Stefano Pascolutti

Boundaries of Moduli Space of Hyperelliptic Curves with a Level 2 Structure

Dottorato in Matematica, Roma «La Sapienza» (2011).

<http://www.bdim.eu/item?id=tesi_2011_PascoluttiStefano_1>

Sapienza
Università di Roma

Boundaries of Moduli Space of Hyperelliptic Curves with a Level 2 Structure

Scuola di Dottorato in Scienze Astronomiche,
Chimiche, Fisiche e Matematiche "Vito Volterra"

Dottorato di Ricerca in Matematica – XXIII Ciclo

Candidate

Stefano Pascolutti
ID number 1185126

Thesis Advisor

Prof. Riccardo Salvati Manni

A thesis submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics

May 2011

Thesis defended on October 2011
in front of a Board of Examiners composed by:

Dott. G. Mondello

Prof. B. van Geemen

Prof. V. Gritsenko

Stefano Pascolutti. *Boundaries of Moduli Space of Hyperelliptic Curves with a Level 2 Structure.*

Ph.D. thesis. Sapienza – University of Rome

VERSION: 10 May 2011

EMAIL: stefano.pascolutti@gmail.com

When life gives you Lemmas
make Theorems

Acknowledgments

I would like to thank Riccardo Salvati Manni, for the proposed problems, the continuous advice, for his constant help and care about this thesis (and many other matters). I would like to express my gratitude to Claudio Fontanari, for all the effort he put in our first joint paper. A special thank for all the people I met in Rome, Jacopo, Paolo, Adriano, Ascanio, Andrea, Marco and all others Ph.D. colleagues and students.

Contents

Introduction

The Schottky problem is the problem of characterizing Jacobians among all abelian varieties. In 1888, for genus 4, Schottky gave a homogeneous polynomial in theta constants which vanishes on $\mathbb{H}_4$ precisely at the Jacobian points; a proof of this was finally published by Igusa in 1981 ([Igu81]). A solution of the Schottky problem in general, such as given by Schottky and Igusa in genus four, would be a set of polynomials in the thetanullwerte which vanish precisely on the Jacobian locus of $\mathbb{H}_g$, the Siegel upper-half space. These equations have proved elusive, whereas other interesting methods of characterizing Jacobians have met with more success.

Along the same lines as the Schottky problem, we may consider other Schottky-type problems, such as the characterization of hyperelliptic Jacobians among all abelian varieties, or characterization of tree-like hyperelliptic curves, which belong to the boundary of the hyperelliptic locus in a suitable compactification. It was known to Schottky by 1880 ([KW22]) that a Jacobian of genus 3 is hyperelliptic precisely when an even thetanullwert vanishes. It goes back to Thomae a complete description of the vanishing and non-vanishing of thetanullwerte on the hyperelliptic locus. Despite of this fact, it took about 100 years, in 1984, to obtain a great progress when Mumford, using the methods of dynamical systems, characterized hyperelliptic Jacobians among all abelian varieties by the vanishing and non-vanishing of these thetanullwerte; by the way, a more classical proof has been given few years later by Poor ([Poo93]) and Salvati Manni ([FSM07]) showed that the characterization is not only set theoretical but also scheme theoretical. This result can hardly be improved, since it is strong and simple at once, but it is exploitable to investigate the boundary components. In fact, we found out that a similar theorem can be proved for boundaries of the hyperelliptic locus. Tsuyumine ([Tsu86]) introduced a valuation on the ring of invariants of a binary forms which classifies the vanishing thetanullwerte on a special divisor; we generalized this valuation, associated to a subset D of the set of the ramification points, with $|D| \geq 3$. Moreover, if we go deeper in the boundary structure, we take any tree-like hyperelliptic curve, which lies in the intersection of some divisors, and the thetanullwerte whose vanishing characterize this type of points among the Jacobian points are classified by this generalized valuation. These are very simple combinatoric conditions. Furthermore, from this point of view we can count how many tree like hyperelliptic curves are there, which share the same tree structure; this is achieved introducing a combinatorial object, the *linegram* which easily classifies the components.

Along with this theory, we found another interesting result. In fact, we learned from [SM00] that there is a modular form that never vanish on the hyperelliptic locus. This, on the moduli space of curves, restrict to a form CHECK CHECK. A Theorem due to Harer ([Har86]) states that the k-th cohomology group of the moduli space of curves of genus g with n maked points vanishes for $k > 4g-4+n$ if $n > 0$ and for $k > 4g-5$ if $n = 0$. On the other hand, it looks like the Harer's vanishing theorem is only the tip of an iceberg of deeper geometrical properties. For instance, an intriguing conjecture of Looijenga says that the moduli space of curves of genus g is a union of $g-1$ affine open subsets, but this is very difficult to verify. There where no advances in this direction, but for $g = 2, 3$ the conjecture is trivially true, since $\mathcal{M}_2$ is affine and the non-hyperelliptic curves of genus 3 can be canonically embedded as quartic plane curve, hence $\mathcal{M}_3 = \mathcal{M}_3 \setminus \{\text{hyperelliptic locus}\} \cup \mathcal{M}_3 \setminus \{\text{locus of plane quartics with at least one hyperflex}\}$ is the union of two affine open subsets. Via modular forms, we can settle the conjecture for $g = 4, 5$. This result will appear on Geometriae Dedicata.

Technical introduction The modular variety of non singular and complete hyperelliptic curves with level 2 structure of genus g is a $2g-1$-dimensional quasi projective variety which admit several standard compactifications. These compactifications arise from different views of this moduli space.

Via period map, we associate to any Riemann surface a complex valued symmetric matrix, with imaginary part definite positive, which is an element of the Siegel upper-half space $\mathbb{H}_g$. Since the period map is determined up to a change of the homology base which preserves the intersection matrix, we have that the matrix associated to the hyperelliptic curve is determined up to the action of the congruence group Γ_g via the usual formula

$$M \cdot Z = (AZ + B)(CZ + D)^{-1}$$

with $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g$ and $Z \in \mathbb{H}_g$. A fundamental domain for the action of Γ_g on $\mathbb{H}_g$ is given by the quotient $\mathbb{H}_g/\Gamma_g$. Hence, we can realize the moduli space of hyperelliptic curves of genus g as an irreducible subvariety of $\mathbb{H}_g/\Gamma_g$. We can introduce a structure level on the hyperelliptic curve, which is essentially given by an ordering of the $2g+2$ ramification points; the period map is the same, but matrices are determined up to the action of the principal congruence subgroup $\Gamma_g[2] \subseteq \Gamma_g$ of level 2, which is given by

$$\Gamma_g[2] = \ker(\Gamma_g \to \mathrm{Sp}_g(\mathbb{Z}/2\mathbb{Z})).$$

We will denote by $\mathcal{I}_g$ the hyperelliptic locus inside $\mathbb{H}_g$: it is the image, via the period map, of Riemann surfaces which are hyperelliptic curves. The quotient $\mathcal{I}_g/\Gamma_g = \mathscr{H}_g$ is the set of representatives of period matrices of hyperelliptic locus. Moreover, we will denote by $\mathscr{H}_g[2] = \mathcal{I}_g/\Gamma_g[2]$ the hyperelliptic locus inside $\mathbb{H}_g/\Gamma_g[2]$. This locus is no more irreducible and it is the union of several irreducible and isomorphic components. Tsuyumine ([Tsu90]) computed the number of components, which is

$$\frac{[\Gamma_g : \Gamma_g[2]]}{(2g+2)!} = 2^{g(2g+1)} \frac{\prod_{k=1}^{g}(1 - 2^{-2k})}{(2g+2)!