On the other hand, for small deformations of $\mathbb{I}_3$ in class *(ii)* or in class *(iii)*, one has

$$h^{1,0}_{BC} + h^{1,0}_A \;=\; \frac{1}{2}\left(h^1_{BC} + h^1_A\right) \;=\; 5 \;>\; 4 \;=\; h^1_{\overline{\partial}} \;=\; h^{1,0}_{\overline{\partial}} + h^{0,1}_{\overline{\partial}} \;=\; h^{1,0}_{\overline{\partial}} + h^{1,0}_{\partial} \;,$$

showing that (1.2.4) is strict, for example, for $(p,q) = (1,0)$.

(For further examples among the small deformations of the Iwasawa manifold, compare the computations in §1.4.4, which are summarized in §1.4.5.)

Remark 1.24. Note that, in the proof of Theorem 1.22, we have actually shown that, for every $k \in \mathbb{N}$,

$$h^k_{BC} + h^k_A \;=\; 2\, h^k_{\overline{\partial}} + a^k + f^k \;.$$

1.2.3 A characterization of the $\partial\overline{\partial}$-Lemma in terms of the Bott-Chern cohomology

This section is devoted to give a characterization of the validity of the $\partial\overline{\partial}$-Lemma in terms of the Bott-Chern cohomology.

Note that, if a compact complex manifold X satisfies the $\partial\overline{\partial}$-Lemma, then, for every $k \in \mathbb{N}$, it holds $h^k_{BC} = h^k_A = h^k_{\overline{\partial}} = h^k_{\partial} = b_k$, and hence (1.2.5) is actually an equality. In fact, we prove now that also the converse holds true: more precisely, the equality in (1.2.5) holds for every $k \in \mathbb{N}$ if and only if the $\partial\overline{\partial}$-Lemma holds; in particular, this gives a characterization of the validity of the $\partial\overline{\partial}$-Lemma just in terms of $\left\{h^k_{BC}\right\}_{k\in\mathbb{N}}$, [AT12b, Theorem B].

Theorem 1.25. *Let X be a compact complex manifold. The equality*

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_A(X)\right) \;=\; 2\, \dim_{\mathbb{C}} H^k_{dR}(X;\mathbb{C})$$

holds for every $k \in \mathbb{N}$ if and only if X satisfies the $\partial\overline{\partial}$-Lemma.

Proof. If X satisfies the $\partial\overline{\partial}$-Lemma, then the natural maps $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$, $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$, and $H^{\bullet,\bullet}_{\overline{\partial}}(X) \to H^{\bullet,\bullet}_{A}(X)$, $H^{\bullet}_{dR}(X;\mathbb{C}) \to H^{\bullet,\bullet}_{A}(X)$ induced by the identity are isomorphisms, [DGMS75, Remark 5.16], and hence, for every $k \in \mathbb{N}$, one has

$$h^k_{BC} \;=\; h^k_A \;=\; h^k_{\overline{\partial}} \;=\; b_k$$

and hence, in particular,

$$h^k_{BC} + h^k_A \;=\; 2\, b_k \;.$$

We split the proof of the converse into the following claims.

`Claim 1` – *If $h^k_{BC} + h^k_A = 2\, b_k$ holds for every $k \in \mathbb{N}$, then the Hodge and Frölicher spectral sequences degenerate at the first step (namely, $E_1 \simeq E_\infty$, that is, $h^k_{\overline{\partial}} = b_k$ for every $k \in \mathbb{N}$) and $a^k = 0 = f^k$ for every $k \in \mathbb{N}$.*

Since, for every $k \in \mathbb{N}$, we have

$$2\, b_k \;=\; h^k_{BC} + h^k_A \;=\; 2\, h^k_{\overline{\partial}} + a^k + f^k \;\geq\; 2\, b_k \;,$$

then $h^k_{\overline{\partial}} = b_k$ and $a^k = 0 = f^k$ for every $k \in \mathbb{N}$.

`Claim 2` – *Fix $k \in \mathbb{N}$. If $a^{k+1} := \sum_{p+q=k+1} \dim_{\mathbb{C}} A^{p,q} = 0$, then the natural map*

$$\bigoplus_{p+q=k} H^{p,q}_{BC}(X) \to H^k_{dR}(X;\mathbb{C})$$

is surjective.

Let $\mathfrak{a} = [\alpha] \in H^k_{dR}(X;\mathbb{C})$. We have to prove that $\mathfrak{a}$ admits a representative whose pure-type components are d-closed. Consider the pure-type decomposition of α:

$$\alpha \;=:\; \sum_{j=0}^{k} (-1)^j \; \alpha^{k-j,j} \;,$$

where $\alpha^{k-j,j} \in \wedge^{k-j,j}X$. Since $\mathrm{d}\,\alpha = 0$, we get that

$$\partial\alpha^{k,0} = 0 \;, \qquad \overline{\partial}\alpha^{k-j,j} - \partial\alpha^{k-j-1,j+1} = 0 \text{ for } j \in \{0,\ldots,k-1\} \;, \qquad \overline{\partial}\alpha^{0.k} = 0 \;;$$

by the hypothesis $a^{k+1} = 0$, for every $j \in \{0,\ldots,k-1\}$, we get that,

$$\overline{\partial}\alpha^{k-j,j} \;=\; \partial\alpha^{k-j-1,j+1} \;\in\; \left(\operatorname{im}\overline{\partial} \cap \operatorname{im}\partial\right) \cap \wedge^{k-j,j+1}X \;=\; \operatorname{im}\partial\overline{\partial} \cap \wedge^{k-j,j+1}X$$

and hence there exists $\eta^{k-j-1,j} \in \wedge^{k-j-1,j}X$ such that

$$\overline{\partial}\alpha^{k-j,j} \;=\; \partial\overline{\partial}\eta^{k-j-1,j} \;=\; \partial\alpha^{k-j-1,j+1} \;.$$

Define

$$\eta \;:=\; \sum_{j=0}^{k-1} (-1)^j \; \eta^{k-j-1,j} \;\in\; \wedge^{k-1}X \otimes_{\mathbb{R}} \mathbb{C} \;.$$

The claim follows noting that

$$\begin{aligned}
\mathfrak{a} \;&=\; [\alpha] \;=\; [\alpha + \mathrm{d}\,\eta] \\
&=\; \left[\left(\alpha^{k,0} + \partial\eta^{k-1,0}\right) + \sum_{j=1}^{k-1} (-1)^j \; \left(\alpha^{k-j,j} + \partial\eta^{k-j-1,j} - \overline{\partial}\eta^{k-j,j-1}\right) + (-1)^k \; \left(\alpha^{0,k} - \overline{\partial}\eta^{0,k-1}\right)\right] \\
&=\; \left[\alpha^{k,0} + \partial\eta^{k-1,0}\right] + \sum_{j=1}^{k-1} (-1)^j \; \left[\alpha^{k-j,j} + \partial\eta^{k-j-1,j} - \overline{\partial}\eta^{k-j,j-1}\right] + (-1)^k \; \left[\alpha^{0,k} - \overline{\partial}\eta^{0,k-1}\right] \;,
\end{aligned}$$

that is, each of the pure-type components of $\alpha + \mathrm{d}\,\eta$ is both ∂-closed and $\overline{\partial}$-closed.

`Claim 3` – *If $h^k_{BC} \geq b_k$ and $h^k_{BC} + h^k_A = 2\, b_k$ for every $k \in \mathbb{N}$, then $h^k_{BC} = b_k$ for every $k \in \mathbb{N}$.*

If n is the complex dimension of X, then, for every $k \in \mathbb{N}$, we have

$$b_k \;\leq\; h^k_{BC} \;=\; h^{2n-k}_A \;=\; 2\, b_{2n-k} - h^{2n-k}_{BC} \;\leq\; b_{2n-k} \;=\; b_k$$

and hence $h^k_{BC} = b_k$ for every $k \in \mathbb{N}$.

Now, by `Claim 1`, we get that $a^k = 0$ for each $k \in \mathbb{N}$; hence, by `Claim 2`, for every $k \in \mathbb{N}$ the natural map

$$\bigoplus_{p+q=k} H^{p,q}_{BC}(X) \to H^k_{dR}(X;\mathbb{C})$$

induced by the identity is surjective, and hence, in particular, $h^k_{BC} \geq b_k$. By `Claim 3` we get therefore that $h^k_{BC} = b_k$ for every $k \in \mathbb{N}$. Hence, the natural map $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$ is actually an isomorphism, which is equivalent to say that X satisfies the $\partial\overline{\partial}$-Lemma. □

Remark 1.26. We note that, using the exact sequences (1.2.2) and (1.2.1), one can prove that, on a compact complex manifold X and for every $k \in \mathbb{N}$,

$$\begin{aligned} e^k &= \left(h^k_{\overline{\partial}} - h^k_{BC}\right) + f^k + c^{k-1} \\ &= \left(h^k_{\overline{\partial}} - h^k_{BC}\right) - \left(h^{k-1}_{\overline{\partial}} - h^{k-1}_{A}\right) + f^k - a^{k-1} + e^{k-2} \,. \end{aligned}$$

Remark 1.27. Note that $E_1 \simeq E_\infty$ is not sufficient to have the equality $h^k_{BC} + h^k_A = 2\,b_k$ for every $k \in \mathbb{N}$ (and hence the $\partial\overline{\partial}$-Lemma): a counterexample is provided by small deformations of the Iwasawa manifold.

Indeed, for small deformations of $\mathbb{I}_3$ in class *(iii)*, since

$$h^1_{\overline{\partial}} = 4 = b_1 \,, \qquad h^2_{\overline{\partial}} = 8 = b_2 \,, h^3_{\overline{\partial}} = 10 = b_3 \,,$$

the Hodge and Frölicher spectral sequences degenerate at the first step, but

$$h^1_{BC} + h^1_A = 10 > 8 = 2\,b_1 \,, \qquad h^2_{BC} + h^2_A = 16 = 2\,b_2 \,, \qquad h^3_{BC} + h^3_A = 28 > 20 = 2\,b_3 \,.$$

Using Theorem 1.25, we get another proof of the stability of the $\partial\overline{\partial}$-Lemma under small deformations of the complex structure, [AT12b, Corollary 2.7]; for different proofs of the same result by means of other techniques see, e.g., [Voi02, Proposition 9.21], [Wu06, Theorem 5.12], [Tom08, §B].

Corollary 1.28. *Satisfying the $\partial\overline{\partial}$-Lemma is a stable property under small deformations of the complex structure.*

Proof. Let $\{X_t\}_{t\in B}$ be a complex-analytic family of compact complex manifolds. Since, for every $k \in \mathbb{N}$, the dimensions $h^k_{BC}(X_t)$ and $h^k_A(X_t)$ are upper-semi-continuous functions in t, [Sch07, Lemme 3.2], while the dimensions $b_k(X_t)$ are constant in t by Ehresmann's theorem, one gets that, if X_{t_0} satisfies the equality $h^k_{BC}(X_{t_0}) + h^k_A(X_{t_0}) = 2\,b_k(X_{t_0})$ for every $k \in \mathbb{N}$, the same holds true for X_t with t near t_0. □

We recall that [DGMS75, 5.21] by P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan characterizes the validity of the $\partial\overline{\partial}$-Lemma on a compact complex manifold in terms of the degeneracy of the Hodge and Frölicher spectral sequence and of the existence of Hodge structures in cohomology. In particular, if follows that, on a compact complex manifold satisfying the $\partial\overline{\partial}$-Lemma, one has the equality $b_k = \sum_{p+q=k} h^{p,q}_{\overline{\partial}}$ for every $k \in \mathbb{N}$ (which is equivalent to the degeneracy of the Hodge and Frölicher spectral sequence) and the symmetry $h^{p,q}_{\overline{\partial}} = h^{q,p}_{\overline{\partial}}$ for every $p, q \in \mathbb{N}$.

Note that, on a compact complex surface X, since the Hodge and Frölicher spectral sequence degenerates at the first step (see, e.g., [BHPVdV04, Theorem IV.2.8]) if $h^{1,0}_{\overline{\partial}} = h^{0,1}_{\overline{\partial}}$ then $b_1 = 2\,h^{1,0}_{\overline{\partial}}$ is even, and hence X is Kähler, by [Kod64, Miy74, Siu83], or [Lam99, Corollaire 5.7], or [Buc99, Theorem 11]. As already remarked, the small deformations of $\mathbb{I}_3$ in class *(iii)* satisfy the degeneracy condition of the Hodge and Frölicher spectral sequence, but they do not satisfy either the $\partial\overline{\partial}$-Lemma, or the symmetry of the Hodge numbers.

It could hence be interesting to construct a compact complex manifold (of any complex dimension greater than or equal to 3) such that $E_1 \simeq E_\infty$ and $h^{p,q}_{\overline{\partial}} = h^{p,q}_{\overline{\partial}}$ for every $p, q \in \mathbb{N}$ but for which the $\partial\overline{\partial}$-Lemma does not hold. A compact complex manifold X whose double complex $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ has the form in Figure 1.1 (where dots denote generators of the $\mathcal{C}^\infty(X;\mathbb{R})$-module $\wedge^{\bullet,\bullet}X$, horizontal arrows are meant as ∂, vertical ones as $\overline{\partial}$ and zero arrows are not depicted) provides such an example.

Remark 1.29. L. Ugarte informed us that M. Ceballos, A. Otal, he himself, and R. Villacampa have found such an example among the 6-dimensional nilmanifolds endowed with left-invariant complex structures: they provided a complete classification, up to equivalence, of the linear integrable complex structures on 6-dimensional nilpotent Lie algebras in [COUV12], where they also studied some applications of their classification.

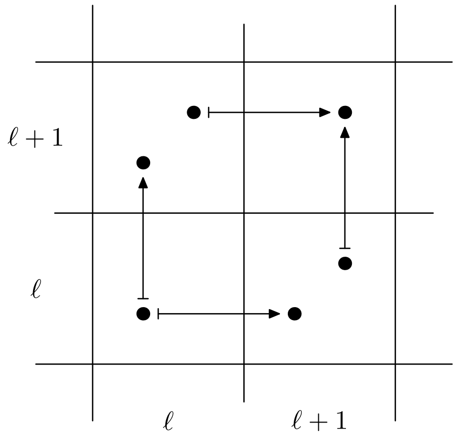

Figure 1.1: An abstract example

1.3 Cohomology computations for special nilmanifolds

We are now interested in studying the Bott-Chern and Aeppli cohomologies in the special case of left-invariant complex structures on nilmanifolds and solvmanifolds.

In this section, we firstly recall some results concerning the computation of the de Rham cohomology and of the Dolbeault cohomology, for nilmanifolds and solvmanifolds, endowed with left-invariant complex structures, §1.3.2, referring to [Nom54, Hat60], respectively [Sak76, CFGU00, CF01, Rol09a, Rol11a]; then, we state and prove the results obtained in [Ang11] about the computation of the Bott-Chern and Aeppli cohomologies, Theorem 1.37, Theorem 1.42. Using these tools, one can compute the de Rham, Dolbeault, Bott-Chern and Aeppli cohomologies for the Iwasawa manifold and for its small deformations, §1.4.2, §1.4.3, §1.4.4.

1.3.1 Left-invariant complex structures on solvmanifolds

We start by recalling some facts and notations concerning left-invariant complex structures on solvmanifolds.

Let $X = \Gamma\backslash G$ be a solvmanifold, that is, a compact quotient of a connected simply-connected solvable Lie group G by a discrete and co-compact subgroup Γ; the Lie algebra naturally associated to G will be denoted by $\mathfrak{g}$ and its complexification by $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We recall that, dealing with G-left-invariant objects on X, we mean objects on X obtained by objects on G that are invariant under the action of G on itself given by left-translations; note that G-left-invariant objects on X are uniquely determined by objects on $\mathfrak{g}$. In particular, a G-left-invariant complex structure J on X is uniquely determined by a linear complex structure J on $\mathfrak{g}$ satisfying the integrability condition $\mathrm{Nij}_J = 0$, [NN57, Theorem 1.1]; the set of G-left-invariant complex structures on X is denoted by

$$\mathcal{C}(\mathfrak{g}) := \left\{ J \in \mathrm{End}(\mathfrak{g}) \;:\; J^2 = -\,\mathrm{id}_{\mathfrak{g}} \;\text{ and }\; \mathrm{Nij}_J = 0 \right\} .$$

Recall that the exterior differential d on X can be written using only the action of $\Gamma(X; TX)$ on $\mathcal{C}^\infty(X)$ and the Lie bracket of the Lie algebra of vector fields on X: more precisely, recall that, if $\varphi \in \wedge^k X$ and $X_0, \ldots, X_k \in \mathcal{C}^\infty(X; TX)$, then

$$\begin{aligned} \mathrm{d}\,\varphi(X_0, \ldots, X_k) \;&=\; \sum_{j=0}^{k} (-1)^j \; X_j\, \varphi(X_0, \ldots, X_{j-1}, X_{j+1}, \ldots, X_k) \\ &+ \sum_{0 \leq j < h \leq k} (-1)^{j+h-1} \; \varphi([X_j, X_h], X_0, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{h-1}, X_{h+1}, \ldots, X_k) \;. \end{aligned}$$

Hence one has a differential complex $(\wedge^\bullet \mathfrak{g}^*, \mathrm{d})$, which is isomorphic, as a differential complex, to the differential subcomplex $\left(\wedge^\bullet_{\mathrm{inv}} X, \mathrm{d}\lfloor_{\wedge^\bullet_{\mathrm{inv}} X}\right)$ of $(\wedge^\bullet X, \mathrm{d})$ given by the G-left-invariant forms on X.

If a G-left-invariant complex structure on X is given, then one also has the double complex $\left(\wedge^{\bullet,\bullet} \mathfrak{g}^*_{\mathbb{C}}, \partial, \overline{\partial}\right)$, which is isomorphic, as a double complex, to the double subcomplex $\left(\wedge^{\bullet,\bullet}_{\mathrm{inv}} X, \partial\lfloor_{\wedge^{\bullet,\bullet}_{\mathrm{inv}} X}, \overline{\partial}\lfloor_{\wedge^{\bullet,\bullet}_{\mathrm{inv}} X}\right)$ of $\left(\wedge^{\bullet,\bullet} X, \partial, \overline{\partial}\right)$ given by the G-left-invariant forms on X.

Finally, given a G-left-invariant complex structure on G and fixed $p, q \in \mathbb{N}$, one also has the following complexes and the following maps of complexes:

$$\begin{tikzcd}
\wedge^{p-1,q-1}\mathfrak{g}^*_{\mathbb{C}} \arrow[r, "\partial\overline{\partial}"] \arrow[d, "\simeq"] & \wedge^{p,q}\mathfrak{g}^*_{\mathbb{C}} \arrow[r, "\mathrm{d}"] \arrow[d, "\simeq"] & \wedge^{p+q+1}\mathfrak{g}^*_{\mathbb{C}} \arrow[d, "\simeq"] \\
\wedge^{p-1,q-1}_{\mathrm{inv}}X \arrow[r, "\partial\overline{\partial}"] \arrow[d, hook, "i"] & \wedge^{p,q}_{\mathrm{inv}}X \arrow[r, "\mathrm{d}"] \arrow[d, hook, "i"] & \wedge^{p+q+1}_{\mathrm{inv}}(X;\mathbb{C}) \arrow[d, hook, "i"] \\
\wedge^{p-1,q-1}X \arrow[r, "\partial\overline{\partial}"] & \wedge^{p,q}X \arrow[r, "\mathrm{d}"] & \wedge^{p+q+1}(X;\mathbb{C})
\end{tikzcd} , \tag{1.3.1}$$

and

$$\begin{tikzcd}
\wedge^{p-1,q}\mathfrak{g}^*_{\mathbb{C}} \oplus \wedge^{p,q-1}\mathfrak{g}^*_{\mathbb{C}} \arrow[r, "\partial+\overline{\partial}"] \arrow[d, "\simeq"] & \wedge^{p,q}\mathfrak{g}^*_{\mathbb{C}} \arrow[r, "\partial\overline{\partial}"] \arrow[d, "\simeq"] & \wedge^{p+1,q+1}\mathfrak{g}^*_{\mathbb{C}} \arrow[d, "\simeq"] \\
\wedge^{p-1,q}_{\mathrm{inv}}X \oplus \wedge^{p,q-1}_{\mathrm{inv}}X \arrow[r, "\partial+\overline{\partial}"] \arrow[d, hook, "i"] & \wedge^{p,q}_{\mathrm{inv}}X \arrow[r, "\partial\overline{\partial}"] \arrow[d, hook, "i"] & \wedge^{p+1,q+1}_{\mathrm{inv}}X \arrow[d, hook, "i"] \\
\wedge^{p-1,q}X \oplus \wedge^{p,q-1}X \arrow[r, "\partial+\overline{\partial}"] & \wedge^{p,q}X \arrow[r, "\partial\overline{\partial}"] & \wedge^{p+1,q+1}X
\end{tikzcd} . \tag{1.3.2}$$

For $\sharp \in \left\{\overline{\partial},\, \partial,\, BC,\, A\right\}$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we will write $H^{\bullet}_{dR}(\mathfrak{g};\mathbb{K}) :=: H^{\bullet}_{dR}(\mathfrak{g};\mathbb{K})$ and $H^{\bullet,\bullet}_{\sharp}(\mathfrak{g}_{\mathbb{C}})$ to denote the cohomology groups of the corresponding complexes of forms on $\mathfrak{g}$, which are isomorphic to the cohomology groups of the corresponding complexes of G-left-invariant forms on X. The rest of this section is devoted to the problem whether these cohomologies are isomorphic to the corresponding cohomologies on X.

1.3.2 Classical results on computations of the de Rham and Dolbeault cohomologies

In this section, we collect some results, by K. Nomizu [Nom54], A. Hattori [Hat60], S. Console and A. Fino [CF01], Y. Sakane [Sak76], L. A. Cordero, M. Fernández, A. Gray, and L. Ugarte [CFGU00], S. Rollenske [Rol09a, Rol11a, Rol09b], concerning the computation of the de Rham cohomology and the Dolbeault cohomology for nilmanifolds and solvmanifolds, endowed with left-invariant complex structures.

First of all, we recall the following result, concerning the de Rham cohomology: it was firstly proven by K. Nomizu for nilmanifolds, and then generalized by A. Hattori to the case of completely-solvable solvmanifolds.

Theorem 1.30 ([Nom54, Theorem 1], [Hat60, Corollary 4.2])**.** *Let $X = \Gamma\backslash G$ be a nilmanifold, or, more in general, a completely-solvable solvmanifold, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. The map of differential complexes $(\wedge^{\bullet}\mathfrak{g}^*,\, \mathrm{d}) \to (\wedge^{\bullet}X,\, \mathrm{d})$ is a quasi-isomorphism:*

$$i\colon H^{\bullet}_{dR}(\mathfrak{g};\mathbb{R}) \stackrel{\simeq}{\to} H^{\bullet}_{dR}(X;\mathbb{R}) .$$

A counterexample in the non-completely-solvable case was provided by P. de Bartolomeis and A. Tomassini in [dBT06, Corollary 4.2, Remark 4.3], studying the Nakamura manifold, [Nak75, §2].

Similar results hold for the Dolbeault cohomology of nilmanifolds endowed with certain left-invariant complex structures; [Con06] and [Rol11a] are recent surveys on the known results. (Some results about the Dolbeault cohomology of solvmanifolds have been recently proven by H. Kasuya, see [Kas12].)

First of all, we recall the following lemma by S. Console and A. Fino, [CF01]: the argument used in the proof can be generalized to Bott-Chern and Aeppli cohomologies, see Lemma 1.36.

Lemma 1.31 ([CF01, Lemma 9])**.** *Let $X = \Gamma\backslash G$ be a nilmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. For any $p \in \mathbb{N}$, the map of complexes $\left(\wedge^{p,\bullet}\mathfrak{g}^*_{\mathbb{C}},\, \overline{\partial}\right) \hookrightarrow \left(\wedge^{p,\bullet}X,\, \overline{\partial}\right)$ induces an injective homomorphism i in cohomology:*

$$i\colon H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}_{\mathbb{C}}) \hookrightarrow H^{\bullet,\bullet}_{\overline{\partial}}(X) .$$

For an arbitrary G-left-invariant complex structure on a nilmanifold $X = \Gamma\backslash G$, it is not known whether $i\colon H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}_{\mathbb{C}}) \hookrightarrow H^{\bullet,\bullet}_{\overline{\partial}}(X)$ actually is an isomorphism, but some results are known for certain classes of G-left-invariant complex structures.

Theorem 1.32 ([Sak76, Theorem 1], [CFGU00, Main Theorem], [CF01, Theorem 2, Remark 4], [Rol09a, Theorem 1.10], [Rol11a, Corollary 3.10]). *Let $X = \Gamma\backslash G$ be a nilmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Then, for every $p \in \mathbb{N}$, the map of complexes*

$$\left(\wedge^{p,\bullet}\mathfrak{g}^*_{\mathbb{C}},\, \overline{\partial}\right) \hookrightarrow \left(\wedge^{p,\bullet}X,\, \overline{\partial}\right) \tag{1.3.3}$$

is a quasi-isomorphism, namely,

$$i\colon H^{\bullet,\bullet}_{\overline{\partial}}\left(\mathfrak{g}_{\mathbb{C}}\right) \stackrel{\simeq}{\to} H^{\bullet,\bullet}_{\overline{\partial}}(X)\ ,$$

provided one of the following conditions holds:

- *X is holomorphically parallelizable;*
- *J is an Abelian complex structure;*
- *J is a nilpotent complex structure;*
- *J is a rational complex structure;*
- *$\mathfrak{g}$ admits a torus-bundle series compatible with J and with the rational structure induced by Γ;*
- *$\dim_{\mathbb{R}}\mathfrak{g} = 6$ and $\mathfrak{g}$ is not isomorphic to $\mathfrak{h}_7 := \left(0^3,\, 12,\, 13,\, 23\right)$.*

We recall, (see, e.g., [Rol09a, Definition 1.5],) that, given a nilpotent Lie algebra $\mathfrak{g}$, a *rational structure* for $\mathfrak{g}$ is a $\mathbb{Q}$-vector space $\mathfrak{g}_{\mathbb{Q}}$ such that $\mathfrak{g}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} = \mathfrak{g}$. A sub-algebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be *rational with respect to a rational structure* $\mathfrak{g}_{\mathbb{Q}}$ if the $\mathbb{Q}$-vector space $\mathfrak{h} \cap \mathfrak{g}_{\mathbb{Q}}$ of $\mathfrak{h}$ is a rational structure for $\mathfrak{h}$. If G is the connected simply-connected Lie group associated to $\mathfrak{g}$, then any discrete co-compact subgroup Γ of G induces a rational structure for $\mathfrak{g}$, given by $\mathbb{Q}\log\Gamma$.

Consider a G-left-invariant complex structure on a nilmanifold $X = \Gamma\backslash G$ with associated Lie algebra $\mathfrak{g}$; we recall that:

- J is called *holomorphically parallelizable* if the the holomorphic tangent bundle is holomorphically trivial, see, e.g., [Wan54, Nak75];
- J is called *Abelian* if $[Jx,\, Jy] = [x,\, y]$ for any $x, y \in \mathfrak{g}$, see, e.g., [BDMM95, ABDM11];
- J is called *nilpotent* if there exists a G-left-invariant co-frame $\left\{\omega^1, \ldots, \omega^n\right\}$ for $\left(T^{1,0}X\right)^*$ with respect to which the structure equations of X are of the form
$$\mathrm{d}\,\omega^j \;=\; \sum_{h<k<j} A^j_{hk}\,\omega^h \wedge \omega^k + \sum_{h,k<j} B^j_{hk}\,\omega^h \wedge \bar\omega^k$$
with $\left\{A^j_{hk},\, B^j_{hk}\right\}_{j,h,k} \subset \mathbb{C}$, see, e.g, [CFGU00];
- J is called *rational* if $J\left(\mathfrak{g}_{\mathbb{Q}}\right) \subseteq \mathfrak{g}_{\mathbb{Q}}$ where $\mathfrak{g}_{\mathbb{Q}}$ is the rational structure for $\mathfrak{g}$ induced by Γ, see, e.g., [CF01].

We recall also the following definitions, [Rol09a, Definition 1.8]. An ascending filtration $\left\{\mathcal{S}^j\mathfrak{g}\right\}_{j\in\{0,\ldots,k\}}$ on $\mathfrak{g}$ is called a *torus-bundle series compatible with a linear complex structure J on $\mathfrak{g}$ and a rational structure $\mathfrak{g}_{\mathbb{Q}}$ for $\mathfrak{g}$* if, for every $j \in \{1, \ldots, k\}$, it holds that *(i)* $\mathcal{S}^j\mathfrak{g}$ is rational with respect to $\mathfrak{g}_{\mathbb{Q}}$ and an ideal in $\mathcal{S}^{j+1}\mathfrak{g}$, *(ii)* $J\mathcal{S}^j\mathfrak{g} = \mathcal{S}^j\mathfrak{g}$, and *(iii)* $\mathcal{S}^{j+1}\mathfrak{g}\big/\mathcal{S}^j\mathfrak{g}$ is Abelian. If, in addition, it holds that *(iv)* $\mathcal{S}^{j+1}\mathfrak{g}\big/\mathcal{S}^j\mathfrak{g}$ is contained in the center of $\mathfrak{g}/\mathcal{S}^j\mathfrak{g}$, then $\left\{\mathcal{S}^j\mathfrak{g}\right\}_{j\in\{0,\ldots,k\}}$ is called a *principal torus-bundle series compatible with J and* $\mathfrak{g}_{\mathbb{Q}}$. Finally, an ascending filtration $\left\{\mathcal{S}^j\mathfrak{g}\right\}_{j\in\{0,\ldots,k\}}$ on $\mathfrak{g}$ is called a *stable (principal) torus-bundle series* if it is a (principal) torus-bundle series compatible with J and $\mathfrak{g}_{\mathbb{Q}}$ for any complex structure J and for any rational structure $\mathfrak{g}_{\mathbb{Q}}$. By S. Rollenske's theorem [Rol09a, Theorem B], every 6-dimensional nilpotent Lie algebra except $\mathfrak{h}_7 := \left(0^3,\, 12,\, 13,\, 23\right)$ admits a stable torus-bundle series.

The property of computing the Dolbeault cohomology using just left-invariant forms turns out to be open along curves of left-invariant complex structures: this was proven by S. Console and A. Fino, [CF01].

Theorem 1.33 ([CF01, Theorem 1]). *Let $X = \Gamma\backslash G$ be a nilmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Let $\mathcal{U} \subseteq \mathcal{C}(\mathfrak{g})$ be the subset containing the G-left-invariant complex structures J on X such that the inclusion i is an isomorphism:*

$$\mathcal{U} \;:=\; \left\{J \in \mathcal{C}\left(\mathfrak{g}\right) \;:\; i\colon H^{\bullet,\bullet}_{\overline{\partial}}\left(\mathfrak{g}_{\mathbb{C}}\right) \stackrel{\simeq}{\hookrightarrow} H^{\bullet,\bullet}_{\overline{\partial}}(X)\right\} \;\subseteq\; \mathcal{C}\left(\mathfrak{g}\right)\ .$$

Then $\mathcal{U}$ is an open set in $\mathcal{C}\left(\mathfrak{g}\right)$.

The strategy of the proof consists in proving that the dimension of the orthogonal of $H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}_{\mathbb{C}})$ in $H^{\bullet,\bullet}_{\overline{\partial}}(X)$ with respect to a given J-Hermitian G-left-invariant metric on $X = \Gamma \backslash G$ is an upper-semi-continuous function in $J \in \mathcal{C}(\mathfrak{g})$ and thus, if it is zero for a given $J \in \mathcal{C}(\mathfrak{g})$, then it remains equal to zero in an open neighbourhood of J in $\mathcal{C}(\mathfrak{g})$. We will use the same argument in proving Theorem 1.42, which is a slight modification of the previous result in the case of the Bott-Chern cohomology.

The aforementioned results suggest the following conjecture.

Conjecture 1.34 ([Rol11a, Conjecture 1]; see also [CFGU00, page 5406], [CF01, page 112]). *Let $X = \Gamma \backslash G$ be a nilmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Then, for any $p \in \mathbb{N}$, the map of complexes* (1.3.3) *is a quasi-isomorphisms, that is,*

$$i \colon H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}_{\mathbb{C}}) \stackrel{\simeq}{\to} H^{\bullet,\bullet}_{\overline{\partial}}(X) .$$

Note that, since i is always injective by [CF01, Lemma 9], this is equivalent to asking that

$$\dim_{\mathbb{C}} \left(H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}_{\mathbb{C}}) \right)^{\perp} = 0 ,$$

where the orthogonality is meant with respect to the inner product induced by a given J-Hermitian G-left-invariant metric g on X.

Finally, as an application of the previous results, we recall the following theorem by S. Rollenske, concerning the deformations of left-invariant complex structures on nilmanifolds.

Theorem 1.35 ([Rol09b, Theorem 2.6]). *Let $X = \Gamma \backslash G$ be a nilmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Suppose that, for $p = 1$, the map of complexes* (1.3.3) *is a quasi-isomorphism: $i \colon H^{1,q}_{\overline{\partial}}(\mathfrak{g}_{\mathbb{C}}) \stackrel{\simeq}{\to} H^{1,q}_{\overline{\partial}}(X)$ for every $q \in \mathbb{N}$. Then all small deformations of the complex structure J are again G-left-invariant complex structures. More precisely, the Kuranishi family of X contains only G-left-invariant complex structures.*

1.3.3 The Bott-Chern cohomology on solvmanifolds

We recall here the results obtained in [Ang11], concerning the computation of the Bott-Chern cohomology for nilmanifolds and solvmanifolds.

Firstly, we prove a slight modification of [CF01, Lemma 9] proven by S. Console and A. Fino for the Dolbeault cohomology: we repeat here their argument for the case of the Bott-Chern cohomology, [Ang11, Lemma 3.6].

Lemma 1.36. *Let $X = \Gamma \backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. The map of complexes* (1.3.1) *induces an injective homomorphism*

$$i \colon H^{\bullet,\bullet}_{BC}(\mathfrak{g}_{\mathbb{C}}) \hookrightarrow H^{\bullet,\bullet}_{BC}(X) .$$

Proof. Fix $p, q \in \mathbb{N}$. Let g be a J-Hermitian G-left-invariant metric on X and consider the induced inner product $\langle \cdot | \cdot\cdot \rangle$ on $\wedge^{\bullet,\bullet}X$. Hence, both ∂, $\overline{\partial}$, and their adjoints ∂^*, $\overline{\partial}^*$ preserve the G-left-invariant forms on X and therefore also $\tilde{\Delta}_{BC}$ does. In such a way, we get a Hodge decomposition also at the level of G-left-invariant forms:

$$\wedge^{p,q}\mathfrak{g}^*_{\mathbb{C}} = \ker \tilde{\Delta}_{BC} \lfloor_{\wedge^{p,q}\mathfrak{g}^*_{\mathbb{C}}} \oplus \operatorname{im} \partial\overline{\partial} \lfloor_{\wedge^{p-1,q-1}\mathfrak{g}^*_{\mathbb{C}}} \oplus \left(\operatorname{im} \partial^* \lfloor_{\wedge^{p+1,q}\mathfrak{g}^*_{\mathbb{C}}} + \operatorname{im} \overline{\partial}^* \lfloor_{\wedge^{p,q+1}\mathfrak{g}^*_{\mathbb{C}}} \right) .$$

Now, take $[\omega] \in H^{p,q}_{BC}(\mathfrak{g}_{\mathbb{C}})$ such that $i[\omega] = 0$ in $H^{p,q}_{BC}(X)$, that is, ω is a G-left-invariant (p,q)-form on X and there exists a (possibly non-G-left-invariant) $(p-1, q-1)$-form η on X such that $\omega = \partial\overline{\partial}\eta$. Up to zero terms in $H^{p,q}_{BC}(\mathfrak{g}_{\mathbb{C}})$, we may assume that $\eta \in (i(\wedge^{p,q}\mathfrak{g}^*_{\mathbb{C}}))^{\perp} \subseteq \wedge^{p,q}X$. Therefore, since $\overline{\partial}^*\partial^*\partial\overline{\partial}\eta$ is a G-left-invariant form (being $\partial\overline{\partial}\eta$ a G-left-invariant form), we have that

$$0 = \left\langle \overline{\partial}^*\partial^*\partial\overline{\partial}\eta \middle| \eta \right\rangle = \left\| \partial\overline{\partial}\eta \right\|^2 = \|\omega\|^2$$

and therefore $\omega = 0$. □

The second general result says that, if the Dolbeault and de Rham cohomologies of a solvmanifold are computed using just left-invariant forms, then also the Bott-Chern cohomology is computed using just left-invariant forms, [Ang11, Theorem 3.7]. The idea of the proof is inspired by [Sch07, §1.c], where M. Schweitzer used a similar argument to explicitly compute the Bott-Chern cohomology in the special case of the Iwasawa manifold.

Theorem 1.37. *Let $X = \Gamma \backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Suppose that*

$$i\colon H^{\bullet}_{dR}(\mathfrak{g};\mathbb{C}) \stackrel{\simeq}{\hookrightarrow} H^{\bullet}_{dR}(X;\mathbb{C}) \qquad \text{and} \qquad i\colon H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}_{\mathbb{C}}) \stackrel{\simeq}{\hookrightarrow} H^{\bullet,\bullet}_{\overline{\partial}}(X) .$$

Then also

$$i\colon H^{\bullet,\bullet}_{BC}(\mathfrak{g}_{\mathbb{C}}) \stackrel{\simeq}{\hookrightarrow} H^{\bullet,\bullet}_{BC}(X) .$$

Proof. Fix $p, q \in \mathbb{N}$. We prove the theorem as a consequence of the following claims.

`Claim 1` – *It suffices to prove that $\frac{\mathrm{im}\,\mathrm{d} \cap \wedge^{p,q} X}{\mathrm{im}\,\partial\overline{\partial}}$ can be computed using just G-left-invariant forms.*
Indeed, we have the exact sequence

$$0 \to \frac{\mathrm{im}\,\mathrm{d} \cap \wedge^{p,q} X}{\mathrm{im}\,\partial\overline{\partial}} \to H^{p,q}_{BC}(X) \to H^{p+q}_{dR}(X;\mathbb{C})$$

and, by hypothesis, $H^{\bullet}_{dR}(X;\mathbb{C})$ can be computed using just G-left-invariant forms.

`Claim 2` – *Under the hypothesis that the Dolbeault cohomology is computed using just G-left-invariant forms, if ψ is a G-left-invariant $\overline{\partial}$-closed form, then every solution ϕ of $\overline{\partial}\phi = \psi$ is G-left-invariant up to $\overline{\partial}$-exact terms.*
Indeed, since $[\psi] = 0$ in $H^{\bullet,\bullet}_{\overline{\partial}}(X)$, there is a G-left-invariant form α such that $\psi = \overline{\partial}\alpha$. Hence, $\phi - \alpha$ defines a class in $H^{\bullet,\bullet}_{\overline{\partial}}(X)$ and hence $\phi - \alpha$ is G-left-invariant up to a $\overline{\partial}$-exact form, and so ϕ is.

`Claim 3` – *Under the hypothesis that the Dolbeault cohomology is computed using just G-left-invariant forms, the space $\frac{\mathrm{im}\,\mathrm{d} \cap \wedge^{p,q} X}{\mathrm{im}\,\partial\overline{\partial}}$ can be computed using just G-left-invariant forms.*
Consider

$$\omega^{p,q} \;=\; \mathrm{d}\,\eta \quad \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} \;\in\; \frac{\mathrm{im}\,\mathrm{d} \cap \wedge^{p,q} X}{\mathrm{im}\,\partial\overline{\partial}} \,. \tag{1.3.4}$$

Decomposing $\eta =: \sum_{p,q} \eta^{p,q}$ in pure-type components, the equality (1.3.4) is equivalent to the system

$$\left\{ \begin{array}{rcccll} & & \partial\eta^{p+q-1,0} & = & 0 & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} \\ \overline{\partial}\eta^{p+q-\ell,\ell-1} & + & \partial\eta^{p+q-\ell-1,\ell} & = & 0 & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} \quad \text{for} \quad \ell \in \{1,\ldots,q-1\} \\ \overline{\partial}\eta^{p,q-1} & + & \partial\eta^{p-1,q} & = & \omega^{p,q} & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} \\ \overline{\partial}\eta^{\ell,p+q-\ell-1} & + & \partial\eta^{\ell-1,p+q-\ell} & = & 0 & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} \quad \text{for} \quad \ell \in \{1,\ldots,p-1\} \\ \overline{\partial}\eta^{0,p+q-1} & & & = & 0 & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} \end{array} \right. .$$

Applying several times `Claim 2`, we may suppose that the forms $\eta^{\ell,p+q-\ell-1}$, with $\ell \in \{0,\ldots,p-1\}$, are G-left-invariant: indeed, they are G-left-invariant up to $\overline{\partial}$-exact terms, but $\overline{\partial}$-exact terms give no contribution in the system, since it is modulo $\mathrm{im}\,\partial\overline{\partial}$. Analogously, using the conjugate version of `Claim 2`, we may suppose that the forms $\eta^{p+q-\ell-1,\ell}$, with $\ell \in \{0,\ldots,q-1\}$, are G-left-invariant. Then we may suppose that $\omega^{p,q} = \overline{\partial}\eta^{p,q-1} + \partial\eta^{p-1,q}$ is G-left-invariant. $\square$

Remark 1.38. Let $X = \Gamma \backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Note that, if the map of complexes $i\colon \left(\wedge^{p,\bullet}\mathfrak{g}^*_{\mathbb{C}},\, \overline{\partial}\right) \to \left(\wedge^{p,\bullet}X,\, \overline{\partial}\right)$ is a quasi-isomorphism for every $p \in \mathbb{N}$, that is,

$$i\colon H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}_{\mathbb{C}}) \stackrel{\simeq}{\hookrightarrow} H^{\bullet,\bullet}_{\overline{\partial}}(X) ,$$

then also the map of complexes $i\colon (\wedge^{\bullet}\mathfrak{g}^*,\, \mathrm{d}) \to (\wedge^{\bullet}X,\, \mathrm{d})$ is a quasi-isomorphism, that is,

$$i\colon H^{\bullet}_{dR}(\mathfrak{g};\mathbb{C}) \stackrel{\simeq}{\hookrightarrow} H^{\bullet}_{dR}(X;\mathbb{C}) .$$

Indeed, the map of double complexes $i\colon \left(\wedge^{\bullet,\bullet}\mathfrak{g}^*_{\mathbb{C}},\, \partial,\, \overline{\partial}\right) \to \left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ induces a map between the corresponding Hodge and Frölicher spectral sequences:

$$i\colon \left\{\left(E^{\bullet,\bullet}_r(\mathfrak{g}_{\mathbb{C}}),\mathrm{d}_r\right)\right\}_{r\in\mathbb{N}} \to \left\{\left(E^{\bullet,\bullet}_r(X),\mathrm{d}_r\right)\right\}_{r\in\mathbb{N}} .$$

Since, see, e.g., [McC01, Theorem 2.15],

$$E^{\bullet,\bullet}_1(\mathfrak{g}_{\mathbb{C}}) \;\simeq\; H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}) \;\Rightarrow\; H^{\bullet}_{dR}(\mathfrak{g}_{\mathbb{C}}) \qquad \text{and} \qquad E^{\bullet,\bullet}_1(X) \;\simeq\; H^{\bullet,\bullet}_{\overline{\partial}}(X) \;\Rightarrow\; H^{\bullet}_{dR}(X;\mathbb{C}) ,$$

one gets that, if $i\colon E^{\bullet,\bullet}_1(\mathfrak{g}_{\mathbb{C}}) \to E^{\bullet,\bullet}_1(X)$ is an isomorphism, then also $i\colon H^{\bullet}_{dR}(\mathfrak{g}_{\mathbb{C}}) \to H^{\bullet}_{dR}(X;\mathbb{C})$ is an isomorphism, see, e.g., [McC01, Theorem 3.5].

As a corollary of [Nom54, Theorem 1], [Sak76, Theorem 1], [CFGU00, Main Theorem], [CF01, Theorem 2, Remark 4], [Rol09a, Theorem 1.10], [Rol11a, Corollary 3.10], and Theorem 1.37, we get the following result, [Ang11, Theorem 3.8].

Theorem 1.39. *Let $X = \Gamma \backslash G$ be a nilmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Suppose that one of the following conditions holds:*

- *X is holomorphically parallelizable;*
- *J is an Abelian complex structure;*
- *J is a nilpotent complex structure;*
- *J is a rational complex structure;*
- *$\mathfrak{g}$ admits a torus-bundle series compatible with J and with the rational structure induced by Γ;*
- *$\dim_{\mathbb{R}} \mathfrak{g} = 6$ and $\mathfrak{g}$ is not isomorphic to $\mathfrak{h}_7 := \left(0^3,\, 12,\, 13,\, 23\right)$.*

Then the de Rham, Dolbeault, Bott-Chern and Aeppli cohomologies can be computed as the cohomologies of the corresponding subcomplexes given by the space of G-left-invariant forms on X; in other words, the inclusions of the several subcomplexes of G-left-invariant forms on X into the corresponding complexes of forms on X are quasi-isomorphisms:

$$i\colon H^{\bullet}_{dR}\left(\mathfrak{g};\mathbb{R}\right) \stackrel{\simeq}{\hookrightarrow} H^{\bullet}_{dR}(X;\mathbb{R}) \qquad \text{and} \qquad i\colon H^{\bullet,\bullet}_{\sharp}\left(\mathfrak{g}_{\mathbb{C}}\right) \stackrel{\simeq}{\hookrightarrow} H^{\bullet,\bullet}_{\sharp}(X) \;,$$

for $\sharp \in \{\partial,\, \overline{\partial},\, BC,\, A\}$.

Remark 1.40. Note that Theorem 1.39, and [Hat60, Corollary 4.2], allow to straightforwardly compute the de Rham, Dolbeault, Bott-Chern, and Aeppli cohomologies of nilmanifolds, endowed with certain left-invariant complex structures, respectively the de Rham cohomology of completely-solvable solvmanifolds, just by computing the space of left-invariant (Δ, or $\overline{\square}$, or $\tilde{\Delta}_{BC}$, or $\tilde{\Delta}_{A}$-)harmonic forms with respect to a left-invariant Riemannian, or Hermitian, metric.

Indeed, suppose that X is a nilmanifold, endowed with a left-invariant complex structure, or a completely-solvable solvmanifold, satisfying $i\colon H^{\bullet}_{dR}\left(\mathfrak{g};\mathbb{R}\right) \stackrel{\simeq}{\hookrightarrow} H^{\bullet}_{dR}(X;\mathbb{R})$, or $i\colon H^{\bullet,\bullet}_{\sharp}\left(\mathfrak{g}_{\mathbb{C}}\right) \stackrel{\simeq}{\hookrightarrow} H^{\bullet,\bullet}_{\sharp}(X)$, for some $\sharp \in \{\partial,\, \overline{\partial},\, BC,\, A\}$. Let g be a left-invariant Riemannian, or Hermitian, metric on X. Hence, the operators Δ, $\overline{\square}$, $\tilde{\Delta}_{BC}$, $\tilde{\Delta}_{A}$ send the subspace of left-invariant forms to the subspace of left-invariant forms, and induce the self-adjoint operators

$$\Delta \in \mathrm{End}\left(\wedge^{\bullet}\mathfrak{g}^*\right) \;, \qquad \overline{\square} \in \mathrm{End}\left(\wedge^{\bullet,\bullet}\mathfrak{g}_{\mathbb{C}}^*\right) \;, \qquad \tilde{\Delta}_{BC} \in \mathrm{End}\left(\wedge^{\bullet,\bullet}\mathfrak{g}_{\mathbb{C}}^*\right) \;, \qquad \tilde{\Delta}_{A} \in \mathrm{End}\left(\wedge^{\bullet,\bullet}\mathfrak{g}_{\mathbb{C}}^*\right) \;,$$

with respect to the inner products $\langle\cdot,\, \cdot\cdot\rangle$ induced by g on the space $\wedge^{\bullet}\mathfrak{g}^*$ and on the space $\wedge^{\bullet,\bullet}\mathfrak{g}_{\mathbb{C}}^*$. Hence, one gets the orthogonal decompositions

$$\wedge^{\bullet}\, \mathfrak{g}^* = \ker\Delta \oplus \mathrm{im}\,\Delta \;, \qquad \wedge^{\bullet,\bullet}\, \mathfrak{g}_{\mathbb{C}}^* = \ker\overline{\square} \oplus \mathrm{im}\,\overline{\square} \;,$$

$$\wedge^{\bullet,\bullet}\, \mathfrak{g}_{\mathbb{C}}^* = \ker\tilde{\Delta}_{BC} \oplus \mathrm{im}\,\tilde{\Delta}_{BC} \;, \qquad \wedge^{\bullet,\bullet}\, \mathfrak{g}_{\mathbb{C}}^* = \ker\tilde{\Delta}_{A} \oplus \mathrm{im}\,\tilde{\Delta}_{A}$$

(one could argue also by using the F. A. Belgun symmetrization trick [Bel00, Theorem 7]). It follows that

$$H^{\bullet}_{dR}\left(\mathfrak{g};\mathbb{R}\right) \simeq \ker\Delta \;, \qquad H^{\bullet,\bullet}_{\overline{\partial}}\left(\mathfrak{g}_{\mathbb{C}}\right) \simeq \ker\overline{\square} \;, \qquad H^{\bullet,\bullet}_{BC}\left(\mathfrak{g}_{\mathbb{C}}\right) \simeq \ker\tilde{\Delta}_{BC} \;, \qquad H^{\bullet,\bullet}_{A}\left(\mathfrak{g}_{\mathbb{C}}\right) \simeq \ker\tilde{\Delta}_{A} \;.$$

Remark 1.41. Let $X = \Gamma \backslash G$ be a $2n$-dimensional solvmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. The map of complexes (1.3.2) induces an injective homomorphism

$$i\colon H^{\bullet,\bullet}_{A}\left(\mathfrak{g}_{\mathbb{C}}\right) \hookrightarrow H^{\bullet,\bullet}_{A}(X) \;.$$

Furthermore, if $i\colon H^{\bullet,\bullet}_{BC}\left(\mathfrak{g}_{\mathbb{C}}\right) \stackrel{\simeq}{\hookrightarrow} H^{\bullet,\bullet}_{BC}(X)$, then the map of complexes (1.3.2) is a quasi-isomorphism, that is,

$$i\colon H^{\bullet,\bullet}_{A}\left(\mathfrak{g}_{\mathbb{C}}\right) \stackrel{\simeq}{\hookrightarrow} H^{\bullet,\bullet}_{A}(X) \;.$$

Indeed, fix a G-left-invariant Hermitian metric g on X. Recall that

$$*\colon H^{\bullet_1,\bullet_2}_{A}(X) \stackrel{\simeq}{\to} H^{n-\bullet_2,n-\bullet_1}_{BC}(X)$$

is an isomorphism, [Sch07, §2.c]. Analogously, note that, by Remark 1.40 and since g is G-left-invariant, the map $*\colon \wedge^{\bullet_1,\bullet_2}\mathfrak{g}_{\mathbb{C}}^* \stackrel{\simeq}{\to} \wedge^{n-\bullet_2,n-\bullet_1}\mathfrak{g}_{\mathbb{C}}^*$ induces an isomorphism

$$*\colon H_A^{\bullet_1,\bullet_2}(\mathfrak{g}_{\mathbb{C}}) \stackrel{\simeq}{\to} H_{BC}^{n-\bullet_2,n-\bullet_1}(\mathfrak{g}_{\mathbb{C}}) .$$

Note also that the diagram

$$\begin{tikzcd} H_A^{\bullet_1,\bullet_2}(\mathfrak{g}_{\mathbb{C}}) \arrow[r, "i"] \arrow[d, "*"', "\simeq"] & H_A^{\bullet_1,\bullet_2}(X) \arrow[d, "\simeq"', "*"] \\ H_{BC}^{n-\bullet_2,n-\bullet_1}(\mathfrak{g}_{\mathbb{C}}) \arrow[r, "i"'] & H_{BC}^{n-\bullet_2,n-\bullet_1}(X) \end{tikzcd}$$

commutes, since g is G-left-invariant. Since the map $i\colon H_{BC}^{n-\bullet_2,n-\bullet_1}(\mathfrak{g}_{\mathbb{C}}) \hookrightarrow H_{BC}^{n-\bullet_2,n-\bullet_1}(X)$ is injective by Lemma 1.36, then also the map $H_A^{\bullet_1,\bullet_2}(\mathfrak{g}_{\mathbb{C}}) \to H_A^{\bullet_1,\bullet_2}(X)$ is injective. If $i\colon H_{BC}^{n-\bullet_2,n-\bullet_1}(\mathfrak{g}_{\mathbb{C}}) \hookrightarrow H_{BC}^{n-\bullet_2,n-\bullet_1}(X)$ is actually an isomorphism, then also $i\colon H_A^{\bullet_1,\bullet_2}(\mathfrak{g}_{\mathbb{C}}) \to H_A^{\bullet_1,\bullet_2}(X)$ is an isomorphism.

A slight modification of [CF01, Theorem 1] by S. Console and A. Fino gives the following result, which says that the property of computing the Bott-Chern cohomology using just left-invariant forms is open in the space of left-invariant complex structures on solvmanifolds, [Ang11, Theorem 3.9].

Theorem 1.42. *Let $X = \Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Let $\sharp \in \{\partial, \overline{\partial}, BC, A\}$. Suppose that*

$$i\colon H_{\sharp_J}^{\bullet,\bullet}(\mathfrak{g}_{\mathbb{C}}) \stackrel{\simeq}{\hookrightarrow} H_{\sharp_J}^{\bullet,\bullet}(X) .$$

Then there exists an open neighbourhood $\mathcal{U}$ of J in $\mathcal{C}(\mathfrak{g})$ such that any $\tilde{J} \in \mathcal{U}$ still satisfies

$$i\colon H_{\sharp_{\tilde{J}}}^{\bullet,\bullet}(\mathfrak{g}_{\mathbb{C}}) \stackrel{\simeq}{\hookrightarrow} H_{\sharp_{\tilde{J}}}^{\bullet,\bullet}(X) .$$

In other words, the set

$$\mathcal{U} := \left\{J \in \mathcal{C}(\mathfrak{g}) \;:\; i\colon H_{\sharp_J}^{\bullet,\bullet}(\mathfrak{g}_{\mathbb{C}}) \stackrel{\simeq}{\hookrightarrow} H_{\sharp_J}^{\bullet,\bullet}(X)\right\}$$

is open in $\mathcal{C}(\mathfrak{g})$.

Proof. As a matter of notation, for $\varepsilon > 0$ small enough, we consider

$$\{(X,\, J_t) \;:\; t \in \Delta(0,\varepsilon)\} \twoheadrightarrow \Delta(0,\varepsilon)$$

a complex-analytic family of G-left-invariant complex structures on X, where $\Delta(0,\varepsilon) := \{t \in \mathbb{C}^m \;:\; |t| < \varepsilon\}$ for some $m \in \mathbb{N}\setminus\{0\}$; moreover, let $\{g_t\}_{t\in\Delta(0,\varepsilon)}$ be a family of J_t-Hermitian G-left-invariant metrics on X depending smoothly on t. We will denote by $\overline{\partial}_t := \overline{\partial}_{J_t}$ and $\overline{\partial}_t^* := -*_{g_t}\partial_{J_t}*_{g_t}$ the $\overline{\partial}$ operator and its g_t-adjoint respectively for the Hermitian structure $(J_t,\, g_t)$ and we set $\Delta_t := \Delta_{\sharp_{J_t}}$ one of the differential operators involved in the definition of the Dolbeault, conjugate Dolbeault, Bott-Chern or Aeppli cohomologies with respect to $(J_t,\, g_t)$; we remark that Δ_t is a self-adjoint elliptic differential operator for all the considered cohomologies.

By hypothesis, we have that $\left(H_{\sharp_{J_0}}^{\bullet,\bullet}(\mathfrak{g}_{\mathbb{C}})\right)^{\perp} = \{0\}$, where the orthogonality is meant with respect to the inner product induced by g_0, and we have to prove the same replacing 0 with $t \in \Delta(0,\varepsilon)$. Therefore, it will suffice to prove that

$$\Delta(0,\varepsilon) \ni t \mapsto \dim_{\mathbb{C}}\left(H_{\sharp_{J_t}}^{\bullet,\bullet}(\mathfrak{g}_{\mathbb{C}})\right)^{\perp} \in \mathbb{N}$$

is an upper-semi-continuous function at 0. For any $t \in \Delta(0,\varepsilon)$, being Δ_t a self-adjoint elliptic differential operator, there exists a complete orthonormal basis $\{e_i(t)\}_{i\in I}$ of eigen-forms for Δ_t spanning $\left(\wedge_{J_t}^{\bullet,\bullet}\mathfrak{g}_{\mathbb{C}}^*\right)^{\perp}$, the orthogonal complement of the space of G-left-invariant forms, see [KS60, Theorem 1]. For any $i \in I$ and $t \in \Delta(0,\varepsilon)$, let $a_i(t)$ be the eigen-value corresponding to $e_i(t)$; Δ_t depending differentiably on $t \in \Delta(0,\varepsilon)$, for any $i \in I$, the function $\Delta(0,\varepsilon) \ni t \mapsto a_i(t) \in \mathbb{C}$ is continuous, see [KS60, Theorem 2]. Therefore, for any $t_0 \in \Delta(0,\varepsilon)$, choosing a constant $c > 0$ such that $c \notin \overline{\{a_i(t_0) \;:\; i \in I\}}$, the function

$$\Psi_c\colon \Delta(0,\varepsilon) \to \mathbb{N} , \qquad t \mapsto \dim \operatorname{span}\{e_i(t) \;:\; a_i(t) < c\}$$

is locally constant at t_0; moreover, for any $t \in \Delta(0,\varepsilon)$ and for any $c > 0$, we have

$$\Psi_c(t) \;\geq\; \dim_{\mathbb{C}}\left(H_{\sharp_{J_t}}^{\bullet,\bullet}(\mathfrak{g}_{\mathbb{C}})\right)^{\perp} .$$

Since the spectrum of Δ_{t_0} has no accumulation point for any $t_0 \in \Delta(0,\varepsilon)$, see [KS60, Theorem 1], the theorem follows choosing $c > 0$ small enough so that $\Psi_c(0) = \dim_{\mathbb{C}}\left(H_{\sharp_{J_0}}^{\bullet,\bullet}(\mathfrak{g}_{\mathbb{C}})\right)^{\perp}$. $\square$

In particular, the left-invariant complex structures on nilmanifolds belonging to the classes of Theorem 1.39, and their small deformations satisfy the following conjecture, [Ang11, Conjecture 3.10], which generalizes Conjecture 1.34.

Conjecture 1.43. *Let $X = \Gamma\backslash G$ be a nilmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Then the de Rham, Dolbeault, Bott-Chern and Aeppli cohomologies can be computed as the cohomologies of the corresponding subcomplexes given by the space of G-left-invariant forms on X, that is,*

$$\dim_{\mathbb{R}} \left(H^{\bullet}_{dR}\left(\mathfrak{g};\mathbb{R}\right)\right)^{\perp} \;=\; 0 \qquad \text{and} \qquad \dim_{\mathbb{C}} \left(H^{\bullet,\bullet}_{\sharp}\left(\mathfrak{g}_{\mathbb{C}}\right)\right)^{\perp} \;=\; 0 \;,$$

where $\sharp \in \{\partial, \overline{\partial}, BC, A\}$, and the orthogonality is meant with respect to the inner product induced by a given J-Hermitian G-left-invariant metric g on X.

1.4 The cohomologies of the Iwasawa manifold and of its small deformations

The Iwasawa manifold is one of the simplest example of non-Kähler complex manifold: as such, it has been studied by several authors, and it has turned out to be a fruitful source of interesting behaviours, see, e.g., [FG86, Nak75, AB90, Bas99, AGS97, KS04, Ye08, Sch07, AT11, Ang11, Fra11].

In this section, we recall the construction of the Iwasawa manifold §1.4.1, see, e.g., [FG86], [Nak75, §2], and of its Kuranishi space, §1.4.1, see [Nak75, §3]; then we write down the de Rham cohomology, §1.4.2, and the Dolbeault cohomology, §1.4.3, (using [Nom54, Theorem 1], and [Sak76, Theorem 1] and [CF01, Theorem 1]), and we compute the Bott-Chern and Aeppli cohomologies, §1.4.4, (using Theorem 1.39 and Theorem 1.42), of the Iwasawa manifold and of its small deformations.

1.4.1 The Iwasawa manifold and its small deformations

The Iwasawa manifold

Let $\mathbb{H}(3;\mathbb{C})$ be the 3-dimensional *Heisenberg group* over $\mathbb{C}$ defined by

$$\mathbb{H}(3;\mathbb{C}) \;:=\; \left\{ \begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3;\mathbb{C}) \;:\; z^1,\, z^2,\, z^3 \in \mathbb{C} \right\} \;,$$

where the product is the one induced by matrix multiplication. (Equivalently, one can consider $\mathbb{H}(3;\mathbb{C})$ as isomorphic to $\left(\mathbb{C}^3,\, *\right)$, where the group structure $*$ on $\mathbb{C}^3$ is defined as

$$(z_1,\; z_2,\; z_3) \;*\; (w_1,\; w_2,\; w_3) \;:=\; (z_1 + w_1,\; z_2 + w_2,\; z_3 + z_1 w_2 + w_3) \;\;.)$$

It is straightforward to prove that $\mathbb{H}(3;\mathbb{C})$ is a connected simply-connected complex 2-step nilpotent Lie group, that is, the Lie algebra $(\mathfrak{h}_3,\, [\cdot, \cdot\cdot])$ naturally associated to $\mathbb{H}(3;\mathbb{C})$ satisfies $[\mathfrak{h}_3, \mathfrak{h}_3] \neq 0$ and $[\mathfrak{h}_3, [\mathfrak{h}_3, \mathfrak{h}_3]] = 0$.

One finds that

$$\begin{cases} \varphi^1 & := & \mathrm{d}\, z^1 \\ \varphi^2 & := & \mathrm{d}\, z^2 \\ \varphi^3 & := & \mathrm{d}\, z^3 - z^1 \,\mathrm{d}\, z^2 \end{cases}$$

is a $\mathbb{H}(3;\mathbb{C})$-left-invariant co-frame for the space of $(1,0)$-forms on $\mathbb{H}(3;\mathbb{C})$, and that the structure equations with respect to this co-frame are

$$\begin{cases} \mathrm{d}\,\varphi^1 & = & 0 \\ \mathrm{d}\,\varphi^2 & = & 0 \\ \mathrm{d}\,\varphi^3 & = & -\varphi^1 \wedge \varphi^2 \end{cases} \;.$$

Consider the action on the left of $\mathbb{H}\,(3;\mathbb{Z}\,[\mathrm{i}]) := \mathbb{H}(3;\mathbb{C}) \cap \mathrm{GL}\,(3;\mathbb{Z}\,[\mathrm{i}])$ on $\mathbb{H}\,(3;\mathbb{C})$ and take the compact quotient

$$\mathbb{I}_3 := \mathbb{H}\,(3;\mathbb{Z}\,[\mathrm{i}]) \backslash \mathbb{H}(3;\mathbb{C}) \;.$$

One gets that $\mathbb{I}_3$ is a 3-dimensional complex nilmanifold, whose ($\mathbb{H}(3;\mathbb{C})$-left-invariant) complex structure J_0 is the one inherited by the standard complex structure on $\mathbb{C}^3$; $\mathbb{I}_3$ is called the *Iwasawa manifold*.

The forms φ^1, φ^2 and φ^3, being $\mathbb{H}(3;\mathbb{C})$-left-invariant, define a co-frame also for $\left(T^{1,0}\mathbb{I}_3\right)^*$. Note that $\mathbb{I}_3$ is a holomorphically parallelizable manifold, that is, its holomorphic tangent bundle is holomorphically trivial. Since, for example, φ^3 is a non-closed holomorphic form, it follows that $\mathbb{I}_3$ admits no Kähler metric. In fact, one can show that $\mathbb{I}_3$ is not formal, having a non-zero Massey triple product, see [FG86, page 158]; therefore the underlying differentiable manifold of $\mathbb{I}_3$ has no complex structure admitting Kähler metrics, see [DGMS75, Main Theorem], even though all the topological obstructions concerning the Betti numbers are satisfied. Nevertheless, $\mathbb{I}_3$ admits the balanced metric $\omega := \sum_{j=1}^{3} \varphi^j \wedge \bar{\varphi}^j$.

We sketch in Figure 1.2 the structure of the finite-dimensional double complex $\left(\wedge^{\bullet,\bullet}\left(\mathfrak{h}_3 \otimes_{\mathbb{R}} \mathbb{C}\right)^*,\, \partial,\, \overline{\partial}\right)$: the dots denote a basis of $\wedge^{\bullet,\bullet}\left(\mathfrak{h}_3 \otimes_{\mathbb{R}} \mathbb{C}\right)^*$, horizontal arrows are meant as ∂, vertical ones as $\overline{\partial}$ and zero arrows are not depicted.

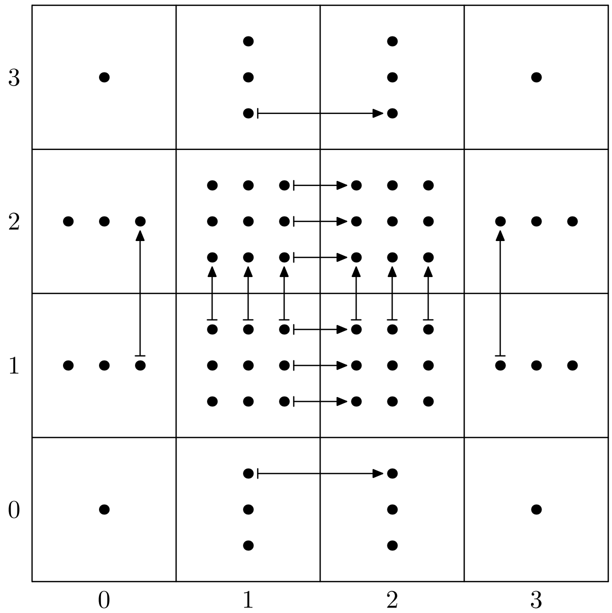

Figure 1.2: The double complex $\left(\wedge^{\bullet,\bullet}\left(\mathfrak{h}_3 \otimes_{\mathbb{R}} \mathbb{C}\right)^*,\, \partial,\, \overline{\partial}\right)$.

Small deformations of the Iwasawa manifold

I. Nakamura classified in [Nak75, §2] the three-dimensional holomorphically parallelizable solvmanifolds into four classes by numerical invariants, giving the Iwasawa manifold $\mathbb{I}_3$ as an example in the second class. Moreover, he explicitly constructed the Kuranishi family of deformations of $\mathbb{I}_3$, showing that it is smooth and depends on 6 effective parameters, [Nak75, pages 94–95], compare also [Rol11b, Corollary 4.9]. In particular, he computed the Hodge numbers of the small deformations of $\mathbb{I}_3$ proving that they have not to remain invariant along a complex-analytic family of complex structures, [Nak75, Theorem 2], compare also [Ye08, §4]; moreover, he proved in this way that the property of being holomorphically parallelizable is not stable under small deformations, [Nak75, page 86], compare also [Rol11b, Theorem 5.1, Corollary 5.2].

Firstly, we recall in the following theorem the results by I. Nakamura concerning the Kuranishi space of the Iwasawa manifold.

Theorem 1.44 ([Nak75, pages 94–96])**.** *Consider the Iwasawa manifold* $\mathbb{I}_3 := \mathbb{H}\left(3;\mathbb{Z}\left[\mathrm{i}\right]\right)\backslash\mathbb{H}(3;\mathbb{C})$. *There exists a locally complete complex-analytic family of complex structures* $\left\{X_{\mathbf{t}} = \left(\mathbb{I}_3,\, J_{\mathbf{t}}\right)\right\}_{\mathbf{t}\in\Delta(\mathbf{0},\varepsilon)}$, *deformations of* $\mathbb{I}_3$, *depending on six parameters*

$$\mathbf{t} \;=\; \left(t_{11},\, t_{12},\, t_{21},\, t_{22},\, t_{31},\, t_{32}\right) \;\in\; \Delta(\mathbf{0},\varepsilon) \;\subset\; \mathbb{C}^6 \;,$$

where $\varepsilon > 0$ *is small enough,* $\Delta(\mathbf{0},\varepsilon) := \left\{\mathbf{s} \in \mathbb{C}^6 \;:\; |\mathbf{s}| < \varepsilon\right\}$, *and* $X_{\mathbf{0}} = \mathbb{I}_3$.

A set of holomorphic coordinates for $X_{\mathbf{t}}$ *is given by*

$$\left\{\begin{array}{lclcl}
\zeta^1 & := & \zeta^1(\mathbf{t}) & := & z^1 + \sum_{k=1}^{2} t_{1k}\, \bar{z}^k \\
\zeta^2 & := & \zeta^2(\mathbf{t}) & := & z^2 + \sum_{k=1}^{2} t_{2k}\, \bar{z}^k \\
\zeta^3 & := & \zeta^3(\mathbf{t}) & := & z^3 + \sum_{k=1}^{2} \left(t_{3k} + t_{2k}\, z^1\right) \bar{z}^k + A\left(\bar{z}^1,\, \bar{z}^2\right) - D\left(\mathbf{t}\right)\, \bar{z}^3
\end{array}\right.$$

where

$$D(\mathbf{t}) := \det \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$$

and

$$A\left(\bar{z}^1, \bar{z}^2\right) := \frac{1}{2}\left(t_{11}\, t_{21}\, \left(\bar{z}^1\right)^2 + 2\, t_{11}\, t_{22}\, \bar{z}^1\, \bar{z}^2 + t_{12}\, t_{22}\, \left(\bar{z}^2\right)^2\right) .$$

For every $\mathbf{t} \in \Delta(\mathbf{0}, \varepsilon)$, the universal covering of $X_{\mathbf{t}}$ is $\mathbb{C}^3$; more precisely,

$$X_{\mathbf{t}} = \Gamma_{\mathbf{t}} \backslash \mathbb{C}^3 ,$$

where $\Gamma_{\mathbf{t}}$ is the subgroup generated by the transformations

$$\left(\zeta^1, \zeta^2, \zeta^3\right) \overset{\left(\omega^1, \omega^2, \omega^3\right)}{\mapsto} \left(\tilde{\zeta}^1, \tilde{\zeta}^2, \tilde{\zeta}^3\right) ,$$

varying $\left(\omega^1, \omega^2, \omega^3\right) \in (\mathbb{Z}[\mathrm{i}])^3$, where

$$\left\{ \begin{array}{rcl} \tilde{\zeta}^1 & := & \zeta^1 + \left(\omega^1 + t_{11}\, \bar{\omega}^1 + t_{12}\, \bar{\omega}^2\right) \\ \tilde{\zeta}^2 & := & \zeta^2 + \left(\omega^2 + t_{21}\, \bar{\omega}^1 + t_{22}\, \bar{\omega}^2\right) \\ \tilde{\zeta}^3 & := & \zeta^3 + \left(\omega^3 + t_{31}\, \bar{\omega}^1 + t_{32}\, \bar{\omega}^2\right) + \omega^1\, \zeta^2 \\ & & + \left(t_{21}\, \bar{\omega}^1 + t_{22}\, \bar{\omega}^2\right)\left(\zeta^1 + \omega^1\right) + A\left(\bar{\omega}^1, \bar{\omega}^2\right) - D(\mathbf{t})\, \bar{\omega}^3 \end{array} \right. .$$

Remark 1.45. Note that, by [Rol11b, Theorem 4.5], if $X = \Gamma \backslash G$ is a holomorphically parallelizable nilmanifold and G is ν-step nilpotent, then $\mathrm{Kur}(X)$ is cut out by polynomial equations of degree at most ν; furthermore, by [Rol11b, Corollary 4.9], the Kuranishi space of X is smooth if and only if the associated Lie algebra $\mathfrak{g}$ to G is a free 2-step nilpotent Lie algebra, i.e., $\mathfrak{g} \simeq \mathfrak{b}_m$ with $m = \dim_{\mathbb{C}} H^{0,1}_{\overline{\partial}}(X)$, where $\mathfrak{b}_m := \mathbb{C}^m \oplus \wedge^2 \mathbb{C}^m$ with Lie bracket $[a_1 + b_1 \wedge c_1,\, a_2 + b_2 \wedge c_2] := a_1 \wedge a_2$ for $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{C}^m$.

According to the classification by I. Nakamura, the small deformations of $\mathbb{I}_3$ are divided into three classes, *(i)*, *(ii)*, and *(iii)*, in terms of their Hodge numbers: such classes are explicitly described by means of polynomial relations in the parameters, see [Nak75, §3]. As we will see in §1.4.4, it turns out that the Bott-Chern cohomology yields a finer classification of the Kuranishi space of $\mathbb{I}_3$; more precisely, $h^{2,2}_{BC}$ assumes different values within class *(ii)*, respectively class *(iii)*, according to the rank of a certain matrix whose entries are related to the complex structure equations with respect to a suitable co-frame, whereas the numbers corresponding to class *(i)* coincide with those for $\mathbb{I}_3$: this allows a further subdivision of classes *(ii)* and *(iii)* into subclasses *(ii.a)*, *(ii.b)*, and *(iii.a)*, *(iii.b)*.

More precisely, the classes and subclasses of this classification are characterized by the following values of the parameters:

class *(i)* $t_{11} = t_{12} = t_{21} = t_{22} = 0$;

class *(ii)* $D(\mathbf{t}) = 0$ and $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0)$:

- **subclass *(ii.a)*** $D(\mathbf{t}) = 0$ and $\mathrm{rk}\, S = 1$;
- **subclass *(ii.b)*** $D(\mathbf{t}) = 0$ and $\mathrm{rk}\, S = 2$;

class *(iii)* $D(\mathbf{t}) \neq 0$:

- **subclass *(iii.a)*** $D(\mathbf{t}) \neq 0$ and $\mathrm{rk}\, S = 1$;
- **subclass *(iii.b)*** $D(\mathbf{t}) \neq 0$ and $\mathrm{rk}\, S = 2$.

The matrix S is defined by

$$S := \begin{pmatrix} \overline{\sigma_{1\bar{1}}} & \overline{\sigma_{2\bar{2}}} & \overline{\sigma_{1\bar{2}}} & \overline{\sigma_{2\bar{1}}} \\ \sigma_{1\bar{1}} & \sigma_{2\bar{2}} & \sigma_{2\bar{1}} & \sigma_{1\bar{2}} \end{pmatrix}$$

where $\sigma_{1\bar{1}}, \sigma_{1\bar{2}}, \sigma_{2\bar{1}}, \sigma_{2\bar{2}} \in \mathbb{C}$ and $\sigma_{12} \in \mathbb{C}$ are complex numbers depending only on $\mathbf{t}$ such that

$$\mathrm{d}\, \varphi^3_{\mathbf{t}} =: \sigma_{12}\, \varphi^1_{\mathbf{t}} \wedge \varphi^2_{\mathbf{t}} + \sigma_{1\bar{1}}\, \varphi^1_{\mathbf{t}} \wedge \bar{\varphi}^1_{\mathbf{t}} + \sigma_{1\bar{2}}\, \varphi^1_{\mathbf{t}} \wedge \bar{\varphi}^2_{\mathbf{t}} + \sigma_{2\bar{1}}\, \varphi^2_{\mathbf{t}} \wedge \bar{\varphi}^1_{\mathbf{t}} + \sigma_{2\bar{2}}\, \varphi^2_{\mathbf{t}} \wedge \bar{\varphi}^2_{\mathbf{t}} ,$$

being

$$\varphi^1_{\mathbf{t}} := \mathrm{d}\, \zeta^1_{\mathbf{t}} , \quad \varphi^2_{\mathbf{t}} := \mathrm{d}\, \zeta^2_{\mathbf{t}} , \quad \varphi^3_{\mathbf{t}} := \mathrm{d}\, \zeta^3_{\mathbf{t}} - z_1\, \mathrm{d}\, \zeta^2_{\mathbf{t}} - \left(t_{21}\, \bar{z}^1 + t_{22}\, \bar{z}^2\right) \mathrm{d}\, \zeta^1_{\mathbf{t}} ,$$

see §1.4.1. As we will show, see §1.4.1, the first order asymptotic behaviour of σ_{12}, $\sigma_{1\bar{1}}$, $\sigma_{1\bar{2}}$, $\sigma_{2\bar{1}}$, $\sigma_{2\bar{2}}$ for $\mathbf{t}$ near 0 is the following:

$$\begin{cases} \sigma_{12} &= -1 + \mathrm{o}\left(|\mathbf{t}|\right) \\ \sigma_{1\bar{1}} &= t_{21} + \mathrm{o}\left(|\mathbf{t}|\right) \\ \sigma_{1\bar{2}} &= t_{22} + \mathrm{o}\left(|\mathbf{t}|\right) \qquad \text{for} \qquad \mathbf{t} \in \text{ classes } \textit{(i)}, \textit{(ii)} \text{ and } \textit{(iii)}\,, \\ \sigma_{2\bar{1}} &= -t_{11} + \mathrm{o}\left(|\mathbf{t}|\right) \\ \sigma_{2\bar{2}} &= -t_{12} + \mathrm{o}\left(|\mathbf{t}|\right) \end{cases} \tag{1.4.1}$$

and, more precisely, for deformations in class *(ii)* we actually have that

$$\begin{cases} \sigma_{12} &= -1 + \mathrm{o}\left(|\mathbf{t}|\right) \\ \sigma_{1\bar{1}} &= t_{21}\left(1 + \mathrm{o}\left(1\right)\right) \\ \sigma_{1\bar{2}} &= t_{22}\left(1 + \mathrm{o}\left(1\right)\right) \qquad \text{for} \qquad \mathbf{t} \in \text{ class } \textit{(ii)}\,. \\ \sigma_{2\bar{1}} &= -t_{11}\left(1 + \mathrm{o}\left(1\right)\right) \\ \sigma_{2\bar{2}} &= -t_{12}\left(1 + \mathrm{o}\left(1\right)\right) \end{cases} \tag{1.4.2}$$

The complex manifold $X_{\mathbf{t}}$ is endowed with the $J_{\mathbf{t}}$-Hermitian $\mathbb{H}(3;\mathbb{C})$-left-invariant metric $g_{\mathbf{t}}$, which is defined as follows:

$$g_{\mathbf{t}} := \sum_{j=1}^{3} \varphi_{\mathbf{t}}^{j} \odot \bar{\varphi}_{\mathbf{t}}^{j} \,.$$

Structure equations for small deformations of the Iwasawa manifold

In this section, we give the structure equations for the small deformations of the Iwasawa manifold; we will use these computations in §1.4.3 and §1.4.4 to write the Bott-Chern cohomology of $X_{\mathbf{t}}$, and in Theorem 2.49 to prove that the cohomological property of being $\mathcal{C}^\infty$-pure-and-full is not stable under small deformations of the complex structure.

Fix $\mathbf{t} \in \Delta(\mathbf{0}, \varepsilon) \subset \mathbb{C}^6$, and consider the small deformation $X_{\mathbf{t}}$ of the Iwasawa manifold $\mathbb{I}_3$. Consider the system of complex coordinates on $X_{\mathbf{t}}$ given by

$$\begin{cases} \zeta_{\mathbf{t}}^{1} &:= z^1 + \sum_{\lambda=1}^{2} t_{1\lambda} \bar{z}^{\lambda} \\ \zeta_{\mathbf{t}}^{2} &:= z^2 + \sum_{\lambda=1}^{2} t_{2\lambda} \bar{z}^{\lambda} \\ \zeta_{\mathbf{t}}^{3} &:= z^3 + \sum_{\lambda=1}^{2} (t_{3\lambda} + t_{2\lambda} z^1) \bar{z}^{\lambda} + A\left(\bar{z}\right) \end{cases} .$$

Consider

$$\begin{cases} \varphi_{\mathbf{t}}^{1} &:= \mathrm{d}\,\zeta_{\mathbf{t}}^{1} \\ \varphi_{\mathbf{t}}^{2} &:= \mathrm{d}\,\zeta_{\mathbf{t}}^{2} \\ \varphi_{\mathbf{t}}^{3} &:= \mathrm{d}\,\zeta_{\mathbf{t}}^{3} - z_1\,\mathrm{d}\,\zeta_{\mathbf{t}}^{2} - \left(t_{21}\,\bar{z}^1 + t_{22}\,\bar{z}^2\right) \mathrm{d}\,\zeta_{\mathbf{t}}^{1} \end{cases}$$

as a co-frame of $(1,0)$-forms on $X_{\mathbf{t}}$ (that is, as a $\Gamma_{\mathbf{t}}$-invariant co-frame of $(1,0)$-forms on $\mathbb{C}^3$). We want to write the structure equations for $X_{\mathbf{t}}$ with respect to this co-frame.

A straightforward computation gives

$$\begin{cases} z^1 &= \gamma\,\left(\zeta_{\mathbf{t}}^{1} + \lambda_1\,\bar{\zeta}_{\mathbf{t}}^{1} + \lambda_2\,\zeta_{\mathbf{t}}^{2} + \lambda_3\,\bar{\zeta}_{\mathbf{t}}^{2}\right) \\ z^2 &= \alpha\,\left(\mu_0\,\zeta_{\mathbf{t}}^{1} + \mu_1\,\bar{\zeta}_{\mathbf{t}}^{1} + \mu_2\,\zeta_{\mathbf{t}}^{2} + \mu_3\,\bar{\zeta}_{\mathbf{t}}^{2}\right) \end{cases} ,$$

where α, β, γ, λ_i (for $i \in \{1,2,3\}$), μ_j (for $j \in \{0,1,2,3\}$) are complex numbers depending just on $\mathbf{t}$, and defined as follows:

$$\left\{\begin{array}{rcl}
\alpha & := & \dfrac{1}{1-\left|t_{22}\right|^2-t_{21}\,\bar{t}_{12}} \\
\beta & := & t_{21}\,\bar{t}_{11}+t_{22}\,\bar{t}_{21} \\
\gamma & := & \dfrac{1}{1-\left|t_{11}\right|^2-\alpha\,\beta\,\left(t_{11}\,\bar{t}_{12}+t_{12}\,\bar{t}_{22}\right)-t_{12}\,\bar{t}_{21}} \\
\lambda_1 & := & -t_{11}\,\left(1+\alpha\,\bar{t}_{12}\,t_{21}+\alpha\,\left|t_{22}\right|^2\right) \\
\lambda_2 & := & \alpha\,\left(t_{11}\,\bar{t}_{12}+t_{12}\,\bar{t}_{22}\right) \\
\lambda_3 & := & -t_{12}\,\left(1+\alpha\,\bar{t}_{12}\,t_{21}+\alpha\,\left|t_{22}\right|^2\right) \\
\mu_0 & := & \beta\,\gamma \\
\mu_1 & := & \lambda_1\,\beta\,\gamma-t_{21} \\
\mu_2 & := & 1+\lambda_2\,\beta\,\gamma \\
\mu_3 & := & \lambda_3\,\beta\,\gamma-t_{22}
\end{array}\right. .$$

For the complex structures in the class *(i)*, one checks that the structure equations (with respect to the co-frame $\left\{\varphi^1_{\mathbf{t}},\,\varphi^2_{\mathbf{t}},\,\varphi^3_{\mathbf{t}}\right\}$) are the same as the ones for $\mathbb{I}_3$, that is,

$$\left\{\begin{array}{rcl}
\mathrm{d}\,\varphi^1_{\mathbf{t}} & = & 0 \\
\mathrm{d}\,\varphi^2_{\mathbf{t}} & = & 0 \\
\mathrm{d}\,\varphi^3_{\mathbf{t}} & = & -\varphi^1_{\mathbf{t}}\wedge\varphi^2_{\mathbf{t}}
\end{array}\right. \qquad \text{for} \qquad \mathbf{t}\in\ \text{class } (i)\,.$$

For small deformations in classes *(ii)* and *(iii)*, we have that

$$\left\{\begin{array}{rcl}
\mathrm{d}\,\varphi^1_{\mathbf{t}} & = & 0 \\
\mathrm{d}\,\varphi^2_{\mathbf{t}} & = & 0 \\
\mathrm{d}\,\varphi^3_{\mathbf{t}} & = & \sigma_{12}\,\varphi^1_{\mathbf{t}}\wedge\varphi^2_{\mathbf{t}} \\
 & & +\sigma_{1\bar{1}}\,\varphi^1_{\mathbf{t}}\wedge\bar{\varphi}^1_{\mathbf{t}}+\sigma_{1\bar{2}}\,\varphi^1_{\mathbf{t}}\wedge\bar{\varphi}^2_{\mathbf{t}} \\
 & & +\sigma_{2\bar{1}}\,\varphi^2_{\mathbf{t}}\wedge\bar{\varphi}^1_{\mathbf{t}}+\sigma_{2\bar{2}}\,\varphi^2_{\mathbf{t}}\wedge\bar{\varphi}^2_{\mathbf{t}}
\end{array}\right. \qquad \text{for} \qquad \mathbf{t}\in\ \text{classes } (ii) \text{ and } (iii)\,,$$

where $\sigma_{12},\,\sigma_{1\bar{1}},\,\sigma_{1\bar{2}},\,\sigma_{2\bar{1}},\,\sigma_{2\bar{2}}\in\mathbb{C}$ are complex numbers depending just on $\mathbf{t}$. The asymptotic behaviour of $\sigma_{12},\,\sigma_{1\bar{1}},\,\sigma_{1\bar{2}},\,\sigma_{2\bar{1}},\,\sigma_{2\bar{2}}\in\mathbb{C}$ is the following:

$$\left\{\begin{array}{rcl}
\sigma_{12} & = & -1+\mathrm{o}\left(|\mathbf{t}|\right) \\
\sigma_{1\bar{1}} & = & t_{21}+\mathrm{o}\left(|\mathbf{t}|\right) \\
\sigma_{1\bar{2}} & = & t_{22}+\mathrm{o}\left(|\mathbf{t}|\right) \\
\sigma_{2\bar{1}} & = & -t_{11}+\mathrm{o}\left(|\mathbf{t}|\right) \\
\sigma_{2\bar{2}} & = & -t_{12}+\mathrm{o}\left(|\mathbf{t}|\right)
\end{array}\right. \qquad \text{for} \qquad \mathbf{t}\in\ \text{classes } (i),\, (ii) \text{ and } (iii)\,, \tag{1.4.3}$$

more precisely, for deformations in class *(ii)* we actually have that

$$\left\{\begin{array}{rcl}
\sigma_{12} & = & -1+\mathrm{o}\left(|\mathbf{t}|\right) \\
\sigma_{1\bar{1}} & = & t_{21}\left(1+\mathrm{o}\left(1\right)\right) \\
\sigma_{1\bar{2}} & = & t_{22}\left(1+\mathrm{o}\left(1\right)\right) \\
\sigma_{2\bar{1}} & = & -t_{11}\left(1+\mathrm{o}\left(1\right)\right) \\
\sigma_{2\bar{2}} & = & -t_{12}\left(1+\mathrm{o}\left(1\right)\right)
\end{array}\right. \qquad \text{for} \qquad \mathbf{t}\in\ \text{class } (ii)\,. \tag{1.4.4}$$

The explicit values of σ_{12}, $\sigma_{1\bar{1}}$, $\sigma_{1\bar{2}}$, $\sigma_{2\bar{1}}$, $\sigma_{2\bar{2}} \in \mathbb{C}$ in the case of class *(ii)* are the following, [AT11, page 416]:

$$\begin{cases} \sigma_{12} & := & -\gamma + t_{21}\bar{\lambda}_3\bar{\gamma} + t_{22}\bar{\alpha}\bar{\mu}_3 \\ \sigma_{1\bar{1}} & := & t_{21}\,\overline{\gamma\left(1 + t_{21}\bar{t}_{12}\alpha + \left|t_{22}\right|^2\alpha\right)} \\ \sigma_{1\bar{2}} & := & t_{22}\,\overline{\gamma\left(1 + t_{21}\bar{t}_{12}\alpha + \left|t_{22}\right|^2\alpha\right)} \\ \sigma_{2\bar{1}} & := & -t_{11}\,\gamma\left(1 + t_{21}\bar{t}_{12}\alpha + \left|t_{22}\right|^2\alpha\right) \\ \sigma_{2\bar{2}} & := & -t_{12}\,\gamma\left(1 + t_{21}\bar{t}_{12}\alpha + \left|t_{22}\right|^2\alpha\right) \end{cases} \qquad \text{for} \qquad \mathbf{t} \in \text{ class } (ii)\,.$$

Note that, for small deformations in class *(ii)*, one has $\sigma_{12} \neq 0$ and $(\sigma_{1\bar{1}}, \sigma_{1\bar{2}}, \sigma_{2\bar{1}}, \sigma_{2\bar{2}}) \neq (0, 0, 0, 0)$.

1.4.2 The de Rham cohomology of the Iwasawa manifold and of its small deformations

Recall that, by Ehresmann's theorem, every complex-analytic family of compact complex manifolds is locally trivial as a differentiable family of compact differentiable manifolds, see, e.g., [MK06, Theorem 4.1]. Therefore the de Rham cohomology of small deformations of the Iwasawa manifold is the same as the de Rham cohomology of $\mathbb{I}_3$, which can be computed by using K. Nomizu's theorem [Nom54, Theorem 1].

In the table below, we list the harmonic representatives with respect to the metric $g_{\mathbf{0}}$ instead of their classes and, as usually, we shorten the notation writing, for example, $\varphi^{A\bar{B}} := \varphi^A \wedge \bar{\varphi}^B$.

$H^k_{dR}(\mathbb{I}_3;\mathbb{C})$	$g_{\mathbf{0}}$-harmonic representatives	$\dim_{\mathbb{C}} H^k_{dR}(\mathbb{I}_3;\mathbb{C})$
$k=1$	φ^1, φ^2, $\bar{\varphi}^1$, $\bar{\varphi}^2$	4
$k=2$	φ^{13}, φ^{23}, $\varphi^{1\bar{1}}$, $\varphi^{1\bar{2}}$, $\varphi^{2\bar{1}}$, $\varphi^{2\bar{2}}$, $\varphi^{\bar{1}\bar{3}}$, $\varphi^{\bar{2}\bar{3}}$	8
$k=3$	φ^{123}, $\varphi^{13\bar{1}}$, $\varphi^{13\bar{2}}$, $\varphi^{23\bar{1}}$, $\varphi^{23\bar{2}}$, $\varphi^{1\bar{1}\bar{3}}$, $\varphi^{1\bar{2}\bar{3}}$, $\varphi^{2\bar{1}\bar{3}}$, $\varphi^{2\bar{2}\bar{3}}$, $\varphi^{\bar{1}\bar{2}\bar{3}}$	10
$k=4$	$\varphi^{123\bar{1}}$, $\varphi^{123\bar{2}}$, $\varphi^{13\bar{1}\bar{3}}$, $\varphi^{13\bar{2}\bar{3}}$, $\varphi^{23\bar{1}\bar{3}}$, $\varphi^{23\bar{2}\bar{3}}$, $\varphi^{1\bar{1}\bar{2}\bar{3}}$, $\varphi^{2\bar{1}\bar{2}\bar{3}}$	8
$k=5$	$\varphi^{123\bar{1}\bar{3}}$, $\varphi^{123\bar{2}\bar{3}}$, $\varphi^{13\bar{1}\bar{2}\bar{3}}$, $\varphi^{23\bar{1}\bar{2}\bar{3}}$	4

Remark 1.46. Note that all the $g_{\mathbf{0}}$-harmonic representatives of $H^{\bullet}_{dR}(\mathbb{I}_3;\mathbb{R})$ are of pure type with respect to $J_{\mathbf{0}}$, that is, they are in $(\wedge^{p,q}\mathbb{I}_3 \oplus \wedge^{q,p}\mathbb{I}_3) \cap \wedge^{p+q}\mathbb{I}_3$ for some $p, q \in \{0,\, 1,\, 2,\, 3\}$; this is no more true for $J_{\mathbf{t}}$ with $\mathbf{t} \neq \mathbf{0}$ small enough, see Theorem 2.49.

1.4.3 The Dolbeault cohomology of the Iwasawa manifold and of its small deformations

The Hodge numbers of the Iwasawa manifold and of its small deformations have been computed by I. Nakamura in [Nak75, page 96]. The $g_{\mathbf{t}}$-harmonic representatives for $H^{\bullet,\bullet}_{\bar{\partial}}(X_{\mathbf{t}})$, for $\mathbf{t}$ small enough, can be computed using the considerations in §1.3.2 and the structure equations given in §1.4.1. We collect here the results of the computations.

In order to reduce the number of cases under consideration, recall that, on a compact complex Hermitian manifold X of complex dimension n, for any $p, q \in \mathbb{N}$, the Hodge-$*$-operator and the conjugation induce an isomorphism

$$H^{p,q}_{\bar{\partial}}(X) \stackrel{\simeq}{\to} H^{n-q,n-p}_{\partial}(X) \stackrel{\simeq}{\to} \overline{H^{n-p,n-q}_{\bar{\partial}}(X)}\,.$$

- **1-forms.** It is straightforward to check that

$$H^{1,0}_{\bar{\partial}}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^1_{\mathbf{t}},\, \varphi^2_{\mathbf{t}},\, \varphi^3_{\mathbf{t}} \right\rangle \qquad \text{for} \qquad \mathbf{t} \in \text{ class } (i)$$

and

$$H^{0,1}_{\bar{\partial}}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \bar{\varphi}^1_{\mathbf{t}},\, \bar{\varphi}^2_{\mathbf{t}} \right\rangle \qquad \text{for} \qquad \mathbf{t} \in \text{ classes } (i),\ (ii) \text{ and } (iii)\,.$$

Since $\overline{\partial}\,\varphi^3_{\mathbf{t}} \neq 0$ for $X_{\mathbf{t}}$ in class *(ii)* or in class *(iii)*, one has

$$H^{1,0}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^1_{\mathbf{t}},\, \varphi^2_{\mathbf{t}} \right\rangle \qquad \text{for} \qquad \mathbf{t} \in \text{ classes } \textit{(ii)} \text{ and } \textit{(iii)} \;:$$

this means in particular that $X_{\mathbf{t}}$ is not holomorphically parallelizable for $\mathbf{t}$ in classes *(ii)* and *(iii)*, [Nak75, pages 86, 96].

Summarizing,

$$\dim_{\mathbb{C}} H^{1,0}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; \begin{cases} 3 & \text{for} \quad \mathbf{t} \in \text{ class } \textit{(i)} \\ 2 & \text{for} \quad \mathbf{t} \in \text{ classes } \textit{(ii)} \text{ and } \textit{(iii)} \end{cases} ,$$

and

$$\dim_{\mathbb{C}} H^{0,1}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; 2 \qquad \text{for} \qquad \mathbf{t} \in \text{ classes } \textit{(i)}, \textit{(ii)} \text{ and } \textit{(iii)}\,.$$

- **2-forms.** A straightforward computation yields

$$H^{2,0}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{12}_{\mathbf{t}},\, \varphi^{13}_{\mathbf{t}},\, \varphi^{23}_{\mathbf{t}} \right\rangle \qquad \text{for} \qquad \mathbf{t} \in \text{ class } \textit{(i)}\,,$$

$$H^{1,1}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{1\bar{1}}_{\mathbf{t}},\, \varphi^{1\bar{2}}_{\mathbf{t}},\, \varphi^{2\bar{1}}_{\mathbf{t}},\, \varphi^{2\bar{2}}_{\mathbf{t}},\, \varphi^{3\bar{1}}_{\mathbf{t}},\, \varphi^{3\bar{2}}_{\mathbf{t}} \right\rangle \qquad \text{for} \qquad \mathbf{t} \in \text{ class } \textit{(i)}\,,$$

and

$$H^{0,2}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{\bar{1}\bar{3}}_{\mathbf{t}},\, \varphi^{\bar{2}\bar{3}}_{\mathbf{t}} \right\rangle \qquad \text{for} \qquad \mathbf{t} \in \text{ classes } \textit{(i)}, \textit{(ii)} \text{ and } \textit{(iii)}\,.$$

We now compute $H^{2,0}_{\overline{\partial}}(X_{\mathbf{t}})$ for $\mathbf{t} \in$ classes *(ii)* and *(iii)*. The $\mathbb{H}(3;\mathbb{C})$-left-invariant $(2,0)$-forms are of the type $A\,\varphi^{12}_{\mathbf{t}} + B\,\varphi^{13}_{\mathbf{t}} + C\,\varphi^{23}_{\mathbf{t}}$ with $A, B, C \in \mathbb{C}$, so one has to solve the linear system

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sigma_{2\bar{1}} & \sigma_{1\bar{1}} \\ 0 & -\sigma_{2\bar{2}} & \sigma_{1\bar{2}} \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \;;$$

since the associated matrix to the system has rank 0 for $\mathbf{t} \in$ class *(i)*, rank 1 for $\mathbf{t} \in$ class *(ii)* and rank 2 for $\mathbf{t} \in$ class *(iii)*, one concludes that

$$\dim_{\mathbb{C}} H^{2,0}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; 2 \qquad \text{for} \qquad \mathbf{t} \in \text{ class } \textit{(ii)}$$

(the generators being $\varphi^{12}_{\mathbf{t}}$ and a linear combination of $\varphi^{13}_{\mathbf{t}}$ and $\varphi^{23}_{\mathbf{t}}$) and

$$\dim_{\mathbb{C}} H^{2,0}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; 1 \qquad \text{for} \qquad \mathbf{t} \in \text{ class } \textit{(iii)}$$

(the generator being $\varphi^{12}_{\mathbf{t}}$).

It remains to compute $H^{1,1}_{\overline{\partial}}(X_{\mathbf{t}})$ for $\mathbf{t} \in$ classes *(ii)* and *(iii)*. For such $\mathbf{t}$, one has that: three independent $\overline{\square}_{J_{\mathbf{t}}}$-harmonic $(1,1)$-forms are of the type $\psi_1 := A\,\varphi^{1\bar{1}}_{\mathbf{t}} + B\,\varphi^{1\bar{2}}_{\mathbf{t}} + C\,\varphi^{2\bar{1}}_{\mathbf{t}} + D\,\varphi^{2\bar{2}}_{\mathbf{t}}$ where A, B, C, $D \in \mathbb{C}$ satisfy the equation

$$\begin{pmatrix} \overline{\sigma_{1\bar{1}}} & -\overline{\sigma_{1\bar{2}}} & -\overline{\sigma_{2\bar{1}}} & \overline{\sigma_{2\bar{2}}} \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \;=\; 0\,,$$

whose matrix has rank 1 for $\mathbf{t} \in$ classes *(ii)* and *(iii)* (while its rank is 0 for $\mathbf{t} \in$ class *(i)*); two other independent $\overline{\square}_{J_{\mathbf{t}}}$-harmonic $(1,1)$-forms are of the type $\psi_2 := E\,\varphi^{1\bar{3}}_{\mathbf{t}} + F\,\varphi^{2\bar{3}}_{\mathbf{t}} + G\,\varphi^{3\bar{1}}_{\mathbf{t}} + H\,\varphi^{3\bar{2}}_{\mathbf{t}}$ where E, F, G, $H \in \mathbb{C}$ are solution of the system

$$\begin{pmatrix} -\overline{\sigma_{12}} & 0 & -\overline{\sigma_{1\bar{2}}} & \overline{\sigma_{1\bar{1}}} \\ 0 & -\overline{\sigma_{12}} & -\overline{\sigma_{2\bar{1}}} & \overline{\sigma_{2\bar{2}}} \end{pmatrix} \cdot \begin{pmatrix} E \\ F \\ G \\ H \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

whose matrix has rank 2 for $\mathbf{t} \in$ classes *(i)*, *(ii)* and *(iii)*; note also that no $(1,1)$-form with a non-zero component in $\varphi^{3\bar{3}}_{\mathbf{t}}$ can be $\overline{\square}_{J_t}$-harmonic. Hence, one can conclude that

$$\dim_{\mathbb{C}} H^{1,1}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; 5 \qquad \text{for} \qquad \mathbf{t} \in \text{ classes } \textit{(ii)} \text{ and } \textit{(iii)}\,.$$

Summarizing,

$$\dim_{\mathbb{C}} H^{2,0}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; \begin{cases} 3 & \text{for} \quad \mathbf{t} \in \text{ class } \textit{(i)} \\ 2 & \text{for} \quad \mathbf{t} \in \text{ class } \textit{(ii)} \\ 1 & \text{for} \quad \mathbf{t} \in \text{ class } \textit{(iii)} \end{cases} ,$$

and

$$\dim_{\mathbb{C}} H^{1,1}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; \begin{cases} 6 & \text{for} \quad \mathbf{t} \in \text{ class } \textit{(i)} \\ 5 & \text{for} \quad \mathbf{t} \in \text{ classes } \textit{(ii)} \text{ and } \textit{(iii)} \end{cases} ,$$

and

$$\dim_{\mathbb{C}} H^{0,2}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; 2 \quad \text{for} \quad \mathbf{t} \in \text{ classes } \textit{(i)}, \textit{(ii)} \text{ and } \textit{(iii)} \,.$$

- **3-forms.** Finally, we have to compute $H^{3,0}_{\overline{\partial}}(X_{\mathbf{t}})$ and $H^{2,1}_{\overline{\partial}}(X_{\mathbf{t}})$. A straightforward linear algebra computation yields to

$$H^{3,0}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{123}_{\mathbf{t}} \right\rangle \quad \text{for} \quad \mathbf{t} \in \text{ classes } \textit{(i)}, \textit{(ii)} \text{ and } \textit{(iii)}$$

and

$$H^{2,1}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{12\bar{1}}_{\mathbf{t}},\, \varphi^{12\bar{2}}_{\mathbf{t}},\, \varphi^{13\bar{1}}_{\mathbf{t}},\, \varphi^{13\bar{2}}_{\mathbf{t}},\, \varphi^{23\bar{1}}_{\mathbf{t}},\, \varphi^{23\bar{2}}_{\mathbf{t}} \right\rangle \quad \text{for} \quad \mathbf{t} \in \text{ class } \textit{(i)} \,.$$

It remains to compute $H^{2,1}_{\overline{\partial}}(X_{\mathbf{t}})$ for $\mathbf{t} \in$ classes *(ii)* and *(iii)*. Firstly, one notes that four of the six generators of the space of $\mathbb{H}(3;\mathbb{C})$-left-invariant $(2,1)$-forms that are $\overline{\square}_{J_{\mathbf{t}}}$-harmonic for $\mathbf{t} \in$ class *(i)* can be slightly modified to get four $\overline{\partial}_{J_{\mathbf{t}}}$-holomorphic $(2,1)$-forms for $\mathbf{t} \in$ class *(ii)* or class *(iii)*: more precisely, one has

$$H^{2,1}_{\overline{\partial}}(X_{\mathbf{t}}) \;\supseteq\; \mathbb{C}\left\langle \varphi^{13\bar{1}}_{\mathbf{t}} - \frac{\sigma_{2\bar{2}}}{\sigma_{12}}\varphi^{12\bar{3}}_{\mathbf{t}},\, \varphi^{13\bar{2}}_{\mathbf{t}} - \frac{\sigma_{2\bar{1}}}{\sigma_{12}}\varphi^{12\bar{3}}_{\mathbf{t}},\, \varphi^{23\bar{1}}_{\mathbf{t}} - \frac{\sigma_{1\bar{2}}}{\sigma_{12}}\varphi^{12\bar{3}}_{\mathbf{t}},\, \varphi^{23\bar{2}}_{\mathbf{t}} - \frac{\sigma_{1\bar{1}}}{\sigma_{12}}\varphi^{12\bar{3}}_{\mathbf{t}} \right\rangle \,;$$

in other words, four independent $\overline{\square}_{J_{\mathbf{t}}}$-harmonic $(2,1)$-forms are of the type $\psi_2 :=: C\,\varphi^{12\bar{3}}_{\mathbf{t}} + D\,\varphi^{13\bar{1}}_{\mathbf{t}} + E\,\varphi^{13\bar{2}}_{\mathbf{t}} + F\,\varphi^{23\bar{1}}_{\mathbf{t}} + G\,\varphi^{23\bar{2}}_{\mathbf{t}}$, where C, D, E, F, $G \in \mathbb{C}$ are solution of the linear system

$$\begin{pmatrix} \overline{\sigma_{12}} & \sigma_{2\bar{2}} & -\sigma_{2\bar{1}} & -\sigma_{1\bar{2}} & \sigma_{1\bar{1}} \end{pmatrix} \cdot \begin{pmatrix} C \\ D \\ E \\ F \\ G \end{pmatrix} \;=\; 0 \,,$$

whose matrix has rank 1 for every $\mathbf{t} \in$ classes *(i)*, *(ii)* and *(iii)*. Note that one can reduce to study the $\overline{\square}$-harmonicity of the $(2,1)$-forms of the type $\psi_1 :=: A\,\varphi^{12\bar{1}}_{\mathbf{t}} + B\,\varphi^{12\bar{2}}_{\mathbf{t}}$: indeed, a $(2,1)$-form $\psi :=: \psi_1 + \psi_2 + H\,\varphi^{13\bar{3}}_{\mathbf{t}} + L\,\varphi^{23\bar{3}}_{\mathbf{t}}$, where $H, L \in \mathbb{C}$, is $\overline{\square}$-harmonic if and only if $H = 0 = L$ and both ψ_1 and ψ_2 are $\overline{\square}$-harmonic. A $(2,1)$-form of the type ψ_1 is $\overline{\square}$-harmonic if and only if A, $B \in \mathbb{C}$ solve the linear system

$$\begin{pmatrix} -\overline{\sigma_{1\bar{1}}} & \overline{\sigma_{1\bar{2}}} \\ -\overline{\sigma_{2\bar{1}}} & \overline{\sigma_{2\bar{2}}} \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

whose matrix has rank 0 for $\mathbf{t} \in$ class *(i)*, rank 1 for $\mathbf{t} \in$ class *(ii)* and rank 2 for $\mathbf{t} \in$ class *(iii)*. In particular, one gets that

$$\dim_{\mathbb{C}} H^{2,1}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; 5 \quad \text{for} \quad \mathbf{t} \in \text{ class } \textit{(ii)}$$

and

$$\dim_{\mathbb{C}} H^{2,1}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; 4 \quad \text{for} \quad \mathbf{t} \in \text{ class } \textit{(iii)} \,.$$

Summarizing,

$$\dim_{\mathbb{C}} H^{3,0}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; 1 \quad \text{for} \quad \mathbf{t} \in \text{ classes } \textit{(i)}, \textit{(ii)} \text{ and } \textit{(iii)} \,,$$

and

$$\dim_{\mathbb{C}} H^{2,1}_{\overline{\partial}}(X_{\mathbf{t}}) \;=\; \begin{cases} 6 & \text{for} \quad \mathbf{t} \in \text{ class } \textit{(i)} \\ 5 & \text{for} \quad \mathbf{t} \in \text{ class } \textit{(ii)} \\ 4 & \text{for} \quad \mathbf{t} \in \text{ class } \textit{(iii)} \end{cases} .$$

1.4.4 The Bott-Chern and Aeppli cohomologies of the Iwasawa manifold and of its small deformations

In this section, using Theorem 1.39 and Theorem 1.42, we explicitly compute the dimensions of $H^{\bullet,\bullet}_{BC}(X_{\mathbf{t}})$, for $\mathbf{t}$ small enough, [Ang11, §5.3]: such numbers are summarized in the tables in §1.4.5.

In order to reduce the number of cases under consideration, recall that, on a compact complex Hermitian manifold X of complex dimension n, for every $p, q \in \mathbb{N}$, the conjugation induces an isomorphism $H^{p,q}_{BC}(X) \stackrel{\simeq}{\to} H^{q,p}_{BC}(X)$, and the Hodge-$*$-operator induces an isomorphism $H^{p,q}_{BC}(X) \stackrel{\simeq}{\to} H^{n-q,n-p}_{A}(X)$; furthermore, note that

$$H^{p,0}_{BC}(X) \;\simeq\; \ker\left(\mathrm{d}\colon\ \wedge^{p,0} X \to \wedge^{p+1}(X;\mathbb{C})\right)$$

and that

$$H^{n,0}_{BC}(X) \;\simeq\; H^{n,0}_{\overline{\partial}}(X)\;.$$

- **1-forms** It is straightforward to check that

$$H^{1,0}_{BC}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^1_{\mathbf{t}},\, \varphi^2_{\mathbf{t}}\right\rangle \qquad \text{for} \qquad \mathbf{t} \in \text{ classes } \textit{(i)},\ \textit{(ii)} \text{ and } \textit{(iii)}\;.$$

- **2-forms** It is straightforward to compute

$$H^{2,0}_{BC}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{12}_{\mathbf{t}},\, \varphi^{13}_{\mathbf{t}},\, \varphi^{23}_{\mathbf{t}}\right\rangle \qquad \text{for} \qquad \mathbf{t} \in \text{ class } \textit{(i)}\;.$$

The computations for $H^{2,0}_{BC}(X_{\mathbf{t}})$ reduce to find $\psi = A\,\varphi^{12}_{\mathbf{t}} + B\,\varphi^{13}_{\mathbf{t}} + C\,\varphi^{23}_{\mathbf{t}}$ where $A,\, B,\, C \in \mathbb{C}$ satisfy the linear system

$$\left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & -\sigma_{2\bar{1}} & \sigma_{1\bar{1}} \\ 0 & -\sigma_{2\bar{2}} & \sigma_{1\bar{2}} \end{array}\right) \cdot \left(\begin{array}{c} A \\ B \\ C \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right)\;,$$

whose matrix has rank 0 for $\mathbf{t} \in$ class *(i)*, rank 1 for $\mathbf{t} \in$ class *(ii)* and rank 2 for $\mathbf{t} \in$ class *(iii)*; so, in particular, we get that

$$\dim_{\mathbb{C}} H^{2,0}_{BC}(X_{\mathbf{t}}) \;=\; 2 \qquad \text{for} \qquad \mathbf{t} \in \text{ class } \textit{(ii)}$$

and

$$\dim_{\mathbb{C}} H^{2,0}_{BC}(X_{\mathbf{t}}) \;=\; 1 \qquad \text{for} \qquad \mathbf{t} \in \text{ class } \textit{(iii)}$$

(more precisely, for $\mathbf{t} \in$ class *(iii)* we have $H^{2,0}_{BC}(X_{\mathbf{t}}) = \mathbb{C}\left\langle \varphi^{12}_{\mathbf{t}}\right\rangle$).

It remains to compute $H^{1,1}_{BC}(X_{\mathbf{t}})$ for $\mathbf{t} \in$ classes *(i)*, *(ii)* and *(iii)*. First of all, it is easy to check that

$$H^{1,1}_{BC}(X_{\mathbf{t}}) \;\supseteq\; \mathbb{C}\left\langle \varphi^{1\bar{1}}_{\mathbf{t}},\, \varphi^{1\bar{2}}_{\mathbf{t}},\, \varphi^{2\bar{1}}_{\mathbf{t}},\, \varphi^{2\bar{2}}_{\mathbf{t}}\right\rangle \qquad \text{for} \qquad \mathbf{t} \in \text{ classes } \textit{(i)},\ \textit{(ii)} \text{ and } \textit{(iii)}\;,$$

and equality holds if $\mathbf{t} \in$ class *(i)*, hence, in particular, if $\mathbf{t} = \mathbf{0}$. This immediately implies that

$$H^{1,1}_{BC}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{1\bar{1}}_{\mathbf{t}},\, \varphi^{1\bar{2}}_{\mathbf{t}},\, \varphi^{2\bar{1}}_{\mathbf{t}},\, \varphi^{2\bar{2}}_{\mathbf{t}}\right\rangle \qquad \text{for} \qquad \mathbf{t} \in \text{ classes } \textit{(i)},\ \textit{(ii)} \text{ and } \textit{(iii)}\;;$$

indeed, the function $\mathbf{t} \mapsto \dim_{\mathbb{C}} H^{1,1}_{BC}(X_{\mathbf{t}})$ is upper-semi-continuous at 0, since $H^{1,1}_{BC}(X_{\mathbf{t}})$ is isomorphic to the kernel of the self-adjoint elliptic differential operator $\tilde{\Delta}_{BC_{J_{\mathbf{t}}}}\lfloor_{\wedge^{1,1}X_{\mathbf{t}}}$. (One can explain this argument saying that the new parts appearing in the computations for $\mathbf{t} \neq \mathbf{0}$ are "too small" to balance out the lack for the ∂-closure or the $\overline{\partial}$-closure.) From another point of view, we can note that $(1,1)$-forms of the type $\psi = A\,\varphi^{1\bar{3}}_{\mathbf{t}} + B\,\varphi^{2\bar{3}}_{\mathbf{t}} + C\,\varphi^{3\bar{1}}_{\mathbf{t}} + D\,\varphi^{3\bar{2}}_{\mathbf{t}} + E\,\varphi^{3\bar{3}}_{\mathbf{t}}$ are $\tilde{\Delta}_{BC_{J_{\mathbf{t}}}}$-harmonic if and only if $E = 0$ and $A,\, B,\, C,\, D \in \mathbb{C}$ satisfy the linear system

$$\left(\begin{array}{cccc} -\overline{\sigma_{12}} & 0 & -\sigma_{1\bar{2}} & -\sigma_{1\bar{1}} \\ 0 & -\overline{\sigma_{12}} & -\sigma_{2\bar{2}} & -\sigma_{2\bar{1}} \\ \hline \overline{\sigma_{1\bar{2}}} & -\overline{\sigma_{1\bar{1}}} & \sigma_{12} & 0 \\ \overline{\sigma_{2\bar{2}}} & -\overline{\sigma_{2\bar{1}}} & 0 & \sigma_{12} \end{array}\right) \cdot \left(\begin{array}{c} A \\ B \\ C \\ D \end{array}\right) \;=\; \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right)\;,$$

whose matrix has rank 4 for every $\mathbf{t} \in$ classes *(i)*, *(ii)* and *(iii)*.

- **3-forms** It is straightforward to compute

$$H^{3,0}_{BC}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{123}_{\mathbf{t}} \right\rangle \qquad \text{for} \quad \mathbf{t}\in \text{ classes } \textit{(i)}, \textit{(ii)} \text{ and } \textit{(iii)}\;.$$

Moreover,

$$H^{2,1}_{BC}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{12\bar1}_{\mathbf{t}},\, \varphi^{12\bar2}_{\mathbf{t}},\, \varphi^{13\bar1}_{\mathbf{t}} - \frac{\sigma_{2\bar2}}{\overline{\sigma_{12}}}\,\varphi^{12\bar3}_{\mathbf{t}},\, \varphi^{13\bar2}_{\mathbf{t}} + \frac{\sigma_{2\bar1}}{\overline{\sigma_{12}}}\,\varphi^{12\bar3}_{\mathbf{t}},\, \varphi^{23\bar1}_{\mathbf{t}} + \frac{\sigma_{1\bar2}}{\overline{\sigma_{12}}}\,\varphi^{12\bar3}_{\mathbf{t}},\, \varphi^{23\bar2}_{\mathbf{t}} - \frac{\sigma_{1\bar1}}{\overline{\sigma_{12}}}\,\varphi^{12\bar3}_{\mathbf{t}} \right\rangle$$

$$\text{for} \quad \mathbf{t}\in \text{ classes } \textit{(i)}, \textit{(ii)} \text{ and } \textit{(iii)}\;;$$

in particular,

$$H^{2,1}_{BC}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{12\bar1}_{\mathbf{t}},\, \varphi^{12\bar2}_{\mathbf{t}},\, \varphi^{13\bar1}_{\mathbf{t}},\, \varphi^{13\bar2}_{\mathbf{t}},\, \varphi^{23\bar1}_{\mathbf{t}},\, \varphi^{23\bar2}_{\mathbf{t}} \right\rangle \qquad \text{for} \quad \mathbf{t}\in \text{ class } \textit{(i)}\;.$$

From another point of view, one can easily check that

$$H^{2,1}_{BC}(X_{\mathbf{t}}) \;\supseteq\; \mathbb{C}\left\langle \varphi^{12\bar1}_{\mathbf{t}},\, \varphi^{12\bar2}_{\mathbf{t}} \right\rangle \qquad \text{for} \quad \mathbf{t}\in \text{ classes } \textit{(i)}, \textit{(ii)} \text{ and } \textit{(iii)}\;,$$

and that the $(2,1)$-forms of the type $\psi = A\,\varphi^{12\bar3}_{\mathbf{t}} + B\,\varphi^{13\bar1}_{\mathbf{t}} + C\,\varphi^{13\bar2}_{\mathbf{t}} + D\,\varphi^{23\bar1}_{\mathbf{t}} + E\,\varphi^{23\bar2}_{\mathbf{t}} + F\,\varphi^{13\bar3}_{\mathbf{t}} + G\,\varphi^{23\bar3}_{\mathbf{t}}$ are $\tilde\Delta_{BC_{J_{\mathbf{t}}}}$-harmonic if and only if $F = 0 = G$ and $A,\, B,\, C,\, D,\, E \in \mathbb{C}$ satisfy the equation

$$\begin{pmatrix} \overline{\sigma_{12}} & \sigma_{2\bar2} & -\sigma_{2\bar1} & \sigma_{1\bar2} & \sigma_{1\bar1} \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \\ D \\ E \end{pmatrix} \;=\; 0\;,$$

whose matrix has rank 1 for every $\mathbf{t}\in$ classes *(i)*, *(ii)* and *(iii)*. Note in particular that the dimensions of $H^{3,0}_{BC}(X_{\mathbf{t}})$ and of $H^{2,1}_{BC}(X_{\mathbf{t}})$ do not depend on $\mathbf{t}$.

- **4-forms** It is straightforward to compute

$$H^{3,1}_{BC}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{123\bar1}_{\mathbf{t}},\, \varphi^{123\bar2}_{\mathbf{t}} \right\rangle \qquad \text{for} \quad \mathbf{t}\in \text{ classes } \textit{(i)}, \textit{(ii)} \text{ and } \textit{(iii)}$$

and

$$H^{2,2}_{BC}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{12\bar1\bar3}_{\mathbf{t}},\, \varphi^{12\bar2\bar3}_{\mathbf{t}},\, \varphi^{13\bar1\bar2}_{\mathbf{t}},\, \varphi^{13\bar1\bar3}_{\mathbf{t}},\, \varphi^{13\bar2\bar3}_{\mathbf{t}},\, \varphi^{23\bar1\bar2}_{\mathbf{t}},\, \varphi^{23\bar1\bar3}_{\mathbf{t}},\, \varphi^{23\bar2\bar3}_{\mathbf{t}} \right\rangle \qquad \text{for} \quad \mathbf{t}\in \text{ class } \textit{(i)}\;.$$

Moreover, one can check that

$$H^{2,2}_{BC}(X_{\mathbf{t}}) \;\supseteq\; \mathbb{C}\left\langle \varphi^{12\bar1\bar3}_{\mathbf{t}},\, \varphi^{12\bar2\bar3}_{\mathbf{t}},\, \varphi^{13\bar1\bar2}_{\mathbf{t}},\, \varphi^{23\bar1\bar2}_{\mathbf{t}} \right\rangle \qquad \text{for} \quad \mathbf{t}\in \text{ classes } \textit{(i)}, \textit{(ii)} \text{ and } \textit{(iii)}\;,$$

and that no $(2,2)$-form with a non-zero component in $\varphi^{12\bar1\bar2}_{\mathbf{t}}$ can be $\tilde\Delta_{BC_{J_t}}$-harmonic. For $H^{2,2}_{BC}(X_{\mathbf{t}})$ with $\mathbf{t}\in$ classes *(ii)* and *(iii)*, we get a new behaviour: there are subclasses in both class *(ii)* and class *(iii)*, which can be distinguished by the dimension of $H^{2,2}_{BC}(X_{\mathbf{t}})$. Indeed, consider $(2,2)$-forms of the type $\psi = A\,\varphi^{13\bar1\bar3}_{\mathbf{t}} + B\,\varphi^{13\bar2\bar3}_{\mathbf{t}} + C\,\varphi^{23\bar1\bar3}_{\mathbf{t}} + D\,\varphi^{23\bar2\bar3}_{\mathbf{t}}$; a straightforward computation shows that such a ψ is $\tilde\Delta_{BC_{J_{\mathbf{t}}}}$-harmonic if and only if $A,\, B,\, C,\, D \in \mathbb{C}$ satisfy the linear system

$$\begin{pmatrix} \overline{\sigma_{2\bar2}} & -\overline{\sigma_{1\bar2}} & -\overline{\sigma_{2\bar1}} & \overline{\sigma_{1\bar1}} \\ \sigma_{2\bar2} & -\sigma_{2\bar1} & -\sigma_{1\bar2} & \sigma_{1\bar1} \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,.$$

As one can straightforwardly note, the rank of the matrix involved is 0 for $\mathbf{t}\in$ class *(i)*, while it is 1 or 2 depending on the values of the parameters within class *(ii)*, or within class *(iii)*. Therefore

$$\dim_{\mathbb{C}} H^{2,2}_{BC}(X_{\mathbf{t}}) \;=\; 7 \qquad \text{for} \quad \mathbf{t}\in \text{ subclasses } \textit{(ii.a)} \text{ and } \textit{(iii.a)}$$

and

$$\dim_{\mathbb{C}} H^{2,2}_{BC}(X_{\mathbf{t}}) \;=\; 6 \qquad \text{for} \quad \mathbf{t}\in \text{ subclasses } \textit{(ii.b)} \text{ and } \textit{(iii.b)}\;.$$

- **5-forms** Finally, let us compute $H^{3,2}_{BC}(X_{\mathbf{t}})$. It is straightforward to check that

$$H^{3,2}_{BC}(X_{\mathbf{t}}) \;=\; \mathbb{C}\left\langle \varphi^{123\bar{1}\bar{2}}_{\mathbf{t}},\, \varphi^{123\bar{1}\bar{3}}_{\mathbf{t}},\, \varphi^{123\bar{2}\bar{3}}_{\mathbf{t}} \right\rangle \qquad \text{for} \quad \mathbf{t} \in \text{ classes } \textit{(i)},\ \textit{(ii)} \text{ and } \textit{(iii)} \ :$$

  in particular, it does not depend on $\mathbf{t} \in \Delta(\mathbf{0}, \varepsilon)$.

We summarize the results of the computations above in the following theorem, [Ang11, Theorem 5.1].

Theorem 1.47. *Consider the Iwasawa manifold $\mathbb{I}_3 := \mathbb{H}(3;\mathbb{Z}[\mathrm{i}])\backslash\mathbb{H}(3;\mathbb{C})$ and the family $\{X_{\mathbf{t}} = (\mathbb{I}_3,\, J_{\mathbf{t}})\}_{\mathbf{t}\in\Delta(\mathbf{0},\varepsilon)}$ of its small deformations, where $\varepsilon > 0$ is small enough and $X_{\mathbf{0}} = \mathbb{I}_3$. Then the dimensions $h^{p,q}_{BC} := h^{p,q}_{BC}(X_{\mathbf{t}}) := \dim_{\mathbb{C}} H^{p,q}_{BC}(X_{\mathbf{t}}) = \dim_{\mathbb{C}} H^{3-p,3-q}_{A}(X_{\mathbf{t}})$ does not depend on $\mathbf{t} \in \Delta(\mathbf{0},\varepsilon)$ whenever $p+q$ is odd or $(p,q) \in \{(1,1),\,(3,1),\,(1,3)\}$, and they are equal to*

$$\begin{array}{ccccccccccc}
h^{1,0}_{BC} & = & h^{0,1}_{BC} & = & 2\,, & & & & & & \\
h^{2,0}_{BC} & = & h^{0,2}_{BC} & \in & \{1,\,2,\,3\}\,, & & & & h^{1,1}_{BC} & = & 4\,, \\
h^{3,0}_{BC} & = & h^{0,3}_{BC} & = & 1\,, & & h^{2,1}_{BC} & = & h^{1,2}_{BC} & = & 6\,, \\
h^{3,1}_{BC} & = & h^{1,3}_{BC} & = & 2\,, & & & & h^{2,2}_{BC} & \in & \{6,\,7,\,8\}\,, \\
h^{3,2}_{BC} & = & h^{2,3}_{BC} & = & 3\,. & & & & & &
\end{array}$$

Remark 1.48. As a consequence of the computations above, we notice that the Bott-Chern cohomology yields a finer classification of the small deformations of $\mathbb{I}_3$ than the Dolbeault cohomology: indeed, note that $\dim_{\mathbb{C}} H^{2,2}_{BC}(X_{\mathbf{t}})$ assumes different values according to different parameters in class *(ii)*, respectively in class *(iii)*; in a sense, this says that the Bott-Chern cohomology "carries more informations" about the complex structure that the Dolbeault one. Note also that most of the dimensions of Bott-Chern cohomology groups are invariant under small deformations: this happens for example for the odd-degree Bott-Chern cohomology groups.

1.4.5 Dimensions of the cohomologies of the Iwasawa manifold and of its small deformations

$\mathbf{H}^{\bullet}_{dR}$	$\mathbf{b_1}$	$\mathbf{b_2}$	$\mathbf{b_3}$	$\mathbf{b_4}$	$\mathbf{b_5}$
$\mathbb{I}_3$ and *(i)*, *(ii)*, *(iii)*	4	8	10	8	4

$\mathbf{H}^{\bullet,\bullet}_{\bar\partial}$	$\mathbf{h}^{1,0}_{\bar\partial}$	$\mathbf{h}^{0,1}_{\bar\partial}$	$\mathbf{h}^{2,0}_{\bar\partial}$	$\mathbf{h}^{1,1}_{\bar\partial}$	$\mathbf{h}^{0,2}_{\bar\partial}$	$\mathbf{h}^{3,0}_{\bar\partial}$	$\mathbf{h}^{2,1}_{\bar\partial}$	$\mathbf{h}^{1,2}_{\bar\partial}$	$\mathbf{h}^{0,3}_{\bar\partial}$	$\mathbf{h}^{3,1}_{\bar\partial}$	$\mathbf{h}^{2,2}_{\bar\partial}$	$\mathbf{h}^{1,3}_{\bar\partial}$	$\mathbf{h}^{3,2}_{\bar\partial}$	$\mathbf{h}^{2,3}_{\bar\partial}$
$\mathbb{I}_3$ and *(i)*	3	2	3	6	2	1	6	6	1	2	6	3	2	3
(ii)	2	2	2	5	2	1	5	5	1	2	5	2	2	2
(iii)	2	2	1	5	2	1	4	4	1	2	5	1	2	2

$\mathbf{H}^{\bullet,\bullet}_{BC}$	$\mathbf{h}^{1,0}_{BC}$	$\mathbf{h}^{0,1}_{BC}$	$\mathbf{h}^{2,0}_{BC}$	$\mathbf{h}^{1,1}_{BC}$	$\mathbf{h}^{0,2}_{BC}$	$\mathbf{h}^{3,0}_{BC}$	$\mathbf{h}^{2,1}_{BC}$	$\mathbf{h}^{1,2}_{BC}$	$\mathbf{h}^{0,3}_{BC}$	$\mathbf{h}^{3,1}_{BC}$	$\mathbf{h}^{2,2}_{BC}$	$\mathbf{h}^{1,3}_{BC}$	$\mathbf{h}^{3,2}_{BC}$	$\mathbf{h}^{2,3}_{BC}$
$\mathbb{I}_3$ and *(i)*	2	2	3	4	3	1	6	6	1	2	8	2	3	3
(ii.a)	2	2	2	4	2	1	6	6	1	2	7	2	3	3
(ii.b)	2	2	2	4	2	1	6	6	1	2	6	2	3	3
(iii.a)	2	2	1	4	1	1	6	6	1	2	7	2	3	3
(iii.b)	2	2	1	4	1	1	6	6	1	2	6	2	3	3

$\mathbf{H}^{\bullet,\bullet}_{A}$	$\mathbf{h}^{1,0}_{A}$	$\mathbf{h}^{0,1}_{A}$	$\mathbf{h}^{2,0}_{A}$	$\mathbf{h}^{1,1}_{A}$	$\mathbf{h}^{0,2}_{A}$	$\mathbf{h}^{3,0}_{A}$	$\mathbf{h}^{2,1}_{A}$	$\mathbf{h}^{1,2}_{A}$	$\mathbf{h}^{0,3}_{A}$	$\mathbf{h}^{3,1}_{A}$	$\mathbf{h}^{2,2}_{A}$	$\mathbf{h}^{1,3}_{A}$	$\mathbf{h}^{3,2}_{A}$	$\mathbf{h}^{2,3}_{A}$
$\mathbb{I}_3$ and *(i)*	3	3	2	8	2	1	6	6	1	3	4	3	2	2
(ii.a)	3	3	2	7	2	1	6	6	1	2	4	2	2	2
(ii.b)	3	3	2	6	2	1	6	6	1	2	4	2	2	2
(iii.a)	3	3	2	7	2	1	6	6	1	1	4	1	2	2
(iii.b)	3	3	2	6	2	1	6	6	1	1	4	1	2	2

1.5 Cohomology of orbifolds

The notion of orbifold has been introduced by I. Satake in [Sat56], with the name of *V-manifold*, and has been studied, among others, by W. L. Baily, [Bai56, Bai54].

In this section, we start by recalling the main definitions and some classical results concerning complex orbifolds and their cohomology, and we are then interested in their Bott-Chern cohomology. Compact complex orbifolds of the type $\tilde{X} = X/G$, where X is a compact complex manifold and G is a finite group of biholomorphisms of X, constitute one of the simplest examples of singular spaces: more precisely, we study the Bott-Chern cohomology for such orbifolds, proving that it can be defined using either currents or forms, or also by computing the G-invariant $\tilde{\Delta}_{BC}$-harmonic forms on X, Theorem 1.55.

1.5.1 Orbifolds and cohomologies

We first recall some classical definitions and results about orbifolds and their cohomologies, referring to [Joy07, Joy00, Sat56, Bai56, Bai54] (see, e.g., [Joy07, Definition 7.4.3]).

Definition 1.49 ([Sat56, Definition 2])**.** A *complex orbifold of complex dimension* n is a singular complex space of complex dimension n whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^n/G$, for finite subgroups $G \subset \mathrm{GL}(n;\mathbb{C})$.

By definition, an object (e.g., a *differential form*, a *Riemannian metric*, a *Hermitian metric*) *on a complex orbifold* $\tilde{X}$ is defined locally at $x \in \tilde{X}$ as a G_x-invariant object on $\mathbb{C}^n$, where $G_x \subseteq \mathrm{GL}(n;\mathbb{C})$ is such that $\tilde{X}$ is locally isomorphic to $\mathbb{C}^n/G_x$ at x.

In particular, one gets a differential complex $\left(\wedge^\bullet \tilde{X},\, \mathrm{d}\right)$, and a double complex $\left(\wedge^{\bullet,\bullet} \tilde{X},\, \partial,\, \overline{\partial}\right)$. Define the de Rham, Dolbeault, Bott-Chern, and Aeppli cohomology groups of $\tilde{X}$ respectively as

$$H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) := \frac{\ker \mathrm{d}}{\mathrm{im}\,\mathrm{d}}\,, \qquad H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) := \frac{\ker \overline{\partial}}{\mathrm{im}\,\overline{\partial}}\,,$$

$$H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) := \frac{\ker \partial \cap \ker \overline{\partial}}{\mathrm{im}\,\partial\overline{\partial}}\,, \qquad H^{\bullet,\bullet}_{A}\left(\tilde{X}\right) := \frac{\ker \partial\overline{\partial}}{\mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial}}\,.$$

The structure of double complex of $\left(\wedge^{\bullet,\bullet}\tilde{X},\, \partial,\, \overline{\partial}\right)$ induces naturally a spectral sequence $\{(E_r^{\bullet,\bullet},\, \mathrm{d}_r)\}_{r\in\mathbb{N}}$, called *Hodge and Frölicher spectral sequence of* $\tilde{X}$, such that $E_1^{\bullet,\bullet} \simeq H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right)$ (see, e.g., [McC01, §2.4]). Hence, one has the *Frölicher inequality*, see [Frö55, Theorem 2],

$$\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}\left(\tilde{X}\right) \;\geq\; \dim_{\mathbb{C}} H^k_{dR}\left(\tilde{X};\mathbb{C}\right)\,,$$

for any $k \in \mathbb{N}$.

Given a Riemannian metric on a complex orbifold $\tilde{X}$ of complex dimension n, one can consider the $\mathbb{R}$-linear Hodge-$*$-operator $*_g\colon\ \wedge^\bullet \tilde{X} \to \wedge^{2n-\bullet}\tilde{X}$, and hence the 2nd order self-adjoint elliptic differential operator $\Delta := [\mathrm{d},\, \mathrm{d}^*] := \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d} \in \mathrm{End}\wedge^\bullet\tilde{X}$.

Analogously, given a Hermitian metric on a complex orbifold $\tilde{X}$ of complex dimension n, one can consider the $\mathbb{C}$-linear Hodge-$*$-operator $*_g\colon\ \wedge^{\bullet_1,\bullet_2} \tilde{X} \to \wedge^{n-\bullet_2,n-\bullet_1}\tilde{X}$, and hence the 2nd order self-adjoint elliptic differential operator $\overline{\square} := \left[\overline{\partial},\, \overline{\partial}^*\right] := \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} \in \mathrm{End}\wedge^{\bullet,\bullet}\tilde{X}$. Furthermore, following [Sch07, §2], see also [KS60, Proposition 5], one can define the 4th order self-adjoint elliptic differential operators

$$\tilde{\Delta}_{BC} := \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\overline{\partial}^*\partial\right)\left(\overline{\partial}^*\partial\right)^* + \left(\overline{\partial}^*\partial\right)^*\left(\overline{\partial}^*\partial\right) + \overline{\partial}^*\overline{\partial} + \partial^*\partial \;\in\; \mathrm{End}\wedge^{\bullet,\bullet}\tilde{X}$$

and

$$\tilde{\Delta}_{A} := \partial\partial^* + \overline{\partial}\overline{\partial}^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\overline{\partial}\partial^*\right)^*\left(\overline{\partial}\partial^*\right) + \left(\overline{\partial}\partial^*\right)\left(\overline{\partial}\partial^*\right)^* \;\in\; \mathrm{End}\wedge^{\bullet,\bullet}\tilde{X}\,.$$

As a matter of notation, given a compact complex orbifold $\tilde{X}$ of complex dimension n, denote the constant sheaf with coefficients in $\mathbb{R}$ over $\tilde{X}$ by $\underline{\mathbb{R}}_{\tilde{X}}$, the sheaf of germs of smooth functions over $\tilde{X}$ by $\mathcal{C}^\infty_{\tilde{X}}$, the sheaf of germs of (p,q)-forms (for $p,q \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{A}^{p,q}_{\tilde{X}}$, the sheaf of germs of k-forms (for $k \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{A}^k_{\tilde{X}}$, the sheaf of germs of bidimension-(p,q)-currents (for $p,q \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{D}_{\tilde{X}\,p,q} =: \mathcal{D}^{n-p,n-q}_{\tilde{X}}$, the sheaf of germs of dimension-k-currents (for $k \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{D}_{\tilde{X}\,k} =: \mathcal{D}^{2n-k}_{\tilde{X}}$, and the sheaf of holomorphic p-forms (for $p \in \mathbb{N}$) over $\tilde{X}$ by $\Omega^p_{\tilde{X}}$.

The following result, concerning the de Rham cohomology of a complex orbifold, was proven by I. Satake, [Sat56], and by W. L. Baily, [Bai56].

Theorem 1.50 ([Sat56, Theorem 1], [Bai56, Theorem H]). *Let $\tilde{X}$ be a compact complex orbifold of complex dimension n. There is a canonical isomorphism*

$$H^{\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;\simeq\; \check{H}^{\bullet}\left(\tilde{X};\underline{\mathbb{R}}_{\tilde{X}}\right) \;,$$

where $\underline{\mathbb{R}}_{\tilde{X}}$ is the constant sheaf with coefficients in $\mathbb{R}$ over $\tilde{X}$.

Furthermore, given a Riemannian metric on $\tilde{X}$, there is a canonical isomorphism

$$H^{\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;\simeq\; \ker\Delta \;.$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;\simeq\; H^{2n-\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;.$$

The isomorphism $H^{\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \simeq \ker\Delta$ can be seen as a consequence of a more general decomposition theorem on orbifolds, [Bai56, Theorem D], which holds for 2^{nd} order self-adjoint elliptic differential operators. In particular, as regards the Dolbeault cohomology, the following result holds.

Theorem 1.51 ([Bai54, page 807], [Bai56, Theorem K]). *Let $\tilde{X}$ be a compact complex orbifold of complex dimension n. There is a canonical isomorphism*

$$H^{\bullet_1,\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) \;\simeq\; \check{H}^{\bullet_2}\left(\tilde{X};\Omega^{\bullet_1}_{\tilde{X}}\right) \;,$$

where $\Omega^{p}_{\tilde{X}}$ is the sheaf of holomorphic p-forms over $\tilde{X}$, for $p\in\mathbb{N}$.

Furthermore, given a Hermitian metric on X, there is a canonical isomorphism

$$H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) \;\simeq\; \ker\overline{\square} \;.$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet_1,\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) \;\simeq\; H^{n-\bullet_1,n-\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) \;.$$

1.5.2 Bott-Chern cohomology of orbifolds of global-quotient-type

Now, we will reduce to study complex orbifolds of the special type

$$\tilde{X} \;=\; X/G \;,$$

where X is a complex manifold and G is a finite group of biholomorphisms of X. Indeed, note that, by the S. Bochner linearization theorem [Boc45, Theorem 1], see, e.g., [DK00, Theorem 2.2.1], see also [Rai06, Theorem 1.7.2], $\tilde{X} = X/G$ is an orbifold according to the above definition.

Orbifolds of global-quotient-type have been considered and studied by D. D. Joyce in constructing examples of compact 7-dimensional manifolds with holonomy G_2, [Joy96b] and [Joy00, Chapters 11-12], and examples of compact 8-dimensional manifolds with holonomy Spin(7), [Joy96a, Joy99] and [Joy00, Chapters 13-14]. See also [FM08, CFM08] for the use of orbifolds of global-quotient-type to construct compact 8-dimensional simply-connected non-formal symplectic manifolds (which do not satisfy, respectively satisfy, the Hard Lefschetz condition), answering to a question by I. K. Babenko and I. A. Taĭmanov, [BT00, Problem].

Since G is a finite group of biholomorphisms, the singular set of $\tilde{X}$ is

$$\operatorname{Sing}\left(\tilde{X}\right) \;=\; \left\{x\,G\in X/G \;:\; x\in X \text{ and } g\cdot x = x \text{ for some } g\in G\setminus\{\mathrm{id}_X\}\right\} \;.$$

Remark 1.52. Not all orbifolds are global quotients X/G: a counterexample is provided by considering weighted projective spaces, see, e.g., [Joy07, Definition 6.5.4].

In particular, for the sake of completeness, we provide in this special case a straightforward proof of [Sat56, Theorem 1] and [Bai56, Theorem H] for the de Rham cohomology, and of [Bai54, page 807] and [Bai56, Theorem K] for the Dolbeault cohomology; furthermore, we extend these results to Bott-Chern and Aeppli cohomologies.

Theorem 1.53 ([Sat56, Theorem 1], [Bai56, Theorem H]). *Let $\tilde{X} = X/G$ be a compact complex orbifold of complex dimension n, where X is a complex manifold and G is a finite group of biholomorphisms of X. There are canonical isomorphisms*

$$H^{\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;\simeq\; \check{H}^{\bullet}\left(\tilde{X};\underline{\mathbb{R}}_{\tilde{X}}\right) \;\simeq\; \frac{\ker\left(\mathrm{d}\colon \mathcal{D}^{\bullet}\tilde{X}\to\mathcal{D}^{\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(\mathrm{d}\colon \mathcal{D}^{\bullet-1}\tilde{X}\to\mathcal{D}^{\bullet}\tilde{X}\right)} \;.$$

Furthermore, given a Riemannian metric on $\tilde{X}$, there is a canonical isomorphism

$$H^{\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;\simeq\; \ker\Delta \;.$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;\simeq\; H^{2n-\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;.$$

Proof. We claim that

$$0 \to \underline{\mathbb{R}}_{\tilde{X}} \to \left(\mathcal{A}^{\bullet}_{\tilde{X}},\, \mathrm{d}\right) \qquad \text{and} \qquad 0 \to \underline{\mathbb{R}}_{\tilde{X}} \to \left(\mathcal{D}^{\bullet}_{\tilde{X}},\, \mathrm{d}\right)$$

are fine resolutions of the constant sheaf $\underline{\mathbb{R}}_{\tilde{X}}$. Indeed, take ϕ a germ of a d-closed k-form on $\tilde{X}$, with $k \in \mathbb{N}\setminus\{0\}$, that is, a germ of a G-invariant k-form on X; by the Poincaré lemma, see, e.g., [Dem12, I.1.22], there exists ψ a germ of a $(k-1)$-form on X such that $\phi = \mathrm{d}\,\psi$; since ϕ is G-invariant, one has

$$\phi \;=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\phi \;=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\left(\mathrm{d}\,\psi\right) \;=\; \mathrm{d}\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\psi\right) \;,$$

that is, taking the germ of the G-invariant $(k-1)$-form

$$\tilde{\psi} \;:=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\psi$$

on X, one gets a germ of a $(k-1)$-form on $\tilde{X}$ such that $\phi = \mathrm{d}\,\tilde{\psi}$. As regards the case $k=0$, it follows straightforwardly since every (G-invariant) d-closed function on X is locally constant. The same argument applies for the sheaves of currents, by using the Poincaré lemma for currents, see, e.g., [Dem12, Theorem I.2.24]. Finally, note that, for every $k \in \mathbb{N}$, the sheaves $\mathcal{A}^k_{\tilde{X}}$ and $\mathcal{D}^k_{\tilde{X}}$ are fine: indeed, they are sheaves of $\mathcal{C}^{\infty}_{\tilde{X}}$-modules over a para-compact space.

Hence, one gets that

$$\check{H}^{\bullet}\left(\tilde{X};\underline{\mathbb{R}}_{\tilde{X}}\right) \;\simeq\; \underbrace{\frac{\ker\left(\mathrm{d}\colon\, \wedge^{\bullet}\tilde{X} \to \wedge^{\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(\mathrm{d}\colon\, \wedge^{\bullet-1}\tilde{X} \to \wedge^{\bullet}\tilde{X}\right)}}_{=:\, H^{\bullet}_{dR}(\tilde{X};\mathbb{R})}$$

$$\simeq\; \frac{\ker\left(\mathrm{d}\colon \mathcal{D}^{\bullet}\tilde{X} \to \mathcal{D}^{\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(\mathrm{d}\colon \mathcal{D}^{\bullet-1}\tilde{X} \to \mathcal{D}^{\bullet}\tilde{X}\right)} \;,$$

see, e.g., [Dem12, Corollary IV.4.19, IV.6.4].

Consider now a Riemannian metric on $\tilde{X}$, that is, a G-invariant Riemannian metric on X. Since the elements of G commute with both d and d^* (the Riemannian metric being G-invariant), and hence with Δ, the decomposition

$$\wedge^{\bullet}X \;=\; \ker\Delta \oplus \mathrm{d}\wedge^{\bullet-1}X \oplus \mathrm{d}^*\wedge^{\bullet+1}X$$

induces a decomposition of the space of G-invariant forms, namely,

$$\wedge^{\bullet}\tilde{X} \;=\; \ker\Delta \oplus \mathrm{d}\wedge^{\bullet-1}\tilde{X} \oplus \mathrm{d}^*\wedge^{\bullet+1}\tilde{X} \;.$$

More precisely, let α be a G-invariant form on X; considering the decomposition $\alpha =: h_\alpha + \mathrm{d}\,\beta + \mathrm{d}^*\gamma$ with $h_\alpha, \beta, \gamma \in \wedge^{\bullet}X$ such that $\Delta h_\alpha = 0$, one has

$$\alpha \;=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\alpha \;=\; \left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*h_\alpha\right) + \mathrm{d}\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\beta\right) + \mathrm{d}^*\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\gamma\right) \;,$$

where $\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*h_\alpha$, $\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\beta$, $\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\gamma \in \wedge^{\bullet}\tilde{X}$ and

$$\Delta\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*h_\alpha\right) \;=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\left(\Delta h_\alpha\right) \;=\; 0 \;.$$

Finally, note that the Hodge-$*$-operator $*\colon\, \wedge^{\bullet}\tilde{X} \to \wedge^{2n-\bullet}\tilde{X}$ sends Δ-harmonic forms to Δ-harmonic forms, and hence it induces an isomorphism

$$*\colon H^{\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \xrightarrow{\simeq} H^{2n-\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;,$$

concluding the proof. □

A similar argument can be repeated for the Dolbeault cohomology; more precisely, the following result holds.

Theorem 1.54 ([Bai54, page 807], [Bai56, Theorem K]). *Let $\tilde{X} = X/G$ be a compact complex orbifold of complex dimension n, where X is a complex manifold and G is a finite group of biholomorphisms of X. There are canonical isomorphisms*

$$H^{\bullet_1,\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) \simeq \check{H}^{\bullet_2}\left(\tilde{X};\Omega^{\bullet_1}_{\tilde{X}}\right) \simeq \frac{\ker\left(\overline{\partial}\colon \mathcal{D}^{\bullet_1,\bullet_2}\tilde{X} \to \mathcal{D}^{\bullet_1,\bullet_2+1}\tilde{X}\right)}{\operatorname{im}\left(\overline{\partial}\colon \mathcal{D}^{\bullet_1,\bullet_2-1}\tilde{X} \to \mathcal{D}^{\bullet_1,\bullet_2}\tilde{X}\right)} .$$

Furthermore, given a Hermitian metric on $\tilde{X}$, there is a canonical isomorphism

$$H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) \simeq \ker \overline{\square} .$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet_1,\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) \simeq H^{n-\bullet_1,n-\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) .$$

Proof. We claim that, for every $p \in \mathbb{N}$,

$$0 \to \Omega^p_{\tilde{X}} \to \left(\mathcal{A}^{p,\bullet}_{\tilde{X}}, \overline{\partial}\right) \qquad \text{and} \qquad 0 \to \Omega^p_{\tilde{X}} \to \left(\mathcal{D}^{p,\bullet}_{\tilde{X}}, \overline{\partial}\right)$$

are fine resolutions of the constant sheaf $\Omega^p_{\tilde{X}}$. Indeed, take ϕ a germ of a $\overline{\partial}$-closed (p,q)-form (respectively, bidimension-(p,q)-current) on $\tilde{X}$, with $q \in \mathbb{N}\setminus\{0\}$, that is, a germ of a G-invariant (p,q)-form (respectively, bidimension-(p,q)-current) on X; by the Dolbeault and Grothendieck lemma, see, e.g., [Dem12, I.3.29], there exists ψ a germ of a $(p,q-1)$-form (respectively, bidimension-$(p,q-1)$-current) on X such that $\phi = \overline{\partial}\psi$; since ϕ is G-invariant, one has

$$\phi = \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\phi = \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\left(\overline{\partial}\psi\right) = \overline{\partial}\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\psi\right) ,$$

that is, taking the germ of the G-invariant $(p,q-1)$-form (respectively, bidimension-$(p,q-1)$-current)

$$\tilde{\psi} := \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\psi$$

on X, one gets a germ of a $(p,q-1)$-form (respectively, bidimension-$(p,q-1)$-current) on $\tilde{X}$ such that $\phi = \overline{\partial}\tilde{\psi}$. As regards the case $q=0$, it follows by the fact that every (G-invariant) $\overline{\partial}$-closed bidimension-$(p,0)$-current on X is locally a holomorphic p-form, see, e.g., [Dem12, I.3.29]. Finally, note that, for every $q \in \mathbb{N}$, the sheaves $\mathcal{A}^{p,q}_{\tilde{X}}$ and $\mathcal{D}^{p,q}_{\tilde{X}}$ are fine: indeed, they are sheaves of $\left(\mathcal{C}^\infty_{\tilde{X}} \otimes_{\mathbb{R}} \mathbb{C}\right)$-modules over a para-compact space.

Hence, one gets that

$$\begin{aligned}\check{H}^{p,\bullet}\left(\tilde{X};\Omega^p_{\tilde{X}}\right) &\simeq \frac{\ker\left(\overline{\partial}\colon \wedge^{p,\bullet}\tilde{X} \to \wedge^{p,\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(\overline{\partial}\colon \wedge^{p,\bullet-1}\tilde{X} \to \wedge^{p,\bullet}\tilde{X}\right)} \\ &\simeq \frac{\ker\left(\overline{\partial}\colon \mathcal{D}^{p,\bullet}\tilde{X} \to \mathcal{D}^{p,\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(\overline{\partial}\colon \mathcal{D}^{p,\bullet-1}\tilde{X} \to \mathcal{D}^{p,\bullet}\tilde{X}\right)} ,\end{aligned}$$

see, e.g., [Dem12, Corollary IV.4.19, IV.6.4].

Consider now a Hermitian metric on $\tilde{X}$, that is, a G-invariant Hermitian metric on X. Since the elements of G commute with both $\overline{\partial}$ and $\overline{\partial}^*$ (the Hermitian metric being G-invariant), and hence with $\overline{\square}$, the decomposition

$$\wedge^{\bullet,\bullet}X = \ker\overline{\square} \oplus \overline{\partial}\wedge^{\bullet,\bullet-1}X \oplus \overline{\partial}^*\wedge^{\bullet,\bullet+1}X$$

induces a decomposition on the space of G-invariant forms, namely,

$$\wedge^{\bullet,\bullet}\tilde{X} = \ker\overline{\square} \oplus \overline{\partial}\wedge^{\bullet,\bullet-1}\tilde{X} \oplus \overline{\partial}^*\wedge^{\bullet,\bullet+1}\tilde{X} .$$

More precisely, let α be a G-invariant form on X; considering the decomposition $\alpha =: h_\alpha + \overline{\partial}\beta + \overline{\partial}^*\gamma$ with $h_\alpha, \beta, \gamma \in \wedge^{\bullet,\bullet}X$ such that $\overline{\square}h_\alpha = 0$, one has

$$\alpha = \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\alpha = \left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*h_\alpha\right) + \overline{\partial}\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\beta\right) + \overline{\partial}^*\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\gamma\right) ,$$

where $\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*h_\alpha$, $\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\beta$, $\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\gamma \in \wedge^{\bullet,\bullet}\tilde{X}$ and

$$\overline{\square}\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*h_\alpha\right) \;=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\left(\overline{\square} h_\alpha\right) \;=\; 0\,.$$

Finally, note that the Hodge-$*$-operator $*\colon\ \wedge^{\bullet_1,\bullet_2}\tilde{X}\to\wedge^{n-\bullet_2,n-\bullet_1}\tilde{X}$ sends $\overline{\square}$-harmonic forms to $\square$-harmonic forms, where $\square:=[\partial,\partial^*]:=\partial\partial^*+\partial^*\partial\in\operatorname{End}\wedge^{\bullet,\bullet}\tilde{X}$, and hence it induces an isomorphism

$$*\colon H^{\bullet_1,\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right)\stackrel{\simeq}{\to}H^{n-\bullet_1,n-\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right)\,,$$

concluding the proof. □

Finally, as done in Theorem 1.53 and Theorem 1.54 for the de Rham cohomology and, respectively, the Dolbeault cohomology, we provide the following result, concerning Bott-Chern and Aeppli cohomologies of compact complex orbifolds of global-quotient-type.

Theorem 1.55. *Let $\tilde{X}=X/G$ be a compact complex orbifold of complex dimension n, where X is a complex manifold and G is a finite group of biholomorphisms of X. For any $p,q\in\mathbb{N}$, there are canonical isomorphisms*

$$H^{p,q}_{BC}\left(\tilde{X}\right)\;\simeq\;\frac{\ker\left(\partial\colon\mathcal{D}^{p,q}\tilde{X}\to\mathcal{D}^{p+1,q}\tilde{X}\right)\cap\ker\left(\overline{\partial}\colon\mathcal{D}^{p,q}\tilde{X}\to\mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon\mathcal{D}^{p-1,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)}\,. \tag{1.5.1}$$

Furthermore, given a Hermitian metric on $\tilde{X}$, there are canonical isomorphisms

$$H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right)\;\simeq\;\ker\tilde{\Delta}_{BC}\qquad\text{and}\qquad H^{\bullet,\bullet}_{A}\left(\tilde{X}\right)\;\simeq\;\ker\tilde{\Delta}_{A}\,.$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet_1,\bullet_2}_{BC}\left(\tilde{X}\right)\;\simeq\;H^{n-\bullet_2,n-\bullet_1}_{A}\left(\tilde{X}\right)\,.$$

Proof. We use the same argument as in the proof of [Ang11, Theorem 3.7] to show that, since the de Rham cohomology and the Dolbeault cohomology of $\tilde{X}$ can be computed using either differential forms or currents, the same holds true for the Bott-Chern and the Aeppli cohomologies.

Indeed, note that, for any $p,q\in\mathbb{N}$, one has the exact sequence

$$0\to\frac{\operatorname{im}\left(\mathrm{d}\colon\ \left(\mathcal{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)\cap\mathcal{D}^{p,q}\tilde{X}}{\operatorname{im}\left(\partial\overline{\partial}\colon\mathcal{D}^{p-1,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)}$$
$$\to\frac{\ker\left(\mathrm{d}\colon\mathcal{D}^{p,q}\tilde{X}\to\mathcal{D}^{p+1,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon\mathcal{D}^{p-1,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)}\to\frac{\ker\left(\mathrm{d}\colon\ \left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q+1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)}{\operatorname{im}\left(\mathrm{d}\colon\ \left(\mathcal{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)}\,,$$

where the maps are induced by the identity. By [Sat56, Theorem 1], see Theorem 1.53, one has

$$\frac{\ker\left(\mathrm{d}\colon\ \left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q+1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)}{\operatorname{im}\left(\mathrm{d}\colon\ \left(\mathcal{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)}\;\simeq\;\frac{\ker\left(\mathrm{d}\colon\ \left(\wedge^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\wedge^{p+q+1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)}{\operatorname{im}\left(\mathrm{d}\colon\ \left(\wedge^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\wedge^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)}\,,$$

therefore it suffices to prove that the space

$$\frac{\operatorname{im}\left(\mathrm{d}\colon\ \left(\mathcal{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)\cap\mathcal{D}^{p,q}\tilde{X}}{\operatorname{im}\left(\partial\overline{\partial}\colon\mathcal{D}^{p-1,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)}$$

can be computed using just differential forms on $\tilde{X}$.

Firstly, we note that, since, by [Bai54, page 807], see Theorem 1.54,

$$\frac{\ker\left(\overline{\partial}\colon\mathcal{D}^{p,q}\tilde{X}\to\mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\overline{\partial}\colon\mathcal{D}^{p,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)}\;\simeq\;\frac{\ker\left(\overline{\partial}\colon\ \wedge^{p,q}\tilde{X}\to\wedge^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\overline{\partial}\colon\ \wedge^{p,q-1}\tilde{X}\to\wedge^{p,q}\tilde{X}\right)}\,,$$

one has that, if $\psi\in\wedge^{r,s}\tilde{X}$ is a $\overline{\partial}$-closed differential form, then every solution $\phi\in\mathcal{D}^{r,s-1}$ of $\overline{\partial}\phi=\psi$ is a differential form up to $\overline{\partial}$-exact terms. Indeed, since $[\psi]=0$ in $\frac{\ker\overline{\partial}\cap\mathcal{D}^{r,s}\tilde{X}}{\operatorname{im}\overline{\partial}}$ and hence in $\frac{\ker\overline{\partial}\cap\wedge^{r,s}\tilde{X}}{\operatorname{im}\overline{\partial}}$, there is a differential form $\alpha \in \wedge^{r,s-1}\tilde{X}$ such that $\psi = \overline{\partial}\alpha$. Hence, $\phi - \alpha \in \mathcal{D}^{r,s-1}\tilde{X}$ defines a class in $\frac{\ker \overline{\partial} \cap \mathcal{D}^{r,s-1}\tilde{X}}{\mathrm{im}\,\overline{\partial}} \simeq \frac{\ker \overline{\partial} \cap \wedge^{r,s-1}\tilde{X}}{\mathrm{im}\,\overline{\partial}}$, and hence $\phi - \alpha$ is a differential form up to a $\overline{\partial}$-exact form, and so ϕ is.

By conjugation, if $\psi \in \wedge^{r,s}\tilde{X}$ is a ∂-closed differential form, then every solution $\phi \in \mathcal{D}^{r-1,s}$ of $\partial\phi = \psi$ is a differential form up to ∂-exact terms.

Now, let

$$\omega^{p,q} \;=\; \mathrm{d}\,\eta \quad \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} \;\in\; \frac{\mathrm{im}\,\mathrm{d} \cap \mathcal{D}^{p,q}X}{\mathrm{im}\,\partial\overline{\partial}} \;.$$

Decomposing $\eta =: \sum_{p,q} \eta^{p,q}$ in pure-type components, where $\eta^{p,q} \in \mathcal{D}^{p,q}\tilde{X}$, the previous equality is equivalent to the system

$$\begin{cases} & \partial\eta^{p+q-1,0} & = & 0 & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} & & \\ \overline{\partial}\eta^{p+q-\ell,\ell-1} & + \;\partial\eta^{p+q-\ell-1,\ell} & = & 0 & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} & \text{for} & \ell \in \{1,\ldots,q-1\} \\ \overline{\partial}\eta^{p,q-1} & + \;\partial\eta^{p-1,q} & = & \omega^{p,q} & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} & & \\ \overline{\partial}\eta^{\ell,p+q-\ell-1} & + \;\partial\eta^{\ell-1,p+q-\ell} & = & 0 & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} & \text{for} & \ell \in \{1,\ldots,p-1\} \\ \overline{\partial}\eta^{0,p+q-1} & & = & 0 & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} & & \end{cases} .$$

By the above argument, we may suppose that, for $\ell \in \{0,\ldots,p-1\}$, the currents $\eta^{\ell,p+q-\ell-1}$ are differential forms: indeed, they are differential forms up to $\overline{\partial}$-exact terms, but $\overline{\partial}$-exact terms give no contribution in the system, which is modulo $\mathrm{im}\,\partial\overline{\partial}$. Analogously, we may suppose that, for $\ell \in \{0,\ldots,q-1\}$, the currents $\eta^{p+q-\ell-1,\ell}$ are differential forms. Then we may suppose that $\omega^{p,q} = \overline{\partial}\eta^{p,q-1} + \partial\eta^{p-1,q}$ is a differential form. Hence (1.5.1) is proven.

Now, we prove that, fixed a G-invariant Hermitian metric on $\tilde{X}$, the Bott-Chern cohomology of $\tilde{X}$ is isomorphic to the space of $\tilde{\Delta}_{BC}$-harmonic G-invariant forms on X. Indeed, since the elements of G commute with ∂, $\overline{\partial}$, ∂^*, and $\overline{\partial}^*$, and hence with $\tilde{\Delta}_{BC}$, the following decomposition, [Sch07, Théorème 2.2],

$$\wedge^{\bullet,\bullet}X \;=\; \ker\tilde{\Delta}_{BC} \oplus \partial\overline{\partial}\wedge^{\bullet-1,\bullet-1}X \oplus \left(\partial^*\wedge^{\bullet+1,\bullet}X + \overline{\partial}^*\wedge^{\bullet,\bullet+1}X\right)$$

induces a decomposition

$$\wedge^{\bullet,\bullet}\tilde{X} \;=\; \ker\tilde{\Delta}_{BC} \oplus \partial\overline{\partial}\wedge^{\bullet-1,\bullet-1}\tilde{X} \oplus \left(\partial^*\wedge^{\bullet+1,\bullet}\tilde{X} + \overline{\partial}^*\wedge^{\bullet,\bullet+1}\tilde{X}\right) \;.$$

More precisely, let $\alpha \in \wedge^{\bullet,\bullet}\tilde{X}$, that is, α is a G-invariant form on X; if α has a decomposition $\alpha = h_\alpha + \partial\overline{\partial}\beta + \left(\partial^*\gamma + \overline{\partial}^*\eta\right)$ with $h_\alpha, \beta, \gamma, \eta \in \wedge^{\bullet,\bullet}X$ such that $\tilde{\Delta}_{BC}h_\alpha = 0$, then one has

$$\begin{aligned}\alpha \;=\; \frac{1}{\mathrm{ord}\,G}\sum_{g\in G} g^*\alpha \;&=\; \left(\frac{1}{\mathrm{ord}\,G}\sum_{g\in G} g^*h_\alpha\right) + \partial\overline{\partial}\left(\frac{1}{\mathrm{ord}\,G}\sum_{g\in G} g^*\beta\right) \\ &\quad + \left(\partial^*\left(\frac{1}{\mathrm{ord}\,G}\sum_{g\in G} g^*\gamma\right) + \overline{\partial}^*\left(\eta\frac{1}{\mathrm{ord}\,G}\sum_{g\in G} g^*\right)\right) ,\end{aligned}$$

where $\frac{1}{\mathrm{ord}\,G}\sum_{g\in G} g^*h_\alpha$, $\frac{1}{\mathrm{ord}\,G}\sum_{g\in G} g^*\beta$, $\frac{1}{\mathrm{ord}\,G}\sum_{g\in G} g^*\gamma$, $\eta\frac{1}{\mathrm{ord}\,G}\sum_{g\in G} g^* \in \wedge^{\bullet,\bullet}\tilde{X}$ and

$$\tilde{\Delta}_{BC}\left(\frac{1}{\mathrm{ord}\,G}\sum_{g\in G} g^*h_\alpha\right) \;=\; \frac{1}{\mathrm{ord}\,G}\sum_{g\in G} g^*\left(\tilde{\Delta}_{BC}h_\alpha\right) \;=\; 0 \;.$$

As regards the Aeppli cohomology, one has the decomposition, [Sch07, §2.c],

$$\wedge^{\bullet,\bullet}X \;=\; \ker\tilde{\Delta}_A \oplus \left(\partial\wedge^{\bullet-1,\bullet}X + \overline{\partial}\wedge^{\bullet,\bullet-1}X\right) \oplus \left(\partial\overline{\partial}\right)^*\wedge^{\bullet+1,\bullet+1}X \;,$$

and hence the decomposition

$$\wedge^{\bullet,\bullet}\tilde{X} \;=\; \ker\tilde{\Delta}_A \oplus \left(\partial\wedge^{\bullet-1,\bullet}\tilde{X} + \overline{\partial}\wedge^{\bullet,\bullet-1}\tilde{X}\right) \oplus \left(\partial\overline{\partial}\right)^*\wedge^{\bullet+1,\bullet+1}\tilde{X} \;,$$

from which one gets the isomorphism $H^{\bullet,\bullet}_A\left(\tilde{X}\right) \simeq \ker\tilde{\Delta}_A$.

Finally, note that the Hodge-$*$-operator $*\colon \wedge^{\bullet_1,\bullet_2}\tilde{X} \to \wedge^{n-\bullet_2,n-\bullet_1}\tilde{X}$ sends $\tilde{\Delta}_{BC}$-harmonic forms to $\tilde{\Delta}_A$-harmonic forms, and hence it induces an isomorphism

$$*\colon H^{\bullet_1,\bullet_2}_{BC}\left(\tilde{X}\right) \stackrel{\simeq}{\to} H^{n-\bullet_2,n-\bullet_1}_{A}\left(\tilde{X}\right) ,$$

concluding the proof. □

Remark 1.56. We note that another proof of the isomorphism

$$H^{p,q}_{BC}\left(\tilde{X}\right) \simeq \frac{\ker\left(\partial\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)} ,$$

and a proof of the isomorphism

$$H^{p,q}_{A}\left(\tilde{X}\right) \simeq \frac{\ker\left(\partial\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\colon \mathcal{D}^{p-1,q}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right) + \operatorname{im}\left(\overline{\partial}\colon \mathcal{D}^{p,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)}$$

follow from the sheaf-theoretic interpretation of the Bott-Chern and Aeppli cohomologies, developed by J.-P. Demailly, [Dem12, §VI.12.1] and M. Schweitzer, [Sch07, §4], see also [Koo11, §3.2].

We recall that, for any $p, q \in \mathbb{N}$, the complex $\left(\mathcal{L}^{\bullet}_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)$ of sheaves is defined as

$$\left(\mathcal{L}^{\bullet}_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right) \colon\quad \mathcal{A}^{0,0}_{\tilde{X}} \stackrel{\mathrm{pr}\circ\mathrm{d}}{\to} \bigoplus_{\substack{r+s=1\\ r<p,\, s<q}} \mathcal{A}^{r,s}_{\tilde{X}} \to \cdots \stackrel{\mathrm{pr}\circ\mathrm{d}}{\to} \bigoplus_{\substack{r+s=p+q-2\\ r<p,\, s<q}} \mathcal{A}^{r,s}_{\tilde{X}} \stackrel{\partial\overline{\partial}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{A}^{r,s}_{\tilde{X}} \stackrel{\mathrm{d}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{A}^{r,s}_{\tilde{X}} \to \cdots ,$$

and the complex $\left(\mathcal{M}^{\bullet}_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{M}^{\bullet}_{\tilde{X}\,p,q}}\right)$ of sheaves is defined as

$$\left(\mathcal{M}^{\bullet}_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{M}^{\bullet}_{\tilde{X}\,p,q}}\right) \colon\quad \mathcal{D}^{0,0}_{\tilde{X}} \stackrel{\mathrm{pr}\circ\mathrm{d}}{\to} \bigoplus_{\substack{r+s=1\\ r<p,\, s<q}} \mathcal{D}^{r,s}_{\tilde{X}} \to \cdots \stackrel{\mathrm{pr}\circ\mathrm{d}}{\to} \bigoplus_{\substack{r+s=p+q-2\\ r<p,\, s<q}} \mathcal{D}^{r,s}_{\tilde{X}} \stackrel{\partial\overline{\partial}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{D}^{r,s}_{\tilde{X}} \stackrel{\mathrm{d}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{D}^{r,s}_{\tilde{X}} \to \cdots ,$$

where pr denotes the projection onto the appropriate space.

By the Poincaré lemma (see, e.g., [Dem12, I.1.22, Theorem I.2.24]) and the Dolbeault and Grothendieck lemma (see, e.g., [Dem12, I.3.29]), one gets M. Schweitzer's lemma [Sch07, Lemme 4.1], which can be extended also to the context of orbifolds by using the same trick as in the proof of Theorem 1.53 and Theorem 1.54; this allows to prove that the map

$$\left(\mathcal{L}^{\bullet}_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right) \to \left(\mathcal{M}^{\bullet}_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{M}^{\bullet}_{\tilde{X}\,p,q}}\right)$$

of complexes of sheaves is a quasi-isomorphism, and hence, see, e.g., [Dem12, Corollary IV.12.6], for every $\ell \in \mathbb{N}$,

$$\mathbb{H}^{\ell}\left(\tilde{X};\, \left(\mathcal{L}^{\bullet}_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)\right) \simeq \mathbb{H}^{\ell}\left(\tilde{X};\, \left(\mathcal{M}^{\bullet}_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)\right) .$$

Since, for every $k \in \mathbb{N}$, the sheaves $\mathcal{L}^{k}_{\tilde{X}\,p,q}$ and $\mathcal{M}^{k}_{\tilde{X}\,p,q}$ are fine (indeed, they are sheaves of $\left(\mathcal{C}^{\infty}_{\tilde{X}} \otimes_{\mathbb{R}} \mathbb{C}\right)$-modules over a para-compact space), one has, see, e.g., [Dem12, Corollary IV.4.19, (IV.12.9)],

$$\mathbb{H}^{p+q-1}\left(\tilde{X};\, \left(\mathcal{L}^{\bullet}_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)\right) \simeq \frac{\ker\left(\partial\colon \wedge^{p,q}\tilde{X} \to \wedge^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \wedge^{p,q}\tilde{X} \to \wedge^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \wedge^{p-1,q-1}\tilde{X} \to \wedge^{p,q}\tilde{X}\right)}$$

and

$$\mathbb{H}^{p+q-1}\left(\tilde{X};\, \left(\mathcal{M}^{\bullet}_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)\right) \simeq \frac{\ker\left(\partial\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)} ,$$

and

$$\mathbb{H}^{p+q-2}\left(\tilde{X};\, \left(\mathcal{L}^{\bullet}_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)\right) \simeq \frac{\ker\left(\partial\overline{\partial}\colon \wedge^{p-1,q-1}\tilde{X} \to \wedge^{p,q}\tilde{X}\right)}{\operatorname{im}\left(\partial\colon \wedge^{p-2,q-1}\tilde{X} \to \wedge^{p-1,q-1}\tilde{X}\right) + \operatorname{im}\left(\overline{\partial}\colon \wedge^{p-1,q-2}\tilde{X} \to \wedge^{p-1,q-1}\tilde{X}\right)}$$

and

$$\mathbb{H}^{p+q-2}\left(\tilde{X};\, \left(\mathcal{M}^{\bullet}_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)\right) \simeq \frac{\ker\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)}{\operatorname{im}\left(\partial\colon \mathcal{D}^{p-2,q-1}\tilde{X} \to \mathcal{D}^{p-1,q-1}\tilde{X}\right) + \operatorname{im}\left(\overline{\partial}\colon \mathcal{D}^{p-1,q-2}\tilde{X} \to \mathcal{D}^{p-1,q-1}\tilde{X}\right)} ,$$

proving the stated isomorphisms.

Chapter 2

Cohomology of almost-complex manifolds

Let X be a $2n$-dimensional (differentiable) manifold endowed with an almost-complex structure J. Note that if J is not integrable, then the Dolbeault cohomology is not defined. In this section, we are concerned with studying some subgroups of the de Rham cohomology related to the almost-complex structure: these subgroups have been introduced by T.-J. Li and W. Zhang in [LZ09], in order to study the relation between the compatible and the tamed symplectic cones on a compact almost-complex manifold, with the aim to throw light on a question by S. K. Donaldson, [Don06, Question 2] (see §2.4.2), and it would be interesting to consider them as a sort of counterpart of the Dolbeault cohomology groups in the non-integrable (or at least in the non-Kähler) case, see [DLZ10, Lemma 2.15, Theorem 2.16]. In particular, we are interested in studying when they let a splitting of the de Rham cohomology, and their relations with cones of metric structures.

More precisely, in §2.1 we introduce the notions of $\mathcal{C}^\infty$-pure-and-full and pure-and-full almost-complex structures, setting the notation and proving some useful relations between them. In §2.2, we study $\mathcal{C}^\infty$-pure-and-fullness on several classes of (almost-)complex manifolds, e.g., solvmanifolds, semi-Kähler manifolds, almost-Kähler manifolds. In §2.3, we study the behaviour of $\mathcal{C}^\infty$-pure-and-fullness under small deformations of the complex structure and along curves of almost-complex structures, investigating properties of stability, and of semi-continuity for the dimensions of the invariant, and anti-invariant subgroups of the de Rham cohomology with respect to the almost-complex structure. In §2.4, we study the cone of semi-Kähler structures on a compact almost-complex manifold and, in particular, we compare the cones of balanced metrics and of strongly-Gauduchon metrics on a compact complex manifold.

The results of this chapter have been obtained jointly with A. Tomassini, in [AT11, AT12a], and with A. Tomassini and W. Zhang, in [ATZ12].

2.1 Subgroups of the de Rham (co)homology of an almost-complex manifold

In this section, we set the notation concerning $\mathcal{C}^\infty$-pure-and-full and pure-and-full almost-complex structures, as introduced in [LZ09], and we study the relations between $\mathcal{C}^\infty$-pure-and-fullness and pure-and-fullness.

2.1.1 $\mathcal{C}^\infty$-pure-and-full and pure-and-full almost-complex structures

In this section, we start by fixing some preliminary notation and recalling some definitions; then we briefly review some results to motivate the study of these topics, see Remark 2.9, which will be further discussed in the next sections.

Let $S \subseteq \mathbb{N} \times \mathbb{N}$ and define

$$H^S_J(X;\mathbb{R}) \;:=\; \left\{[\alpha] \in H^\bullet_{dR}(X;\mathbb{R}) \;:\; \alpha \in \left(\bigoplus_{(p,q)\in S} \wedge^{p,q}X\right) \cap \wedge^\bullet X\right\} \;;$$

note that a real differential form α with a component of type (p,q) has also a component of type (q,p), and hence we are interested in studying the sets S such that whenever $(p,q) \in S$, also $(q,p) \in S$. As a matter of notation, we will usually list the elements of S instead of writing S itself.

Note that, for every $k \in \mathbb{N}$, one has

$$\sum_{\substack{p+q=k \\ p \leq q}} H^{(p,q),(q,p)}_J(X;\mathbb{R}) \;\subseteq\; H^k_{dR}(X;\mathbb{R}) \;,$$

but, in general, the sum is neither direct nor the equality holds: several examples of these facts will be provided in the sequel.

The subgroups $H_J^{(2,0),(0,2)}(X;\mathbb{R})$ and $H_J^{(1,1)}(X;\mathbb{R})$ of $H^2_{dR}(X;\mathbb{R})$ are of special interest for their interpretation as the J-anti-invariant, respectively, J-invariant part of the second de Rham cohomology group. Indeed, note that the endomorphism $J\lfloor_{\wedge^2 X}\in \mathrm{End}\left(\wedge^\bullet X\right)$ naturally extending $J\in\mathrm{End}(TX)$ (that is, $J\alpha := \alpha\left(J\cdot,\, J\cdot\cdot\right)$ for every $\alpha\in\wedge^2 X$) satisfies $\left(J\lfloor_{\wedge^2 X}\right)^2=\mathrm{id}_{\wedge^2 X}$; hence, one has the splitting

$$\wedge^2 X \;=\; \wedge^+_J X\oplus\wedge^-_J X \;,$$

where, for $\pm\in\{+,\,-\}$,

$$\wedge^\pm_J X \;:=\; \left\{\alpha\in\wedge^2 X \;:\; J\alpha=\pm\alpha\right\} \;.$$

Since $H^2_{dR}(X;\mathbb{R})$ contains, in particular, the classes represented by the symplectic forms, and $H_J^{(1,1)}(X;\mathbb{R})$ contains, in particular, the classes represented by the $(1,1)$-forms associated to the Hermitian metrics on X, in [LZ09], T.-J. Li and W. Zhang were interested in studying the *J-invariant subgroup* of $H^2_{dR}(X;\mathbb{R})$, namely,

$$H^+_J(X) \;:=\; H_J^{(1,1)}(X;\mathbb{R}) \;=\; \left\{[\alpha]\in H^2_{dR}(X;\mathbb{R}) \;:\; J\alpha=\alpha\right\} \;,$$

and the *J-anti-invariant subgroup* of $H^2_{dR}(X;\mathbb{R})$, namely,

$$H^-_J(X) \;:=\; H_J^{(2,0),(0,2)}(X;\mathbb{R}) \;=\; \left\{[\alpha]\in H^2_{dR}(X;\mathbb{R}) \;:\; J\alpha=-\alpha\right\} \;.$$

Note that, as in the general case, one has that

$$H^+_J(X)+H^-_J(X) \;\subseteq\; H^2_{dR}(X;\mathbb{R})$$

but, in general, the sum is neither direct nor equal to $H^2_{dR}(X;\mathbb{R})$. The following definition, by T.-J. Li and W. Zhang, singles out the almost-complex structures whose subgroups $H^+_J(X)$ and $H^-_J(X)$ provide a decomposition of $H^2_{dR}(X;\mathbb{R})$.

Definition 2.1 ([LZ09, Definition 2.2, Definition 2.3, Lemma 2.2])**.** An almost-complex structure J on a manifold X is said to be

- $\mathcal{C}^\infty$*-pure* if $H^-_J(X)\cap H^+_J(X)=\{0\}$;
- $\mathcal{C}^\infty$*-full* if $H^-_J(X)+H^+_J(X)=H^2_{dR}(X;\mathbb{R})$;
- $\mathcal{C}^\infty$*-pure-and-full* if it is both $\mathcal{C}^\infty$-pure and $\mathcal{C}^\infty$-full, i.e., if the following cohomology decomposition holds:

$$H^2_{dR}(X;\mathbb{R}) \;=\; H^-_J(X)\oplus H^+_J(X) \;.$$

We will also use the following definition, which is a natural generalization of the notion of $\mathcal{C}^\infty$-pure-and-fullness to higher degree cohomology groups.

Definition 2.2. Let X be a manifold endowed with an almost-complex structure J, and fix $k\in\mathbb{N}$. Consider $H^k_{dR}(X;\mathbb{R})\supseteq\sum_{\substack{p+q=k\\ p\leq q}} H_J^{(p,q),(q,p)}(X;\mathbb{R})$:

- if
$$\bigoplus_{\substack{p+q=k\\ p\leq q}} H_J^{(p,q),(q,p)}(X;\mathbb{R}) \;\subseteq\; H^k_{dR}(X;\mathbb{R})$$
(namely, the sum is direct), then J is called $\mathcal{C}^\infty$*-pure at the k^{th} stage*;
- if
$$H^k_{dR}(X;\mathbb{R}) \;=\; \sum_{\substack{p+q=k\\ p\leq q}} H_J^{(p,q),(q,p)}(X;\mathbb{R}) \;,$$
then J is called $\mathcal{C}^\infty$*-full at the k^{th} stage*;
- if J is both $\mathcal{C}^\infty$-pure at the k^{th} stage and $\mathcal{C}^\infty$-full at the k^{th} stage, that is,
$$H^k_{dR}(X;\mathbb{R}) \;=\; \bigoplus_{\substack{p+q=k\\ p\leq q}} H_J^{(p,q),(q,p)}(X;\mathbb{R}) \;;$$
then J is called $\mathcal{C}^\infty$*-pure-and-full at the k^{th} stage*.

Analogous definitions can be given for the de Rham cohomology with complex coefficients. More precisely, let $S \subseteq \mathbb{N} \times \mathbb{N}$ and define

$$H_J^S(X;\mathbb{C}) := \left\{[\alpha] \in H^\bullet_{dR}(X;\mathbb{C}) \;:\; \alpha \in \bigoplus_{(p,q)\in S} \wedge^{p,q}X\right\}$$

(as previously, we will usually list the elements of S instead of writing S itself); with such notation, one has in particular that $H_J^S(X;\mathbb{R}) = H_J^S(X;\mathbb{C}) \cap H^\bullet_{dR}(X;\mathbb{R})$.

Remark 2.3. Note that, when X is a compact manifold endowed with an integrable almost-complex structure J, then, for any $(p,q) \in \mathbb{N} \times \mathbb{N}$,

$$H_J^{(p,q)}(X;\mathbb{C}) = \mathrm{im}\left(H^{p,q}_{BC}(X) \to H^{p+q}_{dR}(X;\mathbb{C})\right) ,$$

where the map $H^{p,q}_{BC}(X) \to H^{p+q}_{dR}(X;\mathbb{C})$ is the one induced by the identity (note that $\ker\partial \cap \ker\overline{\partial} \subseteq \ker \mathrm{d}$ and $\mathrm{im}\,\partial\overline{\partial} \subseteq \mathrm{im}\,\mathrm{d}$). Indeed, any d-closed (p,q)-form is both ∂-closed and $\overline{\partial}$-closed.

Note that, for every $k \in \mathbb{N}$, one has

$$\sum_{p+q=k} H_J^{(p,q)}(X;\mathbb{C}) \subseteq H^k_{dR}(X;\mathbb{C}) ,$$

but, in general, the sum is neither direct nor the equality holds. We can then give the following definition.

Definition 2.4. Let X be a manifold endowed with an almost-complex structure J, and fix $k \in \mathbb{N}$. Consider $H^k_{dR}(X;\mathbb{C}) \supseteq \sum_{p+q=k} H_J^{(p,q)}(X;\mathbb{C})$:

- if
$$\bigoplus_{p+q=k} H_J^{(p,q)}(X;\mathbb{C}) \subseteq H^k_{dR}(X;\mathbb{C})$$
(namely, the sum is direct), then J is called *complex-$\mathcal{C}^\infty$-pure at the k^{th} stage*;
- if
$$H^k_{dR}(X;\mathbb{C}) = \sum_{p+q=k} H_J^{(p,q)}(X;\mathbb{C}) ,$$
then J is called *complex-$\mathcal{C}^\infty$-full at the k^{th} stage*;
- if J is both complex-$\mathcal{C}^\infty$-pure at the k^{th} stage and complex-$\mathcal{C}^\infty$-full at the k^{th} stage, that is,
$$H^k_{dR}(X;\mathbb{C}) = \bigoplus_{p+q=k} H_J^{(p,q)}(X;\mathbb{C}) ;$$
then J is called *complex-$\mathcal{C}^\infty$-pure-and-full at the k^{th} stage.*

Remark 2.5. In general, being complex-$\mathcal{C}^\infty$-full at the 2^{nd} stage is a stronger condition that being $\mathcal{C}^\infty$-full. Furthermore, if J is integrable, then being complex-$\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage is stronger than being $\mathcal{C}^\infty$-pure-and-full. More precisely, for any (possibly non-integrable) almost-complex structure J, it holds, [DLZ10, Lemma 2.11],

$$\left\{\begin{array}{rcl} H_J^+(X) &=& H_J^{(1,1)}(X;\mathbb{C}) \cap H^2_{dR}(X;\mathbb{R}) \\ H_J^{(1,1)}(X;\mathbb{C}) &=& H_J^+(X) \otimes_{\mathbb{R}} \mathbb{C} \end{array}\right. ,$$

and

$$H_J^{(2,0)}(X;\mathbb{C}) + H_J^{(0,2)}(X;\mathbb{C}) \subseteq H_J^-(X) \otimes_{\mathbb{R}} \mathbb{C} ,$$

and, if J is integrable, it holds

$$\left\{\begin{array}{l} H_J^-(X) = \left(H_J^{(2,0)}(X;\mathbb{C}) + H_J^{(0,2)}(X;\mathbb{C})\right) \cap H^2_{dR}(X;\mathbb{R}) \\ H_J^{(2,0)}(X;\mathbb{C}) + H_J^{(0,2)}(X;\mathbb{C}) = H_J^-(X) \otimes_{\mathbb{R}} \mathbb{C} \end{array}\right. ;$$

indeed, $\mathrm{d}\wedge^{2,0}X \subseteq \wedge^{3,0}X \oplus \wedge^{2,1}X$ and $\mathrm{d}\wedge^{0,2}X \subseteq \wedge^{1,2}X \oplus \wedge^{0,3}X$. (Compare also [DLZ10, Lemma 2.12] for further results in the case of 4-dimensional manifolds.)

Note also that, if J is $\mathcal{C}^\infty$-pure, then

$$H_J^{(1,1)}(X;\mathbb{R}) \cap \left(H_J^{(2,0)}(X;\mathbb{R}) + H_J^{(0,2)}(X;\mathbb{R})\right) = \{0\} .$$

The construction of the subgroups $H^S_J(X;\mathbb{R}) \subseteq H^\bullet_{dR}(X;\mathbb{R})$ and the notion of $\mathcal{C}^\infty$-pure-and-full almost-complex structures can be repeated using the complex of currents $\left(\mathcal{D}_\bullet X := \mathcal{D}^{2n-\bullet}X,\, \mathrm{d}\right)$ instead of the complex of differential forms $(\wedge^\bullet X,\, \mathrm{d})$ and the de Rham homology $H^{dR}_\bullet(X;\mathbb{R})$ instead of the de Rham cohomology $H^\bullet_{dR}(X;\mathbb{R})$. (We refer to §0.5 for notations and references concerning currents and de Rham homology.)

As in the smooth case, accordingly to T.-J. Li and W. Zhang, [LZ09], given $S \subseteq \mathbb{N}\times\mathbb{N}$, let

$$H^J_S(X;\mathbb{R}) \;:=\; \left\{[\alpha] \in H^{dR}_\bullet(X;\mathbb{C}) \;:\; \alpha \in \left(\bigoplus_{(p,q)\in S} \mathcal{D}_{p,q}X\right) \cap \mathcal{D}_\bullet X\right\} \;.$$

In particular, the almost-complex structures on X for which $H^J_{(2,0),(0,2)}(X;\mathbb{R})$ and $H^J_{(1,1)}(X;\mathbb{R})$ provide a decomposition of $H^{dR}_2(X;\mathbb{R})$ are emphasized by the following definition by T.-J. Li and W. Zhang.

Definition 2.6 ([LZ09, Definition 2.15, Lemma 2.16])**.** An almost-complex structure J on a manifold X is said to be:

- *pure* if
$$H^J_{(2,0),(0,2)}(X;\mathbb{R}) \;\cap\; H^J_{(1,1)}(X;\mathbb{R}) \;=\; \{0\} \;;$$
- *full* if
$$H^J_{(2,0),(0,2)}(X;\mathbb{R}) \;+\; H^J_{(1,1)}(X;\mathbb{R}) \;=\; H^{dR}_2(X;\mathbb{R})\;;$$
- *pure-and-full* if it is both pure and full, i.e., if the following decomposition holds:
$$H^J_{(2,0),(0,2)}(X;\mathbb{R}) \oplus H^J_{(1,1)}(X;\mathbb{R}) \;=\; H^{dR}_2(X;\mathbb{R})\;.$$

The following are natural generalizations of the notion of pure-and-fullness.

Definition 2.7. Let X be a manifold endowed with an almost-complex structure J, and fix $k \in \mathbb{N}$. Consider $H^k_{dR}(X;\mathbb{R}) \supseteq \sum_{\substack{p+q=k\\ p\leq q}} H^J_{(p,q),(q,p)}(X;\mathbb{R})$:

- if
$$\bigoplus_{\substack{p+q=k\\ p\leq q}} H^J_{(p,q),(q,p)}(X;\mathbb{R}) \;\subseteq\; H^{dR}_k(X;\mathbb{R})$$
(namely, the sum is direct), then J is called *pure at the k^{th} stage*;
- if
$$H^{dR}_k(X;\mathbb{R}) \;=\; \sum_{\substack{p+q=k\\ p\leq q}} H^J_{(p,q),(q,p)}(X;\mathbb{R})\;,$$
then J is called *full at the k^{th} stage*;
- if J is both pure at the k^{th} stage and full at the k^{th} stage, that is,
$$H^{dR}_k(X;\mathbb{R}) \;=\; \bigoplus_{\substack{p+q=k\\ p\leq q}} H^J_{(p,q),(q,p)}(X;\mathbb{R})\;;$$
then J is called *pure-and-full at the k^{th} stage*.

As regards de Rham homology with complex coefficients, given $S \subseteq \mathbb{N}\times\mathbb{N}$, let

$$H^J_S(X;\mathbb{C}) \;:=\; \left\{[\alpha] \in H^{dR}_\bullet(X;\mathbb{C}) \;:\; \alpha \in \bigoplus_{(p,q)\in S} \mathcal{D}_{p,q}X\right\}\;,$$

so that $H^J_S(X;\mathbb{R}) = H^J_S(X;\mathbb{C}) \cap H^{dR}_\bullet(X;\mathbb{R})$.

Definition 2.8. Let X be a manifold endowed with an almost-complex structure J, and fix $k \in \mathbb{N}$. Consider $H^{dR}_k(X;\mathbb{C}) \supseteq \sum_{p+q=k} H^J_{(p,q)}(X;\mathbb{C})$:

- if
$$\bigoplus_{p+q=k} H^J_{(p,q)}(X;\mathbb{C}) \;\subseteq\; H^{dR}_k(X;\mathbb{C})$$
(namely, the sum is direct), then J is called *complex-pure at the k^{th} stage*;
- if
$$H^{dR}_k(X;\mathbb{C}) \;=\; \sum_{p+q=k} H^J_{(p,q)}(X;\mathbb{C}) \;,$$
then J is called *complex-full at the k^{th} stage*;
- if J is both complex-pure at the k^{th} stage and complex-full at the k^{th} stage, that is,
$$H^{dR}_k(X;\mathbb{C}) \;=\; \bigoplus_{p+q=k} H^J_{(p,q)}(X;\mathbb{C}) \;;$$
then J is called *complex-pure-and-full at the k^{th} stage.*

Remark 2.9. The study of the subgroups $H^{(p,q),(q,p)}_J(X;\mathbb{R})$ and the notion of $\mathcal{C}^\infty$-pure-and-full almost-complex structure have been introduced by T.-J. Li and W. Zhang in [LZ09], in order to study the relations between the compatible and the tamed symplectic cones on a compact almost-complex manifold, and inspired by a question by S. K. Donaldson, [Don06, Question 2]: whether, on a compact 4-dimensional manifold endowed with an almost-complex structure J tamed by a symplectic form, there exists also a symplectic form compatible with J, see §2.4.2. In [DLZ10], T. Drăghici, T.-J. Li, and W. Zhang investigated the 4-dimensional case, proving, in particular, that every almost-complex structure on a compact 4-dimensional manifold is $\mathcal{C}^\infty$-pure-and-full; they also obtained further results for 4-dimensional almost-complex manifolds in [DLZ11], where they studied the dimensions of the subgroups $H^+_J(X)$ and $H^-_J(X)$. In [FT10], A. Fino and A. Tomassini studied the $\mathcal{C}^\infty$-pure-and-fullness in connection with other properties on almost-complex manifolds: in particular, by studying almost-complex solvmanifolds, they provided the first explicit example of a non-$\mathcal{C}^\infty$-pure-and-full almost-complex structure. Jointly with A. Tomassini, we studied in [AT11] the behaviour of $\mathcal{C}^\infty$-pure-and-fullness under small deformations of the complex structure or along curves of almost-complex structures, proving in particular its instability. In [AT12a] we continued the study of the cohomological properties related to the existence of an almost-complex structure, focusing, in particular, on the study of the cone of semi-Kähler structures on a compact semi-Kähler manifold. In [ATZ12], jointly with A. Tomassini and W. Zhang, we further studied cohomological properties of almost-Kähler manifolds, especially in relation with W. Zhang's Lefschetz-type property; in particular, an example of a non-$\mathcal{C}^\infty$-full almost-Kähler structure on a compact manifold is provided. In [DZ11], T. Drăghici and W. Zhang reformulated the S. K. Donaldson "tamed to compatible" question in terms of spaces of exact forms, proving, in particular, that an almost-complex structure J on a compact 4-dimensional manifold admits a compatible symplectic form if and only if it admits tamed symplectic forms with any arbitrarily given J-anti-invariant component. Q. Tan, H. Wang, Y. Zhang, and P. Zhu, in [TWZZ11], continued the study of the dimension of the J-anti-invariant subgroup $H^-_J(X)$ of the de Rham cohomology of a compact almost-complex manifold, considering almost-complex structures being metric related or fundamental form related, showing, for example, that $\dim_{\mathbb{R}} H^-_J(X) = 0$ for a generic almost-complex structure J on a compact 4-dimensional manifold, as conjectured by T. Drăghici, T.-J. Li, and W. Zhang, [DLZ11, Conjecture 2.4]. For further results on the study of J-anti-invariant forms and J-anti-invariant de Rham cohomology classes on a (possibly non-compact) manifold endowed with an almost-complex structure J, see [HMT11] by R. K. Hind, C. Medori, and A. Tomassini, where a result concerning analytic continuation for J-anti-invariant forms is proven. In [LT12], T.-J. Li and A. Tomassini studied the analogue of the above problems for linear (possibly non-integrable) complex structures on 4-dimensional unimodular Lie algebras; in particular, they proved that an analogue of the decomposition in [DLZ10, Theorem 2.3] holds for every 4-dimensional unimodular Lie algebra endowed with a linear (possibly non-integrable) complex structure; furthermore, they considered the linear counterpart of Donaldson's "tamed to compatible" question, and of the tamed and compatible symplectic cones, studying, in particular, a sufficient condition on a 4-dimensional Lie algebra $\mathfrak{g}$ (which holds, for example, for 4-dimensional unimodular Lie algebras) in order that a linear (possibly non-integrable) complex structure admits a taming linear symplectic form if and only if it admits a compatible linear symplectic form. The paper [DLZ12] by T. Drăghici, T.-J. Li, and W. Zhang furnishes a survey on the known results concerning the subgroups $H^+_J(X)$ and $H^-_J(X)$, especially in dimension 4, and their application to S. K. Donaldson's "tamed to compatible" question.

2.1.2 Relations between $\mathcal{C}^\infty$-pure-and-fullness and pure-and-fullness

The following result summarizes the relations between $\mathcal{C}^\infty$-pure-and-fullness and pure-and-fullness, see [AT11, Theorem 2.1], see also [LZ09, Proposition 2.5], and between complex-$\mathcal{C}^\infty$-pure-and-fullness and complex-pure-and-fullness. (Analogous results will be proven in Proposition 3.25 for almost-**D**-complex structures in the sense of F. R. Harvey and H. B. Lawson, and in Proposition 3.12 for symplectic structures.)

Theorem 2.10 (see [LZ09, Proposition 2.5])**.** *Let J be an almost-complex structure on a compact $2n$-dimensional manifold X. The following relations between (complex-)$\mathcal{C}^\infty$-pure-and-full and (complex-)pure-and-full notions hold: for any $k \in \mathbb{N}$,*

$$\begin{array}{ccc} \mathcal{C}^\infty\text{-}\textit{full at the } k^{th} \textit{ stage} & \Longrightarrow & \textit{pure at the } k^{th} \textit{ stage} \\ \Downarrow & & \Downarrow \\ \textit{full at the } (2n-k)^{th} \textit{ stage} & \Longrightarrow & \mathcal{C}^\infty\text{-}\textit{pure at the } (2n-k)^{th} \textit{ stage} \end{array} ,$$

and

$$\begin{array}{ccc} \textit{complex-}\mathcal{C}^\infty\text{-}\textit{full at the } k^{th} \textit{ stage} & \Longrightarrow & \textit{complex-pure at the } k^{th} \textit{ stage} \\ \Downarrow & & \Downarrow \\ \textit{complex-full at the } (2n-k)^{th} \textit{ stage} & \Longrightarrow & \textit{complex-}\mathcal{C}^\infty\text{-}\textit{pure at the } (2n-k)^{th} \textit{ stage} \end{array} .$$

Proof. The horizontal implications follow by considering the non-degenerate duality pairing

$$\langle \cdot, \cdot\cdot \rangle : H^\bullet_{dR}(X;\mathbb{R}) \times H_\bullet^{dR}(X;\mathbb{R}) \to \mathbb{R} , \qquad \text{respectively} \quad \langle \cdot, \cdot\cdot \rangle : H^\bullet_{dR}(X;\mathbb{C}) \times H_\bullet^{dR}(X;\mathbb{C}) \to \mathbb{C} ,$$

and noting that, for any $p, q \in \mathbb{N}$,

$$\ker \left\langle H_J^{(p,q),(p,q)}(X;\mathbb{R}),\ \cdot \right\rangle \supseteq \sum_{\{(r,s),(s,r)\} \neq \{(p,q),(q,p)\}} H^J_{(r,s),(s,r)}(X;\mathbb{R})$$

$$\text{and} \qquad \ker \left\langle \cdot,\ H^J_{(p,q),(q,p)}(X;\mathbb{R}) \right\rangle \supseteq \sum_{\{(r,s),(s,r)\} \neq \{(p,q),(q,p)\}} H_J^{(r,s),(s,r)}(X;\mathbb{R}) ,$$

respectively

$$\ker \left\langle H_J^{(p,q)}(X;\mathbb{C}),\ \cdot \right\rangle \supseteq \sum_{(r,s)\neq(p,q)} H^J_{(r,s)}(X;\mathbb{C}) \qquad \text{and} \qquad \ker \left\langle \cdot,\ H^J_{(p,q)}(X;\mathbb{C}) \right\rangle \supseteq \sum_{(r,s)\neq(p,q)} H_J^{(r,s)}(X;\mathbb{C}) .$$

As an example, we give the details to prove that if J is $\mathcal{C}^\infty$-full at the k^{th} stage then it is also pure at the k^{th} stage, when $k=2$. Let

$$\mathfrak{c} \in H^J_{(2,0),(0,2)}(X;\mathbb{R}) \cap H^J_{(1,1)}(X;\mathbb{R}) ,$$

with $\mathfrak{c} \neq [0]$. Hence,

$$\langle \mathfrak{c}, \cdot \rangle \lfloor_{H_J^{(2,0),(0,2)}(X;\mathbb{R})} = 0 \qquad \text{and} \qquad \langle \mathfrak{c}, \cdot \rangle \lfloor_{H_J^{(1,1)}(X;\mathbb{R})} = 0 ;$$

since J is $\mathcal{C}^\infty$-full, it follows that $\langle \mathfrak{c}, \cdot \rangle \lfloor_{H^2_{dR}(X;\mathbb{R})} = 0$, and hence $\mathfrak{c} = [0]$.

To prove the vertical implications, it is enough to note that the quasi-isomorphism $T_\cdot \colon \wedge^\bullet X \to \mathcal{D}_{2n-\bullet}X$ defined as $T_\varphi := \int_X \varphi \wedge \cdot$ (see §0.5) induces an injective map

$$H_J^{(p,q),(q,p)}(X;\mathbb{R}) \to H^J_{(n-p,n-q),(n-q,n-p)}(X;\mathbb{R}) , \qquad \text{respectively} \qquad H_J^{(p,q)}(X;\mathbb{C}) \to H^J_{(n-p,n-q)}(X;\mathbb{C}) ,$$

for any $p, q \in \mathbb{N}$. □

Remark 2.11. On a compact $2n$-dimensional manifold X endowed with an almost-complex structure J, further linkings between $H^2_{dR}(X;\mathbb{R})$ and $H^{2n-2}_{dR}(X;\mathbb{R})$ could provide further relations between $\mathcal{C}^\infty$-pure-and-full and pure-and-full notions: for example, A. Fino and A. Tomassini proved in [FT10, Theorem 3.7] that, given a J-Hermitian metric g on X, if there exists a basis of g-harmonic representatives for $H^2_{dR}(X;\mathbb{R})$ being of pure type with respect to J, then J is both $\mathcal{C}^\infty$-pure-and-full and pure-and-full. Furthermore, A. Fino and A. Tomassini proved in [FT10, Theorem 4.1] that, given a J-compatible symplectic form ω on X satisfying the Hard Lefschetz Condition (that is, the map $[\omega^k] \smile \cdot \colon H^{n-k}_{dR}(X;\mathbb{R}) \xrightarrow{\simeq} H^{n+k}_{dR}(X;\mathbb{R})$ is an isomorphism for every $k \in \mathbb{N}$), if J is $\mathcal{C}^\infty$-pure-and-full, then J is also pure-and-full (compare Proposition 2.28 for a similar result).

Setting $2n = 4$ and $k = 2$ in Theorem 2.10, it follows that, on compact 4-dimensional almost-complex manifolds, $\mathcal{C}^\infty$-fullness implies $\mathcal{C}^\infty$-pureness. The following result states that, for higher dimensional manifolds, $\mathcal{C}^\infty$-pureness and $\mathcal{C}^\infty$-fullness are not, in general, related properties, [AT12a, Proposition 1.4].

Proposition 2.12. *There exist both examples of compact manifolds endowed with almost-complex structures being $\mathcal{C}^\infty$-full and non-$\mathcal{C}^\infty$-pure, and examples of compact manifolds endowed with almost-complex structures being $\mathcal{C}^\infty$-pure and non-$\mathcal{C}^\infty$-full.*

Proof. The proof follows from the following examples, [AT12a, Example 1.2, Example 1.3].

`Step 1` – *Being $\mathcal{C}^\infty$-full does not imply being $\mathcal{C}^\infty$-pure.* Take a nilmanifold N_1 with associated Lie algebra

$$\mathfrak{h}_{16} := \left(0^3,\ 12,\ 14,\ 24\right) .$$

Consider the left-invariant complex structure on N_1 whose space of $(1,0)$-forms is generated, as a $\mathcal{C}^\infty(N_1;\mathbb{C})$-module, by

$$\begin{cases} \varphi^1 & := & e^1 + \mathrm{i}\, e^2 \\ \varphi^2 & := & e^3 + \mathrm{i}\, e^4 \\ \varphi^3 & := & e^5 + \mathrm{i}\, e^6 \end{cases} .$$

Writing the structure equations in terms of $\left\{\varphi^1,\, \varphi^2,\, \varphi^3\right\}$,

$$\begin{cases} 2\, \mathrm{d}\varphi^1 & = & 0 \\ 2\, \mathrm{d}\varphi^2 & = & \varphi^{1\bar{1}} \\ 2\, \mathrm{d}\varphi^3 & = & -\mathrm{i}\,\varphi^{12} + \mathrm{i}\,\varphi^{1\bar{2}} \end{cases} ,$$

the integrability condition is easily verified.

K. Nomizu's theorem [Nom54, Theorem 1] makes the computation of the cohomology straightforward: in fact, listing the harmonic representatives with respect to the left-invariant Hermitian metric $g := \sum_j \varphi^j \odot \bar{\varphi}^j$ instead of their classes, one finds

$$H^2_{dR}(N_1;\mathbb{C}) \;=\; \mathbb{C}\left\langle \varphi^{13},\, \varphi^{\bar{1}\bar{3}} \right\rangle \oplus \mathbb{C}\left\langle \varphi^{1\bar{3}} - \varphi^{3\bar{1}} \right\rangle \oplus \mathbb{C}\left\langle \varphi^{12} + \varphi^{1\bar{2}},\, \varphi^{2\bar{1}} - \varphi^{\bar{1}\bar{2}} \right\rangle ,$$

where

$$H^{(2,0),(0,2)}_J(N_1;\mathbb{C}) \;=\; \mathbb{C}\left\langle \varphi^{13},\, \varphi^{\bar{1}\bar{3}} \right\rangle \oplus \mathbb{C}\left\langle \varphi^{12} + \varphi^{1\bar{2}},\, \varphi^{2\bar{1}} - \varphi^{\bar{1}\bar{2}} \right\rangle$$

and

$$H^{(1,1)}_J(N_1;\mathbb{C}) \;=\; \mathbb{C}\left\langle \varphi^{1\bar{3}} - \varphi^{3\bar{1}} \right\rangle \oplus \mathbb{C}\left\langle \varphi^{12} + \varphi^{1\bar{2}},\, \varphi^{2\bar{1}} - \varphi^{\bar{1}\bar{2}} \right\rangle .$$

In particular, J is a $\mathcal{C}^\infty$-full, non-$\mathcal{C}^\infty$-pure complex structure.

`Step 2` – *Being $\mathcal{C}^\infty$-pure does not imply being $\mathcal{C}^\infty$-full.* Take a nilmanifold N_2 with associated Lie algebra

$$\mathfrak{h}_2 := \left(0^4,\ 12,\ 34\right) .$$

and consider on it the left-invariant complex structure given requiring that the forms

$$\begin{cases} \varphi^1 & := & e^1 + \mathrm{i}\, e^2 \\ \varphi^2 & := & e^3 + \mathrm{i}\, e^4 \\ \varphi^3 & := & e^5 + \mathrm{i}\, e^6 \end{cases}$$

are of type $(1,0)$.

The integrability condition follows from the structure equations

$$\begin{cases} 2\, \mathrm{d}\varphi^1 & = & 0 \\ 2\, \mathrm{d}\varphi^2 & = & 0 \\ 2\, \mathrm{d}\varphi^3 & = & \mathrm{i}\,\varphi^{1\bar{1}} - \mathrm{i}\,\varphi^{2\bar{2}} \end{cases} .$$

K. Nomizu's theorem [Nom54, Theorem 1] gives

$$H^2_{dR}(N_2;\mathbb{C}) \;=\; \mathbb{C}\left\langle \varphi^{12},\, \varphi^{\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\left\langle \varphi^{1\bar{2}},\, \varphi^{2\bar{1}} \right\rangle \oplus \mathbb{C}\left\langle \varphi^{13} + \varphi^{1\bar{3}},\, \varphi^{3\bar{1}} - \varphi^{\bar{1}\bar{3}},\, \varphi^{3\bar{2}} - \varphi^{\bar{2}\bar{3}},\, \varphi^{23} - \varphi^{2\bar{3}} \right\rangle ,$$

where
$$H_J^{(2,0),(0,2)}(N_2;\mathbb{C}) \;=\; \mathbb{C}\left\langle \varphi^{12},\, \varphi^{\bar{1}\bar{2}} \right\rangle$$
and
$$H_J^{(1,1)}(N_2;\mathbb{C}) \;=\; \mathbb{C}\left\langle \varphi^{1\bar{2}},\, \varphi^{2\bar{1}} \right\rangle \;;$$
this can be proven arguing as follows: with respect to the left-invariant Hermitian metric $g := \sum_j \varphi^j \odot \bar{\varphi}^j$, one computes
$$\partial^*\varphi^{13} \;=\; \partial^*\varphi^{23} \;=\; \partial^*\varphi^{12} \;=\; 0\,,$$
that is, φ^{13}, φ^{12} and φ^{23} are g-orthogonal to the space $\partial\wedge^{1,0}N_2$; in the same way, one computes
$$\partial^*\varphi^{1\bar{2}} \;=\; \overline{\partial}^*\varphi^{1\bar{2}} \;=\; \partial^*\varphi^{1\bar{3}} \;=\; \overline{\partial}^*\varphi^{1\bar{3}} \;=\; 0$$
(compare also Proposition 2.19). In particular, J is a $\mathcal{C}^\infty$-pure, non-$\mathcal{C}^\infty$-full complex structure. □

2.2 $\mathcal{C}^\infty$-pure-and-fullness for special manifolds

In this section, we study the property of being $\mathcal{C}^\infty$-pure-and-full on special classes of (almost-)complex manifolds. After recalling some motivating results by T. Drăghici, T.-J. Li, and W. Zhang, we study $\mathcal{C}^\infty$-pure-and-fullness for left-invariant complex-structures on solvmanifolds, providing some examples in dimension 4 or higher; furthermore, we consider almost-complex manifolds endowed with special metric structures, namely, *semi-Kähler*, and *almost-Kähler* structures.

2.2.1 Special classes of $\mathcal{C}^\infty$-pure-and-full (almost-)complex manifolds

In this section, we recall some results by T. Drăghici, T.-J. Li, and W. Zhang, providing classes of $\mathcal{C}^\infty$-pure-and-full and pure-and-full (almost-)complex manifolds. They could be considered as motivations to study $\mathcal{C}^\infty$-pure-and-fullness: in fact, [DLZ10, Lemma 2.15, Theorem 2.16] suggests that the subgroups $H_J^{(\bullet,\bullet)}(X;\mathbb{C})$ can be viewed as a generalization of the Dolbeault cohomology groups for non-Kähler, and non-integrable, almost-complex manifolds X. On the other hand, [DLZ10, Theorem 2.3] states that, on a compact 4-dimensional almost-complex manifold X, the subgroups $H_J^+(X)$ and $H_J^-(X)$ induce always a decomposition of $H^2_{dR}(X;\mathbb{R})$: this could be intended as a generalization of the Hodge decomposition theorem for compact 4-dimensional almost-complex manifolds.

According to the following result, the groups $H_J^{(\bullet,\bullet)}(X;\mathbb{C})$ can be considered as the counterpart of the Dolbeault cohomology groups in the non-Kähler and non-integrable cases.

Theorem 2.13 ([DLZ10, Lemma 2.15, Theorem 2.16])**.** *Let X be a compact complex manifold. If the Hodge and Frölicher spectral sequence degenerates at the first step and the natural filtration associated with the structure of double complex of $\left(\wedge^{\bullet,\bullet}X,\,\partial,\,\overline{\partial}\right)$ induces a Hodge decomposition of weight k on $H^k_{dR}(X;\mathbb{C})$ for some $k\in\mathbb{N}$, then X is complex-$\mathcal{C}^\infty$-pure-and-full at the k^{th} stage, and*
$$H_J^{(p,q)}(X;\mathbb{C}) \simeq H^{p,q}_{\overline{\partial}}(X)$$
for every $p,q\in\mathbb{N}$ such that $p+q=k$.

A corollary of [DLZ10, Lemma 2.15, Theorem 2.16] is the following result.

Corollary 2.14 ([LZ09, Proposition 2.1], [DLZ10, Theorem 2.16, Proposition 2.17])**.** *One has that:*

(i) every compact complex surface is complex-$\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage, and hence, in particular, $\mathcal{C}^\infty$-pure-and-full and pure-and-full;

(ii) every compact complex manifold satisfying the $\partial\overline{\partial}$-Lemma is complex-$\mathcal{C}^\infty$-pure-and-full at every stage, and hence complex-pure-and-full at every stage;

(iii) every compact complex manifold admitting a Kähler structure is complex-$\mathcal{C}^\infty$-pure-and-full at every stage, and hence complex-pure-and-full at every stage.

Proof. As regards the complex-$\mathcal{C}^\infty$-fullness at the 2^{nd} stage for compact complex surfaces, one has that the assumptions of Theorem 2.13 with $k=2$ hold by [BHPVdV04, Theorem IV.2.8, Proposition IV.2.9].

As regards the complex-$\mathcal{C}^\infty$-fullness at every stage for compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma, one has that the assumptions of Theorem 2.13 for any $k\in\mathbb{N}$ are satisfied by [DGMS75, 5.21].

As regards the complex-$\mathcal{C}^\infty$-fullness at every stage for compact Kähler manifolds, one has that a compact complex manifold admitting a Kähler metric satisfies the $\partial\overline{\partial}$-Lemma, [DGMS75, Lemma 5.11].

Finally, the other statements follow from Remark 2.5 and Theorem 2.10. □

Actually, T. Drăghici, T.-J. Li, and W. Zhang proved in [DLZ10] the following result, which one can consider as a sort of Hodge decomposition theorem in the non-Kähler case.

Theorem 2.15 ([DLZ10, Theorem 2.3])**.** *Every almost-complex structure on a compact* 4*-dimensional manifold is $\mathcal{C}^\infty$-pure-and-full and pure-and-full.*

Proof. The proof of the previous theorem rests on the very special properties of 4-dimensional manifolds. For the sake of completeness, we recall here the argument by T. Drăghici, T.-J. Li, and W. Zhang in [DLZ10]. Firstly, note that, by Theorem 2.10, it suffices to prove that an almost-complex structure J on a compact 4-dimensional manifold is $\mathcal{C}^\infty$-full. Suppose that J is not $\mathcal{C}^\infty$-full. Fix a Hermitian metric g on X, and denote its associated $(1,1)$-form by ω. Recall that the Hodge-$*$-operator $*_g\lfloor_{\wedge^2 X}\colon \wedge^2 X \to \wedge^2 X$ satisfies $\left(*_g\lfloor_{\wedge^2 X}\right)^2 = \mathrm{id}_{\wedge^2 X}$, hence it induces a splitting

$$\wedge^2 X \;=\; \wedge^+_g X \oplus \wedge^-_g X \;,$$

where $\wedge^\pm_g X := \left\{\varphi \in \wedge^2 X \;:\; *_g\varphi = \pm\varphi\right\}$, for $\pm \in \{+,-\}$. Setting $\mathrm{P}\wedge^\bullet X := \ker\Lambda = \ker L^{2-\bullet+1}\lfloor_{\wedge^\bullet X}$ the space of primitive forms, where Λ is the adjoint operator of the Lefschetz operator $L := \omega\wedge\cdot\colon \wedge^\bullet X \to \wedge^{\bullet+2} X$ with respect to the pairing induced by ω (see §0.2), one has

$$\wedge^+_g X \;=\; L\left(\mathcal{C}^\infty(X;\mathbb{R})\right) \oplus \left(\left(\wedge^{2,0}X \oplus \wedge^{0,2}X\right) \cap \wedge^2 X\right) \qquad \text{and} \qquad \wedge^-_g X \;=\; \mathrm{P}\wedge^2 X \cap \wedge^{1,1} X \;;$$

indeed, recall that, on a compact $2n$-dimensional manifold X endowed with an almost-complex structure J and a Hermitian metric g with associated $(1,1)$-form ω, one has, for every $j \in \mathbb{N}$, for every $k \in \mathbb{N}$, the Weil identity, [Wei58, Théorème 2],

$$*_g\, L^j\lfloor_{\mathrm{P}\wedge^k X} \;=\; (-1)^{\frac{k(k+1)}{2}}\; \frac{j!}{(n-k-j)!}\, L^{n-k-j}\, J \;,$$

see, e.g., [Huy05, Proposition 1.2.31]. Since the Laplacian operator Δ and the Hodge-$*$-operator $*_g$ commute, the splitting $\wedge^2 X = \wedge^+_g X \oplus \wedge^-_g X$ induces a decomposition in cohomology,

$$H^2_{dR}(X;\mathbb{R}) \;=\; H^+_g(X) \oplus H^-_g(X) \;,$$

where $H^\pm_g(X) := \left\{[\varphi] \in H^2_{dR}(X;\mathbb{R}) \;:\; \varphi \in \wedge^\pm_g X\right\}$ for $\pm \in \{+,-\}$. Consider the non-degenerate pairing

$$\langle\cdot,\, \cdot\cdot\rangle : H^2_{dR}(X;\mathbb{R}) \times H^2_{dR}(X;\mathbb{R}) \to \mathbb{R} \;, \qquad \langle\varphi,\, \psi\rangle \;:=\; \int_X \varphi\wedge\psi \;,$$

and take $\mathfrak{a} \in \left(H^+_J(X) + H^-_J(X)\right)^\perp \subseteq H^2_{dR}(X;\mathbb{R})$. Since $\wedge^-_g X \subseteq \wedge^{1,1} X$, one can reduce to consider $\mathfrak{a} \in H^+_g(X)$; let $\alpha \in \wedge^+_g X$ be such that $\mathfrak{a} = [\alpha]$. According to the decomposition $\wedge^+_g X = L\left(\mathcal{C}^\infty(X;\mathbb{R})\right) \oplus \left(\left(\wedge^{2,0}X \oplus \wedge^{0,2}X\right)\cap\wedge^2 X\right)$, let $f\,\omega$ be the component of α in $L\left(\mathcal{C}^\infty(X;\mathbb{R})\right)$. Consider the Hodge decomposition

$$f\,\omega \;=\; h_{f\,\omega} + \mathrm{d}\,\vartheta + \mathrm{d}^*\,\eta$$

of $f\,\omega \in \wedge^2 X$, where $h_{f\,\omega} \in \ker\Delta \cap \wedge^2 X$, $\vartheta \in \wedge^1 X$, and $\eta \in \wedge^3 X$. Since $f\,\omega \in \wedge^+_g X$ and by the uniqueness of the Hodge decomposition, one has

$$h_{f\,\omega} + 2\,\mathrm{d}\,\vartheta \;=\; f\,\omega + 2\,\pi_{\wedge^-_g X}\left(\mathrm{d}\,\vartheta\right) \;\in\; \wedge^{1,1} X \cap \wedge^2 X$$

(where $\pi_{\wedge^\pm_g X}\colon \wedge^2 X \to \wedge^\pm_g X$ denotes the natural projection onto $\wedge^\pm_g X$, for $\pm \in \{+,-\}$). Therefore, noting also that $H^+_g(X)$ is orthogonal to $H^-_g(X)$ with respect to $\langle\cdot,\, \cdot\cdot\rangle$, one has

$$0 \;=\; \left\langle\mathfrak{a},\, \left[h_{f\,\omega} + 2\,\mathrm{d}\,\vartheta\right]\right\rangle \;=\; \left\langle\mathfrak{a},\, \left[f\,\omega + 2\pi_{\wedge^-_g X}\left(\mathrm{d}\,\vartheta\right)\right]\right\rangle \;=\; \int_X f^2\,\omega^2 \;,$$

from which it follows that $f = 0$, and hence $\mathfrak{a} = 0$. □

Remark 2.16. The result in [DLZ10, Theorem 2.3] does not hold anymore true in dimension greater than or equal to 6, or without the compactness assumption: the first example of a non-$\mathcal{C}^\infty$-pure almost-complex structure has been provided by A. Fino and A. Tomassini in [FT10, Example 3.3] using a 6-dimensional nilmanifold (for other examples, even in the integrable case, see Proposition 2.12, Example 2.41, Theorem 2.49, Proposition 2.55, Proposition 2.56), while non-$\mathcal{C}^\infty$-pure-and-full almost-complex structures on non-compact 4-dimensional manifolds arise from [DLZ11, Theorem 3.24] by T. Drăghici, T.-J. Li, and W. Zhang.

2.2.2 $\mathcal{C}^\infty$-pure-and-full solvmanifolds

Let $X = \Gamma\backslash G$ be a solvmanifold, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$, and its complexification by $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. (We refer to §0.6 for notations and results concerning solvmanifolds.)

We recall that if X is a nilmanifold or, more in general, a completely-solvable solvmanifold, the inclusion of the sub-complex given by the G-left-invariant differential forms, which is isomorphic to the complex $\wedge^\bullet \mathfrak{g}^*$ of linear forms on the dual of the Lie algebra $\mathfrak{g}$ associated to G, into the de Rham complex of X turns out to be a quasi-isomorphism, in view of K. Nomizu's theorem [Nom54, Theorem 1], respectively A. Hattori's theorem [Hat60, Corollary 4.2].

Let J be a G-left-invariant almost-complex structure on X. In this case, one can study the problem of cohomological decomposition both on X and on $\mathfrak{g}$: in this section, we investigate the relations between the cohomological decompositions at the level of the solvmanifold and at the level of the associated Lie algebra, Proposition 2.19, Corollary 2.20.

Firstly, we set some notations. Consider $H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) := H^\bullet(\wedge^\bullet \mathfrak{g}^*, \mathrm{d})$. Being J a G-left-invariant almost-complex structure, it induces a bi-graded splitting also on the vector space $\wedge^\bullet \mathfrak{g}^*_{\mathbb{C}}$. For every $S \subset \mathbb{N}\times\mathbb{N}$, and for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, set

$$H^S_J(\mathfrak{g};\mathbb{K}) := \left\{ [\alpha] \in H^\bullet_{dR}(\mathfrak{g};\mathbb{K}) \;:\; \alpha \in \bigoplus_{(p,q)\in S} \wedge^{p,q}\mathfrak{g}^*_{\mathbb{C}} \cap \left(\wedge^\bullet \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{K}\right) \right\} ,$$

see [LT12, Definition 0.3].

The following are the natural linear counterparts of the corresponding definitions for manifolds.

Definition 2.17. Let $X = \Gamma\backslash G$ be a solvmanifold, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Fixed $k \in \mathbb{N}$, a G-left-invariant almost-complex structure J on X is called

- *linear-$\mathcal{C}^\infty$-pure at the k^{th} stage* if

$$\bigoplus_{\substack{p+q=k \\ p\leq q}} H^{(p,q),(q,p)}_J(\mathfrak{g};\mathbb{R}) \subseteq H^k_{dR}(\mathfrak{g};\mathbb{R}) ,$$

  namely, if the sum is direct;

- *linear-$\mathcal{C}^\infty$-full at the k^{th} stage* if

$$H^k_{dR}(\mathfrak{g};\mathbb{R}) = \sum_{\substack{p+q=k \\ p\leq q}} H^{(p,q),(q,p)}_J(\mathfrak{g};\mathbb{R}) ,$$

- *linear-$\mathcal{C}^\infty$-pure-and-full at the k^{th} stage* if J is both linear-$\mathcal{C}^\infty$-pure at the k^{th} stage and linear-$\mathcal{C}^\infty$-full at the k^{th} stage, that is, if the cohomological decomposition

$$H^k_{dR}(\mathfrak{g};\mathbb{R}) = \bigoplus_{\substack{p+q=k \\ p\leq q}} H^{(p,q),(q,p)}_J(\mathfrak{g};\mathbb{R})$$

  holds.

Furthermore, J is called

- *linear-complex-$\mathcal{C}^\infty$-pure at the k^{th} stage* if

$$\bigoplus_{p+q=k} H^{(p,q)}_J(\mathfrak{g};\mathbb{C}) \subseteq H^k_{dR}(\mathfrak{g};\mathbb{C}) ,$$

  namely, if the sum is direct;

- *linear-complex-$\mathcal{C}^\infty$-full at the k^{th} stage* if

$$H^k_{dR}(\mathfrak{g};\mathbb{C}) = \sum_{p+q=k} H^{(p,q)}_J(\mathfrak{g};\mathbb{C}) ,$$

- *linear-complex-$\mathcal{C}^\infty$-pure-and-full at the k^{th} stage* if J is both linear-complex-$\mathcal{C}^\infty$-pure at the k^{th} stage and linear-complex-$\mathcal{C}^\infty$-full at the k^{th} stage, that is, if the cohomological decomposition

$$H^k_{dR}(\mathfrak{g};\mathbb{C}) \;=\; \bigoplus_{p+q=k} H^{(p,q)}_J(\mathfrak{g};\mathbb{C})$$

holds.

(In any case, when $k=2$, the specification "at the 2$^{\text{nd}}$ stage" will be understood.)

It is natural to ask what relations link the subgroups $H^{(\bullet,\bullet)}_J(X;\mathbb{R})$ and the subgroups $H^{(\bullet,\bullet)}_J(\mathfrak{g};\mathbb{R})$, and whether a G-left-invariant linear-$\mathcal{C}^\infty$-pure-and-full almost-complex structure on $X=\Gamma\backslash G$ is also $\mathcal{C}^\infty$-pure-and-full.

The following lemma is the F. A. Belgun symmetrization trick, [Bel00, Theorem 7], in the almost-complex setting.

Lemma 2.18 ([Bel00, Theorem 7])**.** *Let $X=\Gamma\backslash G$ be a solvmanifold, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Let J be a G-left-invariant almost-complex structure on X. Let η be the G-bi-invariant volume form on G given by J. Milnor's Lemma, [Mil76, Lemma 6.2], and such that $\int_X \eta = 1$. Up to identifying G-left-invariant forms on X and linear forms over $\mathfrak{g}^*$ through left-translations, consider the Belgun symmetrization map*

$$\mu\colon \wedge^\bullet X \to \wedge^\bullet \mathfrak{g}^* \;, \qquad \mu(\alpha) \;:=\; \int_X \alpha\lfloor_m \eta(m) \;.$$

Then one has that

$$\mu\lfloor_{\wedge^\bullet\mathfrak{g}^*} \;=\; \mathrm{id}\lfloor_{\wedge^\bullet\mathfrak{g}^*} \;,$$

and that

$$\mathrm{d}\left(\mu(\cdot)\right) \;=\; \mu\left(\mathrm{d}\,\cdot\right) \qquad \text{and} \qquad J\left(\mu(\cdot)\right) \;=\; \mu\left(J\cdot\right) \;.$$

Using the previous lemma, we can prove the following Nomizu-type result, which relates the subgroups $H^{(r,s)}_J(X;\mathbb{R})$ with their left-invariant part $H^{(r,s)}_J(\mathfrak{g};\mathbb{R})$. (Analogous results will be proven in Proposition 3.30 for almost-**D**-complex structures in the sense of F. R. Harvey and H. B. Lawson, and in Proposition 3.18 for symplectic structures; compare also with [FT10, Theorem 3.4], by A. Fino and A. Tomassini, for almost-complex structures.)

Proposition 2.19 ([ATZ12, Theorem 5.4])**.** *Let $X=\Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant almost-complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. For any $S\subset\mathbb{N}\times\mathbb{N}$, and for $\mathbb{K}\in\{\mathbb{R},\,\mathbb{C}\}$, the map*

$$j\colon H^S_J(\mathfrak{g};\mathbb{K}) \to H^S_J(X;\mathbb{K})$$

induced by left-translations is injective, and, if $H^\bullet_{dR}(\mathfrak{g};\mathbb{K})\simeq H^\bullet_{dR}(X;\mathbb{K})$ (for instance, if X is a completely-solvable solvmanifold), then $j\colon H^S_J(\mathfrak{g};\mathbb{K})\to H^S_J(X;\mathbb{K})$ is in fact an isomorphism.

Proof. Since J is G-left-invariant, left-translations induce the map $j\colon H^S_J(\mathfrak{g};\mathbb{K})\to H^S_J(X;\mathbb{K})$. Consider the Belgun symmetrization map $\mu\colon\ \wedge^\bullet X\otimes\mathbb{K}\to\wedge^\bullet\mathfrak{g}^*\otimes_{\mathbb{R}}\mathbb{K}$, [Bel00, Theorem 7]: since μ commutes with d by [Bel00, Theorem 7], it induces the map $\mu\colon H^\bullet_{dR}(X;\mathbb{K})\to H^\bullet_{dR}(\mathfrak{g};\mathbb{K})$, and, since μ commutes with J, it preserves the bi-graduation; therefore it induces the map $\mu\colon H^S_J(X;\mathbb{K})\to H^S_J(\mathfrak{g};\mathbb{K})$. Moreover, since μ is the identity on the space of G-left-invariant forms by [Bel00, Theorem 7], we get the commutative diagram

$$\begin{tikzcd} H^S_J(\mathfrak{g};\mathbb{K}) \arrow[r, "j"] \arrow[rr, bend right, "\mathrm{id}"'] & H^S_J(X;\mathbb{K}) \arrow[r, "\mu"] & H^S_J(\mathfrak{g};\mathbb{K}) \end{tikzcd}$$

hence $j\colon H^S_J(\mathfrak{g};\mathbb{K})\to H^S_J(X;\mathbb{K})$ is injective, and $\mu\colon H^S_J(X;\mathbb{K})\to H^S_J(\mathfrak{g};\mathbb{K})$ is surjective.

Furthermore, when $H^\bullet_{dR}(\mathfrak{g};\mathbb{K})\simeq H^\bullet_{dR}(X;\mathbb{K})$ (for instance, when X is a completely-solvable solvmanifold, by A. Hattori's theorem [Hat60, Theorem 4.2]), since

$$\mu\lfloor_{\wedge^\bullet\mathfrak{g}^*\otimes_{\mathbb{R}}\mathbb{K}} \;=\; \mathrm{id}\lfloor_{\wedge^\bullet\mathfrak{g}^*\otimes_{\mathbb{R}}\mathbb{K}}$$

by [Bel00, Theorem 7], we get that $\mu\colon H^\bullet_{dR}(X;\mathbb{K})\to H^\bullet_{dR}(\mathfrak{g};\mathbb{K})$ is the identity map, and hence $\mu\colon H^S_J(X;\mathbb{K})\to H^S_J(\mathfrak{g};\mathbb{K})$ is also injective, and hence an isomorphism. $\square$

As a straightforward consequence, we get the following result.

Corollary 2.20. *Let $X = \Gamma \backslash G$ be a solvmanifold endowed with a G-left-invariant almost-complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Suppose that $H^{\bullet}_{dR}(\mathfrak{g};\mathbb{R}) \simeq H^{\bullet}_{dR}(X;\mathbb{R})$ (for instance, suppose that X is a completely-solvable solvmanifold). For every $k \in \mathbb{N}$, the almost-complex structure J is linear-$\mathcal{C}^\infty$-pure (respectively, linear-$\mathcal{C}^\infty$-full, linear-$\mathcal{C}^\infty$-pure-and-full, linear-complex-$\mathcal{C}^\infty$-pure, linear-complex-$\mathcal{C}^\infty$-full, linear-complex-$\mathcal{C}^\infty$-pure-and-full) at the k^{th} stage if and only if it is $\mathcal{C}^\infty$-pure (respectively, $\mathcal{C}^\infty$-full, $\mathcal{C}^\infty$-pure-and-full, complex-$\mathcal{C}^\infty$-pure, complex-$\mathcal{C}^\infty$-full, complex-$\mathcal{C}^\infty$-pure-and-full) at the k^{th} stage.*

As an example, we provide here an explicit $\mathcal{C}^\infty$-pure-and-full almost-complex structure on a 6-dimensional solvmanifold, [AT11, Example 2.1].

Example 2.21. *A $\mathcal{C}^\infty$-pure-and-full and pure-and-full almost-complex structure on a compact 6-dimensional completely-solvable solvmanifold.*
Let G be the 6-dimensional simply-connected completely-solvable Lie group defined by

$$G := \left\{ \left(\begin{array}{cccccc} e^{x^1} & 0 & x^2\, e^{x^1} & 0 & 0 & x^3 \\ 0 & e^{-x^1} & 0 & x^2\, e^{-x^1} & 0 & x^4 \\ 0 & 0 & e^{x^1} & 0 & 0 & x^5 \\ 0 & 0 & 0 & e^{-x^1} & 0 & x^6 \\ 0 & 0 & 0 & 0 & 1 & x^1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \in \mathrm{GL}(6;\mathbb{R}) \;:\; x^1, \ldots, x^6 \in \mathbb{R} \right\} .$$

According to [FdLS96, §3], there exists a discrete co-compact subgroup $\Gamma \subset G$: therefore $X := \Gamma \backslash G$ is a 6-dimensional completely-solvable solvmanifold.

The G-left-invariant 1-forms on G defined as

$$\begin{array}{ll} e^1 := \mathrm{d}\, x^1 , & e^2 := \mathrm{d}\, x^2 , \\ e^3 := \exp\left(-x^1\right) \cdot \left(\mathrm{d}\, x^3 - x^2\, \mathrm{d}\, x^5\right) , & e^4 := \exp\left(x^1\right) \cdot \left(\mathrm{d}\, x^4 - x^2\, \mathrm{d}\, x^6\right) , \\ e^5 := \exp\left(-x^1\right) \cdot \mathrm{d}\, x^5 ; & e^6 := \exp\left(x^1\right) \cdot \mathrm{d}\, x^6 \end{array}$$

give rise to G-left-invariant 1-forms on X. With respect to the co-frame $\left\{e^1, \ldots, e^6\right\}$, the structure equations are given by

$$\begin{cases} \mathrm{d}\, e^1 = 0 \\ \mathrm{d}\, e^2 = 0 \\ \mathrm{d}\, e^3 = -e^1 \wedge e^3 - e^2 \wedge e^5 \\ \mathrm{d}\, e^4 = e^1 \wedge e^4 - e^2 \wedge e^6 \\ \mathrm{d}\, e^5 = -e^1 \wedge e^5 \\ \mathrm{d}\, e^6 = e^1 \wedge e^6 \end{cases} .$$

Since G is completely-solvable, by A. Hattori's theorem [Hat60, Corollary 4.2], it is straightforward to compute

$$H^2(X;\mathbb{R}) = \mathbb{R}\left\langle e^1 \wedge e^2,\, e^5 \wedge e^6,\, e^3 \wedge e^6 + e^4 \wedge e^5 \right\rangle .$$

Therefore, setting

$$\begin{cases} \varphi^1 := e^1 + \mathrm{i}\, e^2 \\ \varphi^2 := e^3 + \mathrm{i}\, e^4 \\ \varphi^3 := e^5 + \mathrm{i}\, e^6 \end{cases} ,$$

we have that the almost-complex structure J whose $\mathcal{C}^\infty(X;\mathbb{C})$-module of complex $(1,0)$-forms is generated by $\left\{\varphi^1,\, \varphi^2,\, \varphi^3\right\}$ is $\mathcal{C}^\infty$-full: indeed,

$$\begin{array}{rcl} H^{(1,1)}_J(X;\mathbb{R}) &=& \mathbb{R}\left\langle -\frac{1}{2\mathrm{i}}\, \varphi^1 \wedge \bar\varphi^1,\, -\frac{1}{2\mathrm{i}}\, \varphi^3 \wedge \bar\varphi^3 \right\rangle , \\ H^{(2,0),(0,2)}_J(X;\mathbb{R}) &=& \mathbb{R}\left\langle \frac{1}{2\mathrm{i}}\left(\varphi^2 \wedge \varphi^3 - \bar\varphi^2 \wedge \bar\varphi^3\right) \right\rangle . \end{array}$$

Since

$$\mathrm{d} \wedge^1 \mathfrak{g}^*_{\mathbb{C}} = \mathbb{C}\left\langle \varphi^{13} - \varphi^{1\bar3},\, \varphi^{3\bar1} + \varphi^{\bar1\bar3},\, \varphi^{13} + \varphi^{1\bar3},\, \varphi^{3\bar1} - \varphi^{\bar1\bar3},\, \varphi^{12} - \varphi^{2\bar1},\, \varphi^{1\bar2} + \varphi^{\bar1\bar2} \right\rangle ,$$

then J is linear-$\mathcal{C}^\infty$-pure-and-full. Since X is a completely-solvable solvmanifold, one gets that J is also $\mathcal{C}^\infty$-pure by Corollary 2.20. (Note that the $\mathcal{C}^\infty$-pureness of J can be proven also by using a different argument: according to [FT10, Theorem 3.7], since the above basis of harmonic representatives with respect to the G-left-invariant Hermitian metric $\sum_{j=1}^{3} \varphi^j \odot \bar\varphi^j$ consists of pure type forms with respect to the almost-complex structure, J is both $\mathcal{C}^\infty$-pure-and-full and pure-and-full.)

Further results concerning linear (possibly non-integrable) complex structures on 4-dimensional unimodular Lie algebra and their cohomological properties have been obtained by T.-J. Li and A. Tomassini in [LT12]. In particular, they proved an analogous of [DLZ10, Theorem 2.3], namely, that for every 4-dimensional unimodular Lie algebra $\mathfrak{g}$ endowed with a linear (possibly non-integrable) complex structure J, one has the cohomological decomposition $H^2_{dR}(\mathfrak{g};\mathbb{R}) = H^{(2,0),(0,2)}_J(\mathfrak{g};\mathbb{R}) \oplus H^{(1,1)}_J(\mathfrak{g};\mathbb{R})$, [LT12, Theorem 3.3]. Furthermore, they studied the linear counterpart of S. K. Donaldson's question [Don06, Question 2] (see §2.4.2), proving that, on a 4-dimensional Lie algebra $\mathfrak{g}$ satisfying the condition $B \wedge B = 0$, where $B \subseteq \wedge^2\mathfrak{g}$ denotes the space of boundary 2-vectors, a linear (possibly non-integrable) complex structure admits a taming linear symplectic form if and only if it admits a compatible linear symplectic form, [LT12, Theorem 2.5]; note that 4-dimensional unimodular Lie algebras satisfy the assumption $B \wedge B = 0$. Finally, given a linear (possibly non-integrable) complex structure on a 4-dimensional Lie algebra, they studied the convex cones composed of the classes of J-taming, respectively J-compatible, linear symplectic forms, comparing them by means of $H^{(2,0),(0,2)}_J(\mathfrak{g};\mathbb{R})$, [LT12, Theorem 3.10]: this result is the linear counterpart of [LZ09, Theorem 1.1].

2.2.3 Complex-$\mathcal{C}^\infty$-pure-and-fullness for 4-dimensional manifolds

By [DLZ10, Lemma 2.15, Theorem 2.16], or [LZ09, Proposition 2.1], every compact complex surface is complex-$\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage; on the other hand, a compact complex surface is complex-$\mathcal{C}^\infty$-pure-and-full at the 1^{st} stage if and only if its first Betti number b_1 is even, that is, if and only if it admits a Kähler structure, see [Kod64, Miy74, Siu83], or [Lam99, Corollaire 5.7], or [Buc99, Theorem 11].

One may wonder about the relations between being complex-$\mathcal{C}^\infty$-pure-and-full and being integrable for an almost-complex structure on a compact 4-dimensional manifold; this is the matter of the following result, [AT12a, Proposition 1.7].

Proposition 2.22. *There exist*

- *non-complex-$\mathcal{C}^\infty$-pure-and-full at the 1^{st} stage non-integrable almost-complex structures, and*
- *complex-$\mathcal{C}^\infty$-pure-and-full at the 1^{st} stage non-integrable almost-complex structures*

on compact 4-dimensional manifolds with b_1 even.

Proof. The proof follows from the following examples, [AT12a, Example 1.5, Example 1.6].

`Step 1` – *There exists a non-complex-$\mathcal{C}^\infty$-pure-and-full at the 1^{st} stage non-integrable almost-complex structure on a 4-dimensional manifold.* Consider the standard Kähler structure $(J_0,\, \omega_0)$ on the 4-dimensional torus $\mathbb{T}^4$ with coordinates $\left\{x^j\right\}_{j\in\{1,\ldots,4\}}$, that is,

$$J_0 \;:=\; \left(\begin{array}{cc|cc} & & -1 & \\ & & & -1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right) \;\in\; \mathrm{End}\left(\mathbb{T}^4\right) \qquad \text{and} \qquad \omega_0 \;:=\; \mathrm{d}\,x^1\wedge \mathrm{d}\,x^3 + \mathrm{d}\,x^2 \wedge \mathrm{d}\,x^4 \;\in\; \wedge^2\mathbb{T}^4 \;,$$

and, for $\varepsilon > 0$ small enough, let $\{J_t\}_{t\in(-\varepsilon,\,\varepsilon)}$ be the curve of almost-complex structures defined by

$$J_t \;=:\; J_{t,\,\ell} \;:=\; (\mathrm{id} - t\,L)\; J_0\; (\mathrm{id} - t\,L)^{-1} \;=\; \left(\begin{array}{cc|cc} & & -\frac{1-t\,\ell}{1+t\,\ell} & \\ & & & -1 \\ \hline \frac{1+t\,\ell}{1-t\,\ell} & & & \\ & 1 & & \end{array}\right) \;\in\; \mathrm{End}\left(\mathbb{T}^4\right) \;,$$

where

$$L \;=\; \left(\begin{array}{cc|cc} \ell & & & \\ & 0 & & \\ \hline & & -\ell & \\ & & & 0 \end{array}\right) \;\in\; \mathrm{End}\left(\mathbb{T}^4\right)$$

and $\ell = \ell(x_2) \in \mathcal{C}^\infty(\mathbb{R}^4;\, \mathbb{R})$ is a $\mathbb{Z}^4$-periodic non-constant function.

For $t \in (-\varepsilon,\, \varepsilon) \setminus \{0\}$, a straightforward computation yields

$$H^{(1,0)}_{J_t}\left(\mathbb{T}^2_{\mathbb{C}};\mathbb{C}\right) \;=\; \mathbb{C}\left\langle \mathrm{d}\,x^2 + \mathrm{i}\,\mathrm{d}\,x^4\right\rangle \;, \qquad H^{(0,1)}_{J_t}\left(\mathbb{T}^2_{\mathbb{C}};\mathbb{C}\right) \;=\; \mathbb{C}\left\langle \mathrm{d}\,x^2 - \mathrm{i}\,\mathrm{d}\,x^4\right\rangle$$

therefore

$$\dim_{\mathbb{C}} H^{(1,0)}_{J_t}\left(\mathbb{T}^2_{\mathbb{C}};\mathbb{C}\right) + \dim_{\mathbb{C}} H^{(0,1)}_{J_t}\left(\mathbb{T}^2_{\mathbb{C}};\mathbb{C}\right) \;=\; 2 \;<\; 4 \;=\; b_1\left(\mathbb{T}^2_{\mathbb{C}}\right) \;,$$

that is, J_t is not complex-$\mathcal{C}^\infty$-pure-and-full at the 1^{st} stage.

`Step 2` – *There exists a complex-$\mathcal{C}^\infty$-pure-and-full at the 1^{st} stage non-integrable almost-complex structure on a 4-dimensional manifold.* Consider a compact 4-dimensional nilmanifold $X = \Gamma\backslash G$, quotient of the simply-connected nilpotent Lie group G whose associated Lie algebra is

$$\mathfrak{g} \;:=\; \left(0^2,\; 14,\; 12\right) \;;$$

let J be the G-left-invariant almost-complex structure defined by

$$Je^1 \;:=\; -e^2 \;, \qquad Je^3 \;:=\; -e^4 \;;$$

note that J is not integrable, since $\mathrm{Nij}(e_1, e_3) \neq 0$, where $\{e_i\}_{i\in\{1,2,3,4\}}$ is the dual basis of $\left\{e^i\right\}_{i\in\{1,2,3,4\}}$. In fact, X has no integrable almost-complex structure: indeed, since $b_1(X) = 2$ is even, if there were a complex structure on X, then X should carry a Kähler metric; this is not possible for compact non-tori nilmanifolds, by [Has89, Theorem 1, Corollary], or [BG88, Theorem A].

By K. Nomizu's theorem [Nom54, Theorem 1], one computes

$$H^1_{dR}(X;\mathbb{C}) \;=\; \mathbb{C}\left\langle \varphi^1,\, \bar\varphi^1\right\rangle \qquad \text{and} \qquad H^2_{dR}(X;\mathbb{C}) \;=\; \mathbb{C}\left\langle \varphi^{12} + \varphi^{\bar1\bar2},\, \varphi^{1\bar2} - \varphi^{2\bar1}\right\rangle \;;$$

in particular, it follows that J is complex-$\mathcal{C}^\infty$-pure-and-full at the 1^{st} stage. Note that J is not complex-$\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage but just $\mathcal{C}^\infty$-pure-and-full: indeed, using Proposition 2.19, one can prove that the class $\left[\varphi^{12} + \varphi^{\bar1\bar2}\right]$ admits no pure type representative with respect to J. Moreover, observe that the G-left-invariant almost-complex structure

$$J'e^1 \;:=\; -e^3 \;, \qquad J'e^2 \;:=\; -e^4 \;,$$

is complex-$\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage and non-complex-$\mathcal{C}^\infty$-pure-and-full at the 1^{st} stage (obviously, in this case, $h^-_{J'} = 0$, according to [DLZ10, Corollary 2.14]). □

Remark 2.23. T. Drăghici, T.-J. Li, and W. Zhang proved in [DLZ10, Corollary 2.14] that an almost-complex structure on a compact 4-dimensional manifold X is complex-$\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage if and only if J is integrable or $\dim_\mathbb{R} H^-_J(X) = 0$.

2.2.4 Almost-complex manifolds with large anti-invariant cohomology

Given an almost-complex structure J on a compact manifold X, it is natural to ask how large the cohomology subgroup $H^-_J(X)$ can be.

In [DLZ11, Theorem 1.1], T. Drăghici, T.-J. Li, and W. Zhang, starting with a compact complex surface X endowed with the complex structure J, proved that the dimension $h^-_{\tilde J} := \dim_\mathbb{R} H^-_{\tilde J}(X)$ of the $\tilde J$-anti-invariant subgroup $H^-_{\tilde J}(X)$ of $H^2_{dR}(X;\mathbb{R})$ associated to any *metric related* almost-complex structures $\tilde J$ on X (that is, the almost-complex structures $\tilde J$ on X inducing the same orientation as J and with a common compatible metric with J), such that $\tilde J \neq \pm J$, satisfies $h^-_{\tilde J} \in \{0,\, 1,\, 2\}$, and they provided a description of such almost-complex structures $\tilde J$ having $h^-_{\tilde J} \in \{1,\, 2\}$.

In this direction, T. Drăghici, T.-J. Li, and W. Zhang proposed the following conjecture.

Conjecture 2.24 ([DLZ11, Conjecture 2.5])**.** *On a compact 4-dimensional manifold endowed with an almost-complex structure J, if $\dim_\mathbb{R} H^-_J(X) \geq 3$, then J is integrable.*

In [TWZZ11], Q. Tan, H. Wang, Y. Zhang, and P. Zhu proved that, on a compact 4-dimensional manifold endowed with an almost-complex structure J and a J-Hermitian metric g, the dimension $\dim_\mathbb{R} H^-_{\tilde J}(X)$ is constant for all almost-complex structures $\tilde J$ being *fundamental form related* to J, namely, such that $\omega \in \wedge^{1,1}_{\tilde J} X \cap \wedge^2 X$, where $\omega := g\left(J\,\cdot,\, \cdot\cdot\right) \in \wedge^{1,1}_J X \cap \wedge^2 X$ is the $(1,1)$-form with respect to J associated to the J-Hermitian metric g, [TWZZ11, Theorem 1.2]. Then, they proposed to modify [DLZ11, Conjecture 2.5] as follows.

Conjecture 2.25 ([TWZZ11, Question 1.5])**.** *Let X be a compact 4-dimensional manifold endowed with an almost-complex structure J and a J-Hermitian metric g, and denote by $\omega := g\left(J\,\cdot,\, \cdot\cdot\right)$ the $(1,1)$-form associated to g. Suppose that $\dim_\mathbb{R} H^-_J(X) \geq 3$. Does there exist an integrable almost-complex structure $\tilde J$ such that $\omega \in \wedge^{1,1}_{\tilde J} X$?*

Furthermore, in [DLZ11], it was conjectured that $h^-_J = 0$ for a generic almost-complex structure J on a compact 4-dimensional manifold, [DLZ11, Conjecture 2.4]. In [TWZZ11, Theorem 1.1], Q. Tan, H. Wang, Y. Zhang, and P. Zhu proved that this holds true, showing that, on a compact 4-dimensional manifold X admitting almost-complex structures, the set of almost-complex structures J on X with $\dim_{\mathbb{R}} H^-_J(X) = 0$ is an open dense subset of the set of almost-complex structures on X.

In [ATZ12, §5], a 1-parameter family $\{J_t\}_{t\in(-\varepsilon,\,\varepsilon)}$ of (non-integrable) almost-complex structures on the 6-dimensional torus $\mathbb{T}^6$, where $\varepsilon > 0$ is small enough, having $\dim_{\mathbb{R}} H^-_{J_t}\left(\mathbb{T}^6\right)$ greater than 3 has been provided. We recall here the construction, see also [AT11, §4].

Example 2.26. *A family of almost-complex structures on the* 6*-dimensional torus with anti-invariant cohomology of dimension larger than* 3*.*
Consider the 6-dimensional torus $\mathbb{T}^6$, with coordinates $\left\{x^j\right\}_{j\in\{1,\dots,6\}}$. For $\varepsilon > 0$ small enough, choose a function $\alpha\colon (-\varepsilon,\,\varepsilon)\times\mathbb{T}^6 \to \mathbb{R}$ such that $\alpha_t :=: \alpha(t,\cdot) \in \mathcal{C}^\infty\left(\mathbb{T}^6\right)$ depends just on x^3 for any $t \in (-\varepsilon,\,\varepsilon)$, namely $\alpha_t = \alpha_t(x^3)$, and that $\alpha_0(x^3) = 1$. Define the almost-complex structure J_t in such a way that

$$\begin{cases} \varphi^1_t \;:=\; \mathrm{d}\,x^1 \,+\, \mathrm{i}\,\alpha_t\,\mathrm{d}\,x^4 \\ \varphi^2_t \;:=\; \mathrm{d}\,x^2 \,+\, \mathrm{i}\,\mathrm{d}\,x^5 \\ \varphi^3_t \;:=\; \mathrm{d}\,x^3 \,+\, \mathrm{i}\,\mathrm{d}\,x^6 \end{cases}$$

provides a co-frame for the $\mathcal{C}^\infty\left(\mathbb{T}^6;\mathbb{C}\right)$-module of $(1,0)$-forms on $\mathbb{T}^6$ with respect to J_t. In terms of this co-frame, the structure equations are

$$\begin{cases} \mathrm{d}\,\varphi^1_t \;=\; \mathrm{i}\,\mathrm{d}\,\alpha_t \wedge \mathrm{d}\,x^4 \\ \mathrm{d}\,\varphi^2_t \;=\; 0 \\ \mathrm{d}\,\varphi^3_t \;=\; 0 \end{cases} .$$

Straightforward computations give that the J_t-anti-invariant real closed 2-forms are of the type

$$\psi \;=\; \frac{C}{\alpha_t}\left(\mathrm{d}\,x^{13} - \alpha_t\,\mathrm{d}\,x^{46}\right) + D\left(\mathrm{d}\,x^{16} - \alpha_t\,\mathrm{d}\,x^{34}\right) + E\left(\mathrm{d}\,x^{23} - \mathrm{d}\,x^{56}\right) + F\left(\mathrm{d}\,x^{26} - \mathrm{d}\,x^{35}\right) ,$$

where $C,\ D,\ E,\ F \in \mathbb{R}$ (we shorten $\mathrm{d}\,x^{jk} := \mathrm{d}\,x^j \wedge \mathrm{d}\,x^k$). Moreover, the forms $\mathrm{d}\,x^{23} - \mathrm{d}\,x^{56}$ and $\mathrm{d}\,x^{26} - \mathrm{d}\,x^{35}$ are clearly harmonic with respect to the standard Riemannian metric $\sum_{j=1}^{6}\mathrm{d}\,x^j \otimes \mathrm{d}\,x^j$, while the classes of $\mathrm{d}\,x^{16} - \alpha_t\,\mathrm{d}\,x^{34}$ and $\mathrm{d}\,x^{13} - \alpha_t\,\mathrm{d}\,x^{46}$ are non-zero, being their harmonic parts non-zero. Therefore, we get that

$$h^-_{J_t} \;=\; 4 \qquad \text{for small } t \neq 0 ,$$

while $h^-_{J_0} = 6$.

The natural generalization of [DLZ10, Conjecture 2.5] to higher dimensional manifolds yields the following question, [ATZ12, Question 5.2].

Question 2.27. Are there compact $2n$-dimensional manifolds X endowed with non-integrable almost-complex structures J with $\dim_{\mathbb{R}} H^-_J(X) > n\,(n-1)$?

Note that, when $X = \Gamma\backslash G$ is a $2n$-dimensional completely-solvable solvmanifold endowed with a G-left-invariant almost-complex structure J, then, by Proposition 2.19, it follows that

$$\dim_{\mathbb{R}} H^-_J(X) \;\leq\; n\,(n-1) \qquad \text{and} \qquad \dim_{\mathbb{R}} H^+_J(X) \;\leq\; n^2 .$$

2.2.5 Semi-Kähler manifolds

As already recalled, A. Fino and A. Tomassini's [FT10, Theorem 4.1] proves that, given an almost-Kähler structure on a compact manifold, if the almost-complex structure is $\mathcal{C}^\infty$-pure-and-full and the symplectic structure satisfies the Hard Lefschetz Condition, then the almost-complex structure is pure-and-full too; moreover, by [FT10, Proposition 3.2], see also [DLZ10, Proposition 2.8], the almost-complex structure of every almost-Kähler structure on a compact manifold is $\mathcal{C}^\infty$-pure.

To study the cohomology of balanced manifolds X and the duality between $H^{(\bullet,\bullet)}_J(X;\mathbb{C})$ and $H^J_{(\bullet,\bullet)}(X;\mathbb{C})$, we get the following result, [AT12a, Proposition 3.1], which can be considered as the semi-Kähler counterpart of [FT10, Theorem 4.1].

Proposition 2.28. *Let X be a compact $2n$-dimensional manifold endowed with an almost-complex structure J and a semi-Kähler form ω. Suppose that $\left[\omega^{n-1}\right] \smile \cdot\colon H^1_{dR}(X;\mathbb{R}) \to H^{2n-1}_{dR}(X;\mathbb{R})$ is an isomorphism. If J is complex-$\mathcal{C}^\infty$-pure-and-full at the 1^{st} stage, then it is also complex-pure-and-full at the 1^{st} stage, and*

$$H^{(1,0)}_J(X;\mathbb{C}) \simeq H^J_{(0,1)}(X;\mathbb{C}) .$$

Proof. Firstly, note that J is complex-pure at the 1st stage. Indeed, if

$$\mathfrak{a} \in H^J_{(1,0)}(X;\mathbb{C}) \cap H^J_{(0,1)}(X;\mathbb{C}) ,$$

then

$$\mathfrak{a}\lfloor_{H_J^{(1,0)}(X;\mathbb{C})} = 0 = \mathfrak{a}\lfloor_{H_J^{(0,1)}(X;\mathbb{C})} .$$

Therefore, by the assumption

$$H^1_{dR}(X;\mathbb{C}) = H_J^{(1,0)}(X;\mathbb{C}) \oplus H_J^{(0,1)}(X;\mathbb{C}) ,$$

we get that

$$\mathfrak{a} = 0 .$$

Now, note that, since

$$\left[\omega^{n-1}\right] \smile H_J^{(1,0)}(X;\mathbb{C}) \subseteq H_J^{(n,n-1)}(X;\mathbb{C}) \qquad \text{and} \qquad \left[\omega^{n-1}\right] \smile H_J^{(0,1)}(X;\mathbb{C}) \subseteq H_J^{(n-1,n)}(X;\mathbb{C}) ,$$

the isomorphism

$$H^1_{dR}(X;\mathbb{C}) \stackrel{[\omega^{n-1}]\smile\cdot}{\longrightarrow} H^{2n-1}_{dR}(X;\mathbb{C}) \stackrel{T\cdot}{\longrightarrow} H^1_{dR}(X;\mathbb{C})$$

yields the injective maps

$$H_J^{(1,0)}(X;\mathbb{C}) \hookrightarrow H^J_{(0,1)}(X;\mathbb{C}) \qquad \text{and} \qquad H_J^{(0,1)}(X;\mathbb{C}) \hookrightarrow H^J_{(1,0)}(X;\mathbb{C}) .$$

Since, by hypothesis, J is complex-$\mathcal{C}^\infty$-pure-and-full at the 1st stage, namely, $H^1_{dR}(X;\mathbb{C}) = H_J^{(1,0)}(X;\mathbb{C}) \oplus H_J^{(0,1)}(X;\mathbb{C})$, we get the proof. □

We provide here some explicit examples, [AT12a, Example 3.2, Example 3.3], checking the validity of the hypothesis of $\left[\omega^{n-1}\right] \smile \cdot : H^1_{dR}(X;\mathbb{R}) \to H^{2n-1}_{dR}(X;\mathbb{R})$ being an isomorphism in Proposition 2.28.

Example 2.29. *A balanced structure on the Iwasawa manifold.*
On the Iwasawa manifold $\mathbb{I}_3$ (see §1.4.1), consider the balanced structure

$$\omega := \frac{\mathrm{i}}{2}\left(\varphi^1 \wedge \bar\varphi^1 + \varphi^2 \wedge \bar\varphi^2 + \varphi^3 \wedge \bar\varphi^3\right) .$$

Since

$$H^1_{dR}(\mathbb{I}_3;\mathbb{C}) = \mathbb{C}\left\langle \varphi^1,\, \varphi^2,\, \bar\varphi^1,\, \bar\varphi^2\right\rangle \qquad \text{and} \qquad H^5_{dR}(\mathbb{I}_3;\mathbb{C}) = \mathbb{C}\left\langle \varphi^{123\bar1\bar3},\, \varphi^{123\bar2\bar3},\, \varphi^{13\bar1\bar2\bar3},\, \varphi^{23\bar1\bar2\bar3}\right\rangle ,$$

it is straightforward to check that

$$\left[\omega^2\right] \smile \cdot : H^1_{dR}(\mathbb{I}_3;\mathbb{C}) \to H^5_{dR}(\mathbb{I}_3;\mathbb{C})$$

is an isomorphism. Therefore, by Proposition 2.28, $\mathbb{I}_3$ is complex-$\mathcal{C}^\infty$-pure-and-full at the 1st stage and complex-pure-and-full at the 1st stage (the same result follows also arguing as in [FT10, Theorem 3.7], the above harmonic representatives of $H^1_{dR}(\mathbb{I}_3;\mathbb{C})$, with respect to the Hermitian metric $\sum_{j=1}^3 \varphi^j \odot \bar\varphi^j$, being of pure type with respect to the complex structure).

Example 2.30. *A 6-dimensional manifold endowed with a semi-Kähler structure not inducing an isomorphism in cohomology.*
Consider the 6-dimensional nilmanifold

$$X = \Gamma\backslash G := \left(0^4,\ 12,\ 13\right) .$$

In [FT10, Example 3.3], the almost-complex structure

$$J'\, e^1 := -e^2 , \qquad J'\, e^3 := -e^4 , \qquad J'\, e^5 := -e^6$$

is provided as a first example of non-$\mathcal{C}^\infty$-pure almost-complex structure. Note that J' is not even $\mathcal{C}^\infty$-full: indeed, the cohomology class $\left[e^{15}+e^{16}\right]$ admits neither J'-invariant nor J'-anti-invariant G-left-invariant representatives, and hence, by Proposition 2.19, it admits neither J'-invariant nor J'-anti-invariant representatives.

Consider now the almost-complex structure

$$J\,e^1 \;:=\; -e^5\;, \qquad J\,e^2 \;:=\; -e^3\;, \qquad J\,e^4 \;:=\; -e^6$$

and the non-degenerate J-invariant 2-form

$$\omega \;:=\; e^{15}+e^{23}+e^{46}\;.$$

A straightforward computation shows that

$$\mathrm{d}\,\omega \;=\; -e^{134} \;\neq\; 0 \quad \text{and} \quad \mathrm{d}\,\omega^2 \;=\; \mathrm{d}\left(e^{1235}-e^{1456}+e^{2346}\right) \;=\; 0\;.$$

By K. Nomizu's theorem [Nom54, Theorem 1], it is straightforward to compute

$$H^1_{dR}(X;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^1,\, e^2,\, e^3,\, e^4\right\rangle\;.$$

Since

$$\omega^2\,e^1 \;=\; e^{12346} \;=\; \mathrm{d}\,e^{3456}\;,$$

we get that $\left[\omega^2\right] \smile \cdot \colon H^1_{dR}(X;\mathbb{R}) \to H^5_{dR}(X;\mathbb{R})$ is not injective.

We give two explicit examples of $2n$-dimensional complex manifolds endowed with a balanced structure, with $2n=10$, respectively $2n=6$, such that the $(n-1)^{\text{th}}$ power of the associated $(1,1)$-form induces an isomorphism in cohomology, and admitting small balanced deformations, [AT12a, Example 3.4, Example 3.5].

Example 2.31. *A curve of balanced structures on $\eta\beta_5$ inducing an isomorphism in cohomology.*
We recall the construction of the 10-dimensional nilmanifold $\eta\beta_5$, introduced and studied in [AB90] by L. Alessandrini and G. Bassanelli to prove that being p-Kähler is not a stable property under small deformations of the complex structure; more in general, in [AB91], the manifold $\eta\beta_{2n+1}$, for any $n\in\mathbb{N}\setminus\{0\}$, has been provided as a generalization of the Iwasawa manifold $\mathbb{I}_3$, and the existence of p-Kähler structures on $\eta\beta_{2n+1}$ has been investigated. (For definitions and results concerning p-Kähler structures, see [AB91], or, e.g., [Sil96, Ale11].)

For $n\in\mathbb{N}\setminus\{0\}$, consider the complex Lie group

$$G_{2n+1} \;:=\; \left\{ A\in \mathrm{GL}(n+2;\mathbb{C}) \;:\; A \;=\; \left(\begin{array}{c|ccc|c} 1 & x^1 & \cdots & x^n & z \\ \hline 0 & 1 & & & y^1 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & y^n \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right) \right\}\;;$$

equivalently, one can identify G_{2n+1} with $\left(\mathbb{C}^{2n+1},\, *\right)$, where the group structure $*$ is defined as

$$\begin{aligned}&\left(x^1,\,\ldots,\,x^n,\,y^1,\,\ldots,\,y^n,\,z\right) \,*\, \left(u^1,\,\ldots,\,u^n,\,v^1,\,\ldots,\,v^n,\,w\right)\\ &\quad:=\; \left(x^1+u^1,\,\ldots,\,x^n+u^n,\,y^1+v^1,\,\ldots,\,y^n+v^n,\,z+w+x^1\cdot v^1+\cdots+x^n\cdot v^n\right)\;.\end{aligned}$$

Since the subgroup

$$\Gamma_{2n+1} \;:=\; G_{2n+1}\cap \mathrm{GL}\left(n+2;\mathbb{Z}\left[\mathrm{i}\right]\right) \;\subset\; G_{2n+1}$$

is a discrete co-compact subgroup of the nilpotent Lie group G_{2n+1}, one gets a compact complex manifold, of complex dimension $2n+1$,

$$\eta\beta_{2n+1} \;:=\; \Gamma_{2n+1}\backslash G_{2n+1}\;,$$

which is a holomorphically parallelizable nilmanifold and admits no Kähler metric, [Wan54, Corollary 2], or [BG88, Theorem A], or [Has89, Theorem 1, Corollary]; note that $\eta\beta_3=\mathbb{I}_3$ is the Iwasawa manifold (see §1.4.1). In fact, one has that $\eta\beta_{2n+1}$ is not p-Kähler for $1<p\leq n$ and it is p-Kähler for $n+1\leq p\leq 2n+1$, [AB91, Theorem 4.2]; furthermore, $\eta\beta_{2n+1}$ has complex submanifolds of any complex dimension less than or equal to $2n+1$, and hence it follows that the p-Kähler forms on $\eta\beta_{2n+1}$ can never be exact, [AB91, §4.4].

Setting

$$\left\{\begin{array}{lcll} \varphi^{2j-1} & := & \mathrm{d}\,x^j\;, & \text{for} \quad j\in\{1,\ldots,n\}\;,\\ \varphi^{2j} & := & \mathrm{d}\,y^j\;, & \text{for} \quad j\in\{1,\ldots,n\}\;,\\ \varphi^{2n+1} & := & \mathrm{d}\,z-\sum_{j=1}^{n}x^j\,\mathrm{d}\,y^j\;, & \end{array}\right.$$

one gets the global co-frame $\left\{\varphi^j\right\}_{j\in\{1,\ldots,2n+1\}}$ for the space of holomorphic 1-forms, with respect to which the structure equations are

$$\begin{cases} \mathrm{d}\,\varphi^1 \;=\; \cdots \;=\; \mathrm{d}\,\varphi^{2n} \;=\; 0 \\ \mathrm{d}\,\varphi^{2n+1} \;=\; -\sum_{j=1}^{n} \varphi^{2j-1}\wedge\varphi^{2j} \end{cases} .$$

Now, take $2n+1=5$. With respect to the co-frame $\left\{\varphi^j\right\}_{j\in\{1,\ldots,5\}}$ for the space of holomorphic 1-forms on $\eta\beta_5$, the structure equations are

$$\begin{cases} \mathrm{d}\,\varphi^1 \;=\; \mathrm{d}\,\varphi^2 \;=\; \mathrm{d}\,\varphi^3 \;=\; \mathrm{d}\,\varphi^4 \;=\; 0 \\ \mathrm{d}\,\varphi^5 \;=\; -\varphi^{12}-\varphi^{34} \end{cases}$$

(where, as usually, we shorten, e.g., $\varphi^{12} := \varphi^1\wedge\varphi^2$).

Consider on $\eta\beta_5$ the balanced structure

$$\omega \;:=\; \frac{\mathrm{i}}{2}\sum_{j=1}^{5}\varphi^j\wedge\bar\varphi^j \;.$$

By K. Nomizu's theorem [Nom54, Theorem 1], it is straightforward to compute

$$H^1_{dR}\left(\eta\beta_5;\mathbb{C}\right) \;=\; \mathbb{C}\left\langle \varphi^1,\, \varphi^2,\, \varphi^3,\, \varphi^4,\, \bar\varphi^1,\, \bar\varphi^2,\, \bar\varphi^3,\, \bar\varphi^4 \right\rangle$$

and

$$\begin{aligned} H^9_{dR}\left(\eta\beta_5;\mathbb{C}\right) \;&=\; \mathbb{C}\left\langle \varphi^{12345\bar2\bar3\bar4\bar5},\, \varphi^{12345\bar1\bar3\bar4\bar5},\, \varphi^{12345\bar1\bar2\bar4\bar5},\, \varphi^{12345\bar1\bar2\bar3\bar5},\right. \\ &\qquad \left. \varphi^{2345\bar1\bar2\bar3\bar4\bar5},\, \varphi^{1345\bar1\bar2\bar3\bar4\bar5},\, \varphi^{1245\bar1\bar2\bar3\bar4\bar5},\, \varphi^{1235\bar1\bar2\bar3\bar4\bar5} \right\rangle \;; \end{aligned}$$

therefore, $\eta\beta_5$ is complex-$\mathcal{C}^\infty$-pure-and-full at the 1st stage and

$$\left[\omega^4\right] \smile \cdot \colon H^1_{dR}\left(\eta\beta_5;\mathbb{R}\right) \to H^9_{dR}\left(\eta\beta_5;\mathbb{R}\right)$$

is an isomorphism, and so $\eta\beta_5$ is also complex-pure-and-full at the 1st stage by Proposition 2.28 (note that, the above pure type representatives being harmonic with respect to the metric $\sum_{j=1}^{5}\varphi^j\odot\bar\varphi^j$, the same result follows also arguing as in [FT10, Theorem 3.7]).

Now, let $\left\{J_t\right\}_{t\in\Delta(0,\varepsilon)\subset\mathbb{C}}$, where $\varepsilon>0$ is small enough, be a family of small deformations of the complex structure such that

$$\begin{cases} \varphi^1_t \;:=\; \varphi^1 + t\,\bar\varphi^1 \\ \varphi^2_t \;:=\; \varphi^2 \\ \varphi^3_t \;:=\; \varphi^3 \\ \varphi^4_t \;:=\; \varphi^4 \\ \varphi^5_t \;:=\; \varphi^5 \end{cases}$$

is a co-frame for the J_t-holomorphic cotangent bundle. With respect to $\left\{\varphi^j_t\right\}_{j\in\{1,\ldots,5\}}$, the structure equations are written as

$$\begin{cases} \mathrm{d}\,\varphi^1_t \;=\; \mathrm{d}\,\varphi^2_t \;=\; \mathrm{d}\,\varphi^3_t \;=\; \mathrm{d}\,\varphi^4_t \;=\; 0 \\ \mathrm{d}\,\varphi^5_t \;=\; -\frac{1}{1-|t|^2}\,\varphi^{12}_t - \varphi^{34}_t - \frac{t}{1-|t|^2}\,\varphi^{2\bar1}_t \end{cases} .$$

Setting, for $t\in\Delta\left(0,\varepsilon\right)\subset\mathbb{C}$,

$$\omega_t \;:=\; \frac{\mathrm{i}}{2}\sum_{j=1}^{5}\varphi^j_t\wedge\overline{\varphi}^j_t \;,$$

one gets a curve of balanced structures $\left\{\left(J_t,\,\omega_t\right)\right\}_{t\in\Delta(0,\varepsilon)}$ on the smooth manifold underlying $\eta\beta_5$. Furthermore, for every $t\in\Delta\left(0,\varepsilon\right)$, the complex structure J_t is complex-$\mathcal{C}^\infty$-pure-and-full at the 1st stage and

$$\left[\omega^4_t\right] \smile \cdot \colon H^1_{dR}\left(\eta\beta_5;\mathbb{R}\right) \to H^9_{dR}\left(\eta\beta_5;\mathbb{R}\right)$$

is an isomorphism. Therefore, according to Proposition 2.28, it follows that, for every $t\in\Delta\left(0,\varepsilon\right)$, the complex structure J_t is complex-pure-and-full at the 1st stage, and that $H^{(1,0)}_{J_t}\left(\eta\beta_5;\mathbb{C}\right)\simeq H^{J_t}_{(0,1)}\left(\eta\beta_5;\mathbb{C}\right)$.

Example 2.32. *A curve of semi-Kähler structures on a* 6*-dimensional completely-solvable solvmanifold inducing an isomorphism in cohomology.*
Consider a completely-solvable solvmanifold

$$X \;=\; \Gamma\backslash G \;:=\; (0,\; -12,\; 34,\; 0,\; 15,\; 46)$$

endowed with the almost-complex structure J_0 whose holomorphic cotangent bundle has co-frame generated by

$$\begin{cases} \varphi^1 &:= \; e^1 + \mathrm{i}\, e^4 \\ \varphi^2 &:= \; e^2 + \mathrm{i}\, e^5 \\ \varphi^3 &:= \; e^3 + \mathrm{i}\, e^6 \end{cases}$$

and with the J_0-compatible symplectic form

$$\omega_0 \;:=\; e^{14} + e^{25} + e^{36}$$

(see also [FT10, §6.3]). The structure equations with respect to $\left\{\varphi^1,\, \varphi^2,\, \varphi^3\right\}$ are

$$\begin{cases} \mathrm{d}\,\varphi^1 &= \; 0 \\ 2\,\mathrm{d}\,\varphi^2 &= \; -\varphi^{1\bar{2}} - \varphi^{\bar{1}\bar{2}} \\ 2\,\mathrm{i}\,\mathrm{d}\,\varphi^3 &= \; -\varphi^{1\bar{3}} + \varphi^{\bar{1}\bar{3}} \end{cases} \;;$$

using A. Hattori's theorem [Hat60, Corollary 4.2], one computes

$$\begin{aligned} H^1_{dR}(X;\mathbb{R}) \;&=\; \mathbb{R}\left\langle e^1,\, e^4\right\rangle \;, \\ H^5_{dR}(X;\mathbb{R}) \;&=\; \mathbb{R}\left\langle *_{g_0} e^1,\, *_{g_0} e^4\right\rangle \;=\; \mathbb{R}\left\langle e^{23456},\, e^{12356}\right\rangle \;, \end{aligned}$$

where g_0 is the J_0-Hermitian metric induced by $(J_0,\, \omega_0)$.

Now, consider the curve $\{J_t\}_{t\in(-\varepsilon,\,\varepsilon)\subset\mathbb{R}}$ of almost-complex structures on X, where $\varepsilon > 0$ is small enough and J_t is defined requiring that the J_t-holomorphic cotangent bundle is generated by

$$\begin{cases} \varphi^1_t &:= \; \varphi^1 \\ \varphi^2_t &:= \; \varphi^2 + \mathrm{i}\; t\, e^6 \\ \varphi^3_t &:= \; \varphi^3 \end{cases} \;;$$

for every $t \in (-\varepsilon,\, \varepsilon)$, consider also the non-degenerate J_t-compatible 2-form

$$\omega_t \;:=\; e^{14} + e^{25} + e^{36} + t\, e^{26} \;;$$

for $t \neq 0$, one has that $\mathrm{d}\,\omega \neq 0$, but

$$\mathrm{d}\,\omega_t^2 \;=\; \mathrm{d}\left(\omega_0^2 - t\, e^{1246}\right) \;=\; 0 \;,$$

hence $\{(J_t,\, \omega_t)\}_{t\in(-\varepsilon,\,\varepsilon)}$ gives rise to a curve of semi-Kähler structures on X. Moreover, note that

$$\omega_t^2 \wedge e^1 \;=\; e^{12356} \;, \qquad \omega_t^2 \wedge e^4 \;=\; e^{23456} \;,$$

therefore $\left[\omega_t^2\right] \smile \cdot \colon H^1_{dR}(X;\mathbb{R}) \to H^5_{dR}(X;\mathbb{R})$ is an isomorphism, for every $t \in (-\varepsilon,\, \varepsilon)$.

2.2.6 Almost-Kähler manifolds and Lefschetz-type property

Recall that every compact manifold X endowed with a Kähler structure $(J,\, \omega)$ is $\mathcal{C}^\infty$-pure-and-full, in fact, complex-$\mathcal{C}^\infty$-pure-and-full at every stage, [DLZ10, Lemma 2.15, Theorem 2.16], or [LZ09, Proposition 2.1]. A natural question is whether or not the same holds true even for almost-Kähler structures, namely, without the integrability assumption on J.

In this section, we study cohomological properties for almost-Kähler structures, in connection with a Lefschetz-type property, Theorem 2.35, and we describe some explicit examples.

The results in this section have been obtained in a joint work with A. Tomassini and W. Zhang, [ATZ12].

Let X be a compact $2n$-dimensional manifold endowed with an *almost-Kähler structure* $(J,\, \omega,\, g)$, that is, J is an almost-complex structure on X and g is a J-Hermitian metric whose associated $(1,1)$-form $\omega := g\,(J\,\cdot,\, \cdot\cdot) \in \wedge^{1,1}X \cap \wedge^2 X$ is d-closed.

Firstly, we recall the following result on decomposition in cohomology for almost-Kähler manifolds, proven by T. Drăghici, T.-J. Li, and W. Zhang in [DLZ10] and, in a different way, by A. Fino and A. Tomassini in [FT10].

Proposition 2.33 ([DLZ10, Proposition 2.8], [FT10, Proposition 3.2]). *Let X be a compact manifold and let $(J,\, \omega,\, g)$ be an almost-Kähler structure on X. Then J is $\mathcal{C}^\infty$-pure.*

Hence, one is brought to study the $\mathcal{C}^\infty$-fullness of almost-Kähler structures.

Note that ω is in particular a symplectic form on X. We recall that, given a compact $2n$-dimensional manifold X endowed with a symplectic form ω, and fixed $k \in \mathbb{N}$, the Lefschetz-type operator on $(n-k)$-forms associated with ω is the operator

$$L^k \;=:\; L^k_\omega \colon\; \wedge^{n-k} X \to \wedge^{n+k} X\,, \qquad L^k(\alpha) \;:=\; \omega^k \wedge \alpha$$

(see §0.2 for notations concerning symplectic structures); since $\mathrm{d}\,\omega = 0$, the map $L^k \colon\; \wedge^{n-k} X \to \wedge^{n+k} X$ induces a map in cohomology, namely,

$$L^k \colon H^{n-k}_{dR}(X;\mathbb{R}) \to H^{n+k}_{dR}(X;\mathbb{R})\,, \qquad L^k\left(\mathfrak{a}\right) \;:=\; \left[\omega^k\right] \smile \mathfrak{a}\,,$$

Initially motivated by studying, in [Zha11], Taubes currents, which have been introduced by C. H. Taubes in [Tau11] in order to study S. K. Donaldson's "tamed to compatible" question, [Don06, Question 2], W. Zhang considered the following Lefschetz-type property, see also [DLZ12, §2.2].

Definition 2.34. Let X be a compact $2n$-dimensional manifold endowed with an almost-complex structure J and with a J-Hermitian metric g; denote by ω the $(1,1)$-form associated to g. One says that the *Lefschetz-type property (on 2-forms)* holds on X if

$$L^{n-2}_\omega \colon\; \wedge^2 X \to \wedge^{2n-2} X$$

takes g-harmonic 2-forms to g-harmonic $(2n-2)$-forms.

Since the map $L^k \colon\; \wedge^{n-k} X \to \wedge^{n+k} X$ is an isomorphism for every $k \in \mathbb{N}$, [Yan96, Corollary 2.7], it follows that the Lefschetz-type property on 2-forms is stronger than the Hard Lefschetz Condition on 2-classes, namely, the property that $[\omega]^{n-2} \smile \cdot \colon H^2_{dR}(X;\mathbb{R}) \to H^{2n-2}_{dR}(X;\mathbb{R})$ is an isomorphism.

In order to study the relation between the Lefschetz-type property on 2-forms and the $\mathcal{C}^\infty$-fullness, we prove here the following result, [ATZ12, Theorem 2.3], which states that the Lefschetz-type property on 2-forms is satisfied provided that the almost-Kähler structure admits a basis of pure type harmonic representatives for the second de Rham cohomology group. (Recall that A. Fino and A. Tomassini proved in [FT10, Theorem 3.7] that an almost-Kähler manifold admitting a basis of harmonic 2-forms of pure type with respect to the almost-complex structure is $\mathcal{C}^\infty$-pure-and-full and pure-and-full; they also described several examples of non-Kähler solvmanifolds satisfying the above assumption, [FT10, §5, §6].)

Theorem 2.35. *Let X be a compact manifold endowed with an almost-Kähler structure $(J,\, \omega,\, g)$. Suppose that there exists a basis of $H^2_{dR}(X;\mathbb{R})$ represented by g-harmonic 2-forms which are of pure type with respect to J. Then the Lefschetz-type property on 2-forms holds on X.*

Proof. We recall that, on a compact $2n$-dimensional symplectic manifold, using the symplectic form ω instead of a Riemannian metric and miming the Hodge theory for Riemannian manifolds, one can define a symplectic-$\star$-operator $\star_\omega \colon\; \wedge^\bullet X \to \wedge^{2n-\bullet} X$ such that $\alpha \wedge \star_\omega \beta = \left(\omega^{-1}\right)^k (\alpha, \beta)\, \frac{\omega^n}{n!}$ for every $\alpha,\, \beta \in \wedge^k X$, see [Bry88, §2]. (See §0.2 for further details on symplectic structures, and see §3.1.1 for definitions and results concerning the Hodge theory for symplectic manifolds.) In particular, on a compact manifold X endowed with an almost-Kähler structure $(J,\, \omega,\, g)$, the Hodge-$*$-operator $*_g$ and the symplectic-$\star$-operator $\star_\omega$ are related by

$$\star_\omega \;=\; *_g\, J\;,$$

see [Bry88, Theorem 2.4.1, Remark 2.4.4]. Therefore, for forms of pure type with respect to J, the properties of g-harmonic and of being ω-symplectically-harmonic (that is, both d-closed and d^Λ-closed, where d^Λ is the symplectic co-differential operator, which is defined, for every $k \in \mathbb{N}$, as $\mathrm{d}^\Lambda\lfloor_{\wedge^k X} := (-1)^{k+1}\, \star_\omega\, \mathrm{d}\, \star_\omega$) are equivalent. The statement follows noting that

$$[\mathrm{d},\, L] \;=\; 0 \qquad \text{and} \qquad \left[\mathrm{d}^\Lambda,\, L\right] \;=\; -\,\mathrm{d}\;,$$

see, e.g., [Yan96, Lemma 1.2]: hence L sends ω-symplectically-harmonic 2-forms (of pure type with respect to J) to ω-symplectically-harmonic $(2n-2)$-forms (of pure type with respect to J). □

Remark 2.36. Note that, if X is a compact $2n$-dimensional manifold endowed with an almost-Kähler structure $(J,\, \omega,\, g)$ satisfying the Lefschetz-type property on 2-forms and J is $\mathcal{C}^\infty$-full, then J is $\mathcal{C}^\infty$-pure-and-full and pure-and-full, too, [ATZ12, Remark 2.4].

Indeed, we have already noticed that J is $\mathcal{C}^\infty$-pure by [DLZ10, Proposition 2.8] or [FT10, Proposition 3.2]. Moreover, since J is $\mathcal{C}^\infty$-full, J is also pure by [LZ09, Proposition 2.5]. We recall now the argument in [FT10, Theorem 4.1] to prove that J is also full. Firstly, note that if the Lefschetz-type property on 2-forms holds, then $\left[\omega^{n-2}\right] \smile \cdot \colon H^2_{dR}\left(X;\mathbb{R}\right) \to H^{2n-2}_{dR}\left(X;\mathbb{R}\right)$ is an isomorphism. Therefore, we get that

$$H^{2n-2}_{dR}(X;\mathbb{R}) \;=\; H^{(n,n-2),(n-2,n)}_J(X;\mathbb{R}) + H^{(n-1,n-1)}_J(X;\mathbb{R}) \;;$$

indeed, (following the argument in [FT10, Theorem 4.1],) since $\left[\omega^{n-2}\right] \smile \cdot \colon H^2_{dR}(X;\mathbb{R}) \to H^{2n-2}_{dR}(X;\mathbb{R})$ is in particular surjective, we have

$$\begin{aligned} H^{2n-2}_{dR}(X;\mathbb{R}) &= \left[\omega^{n-2}\right] \smile H^2_{dR}(X;\mathbb{R}) \\ &= \left[\omega^{n-2}\right] \smile \left(H^{(2,0),(0,2)}_J(X;\mathbb{R}) \oplus H^{(1,1)}_J(X;\mathbb{R})\right) \\ &\subseteq H^{(n,n-2),(n-2,n)}_J(X;\mathbb{R}) + H^{(n-1,n-1)}_J(X;\mathbb{R}) \;, \end{aligned}$$

yielding the above decomposition of $H^{2n-2}_{dR}(X;\mathbb{R})$. Then, it follows that J is also full by Theorem 2.10.

We describe here some examples, from [ATZ12], of almost-Kähler manifolds, studying Lefschetz-type property and $\mathcal{C}^\infty$-fullness on them.

In the following example, we give a family of $\mathcal{C}^\infty$-full almost-Kähler manifolds satisfying the Lefschetz-type property on 2-forms, [ATZ12, §2.2].

Example 2.37. *A family of $\mathcal{C}^\infty$-full almost-Kähler manifolds satisfying the Lefschetz-type property on 2-forms.* Consider the 6-dimensional Lie algebra

$$\mathfrak{h}_7 \;:=\; \left(0^3,\, 23,\, 13,\, 12\right) \;.$$

By Mal'tsev's theorem [Mal49, Theorem 7], the connected simply-connected Lie group G associated with $\mathfrak{h}_7$ admits a discrete co-compact subgroup Γ: let $N := \Gamma\backslash G$ be the nilmanifold obtained as a quotient of G by Γ. Note that N is not formal by K. Hasegawa's theorem [Has89, Theorem 1, Corollary].

Fix $\alpha > 1$ and consider the G-left-invariant symplectic form ω_α on N defined by

$$\omega_\alpha \;:=\; e^{14} + \alpha\cdot e^{25} + (\alpha-1)\cdot e^{36} \;.$$

Consider the left-invariant almost-complex structure J on N defined by

$$\begin{array}{lll} J_\alpha\, e_1 \;:=\; e_4 \;, & J_\alpha\, e_2 \;:=\; \alpha\, e_5 \;, & J_\alpha\, e_3 \;:=\; (\alpha-1)\, e_6 \;, \\ J_\alpha\, e_4 \;:=\; -e_1 \;, & J_\alpha\, e_5 \;:=\; -\frac{1}{\alpha}\, e_2 \;, & J_\alpha\, e_6 \;:=\; -\frac{1}{\alpha-1}\, e_3 \;, \end{array}$$

where $\{e_1,\,\ldots,\,e_6\}$ denotes the global dual frame of the G-left-invariant co-frame $\{e^1,\,\ldots,\,e^6\}$ associated to the structure equations.

Finally, define the G-left-invariant symmetric tensor

$$g_\alpha(\cdot,\,\cdot\cdot) \;:=\; \omega_\alpha\left(\cdot,\, J_\alpha\cdot\cdot\right) \;.$$

It is straightforward to check that $\left\{(J_\alpha,\,\omega_\alpha,\,g_\alpha)\right\}_{\alpha>1}$ is a family of G-left-invariant almost-Kähler structures on N; moreover, setting

$$\begin{array}{lll} E^1_\alpha \;:=\; e^1 \;, & E^2_\alpha \;:=\; \alpha\, e^2 \;, & E^3_\alpha \;:=\; (\alpha-1)\, e^3 \;, \\ E^4_\alpha \;:=\; e^4 \;, & E^5_\alpha \;:=\; e^5 \;, & E^6_\alpha \;:=\; e^6 \;, \end{array}$$

we get the G-left-invariant g_α-orthonormal co-frame $\left\{E^1_\alpha,\,\ldots,\,E^6_\alpha\right\}$ on N. The structure equations with respect to the co-frame $\left\{E^1_\alpha,\,\ldots,\,E^6_\alpha\right\}$ read as follows:

$$\left\{\begin{array}{rcl} \mathrm{d}\,E^1_\alpha &=& 0 \\ \mathrm{d}\,E^2_\alpha &=& 0 \\ \mathrm{d}\,E^3_\alpha &=& 0 \\ \mathrm{d}\,E^4_\alpha &=& \frac{1}{\alpha(\alpha-1)}\,E^{23}_\alpha \\ \mathrm{d}\,E^5_\alpha &=& \frac{1}{\alpha-1}\,E^{13}_\alpha \\ \mathrm{d}\,E^6_\alpha &=& \frac{1}{\alpha}\,E^{12}_\alpha \end{array}\right. \;.$$

Then

$$\varphi^1_\alpha \;:=\; E^1_\alpha + \mathrm{i}\, E^4_\alpha\,, \qquad \varphi^2_\alpha \;:=\; E^2_\alpha + \mathrm{i}\, E^5_\alpha\,, \qquad \varphi^3_\alpha \;:=\; E^3_\alpha + \mathrm{i}\, E^6_\alpha$$

are $(1,0)$-forms for the almost-complex structure J_α, and

$$\omega_\alpha \;=\; E^1_\alpha \wedge E^4_\alpha + E^2_\alpha \wedge E^5_\alpha + E^3_\alpha \wedge E^6_\alpha\,.$$

By K. Nomizu's theorem [Nom54, Theorem 1], the de Rham cohomology of N is straightforwardly computed:

$$\begin{aligned} H^2_{dR}(N;\mathbb{R}) \;&=\; \mathbb{R}\left\langle E^{15}_\alpha,\, E^{16}_\alpha,\, E^{24}_\alpha,\, E^{26}_\alpha,\, E^{34}_\alpha,\, E^{35}_\alpha,\, E^{14}_\alpha + \frac{1}{\alpha}\, E^{25}_\alpha,\, \frac{1}{\alpha}\, E^{25}_\alpha + \frac{1}{\alpha-1}\, E^{36}_\alpha \right\rangle \\ &=\; \mathbb{R}\left\langle \mathrm{i}\;\alpha\, \varphi^{1\bar{1}}_\alpha + \mathrm{i}\; \varphi^{2\bar{2}}_\alpha,\; \mathrm{i}\,(\alpha-1)\, \varphi^{2\bar{2}}_\alpha + \mathrm{i}\;\alpha\, \varphi^{3\bar{3}}_\alpha,\; \mathfrak{Im}\, \varphi^{1\bar{2}}_\alpha,\; \mathfrak{Im}\, \varphi^{1\bar{3}}_\alpha,\; \mathfrak{Im}\, \varphi^{3\bar{2}}_\alpha \right\rangle \\ &\quad\; \oplus \left\langle \mathfrak{Im}\, \varphi^{12}_\alpha,\; \mathfrak{Im}\, \varphi^{13}_\alpha,\; \mathfrak{Im}\, \varphi^{23}_\alpha \right\rangle\;. \end{aligned}$$

Note that the g_α-harmonic representatives of the above basis of $H^2_{dR}(N;\mathbb{R})$ are of pure type with respect to J_α: hence, the almost-complex structure J_α is $\mathcal{C}^\infty$-pure-and-full and pure-and-full by [FT10, Theorem 3.7]; furthermore, by Theorem 2.35, the Lefschetz-type property on 2-forms holds on N endowed with the almost-Kähler structure $(J_\alpha,\, \omega_\alpha,\, g_\alpha)$, where $\alpha > 1$. Moreover, we get

$$h^+_{J_\alpha}(N) \;=\; 5\,, \qquad h^-_{J_\alpha}(N) \;=\; 3\,.$$

On the other hand, one can explicitly note that

$$\begin{aligned} L_{\omega_\alpha} E^{15}_\alpha \;&=\; E^{1536}_\alpha \;=\; *_{g_\alpha}\, E^{24}_\alpha\,, \\ L_{\omega_\alpha} E^{16}_\alpha \;&=\; E^{1625}_\alpha \;=\; *_{g_\alpha}\, E^{34}_\alpha\,, \\ L_{\omega_\alpha} E^{24}_\alpha \;&=\; E^{2436}_\alpha \;=\; *_{g_\alpha}\, E^{15}_\alpha\,, \\ L_{\omega_\alpha} E^{26}_\alpha \;&=\; E^{2614}_\alpha \;=\; *_{g_\alpha}\, E^{35}_\alpha\,, \\ L_{\omega_\alpha} E^{34}_\alpha \;&=\; E^{3425}_\alpha \;=\; *_{g_\alpha}\, E^{16}_\alpha\,, \\ L_{\omega_\alpha} E^{35}_\alpha \;&=\; E^{3514}_\alpha \;=\; *_{g_\alpha}\, E^{26}_\alpha\,, \end{aligned}$$

and

$$\mathrm{d} *_{g_\alpha} L_{\omega_\alpha} \left(E^{14}_\alpha + \frac{1}{\alpha}\, E^{25}_\alpha \right) \;=\; \mathrm{d}\left(-\frac{\alpha+1}{\alpha}\, E^{36}_\alpha - E^{25}_\alpha - \frac{1}{\alpha}\, E^{14}_\alpha \right) \;=\; 0\,,$$

and

$$\mathrm{d} *_{g_\alpha} L_{\omega_\alpha} \left(e^{25} + e^{36} \right) = 0\,;$$

this proves explicitly that the the Lefschetz-type property on 2-forms holds on N endowed with the almost-Kähler structure $(J_\alpha,\, \omega_\alpha,\, g_\alpha)$, where $\alpha > 1$.

Note that, while $\omega_\alpha \wedge \cdot\colon\; \wedge^2 N \to \wedge^4 N$ induces an isomorphism $[\omega_\alpha] \smile \cdot\colon H^2_{dR}(N;\mathbb{R}) \stackrel{\simeq}{\to} H^4_{dR}(N;\mathbb{R})$ in cohomology, the map $[\omega_\alpha]^2 \smile \cdot\colon H^1_{dR}(N;\mathbb{R}) \to H^5_{dR}(N;\mathbb{R})$ is not an isomorphism, according to [BG88, Theorem A].

We show explicitly that the nilmanifold N is not formal, without using K. Hasegawa's theorem [Has89, Theorem 1, Corollary]. By [DGMS75, Corollary 1], every Massey product on a formal manifold is zero. Since

$$\left[E^1_\alpha\right] \smile \left[E^3_\alpha\right] \;=\; (\alpha-1)\, \left[\mathrm{d}\, E^5_\alpha\right] \;=\; 0 \quad \text{and} \quad \left[E^3_\alpha\right] \smile \left[E^2_\alpha\right] \;=\; -\alpha\, (\alpha-1)\, \left[\mathrm{d}\, E^4_\alpha\right] \;=\; 0\,,$$

the triple Massey product

$$\left\langle [E^1_\alpha],\, [E^3_\alpha],\, [E^2_\alpha] \right\rangle \;=\; -\,(\alpha-1)\, \left[E^{25}_\alpha + \alpha\, E^{14}_\alpha\right]$$

is not zero, and hence N is not formal.

Summarizing, we state the following result, [ATZ12, Proposition 2.5].

Proposition 2.38. *There exists a non-formal* 6*-dimensional nilmanifold endowed with an* 1*-parameter family* $\left\{(J_\alpha,\, \omega_\alpha,\, g_\alpha)\right\}_{\alpha>1}$ *of left-invariant almost-Kähler structures, such that* J_α *is* $\mathcal{C}^\infty$*-pure-and-full and pure-and-full, and for which the Lefschetz-type property on* 2*-forms holds.*

In the following example, we give a $\mathcal{C}^\infty$-pure-and-full almost-Kähler structure on the completely-solvable Nakamura manifold, [ATZ12, §3].

Example 2.39. *A $\mathcal{C}^\infty$-pure-and-full almost-Kähler structure on the completely-solvable Nakamura manifold.* Firstly, we recall the construction of the *completely-solvable Nakamura manifold*: it is a completely-solvable solvmanifold diffeomorphic to the *Nakamura manifold* studied by I. Nakamura in [Nak75, page 90], and it is an example of a cohomologically-Kähler non-Kähler manifold, [dAFdLM92], [FMS03, Example 3.1], [dBT06, §3].

Take $A \in \mathrm{SL}(2;\mathbb{Z})$ with two different real positive eigenvalues e^λ and $\mathrm{e}^{-\lambda}$ with $\lambda > 0$, and fix $P \in \mathrm{GL}(2;\mathbb{R})$ such that $PAP^{-1} = \mathrm{diag}\left(\mathrm{e}^\lambda,\, \mathrm{e}^{-\lambda}\right)$. For example, take

$$A := \left(\begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array}\right) , \qquad \text{and} \qquad P := \left(\begin{array}{cc} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{\sqrt{5}-1}{2} \end{array}\right) ,$$

and consequently $\lambda = \log \frac{3+\sqrt{5}}{2}$.

Let $M^6 :=: M^6(\lambda)$ be the 6-dimensional completely-solvable solvmanifold

$$M^6 := \mathbb{S}^1_{x^2} \times \frac{\mathbb{R}_{x^1} \times \mathbb{T}^2_{\mathbb{C},\,(x^3,\,x^4,\,x^5,\,x^6)}}{\langle T_1 \rangle}$$

where $\mathbb{T}^2_{\mathbb{C}}$ is the 2-dimensional complex torus

$$\mathbb{T}^2_{\mathbb{C}} := \frac{\mathbb{C}^2}{P\mathbb{Z}\left[\mathrm{i}\right]^2}$$

and T_1 acts on $\mathbb{R} \times \mathbb{T}^2_{\mathbb{C}}$ as

$$T_1\left(x^1,\, x^3,\, x^4,\, x^5,\, x^6\right) := \left(x^1+\lambda,\, \mathrm{e}^{-\lambda}\, x^3,\, \mathrm{e}^{\lambda}\, x^4,\, \mathrm{e}^{-\lambda}\, x^5,\, \mathrm{e}^{\lambda}\, x^6\right) .$$

Using coordinates x^2 on $\mathbb{S}^1$, x^1 on $\mathbb{R}$ and $\left(x^3, x^4, x^5, x^6\right)$ on $\mathbb{T}^2_{\mathbb{C}}$, we set

$$\begin{array}{rclrcl} e^1 & := & \mathrm{d}\, x^1 \,, & e^2 & := & \mathrm{d}\, x^2 \,, \\ e^3 & := & \mathrm{e}^{x^1}\, \mathrm{d}\, x^3 \,, & e^4 & := & \mathrm{e}^{-x^1}\, \mathrm{d}\, x^4 \,, \\ e^5 & := & \mathrm{e}^{x^1}\, \mathrm{d}\, x^5 \,, & e^6 & := & \mathrm{e}^{-x^1}\, \mathrm{d}\, x^6 \,. \end{array}$$

as a basis for $\mathfrak{g}^*$, where $\mathfrak{g}$ denotes the Lie algebra naturally associated to M^6; therefore, with respect to $\{e^i\}_{i\in\{1,\ldots,6\}}$, the structure equations are the following:

$$\left\{\begin{array}{rcl} \mathrm{d}\, e^1 & = & 0 \\ \mathrm{d}\, e^2 & = & 0 \\ \mathrm{d}\, e^3 & = & e^1 \wedge e^3 \\ \mathrm{d}\, e^4 & = & -e^1 \wedge e^4 \\ \mathrm{d}\, e^5 & = & e^1 \wedge e^5 \\ \mathrm{d}\, e^6 & = & -e^1 \wedge e^6 \end{array}\right. .$$

Let J be the almost-complex structure on M^6 defined requiring that a co-frame for the space of complex $(1,0)$-forms is given by

$$\left\{\begin{array}{rcl} \varphi^1 & := & \frac{1}{2}\left(e^1 + \mathrm{i}\, e^2\right) \\ \varphi^2 & := & e^3 + \mathrm{i}\, e^5 \\ \varphi^3 & := & e^4 + \mathrm{i}\, e^6 \end{array}\right. .$$

It is straightforward to check that J is integrable.

Being M^6 a compact quotient of a completely-solvable Lie group, one computes the de Rham cohomology of M^6 by A. Hattori's theorem [Hat60, Corollary 4.2]:

$$\begin{array}{rcl} H^0_{dR}\left(M^6;\mathbb{C}\right) & = & \mathbb{C}\left\langle 1 \right\rangle \,, \\ H^1_{dR}\left(M^6;\mathbb{C}\right) & = & \mathbb{C}\left\langle \varphi^1,\, \bar\varphi^1 \right\rangle \,, \\ H^2_{dR}\left(M^6;\mathbb{C}\right) & = & \mathbb{C}\left\langle \varphi^{1\bar1},\, \varphi^{2\bar3},\, \varphi^{3\bar2},\, \varphi^{23},\, \varphi^{\bar2\bar3} \right\rangle \,, \\ H^3_{dR}\left(M^6;\mathbb{C}\right) & = & \mathbb{C}\left\langle \varphi^{12\bar3},\, \varphi^{13\bar2},\, \varphi^{123},\, \varphi^{1\bar2\bar3},\, \varphi^{2\bar1\bar3},\, \varphi^{3\bar1\bar2},\, \varphi^{23\bar1},\, \varphi^{\bar1\bar2\bar3} \right\rangle \end{array}$$

(as usually, for the sake of clearness, we write, for example, $\varphi^{A\bar{B}}$ in place of $\varphi^A \wedge \bar{\varphi}^B$, and we list the harmonic representatives with respect to the metric $g := \sum_{j=1}^{3} \varphi^j \odot \bar{\varphi}^j$ instead of their classes). Therefore, [FMS03, Proposition 3.2]: *(i)* M^6 is *geometrically formal*, that is, the product of g-harmonic forms is still g-harmonic, and therefore it is formal; *(ii)* furthermore,

$$\omega := e^{12} + e^{34} + e^{56}$$

is a symplectic form on M^6 satisfying the Hard Lefschetz Condition.

Note also that $\tilde{\omega} := \frac{\mathrm{i}}{2}\left(\varphi^{1\bar{1}} + \varphi^{2\bar{2}} + \varphi^{3\bar{3}}\right)$ is not closed but $\mathrm{d}\,\tilde{\omega}^2 = 0$, from which it follows that the manifold M^6 admits a balanced metric.

Since M^6 is a compact quotient of a completely-solvable Lie group, by K. Hasegawa's theorem [Has06, Main Theorem], the manifold M^6, endowed with any integrable almost-complex structure (e.g., the J defined above), admits no Kähler structure, and it is not in class $\mathcal{C}$ of Fujiki, see also [FMS03, Theorem 3.3].

Therefore, we consider the (non-integrable) almost-complex structure J' defined by

$$J'\,e^1 := -e^2\,, \qquad J'\,e^3 := -e^4\,, \qquad J'\,e^5 := -e^6\,.$$

Set

$$\begin{cases} \psi^1 := \frac{1}{2}\left(e^1 + \mathrm{i}\,e^2\right) \\ \psi^2 := e^3 + \mathrm{i}\,e^4 \\ \psi^3 := e^5 + \mathrm{i}\,e^6 \end{cases}$$

as a co-frame for the space of $(1,0)$-forms on M^6 with respect to J'; the structure equations with respect to this co-frame are

$$\begin{cases} \mathrm{d}\,\psi^1 = 0 \\ \mathrm{d}\,\psi^2 = \psi^{1\bar{2}} + \psi^{\bar{1}\bar{2}} \\ \mathrm{d}\,\psi^3 = \psi^{1\bar{3}} + \psi^{\bar{1}\bar{3}} \end{cases},$$

from which it is clear that J' is not integrable.

The J'-compatible 2-form

$$\omega' := e^{12} + e^{34} + e^{56}$$

is d-closed; hence $(J',\, \omega')$ is an almost-Kähler structure on M^6.

Moreover, as already remarked, using A. Hattori's theorem [Hat60, Corollary 4.2], one gets

$$\begin{aligned} H^2_{dR}\left(M^6;\mathbb{R}\right) &= \mathbb{R}\left\langle e^{12},\, e^{34},\, e^{56},\, -e^{36}+e^{45},\, e^{36}+e^{45}\right\rangle \\ &= \underbrace{\mathbb{R}\left\langle \mathrm{i}\,\psi^{1\bar{1}},\, \mathrm{i}\,\psi^{2\bar{2}},\, \mathrm{i}\,\psi^{3\bar{3}},\, \mathrm{i}\left(\psi^{2\bar{3}}+\psi^{3\bar{2}}\right)\right\rangle}_{\subseteq H^+_{J'}(M^6)} \oplus \underbrace{\mathbb{R}\left\langle \mathrm{i}\left(\psi^{23}-\psi^{\bar{2}\bar{3}}\right)\right\rangle}_{\subseteq H^-_{J'}(M^6)}\,, \end{aligned}$$

where we have listed the harmonic representatives with respect to the metric $g' := \sum_{j=1}^{6} e^j \odot e^j$ instead of their classes; note that the above g'-harmonic representatives are of pure type with respect to J'. Therefore, J' is obviously $\mathcal{C}^\infty$-full; it is also $\mathcal{C}^\infty$-pure by [FT10, Proposition 3.2], or [DLZ10, Proposition 2.8]. Moreover, since any cohomology class in $H^+_{J'}\left(M^6\right)$, respectively in $H^-_{J'}\left(M^6\right)$, has a d-closed g'-harmonic representative in $\wedge^{1,1}_{J'}M^6 \cap \wedge^2 M^6$, respectively in $\left(\wedge^{2,0}_{J'}M^6 \oplus \wedge^{0,2}_{J'}M^6\right) \cap \wedge^2 M^6$, then J' is also pure-and-full, by [FT10, Theorem 3.7], and the Lefschetz-type property on 2-forms holds, by Theorem 2.35.

One can explicitly check that the Lefschetz-type operator

$$L_{\omega'}\colon \wedge^2 M^6 \to \wedge^4 M^6$$

takes g'-harmonic 2-forms to g'-harmonic 4-forms, since

$$\begin{aligned} L_{\omega'}\,e^{12} &= e^{1234}+e^{1256} &&= *_{g'}\left(e^{34}+e^{56}\right)\,, \\ L_{\omega'}\,e^{34} &= e^{1234}+e^{3456} &&= *_{g'}\left(e^{12}+e^{56}\right)\,, \\ L_{\omega'}\,e^{56} &= e^{1256}+e^{3456} &&= *_{g'}\left(e^{12}+e^{34}\right)\,, \\ L_{\omega'}\,e^{36} &= e^{1236} &&= *_{g'}\,e^{45}\,, \\ L_{\omega'}\,e^{45} &= e^{1245} &&= *_{g'}\,e^{36}\,. \end{aligned}$$

Summarizing, the content of the last example yields the following result, [ATZ12, Proposition 3.3].

Proposition 2.40. *The completely-solvable Nakamura manifold M^6 admits*

- *both a $\mathcal{C}^\infty$-pure-and-full and pure-and-full complex structure J, and*
- *a $\mathcal{C}^\infty$-pure-and-full and pure-and-full almost-Kähler structure $(J',\,\omega',\,g')$, for which the Lefschetz-type property on 2-forms holds.*

Finally, in the following example, we give a non-$\mathcal{C}^\infty$-full almost-Kähler structure, [ATZ12, §4]. In particular, this provides another strong difference between the (non-integrable) almost-Kähler case and the (integrable) Kähler case, all the compact Kähler manifolds being $\mathcal{C}^\infty$-pure-and-full by [DLZ10, Lemma 2.15, Theorem 2.16], or [LZ09, Proposition 2.1].

Example 2.41. *An almost-Kähler non-$\mathcal{C}^\infty$-full structure for which the Lefschetz-type property on 2-forms does not hold.*
Consider the Iwasawa manifold $\mathbb{I}_3 := \mathbb{H}\left(3;\mathbb{Z}\left[\mathrm{i}\right]\right)\backslash\mathbb{H}(3;\mathbb{C})$, see §1.4.1. Recall that, given the standard complex structure induced by the one on $\mathbb{C}^3$ and setting $\left\{\varphi^1,\,\varphi^2,\,\varphi^3\right\}$ as a global co-frame for the $(1,0)$-forms on $\mathbb{I}_3$, by K. Nomizu's theorem [Nom54, Theorem 1] one gets

$$\begin{aligned} H^2_{dR}\left(\mathbb{I}_3;\mathbb{C}\right) \;&=\; \mathbb{R}\left\langle \varphi^{13}+\varphi^{\bar1\bar3},\, \mathrm{i}\left(\varphi^{13}-\varphi^{\bar1\bar3}\right),\, \varphi^{23}+\varphi^{\bar2\bar3},\, \mathrm{i}\left(\varphi^{23}-\varphi^{\bar2\bar3}\right),\, \varphi^{1\bar2}-\varphi^{2\bar1},\right. \\ &\qquad \left. \mathrm{i}\left(\varphi^{1\bar2}+\varphi^{2\bar1}\right),\, \mathrm{i}\,\varphi^{1\bar1},\, \mathrm{i}\,\varphi^{2\bar2}\right\rangle \otimes_{\mathbb{R}} \mathbb{C}\;, \end{aligned}$$

where we have listed the harmonic representatives with respect to the metric $g := \sum_{h=1}^{3}\varphi^h\odot\bar\varphi^h$ instead of their classes. Using the co-frame $\left\{e^1,\,\ldots,\,e^6\right\}$ of the cotangent bundle defined by

$$\varphi^1 =: e^1+\mathrm{i}\,e^2\;,\qquad \varphi^2 =: e^3+\mathrm{i}\,e^4\;,\qquad \varphi^3 =: e^5+\mathrm{i}\,e^6\;,$$

one computes the structure equations

$$\mathrm{d}\,e^1 \;=\; \mathrm{d}\,e^2 \;=\; \mathrm{d}\,e^3 \;=\; \mathrm{d}\,e^4 \;=\; 0\;,\qquad \mathrm{d}\,e^5 \;=\; -e^{13}+e^{24}\;,\qquad \mathrm{d}\,e^6 \;=\; -e^{14}-e^{23}\;.$$

Therefore

$$H^2_{dR}\left(\mathbb{I}_3;\mathbb{R}\right) \;=\; \mathbb{R}\left\langle e^{15}-e^{26},\, e^{16}+e^{25},\, e^{35}-e^{46},\, e^{36}+e^{45},\, e^{13}+e^{24},\, e^{23}-e^{14},\, e^{12},\, e^{34}\right\rangle\;.$$

Consider the almost-complex structure J on X defined by

$$J\,e^1 \;:=\; -e^6\;,\qquad J\,e^2 \;:=\; -e^5\;,\qquad J\,e^3 \;:=\; -e^4\;,$$

and set

$$\omega \;:=\; e^{16}+e^{25}+e^{34}\;.$$

Then $(J,\,\omega,\,g)$ is an almost-Kähler structure on the Iwasawa manifold $\mathbb{I}_3$. We easily get that

$$\mathbb{R}\left\langle e^{16}+e^{25},\, \left(e^{35}-e^{46}\right)+\left(e^{13}+e^{24}\right),\, \left(e^{36}+e^{45}\right)-\left(e^{23}-e^{14}\right),\, e^{34}\right\rangle \subseteq H^+_J\left(\mathbb{I}_3\right)$$

and

$$\mathbb{R}\left\langle e^{15}-e^{26},\, \left(e^{35}-e^{46}\right)-\left(e^{13}+e^{24}\right),\, \left(e^{36}+e^{45}\right)+\left(e^{23}-e^{14}\right)\right\rangle \subseteq H^-_J\left(\mathbb{I}_3\right)\;.$$

We claim that the previous inclusions are actually equalities, and in particular that J is a non-$\mathcal{C}^\infty$-full almost-Kähler structure on $\mathbb{I}_3$. Indeed, we firstly note that, by [FT10, Proposition 3.2] or [DLZ10, Proposition 2.8], J is $\mathcal{C}^\infty$-pure, since it admits a symplectic structure compatible with it. Moreover, we recall that a $\mathcal{C}^\infty$-full almost-complex structure is also pure by [LZ09, Proposition 2.5], and therefore it is also $\mathcal{C}^\infty$-pure at the 4$^{\text{th}}$ stage, by Theorem 2.10, that is,

$$H^{(3,1),(1,3)}_J\left(\mathbb{I}_3;\mathbb{R}\right)\cap H^{(2,2)}_J\left(\mathbb{I}_3;\mathbb{R}\right) \;=\; \{0\}\;.$$

Therefore, our claim reduces to prove that J is not $\mathcal{C}^\infty$-pure at the 4$^{\text{th}}$ stage. Note that

$$\begin{aligned} 0 \;\neq\; \left[e^{3456}\right] \;&=\; \left[e^{3456}-\mathrm{d}\,e^{135}\right] = \left[e^{3456}+e^{1234}\right] \\ &=\; \left[e^{3456}+\mathrm{d}\,e^{135}\right] = \left[e^{3456}-e^{1234}\right]\;, \end{aligned}$$

and that $e^{3456}+e^{1234}\in\left(\wedge^{3,1}_J\mathbb{I}_3\oplus\wedge^{1,3}_J\mathbb{I}_3\right)\cap\wedge^4\mathbb{I}_3$, while $e^{3456}-e^{1234}\in\wedge^{2,2}_J\mathbb{I}_3\cap\wedge^4\mathbb{I}_3$, and so $H^{(3,1),(1,3)}_J\left(\mathbb{I}_3;\mathbb{R}\right)\cap H^{(2,2)}_J\left(\mathbb{I}_3;\mathbb{R}\right)\ni\left[e^{3456}\right]$, therefore J is not $\mathcal{C}^\infty$-pure at the 4$^{\text{th}}$ stage, and hence it is not $\mathcal{C}^\infty$-full.

Let L_ω be the Lefschetz-type operator associated to the almost-Kähler structure $(J,\,\omega,\,g)$. Then, we have

$$L_\omega\left(e^{12}\right) \;=\; e^{1234} \;=\; \mathrm{d}\left(e^{245}\right)\;,$$

namely, L_ω does not take g-harmonic 2-forms to g-harmonic 4-forms.

The previous example proves the following result, [ATZ12, Proposition 4.1].

Proposition 2.42. *The differentiable manifold X underlying the Iwasawa manifold $\mathbb{I}_3 := \mathbb{H}(3;\mathbb{Z}[i])\backslash\mathbb{H}(3;\mathbb{C})$ admits an almost-Kähler structure $(J,\, \omega,\, g)$ which is $\mathcal{C}^\infty$-pure and non-$\mathcal{C}^\infty$-full, and for which the Lefschetz-type property on 2-forms does not hold.*

The argument of the proof of [DLZ10, Theorem 2.3] suggests the following question, [ATZ12, Question 3.4], compare also [DLZ12, §2], in accordance with Proposition 2.42.

Question 2.43. Let X be a compact $2n$-dimensional manifold endowed with an almost-Kähler structure $(J,\, \omega,\, g)$ such that the Lefschetz-type property on 2-forms holds. Is J $\mathcal{C}^\infty$-full?

2.3 $\mathcal{C}^\infty$-pure-and-fullness and deformations of (almost-)complex structures

In this section, we are interested in studying the behaviour of the cohomological decomposition of the de Rham cohomology of an (almost-)complex manifold under small deformations of the complex structure and along curves of almost-complex structures.

More precisely, we prove that being $\mathcal{C}^\infty$-pure-and-full is not a stable property under small deformations of the complex structure, Theorem 2.49, as a consequence of the study of the $\mathcal{C}^\infty$-pure-and-fullness for small deformations of the Iwasawa manifold, Theorem 2.49. Then we study some explicit examples of curves of almost-complex structures on compact manifolds: by using a construction introduced by J. Lee, [Lee04, §1], we construct a curve of almost-complex structures along which the property of being $\mathcal{C}^\infty$-pure-and-full remains satisfied, Theorem 2.53. In §2.3.2, we provide counterexamples to the upper-semi-continuity of $t \mapsto H^-_{J_t}(X)$, Proposition 2.55, and to the lower-semi-continuity of $t \mapsto H^+_{J_t}(X)$, Proposition 2.56, where $\{J_t\}_t$ is a curve of almost-complex structures on a compact manifold X of dimension greater than 4; we also study a stronger semi-continuity problem, §2.3.2.

The results in this section have been obtained in joint work with A. Tomassini, [AT11, AT12a].

2.3.1 Deformations of $\mathcal{C}^\infty$-pure-and-full almost-complex structures

In this section, we consider the problem of the stability of the $\mathcal{C}^\infty$-pure-and-fullness under small deformations of the complex structure and along curves of almost-complex structures.

Instability of $\mathcal{C}^\infty$-pure-and-full property

We recall that a property concerning compact complex (respectively, almost-complex) manifolds (e.g., admitting Kähler metrics, admitting balanced metrics, satisfying the $\partial\overline{\partial}$-Lemma, admitting compatible symplectic structures) is called *stable under small deformations of the complex* (respectively, *almost-complex*) *structure* if, for every complex-analytic family $\{X_t =: (X,\, J_t)\}_{t\in B}$ of compact complex manifolds (respectively, for every smooth curve $\{J_t\}_{t\in B}$ of almost-complex structures on a compact differentiable manifold X), whenever the property holds for $(X,\, J_t)$ for some $t \in B$, it holds also for $(X,\, J_s)$ for any s in a neighbourhood of t in B.

The main result in the context of stability under small deformations of the complex structure is the following classical theorem by K. Kodaira and D. C. Spencer, [KS60], which actually holds for differentiable families of compact complex manifolds.

Theorem 2.44 ([KS60, Theorem 15])**.** *For a compact manifold, admitting a Kähler structure is a stable property under small deformations of the complex structure.*

Remark 2.45. Conditions under which the property of admitting a balanced metric is stable under small deformations of the complex structure have been studied by C.-C. Wu [Wu06, §5], and by J. Fu and S.-T. Yau [FY11].

Note that, by [DLZ11, Theorem 5.4], see also [Don06], on compact almost-complex manifolds of dimension 4, the property of admitting an almost-Kähler structure is stable under small deformations of the almost-complex structure. This result stands on the very special properties of 4-dimensional manifolds, and does not hold true in higher dimension. More precisely, we provide here an explicit example, in dimension 6, showing that, relaxing the integrability condition in the previous theorem (namely, starting with an almost-Kähler structure), we lose the stability under small deformations of the almost-complex structure.

Example 2.46. *A curve $\{J_t\}_t$ of almost-complex structures on a compact 6-dimensional manifold such that J_0 admits an almost-Kähler structure and J_t, for $t \neq 0$, admits no almost-Kähler structure.*
For $c \in \mathbb{R}$, consider the completely-solvable Lie group

$$\mathrm{Sol}(3)_{(x^1,y^1,z^1)} \;:=\; \left\{ \begin{pmatrix} \mathrm{e}^{c\,z^1} & & & x^1 \\ & \mathrm{e}^{-c\,z^1} & & y^1 \\ & & 1 & z^1 \\ & & & 1 \end{pmatrix} \in \mathrm{GL}(4;\mathbb{R}) \;:\; x^1,\, y^1,\, z^1 \in \mathbb{R} \right\} .$$

Choose a suitable $c \in \mathbb{R}$, for which there exists a co-compact discrete subgroup $\Gamma(c) \subset \mathrm{Sol}(3)$ such that

$$M(c)_{(x^1,y^1,z^1)} \;:=\; \Gamma(c) \left\backslash \mathrm{Sol}(3)_{(x^1,y^1,z^1)} \right.$$

is a compact 3-dimensional completely-solvable solvmanifold, [AGH63, §3].

The manifold

$$N^6(c) \;:=\; M(c)_{(x^1,y^1,z^1)} \times M(c)_{(x^2,y^2,z^2)} \;.$$

is cohomologically-Kähler, see [BG90, Example 1], is formal and has a symplectic structure satisfying the Hard Lefschetz Condition, but it admits no Kähler structure, see [FMS03, Theorem 3.5].

Consider $\left\{e^i\right\}_{i\in\{1,\ldots,6\}}$ as a $(\mathrm{Sol}(3)\times\mathrm{Sol}(3))$-left-invariant co-frame for $N^6(c)$, where

$$\begin{aligned} e^1 &:= \mathrm{e}^{-c\,z^1}\,\mathrm{d}\,x^1, & e^2 &:= \mathrm{e}^{-c\,z^1}\,\mathrm{d}\,y^1, & e^3 &:= \mathrm{d}\,z^1, \\ e^4 &:= \mathrm{e}^{-c\,z^2}\,\mathrm{d}\,x^2, & e^5 &:= \mathrm{e}^{-c\,z^2}\,\mathrm{d}\,y^2, & e^6 &:= \mathrm{d}\,z^2; \end{aligned}$$

with respect to it, the structure equations are

$$\begin{cases} \mathrm{d}\,e^1 &= \;\; c\,e^1\wedge e^3 \\ \mathrm{d}\,e^2 &= -c\,e^2\wedge e^3 \\ \mathrm{d}\,e^3 &= \;\; 0 \\ \mathrm{d}\,e^4 &= \;\; c\,e^4\wedge e^6 \\ \mathrm{d}\,e^5 &= -c\,e^5\wedge e^6 \\ \mathrm{d}\,e^6 &= \;\; 0 \end{cases} .$$

By A. Hattori's theorem [Hat60, Corollary 4.2], it is straightforward to compute

$$H^2_{dR}\left(N^6(c);\mathbb{R}\right) \;=\; \mathbb{R}\left\langle e^1\wedge e^2,\, e^3\wedge e^6,\, e^4\wedge e^5\right\rangle \;,$$

hence the space of $(\mathrm{Sol}(3)\times\mathrm{Sol}(3))$-left-invariant d-closed 2-forms is

$$\mathbb{R}\left\langle e^{12},\, e^{36},\, e^{45}\right\rangle \oplus \mathbb{R}\left\langle e^{13},\, e^{23},\, e^{45},\, e^{46}\right\rangle$$

(where, as usually, we shorten $e^{AB} := e^A\wedge e^B$).

Let $J_0 \in \mathrm{End}\left(TN^6(c)\right)$ be the almost-complex structure given, with respect to the frame $\{e_1,\ldots,e_6\}$ dual to $\left\{e^1,\ldots,e^6\right\}$, by

$$J_0 \;:=\; \left(\begin{array}{cc|cccc} & -1 & & & & \\ 1 & & & & & \\ \hline & & & & & -1 \\ & & & & -1 & \\ & & & 1 & & \\ & & 1 & & & \end{array}\right) \;\in\; \mathrm{End}\left(TN^6(c)\right) \;.$$

It is straightforward to check that J_0 admits almost-Kähler structures: more precisely, the cone $\mathcal{K}^c_{J_0,\,\mathrm{inv}}$ of $(\mathrm{Sol}(3)\times\mathrm{Sol}(3))$-left-invariant almost-Kähler structures on $\left(N^6(c),\, J_0\right)$ is

$$\mathcal{K}^c_{J_0,\,\mathrm{inv}} \;=\; \left\{\alpha\,e^1\wedge e^2 + \beta\,e^3\wedge e^6 + \gamma\,e^4\wedge e^5 \;:\; \alpha,\,\beta,\,\gamma > 0\right\} \;.$$

Take now

$$L \;:=\; \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \;\in\; \mathrm{End}\left(TN^6(c)\right)$$

and define, for $t \in \mathbb{R}$, the almost-complex structure

$$J_t \;:=\; \left(\mathrm{id}_{TN^6(c)} - t\,L\right)\, J_0\, \left(\mathrm{id}_{TN^6(c)} - t\,L\right)^{-1} \;=\; \left(\begin{array}{cc|cccc} & 1 & & & 2\,t^2 & -2\,t \\ -1 & & -2\,t & -2\,t^2 & & \\ \hline & & & & -2\,t & 1 \\ & & & & 1 & \\ & & & -1 & & \\ & & -1 & -2\,t & & \end{array}\right) \;\in\; \mathrm{End}\left(T^* N^6(c)\right) \;.$$

We first prove that J_t admits no $(\mathrm{Sol}(3) \times \mathrm{Sol}(3))$-left-invariant almost-Kähler structure for $t \neq 0$. Indeed, for $t \neq 0$, the space of $(\mathrm{Sol}(3) \times \mathrm{Sol}(3))$-left-invariant d-closed J_t-invariant 2-forms is

$$\mathbb{R}\left\langle e^{36} + 2t\, e^{46},\, e^{45} \right\rangle$$

and

$$\left(\beta\, e^3 \wedge e^6 + \gamma\, e^4 \wedge e^5 + 2t\, \beta\, e^4 \wedge e^6\right)^3 \;=\; 0 \qquad \text{for every } \beta,\, \gamma \in \mathbb{R} \;,$$

hence

$$\mathcal{K}^{c}_{J_t,\,\mathrm{inv}} \;=\; \varnothing \qquad \text{for } t \neq 0 \;.$$

Now, using F. A. Belgun's symmetrization trick, [Bel00, Theorem 7], we get that, if J_t admits an almost-Kähler structure ω, then it should admits a $(\mathrm{Sol}(3) \times \mathrm{Sol}(3))$-left-invariant almost-Kähler structure

$$\mu(\omega) \;:=\; \int_{N^6(c)} \omega\lfloor_m\, \eta(m) \;,$$

where η is a $(\mathrm{Sol}(3) \times \mathrm{Sol}(3))$-bi-invariant volume form on $N^6(c)$, whose existence is guaranteed by [Mil76, Lemma 6.2].

We resume the content of the previous example in the following result.

Theorem 2.47. *Being almost-Kähler is not a stable property along curves of almost-complex structures.*

In view of K. Kodaira and D. C. Spencer's theorem [KS60, Theorem 15], a natural question in non-Kähler geometry is what properties, weaker that the property of being Kähler, still remain stable under small deformations of the complex structure. This does not hold true, for example, for the balanced property, as proven in [AB90, Proposition 4.1] by L. Alessandrini and G. Bassanelli; on the other hand, the cohomological property of satisfying the $\partial\overline{\partial}$-Lemma is stable under small deformations of the complex structure, as we have seen in Corollary 1.28, see also [Voi02, Proposition 9.21], or [Wu06, Theorem 5.12], or [Tom08, §B]. We show now that the cohomological property of $\mathcal{C}^\infty$-pure-and-fullness turns out to be non-stable under small deformations of the complex structure, [AT11, Theorem 3.2].

Theorem 2.48. *The properties of being $\mathcal{C}^\infty$-pure-and-full, or $\mathcal{C}^\infty$-pure, or $\mathcal{C}^\infty$-full, or pure-and-full, or pure, or full are not stable under small deformations of the complex structure.*

The proof of Theorem 2.48 follows studying explicitly $\mathcal{C}^\infty$-pure-and-fullness for small deformations of the standard complex structure on the Iwasawa manifold $\mathbb{I}_3$, [AT11, Theorem 3.1]. (We refer to §1.4.1 for notations and results concerning the Iwasawa manifold and its Kuranishi space; we recall here that $\mathbb{I}_3$ is a holomorphically parallelizable nilmanifold of complex dimension 3, and its Kuranishi space is smooth and depends on 6 effective parameters; the small deformations of $\mathbb{I}_3$ can be divided into three classes, *(i)*, *(ii)*, and *(iii)*, according to their Hodge numbers; in particular, the Hodge numbers of the small deformations in class *(i)* are equal to the Hodge numbers of $\mathbb{I}_3$.)

Theorem 2.49. *Let $\mathbb{I}_3 := \mathbb{H}(3; \mathbb{Z}[\mathrm{i}]) \backslash \mathbb{H}(3; \mathbb{C})$ be the Iwasawa manifold, endowed with the complex structure inherited by the standard complex structure on $\mathbb{C}^3$, and consider the small deformations in its Kuranishi space. Then:*

- *the natural complex structure on $\mathbb{I}_3$ is $\mathcal{C}^\infty$-pure-and-full at every stage and pure-and-full at every stage;*
- *the small deformations in class (i) are $\mathcal{C}^\infty$-pure-and-full at every stage and pure-and-full at every stage;*
- *the small deformations in classes (ii) and (iii) are neither $\mathcal{C}^\infty$-pure nor $\mathcal{C}^\infty$-full nor pure nor full.*

Proof. We follow the notation introduced in §1.4.1; in particular, we recall that the structure equations with respect to a certain co-frame $\left\{\varphi^1_{\mathbf{t}},\, \varphi^2_{\mathbf{t}},\, \varphi^3_{\mathbf{t}}\right\}$ of the space of $(1,0)$-forms on $X_{\mathbf{t}} =: (\mathbb{I}_3,\, J_{\mathbf{t}})$, for $\mathbf{t} \in \Delta(0,\varepsilon) \subset \mathbb{C}^6$ with $\varepsilon > 0$ small enough, are the following:

$$\begin{cases} \mathrm{d}\,\varphi^1_{\mathbf{t}} &= 0 \\ \mathrm{d}\,\varphi^2_{\mathbf{t}} &= 0 \\ \mathrm{d}\,\varphi^3_{\mathbf{t}} &= \sigma_{12}\,\varphi^1_{\mathbf{t}} \wedge \varphi^2_{\mathbf{t}} + \sigma_{1\bar{1}}\,\varphi^1_{\mathbf{t}} \wedge \bar{\varphi}^1_{\mathbf{t}} + \sigma_{1\bar{2}}\,\varphi^1_{\mathbf{t}} \wedge \bar{\varphi}^2_{\mathbf{t}} + \sigma_{2\bar{1}}\,\varphi^2_{\mathbf{t}} \wedge \bar{\varphi}^1_{\mathbf{t}} + \sigma_{2\bar{2}}\,\varphi^2_{\mathbf{t}} \wedge \bar{\varphi}^2_{\mathbf{t}} \end{cases} ,$$

where $\sigma_{12},\, \sigma_{1\bar{1}},\, \sigma_{1\bar{2}},\, \sigma_{2\bar{1}},\, \sigma_{2\bar{2}} \in \mathbb{C}$ are complex numbers depending just on $\mathbf{t}$. The asymptotic behaviour of σ_{12}, $\sigma_{1\bar{1}}$, $\sigma_{1\bar{2}}$, $\sigma_{2\bar{1}}$, and $\sigma_{2\bar{2}}$ for $\mathbf{t}$ near $\mathbf{0}$ is the following, see §1.4.1:

$$\begin{cases} \sigma_{12} &= -1 + \mathrm{o}\left(|\mathbf{t}|\right) \\ \sigma_{1\bar{1}} &= t_{21} + \mathrm{o}\left(|\mathbf{t}|\right) \\ \sigma_{1\bar{2}} &= t_{22} + \mathrm{o}\left(|\mathbf{t}|\right) \\ \sigma_{2\bar{1}} &= -t_{11} + \mathrm{o}\left(|\mathbf{t}|\right) \\ \sigma_{2\bar{2}} &= -t_{12} + \mathrm{o}\left(|\mathbf{t}|\right) \end{cases} ;$$

more precisely, for $\mathbf{t}$ in class *(i)*, respectively class *(ii)*, we actually have

$$\begin{cases} \sigma_{12} &= -1 \\ \sigma_{1\bar{1}} &= 0 \\ \sigma_{1\bar{2}} &= 0 \\ \sigma_{2\bar{1}} &= 0 \\ \sigma_{2\bar{2}} &= 0 \end{cases} \qquad \text{for} \qquad \mathbf{t} \in \text{ class } \textit{(i)}\,,$$

and

$$\begin{cases} \sigma_{12} &= -1 + \mathrm{o}\left(|\mathbf{t}|\right) \\ \sigma_{1\bar{1}} &= t_{21}\left(1 + \mathrm{o}\left(1\right)\right) \\ \sigma_{1\bar{2}} &= t_{22}\left(1 + \mathrm{o}\left(1\right)\right) \\ \sigma_{2\bar{1}} &= -t_{11}\left(1 + \mathrm{o}\left(1\right)\right) \\ \sigma_{2\bar{2}} &= -t_{12}\left(1 + \mathrm{o}\left(1\right)\right) \end{cases} \qquad \text{for} \qquad \mathbf{t} \in \text{ class } \textit{(ii)}\,.$$

By K. Nomizu's theorem [Nom54, Theorem 1], one computes straightforwardly the de Rham cohomology of $\mathbb{I}_3$ and of its small deformations; for the sake of clearness, we recall in the following table a basis of the space of the harmonic representatives of the de Rham cohomology classes with respect to the metric $g_{\mathbf{0}} := \sum_{j=1}^{3} \varphi^j_{\mathbf{0}} \odot \bar{\varphi}^j_{\mathbf{0}}$.

k	$\mathbb{K}$	**g_0-harmonic representatives of $H^k_{dR}\left(\mathbb{I}_3;\mathbb{K}\right)$**
1	$\mathbb{C}$	$\varphi^1,\ \varphi^2,\ \bar{\varphi}^1,\ \bar{\varphi}^2$
	$\mathbb{R}$	$\varphi^1 + \bar{\varphi}^1,\ \mathrm{i}\left(\varphi^1 - \bar{\varphi}^1\right),\ \varphi^2 + \bar{\varphi}^2,\ \mathrm{i}\left(\varphi^2 - \bar{\varphi}^2\right)$
2	$\mathbb{C}$	$\varphi^{13},\ \varphi^{23},\ \varphi^{1\bar{1}},\ \varphi^{1\bar{2}},\ \varphi^{2\bar{1}},\ \varphi^{2\bar{2}},\ \varphi^{\bar{1}\bar{3}},\ \varphi^{\bar{2}\bar{3}}$
	$\mathbb{R}$	$\varphi^{13} + \varphi^{\bar{1}\bar{3}},\ \mathrm{i}\left(\varphi^{13} - \varphi^{\bar{1}\bar{3}}\right),\ \varphi^{23} + \varphi^{\bar{2}\bar{3}},\ \mathrm{i}\left(\varphi^{23} - \varphi^{\bar{2}\bar{3}}\right),\ \varphi^{1\bar{2}} - \varphi^{2\bar{1}},\ \mathrm{i}\left(\varphi^{1\bar{2}} + \varphi^{2\bar{1}}\right),\ \mathrm{i}\,\varphi^{1\bar{1}},\ \mathrm{i}\,\varphi^{2\bar{2}}$
3	$\mathbb{C}$	$\varphi^{123},\ \varphi^{13\bar{1}},\ \varphi^{13\bar{2}},\ \varphi^{23\bar{1}},\ \varphi^{23\bar{2}},\ \varphi^{1\bar{1}\bar{3}},\ \varphi^{1\bar{2}\bar{3}},\ \varphi^{2\bar{1}\bar{3}},\ \varphi^{2\bar{2}\bar{3}},\ \varphi^{\bar{1}\bar{2}\bar{3}}$
	$\mathbb{R}$	$\varphi^{123} + \varphi^{\bar{1}\bar{2}\bar{3}},\ \mathrm{i}\left(\varphi^{123} - \varphi^{\bar{1}\bar{2}\bar{3}}\right),\ \varphi^{13\bar{1}} + \varphi^{1\bar{1}\bar{3}},\ \mathrm{i}\left(\varphi^{13\bar{1}} - \varphi^{1\bar{1}\bar{3}}\right),\ \varphi^{13\bar{2}} + \varphi^{2\bar{1}\bar{3}},\ \mathrm{i}\left(\varphi^{13\bar{2}} - \varphi^{2\bar{1}\bar{3}}\right),$ $\varphi^{23\bar{1}} + \varphi^{1\bar{2}\bar{3}},\ \mathrm{i}\left(\varphi^{23\bar{1}} - \varphi^{1\bar{2}\bar{3}}\right),\ \varphi^{23\bar{2}} + \varphi^{2\bar{2}\bar{3}},\ \mathrm{i}\left(\varphi^{23\bar{2}} - \varphi^{2\bar{2}\bar{3}}\right)$
4	$\mathbb{C}$	$\varphi^{123\bar{1}},\ \varphi^{123\bar{2}},\ \varphi^{13\bar{1}\bar{3}},\ \varphi^{13\bar{2}\bar{3}},\ \varphi^{23\bar{1}\bar{3}},\ \varphi^{23\bar{2}\bar{3}},\ \varphi^{1\bar{1}\bar{2}\bar{3}},\ \varphi^{2\bar{1}\bar{2}\bar{3}}$
	$\mathbb{R}$	$\varphi^{123\bar{1}} - \varphi^{1\bar{1}\bar{2}\bar{3}},\ \mathrm{i}\left(\varphi^{123\bar{1}} + \varphi^{1\bar{1}\bar{2}\bar{3}}\right),\ \varphi^{123\bar{2}} - \varphi^{2\bar{1}\bar{2}\bar{3}},\ \mathrm{i}\left(\varphi^{123\bar{2}} + \varphi^{2\bar{1}\bar{2}\bar{3}}\right),\ \varphi^{13\bar{1}\bar{3}},\ \varphi^{13\bar{2}\bar{3}} + \varphi^{23\bar{1}\bar{3}},\ \mathrm{i}\left(\varphi^{13\bar{2}\bar{3}} - \varphi^{23\bar{1}\bar{3}}\right),\ \varphi^{23\bar{2}\bar{3}}$
5	$\mathbb{C}$	$\varphi^{123\bar{1}\bar{3}},\ \varphi^{123\bar{2}\bar{3}},\ \varphi^{13\bar{1}\bar{2}\bar{3}},\ \varphi^{23\bar{1}\bar{2}\bar{3}}$
	$\mathbb{R}$	$\varphi^{123\bar{1}\bar{3}} + \varphi^{13\bar{1}\bar{2}\bar{3}},\ \mathrm{i}\left(\varphi^{123\bar{1}\bar{3}} - \varphi^{13\bar{1}\bar{2}\bar{3}}\right),\ \varphi^{123\bar{2}\bar{3}} + \varphi^{23\bar{1}\bar{2}\bar{3}},\ \mathrm{i}\left(\varphi^{123\bar{2}\bar{3}} - \varphi^{23\bar{1}\bar{2}\bar{3}}\right)$

Note that the above harmonic representatives of the classes in $H^{\bullet}_{dR}(\mathbb{I}_3;\mathbb{R})$ are of pure type with respect to $J_{\mathbf{0}}$ and to $J_{\mathbf{t}}$ with $\mathbf{t}$ in class *(i)*: hence, by Theorem 2.10 (or arguing as in [FT10, Theorem 3.7]), one gets that $\mathbb{I}_3$ and its small deformations in class *(i)* are $\mathcal{C}^\infty$-pure-and-full at every stage and pure-and-full at every stage.

Concerning small deformations $J_{\mathbf{t}}$ in class *(ii)* and in class *(iii)*, using the asymptotic behaviour of the structure equations, we obtain that

$$\left[\sigma_{12}\,\varphi^{12}_{\mathbf{t}}\right] \;=\; \left[\sigma_{1\bar{1}}\,\varphi^{1\bar{1}}_{\mathbf{t}} + \sigma_{1\bar{2}}\,\varphi^{1\bar{2}}_{\mathbf{t}} + \sigma_{2\bar{1}}\,\varphi^{2\bar{1}}_{\mathbf{t}} + \sigma_{2\bar{2}}\,\varphi^{2\bar{2}}_{\mathbf{t}}\right] \;\neq\; 0$$

in $H^2_{dR}(\mathbb{I}_3;\mathbb{C})$. Therefore

$$H^{(1,1)}_{J_{\mathbf{t}}}(\mathbb{I}_3;\mathbb{C}) \cap \left(H^{(2,0)}_{J_{\mathbf{t}}}(\mathbb{I}_3;\mathbb{C}) + H^{(0,2)}_{J_{\mathbf{t}}}(\mathbb{I}_3;\mathbb{C})\right) \;\neq\; \{0\} \;,$$

and in particular $J_{\mathbf{t}}$ is not complex-$\mathcal{C}^\infty$-pure. It follows from Remark 2.5 that $J_{\mathbf{t}}$ cannot be $\mathcal{C}^\infty$-pure; from [LZ09, Proposition 2.30], or Theorem 2.10, it follows that $J_{\mathbf{t}}$ cannot be full.

To prove that small deformations in class *(ii)* and in class *(iii)* are non-pure and non-$\mathcal{C}^\infty$-full, fix $\mathbf{t}$ small enough and choose two positive complex numbers $A =: A(\mathbf{t}) \in \mathbb{C}$ and $B =: B(\mathbf{t}) \in \mathbb{C}$, depending just on $\mathbf{t}$, such that

$$\left(A\,\sigma_{1\bar{2}} - B\,\sigma_{1\bar{1}},\; A\,\sigma_{2\bar{2}} - B\,\sigma_{2\bar{1}}\right) \;\neq\; (0,\,0) \;;$$

computing $-\,\mathrm{d}\left(A\,\varphi^{13\bar{3}}_{\mathbf{t}} + B\,\varphi^{23\bar{3}}_{\mathbf{t}}\right)$, note that

$$\begin{aligned} &\left[\left(A\,\sigma_{2\bar{1}} - B\,\sigma_{1\bar{1}}\right)\varphi^{12\bar{1}\bar{3}}_{\mathbf{t}} + \left(A\,\sigma_{2\bar{2}} - B\,\sigma_{1\bar{2}}\right)\varphi^{12\bar{2}\bar{3}}_{\mathbf{t}} - A\,\bar{\sigma}_{12}\varphi^{13\bar{1}\bar{2}}_{\mathbf{t}} - B\,\bar{\sigma}_{12}\varphi^{23\bar{1}\bar{2}}_{\mathbf{t}}\right] \\ =\;\; &\left[\left(A\,\bar{\sigma}_{1\bar{2}} - B\,\bar{\sigma}_{1\bar{1}}\right)\varphi^{123\bar{1}}_{\mathbf{t}} + \left(A\,\bar{\sigma}_{2\bar{2}} - B\,\bar{\sigma}_{2\bar{1}}\right)\varphi^{123\bar{2}}_{\mathbf{t}}\right] \;\neq\; 0 \;, \end{aligned}$$

in $H^4_{dR}(\mathbb{I}_3;\mathbb{C})$. As before, it follows that $J_{\mathbf{t}}$ is not $\mathcal{C}^\infty$-pure at the 4^{th} stage, and consequently it is neither pure nor $\mathcal{C}^\infty$-full, by Theorem 2.10. □

Curves of $\mathcal{C}^\infty$-pure-and-full almost-complex structures

We study here some explicit examples of curves of almost-complex structures on compact manifolds, along which the property of being $\mathcal{C}^\infty$-pure-and-full remains satisfied. The aim of this section is to better understand the behaviour of $\mathcal{C}^\infty$-pure-and-fullness along curves of almost-complex structures.

Firstly, we recall some general results concerning curves of almost-complex structures on compact manifolds, referring, e.g., to [AL94].

Let J be an almost-complex structure on a compact $2n$-dimensional manifold X. Every curve $\{J_t\}_{t\in(-\varepsilon,\varepsilon)\subset\mathbb{R}}$ of almost-complex structures on X such that $J_0 = J$ can be written, for $\varepsilon > 0$ small enough, as

$$J_t \;=\; (\mathrm{id} - L_t)\; J\; (\mathrm{id} - L_t)^{-1} \;\in\; \mathrm{End}(TX) \;,$$

where $L_t \in \mathrm{End}(TX)$, see, e.g., [AL94, Proposition 1.1.6]; the endomorphism L_t is uniquely determined further requiring that $L_t \in T^{1,0}_J X \otimes \left(T^{0,1}_J X\right)^*$, namely,

$$L_t\,J \,+\, J\,L_t \;=\; 0\;;$$

furthermore, set $L_t \;=:\; t\,L + \mathrm{o}(t)$: if J is compatible with a symplectic form ω, then the curves consisting of ω-compatible almost-complex structures J_t are exactly those ones for which $L^{\mathrm{t}} = L$.

In [dBM10, Proposition 3.3], P. de Bartolomeis and F. Meylan computed $\frac{\mathrm{d}}{\mathrm{d}\,t}\lfloor_{t=0} \mathrm{Nij}_J$, getting a characterization in terms of L of the curves of complex structures starting at a given integrable almost-complex structure J.

A. Fino and A. Tomassini, in [FT10, §6, §7], studied several examples of families of almost-complex structures constructed in such a way. We provide here some further examples, starting with a curve of almost-complex structures on the 4-dimensional torus, [AT11, pages 420–422].

Example 2.50. *A curve of almost-complex structures through the standard Kähler structure on the* 4*-dimensional torus.*
Let $(J_0,\,\omega_0)$ be the standard Kähler structure on the 4-dimensional torus $\mathbb{T}^4$ with coordinates $\{x^j\}_{j\in\{1,\ldots,4\}}$, that is,

$$J_0 \;:=\; \left(\begin{array}{cc|cc} & & -1 & \\ & & & -1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right) \;\in\; \mathrm{End}\left(T\mathbb{T}^4\right) \qquad \text{and} \qquad \omega_0 \;:=\; \mathrm{d}\,x^1\wedge\mathrm{d}\,x^3 + \mathrm{d}\,x^2\wedge\mathrm{d}\,x^4 \;\in\; \wedge^2\mathbb{T}^4 \;.$$

Set

$$L := \left(\begin{array}{cc|cc} \ell & & & \\ & 0 & & \\ \hline & & -\ell & \\ & & & 0 \end{array}\right) \in \operatorname{End}\left(T\mathbb{T}^4\right) ,$$

where $\ell \in \mathcal{C}^\infty(\mathbb{T}^4;\, \mathbb{R})$, that is, $\ell \in \mathcal{C}^\infty(\mathbb{R}^4;\, \mathbb{R})$ is a $\mathbb{Z}^4$-periodic function. For $t \in (-\varepsilon,\, \varepsilon)$ with $\varepsilon > 0$ small enough, define

$$J_{t,\,\ell} := (\mathrm{id} - t\,L)\; J_0\; (\mathrm{id} - t\,L)^{-1} = \left(\begin{array}{cc|cc} & & -\frac{1-t\,\ell}{1+t\,\ell} & \\ & & & -1 \\ \hline \frac{1+t\,\ell}{1-t\,\ell} & & & \\ & 1 & & \end{array}\right) \in \operatorname{End}\left(T\mathbb{T}^4\right) ,$$

obtaining a curve of ω_0-compatible almost-complex structures on $\mathbb{T}^4$, see also Proposition 2.22. To simplify the notation, set

$$\alpha =: \alpha(t,\ell) := \frac{1 - t\,\ell}{1 + t\,\ell} .$$

A co-frame for the holomorphic cotangent bundle of $\mathbb{T}^4$ with respect to $J_{t,\,\ell}$ is given by

$$\begin{cases} \varphi^1_{t,\ell} := \mathrm{d}\,x^1 + \mathrm{i}\,\alpha\,\mathrm{d}\,x^3 \\ \varphi^2_{t,\ell} := \mathrm{d}\,x^2 + \mathrm{i}\,\mathrm{d}\,x^4 \end{cases} ,$$

with respect to which we compute the structure equations

$$\begin{cases} \mathrm{d}\,\varphi^1_{t,\ell} = \mathrm{i}\,\mathrm{d}\alpha \wedge \mathrm{d}\,x^3 \\ \mathrm{d}\,\varphi^2_{t,\ell} = 0 \end{cases} .$$

Note that, taking $\ell = \ell\left(x^1, x^3\right)$, the corresponding almost-complex structure $J_{t,\,\ell}$ is integrable, in fact, $(J_{t,\,\ell},\, \omega_0)$ is a Kähler structure on $\mathbb{T}^4$. Recall that, $\mathbb{T}^4$ being 4-dimensional, $J_{t,\,\ell}$ is $\mathcal{C}^\infty$-pure-and-full by [DLZ10, Theorem 2.3]. For the sake of simplicity, assume $\ell = \ell\left(x^2\right)$ depending just on x^2 and non-constant. Set

$$\begin{aligned} v_1 &:= \mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 - \alpha\;\mathrm{d}\,x^3 \wedge \mathrm{d}\,x^4 , \\ v_2 &:= \mathrm{d}\,x^1 \wedge \mathrm{d}\,x^4 - \alpha\;\mathrm{d}\,x^2 \wedge \mathrm{d}\,x^3 , \end{aligned}$$

$$\begin{aligned} w_1 &:= \alpha\;\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^3 , \\ w_2 &:= \mathrm{d}\,x^2 \wedge \mathrm{d}\,x^4 , \\ w_3 &:= \mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 + \alpha\;\mathrm{d}\,x^3 \wedge \mathrm{d}\,x^4 , \\ w_4 &:= \mathrm{d}\,x^1 \wedge \mathrm{d}\,x^4 + \alpha\;\mathrm{d}\,x^2 \wedge \mathrm{d}\,x^3 . \end{aligned}$$

Using this notation, an arbitrary $J_{t,\,\ell}$-anti-invariant real 2-form $\psi =: A\,v_1 + B\,v_2$, with $A, B \in \mathcal{C}^\infty\left(\mathbb{T}^4;\mathbb{R}\right)$, is d-closed if and only if

$$\begin{cases} \frac{\partial A}{\partial x^3} - \frac{\partial B}{\partial x^1}\,\alpha = 0 \\ \frac{\partial A}{\partial x^4} - \frac{\partial B}{\partial x^2} = 0 \\ -\frac{\partial A}{\partial x^1}\,\alpha - \frac{\partial B}{\partial x^3} = 0 \\ -\frac{\partial B}{\partial x^4}\,\alpha - \frac{\partial A}{\partial x^2}\,\alpha - A\,\frac{\partial \alpha}{\partial x^2} = 0 \end{cases} . \tag{2.3.1}$$

By solving (2.3.1), we obtain the solutions

$$\psi = \frac{A}{\alpha}\,v_1 + B\,v_2 \qquad \text{where} \qquad A,\, B \in \mathbb{R} .$$

Therefore, for $t \in (-\varepsilon,\, \varepsilon)$ with $\varepsilon > 0$ small enough, we have

$$\dim_{\mathbb{R}} H^{(2,0),(0,2)}_{J_{t,\,\ell}}\left(\mathbb{T}^4;\mathbb{R}\right) \leq 2 = \dim_{\mathbb{R}} H^{(2,0),(0,2)}_{J_0}\left(\mathbb{T}^4;\mathbb{R}\right) ,$$

and hence

$$\dim_{\mathbb{R}} H^{(1,1)}_{J_{t,\,\ell}}\left(\mathbb{T}^4;\mathbb{R}\right) \geq 4 = \dim_{\mathbb{R}} H^{(1,1)}_{J_0}\left(\mathbb{T}^4;\mathbb{R}\right) ,$$

accordingly to the upper-semi-continuity, respectively lower-semi-continuity, property proven in [DLZ11, Theorem 2.6] for 4-dimensional almost-complex manifolds.

Now, we turn our attention to the case of dimension greater than 4, [AT11, pages 422–423].

Example 2.51. *A curve of almost-complex structures through the standard Kähler structure on the* 6*-dimensional torus.*
Let $(J_0,\,\omega_0)$ be the standard Kähler structure on the 6-dimensional torus $\mathbb{T}^6$ with coordinates $\{x^j\}_{j\in\{1,\ldots,6\}}$, that is,

$$J_0 \;:=\; \left(\begin{array}{ccc|ccc} & & & -1 & & \\ & & & & -1 & \\ & & & & & -1 \\ \hline 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \end{array}\right) \;\in\; \mathrm{End}\left(T\mathbb{T}^6\right) \qquad \text{and} \qquad \omega_0 \;:=\; \mathrm{d}\,x^1\wedge\mathrm{d}\,x^4+\mathrm{d}\,x^2\wedge\mathrm{d}\,x^5+\mathrm{d}\,x^3\wedge\mathrm{d}\,x^6\;.$$

Set

$$L \;=\; \left(\begin{array}{ccc|ccc} \ell & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ \hline & & & -\ell & & \\ & & & & 0 & \\ & & & & & 0 \end{array}\right) \;\in\; \mathrm{End}\left(T\mathbb{T}^6\right)\;,$$

where $\ell\in\mathcal{C}^\infty(\mathbb{T}^6;\,\mathbb{R})$, that is, $\ell\in\mathcal{C}^\infty(\mathbb{R}^6;\,\mathbb{R})$ is a $\mathbb{Z}^6$-periodic function. For $t\in(-\varepsilon,\,\varepsilon)$ with $\varepsilon>0$ small enough, define

$$J_{t,\,\ell} \;:=\; (\mathrm{id}-t\,L)\;J_0\;(\mathrm{id}-t\,L)^{-1} \;=\; \left(\begin{array}{ccc|ccc} & & & -\frac{1-t\,\ell}{1+t\,\ell} & & \\ & & & & -1 & \\ & & & & & -1 \\ \hline \frac{1+t\,\ell}{1-t\,\ell} & & & & & \\ & 1 & & & & \\ & & 1 & & & \end{array}\right) \;\in\; \mathrm{End}\left(T\mathbb{T}^6\right)\;,$$

obtaining a curve of ω_0-compatible almost-complex structures on $\mathbb{T}^6$, see also Example 2.26. Setting

$$\alpha \;=:\; \alpha(t,\ell) \;:=\; \frac{1\,-\,t\,\ell}{1\,+\,t\,\ell}\;,$$

a co-frame for the holomorphic cotangent bundle of $\mathbb{T}^6$ with respect to $J_{t,\,\ell}$ is given by

$$\left\{\begin{array}{rcl} \varphi^1_{t,\ell} &:=& \mathrm{d}\,x^1\,+\,\mathrm{i}\;\alpha\;\mathrm{d}\,x^4 \\ \varphi^2_{t,\ell} &:=& \mathrm{d}\,x^2\,+\,\mathrm{i}\;\mathrm{d}\,x^5 \\ \varphi^3_{t,\ell} &:=& \mathrm{d}\,x^3\,+\,\mathrm{i}\;\mathrm{d}\,x^6 \end{array}\right.\;,$$

with respect to which the structure equations are

$$\left\{\begin{array}{rcl} \mathrm{d}\,\varphi^1_{t,\ell} &=& \mathrm{i}\;\mathrm{d}\,\alpha\,\wedge\,\mathrm{d}\,x^4 \\ \mathrm{d}\,\varphi^2_{t,\ell} &=& 0 \\ \mathrm{d}\,\varphi^3_{t,\ell} &=& 0 \end{array}\right.\;.$$

Note that if $\ell\;=\;\ell\left(x^1,x^4\right)$, then we get a curve of integrable almost-complex structures, in fact, of Kähler structures, on $\mathbb{T}^6$: in particular, in such a case, $J_{t,\,\ell}$ is $\mathcal{C}^\infty$-pure-and-full. Therefore, as an example, assume that $\ell\,=\,\ell\left(x^3\right)$ depends just on x^3 and is non-constant.

An arbitrary $J_{t,\,\ell}$-anti-invariant real 2-form

$$\begin{aligned}\psi \;\;=:\;\; & A\,\left(\mathrm{d}\,x^1\wedge\mathrm{d}\,x^2-\alpha\;\mathrm{d}\,x^4\wedge\mathrm{d}\,x^5\right)+B\,\left(\mathrm{d}\,x^1\wedge\mathrm{d}\,x^5-\alpha\;\mathrm{d}\,x^2\wedge\mathrm{d}\,x^4\right)+C\,\left(\mathrm{d}\,x^1\wedge\mathrm{d}\,x^3-\alpha\;\mathrm{d}\,x^4\wedge\mathrm{d}\,x^6\right)\\ & +D\,\left(\mathrm{d}\,x^1\wedge\mathrm{d}\,x^6-\alpha\;\mathrm{d}\,x^3\wedge\mathrm{d}\,x^4\right)+E\,\left(\mathrm{d}\,x^2\wedge\mathrm{d}\,x^3-\mathrm{d}\,x^5\wedge\mathrm{d}\,x^6\right)+F\,\left(\mathrm{d}\,x^2\wedge\mathrm{d}\,x^6-\mathrm{d}\,x^3\wedge\mathrm{d}\,x^5\right)\;,\end{aligned}$$

with $A, B, C, D, E, F \in \mathcal{C}^\infty\left(\mathbb{T}^6;\mathbb{R}\right)$, is d-closed if and only if

$$\left\{\begin{array}{rcl}
\frac{\partial A}{\partial x^3} - \frac{\partial C}{\partial x^2} + \frac{\partial E}{\partial x^1} &=& 0 \\
\frac{\partial A}{\partial x^4} - \frac{\partial B}{\partial x^1}\,\alpha &=& 0 \\
\frac{\partial A}{\partial x^5} - \frac{\partial B}{\partial x^2} &=& 0 \\
\frac{\partial A}{\partial x^6} - \frac{\partial D}{\partial x^2} + \frac{\partial F}{\partial x^1} &=& 0 \\
\frac{\partial C}{\partial x^4} - \frac{\partial D}{\partial x^1}\,\alpha &=& 0 \\
-\frac{\partial B}{\partial x^3} + \frac{\partial C}{\partial x^5} - \frac{\partial F}{\partial x^1} &=& 0 \\
\frac{\partial C}{\partial x^6} - \frac{\partial D}{\partial x^3} &=& 0 \\
-\frac{\partial A}{\partial x^1}\,\alpha - \frac{\partial B}{\partial x^4} &=& 0 \\
-\frac{\partial C}{\partial x^1}\,\alpha - \frac{\partial D}{\partial x^4} &=& 0 \\
\frac{\partial B}{\partial x^6} - \frac{\partial D}{\partial x^5} - \frac{\partial E}{\partial x^1} &=& 0 \\
\frac{\partial (B\,\alpha)}{\partial x^3} - \frac{\partial D}{\partial x^2}\,\alpha + \frac{\partial E}{\partial x^4} &=& 0 \\
\frac{\partial E}{\partial x^5} - \frac{\partial F}{\partial x^2} &=& 0 \\
\frac{\partial E}{\partial x^6} - \frac{\partial F}{\partial x^3} &=& 0 \\
-\frac{\partial A}{\partial x^2}\,\alpha - \frac{\partial B}{\partial x^5}\,\alpha &=& 0 \\
-\frac{\partial B}{\partial x^6}\,\alpha - \frac{\partial C}{\partial x^2}\,\alpha - \frac{\partial F}{\partial x^4} &=& 0 \\
-\frac{\partial E}{\partial x^2} - \frac{\partial F}{\partial x^5} &=& 0 \\
-\frac{\partial (A\,\alpha)}{\partial x^3} - \frac{\partial D}{\partial x^5}\,\alpha + \frac{\partial F}{\partial x^4} &=& 0 \\
-\frac{\partial (C\,\alpha)}{\partial x^3} - \frac{\partial D}{\partial x^6}\,\alpha &=& 0 \\
-\frac{\partial E}{\partial x^3} - \frac{\partial F}{\partial x^6} &=& 0 \\
-\frac{\partial A}{\partial x^6}\,\alpha + \frac{\partial C}{\partial x^5}\,\alpha - \frac{\partial E}{\partial x^4} &=& 0
\end{array}\right. . \tag{2.3.2}$$

For $t \neq 0$ small enough, by solving (2.3.2), we obtain that the $J_{t,\,\ell}$-anti-invariant real d-closed 2-forms are

$$\psi \;=\; \frac{C}{\alpha}\left(\mathrm{d}\,x^{13} - \alpha\,\mathrm{d}\,x^{46}\right) + D\,\left(\mathrm{d}\,x^{16} - \alpha\,\mathrm{d}\,x^{34}\right) + E\,\left(\mathrm{d}\,x^{23} - \mathrm{d}\,x^{56}\right) + F\,\left(\mathrm{d}\,x^{26} - \mathrm{d}\,x^{35}\right) \;,$$

where $C,\,D,\,E,\,F \in \mathbb{R}$.

For $t \neq 0$ small enough, we have

$$\dim_{\mathbb{R}} H^{(2,0),(0,2)}_{J_{t,\,\ell}}\left(\mathbb{T}^6;\mathbb{R}\right) \;\leq\; 4 \;<\; 6 \;=\; \dim_{\mathbb{R}} H^{(2,0),(0,2)}_{J_0}\left(\mathbb{T}^6;\mathbb{R}\right) \;,$$

and hence the function $t \mapsto \dim_{\mathbb{R}} H^{(2,0),(0,2)}_{J_{t,\,\ell}}\left(\mathbb{T}^6;\mathbb{R}\right)$ is upper-semi-continuous at 0. On the other hand, the explicit computations for $H^{(1,1)}_{J_{t,\,\ell}}\left(\mathbb{T}^6;\mathbb{R}\right)$ are not so straightforward. In particular, it is not clear if $J_{t,\,\ell}$ remains still $\mathcal{C}^\infty$-full; note that $J_{t,\,\ell}$ is $\mathcal{C}^\infty$-pure by [DLZ10, Proposition 2.7] or [FT10, Proposition 3.2].

We recall here the construction of curves of almost-complex structures through an almost-complex structure J by means of a J-anti-invariant real 2-form, as introduced by J. Lee in [Lee04, §1], in the context of holomorphic curves on symplectic manifolds and Gromov and Witten invariants.

Let J be an almost-complex structure on a compact manifold X; let g be a J-Hermitian metric on X and fix $\gamma \in \left(\wedge^{2,0}X \oplus \wedge^{0,2}X\right) \cap \wedge^2 X$. Define $V_\gamma \in \mathrm{End}\,(TX)$ such that

$$\gamma\,(\cdot,\,\cdot\cdot) \;=\; g\,(V_\gamma\,\cdot,\,\cdot\cdot) \;; \tag{2.3.3}$$

a direct computation shows that $V_\gamma\,J \,+\, J\,V_\gamma \;=\; 0$. Therefore, setting

$$L_\gamma \;:=\; \frac{1}{2}\,V_\gamma\,J \;\in\; \mathrm{End}\,(TX) \;,$$

one gets that $L_\gamma\,J \,+\, J\,L_\gamma \;=\; 0$. For $t \in (-\varepsilon,\,\varepsilon)$ with $\varepsilon > 0$ small enough, define

$$J_{t,\,\gamma} \;:=\; \left(\mathrm{id} - t\,L_\gamma\right)\,J\,\left(\mathrm{id} - t\,L_\gamma\right)^{-1} \;\in\; \mathrm{End}\,(TX) \;,$$

obtaining a curve $\{J_{t,\gamma}\}_{t\in(-\varepsilon,\varepsilon)}$ of almost-complex structures associated with γ.

We give an example of a $\mathcal{C}^\infty$-pure-and-full structure on a non-Kähler manifold such that the stability property of the $\mathcal{C}^\infty$-pure-and-fullness holds along a curve obtained using the construction by J. Lee, [AT11, pages 423–425].

Example 2.52. *A curve of $\mathcal{C}^\infty$-pure-and-full almost-complex structures on the completely-solvable solvmanifold $N^6(c)$.*
We recall that the manifold $N^6(c)$ is a compact 6-dimensional completely-solvable solvmanifold defined, for suitable $c \in \mathbb{R}$, as the product

$$N^6(c) \;:=\; (\Gamma(c)\backslash \mathrm{Sol}(3)) \times (\Gamma(c)\backslash \mathrm{Sol}(3)) \;,$$

where Sol(3) is a completely-solvable Lie group and $\Gamma(c)$ is a co-compact discrete subgroup of Sol(3), [AGH63, §3], see Example 2.46. It has been studied in [BG90, Example 1] as an example of a cohomologically Kähler manifold, and in [FMS03, Example 3.4] by M. Fernández, V. Muñoz, and J. A. Santisteban, as an example of a formal manifold admitting a symplectic structure satisfying the Hard Lefschetz Condition and with no Kähler structure, [FMS03, Theorem 3.5]. A. Fino and A. Tomassini provided in [FT10, §6.3] a family of $\mathcal{C}^\infty$-pure-and-full structures on $N^6(c)$. We construct here a curve of $\mathcal{C}^\infty$-pure-and-full almost-complex structures on $N^6(c)$ using the construction by J. Lee, [Lee04, §1].

Let $\left\{e^i\right\}_{i\in\{1,\dots,6\}}$ be a co-frame for $N^6(c)$ such that the structure equations are

$$\left\{\begin{array}{rcl} \mathrm{d}\,e^1 &=& c\,e^1\wedge e^3 \\ \mathrm{d}\,e^2 &=& -c\,e^2\wedge e^3 \\ \mathrm{d}\,e^3 &=& 0 \\ \mathrm{d}\,e^4 &=& c\,e^4\wedge e^6 \\ \mathrm{d}\,e^5 &=& -c\,e^5\wedge e^6 \\ \mathrm{d}\,e^6 &=& 0 \end{array}\right. .$$

Take the almost-complex structure

$$J \;=\; \left(\begin{array}{cc|cc|cc} & -1 & & & & \\ 1 & & & & & \\ \hline & & & -1 & & \\ & & 1 & & & \\ \hline & & & & & -1 \\ & & & & 1 & \end{array}\right) \;\in\; \mathrm{End}\left(TN^6(c)\right) \;.$$

By A. Hattori's theorem [Hat60, Corollary 4.2], one computes

$$H^2_{dR}\left(N^6(c);\mathbb{R}\right) \;=\; \mathbb{R}\left\langle e^1\wedge e^2,\; e^3\wedge e^6 - e^4\wedge e^5,\; e^3\wedge e^6 + e^4\wedge e^5\right\rangle \;,$$

proving that $\left(N^6(c),\, J\right)$ is $\mathcal{C}^\infty$-pure-and-full and pure-and-full: indeed, the above harmonic representatives with respect to the $(\mathrm{Sol}(3)\times\mathrm{Sol}(3))$-left-invariant metric $g := \sum_{j=1}^{6} e^j \odot e^j$ are of pure type with respect to J, and hence [FT10, Theorem 3.7] assures the $\mathcal{C}^\infty$-pure-and-fullness and the pure-and-fullness. Note that

$$H^{(2,0),(0,2)}_J\left(N^6(c);\mathbb{R}\right) \;=\; \mathbb{R}\left\langle e^3\wedge e^6 + e^4\wedge e^5\right\rangle \;;$$

apply J. Lee's construction [Lee04, §1] to the real J-anti-invariant 2-form

$$\gamma \;:=\; e^3\wedge e^6 + e^4\wedge e^5 \;:$$

the linear map $V \in \mathrm{End}(TX)$ representing γ as in (2.3.3) is

$$V \;=\; \left(\begin{array}{cc|cc|cc} 0 & & & & & \\ & 0 & & & & \\ \hline & & & & & -1 \\ & & & & -1 & \\ \hline & & & 1 & & \\ & & 1 & & & \end{array}\right) \;\in\; \mathrm{End}\left(TN^6(c)\right) \;,$$

and then it is straightforward to compute

$$L \;=\; \left(\begin{array}{cc|cc|cc} 0 & & & & & \\ & 0 & & & & \\ \hline & & & & -\frac{1}{2} & \\ & & & & & \frac{1}{2} \\ \hline & & \frac{1}{2} & & & \\ & & & -\frac{1}{2} & & \end{array}\right) \;\in\; \mathrm{End}\left(TN^6(c)\right) \;,$$

and

$$J_t \;=:\; J_{t,\,\gamma} \;=\; \left(\begin{array}{cc|cc|cc} & -1 & & & & \\ 1 & & & & & \\ \hline & & & -\frac{4-t^2}{4+t^2} & & -\frac{4t}{4+t^2} \\ & & \frac{4-t^2}{4+t^2} & & -\frac{4t}{4+t^2} & \\ \hline & & & \frac{4t}{4+t^2} & & -\frac{4-t^2}{4+t^2} \\ & & \frac{4t}{4+t^2} & & \frac{4-t^2}{4+t^2} & \end{array}\right) \;\in\; \mathrm{End}\left(TN^6(c)\right) \;.$$

To shorten the notation, set

$$\alpha \;=:\; \alpha(t) \;:=\; \frac{4-t^2}{4+t^2}\;, \qquad \beta \;=:\; \beta(t) \;:=\; \frac{4t}{4+t^2}\;.$$

A co-frame for the J_t-holomorphic cotangent bundle is given by

$$\left\{\begin{array}{rcl} \varphi_t^1 &:=& e^1 + \mathrm{i}\, e^2 \\ \varphi_t^2 &:=& e^3 + \mathrm{i}\left(\alpha\, e^4 + \beta\, e^6\right) \\ \varphi_t^3 &:=& e^5 + \mathrm{i}\left(-\beta\, e^4 + \alpha\, e^6\right) \end{array}\right. \;.$$

Since the real d-closed 2-forms

$$\frac{1}{2\,\mathrm{i}}\,\varphi_t^{1\bar{1}}\;, \qquad \frac{1}{2\,\mathrm{i}}\varphi_t^{3\bar{3}} - \frac{\alpha}{c}\,\mathrm{d}\,e^5\;, \qquad \frac{1}{2\,\mathrm{i}}\left(\beta\,\varphi_t^{2\bar{2}} + \alpha\left(\varphi_t^{2\bar{3}} - \varphi_t^{\bar{2}3}\right)\right) + \frac{1}{2\,\mathrm{i}}\varphi_t^{3\bar{3}}$$

generate three different cohomology classes, we get that, for $t \neq 0$ small enough,

$$H^2_{dR}\left(N^6(c);\mathbb{R}\right) \;=\; H^{(1,1)}_{J_t}\left(N^6(c);\mathbb{R}\right)\;,$$

and so, in particular, J is $\mathcal{C}^\infty$-full and pure. A straightforward computation yields

$$\begin{aligned} H^4_{dR}\left(N^6(c);\mathbb{R}\right) \;&=\; \mathbb{R}\left\langle *_g\left(\frac{1}{2\,\mathrm{i}}\,\varphi_t^{1\bar{1}}\right),\, *_g\left(\varphi_t^{3\bar{3}} - \frac{\alpha}{c}\,\mathrm{d}\,e^5\right) + \frac{\alpha}{c}\,\mathrm{d}\left(e^{125}\right), \right. \\ &\qquad \left. \frac{\alpha}{4}\left(\varphi_t^{12\bar{1}\bar{3}} + \varphi_t^{\bar{1}\bar{2}13}\right) + \frac{\beta}{4}\,\varphi_t^{12\bar{1}\bar{2}} + \frac{\alpha\,\beta}{c}\,\mathrm{d}\left(e^{125}\right)\right\rangle \\ &=\; H^{(2,2)}_{J_t}\left(N^6(c);\mathbb{R}\right)\;, \end{aligned}$$

therefore $N^6(c)$ is also $\mathcal{C}^\infty$-full at the 4^{th} stage and hence full and $\mathcal{C}^\infty$-pure.

We resume the content of the last example in the following theorem, [AT11, Theorem 4.1].

Theorem 2.53. *There exists a compact manifold $N^6(c)$ endowed with an almost-complex structure J and a J-Hermitian metric g such that:*

(i) J is $\mathcal{C}^\infty$-pure-and-full;

(ii) each J-anti-invariant g-harmonic form gives rise to a curve $\{J_t\}_{t\in(-\varepsilon,\varepsilon)}$ of $\mathcal{C}^\infty$-pure-and-full almost-complex structures on $N^6(c)$, where $\varepsilon > 0$ is small enough, using J. Lee's construction;

(iii) furthermore, the function

$$(-\varepsilon,\varepsilon) \;\ni\; t \mapsto \dim_{\mathbb{R}} H^{(2,0),(0,2)}_{J_t}\left(N^6(c);\mathbb{R}\right) \;\in\; \mathbb{N}$$

is upper-semi-continuous at 0.

2.3.2 The semi-continuity problem

Given a compact 4-dimensional manifold X and a family $\{J_t\}_t$ of almost-complex structures on X, T. Drăghici, T.-J. Li, and W. Zhang studied in [DLZ11] the semi-continuity properties of the functions $t \mapsto \dim_{\mathbb{R}} H^+_{J_t}(X)$ and $t \mapsto \dim_{\mathbb{R}} H^-_{J_t}(X)$. They proved the following result.

Theorem 2.54 ([DLZ11, Theorem 2.6])**.** *Let X be a compact 4-dimensional manifold and let $\{J_t\}_{t\in I\subseteq\mathbb{R}}$ be a family of ($\mathcal{C}^\infty$-pure-and-full) almost-complex structures on X, for $I \subseteq \mathbb{R}$ an interval. Then the function*

$$I \ \ni\ t \mapsto \dim_{\mathbb{R}} H^-_{J_t}(X) \ \in\ \mathbb{N}$$

is upper-semi-continuous, and therefore the function

$$I \ \ni\ t \mapsto \dim_{\mathbb{R}} H^+_{J_t}(X) \ \in\ \mathbb{N}$$

is lower-semi-continuous.

The previous result is closely related to the geometry of 4-dimensional manifolds; more precisely, it follows from M. Lejmi's result in [Lej10, Lemma 4.1] that a certain operator is a self-adjoint strongly elliptic linear operator with kernel the harmonic J-anti-invariant 2-forms. In this section, we are concerned with establishing if a similar semi-continuity result could occur in dimension higher than 4, possibly assuming further hypotheses.

Counterexamples to semi-continuity

First of all, we provide two examples showing that, in general, no semi-continuity property holds in dimension higher than 4.

The following result provides a counterexample to the upper-semi-continuity of $t \mapsto \dim_{\mathbb{R}} H^-_{J_t}$ in dimension greater than 4, [AT12a, Proposition 4.1].

Proposition 2.55. *The compact 10-dimensional manifold $\eta\beta_5$ is endowed with a $\mathcal{C}^\infty$-pure-and-full complex structure J and a curve $\{J_t\}_{t\in\Delta(0,\varepsilon)\subset\mathbb{C}}$ of complex structures (which are non-$\mathcal{C}^\infty$-pure for $t \neq 0$), with $J_0 = J$, and $\varepsilon > 0$, such that the function*

$$\Delta\left(0,\varepsilon\right) \ \ni\ t \mapsto \dim_{\mathbb{R}} H^-_{J_t}\left(\eta\beta_5\right) \ \in\ \mathbb{N}$$

is not upper-semi-continuous.

Proof. The proof follows from the following example, [AT12a, Example 4.2].

Consider the nilmanifold $\eta\beta_5$ endowed with its natural complex structure J, as described in Example 2.31. We recall that, chosen a suitable co-frame $\left\{\varphi^j\right\}_{j\in\{1,\ldots,5\}}$ of the holomorphic cotangent bundle, the complex structure equations are

$$\left\{\begin{array}{rcl} \mathrm{d}\,\varphi^1 \ =\ \mathrm{d}\,\varphi^2 \ =\ \mathrm{d}\,\varphi^3 \ =\ \mathrm{d}\,\varphi^4 &=& 0 \\ \mathrm{d}\,\varphi^5 &=& -\varphi^{12}-\varphi^{34} \end{array}\right. .$$

By K. Nomizu's theorem [Nom54, Theorem 1], it is straightforward to compute

$$\begin{aligned} H^2_{dR}\left(\eta\beta_5;\mathbb{C}\right) \ &=\ \mathbb{C}\left\langle \varphi^{13},\,\varphi^{14},\,\varphi^{23},\,\varphi^{24},\,\varphi^{\bar1\bar3},\,\varphi^{\bar1\bar4},\,\varphi^{\bar2\bar3},\,\varphi^{\bar2\bar4},\,\varphi^{12}-\varphi^{34},\,\varphi^{\bar1\bar2}-\varphi^{\bar3\bar4}\right\rangle \\ &\oplus\,\mathbb{C}\left\langle \varphi^{1\bar1},\,\varphi^{1\bar2},\,\varphi^{1\bar3},\,\varphi^{1\bar4},\,\varphi^{2\bar1},\,\varphi^{2\bar2},\,\varphi^{2\bar3},\,\varphi^{2\bar4},\,\varphi^{3\bar1},\,\varphi^{3\bar2},\,\varphi^{3\bar3},\,\varphi^{3\bar4},\,\varphi^{4\bar1},\,\varphi^{4\bar2},\,\varphi^{4\bar3},\,\varphi^{4\bar4}\right\rangle \end{aligned}$$

(where, as usually, we have listed the harmonic representatives with respect to the left-invariant Hermitian metric $\sum_{j=1}^5 \varphi^j\odot\bar\varphi^j$ instead of their classes, and we have shortened, e.g., $\varphi^{A\bar B} := \varphi^A\wedge\bar\varphi^B$). Hence the complex structure J is $\mathcal{C}^\infty$-pure-and-full by [FT10, Theorem 3.7], and

$$\dim_{\mathbb{R}} H^-_J\left(\eta\beta_5\right) \ =\ 10\,, \qquad\qquad \dim_{\mathbb{R}} H^+_J\left(\eta\beta_5\right) \ =\ 16\,.$$

Now, for $\varepsilon > 0$ small enough, consider the curve $\{J_t\}_{t\in\Delta(0,\varepsilon)}$ of complex structures such that a co-frame for the J_t-holomorphic cotangent bundle is given by $\left\{\varphi^j_t\right\}_{j\in\{1,\ldots,5\}}$, where, for any $t \in \Delta\left(0,\varepsilon\right)$,

$$\left\{\begin{array}{rcl} \varphi^1_t &:=& \varphi^1+t\,\bar\varphi^1 \\ \varphi^2_t &:=& \varphi^2 \\ \varphi^3_t &:=& \varphi^3 \\ \varphi^4_t &:=& \varphi^4 \\ \varphi^5_t &:=& \varphi^5 \end{array}\right. ,$$

see Example 2.31. The structure equations with respect to $\left\{\varphi^j_t\right\}_{j\in\{1,\ldots,5\}}$ are

$$\left\{\begin{array}{rcl} \mathrm{d}\,\varphi^1_t &=& \mathrm{d}\,\varphi^2_t \;=\; \mathrm{d}\,\varphi^3_t \;=\; \mathrm{d}\,\varphi^4_t \;=\; 0 \\ \mathrm{d}\,\varphi^5_t &=& -\frac{1}{1-|t|^2}\,\varphi^{12}_t - \varphi^{34}_t - \frac{t}{1-|t|^2}\,\varphi^{2\bar1}_t \end{array}\right. .$$

When $\varepsilon > 0$ is small enough, for $t \in \Delta(0,\varepsilon) \setminus \{0\}$, the complex structure J_t is not $\mathcal{C}^\infty$-pure: indeed,

$$H^{(1,1)}_{J_t}(\eta\beta_5;\mathbb{C}) \;\ni\; \left[\frac{t}{1-|t|^2}\,\varphi^{2\bar1}_t + \mathrm{d}\,\varphi^5_t\right] \;=\; \left[-\frac{1}{1-|t|^2}\,\varphi^{12}_t - \varphi^{34}_t\right] \;\in\; H^{(2,0)}_{J_t}(\eta\beta_5;\mathbb{C}) \;,$$

where $\left[\frac{t}{1-|t|^2}\,\varphi^{2\bar1}_t\right] \in H^2_{dR}(\eta\beta_5;\mathbb{C})$ is a non-zero cohomology class by K. Nomizu's theorem [Nom54, Theorem 1]. Moreover, note that

$$H^{(2,0),(0,2)}_{J_t}(\eta\beta_5;\mathbb{C}) \;\supseteq\; \mathbb{C}\left\langle \varphi^{13}_t,\, \varphi^{14}_t,\, \varphi^{23}_t,\, \varphi^{24}_t,\, \varphi^{\bar1\bar3}_t,\, \varphi^{\bar1\bar4}_t,\, \varphi^{\bar2\bar3}_t,\, \varphi^{\bar2\bar4}_t,\, \varphi^{12}_t,\, \varphi^{34}_t,\, \varphi^{\bar1\bar2}_t,\, \varphi^{\bar3\bar4}_t \right\rangle \;,$$

hence, for every $t \in \Delta(0,\varepsilon) \setminus \{0\}$,

$$\dim_{\mathbb{R}} H^-_{J_0}(\eta\beta_5) \;=\; 10 \;<\; 12 \;\leq\; \dim_{\mathbb{R}} H^-_{J_t}(\eta\beta_5) \;,$$

and in particular $t \mapsto h^-_{J_t}$ is not upper-semi-continuous at 0. $\square$

The following result provides a counterexample to the lower-semi-continuity of $t \mapsto \dim_{\mathbb{R}} H^+_{J_t}$ in dimension greater than 4, [AT12a, Proposition 4.3].

Proposition 2.56. *The compact 6-dimensional manifold $\mathbb{S}^3 \times \mathbb{T}^3$ is endowed with a $\mathcal{C}^\infty$-full (non-integrable) almost-complex structure J and a curve $\{J_t\}_{t\in\Delta(0,\varepsilon)\subset\mathbb{C}}$, where $\varepsilon > 0$, of (non-integrable) almost-complex structures (which are not $\mathcal{C}^\infty$-pure), with $J_0 = J$, such that*

$$\Delta(0,\varepsilon) \;\ni\; t \mapsto \dim_{\mathbb{R}} H^+_{J_t}\left(\mathbb{S}^3 \times \mathbb{T}^3\right) \;\in\; \mathbb{N}$$

is not lower-semi-continuous.

Proof. The proof follows from the following example, [AT12a, Example 4.4].

Consider the compact 6-dimensional manifold $\mathbb{S}^3 \times \mathbb{T}^3$, and set a global co-frame $\left\{e^j\right\}_{j\in\{1,\ldots,6\}}$ with respect to which the structure equations are

$$\left(23,\; -13,\; 12,\; 0^3\right) \;;$$

consider the (non-integrable) almost-complex structure J defined requiring that

$$\left\{\begin{array}{rcl} \varphi^1 &:=& e^1 + \mathrm{i}\,e^4 \\ \varphi^2 &:=& e^2 + \mathrm{i}\,e^5 \\ \varphi^3 &:=& e^3 + \mathrm{i}\,e^6 \end{array}\right.$$

generate the $\mathcal{C}^\infty\left(\mathbb{S}^3 \times \mathbb{T}^3;\mathbb{C}\right)$-module of $(1,0)$-forms on $\mathbb{S}^3 \times \mathbb{T}^3$. By the Künneth formula, one computes

$$\begin{array}{rcl} H^2_{dR}(\mathbb{S}^3 \times \mathbb{T}^3;\mathbb{C}) &=& \mathbb{C}\left\langle e^{45},\, e^{46},\, e^{56}\right\rangle \\ &=& \left\langle \varphi^{12} + \varphi^{\bar1\bar2},\, \varphi^{13} + \varphi^{\bar1\bar3},\, \varphi^{23} + \varphi^{\bar2\bar3}\right\rangle = H^-_J\left(\mathbb{S}^3 \times \mathbb{T}^3\right) \\ &=& \left\langle \varphi^{1\bar2} - \varphi^{2\bar1},\, \varphi^{1\bar3} - \varphi^{3\bar1},\, \varphi^{2\bar3} - \varphi^{3\bar2}\right\rangle = H^+_J\left(\mathbb{S}^3 \times \mathbb{T}^3\right) \;. \end{array}$$

For $\varepsilon > 0$ small enough, consider the curve $\{J_t\}_{t\in\Delta(0,\varepsilon)\subset\mathbb{C}}$ of (non-integrable) almost-complex structures defined requiring that, for any $t \in \Delta(0,\varepsilon)$, the J_t-holomorphic cotangent bundle has co-frame

$$\left\{\begin{array}{rcl} \varphi^1_t &:=& \varphi^1 + t\,\bar\varphi^1 \\ \varphi^2_t &:=& \varphi^2 \\ \varphi^3_t &:=& \varphi^3 \end{array}\right. .$$

By using the F. A. Belgun symmetrization trick, [Bel00, Theorem 7], we have that, for $t \in \Delta(0,\varepsilon) \setminus \mathbb{R}$,

$$\left[\varphi^{1\bar{2}} - \varphi^{2\bar{1}}\right] \;=\; \left[\frac{1}{1-|t|^2}\left(\varphi_t^{1\bar{2}} - \varphi_t^{2\bar{1}}\right) - \frac{1}{1-|t|^2}\left(\bar{t}\,\varphi_t^{12} + t\,\varphi_t^{\bar{1}\bar{2}}\right)\right] \;\not\in\; H^+_{J_t}\left(\mathbb{S}^3\times\mathbb{T}^3\right)$$

and

$$\left[\varphi^{1\bar{3}} - \varphi^{3\bar{1}}\right] \;=\; \left[\frac{1}{1-|t|^2}\left(\varphi_t^{1\bar{3}} - \varphi_t^{3\bar{1}}\right) - \frac{1}{1-|t|^2}\left(\bar{t}\,\varphi_t^{13} + t\,\varphi_t^{\bar{1}\bar{3}}\right)\right] \;\not\in\; H^+_{J_t}\left(\mathbb{S}^3\times\mathbb{T}^3\right) \;;$$

indeed, the terms $\psi_1 := \bar{t}\,\varphi_t^{12} + t\,\varphi_t^{\bar{1}\bar{2}}$, respectively $\psi_2 := \bar{t}\,\varphi^{13} + t\,\varphi^{\bar{1}\bar{3}}$, cannot be written as the sum of a J_t-invariant form and a d-exact form: on the contrary, since ψ_1, and ψ_2 are left-invariant, applying Belgun's symmetrization map, [Bel00, Theorem 7], we can suppose that the J_t-anti-invariant component of the d-exact term is actually the J_t-anti-invariant component of the differential of a left-invariant 1-form; but the image of the differential on the space of left-invariant 1-forms is

$$\begin{aligned} \mathrm{d}\wedge^1\mathfrak{g}^*_{\mathbb{C}} \;&=\; \mathbb{C}\left\langle \varphi_t^{23} + \varphi_t^{2\bar{3}} - \varphi_t^{3\bar{2}} + \varphi_t^{\bar{2}\bar{3}},\, (1-\bar{t})\,\varphi_t^{13} + (1-\bar{t})\,\varphi^{1\bar{3}} - (1-t)\,\varphi^{3\bar{1}} + (1-t)\,\varphi^{\bar{1}\bar{3}},\right. \\ &\left. (1-\bar{t})\,\varphi_t^{12} + (1-\bar{t})\,\varphi^{1\bar{2}} - (1-t)\,\varphi^{2\bar{1}} + (1-t)\,\varphi^{\bar{1}\bar{2}} \right\rangle \;, \end{aligned}$$

and hence one should have $t \in \mathbb{R}$. Hence, we have that, for $t \in \Delta(0,\varepsilon) \setminus \mathbb{R}$,

$$\dim_{\mathbb{R}} H^+_{J_t}\left(\mathbb{S}^3\times\mathbb{T}^3\right) \;=\; 1 \;<\; 3 \;=\; \dim_{\mathbb{R}} H^+_{J_0}\left(\mathbb{S}^3\times\mathbb{T}^3\right) \;,$$

and consequently, in particular, $t \mapsto \dim_{\mathbb{R}} H^+_{J_t}\left(\mathbb{S}^3\times\mathbb{T}^3\right)$ is not lower-semi-continuous at 0. □

Semi-continuity in a stronger sense

Note that Proposition 2.55 and Proposition 2.56 force us to consider stronger conditions under which semi-continuity may occur, or to slightly modify the statement of the semi-continuity problem.

We turn our attention to the aim of giving a more precise statement of the semi-continuity problem. We notice that, for a compact 4-dimensional manifold X endowed with a family $\{J_t\}_{t\in\Delta(0,\varepsilon)}$ of almost-complex structures, one does not have only the semi-continuity properties of $t \mapsto \dim_{\mathbb{R}} H^+_{J_t}(X)$ and $t \mapsto \dim_{\mathbb{R}} H^-_{J_t}(X)$, but one gets also that every J_0-invariant class admits a J_t-invariant class *close to it*. This is also a sufficient condition to assure that, if α is a J_0-compatible symplectic structure on X, then there is a J_t-compatible symplectic structure α_t on X for t small enough. Therefore, we are interested in the following problem.

Let X be a compact manifold endowed with an almost-complex structure J and with a curve $\{J_t\}_{t\in(-\varepsilon,\,\varepsilon)\subset\mathbb{R}}$ of almost-complex structures, where $\varepsilon > 0$ is small enough, such that $J_0 = J$. Suppose that

$$H^+_J(X) \;=\; \mathbb{C}\left\langle \left[\alpha^1\right],\, \ldots,\, \left[\alpha^k\right] \right\rangle \;,$$

where $\alpha^1, \ldots, \alpha^k$ are forms of type $(1,1)$ with respect to J. We look for further hypotheses assuring that, for every $t \in (-\varepsilon,\, \varepsilon)$,

$$H^+_{J_t}(X) \;\supseteq\; \mathbb{C}\left\langle \left[\alpha^1_t\right],\, \ldots,\, \left[\alpha^k_t\right] \right\rangle$$

with

$$\alpha^j_t \;=\; \alpha^j + \mathrm{o}(1) \;.$$

In this case, $(-\varepsilon,\, \varepsilon) \ni t \mapsto \dim_{\mathbb{R}} H^+_{J_t}(X) \in \mathbb{N}$ is a lower-semi-continuous function at 0.

Concerning this problem, we have the following result, [AT12a, Proposition 4.5].

Proposition 2.57. *Let X be a compact manifold endowed with an almost-complex structure J. Take $L \in \mathrm{End}(TX)$ and consider the curve $\{J_t\}_{t\in(-\varepsilon,\,\varepsilon)\subset\mathbb{R}}$ of almost-complex structures defined by*

$$J_t \;:=\; (\mathrm{id} - t\,L)\; J\; (\mathrm{id} - t\,L)^{-1} \;\in\; \mathrm{End}(TX) \;,$$

where $\varepsilon > 0$ is small enough. For every $[\alpha] \in H^+_J(X)$ with $\alpha \in \wedge^{1,1}_J(X) \cap \wedge^2 X$, the following conditions are equivalent:

(i) there exists a family $\{\eta_t = \alpha + \mathrm{o}(1)\}_{t\in(-\varepsilon,\,\varepsilon)} \subseteq \wedge^{1,1}_{J_t}(X) \cap \wedge^2 X$ of real 2-forms, with $\varepsilon > 0$ small enough, depending real-analytically in t and such that $\mathrm{d}\,\eta_t = 0$, for every $t \in (-\varepsilon,\, \varepsilon)$;

(ii) there exists $\{\beta_j\}_{j\in\mathbb{N}\setminus\{0\}} \subseteq \wedge^2 X$ *solution of the system*

$$\mathrm{d}\left(\beta_j + 2\,\alpha\left(L^j\cdot,\,\cdot\cdot\right) + 4\sum_{k=1}^{j-1}\alpha\left(L^{j-k}\cdot,\,L^k\cdot\cdot\right) + 2\,\alpha\left(\cdot,\,L^j\cdot\cdot\right)\right.$$
$$\left. + \sum_{h=1}^{j-1}\left(2\,\beta_h\left(L^{j-h}\cdot,\,\cdot\cdot\right) + 4\sum_{k=1}^{j-h-1}\alpha\left(L^{j-h-k}\cdot,\,L^k\cdot\cdot\right) + 2\,\alpha\left(\cdot,\,L^{j-h}\cdot\cdot\right)\right)\right) \;=\; 0\;, \qquad (2.3.4)$$

varying $j \in \mathbb{N}\setminus\{0\}$*, such that* $\sum_{j\geq 1} t^j\,\beta_j$ *converges.*

In particular, the first order obstruction to the existence of η_t *as in (i) reads: there exists* $\beta_1 \in \wedge^2 X$ *such that*

$$\mathrm{d}\left(\beta_1 + 2\,\alpha(L\cdot,\,\cdot\cdot) + 2\,\alpha(\cdot,\,L\cdot\cdot)\right) \;=\; 0\;. \qquad (2.3.5)$$

Proof. Expanding J_t in power series with respect to t, one gets

$$J_t \;=\; J + \sum_{j\geq 1} 2\,t^j\,J\,L^j\;,$$

and then, for every $\varphi \in \wedge^2 X$, one computes

$$J_t\,\varphi(\cdot,\,\cdot\cdot) \;=\; J\,\varphi(\cdot,\,\cdot\cdot) + 2t\,J\,\left(\varphi(L\cdot,\,\cdot\cdot) + \varphi(\cdot,\,L\cdot\cdot)\right) + \mathrm{o}\left(|t|\right)$$

and

$$\mathrm{d}^c_{J_t}\,\varphi \;=\; J_t^{-1}\,\mathrm{d}\,J_t\,\varphi \;=\; \mathrm{d}^c_J\,\varphi + 2t\,J_t\,\mathrm{d}\,J\,\left(\varphi(L\cdot,\,\cdot\cdot) + \varphi(\cdot,\,L\cdot\cdot)\right) + \mathrm{o}\left(|t|\right)\;.$$

Now, given $[\alpha] \in H^+_J(X)$ with $\alpha \in \wedge^{1,1}_J(X) \cap \wedge^2 X$, let $\{\beta_j\}_j$ be such that (2.3.4) holds and $\sum_{j\geq 1} t^j\,\beta_j$ converges, for $t \in (-\varepsilon,\,\varepsilon)$ with $\varepsilon > 0$ small enough; we define

$$\alpha_t \;:=\; \alpha + \sum_{j\geq 1} t^j\,\beta_j \;\in\; \wedge^2 X$$

and

$$\eta_t \;:=\; \frac{\alpha_t + J_t\,\alpha_t}{2} \;\in\; \wedge^{1,1}_{J_t} X \cap \wedge^2 X\;.$$

By construction, η_t is a J_t-invariant real 2-form, real-analytic in t, and such that $\eta_t = \alpha + \mathrm{o}\,(1)$. A straightforward computation yields

$$\mathrm{d}\,\eta_t \;=\; \sum_{j\geq 1} t^j\,\mathrm{d}\left(\beta_j + 2\,\alpha\left(L^j\cdot,\,\cdot\cdot\right) + 4\sum_{k=1}^{j-1}\alpha\left(L^{j-k}\cdot,\,L^k\cdot\cdot\right) + 2\,\alpha\left(\cdot,\,L^j\cdot\cdot\right)\right.$$
$$\left. + \sum_{h=1}^{j-1}\left(2\,\beta_h\left(L^{j-h}\cdot,\,\cdot\cdot\right) + 4\sum_{k=1}^{j-h-1}\alpha\left(L^{j-h-k}\cdot,\,L^k\cdot\cdot\right) + 2\,\alpha\left(\cdot,\,L^{j-h}\cdot\cdot\right)\right)\right)$$

therefore $\mathrm{d}\,\eta_t = 0$.

Conversely, given $[\alpha] \in H^+_J(X)$ with $\alpha \in \wedge^{1,1}_J(X) \cap \wedge^2 X$, let $\eta_t \in \wedge^{1,1}_{J_t}(X) \cap \wedge^2 X$ be real-analytic in t and such that $\eta_t = \alpha + \mathrm{o}\,(1)$ and $\mathrm{d}\,\eta_t = 0$, for every $t \in (-\varepsilon,\,\varepsilon)$ with $\varepsilon > 0$ small enough. Defining $\beta_j \in \wedge^2 X$, for every $j \in \mathbb{N}\setminus\{0\}$, such that

$$\eta_t \;=:\; \alpha + \sum_{j\geq 1} t^j\,\beta_j\;,$$

by the same computation we have that (2.3.4) holds, being $\mathrm{d}\,\eta_t = \mathrm{d}\left(\frac{\eta_t + J_t\,\eta_t}{2}\right) = 0$. □

Remark 2.58. We notice that, if $\mathrm{d}\,J_t \;=\; \pm J_t\,\mathrm{d}$ on $\wedge^2 M$ for any t, then one can simply let

$$\eta_t \;:=\; \frac{\alpha + J_t\,\alpha}{2}$$

so that $\eta_t \in \wedge^{1,1}_{J_t}$ and $\mathrm{d}\,\eta_t \;=\; 0$. This is the case, for example, if any J_t is an Abelian complex structure; C. Maclaughlin, H. Pedersen, Y. S. Poon, and S. Salamon characterized in [MPPS06, Theorem 6] the 2-step nilmanifolds whose complex deformations are Abelian.

Counterexample to the stronger semi-continuity

In the following example, we provide an application of Proposition 2.57, showing a curve of almost-complex structures that does not have the semi-continuity property in the stronger sense described above, [AT12a, Example 4.8].

Example 2.59. *A curve of almost-complex structures that does not satisfy* (2.3.5)*.*
As in Example 2.46 and in Example 2.52, consider, for suitable $c \in \mathbb{R}$, [AGH63, §3], the solvmanifold

$$N^6(c) \;:=\; (\Gamma(c) \backslash \mathrm{Sol}(3)) \times (\Gamma(c) \backslash \mathrm{Sol}(3)) \;,$$

which has been studied in [FMS03, Example 3.4] as an example of a cohomologically Kähler manifold without Kähler structures, see also [BG90, Example 1]. In the following, we consider $N^6 := N^6(1)$. We recall that, with respect to a suitable co-frame, the structure equations of N^6 are

$$(12,\; 0,\; -36,\; 24,\; 56,\; 0) \;.$$

We look for a curve $\{J_t\}_{t\in(-\varepsilon,\varepsilon)\subset\mathbb{R}}$ of almost-complex structures on N^6, where $\varepsilon > 0$ is small enough, and for a J_0-invariant form α that do not satisfy the first-order obstruction (2.3.5) to the stronger semi-continuity problem stated above: therefore, there will not be a J_t-invariant class close to α, for any $t \in (-\varepsilon, \varepsilon)$.

Consider the almost-complex structure represented by

$$J \;=\; \left(\begin{array}{ccc|ccc} & & & -1 & & \\ & & & & -1 & \\ & & & & & -1 \\ \hline 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \end{array}\right) \;\in\; \mathrm{End}\left(TN^6\right) \;,$$

and

$$L \;=\; \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{B} & -\mathbf{A} \end{array}\right) \;\in\; \mathrm{End}\left(TN^6\right) \;,$$

where

$$\mathbf{A} \;=\; \left(a^j_i\right)_{i,j\in\{1,2,3\}} \;, \qquad \mathbf{B} \;=\; \left(b^j_i\right)_{i,j\in\{1,2,3\}}$$

are constant matrices; for

$$\alpha \;=\; e^{14}$$

we have

$$\mathrm{d}\left(\alpha(L\cdot,\,\cdot\cdot) + \alpha(\cdot,\,L\cdot\cdot)\right) \;=\; b^3_1\, e^{123} + a^2_1\, e^{125} - a^3_1\, e^{126} + b^3_1\, e^{136} - a^2_1\, e^{156} + a^3_1\, e^{234} - b^2_1\, e^{245} - b^3_1\, e^{246} + a^3_1\, e^{346} + b^2_1\, e^{456} \;.$$

Then, choosing

$$L \;=\; \left(\begin{array}{ccc|ccc} & & & & & b^3_1 \\ & & & & 0 & \\ & & & 0 & & \\ \hline & & b^3_1 & & & \\ & 0 & & & & \\ 0 & & & & & \end{array}\right) \;\in\; \mathrm{End}\left(TN^6\right)$$

with $b^3_1 \in \mathbb{R} \setminus \{0\}$, it is straightforward to check that there is no $(\mathrm{Sol}(3) \times \mathrm{Sol}(3))$-left-invariant $\beta \in \wedge^2 N^6$ such that

$$\mathrm{d}\,\beta \;=\; b^3_1\, e^{123} + b^3_1\, e^{136} - b^3_1\, e^{246} \;; \tag{2.3.6}$$

hence, by applying the F. A. Belgun symmetrization trick, [Bel00, Theorem 7], there is no (possibly non-$(\mathrm{Sol}(3) \times \mathrm{Sol}(3))$-left-invariant) $\beta \in \wedge^2 N^6$ satisfying (2.3.6).

We resume the content of the last example in the following proposition, [AT12a, Proposition 4.9].

Proposition 2.60. *There exist a compact manifold X endowed with a $\mathcal{C}^\infty$-pure-and-full almost-complex structure J_0, and a curve $\{J_t\}_{t\in(-\varepsilon,\varepsilon)}$ of almost-complex structures on X, where $\varepsilon > 0$ is small enough, such that, for every $t \in (-\varepsilon, \varepsilon)$, there is no J_t-invariant class, real-analytic in t, close to any fixed J_0-invariant class.*

2.4 Cones of metric structures

In introducing and studying the subgroups $H_J^{(\bullet,\bullet)}(X;\mathbb{R})$ on a compact manifold X endowed with an almost-complex structure J, T.-J. Li and W. Zhang were mainly aimed by the problem of investigating the relations between the J-taming and the J-compatible symplectic cones on X. As follows by their theorem [LZ09, Theorem 1.1], whenever J is $\mathcal{C}^\infty$-full, then the subgroup $H_J^-(X)$ measures the difference between the J-taming cone and the J-compatible cone.

In this section, we discuss some results obtained in [AT12a], jointly with A. Tomassini, giving a counterpart of T.-J. Li and W. Zhang's theorem [LZ09, Theorem 1.1] in the semi-Kähler case, Theorem 2.74, and, in particular, comparing the cones of balanced metrics and of strongly-Gauduchon metrics on a compact complex manifold. Furthermore, concerning the search of a holomorphic-tamed non-Kähler example, [LZ09, page 678], [ST10, Question 1.7], we show that no such example can exist among 6-dimensional nilmanifolds endowed with left-invariant complex structures, Theorem 2.67, as proven in a joint work with A. Tomassini, [AT11].

2.4.1 Sullivan's results on cone structures

Firstly, we recall some results by D. P. Sullivan, [Sul76, §I.1], concerning cone structures on a (differentiable) manifold X.

Fixed $p \in \mathbb{N}$, a *cone structure* of p-directions on X is a continuous field $C :=: \{C(x)\}_{x\in X}$, with $C(x)$ a compact convex cone in $\wedge^p(T_xX)$ for every $x \in X$.

A p-form ω on X is called *transverse* to a cone structure C if $\omega\lfloor_x(v) > 0$ for all $v \in C(x) \setminus \{0\}$ and for all $x \in X$; using the partitions of unity, a transverse form could be constructed for any given C, [Sul76, Proposition I.4].

Every cone structure C gives rise to a cone $\mathfrak{C}$ of *structure currents*, which are by definition the currents generated by the Dirac currents associated to the elements in $C(x)$, see [Sul76, Definition I.4]; the set $\mathfrak{C}$ is a compact convex cone in $\mathcal{D}_pX$.

The cone $\mathcal{Z}\mathfrak{C}$ of the *structure cycles* is defined as the sub-cone of $\mathfrak{C}$ consisting of d-closed currents; denote with $\mathcal{B}$ the set of d-exact currents.

Define the cone $H\mathfrak{C}$ in $H_p^{dR}(X;\mathbb{R})$ as the set of the classes of the structure cycles.

The dual cone of $H\mathfrak{C}$ in $H^p_{dR}(X;\mathbb{R})$ is denoted by $\breve{H}\mathfrak{C}$ and is characterized by the relation

$$\left(\breve{H}\mathfrak{C}, H\mathfrak{C}\right) \;\geq\; 0 \;;$$

its interior is denoted by $\operatorname{int}\breve{H}\mathfrak{C}$ and is characterized by the relation

$$\left(\operatorname{int}\breve{H}\mathfrak{C}, H\mathfrak{C}\right) \;>\; 0 \;.$$

A cone structure of 2-directions is said to be *ample* if, for every $x \in X$, it satisfies that

$$C(x) \cap \operatorname{span}\{e \in S_\tau \;:\; \tau \text{ is a 2-plane}\} \;\neq\; \{0\} \;,$$

where S_τ is the Schubert variety, given by the set of 2-planes intersecting τ in at least one line; by [Sul76, Theorem III.2], an ample cone structure admits non-trivial structure cycles.

When the $2n$-dimensional manifold X is endowed with an almost-complex structure J, the following cone structures turn out to be particularly interesting.

For a fixed $p \in \{0,\ldots,n\}$, let $C_{p,J}$ be the cone $\{C_{p,J}(x)\}_{x\in X}$, where, for every $x \in X$, the compact convex cone $C_{p,J}(x)$ is generated by the positive combinations of p-dimensional complex subspaces of $T_xX \otimes_{\mathbb{R}} \mathbb{C}$ belonging to $\wedge^{2p}(T_xX \otimes_{\mathbb{R}} \mathbb{C})$.

The cone $\mathfrak{C}_{p,J}$ of *complex currents* is defined as the compact convex cone, see [Sul76, §III.4], of the structure currents.

The cone $\mathcal{Z}\mathfrak{C}_{p,J}$ of *complex cycles* is defined as the compact convex cone, see [Sul76, §III.7], of the structure cycles.

The structure cone $C_{1,J}$ is ample, [Sul76, p. 249], therefore it admits non-trivial cycles.

We recall the following theorem by D. P. Sullivan, which follows by Hahn and Banach's theorem.

Theorem 2.61 ([Sul76, Theorem I.7]). *Let X be a compact differentiable manifold (with or without boundary) and let C be a cone structure of p-vectors defined on a compact subspace Y in the interior of X.*

(i) There are always non-trivial structure cycles in Y or closed p-forms on X transversal to the cone structure.

(ii) If no closed transverse form exists, some non-trivial structure cycle in Y is homologous to zero in X.

(iii) If no non-trivial structure cycle exists, some transversal closed form is cohomologous to zero.

(iv) If there are both structure cycles and transversal closed forms, then

(a) the natural map

$$\{\textit{structure cycles on } Y\} \to \{\textit{homology classes in } X\}$$

is proper and the image is a compact cone $\mathcal{C} \subseteq H_p^{dR}(X;\mathbb{R})$, and

(b) the interior of the dual cone $\check{\mathcal{C}} \subseteq H^p_{dR}(X;\mathbb{R})$ (that is, $\check{\mathcal{C}}$ is the cone defined by the relation $\left(\check{\mathcal{C}}, \mathcal{C}\right) \geq 0$) consists precisely of the classes of closed forms transverse to C.

2.4.2 The cones of compatible, and tamed symplectic structures

Let X be a manifold endowed with an almost-complex structure J.

We recall that a symplectic form ω is said to *tame* J if it is positive on the J-lines, that is, if $\omega_x\left(v_x,\, J_x v_x\right) > 0$ for every $v_x \in T_xX \setminus \{0\}$ and for every $x \in X$, equivalently, if

$$\tilde{g}_J\left(\cdot,\, \cdot\cdot\right) \;:=\; \frac{1}{2}\left(\omega\left(\cdot,\, J\cdot\cdot\right) - \omega\left(J\cdot,\, \cdot\cdot\right)\right)$$

is a J-Hermitian metric on X with $\pi_{\wedge^{1,1}X}\omega$ as the associated $(1,1)$-form (the map $\pi_{\wedge^{1,1}X} \colon \wedge^\bullet X \to \wedge^{1,1}X$ being the natural projection onto $\wedge^{1,1}X$). A symplectic form ω is called *compatible* with J if it tames J and it is J-invariant, equivalently, if ω is the $(1,1)$-form associated to the J-Hermitian metric $g_J\left(\cdot,\, \cdot\cdot\right) := \omega(\cdot,\, J\cdot\cdot)$. In particular, an integrable almost-complex structure J is called *holomorphic-tamed* if it admits a taming symplectic form; on the other hand, the datum of an integrable almost-complex structure and a compatible symplectic form gives a Kähler structure.

Symplectic cones and Donaldson's question

Consider the *J-tamed cone* $\mathcal{K}^t_J$, which is defined as the set of the cohomology classes of the J-taming symplectic forms, namely,

$$\mathcal{K}^t_J \;:=\; \left\{[\omega] \in H^2_{dR}(X;\mathbb{R}) \;:\; \omega \text{ is a } J\text{-taming symplectic form on } X\right\} \;,$$

and the *J-compatible cone* $\mathcal{K}^c_J$, which is defined as the set of the cohomology classes of the J-compatible symplectic forms, namely,

$$\mathcal{K}^c_J \;:=\; \left\{[\omega] \in H^2_{dR}(X;\mathbb{R}) \;:\; \omega \text{ is a } J\text{-compatible symplectic form on } X\right\} \;.$$

The set $\mathcal{K}^t_J$ is an open convex cone in $H^2_{dR}(X;\mathbb{R})$, and the set $\mathcal{K}^c_J$ is a convex sub-cone of $\mathcal{K}^t_J$ and it is contained in $H^{(1,1)}_J(X;\mathbb{R})$; moreover, both the sets are sub-cones of the *symplectic cone*

$$\mathcal{S} \;:=\; \left\{[\omega] \in H^2_{dR}(X;\mathbb{R}) \;:\; \omega \text{ is a symplectic form on } X\right\}$$

in $H^2_{dR}(X;\mathbb{R})$.

T.-J. Li and W. Zhang proved the following result in [LZ09], concerning the relation between the J-tamed and the J-compatible cones.

Theorem 2.62 ([LZ09, Proposition 3.1, Theorem 1.1, Corollary 1.1])**.** *Let X be a compact manifold endowed with an almost-Kähler structure J (namely, J is an almost-complex structure on X such that $\mathcal{K}^c_J \neq \varnothing$). Then*

$$\mathcal{K}^t_J \cap H^{(1,1)}_J(X;\mathbb{R}) \;=\; \mathcal{K}^c_J \qquad \textit{and} \qquad \mathcal{K}^c_J + H^{(2,0),(0,2)}_J(X;\mathbb{R}) \subseteq \mathcal{K}^t_J \;.$$

Moreover, if J is $\mathcal{C}^\infty$-full, then

$$\mathcal{K}^t_J \;=\; \mathcal{K}^c_J + H^{(2,0),(0,2)}_J(X;\mathbb{R}) \;.$$

In particular, if $\dim X = 4$ *and $b^+(X) = 1$, then $\mathcal{K}^t_J = \mathcal{K}^c_J$.*

The proof is essentially based on [Sul76, Theorem I.7]. Note indeed that the closed forms transverse to the cone $C_{1,J}$ are exactly the J-taming symplectic forms. By [Sul76, Theorem I.7(iv)(b)], it follows that $\mathcal{K}^t_J$ is the interior of the dual cone $\check{H\mathfrak{C}}_{1,J} \subseteq H^2_{dR}(X;\mathbb{R})$ of $H\mathfrak{C}_{1,J} \subseteq H_2^{dR}(X;\mathbb{R})$, [LZ09, Theorem 3.2]. On the other hand, assumed that $\mathcal{K}^c_J$ is non-empty, by the Hahn and Banach separation theorem, $\mathcal{K}^c_J$ is the interior of the dual cone of $H\mathfrak{C}_{1,J} \subseteq H^J_{(1,1)}(X;\mathbb{R})$, [LZ09, Theorem 3.4]. Finally, when $\dim X = 4$, chosen a J-Hermitian metric g on X with associated $(1,1)$-form ω, one has $\wedge^+_g X = \mathbb{R}\langle\omega\rangle \oplus \wedge^-_J X$, hence, in the almost-Kähler case, if $b^+(X) := \dim_\mathbb{R} H^+_g(X) = 1$, then $H^{(2,0),(0,2)}_J(X;\mathbb{R}) = \{0\}$, see [DLZ10, Proposition 3.1].

Whereas the previous theorem by T.-J. Li and W. Zhang could be intended as a "quantitative comparison" between the J-taming and the J-compatible symplectic cones on a compact manifold X endowed with an almost-complex structure J, one could ask what about their "qualitative comparison", namely, one could ask whether $\mathcal{K}^c_J$ being empty implies $\mathcal{K}^t_J$ being empty, too. The following question has been arisen by S. K. Donaldson in [Don06].

Question 2.63 ([Don06, Question 2])**.** *Let X be a compact 4-dimensional manifold endowed with an almost-complex structure J. If J is tamed by a symplectic form, is there a symplectic form compatible with J?*

Remark 2.64. S. K. Donaldson's "tamed to compatible" question has a positive answer for $\mathbb{CP}^2$ by the works by M. Gromov [Gro85] and by C. H. Taubes, [Tau95]. When $b^+(X) = 1$ (where b^+ is the number of positive eigenvalues of the intersection pairing on $H_2(X;\mathbb{R})$), a possible positive answer to [Don06, Question 2], see also [TWY08, Conjecture 1.2], would be provided as a consequence of [Don06, Conjecture 1], see also [TWY08, Conjecture 1.1], concerning the study of the symplectic Calabi and Yau equation, which aims to generalize S.-T. Yau's theorem [Yau77, Yau78], solving the Calabi conjecture, [Cal57], to the non-integrable case. Some results concerning this problem have been recently obtained by several authors, see, e.g., [Wei07, TWY08, TW11a, Tau11, Zha11, LT12, FLSV11, BFV11], see also [TW11b]. More precisely, in [Wei07], all the estimates for the closedness argument of the continuity method applied to the symplectic Calabi and Yau equation, [Don06, Conjecture 1], are reduced to a $\mathcal{C}^0$ *a priori* estimate of a scalar potential function, [Wei07, Theorem 1]; then, the existence of a solution of the symplectic Calabi and Yau equation is proven for compact 4-dimensional manifolds X endowed with an almost-Kähler structure $(J,\,\omega,\,g)$ satisfying $\|\mathrm{Nij}_J\|_{\mathrm{L}^1} < \varepsilon$, where $\varepsilon > 0$ depends just on the data, [Wei07, Theorem 2]. In [TWY08], it is shown that the $\mathcal{C}^\infty$ *a priori* estimates can be reduced to an integral estimate of a scalar function potential, [TWY08, Theorem 1.3]; furthermore, it is shown that [Don06, Conjecture 1] holds under a positive curvature assumption, [TWY08, Theorem 1.4]. In [TW11a], the symplectic Calabi-Yau equation is solved on the Kodaira-Thurston manifold $\mathbb{S}^1 \times (\mathbb{H}(3;\mathbb{Z})\backslash\mathbb{H}(3;\mathbb{R}))$ for any given left-invariant volume form, [TW11a, Theorem 1.1]; further results on the Calabi-Yau equation for torus-bundles over a 2-dimensional torus have been provided in [FLSV11, BFV11]. In [Tau11], it is shown that, on a compact 4-dimensional manifold with $b^+ = 1$ and endowed with a symplectic form ω, a generic ω-tamed almost-complex structure on X is compatible with a symplectic form on X, [Tau11, Theorem 1], which is defined by integrating over a space of currents that are defined by pseudo-holomorphic curves. The Taubes currents have been studied, both in dimension 4 and higher, also by W. Zhang in [Zha11]. In [LZ11], T.-J. Li and W. Zhang were concerned with studying Donaldson's "tamed to compatible" question for almost-complex structures on rational 4-dimensional manifolds; they provided, in particular, an affirmative answer to [Don06, Question 2] for $\mathbb{S}^2 \times \mathbb{S}^2$ and for $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$, see [DLZ12, Theorem 4.11]. In [LT12], a positive answer to S. K. Donaldson's question [Don06, Question 2] is provided in the Lie algebra setting, proving that, given a 4-dimensional Lie algebra $\mathfrak{g}$ such that $B \wedge B = 0$ (where $B \subseteq \wedge^2\mathfrak{g}$ is the space of boundary 2-vectors), e.g., a 4-dimensional unimodular Lie algebra, a linear (possibly non-integrable) complex structure is tamed by a linear symplectic form if and only if it is compatible with a linear symplectic form, [LT12, Theorem 0.2].

In a sense, [LZ09, Corollary 1.1] provides evidences towards an affirmative answer for [Don06, Question 2], especially in the case $b^+ = 1$; confirmed in their opinion by the computations in [DLZ10] in the case $b^+ > 1$, T.-J. Li and W. Zhang speculated in [LZ09, page 655] that the equality $\mathcal{K}^t_J = \mathcal{K}^c_J$ holds for a generic almost-complex structure J on a 4-dimensional manifold.

The analogous of [Don06, Question 2] in dimension higher than 4 has a negative answer: counterexamples in the (non-integrable) almost-complex case can be found in [MT00] by M. Migliorini and A. Tomassini, and in [Tom02] by A. Tomassini. Notwithstanding, since examples of non-Kähler holomorphic-tamed complex structures are not known, T.-J. Li and W. Zhang speculated a negative answer for the following question, also addressed by J. Streets and G. Tian in [ST10].

Question 2.65 ([LZ09, page 678], [ST10, Question 1.7])**.** *Do there exist non-Kähler holomorphic-tamed complex manifolds, of complex dimension greater than 2?*

Tameness conjecture for 6-dimensional nilmanifolds

In view of the speculation in [LZ09, page 678], and of [ST10, Question 1.7], one could ask whether small deformations of the Iwasawa manifold, see §1.4.1, may provide examples of non-Kähler holomorphic-tamed complex structures. In this section, we prove that this is not the case: more precisely, we prove that no example of left-invariant non-Kähler holomorphic-tamed complex structure can be found on 6-dimensional nilmanifolds. The same holds true, more in general, for higher dimensional nilmanifolds, as proven by N. Enrietti, A. Fino, and L. Vezzoni, [EFV12, Theorem 1.3].

We recall that a Hermitian metric g on a complex manifold X is called *pluri-closed* (or *strong Kähler with torsion*, shortly SKT), [Bis89], if the $(1,1)$-form ω associated to g satisfies $\partial\overline{\partial}\omega = 0$.

By the following result, holomorphic-tamed manifolds admit pluri-closed metrics, [AT11, Proposition 3.1].

Proposition 2.66. *Let X be a manifold endowed with a symplectic structure ω and an ω-tamed complex structure J. Then the $(1,1)$-form $\tilde{\omega} := \tilde{g}_J\left(J\,\cdot,\,\cdot\cdot\right)$ associated to the Hermitian metric $\tilde{g}_J\left(\cdot,\,\cdot\cdot\right) := \frac{1}{2}\left(\omega\left(\cdot,\,J\cdot\cdot\right) - \omega\left(J\cdot,\,\cdot\cdot\right)\right)$ is $\partial\overline{\partial}$-closed, namely, $\tilde{g}$ is a pluri-closed metric on X.*

Proof. Decomposing ω in pure type components, set

$$\omega \;=:\; \omega^{2,0} + \omega^{1,1} + \overline{\omega^{2,0}}$$

where $\omega^{2,0} \in \wedge^{2,0}X$ and $\omega^{1,1} = \overline{\omega^{1,1}} \in \wedge^{1,1}X$. Since, by definition, $\tilde{\omega} = \frac{1}{2}\left(\omega + J\omega\right)$, we have $\tilde{\omega} = \omega^{1,1}$. We get that

$$\mathrm{d}\,\omega \;=\; 0 \quad \Leftrightarrow \quad \begin{cases} \partial\omega^{2,0} \;=\; 0 \\ \partial\omega^{1,1} + \overline{\partial}\omega^{2,0} \;=\; 0 \end{cases} ,$$

and hence

$$\partial\overline{\partial}\tilde{\omega} \;=\; \partial\overline{\partial}\omega^{1,1} \;=\; -\overline{\partial}\partial\omega^{1,1} \;=\; \overline{\partial}^2\omega^{2,0} \;=\; 0\,,$$

proving that $\tilde{g}$ is a pluri-closed metric on X. □

Now, we can prove the announced theorem, [AT11, Theorem 3.3].

Theorem 2.67. *Let $X = \Gamma\backslash G$ be a 6-dimensional nilmanifold endowed with a G-left-invariant complex structure J. If X is not a torus, then there is no symplectic structure ω on X taming J.*

Proof. Let ω be a (non-necessarily G-left-invariant) symplectic form on X taming J. By F. A. Belgun's symmetrization trick, [Bel00, Theorem 7], setting

$$\mu(\omega) \;:=\; \int_X \omega\lfloor_m\, \eta(m)\,,$$

where η is a G-bi-invariant volume form on G such that $\int_X \eta = 1$, whose existence follows from J. Milnor's lemma [Mil76, Lemma 6.2], we get a G-left-invariant symplectic form on X taming J. Then, it suffices to prove that, on a non-torus 6-dimensional nilmanifold, there is no left-invariant symplectic structure taming a left-invariant complex structure.

Hence, let ω be such a G-left-invariant symplectic structure. Then, by Proposition 2.66, X should admit a G-left-invariant pluri-closed Hermitian metric g. Hence, by [FPS04, Theorem 1.2], there exists a co-frame $\{\varphi^1,\, \varphi^2,\, \varphi^3\}$ for the J-holomorphic cotangent bundle such that

$$\begin{cases} \mathrm{d}\,\varphi^1 \;=\; 0 \\ \mathrm{d}\,\varphi^2 \;=\; 0 \\ \mathrm{d}\,\varphi^3 \;=\; A\,\overline{\varphi}^1 \wedge \varphi^2 + B\,\overline{\varphi}^2 \wedge \varphi^2 + C\,\varphi^1 \wedge \overline{\varphi}^1 + D\,\varphi^1 \wedge \overline{\varphi}^2 + E\,\varphi^1 \wedge \varphi^2 \end{cases} ,$$

where $A,\, B,\, C,\, D,\, E \in \mathbb{C}$ are complex numbers such that $|A|^2 + |D|^2 + |E|^2 + 2\,\mathfrak{Re}\left(\bar{B}C\right) = 0$. Set

$$\omega \;=:\; \omega^{2,0} + \omega^{1,1} + \overline{\omega^{2,0}}\,,$$

where

$$\omega^{2,0} \;=\; \sum_{i<j} a_{ij}\,\varphi^i \wedge \varphi^j\,, \qquad \omega^{1,1} \;=\; \frac{\mathrm{i}}{2}\sum_{i,j=1}^{3} b_{i\bar{j}}\,\varphi^i \wedge \overline{\varphi}^j\,,$$

with $\left\{a_{ij},\, b_{i\bar{j}}\right\}_{i,j} \subset \mathbb{C}$ such that $\omega^{1,1} = \overline{\omega^{1,1}}$. A straightforward computation yields

$$\mathrm{d}\,\omega \;=\; 0 \qquad \Leftrightarrow \qquad \left(A \;=\; B \;=\; C \;=\; D \;=\; E \;=\; 0 \quad \text{or} \quad b_{3\bar{3}} \;=\; 0\right)\,.$$

Since $b_{3\bar{3}} \neq 0$, we get $A = B = C = D = E = 0$, namely, X is a torus. □

As a corollary, we get the following result, [AT11, Theorem 3.4], concerning the speculation in [LZ09, p. 678], and [ST10, Question 1.7].

Theorem 2.68. *No small deformation of the complex structure of the Iwasawa manifold* $\mathbb{I}_3 := \mathbb{H}(3;\mathbb{Z}[\mathrm{i}])\backslash\mathbb{H}(3;\mathbb{C})$ *can be tamed by any symplectic form.*

2.4.3 The cones of semi-Kähler, and strongly-Gauduchon metrics

Let X be a compact $2n$-dimensional manifold endowed with an almost-complex structure J. We recall that a non-degenerate 2-form ω on X is called *semi-Kähler*, [GH80, page 40], if ω is the $(1,1)$-form associated to a J-Hermitian metric on X (that is, $\omega(\cdot,\ J\cdot) > 0$ and $\omega(J\cdot,\ J\cdot\cdot) = \omega(\cdot,\ \cdot\cdot)$) and $\mathrm{d}\left(\omega^{n-1}\right) = 0$; when J is integrable, a semi-Kähler structure is called *balanced*, [Mic82, Definition 1.4, Theorem 1.6].

We set

$$\begin{aligned}\mathcal{K}b_J^c \;&:=\; \left\{[\Omega] \in H_{dR}^{2n-2}(X;\mathbb{R}) \;:\; \Omega \in \wedge^{n-1,n-1}X \text{ is positive on the}\right.\\ &\qquad \left.\text{complex } (n-1)\text{-subspaces of } T_xX \otimes_{\mathbb{R}} \mathbb{C},\ \text{for every } x \in X\right\} ,\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}b_J^t \;&:=\; \left\{[\Omega] \in H_{dR}^{2n-2}(X;\mathbb{R}) \;:\; \Omega \in \wedge^{2n-2}X \text{ is positive on the}\right.\\ &\qquad \left.\text{complex } (n-1)\text{-subspaces of } T_xX \otimes_{\mathbb{R}} \mathbb{C},\ \text{for every } x \in X\right\} .\end{aligned}$$

We note that $\mathcal{K}b_J^c$ and $\mathcal{K}b_J^t$ are convex cones in $H_{dR}^{2n-2}(X;\mathbb{R})$, and that $\mathcal{K}b_J^c$ is a sub-cone of $\mathcal{K}b_J^t$ and is contained in $H_J^{(n-1,n-1)}(X;\mathbb{R})$.

We recall the following trick by M. L. Michelsohn.

Lemma 2.69 ([Mic82, pp. 279-280])**.** *Let X be a compact $2n$-dimensional manifold endowed with an almost-complex structure J. Let Φ be a real $(n-1,n-1)$-form such that it is positive on the complex $(n-1)$-subspaces of $T_xX \otimes_{\mathbb{R}} \mathbb{C}$, for every $x \in X$. Then Φ can be written as $\Phi = \varphi^{n-1}$, where φ is a J-taming real $(1,1)$-form. In particular, if Φ is* d*-closed, then φ is a semi-Kähler form.*

The previous Lemma allows us to confuse the cone $\mathcal{K}b_J^c$ with the cone generated by the $(n-1)^{\mathrm{th}}$ powers of the semi-Kähler forms, namely,

$$\mathcal{K}b_J^c = \left\{\left[\omega^{n-1}\right] \;:\; \omega \text{ is a semi-Kähler form on } X\right\} .$$

In particular, if J is integrable, then the cone $\mathcal{K}b_J^c$ is just the cone of balanced structures on X. On the other hand, in the integrable case, $\mathcal{K}b_J^t$ is the cone of strongly-Gauduchon metrics on X. We recall that a *strongly-Gauduchon metric* on X, [Pop09, Definition 3.1], is a positive-definite $(1,1)$-form γ on X such that the $(n,n-1)$-form $\partial\left(\gamma^{n-1}\right)$ is $\overline{\partial}$-exact. These metrics have been introduced by D. Popovici in [Pop09] in studying the limits of projective manifolds under deformations of the complex structure, and they turn out to be special cases of *Gauduchon metrics*, [Gau77], for which $\partial\left(\gamma^{n-1}\right)$ is just $\overline{\partial}$-closed; note that the notions of Gauduchon metric and of strongly-Gauduchon metric coincide if the $\partial\overline{\partial}$-Lemma holds, [Pop09, page 15]. D. Popovici proved in [Pop09, Lemma 3.2] that a compact complex manifold X, of complex dimension n, carries a strongly-Gauduchon metric if and only if there exists a real d-closed $(2n-2)$-form Ω such that its component $\Omega^{(n-1,n-1)}$ of type $(n-1,n-1)$ satisfies $\Omega^{(n-1,n-1)} > 0$.

The aim of this section is to compare the cones $\mathcal{K}b_J^c$ and $\mathcal{K}b_J^t$, Theorem 2.74, in the same way as [LZ09, Theorem 1.1] does for $\mathcal{K}_J^c$ and $\mathcal{K}_J^t$ in the almost-Kähler case.

Note that $\mathcal{K}b_J^t$ can be identified with the set of the classes of d-closed $(2n-2)$-forms transverse to $C_{n-1,J}$. On the other hand, we recall the following lemma.

Lemma 2.70 (see, e.g., [Sil96, Proposition I.1.3])**.** *Let X be a compact manifold endowed with an almost-complex structure J, and fix $p \in \mathbb{N}$. A structure current in $\mathfrak{C}_{p,J}$ is a positive current of bi-dimension (p,p).*

As a direct consequence of [Sul76, Theorem I.7], we get the following result, [AT12a, Theorem 2.6].

Theorem 2.71. *Let X be a compact $2n$-dimensional manifold endowed with an almost-complex structure J. Then $\mathcal{K}b_J^t$ is non-empty if and only if there is no non-trivial* d*-closed positive currents of bi-dimension $(n-1,n-1)$ that is a boundary, i.e.,*

$$\mathcal{Z}\mathfrak{C}_{n-1,J} \cap \mathcal{B} \;=\; \{0\} .$$

Furthermore, if we suppose that $0 \notin \mathcal{K}b_J^t$, then $\mathcal{K}b_J^t \subseteq H_{dR}^{2n-2}(X;\mathbb{R})$ is the interior of the dual cone $\check{H}\mathfrak{C}_{n-1,J} \subseteq H_{dR}^{2n-2}(X;\mathbb{R})$ of $H\mathfrak{C}_{n-1,J} \subseteq H_{2n-2}^{dR}(X;\mathbb{R})$.

Proof. Note that if $\omega \in \mathcal{K}b^t_J \neq \varnothing$, and if $\eta :=: \mathrm{d}\,\xi$ is a non-trivial d-closed positive current of bi-dimension $(n-1, n-1)$ being a boundary, then

$$0 \;<\; (\eta,\, \pi_{\wedge^{n-1,n-1}X}\omega) \;=\; (\eta,\, \omega) \;=\; (\mathrm{d}\,\xi,\, \omega) \;=\; (\xi,\, \mathrm{d}\,\omega) \;=\; 0$$

(where $\pi_{\wedge^{n-1,n-1}X} \colon \wedge^\bullet X \to \wedge^{n-1,n-1}X$ is the natural projection onto $\wedge^{n-1,n-1}X$) yields an absurd.

To prove the converse, suppose that no non-trivial d-closed positive currents of bi-dimension $(n-1, n-1)$ is a boundary; then, by [Sul76, Theorem I.7(ii)], there exists a d-closed form that is transverse to $C_{n-1,J}$, that is, $\mathcal{K}b^t_J$ is non-empty.

The last statement follows from [Sul76, Theorem I.7(iv)]: indeed, by the assumption $0 \notin \mathcal{K}b^t_J$, no d-closed transverse form is cohomologous to zero, therefore, by [Sul76, Theorem I.7(iii)], there exists a non-trivial structure cycle. □

We provide a similar characterization for $\mathcal{K}b^c_J$, [AT12a, Theorem 2.7].

Theorem 2.72. *Let X be a compact $2n$-dimensional manifold endowed with an almost-complex structure J. Suppose that $\mathcal{K}b^c_J \neq \varnothing$ and that $0 \notin \mathcal{K}b^c_J$. Then $\mathcal{K}b^c_J \subseteq H^{(n-1,n-1)}_J(X;\mathbb{R})$ is the interior of the dual cone $\breve{H}\mathfrak{C}_{n-1,J} \subseteq H^{(n-1,n-1)}_J(X;\mathbb{R})$ of $H\mathfrak{C}_{n-1,J} \subseteq H^J_{(n-1,n-1)}(X;\mathbb{R})$.*

Proof. By the hypothesis $0 \notin \mathcal{K}^c_J$, we have that $(\mathcal{K}b^c_J, H\mathfrak{C}_{n-1,J}) > 0$, and therefore the inclusion $\mathcal{K}b^c_J \subseteq \operatorname{int}\breve{H}\mathfrak{C}_{n-1,J}$ holds.

To prove the other inclusion, let $e \in H^{(n-1,n-1)}_J(X;\mathbb{R})$ be an element in the interior of the dual cone in $H^{(n-1,n-1)}_J$ of $H\mathfrak{C}_{n-1,J}$, i.e., e is such that $(e, H\mathfrak{C}_{n-1,J}) > 0$. Consider the isomorphism

$$\overline{\sigma}^{n-1,n-1} \colon H^{(n-1,n-1)}_J(X;\mathbb{R}) \stackrel{\simeq}{\to} \left(\frac{\overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{Z}}}{\overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{B}}}\right)^* ,$$

[LZ09, Proposition 2.4] (where $\pi_{\mathcal{D}_{n-1,n-1}X} \colon \mathcal{D}_\bullet X \to \mathcal{D}_{n-1,n-1}X$ denotes the natural projection onto $\mathcal{D}_{n-1,n-1}X$): hence, $\overline{\sigma}^{n-1,n-1}(e)$ gives rise to a functional on $\frac{\overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{Z}}}{\overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{B}}}$, namely, to a functional on $\overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{Z}}$ vanishing on $\overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{B}}$; such a functional, in turn, gives rise to a hyperplane L in $\overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{Z}}$ containing $\overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{B}}$. Being a kernel hyperplane in a closed set, L is closed in $\mathcal{D}_{n-1,n-1}X \cap \mathcal{D}_{2n-2}X$; furthermore, L is disjoint from $\mathfrak{C}_{n-1,J} \setminus \{0\}$, by the choice made for e. Pick a J-Hermitian metric and let φ be its associated $(1,1)$-form; consider

$$K \;:=\; \left\{T \in \mathfrak{C}_{n-1,J} \;:\; T\left(\varphi^{n-1}\right) = 1\right\} ,$$

which is a compact set. Now, in the space $\mathcal{D}_{n-1,n-1}X \cap \mathcal{D}_{2n-2}X$, consider the closed set L, and the compact convex non-empty set K, which have empty intersection. By the Hahn and Banach separation theorem, there exists a hyperplane containing L, and then containing also $\overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{B}}$, and disjoint from K. The functional on $\mathcal{D}_{n-1,n-1}X \cap \mathcal{D}_{2n-2}X$ associated to this hyperplane is a real $(n-1,n-1)$-form being d-closed, since it vanishes on $\overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{B}}$, and positive on the complex $(n-1)$-subspaces of $T_xX \otimes_{\mathbb{R}} \mathbb{C}$, for every $x \in X$, that is, a J-compatible symplectic form. □

The same argument as in [HL83, Proposition 12, Theorem 14] yields the following result, [AT12a, Theorem 2.8], which generalizes [HL83, Proposition 12, Theorem 14], [LZ09, page 671], see also [Mic82, Theorem 4.7].

Theorem 2.73 ([HL83, Proposition 12, Theorem 14], [LZ09, page 671], [AT12a, Theorem 2.8])**.** *Let X be a compact $2n$-dimensional manifold endowed with an almost-complex structure J, and denote by $\pi_{\mathcal{D}_{k,k}X} \colon \mathcal{D}_\bullet X \to \mathcal{D}_{k,k}X$ the natural projection onto $\mathcal{D}_{k,k}X$, for every $k \in \mathbb{N}$.*

(i) If J is integrable, then there exists a Kähler metric if and only if $\mathfrak{C}_{1,J} \cap \pi_{\mathcal{D}_{1,1}X}\mathcal{B} = \{0\}$.

(ii) There exists an almost-Kähler metric if and only if $\mathfrak{C}_{1,J} \cap \overline{\pi_{\mathcal{D}_{1,1}X}\mathcal{B}} = \{0\}$.

(iii) There exists a semi-Kähler metric if and only if $\mathfrak{C}_{n-1,J} \cap \overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{B}} = \{0\}$.

Proof. Note that *(i)*, namely, [HL83, Proposition 12, Theorem 14], is a consequence of *(ii)*: indeed, if J is integrable, then J is closed, [HL83, Lemma 6], that is, $\pi_{\mathcal{D}_{1,1}X}\mathcal{B}$ is a closed set.

The proof of *(ii)*, namely, [LZ09, page 671], being similar, we prove *(iii)*, following closely the proof of *(i)* in [HL83, Proposition 12, Theorem 14].

Firstly, note that if ω is a semi-Kähler form and

$$0 \;\neq\; \eta \;:=:\; \lim_{k\to+\infty} \pi_{\mathcal{D}_{n-1,n-1}X}\left(\mathrm{d}\,\alpha_k\right) \;\in\; \mathfrak{C}_{n-1,J} \cap \overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{B}} \;\neq\; \{0\} ,$$

where $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathcal{D}_{2n-1}X$, then

$$0 \;<\; \left(\eta,\, \omega^{n-1}\right) \;=\; \left(\lim_{k\to+\infty} \pi_{\mathcal{D}_{n-1,n-1}X}\left(\mathrm{d}\,\alpha_k\right),\, \omega^{n-1}\right) \;=\; \lim_{k\to+\infty}\left(\mathrm{d}\,\alpha_k,\, \omega^{n-1}\right) \;=\; \lim_{k\to+\infty}\left(\alpha_k,\, \mathrm{d}\,\omega^{n-1}\right) \;=\; 0\,,$$

yielding an absurd.

For the converse, fix a J-Hermitian metric and let φ be its associated $(1,1)$-form; the set

$$K \;:=\; \left\{T \in \mathfrak{C}_{n-1,J} \;:\; T\left(\varphi^{n-1}\right) = 1\right\}$$

is a compact convex non-empty set in $\mathcal{D}_{n-1,n-1}X \cap \mathcal{D}_{2n-2}X$. By the Hahn and Banach separation theorem, there exists a hyperplane in $\mathcal{D}_{n-1,n-1}X \cap \mathcal{D}_{2n-2}X$ containing the closed subspace $\overline{\pi_{\mathcal{D}_{n-1,n-1}X}\mathcal{B}}$ and disjoint from K; hence, the real $(n-1,n-1)$-form associated to this hyperplane is a real d-closed $(n-1,n-1)$-form and is positive on the complex $(n-1)$-subspaces, namely, it is a semi-Kähler form. □

Now, we can prove the semi-Kähler counterpart, [AT12a, Theorem 2.9], of T.-J. Li and W. Zhang's [LZ09, Proposition 3.1, Theorem 1.1].

Theorem 2.74. *Let X be a compact $2n$-dimensional manifold endowed with an almost-complex structure J. Assume that $\mathcal{K}b_J^c \neq \varnothing$ (that is, there exists a semi-Kähler structure on X) and that $0 \notin \mathcal{K}b_J^t$. Then*

$$\mathcal{K}b_J^t \cap H_J^{(n-1,n-1)}(X;\mathbb{R}) \;=\; \mathcal{K}b_J^c$$

and

$$\mathcal{K}b_J^c + H_J^{(n,n-2),(n-2,n)}(X;\mathbb{R}) \;\subseteq\; \mathcal{K}b_J^t\,.$$

Moreover, if J is $\mathcal{C}^\infty$-full at the $(2n-2)^{th}$ stage, then

$$\mathcal{K}b_J^c + H_J^{(n,n-2),(n-2,n)}(X;\mathbb{R}) \;=\; \mathcal{K}b_J^t\,.$$

Proof. By Theorem 2.71, $\mathcal{K}b_J^t \subseteq H_{dR}^{2n-2}(X;\mathbb{R})$ is the interior of the dual cone $\check{H}\mathfrak{C}_{n-1,J} \subseteq H_{dR}^{2n-2}(X;\mathbb{R})$ of $H\mathfrak{C}_{n-1,J} \subseteq H^{dR}_{2n-2}(X;\mathbb{R})$, and, by Theorem 2.72, $\mathcal{K}b_J^c \subseteq H_J^{(n-1,n-1)}(X;\mathbb{R})$ is the interior of the dual cone $\check{H}\mathfrak{C}_{n-1,J} \subseteq H_J^{(n-1,n-1)}(X;\mathbb{R})$ of $H\mathfrak{C}_{n-1,J} \subseteq H^J_{(n-1,n-1)}(X;\mathbb{R})$; therefore $\mathcal{K}b_J^t \cap H_J^{(n-1,n-1)}(X;\mathbb{R}) = \mathcal{K}b_J^c$.

The inclusion $\mathcal{K}b_J^c + H_J^{(n,n-2),(n-2,n)}(X;\mathbb{R}) \subseteq \mathcal{K}b_J^t$ follows straightforwardly noting that the sum of a semi-Kähler form and a J-anti-invariant $(2n-2)$-form is still d-closed and positive on the complex $(n-1)$-subspaces.

Finally, if J is $\mathcal{C}^\infty$-full at the $(2n-2)^{\mathrm{th}}$ stage, then

$$\begin{aligned}
\mathcal{K}b_J^t &= \operatorname{int}\check{H}\mathfrak{C}_{n-1,J} \;=\; \operatorname{int}\check{H}\mathfrak{C}_{n-1,J} \cap H_{dR}^{2n-2}(X;\mathbb{R}) \\
&= \operatorname{int}\check{H}\mathfrak{C}_{n-1,J} \cap \left(H_J^{(n-1,n-1)}(X;\mathbb{R}) + H_J^{(n,n-2),(n-2,n)}(X;\mathbb{R})\right) \\
&\subseteq \mathcal{K}b_J^c + H_J^{(n,n-2),(n-2,n)}(X;\mathbb{R})\,,
\end{aligned}$$

and hence $\mathcal{K}b_J^c + H_J^{(n,n-2),(n-2,n)}(X;\mathbb{R}) = \mathcal{K}b_J^t$. □

Remark 2.75. We note that, while the de Rham cohomology class of an almost-Kähler metric cannot be trivial, the hypothesis $0 \notin \mathcal{K}b_J^t$ in Theorem 2.74 is not trivial: J. Fu, J. Li, and S.-T. Yau proved in [FLY12, Corollary 1.3] that, for any $k \geq 2$, the connected sum $\left(\mathbb{S}^3 \times \mathbb{S}^3\right)^{\sharp k}$, endowed with the complex structure constructed from the conifold transitions, admits balanced metrics.

Chapter 3

Cohomology of manifolds with special structures

In this chapter, we continue in studying the cohomological properties of (differentiable) manifolds endowed with special structures, other than (almost-)complex structures. More precisely, in Section 3.1, we recall the results obtained jointly with A. Tomassini in [AT12c], concerning the cohomology of symplectic manifolds; in Section 3.2, we study cohomological decompositions on **D**-complex manifolds in the sense of F. R. Harvey and H. B. Lawson: this has been the matter of a joint work with F. A. Rossi, [AR12]; finally, in Section 3.3, we consider domains in $\mathbb{R}^n$ endowed with a smooth proper strictly p-convex exhaustion function, and, using L^2-techniques, we give another proof of a consequence of J.-P. Sha's theorem [Sha86, Theorem 1], and H. Wu's theorem [Wu87, Theorem 1], on the vanishing of the higher degree de Rham cohomology groups, which has been obtained in a joint work with S. Calamai, [AC12].

3.1 Cohomology of symplectic manifolds

The Kähler manifolds have special cohomological properties from both the complex and the symplectic point of view, the Hodge decomposition theorem providing a decomposition of the complex de Rham cohomology in terms of the Dolbeault cohomology groups, and the Lefschetz decomposition theorem providing a decomposition of the de Rham cohomology in terms of primitive cohomology groups. Then, in order to better understand the geometry of non-Kähler manifolds, it may be interesting to investigate both the contribution of the complex structure and the contribution of the symplectic structure.

In this section, we develop the symplectic counterpart of the theory introduced by T.-J. Li and W. Zhang in [LZ09] to study the cohomology of almost-complex manifolds. The results in this section have been obtained jointly with A. Tomassini in [AT12c].

3.1.1 Hodge theory on symplectic manifolds

Cohomological properties of symplectic manifolds have been studied starting from the works by J.-L. Koszul, [Kos85], and by J.-L. Brylinski, [Bry88]. Drawing a parallel between the symplectic and the Riemannian cases, J.-L. Brylinski proposed in [Bry88] a Hodge theory for compact symplectic manifolds $(X,\,\omega)$, introducing a symplectic Hodge-$\star$-operator $\star_\omega$ and the notion of *ω-symplectically-harmonic form* (i.e., a form being both d-closed and d^Λ-closed, where the symplectic co-differential is defined as $\mathrm{d}^\Lambda\lfloor_{\wedge^k X} := (-1)^{k+1}\, \star_\omega\, \mathrm{d}\, \star_\omega$ for every $k \in \mathbb{N}$): in this context, O. Mathieu in [Mat95], and D. Yan in [Yan96], proved that any de Rham cohomology class admits an ω-symplectically-harmonic representative if and only if the Hard Lefschetz Condition is satisfied. Recently, L.-S. Tseng and S.-T. Yau, in [TY12a, TY12b], see also [TY11], introduced new cohomologies for symplectic manifolds $(X,\,\omega)$: among them, in particular, they defined and studied

$$H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) \;:=\; \frac{\ker\left(\mathrm{d}+\mathrm{d}^\Lambda\right)}{\mathrm{im}\,\mathrm{d}\,\mathrm{d}^\Lambda}\;,$$

developing a Hodge theory for this cohomology; furthermore, they studied the dual currents of Lagrangian and co-isotropic submanifolds, and they defined a homology theory on co-isotropic chains, which turns out to be naturally dual to a primitive cohomology. In the context of Generalized Geometry, [Gua04, Gua11, Cav05, Cav07], the cohomology $H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R})$ can be interpreted as the symplectic counterpart of the Bott-Chern cohomology of a complex manifold, see [TY11]. Inspired also by their works, Y. Lin developed in [Lin11] a new approach to the symplectic Hodge theory, proving in particular that, on any compact symplectic manifold satisfying the Hard Lefschetz Condition, there is a Poincaré duality between the primitive homology on co-isotropic chains and the primitive cohomology.

In this section, we recall some notions and results concerning Hodge theory for compact symplectic manifolds; we refer to [Bry88, Mat95, Yan96, Cav05, TY12a, TY12b, Lin11] for further details. (See §0.2 for basic definitions and results on symplectic manifolds.)

Symplectic cohomologies

Let X be a compact $2n$-dimensional manifold endowed with a symplectic structure ω.

We recall, see §0.2, that, ω being non-degenerate, it induces a natural isomorphism $I \colon TX \to T^*X$, namely, $I(\cdot)(\cdot\cdot) = \omega(\cdot, \cdot\cdot)$, and hence a bi-$\mathcal{C}^\infty(X;\mathbb{R})$-linear form $\left(\omega^{-1}\right)^k \colon \wedge^k X \otimes \wedge^k X \to \mathcal{C}^\infty(X;\mathbb{R})$; the *symplectic-$\star$-operator*, is defined, for every α, $\beta \in \wedge^k X$, by, [Bry88, §2],

$$\star_\omega \colon \wedge^\bullet X \to \wedge^{2n-\bullet} X \;, \qquad \alpha \wedge \star_\omega \beta \;=\; \left(\omega^{-1}\right)^k (\alpha, \beta)\, \frac{\omega^n}{n!} \;,$$

and satisfies $\star_\omega^2 = \mathrm{id}_{\wedge^\bullet X}$, [Bry88, Lemma 2.1.2].

We recall that the operators

$$\begin{aligned} L &:= \omega \wedge \cdot \colon \wedge^\bullet X \to \wedge^{\bullet+2} X \;, \\ \Lambda &:= -\iota_\Pi = - \star_\omega L \star_\omega \colon \wedge^\bullet X \to \wedge^{\bullet-2} X \;, \\ H &:= \sum_k (n-k)\, \pi_{\wedge^k X} \colon \wedge^\bullet X \to \wedge^\bullet X \;, \end{aligned}$$

(where $\Pi := \omega^{-1} \in \wedge^2 TX$ is the canonical Poisson bi-vector associated to ω, the interior product with $\xi \in \wedge^2 (TX)$ is denoted by $\iota_\xi \colon \wedge^\bullet X \to \wedge^{\bullet-2} X$, and, for $k \in \mathbb{N}$, the map $\pi_{\wedge^k X} \colon \wedge^\bullet X \to \wedge^k X$ denotes the natural projection onto $\wedge^k X$) yields an $\mathfrak{sl}(2;\mathbb{R})$-representation on $\wedge^\bullet X$ having finite H-spectrum, and hence one has the *Lefschetz decomposition* on differential forms, [Yan96, Corollary 2.6],

$$\wedge^\bullet X \;=\; \bigoplus_{r\in\mathbb{N}} L^r\, \mathrm{P}\wedge^{\bullet-2r} X \;,$$

where the space of *primitive forms* is

$$\mathrm{P}\wedge^\bullet X \;:=\; \ker \Lambda \;=\; \ker L^{n-\bullet+1} \lfloor_{\wedge^\bullet X} \;.$$

Consider now the *symplectic co-differential operator* $\mathrm{d}^\Lambda \colon \wedge^\bullet X \to \wedge^{\bullet-1} X$, defined, for every $k \in \mathbb{N}$, by

$$\mathrm{d}^\Lambda \lfloor_{\wedge^k X} \;:=\; (-1)^{k+1} \star_\omega \mathrm{d} \star_\omega \;;$$

it has been introduced, in general for a Poisson manifold, by J.-L. Koszul, [Kos85], and studied also by J.-L. Brylinski, [Bry88, §1.2]. The basic symplectic identity

$$[\mathrm{d},\, \Lambda] \;=\; \mathrm{d}^\Lambda$$

holds, see, e.g., [Yan96, Corollary 1.3]; by the graded-Jacobi identity, it follows that $\left[\mathrm{d},\, \mathrm{d}^\Lambda\right] = [\mathrm{d},\, [\mathrm{d},\, \Lambda]] = [\mathrm{d},\, [\Lambda,\, \mathrm{d}]] - [\Lambda,\, [\mathrm{d},\, \mathrm{d}]] = -\left[\mathrm{d},\, \mathrm{d}^\Lambda\right]$, since $[\mathrm{d},\, \mathrm{d}] = 0$ and $[\Lambda,\, \mathrm{d}] = -[\mathrm{d},\, \Lambda]$, and hence, [Kos85, page 265], [Bry88, Theorem 1.3.1],

$$\mathrm{d}\,\mathrm{d}^\Lambda + \mathrm{d}^\Lambda\,\mathrm{d} \;=\; 0 \;.$$

Hence, interpreting, as in [Bry88], d^Λ as the symplectic counterpart of the Riemannian co-differential operator d^* associated to a Riemannian metric g on X, then the symplectic counterpart of the Laplacian operator $\Delta := \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d}$ vanishes.

We recall that, if $(J,\, \omega,\, g)$ is an almost-Kähler structure on X, then the symplectic-$\star$-operator $\star_\omega$ and the Hodge-$*$-operator $*_g$ are related by

$$\star_\omega \;=\; J *_g \;,$$

[Bry88, Theorem 2.4.1], and hence d^Λ and $\mathrm{d}^c := J^{-1}\, \mathrm{d}\, J$ are related by

$$\mathrm{d}^\Lambda \;=\; -\left(\mathrm{d}^c\right)^* \;.$$

The previous identity, together with the identity $\mathrm{d}\,\mathrm{d}^\Lambda + \mathrm{d}^\Lambda\,\mathrm{d} = 0$, suggests that d^Λ should be interpreted as the symplectic counterpart of the operator d^c in Complex Geometry; this guess can be made more precise using Complex Generalized Geometry, [Gua04, Gua11, Cav05, Cav07].

The symplectic co-differential operator satisfies $\left(\mathrm{d}^\Lambda\right)^2 = 0$, and hence it gives rise to a differential complex $\left(\wedge^\bullet X,\, \mathrm{d}^\Lambda\right)$. This complex has been introduced, more in general, on a Poisson manifold, with the name of *canonical complex*, by J.-L. Koszul, [Kos85], and studied also by J.-L. Brylinski, [Bry88, §1], and, more recently, by L.-S. Tseng and S.-T. Yau, [TY12a, §3.1]. The homology of the complex $\left(\wedge^\bullet X,\, \mathrm{d}^\Lambda\right)$ is, in J.-L. Koszul's terminology, the *canonical homology* of X,

$$H^\bullet_{\mathrm{d}^\Lambda}(X;\mathbb{R}) \;:=\; \frac{\ker \mathrm{d}^\Lambda}{\mathrm{im}\, \mathrm{d}^\Lambda} \;.$$

Note that, [Bry88, Corollary 2.2.2],

$$\star_\omega \colon H^\bullet_{dR}(X;\mathbb{R}) \stackrel{\simeq}{\to} H^{2n-\bullet}_{\mathrm{d}^\Lambda}(X;\mathbb{R}) \;,$$

hence, for a compact symplectic manifold, the canonical homology groups and the de Rham cohomology groups are isomorphic.

In [TY12a], L.-S. Tseng and S.-T. Yau introduced also the $\left(\mathrm{d}+\mathrm{d}^\Lambda\right)$*-cohomology*, [TY12a, §3.2],

$$H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) \;:=\; \frac{\ker\left(\mathrm{d}+\mathrm{d}^\Lambda\right)}{\mathrm{im}\, \mathrm{d}\,\mathrm{d}^\Lambda} \;,$$

and the $\left(\mathrm{d}\,\mathrm{d}^\Lambda\right)$*-cohomology*, [TY12a, §3.3],

$$H^\bullet_{\mathrm{d}\,\mathrm{d}^\Lambda}(X;\mathbb{R}) \;:=\; \frac{\ker \mathrm{d}\,\mathrm{d}^\Lambda}{\mathrm{im}\,\mathrm{d} + \mathrm{im}\,\mathrm{d}^\Lambda} \;;$$

such cohomologies are, in a sense, the symplectic counterpart of the Aeppli and Bott-Chern cohomologies of complex manifolds, see [TY12a, §5] and [TY11] for further discussions.

Furthermore, they provided a Hodge theory for such cohomologies, proving the following result.

Theorem 3.1 ([TY12a, Theorem 3.5, Corollary 3.6])**.** *Let X be a compact manifold endowed with a symplectic structure ω. Let $(J,\,\omega,\,g)$ be an almost-Kähler structure on X. For a fixed $\lambda > 0$, the 4^{th} order self-adjoint differential operator*

$$\begin{aligned} D_{\mathrm{d}+\mathrm{d}^\Lambda} \quad &:= \quad \left(\mathrm{d}\,\mathrm{d}^\Lambda\right)\left(\mathrm{d}\,\mathrm{d}^\Lambda\right)^* + \left(\mathrm{d}\,\mathrm{d}^\Lambda\right)^*\left(\mathrm{d}\,\mathrm{d}^\Lambda\right) + \left(\mathrm{d}^*\,\mathrm{d}^\Lambda\right)\left(\mathrm{d}^*\,\mathrm{d}^\Lambda\right)^* + \left(\mathrm{d}^*\,\mathrm{d}^\Lambda\right)^*\left(\mathrm{d}^*\,\mathrm{d}^\Lambda\right) \\ &\quad\; +\lambda\left(\mathrm{d}^*\,\mathrm{d} + \left(\mathrm{d}^\Lambda\right)^*\mathrm{d}^\Lambda\right) \;. \end{aligned}$$

is elliptic, with $\ker D_{\mathrm{d}+\mathrm{d}^\Lambda} = \ker \mathrm{d} \cap \ker \mathrm{d}^\Lambda \cap \ker\left(\mathrm{d}\,\mathrm{d}^\Lambda\right)^*$.

Furthermore, there exist an orthogonal decomposition

$$\wedge^\bullet X \;=\; \ker D_{\mathrm{d}+\mathrm{d}^\Lambda} \oplus \mathrm{d}\,\mathrm{d}^\Lambda \wedge^\bullet X \oplus \left(\mathrm{d}^* \wedge^{\bullet+1} X + \left(\mathrm{d}^\Lambda\right)^* \wedge^{\bullet-1} X\right)$$

and an isomorphism

$$H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) \;\simeq\; \ker D_{\mathrm{d}+\mathrm{d}^\Lambda} \;.$$

In particular, $\dim_{\mathbb{R}} H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) < +\infty$.

An analogous statement holds for the $\left(\mathrm{d}\,\mathrm{d}^\Lambda\right)$-cohomology.

Theorem 3.2 ([TY12a, Theorem 3.16, Corollary 3.17])**.** *Let X be a compact manifold endowed with a symplectic structure ω. Let $(J,\,\omega,\,g)$ be an almost-Kähler structure on X. For a fixed $\lambda > 0$, the 4^{th} order self-adjoint differential operator*

$$\begin{aligned} D_{\mathrm{d}\,\mathrm{d}^\Lambda} \quad &:= \quad \left(\mathrm{d}\,\mathrm{d}^\Lambda\right)\left(\mathrm{d}\,\mathrm{d}^\Lambda\right)^* + \left(\mathrm{d}\,\mathrm{d}^\Lambda\right)^*\left(\mathrm{d}\,\mathrm{d}^\Lambda\right) + \left(\mathrm{d}\left(\mathrm{d}^\Lambda\right)^*\right)\left(\mathrm{d}\left(\mathrm{d}^\Lambda\right)^*\right)^* + \left(\mathrm{d}\left(\mathrm{d}^\Lambda\right)^*\right)^*\left(\mathrm{d}\left(\mathrm{d}^\Lambda\right)^*\right) \\ &\quad\; +\lambda\left(\mathrm{d}\,\mathrm{d}^* + \mathrm{d}^\Lambda\left(\mathrm{d}^\Lambda\right)^*\right) \;. \end{aligned}$$

is elliptic, with $\ker D_{\mathrm{d}\,\mathrm{d}^\Lambda} = \ker\left(\mathrm{d}\,\mathrm{d}^\Lambda\right) \cap \ker \mathrm{d}^* \cap \ker\left(\mathrm{d}^\Lambda\right)^*$.

Furthermore, there exist an orthogonal decomposition

$$\wedge^\bullet X \;=\; \ker D_{\mathrm{d}\,\mathrm{d}^\Lambda} \oplus \left(\mathrm{d}\wedge^{\bullet-1}X + \mathrm{d}^\Lambda \wedge^{\bullet+1}X\right) \oplus \left(\mathrm{d}\,\mathrm{d}^\Lambda\right)^* \wedge^\bullet X$$

and an isomorphism

$$H^\bullet_{\mathrm{d}\,\mathrm{d}^\Lambda}(X;\mathbb{R}) \;\simeq\; \ker D_{\mathrm{d}\,\mathrm{d}^\Lambda} \;.$$

In particular, $\dim_{\mathbb{R}} H^\bullet_{\mathrm{d}\,\mathrm{d}^\Lambda}(X;\mathbb{R}) < +\infty$*.*

As for the Bott-Chern and the Aeppli cohomologies, the $\left(\mathrm{d}+\mathrm{d}^\Lambda\right)$-cohomology and the $\left(\mathrm{d}\,\mathrm{d}^\Lambda\right)$-cohomology groups turn out to be isomorphic by means of the Hodge-$*$-operator associated to any Riemannian metric being compatible with ω.

Theorem 3.3 ([TY12a, Lemma 3.23, Proposition 3.24, Corollary 3.25])**.** *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Let $(J,\,\omega,\,g)$ be an almost-Kähler structure on X. The operators $D_{\mathrm{d}+\mathrm{d}^\Lambda}$ and $D_{\mathrm{d}\,\mathrm{d}^\Lambda}$ satisfy*

$$*_g\, D_{\mathrm{d}+\mathrm{d}^\Lambda} \;=\; D_{\mathrm{d}\,\mathrm{d}^\Lambda}\, *_g \;,$$

and hence $*_g$ induces an isomorphism

$$*_g \colon H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) \stackrel{\simeq}{\to} H^{2n-\bullet}_{\mathrm{d}\,\mathrm{d}^\Lambda}(X;\mathbb{R}) \;.$$

Moreover, the cohomology $H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R})$ is invariant under symplectomorphisms and Hamiltonian isotopies, [TY12a, Proposition 2.8].

One has the following commutation relations between the differential operators d, d^Λ, and $\mathrm{d}\,\mathrm{d}^\Lambda$, and the elements L, Λ, and H of the $\mathfrak{sl}(2;\mathbb{R})$-triple, see, e.g., [TY12a, Lemma 2.3]:

$$\begin{array}{rclrclrcl}
[\mathrm{d},\, L] &=& 0\,, & \left[\mathrm{d}^\Lambda,\, L\right] &=& -\,\mathrm{d}\,, & \left[\mathrm{d}\,\mathrm{d}^\Lambda,\, L\right] &=& 0\,,\\
[\mathrm{d},\, \Lambda] &=& \mathrm{d}^\Lambda\,, & \left[\mathrm{d}^\Lambda,\, \Lambda\right] &=& 0\,, & \left[\mathrm{d}\,\mathrm{d}^\Lambda,\, \Lambda\right] &=& 0\,,\\
[\mathrm{d},\, H] &=& \mathrm{d}\,, & \left[\mathrm{d}^\Lambda,\, H\right] &=& -\,\mathrm{d}^\Lambda\,, & \left[\mathrm{d}\,\mathrm{d}^\Lambda,\, H\right] &=& 0\,.
\end{array}$$

Hence, by setting

$$\mathrm{P}H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) \;:=\; \frac{\ker\mathrm{d}\cap\ker\mathrm{d}^\Lambda\cap\mathrm{P}\wedge^\bullet X}{\mathrm{im}\,\mathrm{d}\,\mathrm{d}^\Lambda\cap\mathrm{P}\wedge^\bullet X} \;=\; \frac{\ker\mathrm{d}\cap\mathrm{P}\wedge^\bullet X}{\mathrm{im}\,\mathrm{d}\,\mathrm{d}^\Lambda\lfloor_{\mathrm{P}\wedge^\bullet X}}$$

(where the second equality follows from [TY12a, Lemma 3.9]), one gets the following result.

Theorem 3.4 ([TY12a, Theorem 3.11])**.** *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Then there exist a decomposition*

$$H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) \;=\; \bigoplus_{r\in\mathbb{N}} L^r\,\mathrm{P}H^{\bullet-2r}_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R})$$

and, for every $k\in\mathbb{N}$, an isomorphism

$$L^k \colon H^{n-k}_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) \stackrel{\simeq}{\to} H^{n+k}_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) \;,$$

Analogously, by setting

$$\mathrm{P}H^\bullet_{\mathrm{d}\,\mathrm{d}^\Lambda}(X;\mathbb{R}) \;:=\; \frac{\ker\mathrm{d}\,\mathrm{d}^\Lambda\cap\mathrm{P}\wedge^\bullet X}{\left(\mathrm{im}\,\mathrm{d}+\mathrm{im}\,\mathrm{d}^\Lambda\right)\cap\mathrm{P}\wedge^\bullet X} \;=\; \frac{\ker\mathrm{d}\,\mathrm{d}^\Lambda\cap\mathrm{P}\wedge^\bullet X}{\mathrm{im}\,\left(\mathrm{d}+LH^{-1}\,\mathrm{d}^\Lambda\right)\Big\lfloor_{\mathrm{P}\wedge^{\bullet-1}X}+\mathrm{im}\,\mathrm{d}^\Lambda\Big\lfloor_{\mathrm{P}\wedge^{\bullet+1}X}}$$

(where the second equality follows from [TY12a, Lemma 3.20]), one gets the following result.

Theorem 3.5 ([TY12a, Theorem 3.21])**.** *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Then there exist a decomposition*

$$H^\bullet_{\mathrm{d}\,\mathrm{d}^\Lambda}(X;\mathbb{R}) \;=\; \bigoplus_{r\in\mathbb{N}} L^r\,\mathrm{P}H^{\bullet-2r}_{\mathrm{d}\,\mathrm{d}^\Lambda}(X;\mathbb{R})$$

and, for every $k\in\mathbb{N}$, an isomorphism

$$L^k \colon H^{n-k}_{\mathrm{d}\,\mathrm{d}^\Lambda}(X;\mathbb{R}) \stackrel{\simeq}{\to} H^{n+k}_{\mathrm{d}\,\mathrm{d}^\Lambda}(X;\mathbb{R}) \;,$$

The identity map induces the following natural maps in cohomology:

$$\begin{tikzcd}[column sep=small] & H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \arrow[dl] \arrow[dr] & \\ H^{\bullet}_{dR}(X;\mathbb{R}) \arrow[dr] & & H^{\bullet}_{\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \arrow[dl] \\ & H^{\bullet}_{\mathrm{d}\,\mathrm{d}^{\Lambda}}(X;\mathbb{R}) & \end{tikzcd}$$

Recall that a symplectic manifold is said to satisfy the $\mathrm{d}\,\mathrm{d}^{\Lambda}$*-Lemma* if every d-exact d^{Λ}-closed form is $\mathrm{d}\,\mathrm{d}^{\Lambda}$-exact, [DGMS75], namely, if $H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \to H^{\bullet}_{dR}(X;\mathbb{R})$ is injective.

Remark 3.6. Note that

$$\ker \mathrm{d}^{\Lambda} \cap \operatorname{im} \mathrm{d} \;=\; \operatorname{im} \mathrm{d}\,\mathrm{d}^{\Lambda} \qquad \text{if and only if} \qquad \ker \mathrm{d} \cap \operatorname{im} \mathrm{d}^{\Lambda} \;=\; \operatorname{im} \mathrm{d}\,\mathrm{d}^{\Lambda} \;.$$

Indeed, since $\star_{\omega}^2 = \mathrm{id}_{\wedge^{\bullet}X}$, [Bry88, Lemma 2.1.2], and $\mathrm{d}\,\mathrm{d}^{\Lambda} + \mathrm{d}^{\Lambda}\,\mathrm{d} = 0$, [Bry88, Theorem 1.3.1], one has

$$\star_{\omega} \ker \mathrm{d} \;=\; \ker \mathrm{d}^{\Lambda} \;, \qquad \star_{\omega} \operatorname{im} \mathrm{d} \;=\; \operatorname{im} \mathrm{d}^{\Lambda} \;, \qquad \star_{\omega} \operatorname{im} \mathrm{d}\,\mathrm{d}^{\Lambda} \;=\; \operatorname{im} \mathrm{d}\,\mathrm{d}^{\Lambda} \;.$$

Another cohomological property that can be defined on a $2n$-dimensional compact manifold X endowed with a symplectic form ω is the *Hard Lefschetz Condition*, that is,

$$\text{for every } k \in \mathbb{N} \;, \qquad L^k \colon H^{n-k}_{dR}(X;\mathbb{R}) \stackrel{\simeq}{\to} H^{n+k}_{dR}(X;\mathbb{R}) \;. \tag{HLC}$$

In fact, the following result relates the $\mathrm{d}\,\mathrm{d}^{\Lambda}$-Lemma, the Hard Lefschetz Condition, and the existence of ω-symplectically harmonic representatives in any de Rham cohomology class.

Theorem 3.7 ([Mat95, Corollary 2], [Yan96, Theorem 0.1], [Mer98, Proposition 1.4], [Gui01], [Cav05, Theorem 5.4])**.** *Let X be a compact manifold endowed with a symplectic structure ω. The following conditions are equivalent:*

(i) every de Rham cohomology class admits a representative being both d*-closed and* d^{Λ}*-closed (i.e., Brylinski's conjecture [Bry88, Conjecture 2.2.7] holds on X);*

(ii) the Hard Lefschetz Condition holds on X;

(iii) the natural homomorphism $H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \to H^{\bullet}_{dR}(X;\mathbb{R})$ induced by the identity is actually an isomorphism;

(iv) X satisfies the $\mathrm{d}\,\mathrm{d}^{\Lambda}$*-Lemma.*

Note that, by the Lefschetz decomposition theorem, the compact Kähler manifolds satisfy the Hard Lefschetz Condition; in other terms, note that, given a Kähler structure $(J,\, \omega,\, g)$ on a compact manifold X, one has $\star_{\omega} = J *_g$, [Bry88, Theorem 2.4.1], and hence every de Rham cohomology class admits an ω-symplectically-harmonic representative.

Primitive currents

Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Denote by $\mathcal{D}_{\bullet}X := \mathcal{D}^{2n-\bullet}X$ the space of currents, and consider the de Rham homology $H^{dR}_{\bullet}(X;\mathbb{R}) := H^{\bullet}\left(\mathcal{D}_{\bullet}X,\, \mathrm{d}\right)$. (See §0.5 for definitions and results concerning currents and de Rham homology.)

Following [Lin11, Definition 5.1], set, by duality,

$$\begin{aligned} L \colon \mathcal{D}_{\bullet}X \to \mathcal{D}_{\bullet-2}X \;, &\qquad S \mapsto S\left(L\,\cdot\right) \;, \\ \Lambda \colon \mathcal{D}_{\bullet}X \to \mathcal{D}_{\bullet+2}X \;, &\qquad S \mapsto S\left(\Lambda\,\cdot\right) \;, \\ H \colon \mathcal{D}_{\bullet}X \to \mathcal{D}_{\bullet}X \;, &\qquad S \mapsto S\left(-H\,\cdot\right) \;; \end{aligned}$$

note that

$$[L,\, H] \;=\; 2\,L \;, \qquad [\Lambda,\, H] \;=\; -2\,\Lambda \;, \qquad [L,\, \Lambda] \;=\; H \;.$$

A current $S \in \mathcal{D}^k X$ is said *primitive* if $\Lambda S = 0$, equivalently, if $L^{n-k+1}S = 0$, see, e.g., [Lin11, Proposition 5.3]; denote by $\mathrm{P}\mathcal{D}^{\bullet}X := \mathrm{P}\mathcal{D}_{2n-\bullet}X$ the space of primitive currents on X.

In [Lin11], Y. Lin proved the following result.

Theorem 3.8 ([Lin11, Lemma 5.2, Proposition 5.3, Lemma 5.12])**.** *Let X be a compact manifold endowed with a symplectic structure ω. Then $\langle L,\, \Lambda,\, H\rangle$ gives an $\mathfrak{sl}(2;\mathbb{R})$-module structure on $\mathcal{D}^\bullet X$. In particular, one has the Lefschetz decomposition on the space of currents,*

$$\mathcal{D}^\bullet X \;=\; \bigoplus_{r\in\mathbb{N}} L^r\, \mathrm{P}\mathcal{D}^{\bullet-2r} X \;=:\; \bigoplus_{r\in\mathbb{N}} L^r\, \mathrm{P}\mathcal{D}_{2n-\bullet+2r} X \;.$$

Furthermore, the space of flat currents is an $\mathfrak{sl}(2;\mathbb{R})$-submodule of the space of currents.

Finally, if $j\colon Y \hookrightarrow X$ is a compact oriented submanifold of X of codimension k (possibly with non-empty boundary), then the *dual current* $[Y] \in \mathcal{D}_k X$ associated with Y is defined, by setting, for every $\varphi \in \wedge^k X$,

$$[Y]\,(\varphi) \;:=\; \int_Y j^*(\varphi)\;.$$

If Y is a closed oriented submanifold, then the dual current $[Y]$ is d-closed. According to [TY12a, Lemma 4.1], the dual current $[Y]$ is primitive if and only if Y is co-isotropic.

3.1.2 Symplectic subgroups of (co)homology

In this section, we provide a symplectic counterpart to T.-J. Li and W. Zhang's theory on cohomology of almost-complex manifolds developed in [LZ09]. More precisely, we define the subgroups $H^{(\bullet,\bullet)}_\omega(X;\mathbb{R})$ of the de Rham cohomology $H^\bullet_{dR}(X;\mathbb{R})$ of a symplectic manifold $(X,\,\omega)$, and, analogously, the subgroups $H^\omega_{(\bullet,\bullet)}(X;\mathbb{R})$ of the de Rham homology $H^{dR}_\bullet(X;\mathbb{R})$; then, we study some of their properties: in particular, we prove that, for every compact symplectic manifold, the decomposition $H^2_{dR}(X;\mathbb{R}) = H^{(1,0)}_\omega(X;\mathbb{R}) \oplus H^{(0,2)}_\omega(X;\mathbb{R})$ holds, Theorem 3.14, which provides a symplectic counterpart of [DLZ10, Theorem 2.3].

Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. For any $r, s \in \mathbb{N}$, define

$$H^{(r,s)}_\omega(X;\mathbb{R}) \;:=\; \left\{\left[L^r\,\beta^{(s)}\right] \in H^{2r+s}_{dR}(X;\mathbb{R}) \;:\; \beta^{(s)} \in \mathrm{P}\wedge^s X\right\} \;\subseteq\; H^{2r+s}_{dR}(X;\mathbb{R})\;.$$

Obviously, for every $k \in \mathbb{N}$, one has

$$\sum_{2r+s=k} H^{(r,s)}_\omega(X;\mathbb{R}) \;\subseteq\; H^k_{dR}(X;\mathbb{R}) \;:$$

we are concerned with studying when the above inclusion is actually an equality, and when the sum is actually a direct sum.

Remark 3.9. We underline the relations between the above subgroups and the primitive cohomologies introduced by L.-S. Tseng and S.-T. Yau in [TY12a]. As regards L.-S. Tseng and S.-T. Yau's primitive $\left(\mathrm{d}+\mathrm{d}^\Lambda\right)$-cohomology $\mathrm{P}H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R})$, note that, for every $r, s \in \mathbb{N}$,

$$\mathrm{im}\left(L^r\,\mathrm{P}H^s_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) \to H^\bullet_{dR}(X;\mathbb{R})\right) \;=\; L^r\,H^{(0,s)}_\omega(X;\mathbb{R}) \;\subseteq\; H^{(r,s)}_\omega(X;\mathbb{R})\;.$$

In [TY12a, §4.1], L.-S. Tseng and S.-T. Yau have introduced also the primitive cohomology groups

$$\mathrm{P}H^s_{\mathrm{d}}(X;\mathbb{R}) \;:=\; \frac{\ker\mathrm{d} \cap \ker\mathrm{d}^\Lambda \cap \mathrm{P}\wedge^s X}{\mathrm{im}\,\mathrm{d}\lfloor_{\mathrm{P}\wedge^{s-1}X\cap\ker\mathrm{d}^\Lambda}}\;,$$

where $s \in \mathbb{N}$, proving that the homology on co-isotropic chains is naturally dual to $\mathrm{P}H^{2n-\bullet}_{\mathrm{d}}(X;\mathbb{R})$, see [TY12a, pages 40–41]; in [Lin11, Proposition 2.7], Y. Lin proved that, if the Hard Lefschetz Condition holds on X, then

$$H^{(0,\bullet)}_\omega(X;\mathbb{R}) \;=\; \mathrm{P}H^\bullet_{\mathrm{d}}(X;\mathbb{R})\;.$$

Remark 3.10. In [CT07], D. Conti and A. Tomassini studied the notion of half-flat structure on a 6-dimensional manifold X, see [CS02]. Namely, an $\mathrm{SU}(3)$-*structure* $(\omega,\,\psi)$ on X, where ω is a non-degenerate real 2-form and ψ is a decomposable complex 3-form such that $\psi\wedge\omega = 0$ and $\psi\wedge\bar\psi = -\frac{4\,\mathrm{i}}{3}\,\omega^3$, is called *half-flat* if both $\omega\wedge\omega$ and $\mathfrak{Re}\psi$ are d-closed. Note in particular that, if $(\omega,\,\psi)$ is a symplectic half-flat structure on X, then $[\mathfrak{Re}\psi] \in H^{(0,3)}_\omega(X;\mathbb{R})$.

Remark 3.11. A class of examples of compact symplectic manifolds $(X,\,\omega)$ satisfying the cohomology decomposition by means of the above subgroups $H^{\bullet,\bullet}_{\omega}(X;\mathbb{R})$ (actually, satisfying an even stronger cohomology decomposition) is provided by the compact symplectic manifolds satisfying the $\mathrm{d}\,\mathrm{d}^{\Lambda}$-Lemma, equivalently, as already recalled, the Hard Lefschetz Condition, [Mer98, Proposition 1.4], [Gui01], [Cav05, Theorem 5.4].

More precisely, on a compact manifold X endowed with a symplectic structure ω, the following conditions are equivalent:

- X satisfies the $\mathrm{d}\,\mathrm{d}^{\Lambda}$-Lemma;
- it holds the decomposition

$$H^{\bullet}_{dR}(X;\mathbb{R}) \;=\; \bigoplus_{r\in\mathbb{N}} L^r\, H^{(0,\bullet-2r)}_{\omega}(X;\mathbb{R})\;. \tag{3.1.1}$$

Indeed, recall that the decomposition

$$H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \;=\; \bigoplus_{r\in\mathbb{N}} L^r\,\mathrm{P}H^{\bullet-2r}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R})$$

holds on any compact symplectic manifold, [TY12a, Theorem 3.11]; moreover, the $\mathrm{d}\,\mathrm{d}^{\Lambda}$-Lemma holds on a compact symplectic manifold if and only if the natural homomorphism

$$H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \to H^{\bullet}_{dR}(X;\mathbb{R})$$

induced by the identity is actually an isomorphism; recall also that

$$\mathrm{im}\left(L^r\,\mathrm{P}H^{s}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \to H^{\bullet}_{dR}(X;\mathbb{R})\right) = L^r\, H^{(0,s)}_{\omega}(X;\mathbb{R})\;;$$

hence, if the $\mathrm{d}\,\mathrm{d}^{\Lambda}$-Lemma holds, then one has the decomposition (3.1.1). Conversely, if (3.1.1) holds, then, straightforwardly, X satisfies the Hard Lefschetz condition, and hence also the $\mathrm{d}\,\mathrm{d}^{\Lambda}$-Lemma, [Mer98, Proposition 1.4], [Gui01], [Cav05, Theorem 5.4].

Analogously, considering the space $\mathcal{D}^{\bullet}X :=: \mathcal{D}_{2n-\bullet}X$ of currents and the de Rham homology $H^{dR}_{\bullet}(X;\mathbb{R})$, for every $r, s \in \mathbb{N}$, define

$$H^{\omega}_{(r,s)}(X;\mathbb{R}) \;:=\; \left\{\left[L^r\,B_{(s)}\right] \in H^{dR}_{-2r+s}(X;\mathbb{R}) \;:\; B_{(s)} \in \mathrm{P}\mathcal{D}_s X\right\} \;\subseteq\; H^{dR}_{-2r+s}(X;\mathbb{R})\;;$$

as previously, for every $k\in\mathbb{N}$, we have just the inclusion

$$\sum_{-2r+s=k} H^{\omega}_{(r,s)}(X;\mathbb{R}) \;\subseteq\; H^{dR}_{k}(X;\mathbb{R})\;,$$

but, in general, neither the sum is direct nor the inclusion is an equality.

We prove that, fixed $k\in\mathbb{N}$, if the sum $\sum_{2r+s=2n-k} H^{(r,s)}_{\omega}(X;\mathbb{R})$ gives the whole $(2n-k)^{\text{th}}$ de Rham cohomology group, then the sum of the subgroups of the k^{th} de Rham cohomology group is direct, [AT12c, Proposition 2.4] (this result should be compared with [LZ09, Proposition 2.30] and Theorem 2.10 in the almost-complex case, and with Proposition 3.25 in the **D**-complex case).

Proposition 3.12. *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. For every $k\in\mathbb{N}$, the following implications hold:*

$$\begin{array}{ccc}
H^k_{dR}(X;\mathbb{R}) = \sum_{2r+s=k} H^{(r,s)}_{\omega}(X;\mathbb{R}) & \Longrightarrow & \bigoplus_{-2r+s=k} H^{\omega}_{(r,s)}(X;\mathbb{R}) \;\subseteq\; H^{dR}_{k}(X;\mathbb{R}) \\
\Big\Downarrow & & \Big\Downarrow \\
H^{dR}_{2n-k}(X;\mathbb{R}) = \sum_{-2r+s=2n-k} H^{\omega}_{(r,s)}(X;\mathbb{R}) & \Longrightarrow & \bigoplus_{2r+s=2n-k} H^{(r,s)}_{\omega}(X;\mathbb{R}) \;\subseteq\; H^{2n-k}_{dR}(X;\mathbb{R})\;.
\end{array}$$

Proof. Note that the quasi-isomorphism $T.\colon \wedge^{\bullet} X \ni \varphi \mapsto \int_X \varphi \wedge \cdot \in \mathcal{D}^{\bullet}X$ satisfies

$$T_{L\cdot} \;=\; LT.\;,$$

and hence, in particular, it preserves the bi-graduation,

$$T\left(L^{\bullet_1}\,\mathrm{P}\wedge^{\bullet_2} X\right) \;\subseteq\; L^{\bullet_1}\,\mathrm{P}\mathcal{D}^{\bullet_2} X \;:=:\; L^{\bullet_1}\;\mathrm{P}\mathcal{D}_{2n-\bullet_2} X\;,$$

and it induces, for every $r, s \in \mathbb{N}$, an injective map

$$H^{(r,s)}_{\omega}(X;\mathbb{R}) \hookrightarrow H^{\omega}_{(r,2n-s)}(X;\mathbb{R}) \,.$$

Therefore the two vertical implications are proven.

Consider now the non-degenerate duality pairing

$$\langle\cdot,\, \cdot\cdot\rangle : H^{\bullet}_{dR}(X;\mathbb{R}) \times H^{dR}_{\bullet}(X;\mathbb{R}) \to \mathbb{R} \,,$$

and note that, for every $r, s \in \mathbb{N}$,

$$\ker \left\langle H^{(r,s)}_{\omega}(X;\mathbb{R}),\, \cdot \right\rangle \supseteq \sum_{(p,q)\neq(n-r-s,2n-s)} H^{\omega}_{(p,q)}(X;\mathbb{R}) \,,$$

and, analogously, for every $p, q \in \mathbb{N}$,

$$\ker \left\langle \cdot,\, H^{\omega}_{(p,q)}(X;\mathbb{R}) \right\rangle \supseteq \sum_{(r,s)\neq(n-p-s,2n-q)} H^{(r,s)}_{\omega}(X;\mathbb{R}) \,;$$

this suffices to prove the two horizontal implications. □

A straightforward consequence of [Mat95, Corollary 2], or [Yan96, Theorem 0.1], and Proposition 3.12 is the following result, [AT12c, Corollary 2.5], which should be compared with [DLZ10, Theorem 2.16, Proposition 2.17].

Corollary 3.13. *Let X be a compact manifold endowed with a symplectic structure ω. Suppose that the Hard Lefschetz Condition holds on X, equivalently, that X satisfies the $\mathrm{d}\,\mathrm{d}^{\Lambda}$-Lemma. Then*

$$H^{\bullet}_{dR}(X;\mathbb{R}) \;=\; \bigoplus_{r\in\mathbb{N}} H^{(r,\bullet-2r)}_{\omega}(X;\mathbb{R}) \qquad \text{and} \qquad H^{dR}_{\bullet}(X;\mathbb{R}) \;=\; \bigoplus_{r\in\mathbb{N}} H^{\omega}_{(r,\bullet+2r)}(X;\mathbb{R}) \,.$$

In particular, when $\dim X = 4$ and taking $k = 2$ in Proposition 3.12, one gets that, if $H^2_{dR}(X;\mathbb{R}) = H^{(1,0)}_{\omega}(X;\mathbb{R}) + H^{(0,2)}_{\omega}(X;\mathbb{R})$ holds, then actually $H^2_{dR}(X;\mathbb{R}) = H^{(1,0)}_{\omega}(X;\mathbb{R}) \oplus H^{(0,2)}_{\omega}(X;\mathbb{R})$ holds. In fact, the following result states that $H^2_{dR}(X;\mathbb{R})$ always decomposes as direct sum of $H^{(1,0)}_{\omega}(X;\mathbb{R})$ and $H^{(0,2)}_{\omega}(X;\mathbb{R})$, also in dimension higher than 4, [AT12c, Theorem 2.6]: this gives a symplectic counterpart to T. Drăghici, T.-J. Li and W. Zhang's decomposition theorem [DLZ10, Theorem 2.3] in the complex setting, in fact, without the restriction to dimension 4.

Theorem 3.14. *Let X be a compact manifold endowed with a symplectic structure ω. Then*

$$H^2_{dR}(X;\mathbb{R}) \;=\; H^{(1,0)}_{\omega}(X;\mathbb{R}) \oplus H^{(0,2)}_{\omega}(X;\mathbb{R}) \,.$$

In particular, if $\dim X = 4$*, then*

$$H^{\bullet}_{dR}(X;\mathbb{R}) \;=\; \bigoplus_{r\in\mathbb{N}} H^{(r,\bullet-2r)}_{\omega}(X;\mathbb{R}) \qquad \text{and} \qquad H^{dR}_{\bullet}(X;\mathbb{R}) \;=\; \bigoplus_{r\in\mathbb{N}} H^{\omega}_{(r,\bullet+2r)}(X;\mathbb{R}) \,.$$

Proof. Let $2n := \dim X$. Firstly, we prove that $H^{(1,0)}_{\omega}(X;\mathbb{R}) \cap H^{(0,2)}_{\omega}(X;\mathbb{R}) = \{0\}$. Let

$$\mathfrak{c} =: [f\,\omega] := \left[\beta^{(2)}\right] \in H^{(1,0)}_{\omega}(X;\mathbb{R}) \cap H^{(0,2)}_{\omega}(X;\mathbb{R}) \,,$$

where $f \in \mathcal{C}^{\infty}(X;\mathbb{R})$ and $\beta^{(2)} \in \mathrm{P}\wedge^2 X$. Being $\mathrm{P}\wedge^2 X = \ker L^{n-1}\lfloor_{\wedge^2 X}$, one has

$$0 \;=\; \int_X f\, L^{n-1}\beta^{(2)} \;=\; \int_X f\,\omega \wedge \beta^{(2)} \wedge \omega^{n-2} \;=\; \int_X f\,\omega \wedge f\,\omega \wedge \omega^{n-2} \;=\; \int_X f^2 \omega^n$$

hence $f = 0$, that is, $\mathfrak{c} = 0$.

Now, we prove that $H^2_{dR}(X;\mathbb{R}) = H^{(1,0)}_{\omega}(X;\mathbb{R}) + H^{(0,2)}_{\omega}(X;\mathbb{R})$. Let $\mathfrak{a} =: [\alpha] \in H^2_{dR}(X;\mathbb{R})$. Then $L^{n-1}\mathfrak{a} \in H^{2n}_{dR}(X;\mathbb{R}) = \mathbb{R}\,\langle[\omega^n]\rangle$, that is, there exist $\lambda \in \mathbb{R}$ and $\gamma_{2n-1} \in \wedge^{2n-1}X$ such that $L^{n-1}\alpha = \lambda\,\omega^n + \mathrm{d}\,\gamma_{2n-1}$. Since $L^{n-1}\colon \wedge^1 X \stackrel{\simeq}{\to} \wedge^{2n-1}X$ is an isomorphism, there exists $\gamma_1 \in \wedge^1 X$ such that $L^{n-1}\gamma_1 = \gamma_{2n-1}$. Hence, since $\left[\mathrm{d},\, L^{n-1}\right] = 0$, we get that $L^{n-1}\left(\alpha - \mathrm{d}\,\gamma_1 - \lambda\,\omega\right) = 0$, that is, $\alpha - \mathrm{d}\,\gamma_1 - \lambda\,\omega \in \mathrm{P}\wedge^2 X$; therefore we get that

$$\mathfrak{a} \;=:\; [\alpha] \;=\; [\alpha - \mathrm{d}\,\gamma_1] \;=\; \underbrace{\lambda\,[\omega]}_{\in H^{(1,0)}_{\omega}(X;\mathbb{R})} + \underbrace{[\alpha - \mathrm{d}\,\gamma_1 - \lambda\,\omega]}_{\in H^{(0,2)}_{\omega}(X;\mathbb{R})} \,,$$

concluding the proof. □

Part of the argument in the proof of Theorem 3.14 can be generalized to prove the following result, [AT12c, Remark 2.7].

Proposition 3.15. *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. For every $k \in \left\{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, it holds*

$$H^{(k,0)}_{\omega}(X;\mathbb{R}) \cap H^{(0,2k)}_{\omega}(X;\mathbb{R}) \;=\; \{0\} \;.$$

Proof. Let $\mathfrak{c} :=: \left[f\,\omega^k\right] :=: \left[\beta^{(2k)}\right] \in H^{(k,0)}_{\omega}(X;\mathbb{R}) \cap H^{(0,2k)}_{\omega}(X;\mathbb{R})$, where $f \in \mathcal{C}^{\infty}(X;\mathbb{R})$ and $\beta^{(2k)} \in \mathrm{P}\wedge^{2k}X$. Being $\mathrm{P}\wedge^{2k}X = \ker L^{n-2k+1}\lfloor_{\wedge^{2k}X}$, one has

$$0 \;=\; \int_X f\, L^{n-2k+1}\beta^{(2k)} \wedge \omega^{k-1} \;=\; \int_X f\,\omega^k \wedge \beta^{(2k)} \wedge \omega^{n-2k} \;=\; \int_X f\,\omega^k \wedge f\,\omega^k \wedge \omega^{n-2k} \;=\; \int_X f^2 \omega^n$$

hence $f = 0$, that is, $\mathfrak{c} = 0$. □

In some cases, in studying $H^{(r,s)}_{\omega}(X;\mathbb{R})$, one can reduce to study $H^{(0,s)}_{\omega}(X;\mathbb{R})$: this is the matter of the following result, [AT12c, Proposition 2.8].

Proposition 3.16. *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Then, for every $r, s \in \mathbb{N}$ such that $2r + s \leq n$, one has*

$$H^{(r,s)}_{\omega}(X;\mathbb{R}) \;=\; L^r\, H^{(0,s)}_{\omega}(X;\mathbb{R}) \;.$$

Proof. Since $L\colon \wedge^j X \to \wedge^{j+2}X$ is injective for $j \leq n-1$, [Yan96, Corollary 2.8], (in fact, an isomorphism for $j = n-1$, [Yan96, Corollary 2.7],) and $[\mathrm{d}, L] = 0$, we get that

$$\begin{aligned} H^{(r,s)}_{\omega}(X;\mathbb{R}) \;&=\; \left\{\left[\omega^r\,\beta^{(s)}\right] \in H^{2r+s}_{dR}(X;\mathbb{R}) \;:\; \beta^{(s)} \in \wedge^s X \cap \ker\Lambda \text{ such that } L^r\,\mathrm{d}\,\beta^{(s)} = 0\right\} \\ &=\; \left\{[\omega^r] \smile \left[\beta^{(s)}\right] \in H^{2r+2}_{dR}(X;\mathbb{R}) \;:\; \beta^{(s)} \in \wedge^s X \cap \ker\Lambda\right\} \;, \end{aligned}$$

assumed that $2r + s \leq n$. □

In particular, for every $r \in \left\{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, the spaces $H^{(r,0)}_{\omega}(X;\mathbb{R})$ are 1-dimensional $\mathbb{R}$-vector spaces, more precisely, $H^{(r,0)}_{\omega}(X;\mathbb{R}) = \mathbb{R}\left\langle[\omega^r]\right\rangle$.

Furthermore, by the previous proposition, it follows that, for $k \leq \frac{1}{2}\dim X$, the condition

$$H^k_{dR}(X;\mathbb{R}) = \bigoplus_{r\in\mathbb{N}} H^{(r,k-2r)}_{\omega}(X;\mathbb{R})$$

is in fact equivalent to $H^k_{dR}(X;\mathbb{R}) = \bigoplus_{r\in\mathbb{N}} L^r\, H^{(0,k-2r)}_{\omega}(X;\mathbb{R})$.

3.1.3 Symplectic cohomological decomposition on solvmanifolds

As shown in Corollary 3.13, whenever X is a compact manifold endowed with a symplectic structure ω satisfying the Hard Lefschetz Condition, the de Rham cohomology $H^{\bullet}_{dR}(X;\mathbb{R})$, respectively the de Rham homology $H^{dR}_{\bullet}(X;\mathbb{R})$, decomposes as direct sum of the subgroups $H^{(\bullet,\bullet)}_{\omega}(X;\mathbb{R})$, respectively $H^{\omega}_{(\bullet,\bullet)}(X;\mathbb{R})$. Hence, it should be interesting to study cohomological properties for classes of symplectic manifolds not satisfying the Hard Lefschetz property, e.g., non-tori nilmanifolds, [BG88, Theorem A].

In this section, we study a Nomizu-type theorem for the subgroups $H^{(\bullet,\bullet)}_{\omega}(X;\mathbb{R})$ on completely-solvable solvmanifolds endowed with left-invariant symplectic structures, Proposition 3.18, providing explicit examples and studying their cohomological properties. (As regards notations, definitions, and results concerning nilmanifolds and solvmanifolds, we refer to §0.6.)

Left-invariant symplectic structures on solvmanifolds

Let $X = \Gamma\backslash G$ be a completely-solvable solvmanifold endowed with a G-left-invariant symplectic structure ω.

Recall that, by A. Hattori's theorem [Hat60, Corollary 4.2], the complex $(\wedge^{\bullet}\mathfrak{g}^*, \mathrm{d})$, which is isomorphic to the sub-complex composed of the G-left-invariant forms of $(\wedge^{\bullet}X, \mathrm{d})$, turns out to be quasi-isomorphic to the de Rham complex $(\wedge^{\bullet}X, \mathrm{d})$, that is, $H^{\bullet}_{dR}(\mathfrak{g};\mathbb{R}) \simeq H^{\bullet}_{dR}(X;\mathbb{R})$.

Since ω is G-left-invariant, $\langle L,\, \Lambda,\, H\rangle$ induces a $\mathfrak{sl}(2;\mathbb{R})$-representation both on $\wedge^\bullet X$ and on $\wedge^\bullet \mathfrak{g}^*$. Hence, for any $r, s \in \mathbb{N}$, we can consider both the subgroup $H^{(r,s)}_\omega(X;\mathbb{R})$ of $H^\bullet_{dR}(X;\mathbb{R})$, and the subgroup

$$H^{(r,s)}_\omega(\mathfrak{g};\mathbb{R}) \;:=\; \left\{\left[L^r\,\beta^{(s)}\right] \in H^\bullet_{dR}\left(\mathfrak{g};\mathbb{R}\right) \;:\; \Lambda\beta^{(s)} = 0\right\}$$

of $H^\bullet_{dR}\left(\mathfrak{g};\mathbb{R}\right) \simeq H^\bullet_{dR}(X;\mathbb{R})$, namely, the subgroup constituted of the de Rham cohomology classes admitting G-left-invariant representatives in $L^r\,\mathrm{P}\wedge^s X$.

In this section, we are concerned with studying the linking between $H^{(\bullet,\bullet)}_\omega(X;\mathbb{R})$ and $H^{(\bullet,\bullet)}_\omega(\mathfrak{g};\mathbb{R})$. This will let us study explicit examples in §3.1.3.

In the following lemma, we adapt the F. A. Belgun symmetrization trick, [Bel00, Theorem 7], to the symplectic case, [AT12c, Lemma 3.2].

Lemma 3.17. *Let $X = \Gamma\backslash G$ be a solvmanifold, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Let ω be a G-left-invariant symplectic structure on X. Let η be a G-bi-invariant volume form on G, given by J. Milnor's Lemma [Mil76, Lemma 6.2], such that $\int_X \eta = 1$. Up to identifying G-left-invariant forms on X and linear forms over $\mathfrak{g}^*$ through left-translations, consider the F. A. Belgun symmetrization map, [Bel00, Theorem 7],*

$$\mu\colon\; \wedge^\bullet X \to \wedge^\bullet \mathfrak{g}^*\;, \qquad \mu(\alpha) \;:=\; \int_X \alpha\lfloor_m\, \eta(m)\;.$$

One has that

$$\mu\lfloor_{\wedge^\bullet\mathfrak{g}^*} \;=\; \mathrm{id}\lfloor_{\wedge^\bullet\mathfrak{g}^*}$$

and that

$$\mathrm{d}\left(\mu(\cdot)\right) \;=\; \mu\left(\mathrm{d}\,\cdot\right) \qquad \textit{and} \qquad L\left(\mu(\cdot)\right) \;=\; \mu\left(L\cdot\right)\;.$$

In particular, μ sends primitive forms to G-left-invariant primitive forms.

Proof. It has to be shown just that $\mu\left(L\,\alpha\right) = L\,\mu\left(\alpha\right)$ for every $\alpha \in \wedge^\bullet X$. Note that, ω being a G-left-invariant form, one has $\mu\left(L\,\alpha\right) = \int_X \left(\omega\wedge\alpha\right)\lfloor_m\,\eta(m) = \int_X \omega\lfloor_m \wedge \alpha\lfloor_m\,\eta(m) = \omega\wedge\int_X \alpha\lfloor_m\,\eta(m) = L\,\mu\left(\alpha\right)$, for every $\alpha \in \wedge^\bullet X$. □

As a consequence of the previous lemma, we can prove the following result, which relates the subgroups $H^{(r,s)}_\omega(X;\mathbb{R})$ with their G-left-invariant part $H^{(r,s)}_\omega(\mathfrak{g};\mathbb{R})$, [AT12c, Proposition 3.3] (compare with Proposition 2.19, and also with [FT10, Theorem 3.4], for almost-complex structures, and with Proposition 3.30 for almost-$\mathbf{D}$-complex structures in the sense of F. R. Harvey and H. B. Lawson).

Proposition 3.18. *Let $X = \Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant symplectic structure ω, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. For every $r, s \in \mathbb{N}$, the map*

$$j\colon H^{(r,s)}_\omega(\mathfrak{g};\mathbb{R}) \to H^{(r,s)}_\omega(X;\mathbb{R})$$

induced by left-translations is injective, and, if $H^\bullet_{dR}\left(\mathfrak{g};\mathbb{R}\right) \simeq H^\bullet_{dR}(X;\mathbb{R})$ (for instance, if X is a completely-solvable solvmanifold), then it is in fact an isomorphism.

Proof. Left-translations induce the map $j\colon H^{(r,s)}_\omega(\mathfrak{g};\mathbb{R}) \to H^{(r,s)}_\omega(X;\mathbb{R})$. Consider the F. A. Belgun's symmetrization map $\mu\colon\; \wedge^\bullet X \to \wedge^\bullet\mathfrak{g}^*$: since it commutes with d by [Bel00, Theorem 7], it induces the map $\mu\colon H^\bullet_{dR}(X;\mathbb{R}) \to H^\bullet_{dR}\left(\mathfrak{g};\mathbb{R}\right)$, and, since it commutes with L by Lemma 3.17, it induces the map $\mu\colon H^{(r,s)}_\omega(X;\mathbb{R}) \to H^{(r,s)}_\omega(\mathfrak{g};\mathbb{R})$. Moreover, since μ is the identity on the space of G-left-invariant forms, we get the commutative diagram

$$\begin{tikzcd} H^{(r,s)}_\omega(\mathfrak{g};\mathbb{R}) \arrow[r, "j"] \arrow[rr, bend right, "\mathrm{id}"'] & H^{(r,s)}_\omega(X;\mathbb{R}) \arrow[r, "\mu"] & H^{(r,s)}_\omega(\mathfrak{g};\mathbb{R})\;. \end{tikzcd}$$

Hence $j\colon H^{(r,s)}_\omega(\mathfrak{g};\mathbb{R}) \to H^{(r,s)}_\omega(X;\mathbb{R})$ is injective, and $\mu\colon H^{(r,s)}_\omega(X;\mathbb{R}) \to H^{(r,s)}_\omega(\mathfrak{g};\mathbb{R})$ is surjective.

Furthermore, when $H^\bullet_{dR}\left(\mathfrak{g};\mathbb{R}\right) \simeq H^\bullet_{dR}(X;\mathbb{R})$ (for instance, when X is a completely-solvable solvmanifold, by A. Hattori's theorem [Hat60, Theorem 4.2]), since $\mu\lfloor_{\wedge^\bullet\mathfrak{g}^*} = \mathrm{id}\lfloor_{\wedge^\bullet\mathfrak{g}^*}$ by [Bel00, Theorem 7], we get that the map $\mu\colon H^\bullet_{dR}(X;\mathbb{R}) \to H^\bullet_{dR}\left(\mathfrak{g};\mathbb{R}\right)$ is the identity map, and hence $\mu\colon H^{(r,s)}_\omega(X;\mathbb{R}) \to H^{(r,s)}_\omega(\mathfrak{g};\mathbb{R})$ is also injective, hence an isomorphism. □

Symplectic (co)homology decomposition on solvmanifolds

Proposition 3.18 is a useful tool to study explicit examples, [AT12c, Example 3.4, Example 3.5, Example 3.6].

Example 3.19. *A* 6*-dimensional symplectic nilmanifold such that* $H^{(0,3)}_{\omega}(X;\mathbb{R}) + H^{(1,1)}_{\omega}(X;\mathbb{R}) \subsetneq H^3_{dR}(X;\mathbb{R})$ *and* $H^{(0,3)}_{\omega}(X;\mathbb{R}) \cap H^{(1,1)}_{\omega}(X;\mathbb{R}) \neq \{0\}$.
Take a 6-dimensional nilmanifold

$$X \;=\; \Gamma\backslash G \;:=\; \left(0^3,\; 12,\; 14-23,\; 15+34\right)$$

endowed with the G-left-invariant symplectic structure

$$\omega \;:=\; e^{16}+e^{35}+e^{24} \;.$$

By K. Nomizu's theorem [Nom54, Theorem 1], one computes

$$\begin{aligned}
H^1_{dR}(X;\mathbb{R}) &= \underbrace{\mathbb{R}\left\langle e^1,\; e^2,\; e^3\right\rangle}_{=H^{(0,1)}_{\omega}(X;\mathbb{R})} ,\\
H^2_{dR}(X;\mathbb{R}) &= \underbrace{\mathbb{R}\left\langle e^{16}+e^{35}+e^{24}\right\rangle}_{=H^{(1,0)}_{\omega}(X;\mathbb{R})} \oplus \underbrace{\mathbb{R}\left\langle e^{13},\; e^{14}+e^{23},\; 2\cdot e^{24}-e^{16}-e^{35}\right\rangle}_{=H^{(0,2)}_{\omega}(X;\mathbb{R})} ,\\
H^3_{dR}(X;\mathbb{R}) &= \mathbb{R}\left\langle e^{126}-e^{145}-2\cdot e^{235},\; e^{136},\; e^{146}+\frac{1}{2}\cdot e^{236}+\frac{1}{2}\cdot e^{345},\; e^{245}\right\rangle
\end{aligned}$$

(where, as usual, we have listed the harmonic representatives with respect to the G-left-invariant metric $\sum_{j=1}^{6} e^j \odot e^j$ instead of their classes, and we have shortened, for example, $e^{hk} := e^h \wedge e^k$).

Since the Lefschetz decompositions of the g-harmonic representatives of $H^3_{dR}(X;\mathbb{R})$ are

$$\begin{aligned}
e^{126}-e^{145}-2\cdot e^{235} &= \underbrace{\left(-\frac{1}{2}\cdot e^{126}-\frac{1}{2}\cdot e^{235}-e^{145}\right)}_{\in \mathrm{P}\wedge^3 X} + \underbrace{\left(\frac{3}{2}\cdot e^{126}-\frac{3}{2}\cdot e^{235}\right)}_{=L\left(-\frac{3}{2}\cdot e^2\right)} ,\\
e^{136} &= \underbrace{\left(\frac{1}{2}\cdot e^{136}-\frac{1}{2}\cdot e^{234}\right)}_{\in \mathrm{P}\wedge^3 X} + \underbrace{\left(\frac{1}{2}\cdot e^{136}+\frac{1}{2}\cdot e^{234}\right)}_{=L\left(-\frac{1}{2}\cdot e^3\right)} ,\\
e^{146}+\frac{1}{2}\cdot e^{236}+\frac{1}{2}\cdot e^{345} &= \underbrace{\left(\frac{1}{4}\cdot e^{146}-\frac{1}{4}\cdot e^{345}+\frac{1}{2}\cdot e^{236}\right)}_{\in \mathrm{P}\wedge^3 X} + \underbrace{\left(\frac{3}{4}\cdot e^{146}+\frac{3}{4}\cdot e^{345}\right)}_{=L\left(-\frac{3}{4}\cdot e^4\right)} ,\\
e^{245} &= \underbrace{\left(\frac{1}{2}\cdot e^{156}+\frac{1}{2}\cdot e^{245}\right)}_{\in \mathrm{P}\wedge^3 X} + \underbrace{\left(-\frac{1}{2}\cdot e^{156}+\frac{1}{2}\cdot e^{245}\right)}_{=L\left(\frac{1}{2}\cdot e^5\right)} ,
\end{aligned}$$

and since

$$\mathrm{d}\wedge^2\mathfrak{g}^* \;=\; \mathbb{R}\left\langle e^{123},\, e^{124},\; e^{125},\; e^{126}+e^{145},\; e^{134},\; e^{135},\; e^{146}-e^{236}-e^{345},\; e^{234}\right\rangle \;,$$

we get that

$$\left[e^{126}-e^{145}-2\cdot e^{235}\right] \;=\; \left[e^{126}-e^{145}-2\cdot e^{235}+\mathrm{d}\,e^{46}\right] \;=\; \left[2\cdot e^{126}-2\cdot e^{235}\right] \;=\; \left[L\left(-2\cdot e^2\right)\right] \;\in\; H^{(1,1)}_{\omega}(X;\mathbb{R})$$

and

$$\begin{aligned}
\left[e^{136}\right] &= \left[e^{136}+\mathrm{d}\left(\frac{1}{2}\cdot e^{45}-\frac{1}{2}\cdot e^{26}\right)\right] \;=\; \left[e^{136}+e^{234}\right] \;=\; \left[L\left(-e^3\right)\right] \;\in\; H^{(1,1)}_{\omega}(X;\mathbb{R}) ,\\
\left[e^{136}\right] &= \left[e^{136}-\mathrm{d}\left(\frac{1}{2}\cdot e^{45}-\frac{1}{2}\cdot e^{26}\right)\right] \;=\; \left[e^{136}-e^{234}\right] \;\in\; H^{(0,3)}_{\omega}(X;\mathbb{R}) ,
\end{aligned}$$

while it is straightforward to check that

$$\mathbb{R}\left\langle \left[e^{146}+\frac{1}{2}\cdot e^{236}+\frac{1}{2}\cdot e^{345}\right],\ \left[e^{245}\right] \right\rangle \cap \left(H^{(0,3)}_{\omega}(X;\mathbb{R})+H^{(1,1)}_{\omega}(X;\mathbb{R})\right) \;=\; \{0\}\ ;$$

in particular, $H^{(0,3)}_{\omega}(X;\mathbb{R})+H^{(1,1)}_{\omega}(X;\mathbb{R}) \subsetneq H^3_{dR}(X;\mathbb{R})$ and $H^{(0,3)}_{\omega}(X;\mathbb{R}) \cap H^{(1,1)}_{\omega}(X;\mathbb{R}) \neq \{0\}$.

Example 3.20. *A* 6*-dimensional symplectic solvmanifold satisfying the decomposition*

$$H^{\bullet}_{dR}(X;\mathbb{R}) = \bigoplus_{r\in\mathbb{N}} L^r\, H^{(0,\bullet-2r)}_{\omega}(X;\mathbb{R})\ .$$

Take the 6-dimensional solvable Lie algebra

$$\mathfrak{g}^{-1}_{3.4}\oplus\mathfrak{g}^{0}_{3.5} \;:=\; (-13,\ 23,\ 0,\ -56,\ 46,\ 0)$$

endowed with the linear symplectic structure

$$\omega \;:=\; e^{12}+e^{36}+e^{45}\ .$$

The corresponding connected simply-connected Lie group $G^{-1}_{3.4}\times G^{0}_{3.5}$ admits a compact quotient X, whose de Rham cohomology is the same as the cohomology of $\left(\wedge^{\bullet}\left(\mathfrak{g}^{-1}_{3.4}\oplus\mathfrak{g}^{0}_{3.5}\right)^*,\ \mathrm{d}\right)$, see [Boc09, Table 5]: indeed, note that $\dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R}) = \dim_{\mathbb{R}} H^k\left(\mathfrak{g}^{-1}_{3.4}\oplus\mathfrak{g}^{0}_{3.5};\mathbb{R}\right)$ for every $k\in\mathbb{N}$.

It is straightforward to compute

$$\begin{aligned}
H^1_{dR}(X;\mathbb{R}) &= \underbrace{\mathbb{R}\left\langle e^3,\ e^6\right\rangle}_{=H^{(0,1)}_{\omega}(X;\mathbb{R})}\ ,\\
H^2_{dR}(X;\mathbb{R}) &= \underbrace{\mathbb{R}\left\langle e^{12}+e^{36}+e^{45}\right\rangle}_{=H^{(1,0)}_{\omega}(X;\mathbb{R})}\oplus\underbrace{\mathbb{R}\left\langle e^{12}-e^{36},\ e^{12}-e^{45}\right\rangle}_{=H^{(0,2)}_{\omega}(X;\mathbb{R})}\ ,\\
H^3_{dR}(X;\mathbb{R}) &= \underbrace{\mathbb{R}\left\langle e^{123}+e^{345},\ e^{126}+e^{456}\right\rangle}_{=H^{(1,0)}_{\omega}(X;\mathbb{R})=L\,H^{(0,1)}_{\omega}(X;\mathbb{R})}\oplus\underbrace{\mathbb{R}\left\langle e^{123}-e^{345},\ e^{126}-e^{456}\right\rangle}_{=H^{(0,3)}_{\omega}(X;\mathbb{R})}\ ,\\
H^4_{dR}(X;\mathbb{R}) &= \underbrace{\mathbb{R}\left\langle e^{1236}+e^{1245}+e^{3456}\right\rangle}_{=H^{(2,0)}_{\omega}(X;\mathbb{R})}\oplus\underbrace{\mathbb{R}\left\langle e^{1236}-e^{1245},\ e^{1236}-e^{3456}\right\rangle}_{=H^{(1,2)}_{\omega}(X;\mathbb{R})=L\,H^{(0,2)}_{\omega}(X;\mathbb{R})}\ ,\\
H^5_{dR}(X;\mathbb{R}) &= \underbrace{\mathbb{R}\left\langle e^{12456},\ e^{12345}\right\rangle}_{=H^{(2,1)}_{\omega}(X;\mathbb{R})=L^2\,H^{(0,1)}_{\omega}(X;\mathbb{R})}\ ,
\end{aligned}$$

hence we get a decomposition

$$H^{\bullet}_{dR}(X;\mathbb{R}) \;=\; \bigoplus_{r\in\mathbb{N}} L^r\, H^{(0,\bullet-2r)}_{\omega}(X;\mathbb{R})\ .$$

In particular, it follows that the Hard Lefschetz Condition holds on $(X,\ \omega)$.

Example 3.21. *A* 6*-dimensional completely-solvable solvmanifold such that* $H^{(0,3)}_{\omega}(X;\mathbb{R})+H^{(1,1)}_{\omega}(X;\mathbb{R}) \subsetneq H^3_{dR}(X;\mathbb{R})$.
Take a 6-dimensional completely-solvable solvmanifold with Lie algebra $\mathfrak{g}_{3.1}\oplus\mathfrak{g}^{-1}_{3.4}$,

$$X \;=\; \Gamma\backslash G \;:=\; (-23,\ 0,\ 0,\ -46,\ 56,\ 0)\ ,$$

see [Boc09, Table 5], endowed with the G-left-invariant symplectic structure

$$\omega \;:=\; e^{12}+e^{36}+e^{45}\ .$$

By A. Hattori's theorem [Hat60, Corollary 4.2], one computes

$$\begin{aligned}
H^1_{dR}(X;\mathbb{R}) &= \underbrace{\mathbb{R}\left\langle e^2,\, e^3,\, e^6\right\rangle}_{=H^{(0,1)}_\omega(X;\mathbb{R})} ,\\
H^2_{dR}(X;\mathbb{R}) &= \underbrace{\mathbb{R}\left\langle e^{12}+e^{36}+e^{45}\right\rangle}_{=H^{(1,0)}_\omega(X;\mathbb{R})} \oplus \underbrace{\mathbb{R}\left\langle e^{12}-e^{36},\, e^{12}-e^{45},\, e^{13},\, e^{26}\right\rangle}_{=H^{(0,2)}_\omega(X;\mathbb{R})} ,\\
H^3_{dR}(X;\mathbb{R}) &= \mathbb{R}\left\langle e^{123},\, e^{126},\, e^{136},\, e^{245},\, e^{345},\, e^{456}\right\rangle .
\end{aligned}$$

Note that $e^{123}-e^{345}$, $e^{126}-e^{456}$ and $e^{245}+\mathrm{d}\,e^{16}$ are primitive, and consequently

$$H^{(0,3)}_\omega(X;\mathbb{R}) \supseteq \mathbb{R}\left\langle e^{123}-e^{345},\, e^{126}-e^{456},\, e^{245}\right\rangle ;$$

since $e^{123}+e^{345}=Le^3$, $e^{126}+e^{456}=Le^6$, and $e^{245}-\mathrm{d}\,e^{16}=L\,e^2$, it follows that

$$H^{(1,1)}_\omega(X;\mathbb{R}) = L\,H^{(0,3)}_\omega(X;\mathbb{R}) \supseteq \mathbb{R}\left\langle e^{123}+e^{345},\, e^{126}+e^{456},\, e^{245}\right\rangle ;$$

since

$$e^{136} = \underbrace{\frac{1}{2}\left(e^{136}+e^{145}\right)}_{\in L\,\mathrm{P}\wedge^1 X} + \underbrace{\frac{1}{2}\left(e^{136}-e^{145}\right)}_{\in \mathrm{P}\wedge^3 X} ,$$

and

$$\mathrm{d}\wedge^2\mathfrak{g}^* = \mathbb{R}\left\langle e^{146}-e^{234},\, e^{156}+e^{235},\, e^{236},\, e^{246},\, e^{256},\, e^{346},\, e^{356}\right\rangle ,$$

it follows that

$$\left\langle e^{136}\right\rangle \not\in H^{(0,3)}_\omega(X;\mathbb{R}) + H^{(1,1)}_\omega(X;\mathbb{R}) ,$$

and hence $H^{(0,3)}_\omega(X;\mathbb{R}) + H^{(1,1)}_\omega(X;\mathbb{R}) \subsetneq H^3_{dR}(X;\mathbb{R})$.

Finally, we give explicit examples of dual currents on a compact symplectic half-flat manifold, [AT12c, Example 3.7].

Example 3.22. *Dual currents of oriented special Lagrangian submanifolds of the Nakamura manifold.* Let $\mathbb{C}^3$ be endowed with the product $*$ defined by

$$\left(w^1,\, w^2,\, w^3\right) * \left(z^1,\, z^2,\, z^3\right) := \left(z^1+w^1,\, \mathrm{e}^{-w^1} z^2+w^2,\, \mathrm{e}^{w^1} z^3+w^3\right)$$

for every $\left(w^1,\, w^2,\, w^3\right),\, \left(z^1,\, z^2,\, z^3\right) \in \mathbb{C}^3$. Then $\left(\mathbb{C}^3,\, *\right)$ is a complex solvable (non-nilpotent) Lie group and, according to [Nak75], it admits a lattice $\Gamma\subset\mathbb{C}^3$, such that $X:=\Gamma\backslash(\mathbb{C}^3,*)$ is a solvmanifold, which is called the *Nakamura manifold*, see also [dBT06, §3]. Setting

$$\varphi^1 := \mathrm{d}\,z^1 , \qquad \varphi^2 := \mathrm{e}^{z^1}\,\mathrm{d}\,z^2 , \qquad \varphi^3 := \mathrm{e}^{-z^1}\,\mathrm{d}\,z^3 ,$$

then $\left\{\varphi^1,\, \varphi^2,\, \varphi^3\right\}$ is a global complex co-frame on X satisfying the following complex structure equations:

$$\begin{cases} \mathrm{d}\,\varphi^1 = 0 \\ \mathrm{d}\,\varphi^2 = \varphi^{12} \\ \mathrm{d}\,\varphi^3 = -\varphi^{13} \end{cases} .$$

If we set $\varphi^j =: e^j + \mathrm{i}\,e^{3+j}$, for $j\in\{1,\,2,\,3\}$, then the last equations yield to

$$\begin{cases} \mathrm{d}\,e^1 = 0 \\ \mathrm{d}\,e^2 = e^{12}-e^{45} \\ \mathrm{d}\,e^3 = -e^{13}+e^{46} \\ \mathrm{d}\,e^4 = 0 \\ \mathrm{d}\,e^5 = e^{15}-e^{24} \\ \mathrm{d}\,e^6 = -e^{16}+e^{34} \end{cases} . \tag{3.1.2}$$

Then, [dBT06, §5],

$$\omega \;:=\; e^{14}+e^{35}+e^{62}\;,$$

and

$$\begin{aligned} Je^1 &:= -e^4\;, & Je^3 &:= -e^5\;, & Je^6 &:= -e^2\;,\\ Je^4 &:= e^1\;, & Je^5 &:= e^3\;, & Je^2 &:= e^6\;, \end{aligned}$$

and

$$\psi \;:=\; \left(e^1+\mathrm{i}\,e^4\right)\wedge\left(e^3+\mathrm{i}\,e^5\right)\wedge\left(e^6+\mathrm{i}\,e^2\right)$$

give rise to a symplectic half-flat structure on X, where

$$\mathfrak{Re}\,\psi \;=\; e^{136}+e^{125}+e^{234}-e^{456}\;.$$

Note that the Hard Lefschetz Condition holds on $(X,\,\omega)$, [dBT06, Theorem 5.1].

Setting $z^j =: x^j+\mathrm{i}\,y^j$, for $j\in\{1,\,2,\,3\}$, and denoting by $\pi\colon\mathbb{C}^3\to X$ the canonical projection, we easily check that

$$\begin{aligned} Y_1 &:= \pi\left(\left\{\left(x^1,\,x^2,\,x^3,\,y^1,\,y^2,\,y^3\right)\in\mathbb{C}^3 \;:\; x^2=y^4=y^5=0\right\}\right)\;,\\ Y_2 &:= \pi\left(\left\{\left(x^1,\,x^2,\,x^3,\,y^1,\,y^2,\,y^3\right)\in\mathbb{C}^3 \;:\; x^3=y^4=y^6=0\right\}\right) \end{aligned}$$

are special Lagrangian submanifolds of $(X,\,\omega,\,\psi)$, namely, for $j\in\{1,\,2\}$, it holds $\mathfrak{Re}\,\psi\lfloor_{Y_j}=\mathrm{Vol}_{Y_j}$, and, consequently, the associated dual currents $[Y_j]$ are primitive.

3.2 Cohomology of D-complex manifolds

In this section, we provide some results obtained in a joint work with F. A. Rossi, [AR12], concerning the de Rham cohomology of almost-**D**-complex manifolds. **D**-complex Geometry is, in a sense, the "hyperbolic analogue" of Complex Geometry: an almost-**D**-complex structure on a manifold X is given by an endomorphism $K\in\mathrm{End}(TX)$ such that $K^2=\mathrm{id}_{TX}$ and the eigen-bundles of TX with respect to the eigenvalues 1 and -1 of $K\in\mathrm{End}(TX)$ have the same rank. Recently, **D**-complex Geometry appeared to be related with many other problems and notions in Mathematics and Physics (in particular, with product structures, bi-Lagrangian geometry, and optimal transport theory).

It is natural to ask what properties from Complex Geometry can be translated in the **D**-complex setting. We refer to the work by F. A. Rossi, e.g., [Ros12a, Ros12b], for problems and results in this direction. Here, we are mainly concerned in cohomological properties. In fact, it turns out that the **D**-complex counterpart of the Dolbeault cohomology is not well-behaved, not being finite-dimensional. This fact leads us to study some subgroups of the de Rham cohomology related to the almost-**D**-complex structure, miming the theory introduced by T.-J. Li and W. Zhang in [LZ09] for almost-complex manifolds. More precisely, we study the subgroups of the de Rham cohomology of an almost-**D**-complex manifold consisting of the classes admitting invariant, respectively anti-invariant, representatives with respect to the almost-**D**-complex structure; in particular, we prove that, on a 4-dimensional nilmanifold endowed with a left-invariant **D**-complex structure, such subgroups provide a decomposition at the level of the real second de Rham cohomology group, Theorem 3.47; counterexamples without the hypothesis on dimension, respectively nilpotency, respectively integrability, are provided. Moreover, we consider deformations of **D**-complex structures: in particular, we show that admitting a **D**-Kähler structure is not a stable property under small deformations of the **D**-complex structure, Theorem 3.50, providing another strong difference with the complex case (indeed, recall that admitting a Kähler structure is a stable property under small deformations of the complex structure by K. Kodaira and D. C. Spencer's theorem [KS60, Theorem 15]).

3.2.1 D-complex structures on manifolds

We start by recalling the basic definitions in **D**-complex Geometry. We refer, e.g., to [HL83, AMT09, CMMS04, CMMS05, CM09, CFAG96, KMW10, ABDMO05, AS05, Kra10, Ros12a, Ros12b] and the references therein for more results about (almost-)**D**-complex structures and for motivations for their study.

Let X be a $2n$-dimensional manifold. Consider $K\in\mathrm{End}(TX)$ such that $K^2=\lambda\,\mathrm{id}_{TX}$ where $\lambda\in\{-1,\,0,\,1\}$: if $\lambda=-1$, then by definition K is an *almost-complex* structure; if $\lambda=0$, then the structure K is called an *almost-subtangent* structure; if $\lambda=1$, then K is said to be an *almost-product* structure; according to [Vai07, §1], these three structures are called *almost-c.p.s.* structures.

In the case $K^2=\mathrm{id}_{TX}$, one gets that K has eigen-values $\{1,\,-1\}$ and hence there is a decomposition $TX=T^+X\oplus T^-X$ where $T^\pm X$ is given, point by point, by the eigen-space of K corresponding to the eigen-value

± 1, where $\pm \in \{+, -\}$. By definition, an *almost-$\mathbf{D}$-complex* structure (also called *almost-para-complex* structure) on X is an endomorphism $K \in \mathrm{End}(TX)$ such that

$$K^2 \;=\; \mathrm{id}_{TX} \qquad \text{and} \qquad \mathrm{rk}\, T^+X \;=\; \mathrm{rk}\, T^-X \;=\; \frac{1}{2}\dim X \;;$$

a $\mathbf{D}$-*holomorphic map* between two almost-$\mathbf{D}$-complex manifolds (X_1, K_1) and (X_2, K_2) is a smooth map $f\colon X_1 \to X_2$ such that $\mathrm{d}\, f \circ K_1 = K_2 \circ \mathrm{d}\, f$.

An almost-$\mathbf{D}$-complex structure is said to be *integrable* (and hence it is called $\mathbf{D}$-*complex*, or also *para-complex*) if

$$\left[T^+X,\, T^+X\right] \;\subseteq\; T^+X \qquad \text{and} \qquad \left[T^-X,\, T^-X\right] \;\subseteq\; T^-X \;.$$

The integrability condition is, straightforwardly, equivalent to the vanishing of the *Nijenhuis tensor* N_K of K, where

$$N_K(\cdot,\, \cdot\cdot) \;:=\; [\cdot,\, \cdot\cdot] + [K\cdot,\, K\cdot\cdot] - K\,[K\cdot,\, \cdot\cdot] - K\,[\cdot,\, K\cdot\cdot] \;;$$

furthermore, as in the complex case, one has that an almost-$\mathbf{D}$-complex structure on an n-dimensional manifold X is integrable if and only if it is naturally associated to a structure on X defined in terms of local homeomorphisms with open sets in $\mathbf{D}^n$ and $\mathbf{D}$-holomorphic changes of coordinates, see, e.g., [CMMS04, Proposition 3], where $\mathbf{D}^n := \mathbb{R}^n + \tau\,\mathbb{R}^n$, with $\tau^2 = 1$, is the algebra of *double numbers*.

We recall that, given a $2n$-dimensional manifold endowed with an almost-$\mathbf{D}$-complex structure K, a $\mathbf{D}$-*Hermitian metric* on X is a pseudo-Riemannian metric of signature (n, n) such that $g(K\cdot,\, K\cdot\cdot) = -g(\cdot,\, \cdot\cdot)$. A $\mathbf{D}$-*Kähler* structure on a manifold X is the datum of an integrable $\mathbf{D}$-complex structure K and a $\mathbf{D}$-Hermitian metric g such that its associated K-anti-invariant form $\omega := g\,(K\,\cdot,\, \cdot\cdot)$ is d-closed, equivalently, the datum of a K-*compatible* (that is, a K-anti-invariant) symplectic form on X, see, e.g., [AMT09, §5.1], [CMMS04, Theorem 1].

The basic example of $\mathbf{D}$-complex structure is given on the product of two manifolds of the same dimension: given X^+ and X^- two manifolds with $\dim X^+ = \dim X^-$, the product $X^+ \times X^-$ inherits a natural $\mathbf{D}$-complex structure K, given by the decomposition

$$T\left(X^+ \times X^-\right) \;=\; TX^+ \oplus TX^- \,,$$

where $K\lfloor_{TX^+} = \mathrm{id}_{T(X^+\times X^-)}$ and $K\lfloor_{TX^-} = -\,\mathrm{id}_{T(X^+\times X^-)}$. Every $\mathbf{D}$-complex manifold is locally of this form, see, e.g., [CMMS04, Proposition 2].

Starting from $K \in \mathrm{End}(TX)$ such that $K^2 = \mathrm{id}_{TX}$, one can define, by duality, an endomorphism $K \in \mathrm{End}(T^*X)$ such that $K^2 = \mathrm{id}_{T^*X}$, and hence one gets a natural decomposition $T^*X = \left(T^+X\right)^* \oplus \left(T^-X\right)^*$ into eigen-bundles. Extending $K \in \mathrm{End}(T^*X)$ to $K \in \mathrm{End}\left(\wedge^\bullet X\right)$, one gets the following decomposition on the bundle of differential ℓ-forms, for $\ell \in \mathbb{N}$:

$$\wedge^\ell X \;=\; \bigoplus_{p+q=\ell} \wedge^{p,\,q}_{+\,-} X \qquad \text{where} \qquad \wedge^{p,\,q}_{+\,-}\, X \;:=\; \wedge^p\left(T^+X\right)^* \otimes \wedge^q\left(T^-X\right)^* \;;$$

note that, for any $p, q \in \mathbb{N}$, the structure K acts on $\wedge^{p,\,q}_{+\,-}\, X$ as $(+1)^p\,(-1)^q\,\mathrm{id}_{\wedge^{p,\,q}_{+\,-}\, X}$. In particular, for any $\ell \in \mathbb{N}$, one has

$$\wedge^\ell X \;=\; \wedge^{\ell\,+}_K X \oplus \wedge^{\ell\,-}_K X$$

where

$$\wedge^{\ell\,+}_K X \;:=\; \bigoplus_{p+q=\ell,\; q=0 \bmod 2} \wedge^{p,\,q}_{+\,-} X \qquad \text{and} \qquad \wedge^{\ell\,-}_K X \;:=\; \bigoplus_{p+q=\ell,\; q=1 \bmod 2} \wedge^{p,\,q}_{+\,-} X \;;$$

note that $K\lfloor_{\wedge^{\ell\,+}_K X} = \mathrm{id}_{\wedge^{\ell\,+}_K X}$ and $K\lfloor_{\wedge^{\ell\,-}_K X} = -\,\mathrm{id}_{\wedge^{\ell\,-}_K X}$.

If a $\mathbf{D}$-complex structure K is given, then the exterior differential splits as

$$\mathrm{d} \;=\; \partial_+ + \partial_-$$

where

$$\partial_+ \;:=\; \pi_{\wedge^{p+1,\,q}_{+\,-} X} \circ \mathrm{d}\colon\; \wedge^{p,\,q}_{+\,-}\, X \to \wedge^{p+1,\,q}_{+\,-} X$$

and

$$\partial_- \;:=\; \pi_{\wedge^{p,\,q+1}_{+\,-} X} \circ \mathrm{d}\colon\; \wedge^{p,\,q}_{+\,-}\, X \to \wedge^{p,\,q+1}_{+\,-} X$$

(where $\pi_{\wedge^{r,\,s}_{+\,-}X}\colon \wedge^{\bullet,\,\bullet}_{+\,-}X \to \wedge^{r,\,s}_{+\,-}X$ denotes the natural projection onto $\wedge^{r,\,s}_{+\,-}X$, for every $r, s \in \mathbb{N}$). In particular, the condition $\mathrm{d}^2 = 0$ can be rewritten as

$$\begin{cases} \partial_+^2 &= 0 \\ \partial_+\partial_- + \partial_-\partial_+ &= 0 \\ \partial_-^2 &= 0 \end{cases}$$

and hence one can define a **D**-complex counterpart of the Dolbeault cohomology by considering the cohomology of the differential complex $\left(\wedge^{\bullet,\,q}_{+\,-}X,\, \partial_+\right)$ for every $q \in \mathbb{N}$, that is,

$$H^{\bullet,\bullet}_{\partial_+}(X;\mathbb{R}) \;:=\; \frac{\ker \partial_+}{\mathrm{im}\,\partial_+} \;.$$

Unfortunately, one cannot hope to adjust the Hodge theory of the complex case to this non-elliptic context. For example, take X^+ and X^- two manifolds having the same dimension and consider the natural **D**-complex structure on $X^+ \times X^-$; one has that

$$H^{0,0}_{\partial_+}\left(X^+ \times X^-\right) \;\simeq\; \mathcal{C}^\infty\left(X^-\right) \;,$$

hence the space $H^{0,0}_{\partial_+}\left(X^+ \times X^-\right)$ of ∂_+-closed functions on $X^+ \times X^-$ is not finite-dimensional, even if X^+ and X^- are compact.

3.2.2 D-complex subgroups of (co)homology

In this section, we adapt T.-J. Li and W. Zhang's theory on cohomology of almost-complex manifolds, [LZ09], to the almost-**D**-complex case. More precisely, let X be a $2n$-dimensional compact manifold endowed with an almost-**D**-complex structure K; we are interested in studying when the decomposition

$$\wedge^\bullet X \;=\; \bigoplus_{p,q} \wedge^{p,\,q}_{+\,-}X \;=\; \wedge^{\bullet\,+}_K X \oplus \wedge^{\bullet\,-}_K X$$

gives rise to a cohomological decomposition.

We start by giving some definitions. For any $p, q \in \mathbb{N}$, we define the subgroup

$$H^{(p,q)}_K(X;\mathbb{R}) \;:=\; \left\{[\alpha] \in H^{p+q}_{dR}(X;\mathbb{R}) \;:\; \alpha \in \wedge^{p,\,q}_{+\,-}X\right\} \;\subseteq\; H^\bullet_{dR}(X;\mathbb{R}) \;,$$

and, for any $\ell \in \mathbb{N}$ and for $\pm \in \{+,-\}$, the subgroup

$$H^{\ell\,\pm}_K(X;\mathbb{R}) \;:=\; \left\{[\alpha] \in H^\ell_{dR}(X;\mathbb{R}) \;:\; K\alpha = \pm\alpha\right\} \;=\; \left\{[\alpha] \in H^\ell_{dR}(X;\mathbb{R}) \;:\; \alpha \in \wedge^{\ell\,\pm}_K X\right\} \;\subseteq\; H^\bullet_{dR}(X;\mathbb{R}) \;.$$

Note, that, if K is integrable, then, for any $\ell \in \mathbb{N}$,

$$H^{\ell\,+}_K \;=\; \bigoplus_{p+q=\ell,\; q=0 \bmod 2} H^{(p,q)}_K(X;\mathbb{R}) \qquad \text{and} \qquad H^{\ell\,-}_K \;=\; \bigoplus_{p+q=\ell,\; q=1 \bmod 2} H^{(p,q)}_K(X;\mathbb{R}) \;.$$

As in [LZ09, Definition 2.2, Definition 2.3, Lemma 2.2], see §2.1.1, for almost-complex structures, we introduce the following definition, [AR12, Definition 1.2].

Definition 3.23. For $\ell \in \mathbb{N}$, an almost-**D**-complex structure K on the manifold X is said to be

- $\mathcal{C}^\infty$*-pure at the* ℓ^{th} *stage* if
$$H^{\ell\,+}_K(X;\mathbb{R}) \cap H^{\ell\,-}_K(X;\mathbb{R}) \;=\; \{0\} \;;$$
- $\mathcal{C}^\infty$*-full at the* ℓ^{th} *stage* if
$$H^{\ell\,+}_K(X;\mathbb{R}) + H^{\ell\,-}_K(X;\mathbb{R}) \;=\; H^\ell_{dR}(X;\mathbb{R}) \;;$$
- $\mathcal{C}^\infty$*-pure-and-full at the* ℓ^{th} *stage* if it is both $\mathcal{C}^\infty$-pure at the ℓ^{th} stage and $\mathcal{C}^\infty$-full at the ℓ^{th} stage, namely, if it satisfies the cohomological decomposition
$$H^\ell_{dR}(X;\mathbb{R}) \;=\; H^{\ell\,+}_K(X;\mathbb{R}) \oplus H^{\ell\,-}_K(X;\mathbb{R}) \;.$$

Consider now the space $\mathcal{D}^\bullet X :=: \mathcal{D}_{2n-\bullet} X$ of currents on X and the de Rham homology $H^{dR}_\bullet(X;\mathbb{R})$ (we refer to §0.5, and references therein, for definitions and results concerning currents and de Rham homology). The action of K on $\wedge^\bullet X$ induces, by duality, an action, still denoted by K, on $\mathcal{D}_\bullet X$, and hence, for any $\ell \in \mathbb{N}$, a decomposition

$$\mathcal{D}_\ell X \;=:\; \bigoplus_{p+q=\ell} \mathcal{D}^{+\,-}_{p,\,q} X \;.$$

For any $p,q \in \mathbb{N}$, note that the space $\mathcal{D}^{+\,-}_{p,\,q} X :=: \mathcal{D}^{n-p,\,n-q}_{+\,-}$ is the topological dual space of the topological subspace $\wedge^{p,\,q}_{+\,-} X$ of $\wedge^\bullet X$, and that the quasi-isomorphism $T.\colon \wedge^\bullet X \ni \alpha \mapsto \int_X \alpha \wedge \cdot \in \mathcal{D}^\bullet X$ yields the inclusion $T.\colon \wedge^{p,\,q}_{+\,-} \hookrightarrow \mathcal{D}^{p,\,q}_{+\,-} X$. As before, we have

$$\mathcal{D}_\bullet X \;=\; \mathcal{D}^K_{\bullet +} X \,\oplus\, \mathcal{D}^K_{\bullet -} X$$

where

$$\mathcal{D}^K_{\bullet +} X \;:=\; \bigoplus_{q=0 \bmod 2} \mathcal{D}^{+\,-}_{\bullet,\,q} X \qquad \text{and} \qquad \mathcal{D}^K_{\bullet -} X \;:=\; \bigoplus_{q=1 \bmod 2} \mathcal{D}^{+\,-}_{\bullet,\,q} X \;,$$

and $K\lfloor_{\mathcal{D}^K_{\bullet\pm} X} = \pm\, \mathrm{id}_{\mathcal{D}^K_{\bullet\pm} X}$ for $\pm \in \{+,\,-\}$.

For any $p,q \in \mathbb{N}$, we define the subgroup

$$H^K_{(p,q)}\,(X;\mathbb{R}) \;:=\; \left\{[\alpha] \in H^{dR}_{p+q}\,(X;\mathbb{R}) \;:\; \alpha \in \mathcal{D}^{+\,-}_{p,\,q} X\right\} \;\subseteq\; H^{dR}_\bullet(X;\mathbb{R}) \;,$$

and, for any $\ell \in \mathbb{N}$ and for $\pm \in \{+,-\}$, the subgroup

$$H^K_{\ell\pm}\,(X;\mathbb{R}) \;:=\; \left\{[\alpha] \in H^{dR}_\ell\,(X;\mathbb{R}) \;:\; K\alpha = \pm\alpha\right\} \;=\; \left\{[\alpha] \in H^{dR}_\ell\,(X;\mathbb{R}) \;:\; \alpha \in \mathcal{D}^K_{\ell\pm} X\right\} \;\subseteq\; H^{dR}_\bullet(X;\mathbb{R}) \,.$$

We are particularly interested in the almost-**D**-complex structures admitting a homological decomposition through the subgroups $H^K_{\bullet +}\,(X;\mathbb{R})$ and $H^K_{\bullet -}\,(X;\mathbb{R})$, [AR12, Definition 1.3].

Definition 3.24. For $\ell \in \mathbb{N}$, an almost-**D**-complex structure K on the manifold X is said to be

- *pure at the ℓ^{th} stage* if
$$H^K_{\ell +}\,(X;\mathbb{R}) \cap H^K_{\ell -}\,(X;\mathbb{R}) \;=\; \{0\} \;;$$
- *full at the ℓ^{th} stage* if
$$H^K_{\ell +}\,(X;\mathbb{R}) \,+\, H^K_{\ell -}\,(X;\mathbb{R}) \;=\; H_\ell\,(X;\mathbb{R}) \;;$$
- *pure-and-full at the ℓ^{th} stage* if it is both pure at the ℓ^{th} stage and full at the ℓ^{th} stage, namely, if it satisfies the homological decomposition
$$H_\ell\,(X;\mathbb{R}) \;=\; H^K_{\ell +}\,(X;\mathbb{R}) \,\oplus\, H^K_{\ell -}\,(X;\mathbb{R}) \;.$$

The introduced notions are not completely independent. Using the same argument as in Theorem 2.10, see [LZ09, Proposition 2.5], and in Proposition 3.12, we prove the following relations between $\mathcal{C}^\infty$-pure-and-fullness and pure-and-fullness for almost-**D**-complex structures, [AR12, Proposition 1.4].

Proposition 3.25. *Let K be an almost-**D**-complex structure on a $2n$-dimensional compact manifold X. Then, for every $\ell \in \mathbb{N}$, the following implications hold:*

$$\begin{array}{ccc} \mathcal{C}^\infty\text{-}\textit{full at the } \ell^{th} \textit{ stage} & \Longrightarrow & \textit{pure at the } \ell^{th} \textit{ stage} \\ \Downarrow & & \Downarrow \\ \textit{full at the } (2n-\ell)^{th} \textit{ stage} & \Longrightarrow & \mathcal{C}^\infty\text{-}\textit{pure at the } (2n-\ell)^{th} \textit{ stage} \end{array}$$

Proof. We recall that the quasi-isomorphism $T.\colon \wedge^\bullet X \ni \alpha \mapsto \int_X \alpha \wedge \cdot \in \mathcal{D}_{2n-\bullet} X$ induces, for every $p,q \in \mathbb{N}$, the inclusion

$$H^{(p,q)}_K(X;\mathbb{R}) \hookrightarrow H^K_{(n-p,n-q)}(X;\mathbb{R}) \;:$$

this fact proves the two vertical implications.

To prove the horizontal implications, consider the duality paring $\langle\cdot,\,\cdot\cdot\rangle : D_\ell X \times \wedge^\ell X \to \mathbb{R}$ and the induced non-degenerate pairing

$$\langle\cdot,\,\cdot\cdot\rangle : H^\ell_{dR}(X;\mathbb{R}) \times H^{dR}_\ell(X;\mathbb{R}) \to \mathbb{R} \;.$$

Suppose that K is $\mathcal{C}^\infty$-full at the ℓ^{th} stage, that is, $H^\ell_{dR}(X;\mathbb{R}) = H^{\ell+}_K(X;\mathbb{R}) + H^{\ell-}_K(X;\mathbb{R})$, and let $\mathfrak{c} = [\gamma_+] = [\gamma_-] \in H^K_{\ell+}(X;\mathbb{R}) \cap H^K_{\ell-}(X;\mathbb{R})$ with $\gamma_+ \in D^K_{\ell+}X$ and $\gamma_- \in D^K_{\ell-}X$; since

$$\left\langle H^\ell(X;\mathbb{R}),\, \mathfrak{c}\right\rangle \;=\; \left\langle H^{\ell+}_K(X;\mathbb{R}) + H^{\ell-}_K(X;\mathbb{R}),\, \mathfrak{c}\right\rangle \;=\; \left\langle H^{\ell+}_K(X;\mathbb{R}),\, [\gamma_-]\right\rangle + \left\langle H^{\ell-}_K(X;\mathbb{R}),\, [\gamma_+]\right\rangle \;=\; 0\,,$$

one has $\mathfrak{c} = 0$ in $H^{dR}_\ell(X;\mathbb{R})$; hence K is pure at the ℓ^{th} stage.

Similarly, since

$$\left\langle H^{\ell+}_K(X;\mathbb{R}) \cap H^{\ell-}_K(X;\mathbb{R}),\, H^K_{\ell+}(X;\mathbb{R}) + H^K_{\ell-}(X;\mathbb{R})\right\rangle \;=\; 0\,,$$

we get that, if K is full at the ℓ^{th} stage, then it is $\mathcal{C}^\infty$-pure at the ℓ^{th} stage. □

In particular, by applying Proposition 3.25 with $2n = 4$ and $k = 2$, one gets that, on a compact 4-dimensional manifold endowed with an almost-**D**-complex structure, being $\mathcal{C}^\infty$-full at the 2^{nd} stage is stronger than being $\mathcal{C}^\infty$-pure at the 2^{nd} stage.

A straightforward consequence of Proposition 3.25 is the following result, [AR12, Corollary 1.5].

Corollary 3.26. *Let K be an almost-**D**-complex structure on a compact manifold X. If K is $\mathcal{C}^\infty$-full at every stage, then it is also $\mathcal{C}^\infty$-pure-and-full at every stage and pure-and-full at every stage.*

As an application of the Künneth formula, T. Drăghici, T.-J. Li, and W. Zhang noted that, given X_1, respectively X_2, a compact manifold endowed with a $\mathcal{C}^\infty$-pure-and-full almost-complex structure J_1, respectively J_2, and assuming that $b_1(X_1) = 0$, or $b_1(X_2) = 0$, then the almost-complex structure $J_1 + J_2$ on $X_1 \times X_2$ is $\mathcal{C}^\infty$-pure-and-full, [DLZ12, Proposition 2.6]. In the **D**-complex case, we have the following, [AR12, Theorem 1.6].

Theorem 3.27. *Let X^+ and X^- be two compact manifolds of the same dimension. Then the natural **D**-complex structure on the product $X^+ \times X^-$ is $\mathcal{C}^\infty$-pure-and-full at every stage and pure-and-full at every stage.*

Proof. For any $\ell \in \mathbb{N}$, using the Künneth formula, one gets

$$\begin{aligned}
H^\ell_{dR}\left(X^+ \times X^-;\mathbb{R}\right) \;&\simeq\; \bigoplus_{p+q=\ell} H^p_{dR}\left(X^+;\mathbb{R}\right) \otimes H^q_{dR}\left(X^-;\mathbb{R}\right)\\
&=\; \underbrace{\left(\bigoplus_{p+q=\ell,\ q=0 \bmod 2} H^p_{dR}\left(X^+;\mathbb{R}\right) \otimes H^q_{dR}\left(X^-;\mathbb{R}\right)\right)}_{\subseteq H^{\ell+}_K(X^+\times X^-;\mathbb{R})} \oplus \underbrace{\left(\bigoplus_{p+q=\ell,\ q=1 \bmod 2} H^p_{dR}\left(X^+;\mathbb{R}\right) \otimes H^q_{dR}\left(X^-;\mathbb{R}\right)\right)}_{\subseteq H^{\ell-}_K(X^+\times X^-;\mathbb{R})}\\
&\subseteq\; H^{\ell+}_K\left(X^+ \times X^-;\mathbb{R}\right) + H^{\ell-}_K\left(X^+ \times X^-;\mathbb{R}\right)\;;
\end{aligned}$$

hence, by using Corollary 3.26, one gets the theorem. □

3.2.3 D-complex cohomological decomposition on solvmanifolds

In this section, we consider left-invariant **D**-complex structures on solvmanifolds, as in §2.2.3 for almost-complex structures, and in §3.1.3 for symplectic structures. We recall that, given a Lie algebra $\mathfrak{g}$, one has the differential operator $\mathrm{d}\colon \wedge^\bullet \mathfrak{g}^* \to \wedge^{\bullet+1}\mathfrak{g}^*$ naturally induced by the Lie bracket $[\cdot,\, \cdot\cdot]$, and hence the cohomology $H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) := H^\bullet\left(\wedge^\bullet\mathfrak{g}^*,\, \mathrm{d}\right)$. Hence, we are concerned with studying the linear counterpart of **D**-complex structures on Lie algebras, and the corresponding decomposition problem for the cohomology $H^\bullet_{dR}(\mathfrak{g};\mathbb{R})$. In particular, we prove a Nomizu-type result for the subgroups $H^{\ell\pm}_K(X;\mathbb{R})$, Proposition 3.30; it will allow to explicitly study several examples of **D**-complex solvmanifolds: in §3.2.3, we provide some examples of **D**-complex structures on solvmanifolds, even admitting a **D**-Kähler structure, that do not satisfy the cohomology decomposition by means of the subgroups $H^{\ell\pm}_K(X;\mathbb{R})$; then, we prove that, for every left-invariant **D**-complex structure on a 4-dimensional nilmanifold, it holds $H^2_{dR}(X;\mathbb{R}) = H^{2+}_K(X;\mathbb{R}) \oplus H^{2-}_K(X;\mathbb{R})$, Theorem 3.47, which provides a partial **D**-complex counterpart of [DLZ10, Theorem 2.3]. (We refer to §0.6 for definitions and results concerning solvmanifolds.)

We recall that a *linear almost-**D**-complex* structure K on $\mathfrak{g}$ is an endomorphism $K \in \mathrm{End}(\mathfrak{g})$ such that

$$K^2 \;=\; \mathrm{id}_{\mathfrak{g}} \qquad \text{and} \qquad \dim_{\mathbb{R}} \mathfrak{g}^+ \;=\; \dim_{\mathbb{R}} \mathfrak{g}^- \;=\; \frac{1}{2}\dim_{\mathbb{R}} \mathfrak{g}\,,$$

where $\mathfrak{g}^\pm$ is the eigen-space of K corresponding to the eigen-value ± 1, for $\pm \in \{+,\, -\}$. A linear almost-**D**-complex structure on $\mathfrak{g}$ is said to be *integrable* (and hence it is called a *linear **D**-complex* structure on $\mathfrak{g}$) if $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are Lie subalgebras of $\mathfrak{g}$, that is,

$$\left[\mathfrak{g}^+,\, \mathfrak{g}^+\right] \;\subseteq\; \mathfrak{g}^+ \qquad \text{and} \qquad \left[\mathfrak{g}^-,\, \mathfrak{g}^-\right] \;\subseteq\; \mathfrak{g}^-\,.$$

As a matter of notation, with respect to a given basis $\{e_j\}_{j\in\{1,\ldots,\dim_{\mathbb{R}}\mathfrak{g}\}}$ of $\mathfrak{g}$, in writing a(n almost-)**D**-complex structure K, e.g., (suppose $\dim_{\mathbb{R}}\mathfrak{g}=6$,) as

$$K \;:=\; (-\;+\;+\;-\;-\;+)$$

we mean that

$$\mathfrak{g}^{+} \;:=\; \mathbb{R}\left\langle e_2,\; e_3,\; e_6\right\rangle \qquad \text{and} \qquad \mathfrak{g}^{-} \;:=\; \mathbb{R}\left\langle e_1,\; e_4,\; e_5\right\rangle \;.$$

By considering the dual map $K \in \mathrm{End}\left(\mathfrak{g}^*\right)$ of $K\in\mathrm{End}(\mathfrak{g})$ and by extending it to $K\in\mathrm{End}\left(\wedge^\bullet\mathfrak{g}^*\right)$, the splitting $\mathfrak{g}=\mathfrak{g}^+\oplus\mathfrak{g}^-$ into eigen-spaces given by the linear almost-**D**-complex structure K on $\mathfrak{g}$ induces also a splitting $\mathfrak{g}^*=\left(\mathfrak{g}^+\right)^*\oplus\left(\mathfrak{g}^-\right)^*$, and hence, for every $\ell\in\mathbb{N}$, a splitting on the space of ℓ-forms on $\mathfrak{g}^*$:

$$\wedge^{\ell}\mathfrak{g}^* \;=\; \bigoplus_{p+q=\ell}\wedge^{p,\,q}_{+\,-}\,\mathfrak{g}^* \qquad\text{where}\qquad \wedge^{p,\,q}_{+\,-}\,\mathfrak{g}^* \;:=\; \bigoplus_{p+q=\ell}\wedge^{p}\left(\mathfrak{g}^+\right)^*\otimes\wedge^{q}\left(\mathfrak{g}^-\right)^* \;;$$

for any $p,q\in\mathbb{N}$, one has $K\lfloor_{\wedge^{p,\,q}_{+\,-}\,\mathfrak{g}^*}=(+1)^p\,(-1)^q\,\mathrm{id}_{\wedge^{p,\,q}_{+\,-}\,\mathfrak{g}^*}$. Consider also the splitting of the space of forms into its K-invariant and K-anti-invariant components:

$$\wedge^\bullet\mathfrak{g}^* \;=\; \wedge^{\bullet\,+}_{K}\mathfrak{g}^*\,\oplus\,\wedge^{\bullet\,-}_{K}\mathfrak{g}^*$$

where

$$\wedge^{\bullet\,+}_{K}\mathfrak{g}^* \;:=\; \bigoplus_{q=0\bmod 2}\wedge^{\bullet,\,q}_{+\,-}\,\mathfrak{g}^* \qquad\text{and}\qquad \wedge^{\bullet\,-}_{K}\,\mathfrak{g}^* \;:=\; \bigoplus_{q=1\bmod 2}\wedge^{\bullet,\,q}_{+\,-}\,\mathfrak{g}^* \;.$$

As for manifolds, we define, for every $p,q\in\mathbb{N}$, the subgroup

$$H^{(p,q)}_{K}\left(\mathfrak{g};\mathbb{R}\right) \;:=\; \left\{[\alpha]\in H^{p+q}_{dR}\left(\mathfrak{g};\mathbb{R}\right)\;:\;\alpha\in\wedge^{p,\,q}_{+\,-}\,\mathfrak{g}^*\right\}\;\subseteq\;H^{\bullet}_{dR}\left(\mathfrak{g};\mathbb{R}\right)\;,$$

and, for any $\ell\in\mathbb{N}$ and for $\pm\in\{+,-\}$, the subgroup

$$H^{\ell\,\pm}_{K}\left(\mathfrak{g};\mathbb{R}\right) \;:=\; \left\{[\alpha]\in H^{\ell}\left(\mathfrak{g};\mathbb{R}\right)\;:\;K\alpha=\pm\alpha\right\} \;=\; \left\{[\alpha]\in H^{\ell}\left(\mathfrak{g};\mathbb{R}\right)\;:\;\alpha\in\wedge^{\bullet\,\pm}_{K}\mathfrak{g}^*\right\}\;\subseteq\;H^{\bullet}_{dR}\left(\mathfrak{g};\mathbb{R}\right)\;,$$

and we give the following definition, [AR12, Definition 2.1].

Definition 3.28. For $\ell\in\mathbb{N}$, a linear almost-**D**-complex structure on the Lie algebra $\mathfrak{g}$ is said to be

- *linear-$\mathcal{C}^\infty$-pure at the ℓ^{th} stage* if

$$H^{\ell\,+}_{K}\left(\mathfrak{g};\mathbb{R}\right)\cap H^{\ell\,-}_{K}\left(\mathfrak{g};\mathbb{R}\right) \;=\; \{0\}\;;$$

- *linear-$\mathcal{C}^\infty$-full at the ℓ^{th} stage* if

$$H^{\ell\,+}_{K}\left(\mathfrak{g};\mathbb{R}\right)+H^{\ell\,-}_{K}\left(\mathfrak{g};\mathbb{R}\right) \;=\; H^{\ell}_{dR}\left(\mathfrak{g};\mathbb{R}\right)\;;$$

- *linear-$\mathcal{C}^\infty$-pure-and-full at the ℓ^{th} stage* if it is both $\mathcal{C}^\infty$-pure at the ℓ^{th} stage and $\mathcal{C}^\infty$-full at the ℓ^{th} stage, namely, if it satisfies the cohomological decomposition

$$H^{\ell}_{dR}\left(\mathfrak{g};\mathbb{R}\right) \;=\; H^{\ell\,+}_{K}\left(\mathfrak{g};\mathbb{R}\right)\oplus H^{\ell\,-}_{K}\left(\mathfrak{g};\mathbb{R}\right)\;.$$

Given a $2n$-dimensional solvmanifold $X=\Gamma\backslash G$, one can consider the associated Lie algebra $\mathfrak{g}$ to the Lie group G. Note that a G-left-invariant almost-**D**-complex structure on X is uniquely determined, through left-translations on G, by a linear almost-**D**-complex structure on $\mathfrak{g}$; furthermore, a G-left-invariant almost-**D**-complex structure on X is integrable if and only if the corresponding linear almost-**D**-complex structure on $\mathfrak{g}$ is integrable. Hence, in the following we will confuse a G-left-invariant (almost-)**D**-complex structure K on the solvmanifold $X=\Gamma\backslash G$ and the corresponding linear (almost-)**D**-complex structure on the naturally associated Lie algebra $\mathfrak{g}$.

We recall that the left-translations induce an injective map in cohomology,

$$H^{\bullet}_{dR}\left(\mathfrak{g};\mathbb{R}\right)\;\hookrightarrow\;H^{\bullet}_{dR}(X;\mathbb{R})\;,$$

where $H^{\bullet}_{dR}\left(\mathfrak{g};\mathbb{R}\right)$ can be interpreted as the cohomology of the sub-complex composed of the G-left-invariant forms of $\left(\wedge^\bullet X,\,\mathrm{d}\right)$, and that this map is actually an isomorphism if G is nilpotent, respectively completely-solvable, by K. Nomizu's theorem [Nom54, Theorem 1], respectively by A. Hattori's theorem [Hat60, Corollary 4.2].

Hence, given a G-left-invariant $\mathbf{D}$-complex structure on X, we may study $\mathbf{D}$-complex decomposition in cohomology both on $H^\bullet_{dR}(\mathfrak{g};\mathbb{R})$ and on $H^\bullet_{dR}(X;\mathbb{R})$. The aim of this section is to make clear the connection between the $\mathcal{C}^\infty$-pure-and-fullness of a left-invariant almost-$\mathbf{D}$-complex structure on a completely-solvable solvmanifold and the linear-$\mathcal{C}^\infty$-pure-and-fullness of the corresponding linear almost-$\mathbf{D}$-complex structure on the associated Lie algebra.

The following lemma adapt F. A. Belgun's symmetrization trick, [Bel00, Theorem 7], to the $\mathbf{D}$-complex case, [AR12, Lemma 2.3].

Lemma 3.29. *Let $X = \Gamma\backslash G$ be a solvmanifold, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Let K be a G-left-invariant almost-$\mathbf{D}$-complex structure on X, or equivalently the associated linear almost-$\mathbf{D}$-complex structure on $\mathfrak{g}$. Let η be the G-bi-invariant volume form on G given by J. Milnor's Lemma [Mil76, Lemma 6.2], and such that $\int_X \eta = 1$. Up to identifying G-left-invariant forms on X and linear forms over $\mathfrak{g}^*$ through left-translations, consider the Belgun symmetrization map, [Bel00, Theorem 7],*

$$\mu\colon \wedge^\bullet X \to \wedge^\bullet \mathfrak{g}^* , \qquad \mu(\alpha) := \int_X \alpha\lfloor_m \eta(m) .$$

One has that

$$\mu\lfloor_{\wedge^\bullet \mathfrak{g}^*} = \mathrm{id}\lfloor_{\wedge^\bullet \mathfrak{g}^*}$$

and that

$$\mathrm{d}\left(\mu(\cdot)\right) = \mu\left(\mathrm{d}\,\cdot\right) \qquad \text{and} \qquad K\left(\mu(\cdot)\right) = \mu\left(K\cdot\right) .$$

As a consequence, we get the following result, [AR12, Proposition 2.4] (compare with Proposition 2.19, see also [FT10, Theorem 3.4], in the almost-complex case, and with Proposition 3.18 in the symplectic case).

Proposition 3.30. *Let $X = \Gamma\backslash G$ be a solvmanifold, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Suppose that $H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) \simeq H^\bullet_{dR}(X;\mathbb{R})$ (e.g., suppose that X is a completely-solvable solvmanifold). Let K be a G-left-invariant almost-$\mathbf{D}$-complex structure on X. Then, for every $\ell \in \mathbb{N}$ and for $\pm \in \{+,-\}$, the injective map*

$$H^{\ell\,\pm}_K(\mathfrak{g};\mathbb{R}) \to H^{\ell\,\pm}_K(X;\mathbb{R})$$

induced by left-translations is an isomorphism.

Proof. Consider the F. A. Belgun symmetrization map $\mu\colon \wedge^\bullet X \to \wedge^\bullet \mathfrak{g}^*$, [Bel00, Theorem 7]. It is enough to observe the following three facts.

(i) Since $\mathrm{d}\left(\mu(\cdot)\right) = \mu\left(\mathrm{d}\,\cdot\right)$, [Bel00, Theorem 7], one has that μ sends d-closed, respectively d-exact, forms to d-closed, respectively d-exact, G-left-invariant forms, and so it induces a map

$$\mu\colon H^\bullet_{dR}(X;\mathbb{R}) \to H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) .$$

(ii) Since $K\left(\mu(\cdot)\right) = \mu\left(K\cdot\right)$, Lemma 3.29, for $\pm \in \{+,-\}$, one has

$$\mu\lfloor_{\wedge^{\bullet\,\pm}_K X}\colon \wedge^{\bullet\,\pm}_K X \to \wedge^{\bullet\,\pm}_K \mathfrak{g}^* ,$$

and hence

$$\mu\lfloor_{H^{\bullet\,\pm}_K(X;\mathbb{R})}\colon H^{\bullet\,\pm}_K(X;\mathbb{R}) \to H^{\bullet\,\pm}_K(\mathfrak{g};\mathbb{R}) .$$

(iii) Finally, if $H^\bullet_{dR}(X;\mathbb{R}) \simeq H^\bullet_{dR}(\mathfrak{g};\mathbb{R})$ (e.g., if X is a completely-solvable solvmanifold, [Hat60, Corollary 4.2]), then the condition $\mu\lfloor_{\wedge^\bullet_{\mathrm{inv}} X} = \mathrm{id}\lfloor_{\wedge^\bullet_{\mathrm{inv}} X}$, [Bel00, Theorem 7], gives that μ is the identity in cohomology. $\square$

As a straightforward corollary, we get the following result, [AR12, Proposition 2.4] (compare with Corollary 2.20 in the almost-complex case).

Corollary 3.31. *Let $X = \Gamma\backslash G$ be a solvmanifold such that $H^\bullet_{dR}(X;\mathbb{R}) \simeq H^\bullet_{dR}(\mathfrak{g};\mathbb{R})$ (e.g., a completely-solvable solvmanifold), and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Let K be a G-left-invariant almost-$\mathbf{D}$-complex structure on X. For every $\ell \in \mathbb{N}$, the associated linear almost-$\mathbf{D}$-complex structure $K \in \mathrm{End}(\mathfrak{g})$ is linear-$\mathcal{C}^\infty$-pure (respectively, linear-$\mathcal{C}^\infty$-full, linear-$\mathcal{C}^\infty$-pure-and-full) at the ℓ^{th} stage if and only if the G-left-invariant almost-$\mathbf{D}$-complex structure $K \in \mathrm{End}(TX)$ is $\mathcal{C}^\infty$-pure (respectively, $\mathcal{C}^\infty$-full, $\mathcal{C}^\infty$-pure-and-full) at the ℓ^{th} stage.*

Non-$\mathcal{C}^\infty$-pure-and-full (almost-)D-complex nilmanifolds

We provide here some explicit examples of left-invariant (almost-)**D**-complex structures on nilmanifolds, studying the corresponding subgroups in cohomology, and providing differences between the **D**-complex and the complex cases, Proposition 3.34.

More precisely, recall that every Kähler structure on a compact manifold is $\mathcal{C}^\infty$-pure-and-full, [DLZ10, Lemma 2.15, Theorem 2.16], or [LZ09, Proposition 2.1], and that every almost-complex structure on a 4-dimensional compact manifold is $\mathcal{C}^\infty$-pure-and-full, [DLZ10, Theorem 2.3]: we give instead an example of a **D**-complex structure on a 6-dimensional nilmanifold such that it is non-$\mathcal{C}^\infty$-full at the 2$^{\text{nd}}$ stage, [AR12, Example 3.1], respectively non-$\mathcal{C}^\infty$-pure at the 2$^{\text{nd}}$ stage, [AR12, Example 3.2], despite it admits a **D**-Kähler structure; furthermore, we provide a non-$\mathcal{C}^\infty$-pure-and-full at the 2$^{\text{nd}}$ stage almost-**D**-complex structure on a 4-dimensional manifold, proving that no almost-**D**-complex counterpart of [DLZ10, Theorem 2.3] could exist.

Example 3.32. *A* **D**-*complex structure on a* 6-*dimensional nilmanifold that is* $\mathcal{C}^\infty$-*pure at the* 2^{nd} *stage and non-*$\mathcal{C}^\infty$-*full at the* 2^{nd} *stage and that admits a* **D**-*Kähler structure.*
Consider a nilmanifold

$$X \;=\; \Gamma\backslash G \;:=\; \left(0^4,\; 12,\; 13\right)$$

and define the G-left-invariant **D**-complex structure K by setting

$$K \;:=\; \left(-\,+\,+\,-\,-\,+\right)\;.$$

By K. Nomizu's theorem [Nom54, Theorem 1], the de Rham cohomology of X is given by

$$H^2_{dR}(X;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\, e^{15},\, e^{16},\, e^{23},\, e^{24},\, e^{25},\, e^{34},\, e^{36},\, e^{26}+e^{35}\right\rangle$$

(where, as usual, we list the harmonic representatives with respect to the G-left-invariant metric $\sum_{j=1}^{6} e^j \odot e^j$ instead of their classes, and we write, e.g., e^{hk} to shorten $e^h \wedge e^k$). Note that

$$H^{2\,+}_{K}(\mathfrak{g};\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\, e^{15},\, e^{23},\, e^{36}\right\rangle \qquad \text{and} \qquad H^{2\,-}_{K}(\mathfrak{g};\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{16},\, e^{24},\, e^{25},\, e^{34}\right\rangle\;,$$

since the space of G-left-invariant d-exact 2-forms is $\mathbb{R}\left\langle e^{12},\, e^{13}\right\rangle$, and hence no G-left-invariant representative in the class $\left[e^{26}+e^{35}\right]$ is of pure type with respect to K. It follows that $K \in \mathrm{End}\left(\mathfrak{g}\right)$ is linear-$\mathcal{C}^\infty$-pure at the 2$^{\text{nd}}$ stage and linear non-$\mathcal{C}^\infty$-full at the 2$^{\text{nd}}$ stage, and hence $K \in \mathrm{End}(TX)$ is $\mathcal{C}^\infty$-pure at the 2$^{\text{nd}}$ stage and non-$\mathcal{C}^\infty$-full at the 2$^{\text{nd}}$ stage, by Corollary 3.31. (Note that, K being Abelian, one can deduce the $\mathcal{C}^\infty$-pureness at the 2$^{\text{nd}}$ stage also by Corollary 3.43.)

Moreover, we observe that

$$\omega \;:=\; e^{16}+e^{25}+e^{34}$$

is a (G-left-invariant) symplectic form compatible with K, hence $(K,\,\omega)$ is a **D**-Kähler structure on X.

Example 3.33. *A* **D**-*complex structure on a* 6-*dimensional nilmanifold that is non-*$\mathcal{C}^\infty$-*pure at the* 2^{nd} *stage, and hence non-*$\mathcal{C}^\infty$-*full at the* 4^{th} *stage, and that admits a* **D**-*Kähler structure.*
Consider a nilmanifold

$$X \;=\; \Gamma\backslash G \;:=\; \left(0^3,\; 12,\; 13+14,\; 24\right)$$

and define the G-left-invariant **D**-complex structure

$$K \;:=\; \left(+\,-\,+\,-\,+\,-\right)\;.$$

(Note that $[\mathfrak{g}^-,\,\mathfrak{g}^-] \neq \{0\}$, since $[e_2,e_4] = -e_6$, hence K is not Abelian.)
We have

$$H^{2\,+}_{K}(\mathfrak{g};\mathbb{R}) \;\ni\; \left[e^{13}\right] \;=\; \left[e^{13}-\mathrm{d}\,e^{5}\right] \;=\; -\left[e^{14}\right] \;\in\; H^{2\,-}_{K}(\mathfrak{g};\mathbb{R})$$

and therefore $0 \neq \left[e^{13}\right] \in H^{2\,+}_{K}(\mathfrak{g};\mathbb{R}) \cap H^{2\,-}_{K}(\mathfrak{g};\mathbb{R})$, namely, $K \in \mathrm{End}(\mathfrak{g})$ is not linear-$\mathcal{C}^\infty$-pure at the 2$^{\text{nd}}$ stage, hence, by Corollary 3.31, $K \in \mathrm{End}(TX)$ is not $\mathcal{C}^\infty$-pure at the 2$^{\text{nd}}$ stage; moreover, by Proposition 3.25, we have also that K is not $\mathcal{C}^\infty$-full at the 4$^{\text{th}}$ stage.

Furthermore,

$$\omega \;:=\; e^{16}+e^{25}+e^{34}$$

is a (G-left-invariant) symplectic form compatible with K, hence $(K,\,\omega)$ is a **D**-Kähler structure on X.

It is straightforward to obtain higher-dimensional examples of **D**-Kähler non-$\mathcal{C}^\infty$-full, respectively non-$\mathcal{C}^\infty$-pure, at the 2$^{\text{nd}}$ stage structures, taking products with standard **D**-complex tori.

The contents of the previous two examples are resumed in the following result, [AR12, Proposition 3.3], which gives a difference with the complex case, [LZ09, Proposition 2.1], or [DLZ10, Lemma 2.15, Theorem 2.16].

Proposition 3.34. *Admitting a* $\mathbf{D}$*-Kähler structure is not a sufficient condition for either being* $\mathcal{C}^\infty$*-pure at the* 2^{nd} *stage or being* $\mathcal{C}^\infty$*-full at the* 2^{nd} *stage.*

We provide now a counterexample showing that T. Drăghici, T.-J. Li, and W. Zhang's decomposition theorem for compact 4-dimensional almost-complex manifolds, [DLZ10, Theorem 2.3], does not hold, in general, in the almost-$\mathbf{D}$-complex case, [AR12, Example 3.4].

Example 3.35. *An almost-*$\mathbf{D}$*-complex structure on a* 4*-dimensional nilmanifold that is non-*$\mathcal{C}^\infty$*-pure-and-full at the* 2^{nd} *stage.*
Consider a nilmanifold

$$X \;=\; \Gamma\backslash G \;:=\; (0,\; 0,\; 12,\; 0)$$

and define the G-left-invariant almost-$\mathbf{D}$-complex structure K requiring that $K\lfloor_{\mathfrak{g}^+} = \mathrm{id}_{\mathfrak{g}^+}$ and $K\lfloor_{\mathfrak{g}^-} = -\,\mathrm{id}_{\mathfrak{g}^-}$ where

$$\mathfrak{g}^+ \;:=\; \mathbb{R}\left\langle e_1,\, e_4 - e_2\right\rangle \qquad \text{and} \qquad \mathfrak{g}^- \;:=\; \mathbb{R}\left\langle e_2,\, e_3\right\rangle .$$

Note that K is not integrable, since $[\mathfrak{g}^+,\, \mathfrak{g}^+] \ni [e_1, e_4 - e_2] = e_3 \notin \mathfrak{g}^+$.

Note that we have

$$H^{2\,+}_K(\mathfrak{g};\mathbb{R}) \ni \left[e^{14}\right] = \left[e^{14} + \mathrm{d}\,e^3\right] = \left[e^{14} + e^{12}\right] = \left[e^1 \wedge \left(e^4 + e^2\right)\right] \in H^{2\,-}_K(\mathfrak{g};\mathbb{R}) \;,$$

and therefore we get that $0 \neq \left[e^{14}\right] \in H^{2\,+}_K(\mathfrak{g};\mathbb{R}) \cap H^{2\,-}_K(\mathfrak{g};\mathbb{R})$; then, K is non-$\mathcal{C}^\infty$-pure at the 2^{nd} stage and non-$\mathcal{C}^\infty$-full at the 2^{nd} stage, by Corollary 3.31, and Proposition 3.25.

$\mathcal{C}^\infty$-pure-and-fullness of low-dimensional D-complex solvmanifolds

In this section, we state and prove Theorem 3.47, providing a partial $\mathbf{D}$-complex counterpart of [DLZ10, Theorem 2.3] in the almost-complex case. We start by fixing some notations and by proving some preliminary results.

Given a linear $\mathbf{D}$-complex structure K on a Lie algebra $\mathfrak{g}$, consider the induced eigen-spaces decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$, and consider the nilpotent steps

$$s^+ \;:=\; s\left(\mathfrak{g}^+\right) \qquad \text{and} \qquad s^- \;:=\; s\left(\mathfrak{g}^-\right) \;.$$

(As a matter of notation, recall that, given a Lie algebra $(\mathfrak{a},\, [\cdot,\cdot\cdot])$, the *lower central series* $\{\mathfrak{a}^n\}_{n\in\mathbb{N}}$ is defined, by induction on $n \in \mathbb{N}$, as

$$\begin{cases} \mathfrak{a}^0 & := \;\; \mathfrak{a} \\ \mathfrak{a}^{n+1} & := \;\; [\mathfrak{a}^n,\, \mathfrak{a}] \quad \text{for } n \in \mathbb{N} \end{cases} \;;$$

note that $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ is a descending sequence of Lie algebras:

$$\mathfrak{a} \;=\; \mathfrak{a}^0 \;\supseteq \mathfrak{a}^1 \;\supseteq\; \cdots \;\supseteq\; \mathfrak{a}^{j-1} \;\supseteq\; \mathfrak{a}^j \;\supseteq\; \cdots \;;$$

recall that the *nilpotent step* of $\mathfrak{a}$ is defined as

$$s\left(\mathfrak{a}\right) \;:=\; \inf\left\{m \in \mathbb{N} \;:\; \mathfrak{a}^m = 0\right\} \;;$$

in particular, if $s\left(\mathfrak{a}\right) < +\infty$, then, by definition, $\mathfrak{a}$ is nilpotent.)

Since $\mathfrak{g}^+ \subset \mathfrak{g}$ and $\mathfrak{g}^- \subset \mathfrak{g}$, we have obviously that

$$s^+ \;\leq\; s\left(\mathfrak{g}\right) \qquad \text{and} \qquad s^- \;\leq\; s\left(\mathfrak{g}\right) \;.$$

In fact, we have the following lemma, [AR12, Lemma 3.5].

Lemma 3.36. *Let* $\mathfrak{g}$ *be a* $2n$*-dimensional nilpotent Lie algebra, namely,* $s\left(\mathfrak{g}\right) < +\infty$*. Let* K *be a linear* $\mathbf{D}$*-complex structure on* $\mathfrak{g}$*, inducing the eigen-spaces decomposition* $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$*. Then, setting* $s^\pm := s\left(\mathfrak{g}^\pm\right)$ *for* $\pm \in \{+,\, -\}$*, we have*

$$1 \;\leq\; s^+ \;\leq\; n-1 \qquad \text{and} \qquad 1 \;\leq\; s^- \;\leq\; n-1 \,.$$

Proof. It suffices to note that, for $\pm \in \{+,\; -\}$, we have

$$\begin{cases} \dim_{\mathbb{R}} \left(\mathfrak{g}^\pm\right)^0 & = \quad n \\ \dim_{\mathbb{R}} \left(\mathfrak{g}^\pm\right)^k & \leq \quad \max\left\{n-k-1,\, 0\right\} \quad \text{for } k \in \mathbb{N}\setminus\{0\} \end{cases} \;,$$

as a consequence of the nilpotency and of the integrability properties. □

The following result, [AR12, Proposition 3.6], should be compared with Theorem 3.27.

Proposition 3.37. *Let $\mathfrak{g}$ be a Lie algebra. If K is a linear* **D***-complex structure on $\mathfrak{g}$ with eigen-spaces $\mathfrak{g}^+$ and $\mathfrak{g}^-$ such that $[\mathfrak{g}^+,\, \mathfrak{g}^-] = \{0\}$, then K is linear-$\mathcal{C}^\infty$-pure-and-full at every stage.*

Proof. Since $[\mathfrak{g}^+,\, \mathfrak{g}^-] = \{0\}$, one has $\mathfrak{g} = \mathfrak{g}^+ \times \mathfrak{g}^-$ and, using the Künneth formula, one gets the statement, as in the proof of Theorem 3.27. □

Therefore, from Corollary 3.31, one gets the following corollary, [AR12, Corollary 3.7].

Corollary 3.38. *Let $X = \Gamma\backslash G$ be a completely-solvable solvmanifold endowed with a G-left-invariant* **D***-complex structure K, and denote the Lie algebra naturally associated to the Lie group G by $\mathfrak{g}$. Consider the linear* **D***-complex structure $K \in \mathrm{End}(\mathfrak{g})$ induced by $K \in \mathrm{End}(TX)$. Suppose that the eigen-spaces $\mathfrak{g}^+$ and $\mathfrak{g}^-$ of $K \in \mathrm{End}(\mathfrak{g})$ satisfy $[\mathfrak{g}^+,\, \mathfrak{g}^-] = \{0\}$. Then K is $\mathcal{C}^\infty$-pure-and-full at every stage and pure-and-full at every stage.*

We recall that a **D**-complex structure on a manifold X is said to be *Abelian* if the induced eigen-bundle decomposition $TX = T^+X \oplus T^-X$ satisfies $[T^+X, T^+X] = \{0\} = [T^-X, T^-X]$; analogously, a linear **D**-complex structure on a Lie algebra $\mathfrak{g}$ is said to be *Abelian* if the induced decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ satisfies $[\mathfrak{g}^+,\, \mathfrak{g}^+] = \{0\} = [\mathfrak{g}^-,\, \mathfrak{g}^-]$, namely, $s\,(\mathfrak{g}^+) = 1 = s\,(\mathfrak{g}^-)$. Obviously, if $X = \Gamma\backslash G$ is a solvmanifold endowed with a G-left-invariant **D**-complex structure, then $K \in \mathrm{End}\,(TX)$ is Abelian if and only if the associated linear **D**-complex structure $K \in \mathrm{End}\,(\mathfrak{g})$ is Abelian.

Remark 3.39. Note that every linear **D**-complex structure on a 4-dimensional nilpotent Lie algebra is Abelian, as a consequence of Lemma 3.36.

We prove that every linear Abelian **D**-complex structure is linear-$\mathcal{C}^\infty$-pure at the 2$^{\text{nd}}$ stage, [AR12, Theorem 3.10].

Theorem 3.40. *Let $\mathfrak{g}$ be a Lie algebra and K be a linear Abelian* **D***-complex structure on $\mathfrak{g}$. Then K is linear-$\mathcal{C}^\infty$-pure at the 2nd stage.*

Proof. Denote by $\pi_{\wedge^{\bullet+}_K \mathfrak{g}^*} \colon \wedge^\bullet \mathfrak{g}^* \to \wedge^{\bullet+}_K \mathfrak{g}^*$ the natural projection onto the space $\wedge^{\bullet+}_K \mathfrak{g}^*$. Recall that $\mathrm{d}\,\eta := -\eta\,([\cdot,\, \cdot\cdot])$ for every $\eta \in \wedge^1 \mathfrak{g}^*$; therefore, since $[\mathfrak{g}^+,\, \mathfrak{g}^+] = 0$ and $[\mathfrak{g}^-,\, \mathfrak{g}^-] = 0$ by hypothesis, we have that

$$\pi_{\wedge^{\bullet+}_K \mathfrak{g}^*}\left(\mathrm{im}\left(\mathrm{d}\colon\ \wedge^1 \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*\right)\right) \;=\; \{0\}\ .$$

Suppose that there exists $[\gamma^+] = [\gamma^-] \in H^{2+}_K(\mathfrak{g};\mathbb{R}) \cap H^{2-}_K(\mathfrak{g};\mathbb{R})$, where $\gamma^+ \in \wedge^{2+}_K \mathfrak{g}^*$ and $\gamma^- \in \wedge^{2-}_K \mathfrak{g}^*$, and $\gamma^+ = \gamma^- + \mathrm{d}\,\alpha$ for some $\alpha \in \wedge^1 \mathfrak{g}^*$. Since $\pi_{\wedge^{\bullet+}_K \mathfrak{g}^*}(\mathrm{d}\,\alpha) = 0$, we have that $\gamma^+ = 0$ and hence $[\gamma^+] = 0$, so K is linear-$\mathcal{C}^\infty$-pure at the 2$^{\text{nd}}$ stage. □

Remark 3.41. We note that the condition of K being Abelian in Theorem 3.40 cannot be dropped or weakened, in general. In fact, Example 3.49 shows that the Abelian assumption just on $\mathfrak{g}^-$ is not sufficient to have $\mathcal{C}^\infty$-pureness at the 2$^{\text{nd}}$ stage. Another example of this fact, on a (non-unimodular) solvable Lie algebra, is given below, [AR12, Example 3.12].

Example 3.42. *A 4-dimensional (non-unimodular) solvable Lie algebra with a non-Abelian* **D***-complex structure that is not linear-$\mathcal{C}^\infty$-pure at the 2nd stage.*
Consider the 4-dimensional solvable Lie algebra defined by

$$\mathfrak{g} \;:=\; \left(0^3,\; 13+34\right)\ ;$$

note that $\mathfrak{g}$ is not unimodular, since $\mathrm{d}\,e^{124} = e^{1234}$, see Lemma 3.45, [Kos50, §III].

Set the linear **D**-complex structure

$$K \;:=\; (+\;+\;-\;-)\ ;$$

K is not Abelian, since $[\mathfrak{g}^+,\, \mathfrak{g}^+] = 0$ but $[\mathfrak{g}^-,\, \mathfrak{g}^-] = \mathbb{R}\,\langle e_4\rangle \neq \{0\}$.

It is straightforward to check that K is linear-$\mathcal{C}^\infty$-full at the 2$^{\text{nd}}$ stage: in fact,

$$H^2_{dR}(\mathfrak{g};\mathbb{R}) \;=\; \underbrace{\mathbb{R}\left\langle e^{12},\, e^{34}\right\rangle}_{H^+_K(\mathfrak{g};\mathbb{R})} \oplus \underbrace{\left\langle e^{23}\right\rangle}_{H^-_K(\mathfrak{g};\mathbb{R})}\ ;$$

on the other hand, K is not linear-$\mathcal{C}^\infty$-pure at the 2$^{\text{nd}}$ stage, since

$$H^{2+}_K(\mathfrak{g};\mathbb{R}) \;\ni\; \left[e^{34}\right] \;=\; \left[e^{34} - \mathrm{d}\,e^4\right] \;=\; -\left[e^{13}\right] \;\in\; H^{2-}_K(\mathfrak{g};\mathbb{R})$$

and $\left[e^{34}\right] \neq 0$.

A direct consequence of Theorem 3.40 and Corollary 3.31 is the following result, [AR12, Corollary 3.13].

Corollary 3.43. *Let $X = \Gamma\backslash G$ be a completely-solvable solvmanifold endowed with a G-left-invariant Abelian $\mathbf{D}$-complex structure K. Then K is $\mathcal{C}^\infty$-pure at the 2^{nd} stage.*

Remark 3.44. We remark that, for a **D**-complex structure on a compact manifold of dimension greater than 4, being Abelian or being $\mathcal{C}^\infty$-pure at the 2^{nd} stage is not a sufficient condition to have $\mathcal{C}^\infty$-fullness at the 2^{nd} stage. Indeed, Example 3.32 provides a G-left-invariant **D**-complex structure K on a 6-dimensional nilmanifold $X = \Gamma\backslash G$ such that K is Abelian, $\mathcal{C}^\infty$-pure at the 2^{nd} stage and non-$\mathcal{C}^\infty$-full at the 2^{nd} stage.

As observed in Remark 3.39, any left-invariant **D**-complex structure on a 4-dimensional nilmanifold is Abelian, and hence $\mathcal{C}^\infty$-pure at the 2^{nd} stage by Corollary 3.43. In general, a left-invariant Abelian **D**-complex structure on a nilmanifold of dimension greater than 4 may be non-$\mathcal{C}^\infty$-full at the 2^{nd} stage, Example 3.32. Notwithstanding, we prove that every left-invariant **D**-complex structure on a 4-dimensional nilmanifold is $\mathcal{C}^\infty$-full, in fact $\mathcal{C}^\infty$-pure-and-full, at the 2^{nd} stage, Theorem 3.47. To prove this fact, we need the following lemmata: the first one is a classical result, the second one is [AR12, Lemma 3.16].

Lemma 3.45 ([Kos50, §III])**.** *Let $\mathfrak{g}$ be a unimodular Lie algebra of dimension n. Then*

$$\mathrm{d}\lfloor_{\wedge^{n-1}\mathfrak{g}^*} \;=\; 0\,.$$

Lemma 3.46. *Let $\mathfrak{g}$ be a unimodular Lie algebra of dimension $2n$ endowed with an Abelian linear $\mathbf{D}$-complex structure K. Then*

$$\mathrm{d}\lfloor_{\wedge^{n,0}_{+\,-}\mathfrak{g}^*\,\oplus\,\wedge^{0,n}_{+\,-}\mathfrak{g}^*} \;=\; 0\,.$$

Proof. Consider the eigen-spaces decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ induced by K, and fix two bases for $\left(\mathfrak{g}^+\right)^*$ and $\left(\mathfrak{g}^-\right)^*$:

$$\left(\mathfrak{g}^+\right)^* \;=\; \mathbb{R}\left\langle e^1,\,\ldots,\,e^n\right\rangle \qquad \text{and} \qquad \left(\mathfrak{g}^-\right)^* \;=\; \mathbb{R}\left\langle f^1,\,\ldots,\,f^n\right\rangle\,.$$

Since K is Abelian, the general structure equations, in terms of these bases, are

$$\begin{cases} \mathrm{d}\,e^j &=: \;\sum_{h,\,k=1}^n a^j_{hk}\,e^h\wedge f^k \qquad \text{for } j\in\{1,\ldots,n\} \\ \mathrm{d}\,f^j &=: \;\sum_{h,\,k=1}^n b^j_{hk}\,e^h\wedge f^k \qquad \text{for } j\in\{1,\ldots,n\} \end{cases},$$

where $\left\{a^j_{hk},\, b^j_{hk}\right\}_{j,h,k\in\{1,\ldots,n\}} \subset \mathbb{R}$.

By [Kos50, §III], see Lemma 3.45, for any $k\in\{1,\ldots,n\}$, one has that

$$\mathrm{d}\left(e^1\wedge\cdots\wedge e^n\wedge f^1\wedge\cdots\wedge f^{k-1}\wedge f^{k+1}\wedge\cdots\wedge f^n\right) \;=\; 0\,;$$

by a straightforward computation, we get that,

$$\sum_{\ell=1}^n a^\ell_{\ell k} \;=\; 0\,,$$

Hence, we get that

$$\mathrm{d}\left(e^1\wedge\cdots\wedge e^n\right) \;=\; (-1)^n\sum_{k=1}^n\left(\sum_{\ell=1}^n a^\ell_{\ell k}\right)e^1\wedge\cdots\wedge e^n\wedge f^k \;=\; 0\,.$$

Arguing in the same way, we prove also that

$$\mathrm{d}\left(f^1\wedge\cdots\wedge f^n\right) \;=\; 0\,,$$

completing the proof. □

We can now prove the following result, [AR12, Theorem 3.17].

Theorem 3.47. *Every left-invariant $\mathbf{D}$-complex structure on a 4-dimensional nilmanifold is $\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage, and hence also pure-and-full at the 2^{nd} stage.*

Proof. By Remark 3.39 and Corollary 3.43, we get the $\mathcal{C}^\infty$-pureness at the 2^{nd} stage.

We recall that, by J. Milnor's lemma [Mil76, Lemma 6.2], the Lie algebra associated to any nilmanifold is unimodular. From Lemma 3.46 one gets that, on every 4-dimensional **D**-complex nilmanifold, the **D**-complex invariant component of a left-invariant 2-form is d-closed. Hence both the **D**-complex invariant component and the **D**-complex anti-invariant component of a d-closed left-invariant 2-form is d-closed. Hence the linear **D**-complex structure is linear-$\mathcal{C}^\infty$-full at the 2^{nd} stage. Therefore, by Corollary 3.31, the left-invariant **D**-complex structure is $\mathcal{C}^\infty$-full at the 2^{nd} stage.

Finally, Proposition 3.25 gives the pure-and-fullness at the 2^{nd} stage. □

Remark 3.48. We note that Theorem 3.47 is optimal. Indeed, we cannot grow the dimension, Example 3.32 and Example 3.33, nor change the nilpotent hypothesis with a solvable condition, Example 3.49, nor drop the integrability condition on the **D**-complex structure, Example 3.35.

3.2.4 Small deformations of D-complex structures

In this section, we study explicit examples of small deformations of the **D**-complex structure on nilmanifolds and solvmanifolds, studying the behaviour of being **D**-Kähler, Theorem 3.50, the behaviour of $\mathcal{C}^\infty$-pure-and-fullness, Proposition 3.51, and the semi-continuity problem for the dimensions of the **D**-(anti-)invariant subgroups of the second de Rham cohomology group, Proposition 3.54.

We refer to [MT11, Ros12a] for more results concerning deformations of (almost-)**D**-complex structures.

In the following example, [AR12, Example 4.1], we construct a curve $\{K_t\}_{t\in\mathbb{R}}$ of left-invariant **D**-complex structures on a 4-dimensional solvmanifold such that (*i*) K_0 is $\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage and admits a **D**-Kähler structure, and (*ii*) K_t, for $t \neq 0$, is neither $\mathcal{C}^\infty$-pure at the 2^{nd} stage nor $\mathcal{C}^\infty$-full at the 2^{nd} stage and does not admit any **D**-Kähler structure. In particular, this example proves that being **D**-Kähler is not a stable property under small deformations of the **D**-complex structure, Theorem 3.50, and it shows also that the nilpotency condition in Theorem 3.47 cannot be dropped out, Remark 3.48.

Example 3.49. *There exists a 4-dimensional solvmanifold endowed with a left-invariant* **D***-complex structure K such that (i) K is $\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage, (ii) it admits a* **D***-Kähler structure, and (iii) it has small* **D***-complex deformations that are neither* **D***-Kähler nor $\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage.*
Consider a 4-dimensional completely-solvable solvmanifold

$$X \;=\; \Gamma\backslash G \;:=\; \left(0^2,\; 23,\; -24\right)$$

(for its existence, see, e.g., [Boc09, Table 8]).

By A. Hattori's theorem [Hat60, Corollary 4.2], it is straightforward to compute

$$H^2_{dR}(X;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{12},\, e^{34}\right\rangle \;.$$

For every $t \in \mathbb{R}$, consider the G-left-invariant **D**-complex structure

$$K_t \;:=\; \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \;.$$

For every $t \in \mathbb{R}$, we have that

$$\mathfrak{g}^+_{K_t} \;=\; \mathbb{R}\left\langle e_2,\, e_3\right\rangle \qquad \text{and} \qquad \mathfrak{g}^-_{K_t} \;=\; \mathbb{R}\left\langle e_1,\, e_4 + t\, e_2\right\rangle \;:$$

in particular, $\left[\mathfrak{g}^+_{K_t},\, \mathfrak{g}^+_{K_t}\right] = \mathbb{R}\left\langle e_3\right\rangle \subseteq \mathfrak{g}^+_{K_t}$ and $\left[\mathfrak{g}^-_{K_t},\, \mathfrak{g}^-_{K_t}\right] = \{0\}$, which proves the integrability of K_t, for every $t \in \mathbb{R}$.

In particular, for $t = 0$, we have the (non-Abelian) **D**-complex structure

$$K_0 \;=\; (-\,+\,+\,-) \;.$$

It is straightforward to check that K_0 is linear-$\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage, and hence $\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage by Corollary 3.31: in fact, by Proposition 3.30,

$$H^{2\,+}_{K_0}(X;\mathbb{R}) \;=\; \{0\} \qquad \text{and} \qquad H^{2\,-}_{K_0}(X;\mathbb{R}) \;=\; H^2_{dR}(X;\mathbb{R}) \;;$$

in particular, we have

$$\dim_{\mathbb{R}} H^{2\,+}_{K_0}(X;\mathbb{R}) \;=\; 0 \;, \qquad \dim_{\mathbb{R}} H^{2\,-}_{K_0}(X;\mathbb{R}) \;=\; 2.$$

Furthermore, for $t \neq 0$, we get

$$H^{2\,-}_{K_t}(X;\mathbb{R}) \ni \left[e^{34}\right] \;=\; \left[e^{34} + \frac{1}{t}\,\mathrm{d}\,e^{3}\right] \;=\; \left[e^{34} + \frac{1}{t}\left(e^{23} + t\,e^{43} - t\,e^{43}\right)\right] \;=\; \left[\frac{1}{t}\left(e^{2} - t\,e^{4}\right) \wedge e^{3}\right] \in H^{2\,+}_{K_t}(X;\mathbb{R}) \;,$$

and therefore $0 \neq \left[e^{34}\right] \in H^{2\,-}_{K_t}(X;\mathbb{R}) \cap H^{2\,+}_{K_t}(X;\mathbb{R})$, namely, K_t is neither $\mathcal{C}^\infty$-pure at the 2$^{\text{nd}}$ stage nor $\mathcal{C}^\infty$-full at the 2$^{\text{nd}}$ stage, by Proposition 3.25 (in fact, since the space of G-left-invariant d-exact 2-forms is

$$\mathrm{d} \wedge^1 \mathfrak{g}^* \;=\; \mathbb{R}\left\langle \left(e^{2} - t\,e^{4}\right) \wedge e^{3} - t\,e^{34},\, \left(e^{2} - t\,e^{4}\right) \wedge e^{4} \right\rangle \;,$$

no G-left-invariant representative of the class $\left[e^{12}\right] = \left[e^{1} \wedge \left(e^{2} - te^{4}\right) + te^{14}\right]$ is of pure type with respect to K_t). Therefore, for $t \neq 0$, we have

$$\dim_{\mathbb{R}} H^{2+}_{K_t}(X;\mathbb{R}) \;=\; 1 \;, \qquad \dim_{\mathbb{R}} H^{2-}_{K_t}(X;\mathbb{R}) \;=\; 1 .$$

Note that, for every $t \in \mathbb{R}$, one has $s\left(\mathfrak{g}^{-}_{K_t}\right) = 0$ and $s\left(\mathfrak{g}^{+}_{K_t}\right) = 1$, but, for $t \neq 0$, the $\mathbf{D}$-complex structure K_t is not $\mathcal{C}^\infty$-pure at the 2$^{\text{nd}}$ stage: therefore the Abelian condition on just $\mathfrak{g}^-$ in Theorem 3.40 is not sufficient to have $\mathcal{C}^\infty$-pureness at the 2$^{\text{nd}}$ stage, as observed in Remark 3.41.

Note moreover that, in this example, the functions

$$\mathbb{R} \ni t \mapsto \dim_{\mathbb{R}} H^{2+}_{K_t}(X;\mathbb{R}) \in \mathbb{N} \qquad \text{and} \qquad \mathbb{R} \ni t \mapsto \dim_{\mathbb{R}} H^{2-}_{K_t}(X;\mathbb{R}) \in \mathbb{N}$$

are, respectively, lower-semi-continuous and upper-semi-continuous.

Furthermore, we note that X admits a (G-left-invariant) symplectic form

$$\omega \;:=\; e^{12} + e^{34} \;,$$

which is compatible with the $\mathbf{D}$-complex structure K_0: therefore, $(K_0,\, \omega)$ is a $\mathbf{D}$-Kähler structure on X. On the other hand, for $t \neq 0$, one has $H^{-}_{K_t}(X;\mathbb{R}) = \mathbb{R}\left\langle e^{34} \right\rangle$ and therefore, if a K_t-compatible symplectic form ω_t existed, then it should be in the same cohomology class as e^{34}, and then it should satisfy

$$\mathrm{Vol}(X) \;=\; \int_X \omega_t \wedge \omega_t \;=\; \int_X e^{34} \wedge e^{34} \;=\; 0 \;,$$

which is not possible; therefore, for $t \neq 0$, there is no symplectic structure compatible with the $\mathbf{D}$-complex structure K_t: in particular, $(X,\, K_t)$ admits no $\mathbf{D}$-Kähler structure.

In particular, the previous example proves the following result, [AR12, Theorem 4.2], providing another strong difference between the $\mathbf{D}$-complex and the complex cases (recall that being Kähler is a stable property under small deformations of the complex structure by K. Kodaira and D. C. Spencer's stability theorem [KS60, Theorem 15]).

Theorem 3.50. *The property of being $\mathbf{D}$-Kähler is not stable under small deformations of the $\mathbf{D}$-complex structure.*

Furthermore, Example 3.49 proves also the following instability result, [AR12, Proposition 4.3], analogous to Theorem 2.48, which proves the instability of $\mathcal{C}^\infty$-pure-and-fullness in the complex case.

Proposition 3.51. *The properties of being $\mathcal{C}^\infty$-pure at the 2^{nd} stage, or $\mathcal{C}^\infty$-full at the 2^{nd} stage are not stable under small deformations of the $\mathbf{D}$-complex structure.*

We have already recalled, see §2.3.2, that T. Drăghici, T.-J. Li, and W. Zhang proved in [DLZ11, Theorem 2.6] that, given a curve $\left\{J_t\right\}_t$ of ($\mathcal{C}^\infty$-pure-and-full) almost-complex structures on a 4-dimensional compact manifold X, the dimension of $H^{+}_{J_t}(X;\mathbb{R})$ is upper-semi-continuous in t and hence, as a consequence of [DLZ10, Theorem 2.3], the dimension of $H^{-}_{J_t}(X;\mathbb{R})$ is lower-semi-continuous in t; this result holds no more true for almost-complex manifolds of higher dimension, Proposition 2.56, Proposition 2.55. In the next two examples, [AR12, Example 4.4], respectively [AR12, Example 4.5], we study the behaviour of the dimensions of the $\mathbf{D}$-complex invariant and $\mathbf{D}$-complex anti-invariant subgroups of the cohomology along curves of $\mathbf{D}$-complex structures.

Example 3.52. *A curve of $\mathbf{D}$-complex structures on a 6-dimensional nilmanifold such that the dimensions of the $\mathbf{D}$-complex invariant and anti-invariant subgroups of the second de Rham cohomology group jump (lower-semi-continuously) along the curve.*

Consider a 6-dimensional nilmanifold

$$X \;=\; \Gamma \backslash G \;:=\; \left(0^3,\; 12,\; 13,\; 24\right) \;.$$

By K. Nomizu's theorem [Nom54, Theorem 1], it is straightforward to compute

$$H^2_{dR}(X;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\, e^{15},\, e^{23},\, e^{26},\, e^{35},\, e^{25}+e^{34}\right\rangle \;.$$

For every $t \in [0,\,1]$, consider the left-invariant **D**-complex structure

$$K_t \;:=\; \left(\begin{array}{cc|cc|cc} 1 & & & & & \\ & -1 & & & & \\ \hline & & \frac{(1-t)^2-t^2}{(1-t)^2+t^2} & \frac{2t(1-t)}{(1-t)^2+t^2} & & \\ & & \frac{2t(1-t)}{(1-t)^2+t^2} & -\frac{(1-t)^2-t^2}{(1-t)^2+t^2} & & \\ \hline & & & & 1 & \\ & & & & & -1 \end{array}\right) \;.$$

For $0 \leq t \leq 1$, one checks that

$$\mathfrak{g}^+_{K_t} \;=\; \mathbb{R}\left\langle e_1,\, (1-t)\, e_3 + t\, e_4,\, e_5\right\rangle \qquad \text{and} \qquad \mathfrak{g}^-_{K_t} \;=\; \mathbb{R}\left\langle e_2,\, t\, e_3 - (1-t)\, e_4,\, e_6\right\rangle \;;$$

therefore, it is straightforwardly checked that the integrability condition of K_t is satisfied for every $t \in [0,\,1]$.

[**Case** $t=0$] For $t=0$, the **D**-complex structure

$$K_0 \;=\; (+\,-\,+\,-\,+\,-)$$

is $\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage: in fact,

$$H^2_{dR}(X;\mathbb{R}) \;=\; \underbrace{\mathbb{R}\left\langle e^{15},\, e^{26},\, e^{35}\right\rangle}_{=\,H^{2\,+}_{K_0}(X;\mathbb{R})} \oplus \underbrace{\mathbb{R}\left\langle e^{14},\, e^{23},\, e^{25}+e^{34}\right\rangle}_{=\,H^{2\,-}_{K_0}(X;\mathbb{R})} \;;$$

therefore

$$\dim_{\mathbb{R}} H^{2\,+}_{K_0}(X;\mathbb{R}) \;=\; 3 \qquad \text{and} \qquad \dim H^{2\,-}_{K_0}(X;\mathbb{R}) \;=\; 3 \;.$$

[**Case** $0<t<1$] For $0<t<1$, one has

$$H^{2\,+}_{K_t}(X;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\, e^{15},\, e^{23},\, e^{26}\right\rangle$$

and

$$H^{2\,-}_{K_t}(X;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\, e^{23},\, e^{25}+e^{34}\right\rangle \;;$$

it follows that the **D**-complex structure K_t is neither $\mathcal{C}^\infty$-pure at the 2^{nd} stage nor $\mathcal{C}^\infty$-full at the 2^{nd} stage; moreover,

$$\dim_{\mathbb{R}} H^{2\,+}_{K_t}(X;\mathbb{R}) \;=\; 4 \qquad \text{and} \qquad \dim H^{2\,-}_{K_t}(X;\mathbb{R}) \;=\; 3 \;.$$

[**Case** $t=1$] For $t=1$, the **D**-complex structure

$$K_1 \;=\; (+\,-\,-\,+\,+\,-)$$

is $\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage: in fact,

$$H^2_{dR}(X;\mathbb{R}) \;=\; \underbrace{\mathbb{R}\left\langle e^{14},\, e^{15},\, e^{23},\, e^{26}\right\rangle}_{=\,H^{2\,+}_{K_1}(X;\mathbb{R})} \oplus \underbrace{\mathbb{R}\left\langle e^{35},\, e^{25}+e^{34}\right\rangle}_{=\,H^{2\,-}_{K_1}(X;\mathbb{R})} \;;$$

therefore

$$\dim_{\mathbb{R}} H^{2\,+}_{K_1}(X;\mathbb{R}) \;=\; 4 \qquad \text{and} \qquad \dim H^{2\,-}_{K_1}(X;\mathbb{R}) \;=\; 2 \;.$$

In particular, it follows that the functions

$$[0,\,1] \ni t \mapsto \dim_{\mathbb{R}} H^{2\,+}_{K_t}(X;\mathbb{R}) \in \mathbb{N} \qquad \text{and} \qquad [0,\,1] \ni t \mapsto \dim_{\mathbb{R}} H^{2\,-}_{K_t}(X;\mathbb{R}) \in \mathbb{N}$$

are non-constant and lower-semi-continuous.

Example 3.49 and Example 3.52 show that the dimension of the **D**-complex anti-invariant subgroup of the de Rham cohomology in general is not upper-semi-continuous (as it is in Example 3.49) or lower-semi-continuous (as it is in Example 3.52) along curves of **D**-complex structures. We give now an example showing that also the dimension of the **D**-complex invariant subgroup of the de Rham cohomology in general is not lower-semi-continuous (as it is in Example 3.49 and in Example 3.52) along curves of **D**-complex structures, [AR12, Example 4.5].

Example 3.53. *A curve of* $\mathbf{D}$*-complex structures on a* 6*-dimensional nilmanifold such that the dimensions of the* $\mathbf{D}$*-complex invariant and anti-invariant subgroups of the second de Rham cohomology group jump (upper-semi-continuously) along the curve.*
Consider a 6-dimensional nilmanifold

$$X \;=\; \Gamma\backslash G \;:=\; \left(0^3,\; 12,\; 13,\; 24\right) \;.$$

By K. Nomizu's theorem [Nom54, Theorem 1], it is straightforward to compute

$$H^2_{dR}(X;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\; e^{15},\; e^{23},\; e^{26},\; e^{35},\; e^{25}+e^{34}\right\rangle \;.$$

For every $t\in[0,\,1]$, consider the G-left-invariant $\mathbf{D}$-complex structure

$$K_t \;:=\; \left(\begin{array}{cccc|cc} 1 &&&&& \\ & -1 &&&& \\ && -1 &&& \\ &&& 1 && \\ \hline &&&& \frac{(1-t)^2-t^2}{(1-t)^2+t^2} & \frac{2t(1-t)}{(1-t)^2+t^2} \\ &&&& \frac{2t(1-t)}{(1-t)^2+t^2} & -\frac{(1-t)^2-t^2}{(1-t)^2+t^2} \end{array}\right) \;.$$

For $0\leq t\leq 1$, one checks that

$$\mathfrak{g}^+_{K_t} \;=\; \mathbb{R}\left\langle e_1,\, e_4,\, (1-t)\,e_5+t\,e_6\right\rangle \qquad\text{and}\qquad \mathfrak{g}^-_{K_t} \;=\; \mathbb{R}\left\langle e_2,\, e_3,\, t\,e_5-(1-t)\,e_6\right\rangle \;.$$

Therefore one straightforwardly checks that, for every $t\in[0,\,1]$, the structure K_t is integrable, in fact Abelian: hence it is in particular $\mathcal{C}^\infty$-pure at the 2^{nd} stage by Corollary 3.43.

[**Case** $t=0$] For $t=0$, the $\mathbf{D}$-complex structure

$$K_0 \;=\; (+\,-\,-\,+\,+\,-)$$

is $\mathcal{C}^\infty$-pure-and-full at the 2^{nd} stage, and

$$H^2_{dR}(X;\mathbb{R}) \;=\; \underbrace{\mathbb{R}\left\langle e^{14},\, e^{15},\, e^{23},\, e^{26}\right\rangle}_{=H^{2\,+}_{K_0}(X;\mathbb{R})} \oplus \underbrace{\mathbb{R}\left\langle e^{35},\, e^{25}+e^{34}\right\rangle}_{=H^{2\,-}_{K_0}(X;\mathbb{R})} \;;$$

in particular,

$$\dim_{\mathbb{R}} H^{2\,+}_{K_0}(X;\mathbb{R}) \;=\; 4 \qquad\text{and}\qquad \dim H^{2\,-}_{K_0}(X;\mathbb{R}) \;=\; 2\;.$$

[**Case** $0<t<1$] For $0<t<1$, one has

$$H^{2\,+}_{K_t}(X;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\, e^{23}\right\rangle$$

and

$$H^{2\,-}_{K_t}(X;\mathbb{R}) \;=\; \mathbb{R}\left\langle t\,e^{26}+(1-t)\,e^{25}+(1-t)\,e^{34}\right\rangle \;,$$

while

$$\mathbb{R}\left\langle e^{15},\, e^{35},\, e^{26}\right\rangle \cap \left(H^{2\,+}_{K_t}(X;\mathbb{R})\oplus H^{2\,-}_{K_t}(X;\mathbb{R})\right) \;=\; \{0\} \;;$$

it follows that the $\mathbf{D}$-complex structure K_t is $\mathcal{C}^\infty$-pure at the 2^{nd} stage and non-$\mathcal{C}^\infty$-full at the 2^{nd} stage, and that

$$\dim_{\mathbb{R}} H^{2\,+}_{K_t}(X;\mathbb{R}) \;=\; 2 \qquad\text{and}\qquad \dim H^{2\,-}_{K_t}(X;\mathbb{R}) \;=\; 1\;.$$

[**Case** $t=1$] For $t=1$, the $\mathbf{D}$-complex structure

$$K_1 \;=\; (+\,-\,-\,+\,-\,+) \;.$$

is $\mathcal{C}^\infty$-pure at the 2^{nd} stage and non-$\mathcal{C}^\infty$-full at the 2^{nd} stage, and

$$H^2_{dR}(X;\mathbb{R}) \;=\; \underbrace{\mathbb{R}\left\langle e^{14},\, e^{23},\, e^{35}\right\rangle}_{=H^{2\,+}_{K_1}(X;\mathbb{R})} \oplus \underbrace{\mathbb{R}\left\langle e^{15},\, e^{26}\right\rangle}_{=H^{2\,-}_{K_1}(X;\mathbb{R})} \oplus \mathbb{R}\left\langle e^{25}+e^{34}\right\rangle \;,$$

where

$$\mathbb{R}\left\langle e^{25}+e^{34}\right\rangle \cap \left(H^{2\,+}_{K_1}(X;\mathbb{R})\oplus H^{2\,-}_{K_1}(X;\mathbb{R})\right) \;=\; \{0\} \;;$$

in particular,

$$\dim_{\mathbb{R}} H^{2\,+}_{K_1}(X;\mathbb{R}) \;=\; 3 \qquad\text{and}\qquad \dim H^{2\,-}_{K_1}(X;\mathbb{R}) \;=\; 2\;.$$

In particular, the functions

$$[0,\,1] \ni t \mapsto \dim_{\mathbb{R}} H^{2\,+}_{K_t}(X;\mathbb{R}) \in \mathbb{N} \quad \text{and} \quad [0,\,1] \ni t \mapsto \dim_{\mathbb{R}} H^{2\,-}_{K_t}(X;\mathbb{R}) \in \mathbb{N}$$

are non-constant and upper-semi-continuous.

Example 3.52 and Example 3.53 yield the following result, [AR12, Proposition 4.6].

Proposition 3.54. *Let X be a compact manifold and let $\{K_t\}_{t\in I\subseteq\mathbb{R}}$ be a curve of $\mathbf{D}$-complex structures on X. Then, in general, the functions*

$$I \ni t \mapsto \dim_{\mathbb{R}} H^{2\,+}_{K_t}(X;\mathbb{R}) \in \mathbb{N} \qquad \textit{and} \qquad I \ni t \mapsto \dim_{\mathbb{R}} H^{2\,-}_{K_t}(X;\mathbb{R}) \in \mathbb{N}$$

are not upper-semi-continuous or lower-semi-continuous.

3.3 Cohomology of strictly p-convex domains in $\mathbb{R}^n$

In Complex Analysis, properties concerning the existence of exhaustion functions with convexity properties may have consequences on the vanishing of the cohomology. Indeed, recall, for example, that the Dolbeault cohomology groups $H^{p,q}_{\overline{\partial}}(D)$ of a *strictly pseudo-convex* domain D in $\mathbb{C}^n$ (that is, a domain admitting a smooth proper strictly pluri-sub-harmonic exhaustion function) vanish for $q \geq 1$, for any $p \in \mathbb{N}$. In fact, the following result holds.

Theorem 3.55 ([Hör65, Theorem 2.2.4, Theorem 2.2.5], [Hör90, Theorem 4.2.2, Corollary 4.2.6])**.** *Let $D \subseteq \mathbb{C}^n$ be a strictly pseudo-convex domain. Then, for any $q > 0$, every $\overline{\partial}$-closed (p,q)-form $\eta \in \mathrm{L}^2_{loc}(X; \wedge^{p,q}T^*X)$ (respectively, $\eta \in \wedge^{p,q}X$) is $\overline{\partial}$-exact, namely, there exists $\alpha \in \mathrm{L}^2_{loc}\left(X; \wedge^{p,q-1}T^*X\right)$ (respectively, $\alpha \in \wedge^{p,q-1}X$) such that $\eta = \overline{\partial}\alpha$.*

Generalizing the notion of strictly pseudo-convex domain, A. Andreotti and H. Grauert, [AG62], studied *q-complete* domains in $\mathbb{C}^n$ (that is, domains in $\mathbb{C}^n$ admitting a smooth proper exhaustion function whose Levi form has at least $n-q+1$ positive eigen-values), and provided an analogue of the L. Hörmander theorem.

Theorem 3.56 ([AG62, Proposition 27], [AV65a, Theorem 5])**.** *Let $D \Subset \mathbb{C}^n$ be a q-complete domain. Then $H^{r,s}_{\overline{\partial}}(X) = \{0\}$, for any $r \in \mathbb{N}$ and for any $s \geq q$.*

Recently, F. R. Harvey and H. B. Lawson, [HL12, HL11], and references therein, raised the interest on generalizations of the concept of convexity for Riemannian manifolds, studying the existence of exhaustion functions whose Hessian is positive definite or satisfies weaker positivity conditions; in this context, holomorphic convexity and q-completeness motivate the notion of *geometric convexity*.

J.-P. Sha, [Sha86, Theorem 1], and H. Wu, [Wu87, Theorem 1], (see also [HL11, Proposition 5.7],) proved, using Morse theory, that the existence of a smooth proper strictly p-pluri-sub-harmonic exhaustion function on a domain in $\mathbb{R}^n$ has consequences on the homotopy type of the domain; hence, vanishing results for the de Rham cohomology hold for strictly p-convex domains in $\mathbb{R}^n$ in the sense of F. R. Harvey and H. B. Lawson.

In this section, we re-prove, using different techniques, the vanishing result by J.-P. Sha, and H. Wu for the de Rham cohomology of strictly p-convex domains in $\mathbb{R}^n$ in the sense of F. R. Harvey and H. B. Lawson; more precisely, we use the L^2-techniques developed by L. Hörmander, [Hör65], and used also by A. Andreotti and E. Vesentini, [AV65a, AV65b] (see also [Ves67]); such L^2-techniques could be hopefully applied in a wider context.

The results in this section have been obtained in a joint work with S. Calamai, [AC12].

3.3.1 The notion of p-convexity by F. R. Harvey and H. B. Lawson

In this section, following F. R. Harvey and H. B. Lawson, [HL11, HL12], we recall the basic notions and results concerning p-convexity, starting from the definition of p-positive symmetric endomorphism, and then recalling the notions of (strictly) p-pluri-sub-harmonic function and (strictly) p-convex domain.

Let V be an n-dimensional $\mathbb{R}$-vector space endowed with an inner product $\langle \cdot \mid \cdot\cdot \rangle$.

Let $G\colon V \stackrel{\simeq}{\to} V^*$ denote the isomorphism induced by $\langle \cdot \mid \cdot\cdot \rangle$, defined as

$$G\colon V \ \ni\ v \mapsto \langle v \mid \cdot \rangle \ \in\ V^* \,.$$

One gets an isomorphism

$$G^{-1}\colon V^* \otimes V^* \stackrel{\simeq}{\to} \mathrm{Hom}\,(V,\, V) \ ;$$

this isomorphism sends the space of the symmetric elements of $(V \otimes V)^*$, namely,

$$\mathrm{Sym}^2(V) \;:=\; \left\{ A \in (V \otimes V)^* \;:\; A(v \otimes w) = A(w \otimes v) \;,\; \text{for any } v, w \in V \right\} \;.$$

to the space of the $\langle \cdot \mid \cdot\cdot \rangle$-symmetric endomorphisms of V.

Given $A \in V^* \otimes V^*$, the endomorphism $G^{-1}A \in \mathrm{Hom}(V, V)$ extends to

$$D^{[p]}_{G^{-1}A} \in \mathrm{Hom}(\wedge^p V, \wedge^p V)$$

by setting, for any simple vector $v_{i_1} \wedge \cdots \wedge v_{i_p} \in \wedge^p V$,

$$D^{[p]}_{G^{-1}A}\left(v_{i_1} \wedge \cdots \wedge v_{i_p}\right) \;:=\; \sum_{\ell=1}^{p} v_{i_1} \wedge \cdots \wedge v_{i_{\ell-1}} \wedge G^{-1}A\left(v_{i_\ell}\right) \wedge v_{i_{\ell+1}} \wedge \cdots \wedge v_{i_p} \;;$$

note that $D^{[p]}_{G^{-1}A} \in \mathrm{Hom}(\wedge^p V, \wedge^p V)$ is a symmetric endomorphism with respect to the inner product on $\wedge^p V$ induced by $\langle \cdot \mid \cdot\cdot \rangle$.

Note that, given $A \in \mathrm{Sym}^2(V)$, if the set of the eigenvalues of $G^{-1}A$ is

$$\mathrm{spec}\left(G^{-1}A\right) \;=\; \{\lambda_1, \ldots, \lambda_n\} \;,$$

then the set of the eigenvalues of $D^{[p]}_{G^{-1}A}$ is

$$\mathrm{spec}\left(D^{[p]}_{G^{-1}A}\right) \;=\; \left\{ \lambda_{i_1} + \cdots + \lambda_{i_p} \;:\; i_1, \ldots, i_p \in \{1, \ldots, n\} \text{ s.t. } i_1 < \cdots < i_p \right\} \;.$$

Finally, given a $\langle \cdot \mid \cdot\cdot \rangle$-symmetric endomorphism $E \in \mathrm{Hom}(V, V)$, let $\mathrm{sgn}(E)$ denote the number of non-negative eigenvalues of E:

$$\mathrm{sgn}(E) \;:=\; \mathrm{card}\,\{\lambda \in \mathrm{spec}(E) \;:\; \lambda \geq 0\} \;.$$

Note that, given two inner products on V inducing the isomorphisms $G_1 \colon V \overset{\simeq}{\to} V^*$ and $G_2 \colon V \overset{\simeq}{\to} V^*$ respectively, then there holds $\mathrm{sgn}\left(G_1^{-1}A\right) = \mathrm{sgn}\left(G_2^{-1}A\right)$, but, for $p > 1$, it might hold

$$\mathrm{sgn}\left(D^{[p]}_{G_1^{-1}A}\right) \;\neq\; \mathrm{sgn}\left(D^{[p]}_{G_2^{-1}A}\right) \;.$$

As said, $\mathrm{sgn}\left(D^{[p]}_{G^{-1}A}\right)$ counts the non-negative sums of p eigenvalues of $G^{-1}A \in \mathrm{Hom}(V, V)$. As a natural generalization of the notion of convexity, one is interested in studying symmetric endomorphisms having at least a certain number of non-negative sums of p eigenvalues. (Compare also [HL12, Definition 2.1], concerning the notion of positivity with respect to a sub-bundle of the Grassmannian bundle $\mathrm{G}_{\mathbb{R}}(p, TX)$ over a Riemannian manifold X.)

Definition 3.57 ([HL11, Definition 2.1, §3])**.**

- Let V be an n-dimensional $\mathbb{R}$-vector space endowed with an inner product $\langle \cdot \mid \cdot\cdot \rangle$. For $p \in \{1, \ldots, n\}$, and for $k \in \left\{1, \ldots, \binom{n}{p}\right\}$, define the space of *p-positive forms of k^{th} branch* on V as

$$\mathcal{P}^{(k)}_p(V, \langle \cdot \mid \cdot\cdot \rangle) := \left\{ A \in \mathrm{Sym}^2(V) \;:\; \mathrm{sgn}\left(D^{[p]}_{G^{-1}A}\right) \geq \binom{n}{p} - k + 1 \right\} \;.$$

- Let X be an n-dimensional manifold endowed with a Riemannian metric g. For $p \in \{1, \ldots, n\}$, and for $k \in \left\{1, \ldots, \binom{n}{p}\right\}$, define the space of *p-positive sections of k^{th} branch* of the bundle $\mathrm{Sym}^2(TX)$ of symmetric endomorphisms of TX as

$$\mathcal{P}^{(k)}_p(X, g) \;:=\; \left\{ A \in \mathrm{Sym}^2(TX) \;:\; \forall x \in X,\; A_x \in \mathcal{P}^{(k)}_p(T_xX, g_x) \right\}.$$

In order to introduce exhaustion functions on a given Riemannian manifold, we focus on special p-positive symmetric 2-forms: those arising from the Hessian of smooth functions.

Let (X, g) be a Riemannian manifold, and denote the Levi Civita connection associated to the Riemannian metric g by ∇^{LC}. For every $u \in \mathcal{C}^\infty(X; \mathbb{R})$, let

$$\mathrm{Hess}\, u \;\in\; \mathrm{Sym}^2(TX)$$

be defined, for any $V, W \in \mathcal{C}^\infty(X; TX)$, as

$$\mathrm{Hess}\, u\,(V, W) \;:=\; V\,W\,u - \left(\nabla^{LC}_V W\right)\, u \;.$$

Definition 3.58 ([HL11, Definition 2.2', §3])**.** Let X be an n-dimensional manifold endowed with a Riemannian metric g. Fix $p \in \{1, \ldots, n\}$, and $k \in \left\{1, \ldots, \binom{n}{k}\right\}$.

- The space
$$\mathrm{PSH}_p^{(k)}(X,\, g) \;:=\; \left\{u \in \mathcal{C}^\infty(X;\mathbb{R}) \;:\; \mathrm{Hess}\, u \in \mathcal{P}_p^{(k)}(X,\, g)\right\}\,,$$
is called the space of *p-pluri-sub-harmonic functions of k^{th} branch* on X.

- The space
$$\mathrm{int}\left(\mathrm{PSH}_p^{(k)}(X,\, g)\right) \;:=\; \left\{u \in \mathcal{C}^\infty(X;\mathbb{R}) \;:\; \mathrm{Hess}\, u \in \mathrm{int}\left(\mathcal{P}_p^{(k)}(X,\, g)\right)\right\}\,,$$
(where $\mathrm{int}\left(\mathcal{P}_p^{(k)}(X,\, g)\right)$ denotes the interior of $\mathcal{P}_p^{(k)}(X,\, g)$) is called the space of *strictly p-pluri-sub-harmonic functions of k^{th} branch* on X.

(Compare also [HL12, Definition 2.1] for the notion of (strictly) p-pluri-sub-harmonicity with respect to a sub-bundle of the Grassmannian bundle $\mathrm{G}_{\mathbb{R}}(p,\, TX)$ over a Riemannian manifold X.)

We can now define (strictly) p-convexity in terms of the p-convex hulls (and of the p-core).

Let X be an n-dimensional Riemannian manifold endowed with a Riemannian metric g, and fix $p \in \{1, \ldots, n\}$. Let $K \subseteq X$ be a subset of X; the *p-convex hull* of K, [HL11, Definition 4.1], is defined as
$$\widetilde{K}^{\mathrm{PSH}_p^{(1)}(X,\, g)} \;:=\; \left\{x \in X \;:\; \forall \phi \in PSH_p^{(1)}(X,\, g)\,,\; \phi(x) \leq \max_{y\in K} \phi(y)\right\}\,.$$
(Compare also [HL12, Definition 4.3] for the notion of convex hull with respect to a sub-bundle of the Grassmannian bundle $\mathrm{G}_{\mathbb{R}}(p,\, TX)$ over a Riemannian manifold X.)

Definition 3.59 ([HL11, Definition 4.3])**.** Let X be an n-dimensional Riemannian manifold endowed with a Riemannian metric g, and fix $p \in \{1, \ldots, n\}$. One says that X is *p-convex* if, for any subset $K \subseteq X$ that is relatively compact in X, then $\widetilde{K}^{\mathrm{PSH}_p^{(1)}(X,\, g)}$ is relatively compact in X.

(Compare also [HL12, Definition 4.5] for the notion of convexity with respect to a sub-bundle of the Grassmannian bundle $\mathrm{G}_{\mathbb{R}}(p,\, TX)$ over a Riemannian manifold X.)

Define the *p-core* of X, [HL11, Definition 5.3], as
$$\mathrm{Core}_p(X,\, g) \;:=\; \left\{x \in X \;:\; \text{for all } u \in \mathrm{PSH}_p^{(1)}(X,\, g)\,,\; \mathrm{Hess}\, u(x) \not\in \mathrm{int}\left(\mathcal{P}_p^{(1)}(T_xX,\, g_x)\right)\right\}\,.$$
(Compare also [HL12, Definition 4.1] for the notion of core with respect to a sub-bundle of the Grassmannian bundle $\mathrm{G}_{\mathbb{R}}(p,\, TX)$ over a Riemannian manifold X.)

Definition 3.60 ([HL11, Definition 5.2, Theorem 5.4])**.** Let X be an n-dimensional Riemannian manifold endowed with a Riemannian metric g, and fix $p \in \{1, \ldots, n\}$. One says that the manifold X is *strictly p-convex* if (i) $\mathrm{Core}_p(X,\, g) = \varnothing$, and ($ii$) for any subset $K \subseteq X$ that is relatively compact in X, then $\widetilde{K}^{\mathrm{PSH}_p^{(1)}(X,\, g)}$ is relatively compact in X.

(Compare also [HL12, Definition 4.9] for the notion of strictly convexity with respect to a sub-bundle of the Grassmannian bundle $\mathrm{G}_{\mathbb{R}}(p,\, TX)$ over a Riemannian manifold X.)

The relations between (strictly) p-convexity and the existence of smooth proper (strictly) p-pluri-sub-harmonic exhaustion functions were proven by F. R. Harvey and H. B. Lawson in [HL11, HL12]. Namely, the following result holds.

Theorem 3.61 ([HL11, Theorem 4.4, Theorem 5.4])**.** *Let X be an n-dimensional Riemannian manifold endowed with a Riemannian metric g, and fix $p \in \{1, \ldots, n\}$. Then X is p-convex (respectively, strictly p-convex) if and only if X admits a smooth proper exhaustion function $u \in \mathrm{PSH}_p^{(1)}(X,\, g)$ (respectively, $u \in \mathrm{int}\left(\mathrm{PSH}_p^{(1)}(X,\, g)\right)$).*

(Compare also [HL12, Theorem 4.4, Theorem 4.8] for the relations between (strictly) convexity and the existence of smooth proper (strictly) pluri-sub-harmonic exhaustion functions with respect to a sub-bundle of the Grassmannian bundle $\mathrm{G}_{\mathbb{R}}(p,\, TX)$ over a Riemannian manifold X.)

(We recall that a function $u\colon X \to \mathbb{R}$, where X is a manifold, is said to be an *exhaustion* if, for any $c \in \mathbb{R}$, the set $u^{-1}((-\infty,\, c)) = \{x \in X \;:\; u(x) < c\} \subseteq X$ is relatively compact in X.)

The previous definitions are motivated by the classical notions of (strictly) (q-)pseudo-convex functions, and q-complete and pseudo-convex domains, in Complex Analysis.

Definition 3.62 ([And63, §4], [AG62, §10]). Let $D \subseteq \mathbb{C}^n$ be a domain, and let ϕ be a smooth real-valued function on D. The function ϕ is called *q-pseudo-convex* or *q-pluri-sub-harmonic* (respectively, *strictly q-pseudo-convex* or *strictly-q-pluri-sub-harmonic*), if and only if, for any $z \in D$, the Hermitian form $\mathrm{L}(\phi)_z$ defined, for $\xi :=: (\xi^a)_{a\in\{1,\ldots,n\}} \in \mathbb{C}^n$, as

$$\mathrm{L}(\phi)_z(\xi) := \sum_{a,b=1}^{n} \frac{\partial^2\phi}{\partial z^a \partial \bar{z}^b}(z)\,\xi^a\,\overline{\xi^b}\,,$$

has, at least, $n-q+1$ non-negative (respectively, positive) eigenvalues. When $q=1$, (strictly) 1-pseudo-convex functions are called *(strictly) pseudo-convex*, or *(strictly) pluri-sub-harmonic*.

Definition 3.63 ([Rot55], [AG62, §16.c]). A domain $D \subseteq \mathbb{C}^n$ is called *q-complete* if there exists a smooth proper strictly q-pseudo-convex exhaustion function. When $q=1$, the 1-complete domains are called *strictly pseudo-convex.*

A. Andreotti and H. Grauert, in [AG62], proved a vanishing theorem for the higher-degree Dolbeault cohomology groups of q-complete domains; the same result was proven by A. Andreotti and E. Vesentini, in [AV65a], (see also [Ves67, Theorem 4.2],) extending the L^2-techniques by L. Hörmander, [Hör65]. More precisely, [AG62, Proposition 27], and [AV65a, Theorem 5], state that, given a q-complete domain $D \Subset \mathbb{C}^n$, it holds $H^{r,s}_{\overline{\partial}}(X) = \{0\}$, for any $r \in \mathbb{N}$ and for any $s \geq q$.

3.3.2 Vanishing of the de Rham cohomology for strictly p-convex domains

In this section, motivated by A. Andreotti and H. Grauert's vanishing result for the Dolbeault cohomology of q-complete domains in $\mathbb{C}^n$, [AG62], and by A. Andreotti and E. Vesentini's proof using L^2-techniques, [AV65a], we consider domains X in $\mathbb{R}^n$ endowed with a proper exhaustion function $u \in \mathcal{C}^\infty(X;\mathbb{R})$ whose Hessian is in $\operatorname{int}\left(\mathcal{P}^{(1)}_p(X,g)\right)$, re-proving, with L^2-techniques, the vanishing result for the higher-degree de Rham cohomology groups for strictly p-convex domains in the sense of F. R. Harvey and H. B. Lawson, Theorem 3.68, yet shown by J.-P. Sha, [Sha86], and by H. Wu, [Wu87, Theorem 1], using Morse theory, as a consequence of results on the homotopy type of X. Firstly, we recall some definitions and we set some notations; then, we prove some preliminary lemmata and estimates; finally we prove Theorem 3.67, stating that, on a strictly p-convex domain in $\mathbb{R}^n$, every d-closed k-form with $k \geq p$ is d-exact.

Let X be an oriented Riemannian manifold of dimension n, and denote its Riemannian metric by g and its volume by vol. The Riemannian metric g induces, for every $x \in X$, a point-wise inner product $\langle \cdot \,|\cdot\cdot\rangle_{g_x} : \wedge^\bullet T^*_x X \times \wedge^\bullet T^*_x X \to \mathbb{R}$.

Fix $\phi \in \mathcal{C}^0(X;\mathbb{R})$ a continuous function. For every φ, $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(X;\wedge^\bullet T^*X)$, let

$$\langle \varphi \,|\, \psi\rangle_{\mathrm{L}^2_\phi} := \int_X \langle \varphi\,|\psi\rangle_{g_x} \exp(-\phi)\,\mathrm{vol} \in \mathbb{R}\,,$$

and, for $k \in \mathbb{N}$, define $\mathrm{L}^2_\phi\left(X;\wedge^k T^*X\right)$ as the completion of the space $\mathcal{C}^\infty_{\mathrm{c}}\left(X;\wedge^k T^*X\right)$ of smooth k-forms with compact support with respect to the metric induced by $\|\cdot\|_{\mathrm{L}^2_\phi} := \langle \cdot \,|\, \cdot\rangle_{\mathrm{L}^2_\phi}$. Therefore, the space $\mathrm{L}^2_\phi\left(X;\wedge^k T^*X\right)$ is a Hilbert space, endowed with the inner product $\langle \cdot \,|\cdot\cdot\rangle_{\mathrm{L}^2_\phi}$, and $\mathcal{C}^\infty_{\mathrm{c}}\left(X;\wedge^k T^*X\right)$ is dense in $\mathrm{L}^2_\phi\left(X;\wedge^k T^*X\right)$. For any $k \in \mathbb{N}$, let $\mathrm{L}^2_{\mathrm{loc}}\left(X;\wedge^k T^*X\right)$ denote the space of k-forms φ whose restriction $\varphi\lfloor_K$ to every compact set $K \subseteq X$ belongs to $\mathrm{L}^2\left(K;\wedge^k T^*K\right)$.

For every ϕ_1, $\phi_2 \in \mathcal{C}^0(X;\mathbb{R})$, the operator

$$\mathrm{d}\colon \mathrm{L}^2_{\phi_1}(X;\wedge^\bullet T^*X) \dashrightarrow \mathrm{L}^2_{\phi_2}\left(X;\wedge^{\bullet+1} T^*X\right)$$

is densely-defined and closed; denote by

$$\mathrm{d}^*_{\phi_2,\,\phi_1}\colon \mathrm{L}^2_{\phi_2}\left(X;\wedge^{\bullet+1} T^*X\right) \dashrightarrow \mathrm{L}^2_{\phi_1}(X;\wedge^\bullet T^*X)$$

its adjoint, which is a densely-defined closed operator, see, e.g., [dSSST06, Theorem 7.55].

Moreover, for a domain X in $\mathbb{R}^n$, with set of coordinates $\left\{x^1,\ldots,x^n\right\}$, fixed $k \in \mathbb{N}$, $s \in \mathbb{N}$, and $\phi \in \mathcal{C}^\infty(X;\mathbb{R})$, one can consider the Sobolev space $\mathrm{W}^{s,2}_\phi\left(X;\wedge^k T^*X\right)$, which is defined as the space of k-forms $\varphi :=: \widetilde{\sum_{|I|=k}}\varphi_I\,\mathrm{d}\,x^I$ such that $\frac{\partial^{\ell_1+\cdots+\ell_n}\varphi_I}{\partial^{\ell_1}x^1\cdots\partial^{\ell_n}x^n} \in \mathrm{L}^2_\phi\left(X;\wedge^k T^*X\right)$ for every multi-index $(\ell_1,\ldots,\ell_n) \in \mathbb{N}^n$ satisfying $\ell_1+\cdots+\ell_n \leq s$ and for every strictly increasing multi-index I such that $|I| = k$. The space $\mathrm{W}^{s,2}_{\mathrm{loc}}\left(X; \wedge^k T^* X\right)$ is defined as the space of k-forms φ whose restriction $\varphi\lfloor_K$ to every compact set $K \subseteq X$ belongs to $\mathrm{W}^{s,2}\left(K; \wedge^k T^* K\right)$.

As a matter of notation, the symbol $\widetilde{\sum}_{|I|=k}$ denotes the sum over the strictly increasing multi-indices $I :=: (i_1, \ldots, i_k) \in \mathbb{N}^k$ (that is, the multi-indices such that $0 < i_1 < \cdots < i_k$) of length k. We use $\left\{x^1, \ldots, x^n\right\}$ as a set of coordinates on $\mathbb{R}^n$, and, given a multi-index $I :=: (i_1, \ldots, i_k) \in \mathbb{N}^k$, we shorten $\mathrm{d}\, x^I := \mathrm{d}\, x^{i_1} \wedge \cdots \wedge \mathrm{d}\, x^{i_k}$. Given I_1 and I_2 two multi-indices of length k, let $\operatorname{sign}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$ be the sign of the permutation $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$ if I_1 is a permutation of I_2, and zero otherwise.

Let X be a domain in $\mathbb{R}^n$, that is, an open connected subset of $\mathbb{R}^n$, endowed with the metric and the volume induced, respectively, by the Euclidean metric and the standard volume of $\mathbb{R}^n$.

For $\phi_1, \phi_2 \in \mathcal{C}^\infty(X; \mathbb{R})$, consider $\mathrm{d}\colon \mathrm{L}^2_{\phi_1}\left(X; \wedge^{k-1} T^* X\right) \dashrightarrow \mathrm{L}^2_{\phi_2}\left(X; \wedge^k T^* X\right)$. The following lemma gives an explicit expression of the adjoint $\mathrm{d}^*_{\phi_2, \phi_1} \colon \mathrm{L}^2_{\phi_2}\left(X; \wedge^k T^* X\right) \dashrightarrow \mathrm{L}^2_{\phi_1}\left(X; \wedge^{k-1} T^* X\right)$, [AC12, Lemma 2.1] (compare with, e.g., [dSSST06, §8.2.1], [Gun90, Lemma O.2] in the complex case).

Lemma 3.64. *Let X be a domain in $\mathbb{R}^n$. Let $\phi_1, \phi_2 \in \mathcal{C}^\infty(X; \mathbb{R})$ and consider*

$$\begin{tikzcd} \mathrm{L}^2_{\phi_1}\left(X; \wedge^{k-1} T^* X\right) \arrow[r, dashed, bend left, "\mathrm{d}"] & \mathrm{L}^2_{\phi_2}\left(X; \wedge^k T^* X\right) \arrow[l, dashed, bend left, "\mathrm{d}^*_{\phi_2, \phi_1}"] \end{tikzcd} .$$

Let

$$v :=: \widetilde{\sum_{|I|=k}} v_I \,\mathrm{d}\, x^I \in \mathrm{L}^2_{\phi_2}\left(X; \wedge^k T^* X\right)$$

and suppose that $v \in \operatorname{dom} \mathrm{d}^*_{\phi_2, \phi_1}$. Then

$$\begin{aligned} \mathrm{d}^*_{\phi_2, \phi_1} v &= \exp(\phi_1)\, \mathrm{d}^*_{0,0}\left(\exp(-\phi_2)\, v\right) \\ &= \widetilde{\sum_{|J|=k-1}} \left(-\exp(\phi_1) \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix} \ell J \\ I \end{pmatrix} \frac{\partial\left(v_I \exp(-\phi_2)\right)}{\partial x^\ell} \right) \mathrm{d}\, x^J . \end{aligned}$$

Proof. By definition of $\mathrm{d}^*_{\phi_2, \phi_1}$, for every $u \in \operatorname{dom} \mathrm{d}$, one has $\langle \mathrm{d}\, u \mid v \rangle_{\mathrm{L}^2_{\phi_2}} = \left\langle u \,\middle|\, \mathrm{d}^*_{\phi_2, \phi_1} v \right\rangle_{\mathrm{L}^2_{\phi_1}}$. Hence, consider

$$u :=: \widetilde{\sum_{|J|=k-1}} u_J \,\mathrm{d}\, x^J \in \mathcal{C}^\infty_{\mathrm{c}}\left(X; \wedge^{k-1} T^* X\right) ,$$

and compute

$$\mathrm{d}\, u = \widetilde{\sum_{\substack{|J|=k-1 \\ |I|=k}}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix} \ell J \\ I \end{pmatrix} \frac{\partial u_J}{\partial x^\ell} \,\mathrm{d}\, x^I .$$

The statement follows by computing

$$\begin{aligned} \langle \mathrm{d}\, u \mid v \rangle_{\mathrm{L}^2_{\phi_2}} &= \int_X \widetilde{\sum_{\substack{|J|=k-1 \\ |I|=k}}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix} \ell J \\ I \end{pmatrix} \frac{\partial u_J}{\partial x^\ell} v_I \exp(-\phi_2) \operatorname{vol} \\ &= -\int_X \widetilde{\sum_{\substack{|J|=k-1 \\ |I|=k}}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix} \ell J \\ I \end{pmatrix} \frac{\partial\left(v_I \exp(-\phi_2)\right)}{\partial x^\ell} u_J \operatorname{vol} \end{aligned}$$

and

$$\left\langle u \,\middle|\, \mathrm{d}^*_{\phi_2, \phi_1} v \right\rangle_{\mathrm{L}^2_{\phi_1}} = \int_X \widetilde{\sum_{|J|=k-1}} \left(\mathrm{d}^*_{\phi_2, \phi_1} v\right)_J u_J \exp(-\phi_1) \operatorname{vol} ,$$

where $\mathrm{d}^*_{\phi_2, \phi_1} v =: \widetilde{\sum}_{|J|=k-1} \left(\mathrm{d}^*_{\phi_2, \phi_1} v\right)_J \mathrm{d}\, x^J$. $\square$

For any fixed $\phi \in \mathcal{C}^\infty(X;\mathbb{R})$ and for any $j \in \{1,\ldots,n\}$, define the operator

$$\delta_j^\phi \colon \mathcal{C}^\infty(X;\mathbb{R}) \to \mathcal{C}^\infty(X;\mathbb{R}) \ ,$$

where

$$\delta_j^\phi(f) \ := \ -\exp(\phi)\ \frac{\partial\left(f\ \exp(-\phi)\right)}{\partial x^j} \ = \ \frac{\partial \phi}{\partial x^j}\cdot f - \frac{\partial f}{\partial x^j} \ .$$

The following lemma states that δ_j^ϕ is the adjoint of $\frac{\partial}{\partial x^j}$ in $\mathrm{L}^2_\phi\left(X;\wedge^0 T^*X\right)$, and computes the commutator between δ_j^ϕ and $\frac{\partial}{\partial x^k}$, [AC12, Lemma 2.2] (compare with, e.g., [Hör90, pages 83–84]).

Lemma 3.65. *Let X be a domain in $\mathbb{R}^n$. Let $\phi \in \mathcal{C}^\infty(X;\mathbb{R})$ and $j \in \{1,\ldots,n\}$, and consider the operator $\delta_j^\phi \colon \mathcal{C}^\infty(X;\mathbb{R}) \to \mathcal{C}^\infty(X;\mathbb{R})$. Then:*

- *for every $w_1, w_2 \in \mathcal{C}^\infty_{\mathrm{c}}(X;\mathbb{R})$,*

$$\int_X w_1 \cdot \frac{\partial w_2}{\partial x^k}\ \exp(-\phi)\ \mathrm{vol} \ = \ \int_X \delta_k^\phi(w_1)\cdot w_2\ \exp(-\phi)\ \mathrm{vol}\ ;$$

- *for any $k \in \{1,\ldots,n\}$, the following commutation formula holds in* $\mathrm{End}\left(\mathcal{C}^\infty_{\mathrm{c}}(X;\mathbb{R})\right)$*:*

$$\left[\delta_j^\phi,\ \frac{\partial}{\partial x^k}\right] \ = \ -\frac{\partial^2\phi}{\partial x^j \partial x^k}\cdot \ .$$

Proof. As regards the first item, one has

$$\begin{aligned}
\int_X w_1 \cdot \frac{\partial w_2}{\partial x^k}\ \exp(-\phi)\ \mathrm{vol} \ &= \ -\int_X w_2 \cdot \frac{\partial}{\partial x^k}\left(w_1\ \exp(-\phi)\right)\mathrm{vol} \\
&= \ \int_X w_2 \cdot \left(w_1\, \frac{\partial \phi}{\partial x^k} - \frac{\partial w_1}{\partial x^k}\right)\ \exp(-\phi)\ \mathrm{vol} \\
&= \ \int_X \delta_k^\phi(w_1)\cdot w_2\ \exp(-\phi)\ \mathrm{vol}\ .
\end{aligned}$$

As regards the second item, one has, for every $f \in \mathcal{C}^\infty(X;\mathbb{R})$,

$$\begin{aligned}
\left[\delta_j^\phi,\ \frac{\partial}{\partial x^k}\right](f) \ &= \ \delta_j^\phi\left(\frac{\partial f}{\partial x^k}\right) - \frac{\partial}{\partial x^k}\left(\delta_j^\phi(f)\right) \\
&= \ \frac{\partial \phi}{\partial x^j}\cdot\frac{\partial f}{\partial x^k} - \frac{\partial^2 f}{\partial x^j \partial x^k} - \frac{\partial}{\partial x^k}\left(\frac{\partial \phi}{\partial x^j}\cdot f - \frac{\partial f}{\partial x^j}\right) \\
&= \ \frac{\partial \phi}{\partial x^j}\cdot\frac{\partial f}{\partial x^k} - \frac{\partial^2 f}{\partial x^j \partial x^k} - \frac{\partial^2 \phi}{\partial x^k \partial x^j}\cdot f - \frac{\partial \phi}{\partial x^j}\cdot\frac{\partial f}{\partial x^k} + \frac{\partial^2 f}{\partial x^k \partial x^j} \\
&= \ -\frac{\partial^2 \phi}{\partial x^k \partial x^j}\cdot f\ ,
\end{aligned}$$

concluding the proof of the lemma. □

Finally, we prove the following estimate, [AC12, Proposition 2.3, Remark 2.4], which will be used in the proof of Theorem 3.67 (we refer to [Hör90, §4.2], or, e.g., [Gun90, Lemma O.3] and [dSSST06, §8.3.1] for its complex counterpart).

Proposition 3.66. *Let X be a domain in $\mathbb{R}^n$ and $\phi,\, \psi \in \mathcal{C}^\infty(X;\mathbb{R})$. Consider*

$$\begin{tikzcd}
\mathrm{L}^2_{\phi-2\psi}\left(X;\wedge^{k-1}T^*X\right) \arrow[r, dashed, shift left, "\mathrm{d}"] & \mathrm{L}^2_{\phi-\psi}\left(X;\wedge^{k}T^*X\right) \arrow[l, dashed, shift left, "\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}"] \arrow[r, dashed, shift left, "\mathrm{d}"] & \mathrm{L}^2_{\phi}\left(X;\wedge^{k+1}T^*X\right) \arrow[l, dashed, shift left, "\mathrm{d}^*_{\phi,\,\phi-\psi}"]
\end{tikzcd} \ .$$

Then, for any $\eta :=: \widetilde{\sum_{|I|=k}}\eta_I \, \mathrm{d}\, x^I \in \mathcal{C}^\infty_{\mathrm{c}}\left(X; \wedge^k T^* X\right)$*, one has*

$$\begin{aligned}
&\int_X \widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_2|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \mathrm{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix} \mathrm{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix} \frac{\partial^2\phi}{\partial x^{\ell_1}\,\partial x^{\ell_2}}\,\eta_{I_1}\,\eta_{I_2}\,\exp(-\phi)\,\mathrm{vol}\\
&\leq \int_X \left(\widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_2|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \mathrm{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix} \mathrm{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix} \frac{\partial^2\phi}{\partial x^{\ell_1}\,\partial x^{\ell_2}}\,\eta_{I_1}\,\eta_{I_2} + \widetilde{\sum_{|I|=k}}\sum_{\ell=1}^{n}\left|\frac{\partial \eta_I}{\partial x^\ell}\right|^2 \right)\\
&\quad\cdot\exp(-\phi)\,\mathrm{vol}\\
&\leq C\cdot\left(\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right\|^2_{\mathrm{L}^2_{\phi-2\psi}} + \left\|\mathrm{d}\,\eta\right\|^2_{\mathrm{L}^2_{\phi}} + \int_X \widetilde{\sum_{|I|=k}}\sum_{\ell=1}^{n}\left|\frac{\partial\psi}{\partial x^\ell}\right|^2 |\eta_I|^2 \exp(-\phi)\,\mathrm{vol}\right) ,
\end{aligned}$$

where $C :=: C(k,n) \in \mathbb{N}$ *is a constant depending just on* k *and* n*.*

Proof. It is straightforward to compute

$$\mathrm{d}\,\eta = \widetilde{\sum_{\substack{|I|=k\\|H|=k+1}}} \sum_{\ell=1}^{n} \mathrm{sign}\begin{pmatrix}\ell I\\ H\end{pmatrix} \frac{\partial\eta_I}{\partial x^\ell}\,\mathrm{d}\,x^H$$

and, using Lemma 3.64,

$$\begin{aligned}
\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta &= -\exp(-\psi) \widetilde{\sum_{\substack{|J|=k-1\\|I|=k}}} \sum_{\ell=1}^{n} \mathrm{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \left(\frac{\partial\eta_I}{\partial x^\ell} - \frac{\partial(\phi-\psi)}{\partial x^\ell}\,\eta_I\right)\mathrm{d}\,x^J\\
&= \exp(-\psi) \widetilde{\sum_{\substack{|J|=k-1\\|I|=k}}} \sum_{\ell=1}^{n} \mathrm{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \left(\delta^\phi_\ell(\eta_I) - \frac{\partial\psi}{\partial x^\ell}\,\eta_I\right)\mathrm{d}\,x^J .
\end{aligned}$$

For every J such that $|J| = k-1$, the previous equality gives

$$\widetilde{\sum_{|I|=k}}\sum_{\ell=1}^{n} \mathrm{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \delta^\phi_\ell(\eta_I) = \exp(\psi)\left(\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right)_J + \widetilde{\sum_{|I|=k}}\sum_{\ell=1}^{n} \mathrm{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \frac{\partial\psi}{\partial x^\ell}\,\eta_I ,$$

where $\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta =: \widetilde{\sum_{|J|=k-1}}\left(\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right)_J \mathrm{d}\,x^J$.

By the inequality between the geometric mean and the arithmetic mean, one gets

$$\begin{aligned}
&\int_X \widetilde{\sum_{|J|=k-1}} \left| \widetilde{\sum_{|I|=k}}\sum_{\ell=1}^{n} \mathrm{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \delta^\phi_\ell(\eta_I)\right|^2 \exp(-\phi)\,\mathrm{vol}\\
&\leq 2\int_X \widetilde{\sum_{|J|=k-1}} \left(\left|\left(\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right)_J\right|^2 \exp(2\psi) + \left|\widetilde{\sum_{|I|=k}}\sum_{\ell=1}^{n} \mathrm{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \frac{\partial\psi}{\partial x^\ell}\,\eta_I\right|^2\right) \exp(-\phi)\,\mathrm{vol}\\
&\leq C\left(\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right\|^2_{\mathrm{L}^2_{\phi-2\psi}} + \int_X \widetilde{\sum_{|I|=k}}\sum_{\ell=1}^{n}\left|\frac{\partial\psi}{\partial x^\ell}\right|^2 \cdot |\eta_I|^2 \exp(-\phi)\,\mathrm{vol}\right) , \qquad (3.3.1)
\end{aligned}$$

where $C :=: C(k,n) \in \mathbb{N}$ depends on k and n only.

Now, using Lemma 3.65, one computes

$$
\begin{aligned}
&\int_X \widetilde{\sum_{|J|=k-1}} \left| \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix} \ell J \\ I \end{pmatrix} \delta^{\phi}_{\ell}\left(\eta_I\right) \right|^2 \exp\left(-\phi\right)\, \mathrm{vol} \\
&\quad = \widetilde{\sum_{|J|=k-1}} \widetilde{\sum_{\substack{|I_1|=k \\ |I_2|=k}}} \sum_{\ell_1,\, \ell_2=1}^{n} \operatorname{sign}\begin{pmatrix} \ell_1 J \\ I_1 \end{pmatrix} \operatorname{sign}\begin{pmatrix} \ell_2 J \\ I_2 \end{pmatrix} \int_X \delta^{\phi}_{\ell_1}\left(\eta_{I_1}\right) \cdot \delta^{\phi}_{\ell_2}\left(\eta_{I_2}\right) \exp\left(-\phi\right)\, \mathrm{vol} \\
&\quad = \widetilde{\sum_{\substack{|J|=k-1 \\ |I_1|=k \\ |I_2|=k}}} \sum_{\ell_1,\, \ell_2=1}^{n} \operatorname{sign}\begin{pmatrix} \ell_1 J \\ I_1 \end{pmatrix} \operatorname{sign}\begin{pmatrix} \ell_2 J \\ I_2 \end{pmatrix} \\
&\qquad \cdot \int_X \left(\frac{\partial \eta_{I_1}}{\partial x^{\ell_2}} \frac{\partial \eta_{I_2}}{\partial x^{\ell_1}} + \frac{\partial^2 \phi}{\partial x^{\ell_1}\, \partial x^{\ell_2}}\, \eta_{I_1}\, \eta_{I_2} \right) \exp\left(-\phi\right)\, \mathrm{vol} \;.
\end{aligned}
\tag{3.3.2}
$$

Now, note that

$$
\begin{aligned}
\left|\mathrm{d}\,\eta\right|^2 &= \widetilde{\sum_{|H|=k+1}} \left| \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix} \ell I \\ H \end{pmatrix} \frac{\partial \eta_I}{\partial x^{\ell}} \right|^2 \\
&= \widetilde{\sum_{|H|=k+1}} \left(\widetilde{\sum_{\substack{|I_1|=k \\ |I_2|=k}}} \sum_{\ell_1,\, \ell_2=1}^{n} \operatorname{sign}\begin{pmatrix} \ell_1 I_1 \\ H \end{pmatrix} \operatorname{sign}\begin{pmatrix} \ell_2 I_2 \\ H \end{pmatrix} \frac{\partial \eta_{I_1}}{\partial x^{\ell_1}} \frac{\partial \eta_{I_2}}{\partial x^{\ell_2}} \right) \\
&= \widetilde{\sum_{\substack{|I_1|=k \\ |I_2|=k}}} \sum_{\ell_1,\, \ell_2=1}^{n} \operatorname{sign}\begin{pmatrix} \ell_1 I_1 \\ \ell_2 I_2 \end{pmatrix} \frac{\partial \eta_{I_1}}{\partial x^{\ell_1}} \frac{\partial \eta_{I_2}}{\partial x^{\ell_2}} \\
&= \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \left| \frac{\partial \eta_I}{\partial x^{\ell}} \right|^2 - \widetilde{\sum_{\substack{|J|=k-1 \\ |I_1|=k \\ |I_2|=k}}} \sum_{\ell_1,\, \ell_2=1}^{n} \operatorname{sign}\begin{pmatrix} \ell_1 J \\ I_1 \end{pmatrix} \operatorname{sign}\begin{pmatrix} \ell_2 J \\ I_2 \end{pmatrix} \frac{\partial \eta_{I_1}}{\partial x^{\ell_2}} \frac{\partial \eta_{I_2}}{\partial x^{\ell_1}} \;.
\end{aligned}
\tag{3.3.3}
$$

Hence, in view of (3.3.3), (3.3.2), (3.3.1), we get

$$
\begin{aligned}
&\int_X \widetilde{\sum_{\substack{|J|=k-1 \\ |I_1|=k \\ |I_2|=k}}} \sum_{\ell_1,\, \ell_2=1}^{n} \operatorname{sign}\begin{pmatrix} \ell_1 J \\ I_1 \end{pmatrix} \operatorname{sign}\begin{pmatrix} \ell_2 J \\ I_2 \end{pmatrix} \frac{\partial^2 \phi}{\partial x^{\ell_1}\, \partial x^{\ell_2}}\, \eta_{I_1}\, \eta_{I_2} \exp\left(-\phi\right)\, \mathrm{vol} \\
&\quad \leq \int_X \left(\widetilde{\sum_{\substack{|J|=k-1 \\ |I_1|=k \\ |I_2|=k}}} \sum_{\ell_1,\, \ell_2=1}^{n} \operatorname{sign}\begin{pmatrix} \ell_1 J \\ I_1 \end{pmatrix} \operatorname{sign}\begin{pmatrix} \ell_2 J \\ I_2 \end{pmatrix} \frac{\partial^2 \phi}{\partial x^{\ell_1}\, \partial x^{\ell_2}}\, \eta_{I_1}\, \eta_{I_2} + \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \left| \frac{\partial \eta_I}{\partial x^{\ell}} \right|^2 \right) \exp\left(-\phi\right)\, \mathrm{vol} \\
&\quad = \int_X \left(\widetilde{\sum_{|J|=k-1}} \left| \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix} \ell J \\ I \end{pmatrix} \delta^{\phi}_{\ell}\left(\eta_I\right) \right|^2 + \widetilde{\sum_{|H|=k+1}} \left|\left(\mathrm{d}\,\eta\right)_H\right|^2 \right) \exp\left(-\phi\right)\, \mathrm{vol} \\
&\quad \leq C \cdot \left(\left\| \mathrm{d}^*_{\phi-\psi,\, \phi-2\psi} \eta \right\|^2_{\mathrm{L}^2_{\phi-2\psi}} + \left\| \mathrm{d}\,\eta \right\|^2_{\mathrm{L}^2_{\phi}} + \int_X \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \left| \frac{\partial \psi}{\partial x^{\ell}} \right|^2 \left|\eta_I\right|^2 \exp\left(-\phi\right)\, \mathrm{vol} \right) \;,
\end{aligned}
$$

concluding the proof. □

Using the previous result, we prove here the following theorem, [AC12, Theorem 3.1].

Theorem 3.67. *Let X be a strictly p-convex domain in $\mathbb{R}^n$, and fix $k \in \mathbb{N}$ such that $k \geq p$. Then, every* d*-closed k-form $\eta \in \wedge^k X$ is* d*-exact, namely, there exists $\alpha \in \wedge^{k-1} X$ such that $\eta = \mathrm{d}\,\alpha$.*

Proof. Let us split the proof in the following steps.

`Step 1` – *Preliminary definitions.* X being a strictly p-convex domain in $\mathbb{R}^n$, by F. R. Harvey and H. B. Lawson's [HL12, Theorem 4.8] (see also [HL11, Theorem 5.4]), there exists a smooth proper strictly p-pluri-sub-harmonic exhaustion function

$$\rho \;\in\; \mathrm{int}\left(\mathrm{PSH}^{(1)}_p\,(X,\,g)\right) \cap \mathcal{C}^\infty\,(X;\mathbb{R}) \;,$$

where g is the metric on X induced by the Euclidean metric on $\mathbb{R}^n$.

For every $m \in \mathbb{N}$, consider the compact set

$$K^{(m)} \;:=\; \{x \in X \;:\; \rho(x) \leq m\} \;,$$

and define

$$L^{(m)} \;:=\; \min_{K^{(m)}} \lambda^{[k]}_1 \;>\; 0 \,,$$

where, for every $x \in X$, the real numbers $\lambda^{[k]}_1(x) \leq \cdots \leq \lambda^{[k]}_{\binom{n}{k}}(x)$ are the ordered eigen-values of the endomorphism $D^{[k]}_{g^{-1}\,\mathrm{Hess}\,\rho(x)} \in \mathrm{Hom}\left(\wedge^k T_x X,\, \wedge^k T_x X\right)$, and $\lambda_1(x) \leq \cdots \leq \lambda_n(x)$ are the ordered eigen-values of the endomorphism $g^{-1}\,\mathrm{Hess}\,\rho(x) \in \mathrm{Hom}\,(T_x X,\, T_x X)$; indeed, note that, for every $x \in X$, since ρ is strictly p-pluri-sub-harmonic,

$$\lambda^{[k]}_1(x) \;=\; \lambda_1(x) + \cdots + \lambda_k(x) \;\geq\; \lambda_1(x) + \cdots + \lambda_p(x) \;>\; 0 \;,$$

and that the function $X \ni x \mapsto \lambda^{[k]}_1(x) \in \mathbb{R}$ is continuous.

Fix $\{\rho_\nu\}_{\nu\in\mathbb{N}} \subset \mathcal{C}^\infty_{\mathrm{c}}\,(X;\mathbb{R})$ such that (i) $0 \leq \rho_\nu \leq 1$ for every $\nu \in \mathbb{N}$, and (ii) for every compact set $K \subseteq X$, there exists $\nu_0 :=: \nu_0(K) \in \mathbb{N}$ such that $\rho_\nu\lfloor_K = 1$ for every $\nu \geq \nu_0$.

Then, we can choose $\psi \in \mathcal{C}^\infty\,(X;\mathbb{R})$ such that, for every $\nu \in \mathbb{N}$,

$$\left|\mathrm{d}\,\rho_\nu\right|^2 \;\leq\; \exp\,(\psi) \;\;.$$

For every $m \in \mathbb{N}$, set

$$\gamma^{(m)} \;:=\; \max_{K^{(m)}} \left(C \cdot |\mathrm{d}\,\psi|^2 + \exp\,(\psi)\right) \;,$$

where $C :=: C(n,k)$ is the constant in Proposition 3.66.

Fix $\chi \in \mathcal{C}^\infty\,(\mathbb{R};\mathbb{R})$ such that (i) $\chi' > 0$, (ii) $\chi'' > 0$, and (iii) $\chi'\lfloor_{(-\infty,\,m]} > \frac{\gamma^{(m)}}{L^{(m)}}$, for every $m \in \mathbb{N}$. Define

$$\phi \;:=\; \chi \circ \rho \,;$$

then, $\phi \in \mathrm{int}\left(\mathrm{PSH}^{(1)}_p\,(X,\,g)\right) \cap \mathcal{C}^\infty\,(X;\mathbb{R})$; furthermore

$$\frac{\partial^2\phi}{\partial x^{\ell_1}\partial x^{\ell_2}} \;=\; \chi'' \circ \rho \cdot \frac{\partial\rho}{\partial x^{\ell_1}} \cdot \frac{\partial\rho}{\partial x^{\ell_2}} + \chi' \circ \rho \cdot \frac{\partial^2\rho}{\partial x^{\ell_1}\partial x^{\ell_2}} \;.$$

Choose $\mu \in \mathcal{C}^\infty\,(X;\mathbb{R})$ such that, for every $m \in \mathbb{N}$,

$$\chi' \circ \rho\lfloor_{K^{(m)}} \cdot L^{(m)} \;\geq\; \mu\lfloor_{K^{(m)}} \;\geq\; \gamma^{(m)} \;.$$

`Step 2` – *For every $\eta \in \mathcal{C}^\infty_{\mathrm{c}}\left(X;\wedge^k T^*X\right)$, it holds $\|\eta\|^2_{\mathrm{L}^2_{\phi-\psi}} \leq C \cdot \left(\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right\|^2_{\mathrm{L}^2_{\phi-2\psi}} + \|\mathrm{d}\,\eta\|^2_{\mathrm{L}^2_{\phi}}\right)$.* Since

$$D^{[k]}_{g^{-1}\,\mathrm{Hess}\,\rho} \;=\; \left(\widetilde{\sum_{|J|=k-1}} \sum_{\ell_1,\,\ell_2=1}^{n} \mathrm{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix} \mathrm{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix} \frac{\partial^2\rho}{\partial x^{\ell_1}\partial x^{\ell_2}}\right)_{I_1,I_2} \;\in\; \mathrm{Hom}\left(\wedge^k TX,\, \wedge^k TX\right) \;,$$

then, by `Step 1`, one has the estimate

$$
\begin{aligned}
&\widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_1|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\left(\begin{array}{c}\ell_1 J\\ I_1\end{array}\right) \operatorname{sign}\left(\begin{array}{c}\ell_2 J\\ I_2\end{array}\right) \frac{\partial^2\phi}{\partial x^{\ell_1}\partial x^{\ell_2}}\,\eta_{I_1}\,\eta_{I_2}\\
&\quad = \widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_1|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\left(\begin{array}{c}\ell_1 J\\ I_1\end{array}\right) \operatorname{sign}\left(\begin{array}{c}\ell_2 J\\ I_2\end{array}\right) \chi''\circ\rho\cdot\frac{\partial\rho}{\partial x^{\ell_1}}\,\frac{\partial\rho}{\partial x^{\ell_2}}\,\eta_{I_1}\,\eta_{I_2}\\
&\qquad + \widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_1|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\left(\begin{array}{c}\ell_1 J\\ I_1\end{array}\right) \operatorname{sign}\left(\begin{array}{c}\ell_2 J\\ I_2\end{array}\right) \chi'\circ\rho\cdot\frac{\partial^2\rho}{\partial x^{\ell_1}\partial x^{\ell_2}}\,\eta_{I_1}\,\eta_{I_2}\\
&\quad = \widetilde{\sum_{|J|=k-1}} \chi''\circ\rho\cdot\left|\widetilde{\sum_{|I|=k}}\sum_{\ell=1}^{n}\operatorname{sign}\left(\begin{array}{c}\ell J\\ I\end{array}\right)\frac{\partial\rho}{\partial x^{\ell}}\,\eta_I\right|^2\\
&\qquad + \chi'\circ\rho\cdot\widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_1|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\left(\begin{array}{c}\ell_1 J\\ I_1\end{array}\right) \operatorname{sign}\left(\begin{array}{c}\ell_2 J\\ I_2\end{array}\right) \frac{\partial^2\rho}{\partial x^{\ell_1}\partial x^{\ell_2}}\,\eta_{I_1}\,\eta_{I_2}\\
&\quad \geq \chi'\circ\rho\cdot\lambda_1^{[k]}(x)\cdot\widetilde{\sum_{|I|=k}}\left|\eta_I\right|^2\\
&\quad \geq \mu\cdot\widetilde{\sum_{|I|=k}}\left|\eta_I\right|^2 \;.
\end{aligned}
$$

Hence, using Proposition 3.66, we get that, for every $\eta\in\mathcal{C}^\infty_{\mathrm{c}}\left(X;\wedge^k T^*X\right)$,

$$
\begin{aligned}
\|\eta\|^2_{\mathrm{L}^2_{\phi-\psi}} &= \int_X \widetilde{\sum_{|I|=k}}\left|\eta_I\right|^2\exp\left(-\left(\phi-\psi\right)\right)\,\mathrm{vol}\\
&\leq \int_X \widetilde{\sum_{|I|=k}}\left(\mu - C\cdot\sum_{\ell=1}^{n}\left|\frac{\partial\psi}{\partial x^\ell}\right|^2\right)\cdot\left|\eta_I\right|^2\exp\left(-\phi\right)\,\mathrm{vol}\\
&\leq \int_X \left(\widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_2|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\left(\begin{array}{c}\ell_1 J\\ I_1\end{array}\right) \operatorname{sign}\left(\begin{array}{c}\ell_2 J\\ I_2\end{array}\right) \frac{\partial^2\phi}{\partial x^{\ell_1}\,\partial x^{\ell_2}}\,\eta_{I_1}\,\eta_{I_2}\right.\\
&\qquad\left. -C\cdot\widetilde{\sum_{|I|=k}}\sum_{\ell=1}^{n}\left|\frac{\partial\psi}{\partial x^\ell}\right|^2\left|\eta_I\right|^2\right)\exp\left(-\phi\right)\,\mathrm{vol}\\
&\leq C\cdot\left(\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right\|^2_{\mathrm{L}^2_{\phi-2\psi}}+\left\|\mathrm{d}\,\eta\right\|^2_{\mathrm{L}^2_{\phi}}\right)\;,
\end{aligned}
$$

where $C:=:C(k,n)\in\mathbb{N}$ is the constant in Proposition 3.66, depending just on k and n.

`Step 3` – *The space* $\mathcal{C}^\infty_{\mathrm{c}}\left(X;\wedge^k T^*X\right)$ *is dense in the space* $\operatorname{dom}\mathrm{d}\cap\operatorname{dom}\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}$ *endowed with the norm* $\|\cdot\|_{\mathrm{L}^2_{\phi-\psi}}+\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\cdot\right\|_{\mathrm{L}^2_{\phi-2\psi}}+\|\mathrm{d}\cdot\|_{\mathrm{L}^2_{\phi}}$. Consider

$$
\begin{tikzcd}[column sep=huge]
\mathrm{L}^2_{\phi-2\psi}\left(X;\wedge^{k-1}T^*X\right) \arrow[r, dashed, shift left, "\mathrm{d}"] & \mathrm{L}^2_{\phi-\psi}\left(X;\wedge^{k}T^*X\right) \arrow[l, dashed, shift left, "\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}"] \arrow[r, dashed, shift left, "\mathrm{d}"] & \mathrm{L}^2_{\phi}\left(X;\wedge^{k+1}T^*X\right) \arrow[l, dashed, shift left, "\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}"]
\end{tikzcd}\;.
$$

Fix $\eta \in \operatorname{dom} \mathrm{d} \cap \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \subseteq \mathrm{L}^2_{\phi-\psi}\left(X; \wedge^k T^* X\right)$.

Firstly, we prove that $\{\rho_\nu\, \eta\}_{\nu\in\mathbb{N}} \subset \operatorname{dom} \mathrm{d} \cap \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \subseteq \mathrm{L}^2_{\phi-\psi}\left(X; \wedge^k T^* X\right)$, where $\{\rho_\nu\}_{\nu\in\mathbb{N}} \subset \mathcal{C}^\infty_{\mathrm{c}}(X;\mathbb{R})$ has been defined in `Step 1`, is a sequence of functions having compact support and converging to η in the graph norm $\|\cdot\|_{\mathrm{L}^2_{\phi-\psi}} + \left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\cdot\right\|_{\mathrm{L}^2_{\phi-2\psi}} + \|\mathrm{d}\,\cdot\|_{\mathrm{L}^2_{\phi}}$. Indeed,

$$\begin{aligned} \left|\mathrm{d}\left(\rho_\nu\, \eta\right) - \rho_\nu\, \mathrm{d}\,\eta\right|^2 \exp\left(-\phi\right) &= |\eta|^2 \cdot \left|\mathrm{d}\,\rho_\nu\right|^2 \exp\left(-\phi\right) \\ &\leq |\eta|^2 \exp\left(-\left(\phi-\psi\right)\right) \;\in\; \mathrm{L}^2\left(X; \wedge^k T^* X\right) \;, \end{aligned}$$

hence, by the Lebesgue dominated convergence theorem, $\left\|\mathrm{d}\left(\rho_\nu\, \eta\right) - \rho_\nu\, \mathrm{d}\,\eta\right\|_{\mathrm{L}^2_\phi} \to 0$ as $\nu \to +\infty$. Furthermore, for every $\nu \in \mathbb{N}$, note that $\rho_\nu\, \eta \in \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}$: indeed, the map

$$\mathrm{L}^2_{\phi-2\psi}\left(X; \wedge^{k-1} T^* X\right) \supseteq \operatorname{dom} \mathrm{d} \ni u \mapsto \langle \rho_\nu\, \eta \mid \mathrm{d}\, u\rangle_{\mathrm{L}^2_{\phi-\psi}} \in \mathbb{R}$$

is continuous, being

$$\begin{aligned} \langle \rho_\nu\, \eta \mid \mathrm{d}\, u\rangle_{\mathrm{L}^2_{\phi-\psi}} &= \langle \eta \mid \mathrm{d}\left(\rho_\nu\, u\right)\rangle_{\mathrm{L}^2_{\phi-\psi}} - \langle \eta \mid \mathrm{d}\,\rho_\nu \wedge u\rangle_{\mathrm{L}^2_{\phi-\psi}} \\ &= \left\langle \rho_\nu\, \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta \mid u\right\rangle_{\mathrm{L}^2_{\phi-2\psi}} - \langle \eta \mid \mathrm{d}\,\rho_\nu \wedge u\rangle_{\mathrm{L}^2_{\phi-\psi}} \;, \end{aligned}$$

hence, by the Riesz representation theorem, there exists

$$\tilde{\eta} \;=:\; \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\left(\rho_\nu\, \eta\right) \in \mathrm{L}^2_{\phi-2\psi}\left(X; \wedge^{k-1} T^* X\right)$$

such that, for every $u \in \operatorname{dom} \mathrm{d} \subseteq \mathrm{L}^2_{\phi-2\psi}\left(X; \wedge^{k-1} T^* X\right)$, it holds

$$\langle \rho_\nu\, \eta \mid \mathrm{d}\, u\rangle_{\mathrm{L}^2_{\phi-\psi}} = \langle \tilde{\eta} \mid u\rangle_{\mathrm{L}^2_{\phi-2\psi}} \;.$$

Finally, note that, for every $u \in \operatorname{dom} \mathrm{d} \subseteq \mathrm{L}^2_{\phi-2\psi}\left(X; \wedge^{k-1} T^* X\right)$,

$$\begin{aligned} \left|\left\langle \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\left(\rho_\nu\, \eta\right) - \rho_\nu\, \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\,\eta \mid u\right\rangle_{\mathrm{L}^2_{\phi-2\psi}}\right| &= \left|\langle \rho_\nu\, \eta \mid \mathrm{d}\, u\rangle_{\mathrm{L}^2_{\phi-\psi}} - \left\langle \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta \mid \rho_\nu\, u\right\rangle_{\mathrm{L}^2_{\phi-2\psi}}\right| \\ &= \left|\langle \eta \mid \mathrm{d}\,\rho_\nu \wedge u\rangle_{\mathrm{L}^2_{\phi-\psi}}\right| \\ &\leq \|\eta\|_{\mathrm{L}^2_{\phi-\psi}} \cdot \left\|\mathrm{d}\,\rho_\nu \wedge u\right\|_{\mathrm{L}^2_{\phi-\psi}} \;, \end{aligned}$$

hence, by the Lebesgue dominated convergence theorem, $\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\left(\rho_\nu\, \eta\right) - \rho_\nu\, \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right\|_{\mathrm{L}^2_{\phi-2\psi}} \to 0$ as $\nu \to +\infty$. This shows that $\rho_\nu\, \eta \to \eta$ as $\nu \to +\infty$ with respect to the graph norm.

Hence, we may suppose that $\eta \in \operatorname{dom} \mathrm{d} \cap \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \subseteq \mathrm{L}^2_{\phi-\psi}\left(X; \wedge^k T^* X\right)$ has compact support. Let $\{\Phi_\varepsilon\}_{\varepsilon\in\mathbb{R}\setminus\{0\}} \subseteq \mathcal{C}^\infty\left(\mathbb{R}^n;\mathbb{R}\right)$ be a family of positive mollifiers, that is, for every $\varepsilon \in \mathbb{R}\setminus\{0\}$,

$$\Phi_\varepsilon \;:=\; \varepsilon^{-n}\, \Phi\left(\frac{\cdot}{\varepsilon}\right) \;\in\; \mathcal{C}^\infty\left(\mathbb{R}^n;\mathbb{R}\right) \;,$$

where (*i*) $\Phi \in \mathcal{C}^\infty_{\mathrm{c}}\left(\mathbb{R}^n;\mathbb{R}\right)$, (*ii*) $\int_{\mathbb{R}^n} \Phi \operatorname{vol}_{\mathbb{R}^n} = 1$, (*iii*) $\lim_{\varepsilon\to 0} \Phi_\varepsilon = \delta$, where δ is the Dirac delta function, and (*iv*) $\Phi \geq 0$. Consider the convolution

$$\{\eta * \Phi_\varepsilon\}_{\varepsilon\in\mathbb{R}} \subset \mathcal{C}^\infty_{\mathrm{c}}\left(X; \wedge^k T^* X\right) \;;$$

we prove that $\eta * \Phi_\varepsilon \to \eta$ as $\varepsilon \to 0$ with respect to the graph norm. Clearly,

$$\lim_{\varepsilon\to 0} \|\eta - \eta * \Phi_\varepsilon\|_{\mathrm{L}^2_{\phi-\psi}} \;=\; 0 \;.$$

Since $\mathrm{d}\left(\eta * \Phi_\varepsilon\right) = \mathrm{d}\,\eta * \Phi_\varepsilon$, one has that

$$\lim_{\varepsilon\to 0} \left\|\mathrm{d}\left(\eta * \Phi_\varepsilon\right) - \mathrm{d}\,\eta\right\|_{\mathrm{L}^2_\phi} \;=\; 0 \;.$$

Finally, write

$$\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \;=:\; \exp\left(-\psi\right)\left(\mathrm{d}^*_{0,\,0} + A_{\phi-\psi,\,\phi-2\psi}\right) \;,$$

where $\mathrm{d}^*_{0,\,0}$ is a differential operator with constant coefficients, and $A_{\phi-\psi,\,\phi-2\psi}$ is a differential operator of order zero defined, for every $v \in \mathrm{L}^2_{\phi-\psi}\left(X; \wedge^k T^* X\right)$, as

$$A_{\phi-\psi,\,\phi-2\psi}\,(v) \;:=\; \widetilde{\sum_{\substack{|J|=k-1\\|I|=k}}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix} \ell J \\ I \end{pmatrix} \frac{\partial\,(\phi-\psi)}{\partial x^\ell} \cdot \eta \,\mathrm{d}\, x^J \;;$$

hence

$$\begin{aligned}\left(\mathrm{d}^*_{0,\,0} + A_{\phi-\psi,\,\phi-2\psi}\right)(\eta * \Phi_\varepsilon) \quad &= \quad \left(\left(\mathrm{d}^*_{0,\,0} + A_{\phi-\psi,\,\phi-2\psi}\right)(\eta)\right) * \Phi_\varepsilon - \left(A_{\phi-\psi,\,\phi-2\psi}\eta\right) * \Phi_\varepsilon + A_{\phi-\psi,\,\phi-2\psi}\,(\eta * \Phi_\varepsilon) \\ &\to \quad \left(\mathrm{d}^*_{0,\,0} + A_{\phi-\psi,\,\phi-2\psi}\right)(\eta)\end{aligned}$$

as $\varepsilon \to 0$ in $\mathrm{L}^2_{\phi-2\psi}\left(X; \wedge^{k-1} T^* X\right)$; since η has compact support, it follows that

$$\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\,(\eta * \Phi_\varepsilon) \to \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\,(\eta)$$

as $\varepsilon \to 0$ in $\mathrm{L}^2_{\phi-2\psi}\left(X; \wedge^{k-1} T^* X\right)$, that is,

$$\lim_{\varepsilon\to 0} \left\| \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\,(\eta * \Phi_\varepsilon) - \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\,(\eta) \right\|_{\mathrm{L}^2_{\phi-2\psi}} \;=\; 0 \,.$$

`Step 4` – *If* $\|\eta\|^2_{\mathrm{L}^2_{\phi-\psi}} \leq C \cdot \left(\left\| \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta \right\|^2_{\mathrm{L}^2_{\phi-2\psi}} + \|\mathrm{d}\,\eta\|^2_{\mathrm{L}^2_\phi} \right)$ *holds for every* $\mathcal{C}^\infty_{\mathrm{c}}\left(X;\, \wedge^k T^* X\right)$*, then it holds for every* $\eta \in \operatorname{dom} \mathrm{d} \cap \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}$. Let $\eta \in \operatorname{dom} \mathrm{d} \cap \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}$. By `Step 3`, take $\left\{\eta_j\right\}_{j\in\mathbb{N}} \subset \mathcal{C}^\infty_{\mathrm{c}}\left(X; \wedge^k T^* X\right)$ such that $\eta_j \to \eta$ as $j \to +\infty$ in the graph norm. Since, for every $j \in \mathbb{N}$, one has

$$\|\eta_j\|^2_{\mathrm{L}^2_{\phi-\psi}} \;\leq\; C \cdot \left(\left\| \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta_j \right\|^2_{\mathrm{L}^2_{\phi-2\psi}} + \|\mathrm{d}\,\eta_j\|^2_{\mathrm{L}^2_\phi} \right) \,,$$

and since, for $j \to +\infty$,

$$\|\eta_j - \eta\|_{\mathrm{L}^2_{\phi-\psi}} \to 0 \,, \qquad \left\| \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta_j - \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta \right\|_{\mathrm{L}^2_{\phi-2\psi}} \to 0 \,, \qquad \text{and} \qquad \|\mathrm{d}\,\eta_j - \mathrm{d}\,\eta\|_{\mathrm{L}^2_\phi} \to 0 \,,$$

we get that also

$$\|\eta\|^2_{\mathrm{L}^2_{\phi-\psi}} \;\leq\; C \cdot \left(\left\| \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta \right\|^2_{\mathrm{L}^2_{\phi-2\psi}} + \|\mathrm{d}\,\eta\|^2_{\mathrm{L}^2_\phi} \right) \,.$$

`Step 5` – *Existence of a solution in* $\mathrm{L}^2_{\mathrm{loc}}\left(X; \wedge^k T^* X\right)$. We prove here that the operator

$$\mathrm{d} \colon \mathrm{L}^2_{\phi-2\psi}\left(X; \wedge^{k-1} T^* X\right) \dashrightarrow \ker\left(\mathrm{d} \colon \mathrm{L}^2_{\phi-\psi}\left(X; \wedge^k T^* X\right) \dashrightarrow \mathrm{L}^2_\phi\left(X; \wedge^{k+1} T^* X\right)\right)$$

is surjective, hence, for every $\eta \in \ker\left(\mathrm{d} \colon \mathrm{L}^2_{\phi-\psi}\left(X; \wedge^k T^* X\right) \dashrightarrow \mathrm{L}^2_\phi\left(X; \wedge^{k+1} T^* X\right)\right)$, the equation $\mathrm{d}\,\alpha = \eta$ has a solution α in $\mathrm{L}^2_{\phi-\psi}\left(X; \wedge^{k-1} T^* X\right) \subseteq \mathrm{L}^2_{\mathrm{loc}}\left(X; \wedge^{k-1} T^* X\right)$.

We recall, see, e.g., [Hör90, Lemma 4.1.1], that, given the Hilbert spaces $\left(H_1,\, \langle\cdot,\, \cdot\cdot\rangle_{H_1}\right)$ and $\left(H_2,\, \langle\cdot,\, \cdot\cdot\rangle_{H_2}\right)$, and a densely-defined closed operator $T \colon H_1 \dashrightarrow H_2$, whose adjoint is $T^* \colon H_2 \dashrightarrow H_1$, if $F \subseteq H_2$ is a closed subspace such that $\operatorname{im} T \subseteq F$, then the following conditions are equivalent:

(i) $\operatorname{im} T = F$;

(ii) there exists $C > 0$ such that, for every $y \in \operatorname{dom} T^* \cap F$,

$$\|y\|_{H_2} \;\leq\; C \cdot \|T^* y\|_{H_1} \,.$$

Hence, consider

$$\mathrm{d} \colon \mathrm{L}^2_{\phi-2\psi}\left(X; \wedge^{k-1} T^* X\right) \dashrightarrow \mathrm{L}^2_{\phi-\psi}\left(X; \wedge^k T^* X\right)$$

and

$$\begin{aligned}\mathrm{L}^2_{\phi-\psi}\left(X; \wedge^k T^* X\right) \;\supseteq\; F \quad &:= \quad \ker\left(\mathrm{d} \colon \mathrm{L}^2_{\phi-\psi}\left(X; \wedge^k T^* X\right) \dashrightarrow \mathrm{L}^2_\phi\left(X; \wedge^{k+1} T^* X\right)\right) \\ &\supseteq \quad \operatorname{im}\left(\mathrm{d} \colon \mathrm{L}^2_{\psi-2\psi}\left(X; \wedge^{k-1} T^* X\right) \dashrightarrow \mathrm{L}^2_{\phi-\psi}\left(X; \wedge^k T^* X\right)\right) \,.\end{aligned}$$

By `Step 4`, for every $\eta \in \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \cap F \subseteq \operatorname{dom} \mathrm{d} \cap \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}$, it holds that

$$\|\eta\|^2_{\mathrm{L}^2_{\phi-\psi}} \;\leq\; C\, \left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right\|^2_{\mathrm{L}^2_{\phi-2\psi}} \;,$$

from which it follows that

$$F \;=\; \operatorname{im}\left(\mathrm{d}\colon \mathrm{L}^2_{\psi-2\psi}\left(X;\wedge^{k-1}T^*X\right) \dashrightarrow \mathrm{L}^2_{\phi-\psi}\left(X;\wedge^{k}T^*X\right)\right) \;.$$

`Step 6` – *Sobolev regularity of the solutions with compact support.* We prove that, for every $\alpha \in \mathrm{L}^2\left(X;\wedge^{k-1}T^*X\right)$ with compact support, if $\mathrm{d}\,\alpha \in \mathrm{L}^2\left(X;\wedge^{k}T^*X\right)$ and $\mathrm{d}^*_{0,0}\alpha \in \mathrm{L}^2\left(X;\wedge^{k-2}T^*X\right)$, then $\alpha \in \mathrm{W}^{1,2}\left(X;\wedge^{k-1}T^*X\right)$. Indeed, take $\{\Phi_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ a family of positive mollifiers and, for every $\varepsilon \in \mathbb{R}$, consider $\alpha * \Phi_\varepsilon \in \mathcal{C}^\infty_{\mathrm{c}}\left(X;\wedge^{k-1}T^*X\right)$; by Proposition 3.66 with $\phi := 0$ and $\psi := 0$, we get that, for any multi-index I such that $|I| = k-1$ and for any $\ell \in \{1,\ldots,n\}$,

$$\int_X \left|\frac{\partial\left(\alpha_I * \Phi_\varepsilon\right)}{\partial x^\ell}\right|^2 \mathrm{vol} \;\leq\; C\cdot\left(\left\|\mathrm{d}^*_{0,\,0}\left(\alpha*\Phi_\varepsilon\right)\right\|^2_{\mathrm{L}^2} + \left\|\mathrm{d}\left(\alpha*\Phi_\varepsilon\right)\right\|^2_{\mathrm{L}^2}\right) \;,$$

where $C =: C(k,n)$ is a constant depending just on k and n; since, for every multi-index I such that $|I| = k-1$, and for every $\ell \in \{1,\ldots,n\}$, it holds that

$$\lim_{\varepsilon\to0}\int_X \left|\frac{\partial\left(\alpha_I * \Phi_\varepsilon\right)}{\partial x^\ell} - \frac{\partial\alpha_I}{\partial x^\ell}\right|^2 \mathrm{vol} \;=\; \lim_{\varepsilon\to0}\left\|\mathrm{d}^*_{0,\,0}\left(\alpha*\Phi_\varepsilon\right) - \mathrm{d}^*_{0,\,0}\alpha\right\|_{\mathrm{L}^2} \;=\; \lim_{\varepsilon\to0}\left\|\mathrm{d}\left(\alpha*\Phi_\varepsilon\right) - \mathrm{d}\,\alpha\right\|_{\mathrm{L}^2} \;=\; 0\,,$$

we get that

$$\int_X \left|\frac{\partial\alpha_I}{\partial x^\ell}\right|^2 \mathrm{vol} \;\leq\; C\cdot\left(\left\|\mathrm{d}^*_{0,\,0}\alpha\right\|^2_{\mathrm{L}^2} + \left\|\mathrm{d}\,\alpha\right\|^2_{\mathrm{L}^2}\right) \;,$$

proving the claim.

`Step 7` – *Regularization of the solution.* By `Step 5`, if $\eta \in \wedge^k X$ is such that $\mathrm{d}\,\eta = 0$, then the equation $\mathrm{d}\,\alpha = \eta$ has a solution $\alpha \in \mathrm{L}^2_{\mathrm{loc}}\left(X;\wedge^{k-1}T^*X\right)$; we prove that actually $\alpha \in \wedge^{k-1}X$.

Note that we may suppose that the solution $\alpha \in \mathrm{L}^2_{\mathrm{loc}}\left(X;\wedge^{k-1}T^*X\right)$ satisfies

$$\alpha \;\in\; \left(\ker\mathrm{d}\right)^{\perp_{\mathrm{L}^2_{\mathrm{loc}}\left(X;\wedge^{k-1}T^*X\right)}} \;=\; \overline{\operatorname{im}\mathrm{d}^*_{0,\,0}} \;=\; \operatorname{im}\mathrm{d}^*_{0,\,0} \;\subseteq\; \ker\mathrm{d}^*_{0,\,0}\;;$$

hence, α satisfies the system of differential equation

$$\begin{cases} \mathrm{d}\,\alpha &=& \eta \\ \mathrm{d}^*_{0,\,0}\alpha &=& 0 \end{cases} \;.$$

We prove, by induction on $s \in \mathbb{N}$, that $\alpha \in \mathrm{W}^{s,2}_{\mathrm{loc}}\left(X;\wedge^{k-1}T^*X\right)$ for every $s \in \mathbb{N}$. Indeed, we have by `Step 5` that $\alpha \in \mathrm{W}^{0,2}_{\mathrm{loc}}\left(X;\wedge^{k-1}T^*X\right) = \mathrm{L}^2_{\mathrm{loc}}\left(X;\wedge^{k-1}T^*X\right)$. Suppose now that $\alpha \in \mathrm{W}^{s,2}_{\mathrm{loc}}\left(X;\wedge^{k-1}T^*X\right)$ and prove that $\alpha \in \mathrm{W}^{s+1,2}_{\mathrm{loc}}\left(X;\wedge^{k-1}T^*X\right)$. Clearly, $\eta \in \wedge^k X \subseteq \mathrm{W}^{\sigma,2}_{\mathrm{loc}}\left(X;\wedge^{k}T^*X\right)$ for every $\sigma \in \mathbb{N}$. Take K a compact subset of X, and choose $\widehat{\chi} \in \mathcal{C}^\infty_{\mathrm{c}}\left(X;\mathbb{R}\right)$ such that $\operatorname{supp}\widehat{\chi} \supset K$. For any multi-index $L =: (\ell_1,\ldots,\ell_n) \in \mathbb{N}^n$ such that $\ell_1+\cdots+\ell_n = s$, being

$$\mathrm{d}\left(\widehat{\chi}\cdot\frac{\partial^s\alpha}{\partial^{\ell_1}x^1\cdots\partial^{\ell_n}x^n}\right) \;=\; \mathrm{d}\,\widehat{\chi}\wedge\frac{\partial^s\alpha}{\partial^{\ell_1}x^1\cdots\partial^{\ell_n}x^n} + \widehat{\chi}\cdot\frac{\partial^s\eta}{\partial^{\ell_1}x^1\cdots\partial^{\ell_n}x^n} \;\in\; \mathrm{L}^2\left(K;\wedge^kT^*K\right)$$

and

$$\mathrm{d}^*_{0,\,0}\left(\widehat{\chi}\cdot\frac{\partial^s\alpha}{\partial^{\ell_1}x^1\cdots\partial^{\ell_n}x^n}\right) \;=\; -\widetilde{\sum_{\substack{|J|=k-1\\|I|=k}}}\sum_{\ell=1}^{n}\operatorname{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix}\frac{\partial\widehat{\chi}}{\partial x^\ell}\cdot\frac{\partial^s\alpha_I}{\partial^{\ell_1}x^1\cdots\partial^{\ell_n}x^n}\,\mathrm{d}\,x^J \;\in\; \mathrm{L}^2\left(K;\wedge^{k-2}T^*K\right)$$

we get that $\widehat{\chi}\cdot\frac{\partial^s\alpha}{\partial^{\ell_1}x^1\cdots\partial^{\ell_n}x^n} \in \mathrm{W}^{1,2}\left(K;\wedge^{k-1}T^*K\right)$, that is, $\alpha \in \mathrm{W}^{s+1,2}\left(K;\wedge^{k-1}T^*K\right)$. Hence, we get that $\alpha \in \mathrm{W}^{s+1,2}_{\mathrm{loc}}\left(X;\wedge^{k-1}T^*X\right)$.

Since $\mathrm{W}^{\sigma,2}_{\mathrm{loc}}\left(X;\wedge^{k-1}T^*X\right) \hookrightarrow \mathcal{C}^m\left(X;\wedge^{k-1}T^*X\right)$ for every $0 \leq m < \sigma - \frac{n}{2}$, see, e.g., [GT01, Corollary 7.11], we get that $\alpha \in \wedge^{k-1}X$, concluding the proof of the theorem. □

As a straightforward corollary, we get the following vanishing theorem for the higher-degree de Rham cohomology groups of a strictly p-convex domain in $\mathbb{R}^n$, [AC12, Theorem 3.1]; for a different proof, involving Morse theory, compare [Sha86, Theorem 1] by J.-P. Sha, and [Wu87, Theorem 1] by H. Wu, see also [HL11, Proposition 5.7].

Theorem 3.68 ([AC12, Theorem 3.1], see [Sha86, Theorem 1], [Wu87, Theorem 1], [HL11, Proposition 5.7])**.** *Let X be a strictly p-convex domain in $\mathbb{R}^n$. Then $H^k_{dR}(X;\mathbb{R}) = \{0\}$ for every $k \geq p$.*

Bibliography

[AB90] L. Alessandrini and G. Bassanelli, *Small deformations of a class of compact non-Kähler manifolds*, Proc. Amer. Math. Soc. **109** (1990), no. 4, 1059–1062.

[AB91] ———, *Compact p-Kähler manifolds*, Geom. Dedicata **38** (1991), no. 2, 199–210.

[AB96] ———, *The class of compact balanced manifolds is invariant under modifications*, Complex analysis and geometry (Trento, 1993), Lecture Notes in Pure and Appl. Math., vol. 173, Dekker, New York, 1996, pp. 1–17.

[Aba88] M. Abate, *Annular bundles*, Pacific J. Math. **134** (1988), no. 1, 1–26.

[ABDM11] A. Andrada, M. L. Barberis, and I. G. Dotti Miatello, *Classification of abelian complex structures on* 6*-dimensional Lie algebras*, J. Lond. Math. Soc. (2) **83** (2011), no. 1, 232–255.

[ABDMO05] A. Andrada, M. L. Barberis, I. G. Dotti Miatello, and G. P. Ovando, *Product structures on four dimensional solvable Lie algebras*, Homology Homotopy Appl. **7** (2005), no. 1, 9–37.

[AC12] D. Angella and S. Calamai, *A vanishing result for strictly p-convex domains*, Ann. Mat. Pura Appl. **Online First,** `http://link.springer.com/article/10.1007/s10231-012-0315-5` (2012), 16 pages, DOI: 10.1007/s00222-012-0406-3.

[Aep65] A. Aeppli, *On the cohomology structure of Stein manifolds*, Proc. Conf. Complex Analysis (Minneapolis, Minn., 1964), Springer, Berlin, 1965, pp. 58–70.

[AG62] A. Andreotti and H. Grauert, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.

[AGH63] L. Auslander, L. W. Green, and F. J. Hahn, *Flows on homogeneous spaces*, With the assistance of L. Markus and W. Massey, and an appendix by L. Greenberg. Annals of Mathematics Studies, No. 53, Princeton University Press, Princeton, N.J., 1963.

[AGS97] E. Abbena, S. Garbiero, and S. Salamon, *Hermitian geometry on the Iwasawa manifold*, Boll. Un. Mat. Ital. B (7) **11** (1997), no. 2, suppl., 231–249.

[AL94] M. Audin and J. Lafontaine (eds.), *Holomorphic curves in symplectic geometry*, Progress in Mathematics, vol. 117, Birkhäuser Verlag, Basel, 1994.

[Ale98] L. Alessandrini, *Correnti positive: uno strumento per l'analisi globale su varietà complesse*, Rend. Sem. Mat. Fis. Milano **68** (1998), no. 1, 59–120 (2001).

[Ale10] ———, *Correnti positive e varietà complesse*, Rend. Mat. Appl. (7) **30** (2010), no. 2, 145–181.

[Ale11] ———, *Classes of compact non-Kähler manifolds*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 19-20, 1089–1092.

[AMT09] D. V. Alekseevskiĭ, K. Medori, and A. Tomassini, *Homogeneous para-Kählerian Einstein manifolds*, Uspekhi Mat. Nauk **64** (2009), no. 1(385), 3–50, Translation in *Russian Math. Surveys* **64** (2009), no. 1, 1–43.

[AN54] Y. Akizuki and S. Nakano, *Note on Kodaira-Spencer's proof of Lefschetz theorems*, Proc. Japan Acad. **30** (1954), no. 4, 266–272.

[AN71] A. Andreotti and F. Norguet, *Cycles of algebraic manifolds and $\partial\bar{\partial}$-cohomology*, Ann. Scuola Norm. Sup. Pisa (3) **25** (1971), no. 1, 59–114.

[And63] A. Andreotti, *Théorèmes de dépendance algébrique sur les espaces complexes pseudo-concaves*, Bull. Soc. Math. France **91** (1963), 1–38.

[Ang11] D. Angella, *The Cohomologies of the Iwasawa Manifold and of Its Small Deformations*, J. Geom. Anal. **Online First,** `http://www.springerlink.com/content/m7547621g7722265/` (2011), 24 pages, DOI: 10.1007/s12220-011-9291-z.

[AR12] D. Angella and F. A. Rossi, *Cohomology of* $\mathbf{D}$*-complex manifolds*, Differ. Geom. Appl. **30** (2012), no. 5, 530–547.

[AS05] A. Andrada and S. Salamon, *Complex product structures on Lie algebras*, Forum Math. **17** (2005), no. 2, 261–295.

[AT11] D. Angella and A. Tomassini, *On cohomological decomposition of almost-complex manifolds and deformations*, J. Symplectic Geom. **9** (2011), no. 3, 403–428.

[AT12a] ———, *On the cohomology of almost-complex manifolds*, Internat. J. Math. **23** (2012), no. 2, 1250019, 25.

[AT12b] ———, *On the* $\partial\overline{\partial}$*-Lemma and Bott-Chern cohomology*, Invent. Math. **Online First,** `http://www.springerlink.com/content/h2t746g084105521/` (2012), 11 pages, DOI: 10.1007/s00222-012-0406-3.

[AT12c] ———, *Symplectic manifolds and cohomological decomposition*, `arXiv:1211.2565v1 [math.SG]`, 2012.

[ATZ12] D. Angella, A. Tomassini, and W. Zhang, *On cohomological decomposition of almost-Kähler structures*, `arXiv:1211.2928v1 [math.DG]`, to appear in *Proc. Amer. Math. Soc.*, 2012.

[AV65a] A. Andreotti and E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 25, 81–130.

[AV65b] ———, *Erratum to: Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 27, 153–155.

[Bai54] W. L. Baily, Jr., *On the quotient of an analytic manifold by a group of analytic homeomorphisms*, Proc. Nat. Acad. Sci. U. S. A. **40** (1954), no. 9, 804–808.

[Bai56] ———, *The decomposition theorem for* V*-manifolds*, Amer. J. Math. **78** (1956), no. 4, 862–888.

[Bal06] W. Ballmann, *Lectures on Kähler manifolds*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2006.

[Bas99] G. Bassanelli, *Area-minimizing Riemann surfaces on the Iwasawa manifold*, J. Geom. Anal. **9** (1999), no. 2, 179–201.

[BC65] R. Bott and S. S. Chern, *Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections*, Acta Math. **114** (1965), no. 1, 71–112.

[BDMM95] M. L. Barberis, I. G. Dotti Miatello, and R. J. Miatello, *On certain locally homogeneous Clifford manifolds*, Ann. Global Anal. Geom. **13** (1995), no. 3, 289–301.

[Bel00] F. A. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), no. 1, 1–40.

[BFV11] E. Buzano, A. Fino, and L. Vezzoni, *The Calabi-Yau equation for* T^2*-bundles over* $\mathbb{T}^2$*: the non-Lagrangian case*, Rend. Semin. Mat. Univ. Politec. Torino **69** (2011), no. 3, 281–298.

[BG88] Ch. Benson and C. S. Gordon, *Kähler and symplectic structures on nilmanifolds*, Topology **27** (1988), no. 4, 513–518.

[BG90] ———, *Kähler structures on compact solvmanifolds*, Proc. Amer. Math. Soc. **108** (1990), no. 4, 971–980.

[BHPVdV04] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 4, Springer-Verlag, Berlin, 2004.

[Big69] B. Bigolin, *Gruppi di Aeppli*, Ann. Scuola Norm. Sup. Pisa (3) **23** (1969), no. 2, 259–287.

[Big70] ———, *Osservazioni sulla coomologia del $\partial\bar{\partial}$*, Ann. Scuola Norm. Sup. Pisa (3) **24** (1970), no. 3, 571–583.

[Bis89] J.-M. Bismut, *A local index theorem for non-Kähler manifolds*, Math. Ann. **284** (1989), no. 4, 681–699.

[Bis11a] ———, *Hypoelliptic Laplacian and Bott-Chern cohomology*, preprint (Orsay), 2011.

[Bis11b] ———, *Laplacien hypoelliptique et cohomologie de Bott-Chern*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 1-2, 75–80.

[Boc45] S. Bochner, *Compact groups of differentiable transformations*, Ann. of Math. (2) **46** (1945), no. 3, 372–381.

[Boc09] C. Bock, *On low-dimensional solvmanifolds*, `arXiv:0903.2926v4 [math.DG]`, 2009.

[Bog78] F. A. Bogomolov, *Hamiltonian Kählerian manifolds*, Dokl. Akad. Nauk SSSR **243** (1978), no. 5, 1101–1104.

[Bou04] S. Boucksom, *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 1, 45–76.

[Bry88] J.-L. Brylinski, *A differential complex for Poisson manifolds*, J. Differential Geom. **28** (1988), no. 1, 93–114.

[BT00] I. K. Babenko and I. A. Taĭmanov, *On nonformal simply connected symplectic manifolds*, Sibirsk. Mat. Zh. **41** (2000), no. 2, 253–269, Translation in *Siberian Math. J.* **41** (2000), no. 2, 204–217.

[Buc99] N. Buchdahl, *On compact Kähler surfaces*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 1, 287–302.

[Buc08] ———, *Algebraic deformations of compact Kähler surfaces. II*, Math. Z. **258** (2008), no. 3, 493–498.

[Cal57] E. Calabi, *On Kähler manifolds with vanishing canonical class*, Algebraic geometry and topology. A symposium in honor of S. Lefschetz, Princeton University Press, Princeton, N. J., 1957, pp. 78–89.

[Cam91] F. Campana, *The class $\mathcal{C}$ is not stable by small deformations*, Math. Ann. **290** (1991), no. 1, 19–30.

[Cav05] G. R. Cavalcanti, *New aspects of the dd^c-lemma*, `arXiv:math/0501406 [math.DG]`, 2005, Oxford University D. Phil thesis.

[Cav07] ———, *Introduction to generalized complex geometry*, Publicações Matemáticas do IMPA. [IMPA Mathematical Publications], Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2007, 26o Colóquio Brasileiro de Matemática. [26th Brazilian Mathematics Colloquium].

[CdS01] A. Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics, vol. 1764, Springer-Verlag, Berlin, 2001.

[CF01] S. Console and A. Fino, *Dolbeault cohomology of compact nilmanifolds*, Transform. Groups **6** (2001), no. 2, 111–124.

[CF11] ———, *On the de Rham cohomology of solvmanifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **X** (2011), no. 4, 801–818.

[CFAG96] V. Cruceanu, P. Fortuny Ayuso, and P. M. Gadea, *A survey on paracomplex geometry*, Rocky Mountain J. Math. **26** (1996), no. 1, 83–115.

[CFGU00] L. A. Cordero, M. Fernández, A. Gray, and L. Ugarte, *Compact nilmanifolds with nilpotent complex structures: Dolbeault cohomology*, Trans. Amer. Math. Soc. **352** (2000), no. 12, 5405–5433.

[CFM08] G. R. Cavalcanti, M. Fernández, and V. Muñoz, *Symplectic resolutions, Lefschetz property and formality*, Adv. Math. **218** (2008), no. 2, 576–599.

[CFUG97] L. A. Cordero, M. Fernández, L. Ugarte, and A. Gray, *A general description of the terms in the Frölicher spectral sequence*, Differential Geom. Appl. **7** (1997), no. 1, 75–84.

[Cho49] W.-L. Chow, *On compact complex analytic varieties*, Amer. J. Math. **71** (1949), no. 4, 893–914.

[CM09] V. Cortés and T. Mohaupt, *Special geometry of Euclidean supersymmetry. III. The local r-map, instantons and black holes*, J. High Energy Phys. (2009), no. 7, 066, 64.

[CMMS04] V. Cortés, C. Mayer, T. Mohaupt, and F. Saueressig, *Special geometry of Euclidean supersymmetry. I. Vector multiplets*, J. High Energy Phys. (2004), no. 3, 028, 73 pp. (electronic).

[CMMS05] ———, *Special geometry of Euclidean supersymmetry. II. Hypermultiplets and the c-map*, J. High Energy Phys. (2005), no. 6, 025, 37 pp. (electronic).

[Con06] S. Console, *Dolbeault cohomology and deformations of nilmanifolds*, Rev. Un. Mat. Argentina **47** (2006), no. 1, 51–60 (2007).

[COUV12] M. Ceballos, A. Otal, L. Ugarte, and R. Villacampa, *Classification of complex structures on 6-dimensional nilpotent Lie algebras*, `arXiv:1111.5873v3 [math.DG]`, 2012.

[CS02] S. Chiossi and S. Salamon, *The intrinsic torsion of* $\mathrm{SU}(3)$ *and* G_2 *structures*, Differential geometry, Valencia, 2001, World Sci. Publ., River Edge, NJ, 2002, pp. 115–133.

[CT07] D. Conti and A. Tomassini, *Special symplectic six-manifolds*, Q. J. Math. **58** (2007), no. 3, 297–311.

[dAFdLM92] L. C. de Andrés, M. Fernández, M. de León, and J. J. Mencía, *Some six-dimensional compact symplectic and complex solvmanifolds*, Rend. Mat. Appl. (7) **12** (1992), no. 1, 59–67.

[Dar82] G. Darboux, *Sur le problème de Pfaff*, Bull. Sci. Math. **6** (1882), 14–36, 49–68.

[dBM10] P. de Bartolomeis and F. Meylan, *Intrinsic deformation theory of* CR *structures*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **9** (2010), no. 3, 459–494.

[dBT06] P. de Bartolomeis and A. Tomassini, *On solvable generalized Calabi-Yau manifolds*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 5, 1281–1296.

[dBT12] ———, *Exotic deformations of Calabi-Yau manifolds*, to appear in *Ann. Inst. Fourier (Grenoble)*, 2012.

[Dem86] J.-P. Demailly, *Sur l'identité de Bochner-Kodaira-Nakano en géométrie hermitienne*, Séminaire d'analyse P. Lelong-P. Dolbeault-H. Skoda, années 1983/1984, Lecture Notes in Math., vol. 1198, Springer, Berlin, 1986, pp. 88–97.

[Dem12] ———, *Complex Analytic and Differential Geometry*, `http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`, 2012, Version of Thursday June 21, 2012.

[DGMS75] P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), no. 3, 245–274.

[DK00] J. J. Duistermaat and J. A. C. Kolk, *Lie groups*, Universitext, Springer-Verlag, Berlin, 2000.

[DLZ10] T. Drăghici, T.-J. Li, and W. Zhang, *Symplectic forms and cohomology decomposition of almost complex four-manifolds*, Int. Math. Res. Not. IMRN (2010), no. 1, 1–17.

[DLZ11] ———, *On the J-anti-invariant cohomology of almost complex* 4*-manifolds*, to appear in *Q. J. Math.*, `http://qjmath.oxfordjournals.org/content/early/2011/12/06/qmath.har034.`, 2011, DOI: 10.1093/qmath/har034.

[DLZ12] ———, *Geometry of tamed almost complex structures on* 4*-dimensional manifolds*, Fifth International Congress of Chinese Mathematicians. Part 1, 2, AMS/IP Stud. Adv. Math., 51, pt. 1, vol. 2, Amer. Math. Soc., Providence, RI, 2012, pp. 233–251.

[Dol53] P. Dolbeault, *Sur la cohomologie des variétés analytiques complexes*, C. R. Acad. Sci. Paris **236** (1953), 175–177.

[Don06] S. K. Donaldson, *Two-forms on four-manifolds and elliptic equations*, Inspired by S. S. Chern, Nankai Tracts Math., vol. 11, World Sci. Publ., Hackensack, NJ, 2006, pp. 153–172.

[DP04] J.-P. Demailly and M. Păun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. (2) **159** (2004), no. 3, 1247–1274.

[dR84] G. de Rham, *Differentiable manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 266, Springer-Verlag, Berlin, 1984, Forms, currents, harmonic forms, Translated from the French by F. R. Smith, With an introduction by S. S. Chern.

[dSSST06] G. della Sala, A. Saracco, A. Simioniuc, and G. Tomassini, *Lectures on complex analysis and analytic geometry*, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], vol. 3, Edizioni della Normale, Pisa, 2006.

[DZ11] T. Drăghici and W. Zhang, *A note on exact forms on almost complex manifolds*, `arXiv:1111.7287v1 [math.SG]`, 2011.

[EFV12] N. Enrietti, A. Fino, and L. Vezzoni, *Tamed Symplectic forms and Strong Kahler with torsion metrics*, J. Symplectic Geom. **10** (2012), no. 2, 203–224.

[Ehr47] C. Ehresmann, *Sur les espaces fibrés différentiables*, C. R. Acad. Sci. Paris **224** (1947), 1611–1612.

[FdLS96] M. Fernández, M. de León, and M. Saralegui, *A six-dimensional compact symplectic solvmanifold without Kähler structures*, Osaka J. Math. **33** (1996), no. 1, 19–35.

[Fed69] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[FG86] M. Fernández and A. Gray, *The Iwasawa manifold*, Differential geometry, Peñíscola 1985, Lecture Notes in Math., vol. 1209, Springer, Berlin, 1986, pp. 157–159.

[FG04] A. Fino and G. Grantcharov, *Properties of manifolds with skew-symmetric torsion and special holonomy*, Adv. Math. **189** (2004), no. 2, 439–450.

[FLSV11] A. Fino, Y. Y. Li, S. Salamon, and L. Vezzoni, *The Calabi-Yau equation on* 4*-manifolds over* 2*-tori*, to appear in *Trans. Am. Math. Soc.*, 2011.

[FLY12] J. Fu, J. Li, and S.-T. Yau, *Balanced metrics on non-Kähler Calabi-Yau threefolds*, J. Differential Geom. **90** (2012), no. 1, 81–129.

[FM08] M. Fernández and V. Muñoz, *An* 8*-dimensional nonformal, simply connected, symplectic manifold*, Ann. of Math. (2) **167** (2008), no. 3, 1045–1054.

[FMS03] M. Fernández, V. Muñoz, and J. A. Santisteban, *Cohomologically Kähler manifolds with no Kähler metrics*, Int. J. Math. Math. Sci. (2003), no. 52, 3315–3325.

[FOT08] Y. Félix, J. Oprea, and D. Tanré, *Algebraic models in geometry*, Oxford Graduate Texts in Mathematics, vol. 17, Oxford University Press, Oxford, 2008.

[FPS04] A. Fino, M. Parton, and S. Salamon, *Families of strong KT structures in six dimensions*, Comment. Math. Helv. **79** (2004), no. 2, 317–340.

[Fra11] M. G. Franzini, *Deformazioni di varietà bilanciate e loro proprietà coomologiche*, Tesi di Laurea Magistrale in Matematica, Università degli Studi di Parma, 2011.

[Frö55] A. Frölicher, *Relations between the cohomology groups of Dolbeault and topological invariants*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), no. 9, 641–644.

[FT10] A. Fino and A. Tomassini, *On some cohomological properties of almost complex manifolds*, J. Geom. Anal. **20** (2010), no. 1, 107–131.

[Fub04] G. Fubini, *Sulle metriche definite da una forma Hermitiana. Nota*, Ven. Ist. Atti 63 [(8) 6] (1904), 501–513.

[Fuj78] A. Fujiki, *On automorphism groups of compact Kähler manifolds*, Invent. Math. **44** (1978), no. 3, 225–258.

[FY11] J. Fu and S.-T. Yau, *A note on small deformations of balanced manifolds*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 13-14, 793–796.

[Gau77] P. Gauduchon, *Le théorème de l'excentricité nulle*, C. R. Acad. Sci. Paris Sér. A-B **285** (1977), no. 5, A387–A390.

[GH80] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (4) **123** (1980), no. 1, 35–58.

[GH94] Ph. A. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1994, Reprint of the 1978 original.

[Gri66] Ph. A. Griffiths, *The extension problem in complex analysis. II. Embeddings with positive normal bundle*, Amer. J. Math. **88** (1966), no. 2, 366–446.

[Gro85] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347.

[GT01] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition.

[Gua04] M. Gualtieri, *Generalized complex geometry*, `arXiv:math/0401221v1 [math.DG]`, 2004, Oxford University DPhil thesis.

[Gua07] D. Guan, *Modification and the cohomology groups of compact solvmanifolds*, Electron. Res. Announc. Amer. Math. Soc. **13** (2007), 74–81.

[Gua11] M. Gualtieri, *Generalized complex geometry*, Ann. of Math. (2) **174** (2011), no. 1, 75–123.

[Gui01] V. Guillemin, *Symplectic Hodge theory and the* $\mathrm{d}\,\delta$*-lemma*, preprint (Massachusetts Insitute of Technology), 2001.

[Gun90] R. C. Gunning, *Introduction to holomorphic functions of several variables. Vol. I*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1990, Function theory.

[Han57] J.-i. Hano, *On Kaehlerian homogeneous spaces of unimodular Lie groups*, Amer. J. Math. **79** (1957), 885–900.

[Has89] K. Hasegawa, *Minimal models of nilmanifolds*, Proc. Amer. Math. Soc. **106** (1989), no. 1, 65–71.

[Has06] ———, *A note on compact solvmanifolds with Kähler structures*, Osaka J. Math. **43** (2006), no. 1, 131–135.

[Hat60] A. Hattori, *Spectral sequence in the de Rham cohomology of fibre bundles*, J. Fac. Sci. Univ. Tokyo Sect. I **8** (1960), no. 2, 289–331 (1960).

[Hir62] H. Hironaka, *An example of a non-Kählerian complex-analytic deformation of Kählerian complex structures*, Ann. of Math. (2) **75** (1962), no. 1, 190–208.

[Hit03] N. J. Hitchin, *Generalized Calabi-Yau manifolds*, Q. J. Math. **54** (2003), no. 3, 281–308.

[Hit10] ———, *Generalized geometry—an introduction*, Handbook of pseudo-Riemannian geometry and supersymmetry, IRMA Lect. Math. Theor. Phys., vol. 16, Eur. Math. Soc., Zürich, 2010, pp. 185–208.

[HL83] F. R. Harvey and H. B. Lawson, Jr., *An intrinsic characterization of Kähler manifolds*, Invent. Math. **74** (1983), no. 2, 169–198.

[HL11] ———, *The foundations of p-convexity and p-plurisubharmonicity in riemannian geometry*, `arXiv:1111.3895 [math.DG]`, 2011.

[HL12] ———, *Geometric plurisubharmonicity and convexity: An introduction*, Adv. Math. **230** (2012), no. 4-6, 2428–2456.

[HMT11] R. K. Hind, C. Medori, and A. Tomassini, *On non-pure forms on almost complex manifolds*, to appear in *Proc. Amer. Math. Soc.*, 2011.

[Hod35] W. V. D. Hodge, *Harmonic integrals associated with algebraic varieties*, Proc. London math. Soc. (2) **39** (1935), 249–271.

[Hod89] W. V. D. Hodge, *The theory and applications of harmonic integrals*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1989, Reprint of the 1941 original, With a foreword by Michael Atiyah.

[Hör65] L. Hörmander, *L^2 estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math. **113** (1965), no. 1, 89–152.

[Hör90] ______, *An introduction to complex analysis in several variables*, third ed., North-Holland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990.

[Hum78] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York, 1978, Second printing, revised.

[Huy05] D. Huybrechts, *Complex geometry*, Universitext, Springer-Verlag, Berlin, 2005.

[Joy96a] D. D. Joyce, *Compact 8-manifolds with holonomy* Spin(7), Invent. Math. **123** (1996), no. 3, 507–552.

[Joy96b] ______, *Compact Riemannian 7-manifolds with holonomy G_2. I, II*, J. Differ. Geom. **43** (1996), no. 2, 291–328, 329–375.

[Joy99] ______, *A new construction of compact 8-manifolds with holonomy* Spin(7), J. Differ. Geom. **53** (1999), no. 1, 89–130.

[Joy00] ______, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[Joy07] ______, *Riemannian holonomy groups and calibrated geometry*, Oxford Graduate Texts in Mathematics, vol. 12, Oxford University Press, Oxford, 2007.

[JY93] J. Jost and S.-T. Yau, *A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry*, Acta Math. **170** (1993), no. 2, 221–254.

[JY94] ______, *Correction to: "A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry" [Acta Math.* **170** *(1993), no. 2, 221–254; MR1226528 (94g:58053)]*, Acta Math. **173** (1994), no. 2, 307.

[Käh33] E. Kähler, *Über eine bemerkenswerte Hermitesche Metrik*, Abh. Math. Sem. Univ. Hamburg **9** (1933), no. 1, 176–186.

[Kas12] H. Kasuya, *Techniques of computations of Dolbeault cohomology of solvmanifolds*, to appear in *Math. Z.*, `arXiv:1107.4761v3 [math.DG]`, 2012, DOI: 10.1007/s00209-012-1013-0.

[KMW10] Y.-H. Kim, R. J. McCann, and M. Warren, *Pseudo-Riemannian geometry calibrates optimal transportation*, Math. Res. Lett. **17** (2010), no. 6, 1183–1197.

[KNS58] K. Kodaira, L. Nirenberg, and D. C. Spencer, *On the existence of deformations of complex analytic structures*, Ann. of Math. (2) **68** (1958), no. 2, 450–459.

[Kod54] K. Kodaira, *On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties)*, Ann. of Math. (2) **60** (1954), 28–48.

[Kod63] ______, *On compact analytic surfaces. III*, Ann. of Math. (2) **78** (1963), no. 1, 1–40.

[Kod64] ______, *On the structure of compact complex analytic surfaces. I*, Amer. J. Math. **86** (1964), 751–798.

[Kod05] ______, *Complex manifolds and deformation of complex structures*, english ed., Classics in Mathematics, Springer-Verlag, Berlin, 2005, Translated from the 1981 Japanese original by Kazuo Akao.

[Koo11] R. Kooistra, *Regulator currents on compact complex manifolds*, ProQuest LLC, Ann Arbor, MI, 2011, Thesis (Ph.D.)–University of Alberta (Canada).

[Kos50] J.-L. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France **78** (1950), 65–127.

[Kos85] ______, *Crochet de Schouten-Nijenhuis et cohomologie*, Astérisque (1985), no. Numero Hors Serie, 257–271, The mathematical heritage of Élie Cartan (Lyon, 1984).

[Kra10] M. Krahe, *Para-pluriharmonic maps and twistor spaces*, Handbook of pseudo-Riemannian geometry and supersymmetry, IRMA Lect. Math. Theor. Phys., vol. 16, Eur. Math. Soc., Zürich, 2010, pp. 497–557.

[KS58] K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures. I, II*, Ann. of Math. (2) **67** (1958), no. 2, 3, 328–401, 403–466.

[KS60] ______, *On deformations of complex analytic structures. III. Stability theorems for complex structures*, Ann. of Math. (2) **71** (1960), no. 1, 43–76.

[KS04] G. Ketsetzis and S. Salamon, *Complex structures on the Iwasawa manifold*, Adv. Geom. **4** (2004), no. 2, 165–179.

[Kur62] M. Kuranishi, *On the locally complete families of complex analytic structures*, Ann. of Math. (2) **75** (1962), no. 3, 536–577.

[Kur65] ______, *New proof for the existence of locally complete families of complex structures*, Proc. Conf. Complex Analysis (Minneapolis, 1964), Springer, Berlin, 1965, pp. 142–154.

[Lam99] A. Lamari, *Courants kählériens et surfaces compactes*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 1, vii, x, 263–285.

[Lee04] J. Lee, *Family Gromov-Witten invariants for Kähler surfaces*, Duke Math. J. **123** (2004), no. 1, 209–233.

[Lej10] M. Lejmi, *Stability under deformations of extremal almost-Kähler metrics in dimension* 4, Math. Res. Lett. **17** (2010), no. 4, 601–612.

[Lin11] Y. Lin, *Currents, primitive cohomology classes and symplectic Hodge theory*, `arXiv:1112.2442v1 [math.SG]`, 2011.

[LO94] G. Lupton and J. Oprea, *Symplectic manifolds and formality*, J. Pure Appl. Algebra **91** (1994), no. 1-3, 193–207.

[LP92] C. LeBrun and Y. S. Poon, *Twistors, Kähler manifolds, and bimeromorphic geometry. II*, J. Amer. Math. Soc. **5** (1992), no. 2, 317–325.

[LT12] T.-J. Li and A. Tomassini, *Almost Kähler structures on four dimensional unimodular Lie algebras*, J. Geom. Phys. **62** (2012), no. 7, 1714–1731.

[Lus10] L. Lussardi, *A Stampacchia-type inequality for a fourth-order elliptic operator on Kähler manifolds and applications*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **21** (2010), no. 2, 159–173.

[LZ09] T.-J. Li and W. Zhang, *Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds*, Comm. Anal. Geom. **17** (2009), no. 4, 651–683.

[LZ11] ______, *J-symplectic cones of rational four manifolds*, preprint, 2011.

[Mal49] A. I. Mal'tsev, *On a class of homogeneous spaces*, Izvestiya Akad. Nauk. SSSR. Ser. Mat. **13** (1949), 9–32, Translation in *Amer. Math. Soc. Translation* **1951** (1951), no. 39, 33 pp.

[Man04] M. Manetti, *Lectures on deformations of complex manifolds (deformations from differential graded viewpoint)*, Rend. Mat. Appl. (7) **24** (2004), no. 1, 1–183.

[Mat51] Y. Matsushima, *On the discrete subgroups and homogeneous spaces of nilpotent Lie groups*, Nagoya Math. J. **2** (1951), 95–110.

[Mat95] O. Mathieu, *Harmonic cohomology classes of symplectic manifolds*, Comment. Math. Helv. **70** (1995), no. 1, 1–9.

[McC01] J. McCleary, *A user's guide to spectral sequences*, second ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001.

[Mer98] S. A. Merkulov, *Formality of canonical symplectic complexes and Frobenius manifolds*, Internat. Math. Res. Notices (1998), no. 14, 727–733.

[Mic82] M. L. Michelsohn, *On the existence of special metrics in complex geometry*, Acta Math. **149** (1982), no. 3-4, 261–295.

[Mil76] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), no. 3, 293–329.

[Miy74] Y. Miyaoka, *Kähler metrics on elliptic surfaces*, Proc. Japan Acad. **50** (1974), no. 8, 533–536.

[MK06] J. Morrow and K. Kodaira, *Complex manifolds*, AMS Chelsea Publishing, Providence, RI, 2006, Reprint of the 1971 edition with errata.

[Moı66] B. G. Moıšezon, *On n-dimensional compact complex manifolds having n algebraically independent meromorphic functions. I, II, III*, Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), no. 1, 2, 3, 133–174, 345–386, 621–656, Translation in *Am. Math. Soc., Transl., II. Ser.*, **63** (1967), 51–177.

[Mor07] A. Moroianu, *Lectures on Kähler geometry*, London Mathematical Society Student Texts, vol. 69, Cambridge University Press, Cambridge, 2007.

[Mos54] G. D. Mostow, *Factor spaces of solvable groups*, Ann. of Math. (2) **60** (1954), no. 1, 1–27.

[Mos57] ———, *Errata, "Factor spaces of solvable groups."*, Ann. of Math. (2) **66** (1957), no. 3, 590.

[MPPS06] C. Maclaughlin, H. Pedersen, Y. S. Poon, and S. Salamon, *Deformation of 2-step nilmanifolds with abelian complex structures*, J. London Math. Soc. (2) **73** (2006), no. 1, 173–193.

[MT00] M. Migliorini and A. Tomassini, *Local calibrations of almost complex structures*, Forum Math. **12** (2000), no. 6, 723–730.

[MT11] C. Medori and A. Tomassini, *On small deformations of paracomplex manifolds*, J. Noncommut. Geom. **5** (2011), no. 4, 507–522.

[Nak75] I. Nakamura, *Complex parallelisable manifolds and their small deformations*, J. Differ. Geom. **10** (1975), 85–112.

[NN57] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. (2) **65** (1957), no. 3, 391–404.

[Nom54] K. Nomizu, *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*, Ann. of Math. (2) **59** (1954), 531–538.

[Ofm85a] S. Ofman, *Résultats sur les $d'd''$ et d''-cohomologies. Application à l'intégration sur les cycles analytiques. I*, C. R. Acad. Sci. Paris Sér. I Math. **300** (1985), no. 2, 43–45.

[Ofm85b] ———, *Résultats sur les $d'd''$ et d''-cohomologies. Application à l'intégration sur les cycles analytiques. II*, C. R. Acad. Sci. Paris Sér. I Math. **300** (1985), no. 5, 133–135.

[Ofm88] ———, *$d'd''$, d''-cohomologies et intégration sur les cycles analytiques*, Invent. Math. **92** (1988), no. 2, 389–402.

[Pop09] D. Popovici, *Limits of Projective Manifolds Under Holomorphic Deformations*, `arXiv:0910.2032v1 [math.AG]`, 2009.

[Pop10] ———, *Limits of Moishezon Manifolds Under Holomorphic Deformations*, `arXiv:1003.3605v1 [math.AG]`, 2010.

[Pop11] ———, *Deformation Openness and Closedness of Various Classes of Compact Complex Manifolds; Examples*, `arXiv:1102.1687v1 [math.AG]`, 2011, to appear in *Ann. Sc. Norm. Super. Pisa Cl. Sci.*

[Rai06] J. Raissy, *Normalizzazione di campi vettoriali olomorfi*, Tesi di Laurea Specialistica in Matematica, Università di Pisa, 2006, `http://etd.adm.unipi.it/theses/available/etd-06022006-141206/`.

[Rol09a] S. Rollenske, *Geometry of nilmanifolds with left-invariant complex structure and deformations in the large*, Proc. Lond. Math. Soc. (3) **99** (2009), no. 2, 425–460.

[Rol09b] ———, *Lie-algebra Dolbeault-cohomology and small deformations of nilmanifolds*, J. Lond. Math. Soc. (2) **79** (2009), no. 2, 346–362.

[Rol11a] ———, *Dolbeault cohomology of nilmanifolds with left-invariant complex structure*, Complex and Differential Geometry (Wolfgang Ebeling, Klaus Hulek, and Knut Smoczyk, eds.), Springer Proceedings in Mathematics, vol. 8, Springer Berlin Heidelberg, 2011, pp. 369–392.

[Rol11b] ———, *The Kuranishi space of complex parallelisable nilmanifolds*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 3, 513–531.

[Ros12a] F. A. Rossi, *On deformations of* $\mathbf{D}$*-manifolds and CR* $\mathbf{D}$*-manifolds*, J. Geom. Phys. **62** (2012), no. 2, 464–478.

[Ros12b] ———, *On Ricci-flat* $\mathbf{D}$*-Kähler manifolds*, preprint, 2012.

[Rot55] W. Rothstein, *Zur Theorie der analytischen Mannigfaltigkeiten im Raume von n komplexen Veränderlichen*, Math. Ann. **129** (1955), no. 1, 96–138.

[Sak76] Y. Sakane, *On compact complex parallelisable solvmanifolds*, Osaka J. Math. **13** (1976), no. 1, 187–212.

[Sal01] S. M. Salamon, *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra **157** (2001), no. 2-3, 311–333.

[Sat56] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), no. 6, 359–363.

[Sch29] J. A. Schouten, *Über unitäre Geometrie*, Proceedings Amsterdam **32** (1929), 457–465.

[Sch07] M. Schweitzer, *Autour de la cohomologie de Bott-Chern*, `arXiv:0709.3528 [math.AG]`, 2007.

[Ser55] J.-P. Serre, *Un théorème de dualité*, Comment. Math. Helv. **29** (1955), no. 9, 9–26.

[Ser56] ———, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 1–42.

[Sha86] J.-P. Sha, *p-convex Riemannian manifolds*, Invent. Math. **83** (1986), no. 3, 437–447.

[Sil96] A. Silva, $\partial\overline{\partial}$*-closed positive currents and special metrics on compact complex manifolds*, Complex analysis and geometry (Trento, 1993), Lecture Notes in Pure and Appl. Math., vol. 173, Dekker, New York, 1996, pp. 377–441.

[Siu83] Y. T. Siu, *Every $K3$ surface is Kähler*, Invent. Math. **73** (1983), no. 1, 139–150.

[ST10] J. Streets and G. Tian, *A parabolic flow of pluriclosed metrics*, Int. Math. Res. Not. IMRN (2010), no. 16, 3101–3133.

[Stu05] E. Study, *Kürzeste Wege im komplexen Gebiet*, Math. Ann. **60** (1905), no. 3, 321–378.

[Sul76] D. P. Sullivan, *Cycles for the dynamical study of foliated manifolds and complex manifolds*, Invent. Math. **36** (1976), no. 1, 225–255.

[Sul77] ———, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 269–331 (1978).

[SvD30] J. A. Schouten and D. van Dantzig, *Über unitäre Geometrie*, Math. Ann. **103** (1930), no. 1, 319–346.

[Tau95] C. H. Taubes, *The Seiberg-Witten and Gromov invariants*, Math. Res. Lett. **2** (1995), no. 2, 221–238.

[Tau11] ———, *Tamed to compatible: symplectic forms via moduli space integration*, J. Symplectic Geom. **9** (2011), no. 2, 161–250.

[Tia87] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, Singapore, 1987, pp. 629–646.

[Tia00] ———, *Canonical metrics in Kähler geometry*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2000, Notes taken by Meike Akveld.

[TK97] A. Tralle and J. Kedra, *Compact completely solvable Kähler solvmanifolds are tori*, Internat. Math. Res. Notices (1997), no. 15, 727–732.

[Tod89] A. N. Todorov, *The Weil-Petersson geometry of the moduli space of* $\mathrm{SU}(n \geq 3)$ *(Calabi-Yau) manifolds. I*, Comm. Math. Phys. **126** (1989), no. 2, 325–346.

[Tom02] A. Tomassini, *Some examples of non calibrable almost complex structures*, Forum Math. **14** (2002), no. 6, 869–876.

[Tom08] A. Tomasiello, *Reformulating supersymmetry with a generalized Dolbeault operator*, J. High Energy Phys. (2008), no. 2, 010, 25.

[TW11a] V. Tosatti and B. Weinkove, *The Calabi-Yau equation on the Kodaira-Thurston manifold*, J. Inst. Math. Jussieu **10** (2011), no. 2, 437–447.

[TW11b] ______, *The Calabi-Yau equation, symplectic forms and almost complex structures*, Geometry and analysis. No. 1, Adv. Lect. Math. (ALM), vol. 17, Int. Press, Somerville, MA, 2011, pp. 475–493.

[TWY08] V. Tosatti, B. Weinkove, and S.-T. Yau, *Taming symplectic forms and the Calabi-Yau equation*, Proc. Lond. Math. Soc. (3) **97** (2008), no. 2, 401–424.

[TWZZ11] Q. Tan, H. Wang, Y. Zhang, and P. Zhu, *On cohomology of almost complex 4-manifolds*, `arXiv:1112.0768v1 [math.SG]`, 2011.

[TY11] L.-S. Tseng and S.-T. Yau, *Generalized Cohomologies and Supersymmetry*, `arXiv:1111.6968v1 [hep-th]`, 2011.

[TY12a] ______, *Cohomology and Hodge Theory on Symplectic Manifolds: I*, J. Differ. Geom. **91** (2012), no. 3, 383–416.

[TY12b] ______, *Cohomology and Hodge Theory on Symplectic Manifolds: II*, J. Differ. Geom. **91** (2012), no. 3, 417–443.

[Vai07] I. Vaisman, *Reduction and submanifolds of generalized complex manifolds*, Differential Geom. Appl. **25** (2007), no. 2, 147–166.

[Var86] J. Varouchas, *Propriétés cohomologiques d'une classe de variétés analytiques complexes compactes*, Séminaire d'analyse P. Lelong-P. Dolbeault-H. Skoda, années 1983/1984, Lecture Notes in Math., vol. 1198, Springer, Berlin, 1986, pp. 233–243.

[Ves67] E. Vesentini, *Lectures on Levi convexity of complex manifolds and cohomology vanishing theorems*, Notes by M. S. Raghunathan. Tata Institute of Fundamental Research Lectures on Mathematics, No. 39, Tata Institute of Fundamental Research, Bombay, 1967.

[Voi02] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés [Specialized Courses], vol. 10, Société Mathématique de France, Paris, 2002.

[Voi04] ______, *On the homotopy types of compact Kähler and complex projective manifolds*, Invent. Math. **157** (2004), no. 2, 329–343.

[Voi06] ______, *On the homotopy types of Kähler manifolds and the birational Kodaira problem*, J. Differ. Geom. **72** (2006), no. 1, 43–71.

[Wan54] H.-C. Wang, *Complex parallisable manifolds*, Proc. Amer. Math. Soc. **5** (1954), no. 5, 771–776.

[Wei58] A. Weil, *Introduction à l'étude des variétés kählériennes*, Publications de l'Institut de Mathématique de l'Université de Nancago, VI. Actualités Sci. Ind. no. 1267, Hermann, Paris, 1958.

[Wei07] B. Weinkove, *The Calabi-Yau equation on almost-Kähler four-manifolds*, J. Differ. Geom. **76** (2007), no. 2, 317–349.

[Wel74] R. O. Wells, Jr., *Comparison of de Rham and Dolbeault cohomology for proper surjective mappings*, Pacific J. Math. **53** (1974), no. 1, 281–300.

[Wel08] ______, *Differential analysis on complex manifolds*, third ed., Graduate Texts in Mathematics, vol. 65, Springer, New York, 2008, With a new appendix by Oscar Garcia-Prada.

[Wey97] H. Weyl, *The classical groups*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997, Their invariants and representations, Fifteenth printing, Princeton Paperbacks.

[Wu87] H. Wu, *Manifolds of partially positive curvature*, Indiana Univ. Math. J. **36** (1987), no. 3, 525–548.

[Wu06] C.-C. Wu, *On the geometry of superstrings with torsion*, ProQuest LLC, Ann Arbor, MI, 2006, Thesis (Ph.D.)–Harvard University.

[Yan96] D. Yan, *Hodge structure on symplectic manifolds*, Adv. Math. **120** (1996), no. 1, 143–154.

[Yau77] S.-T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 5, 1798–1799.

[Yau78] ———, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[Ye08] X. Ye, *The jumping phenomenon of Hodge numbers*, Pacific J. Math. **235** (2008), no. 2, 379–398.

[Zha11] W. Zhang, *From Taubes currents to almost Kähler forms*, preprint, 2011.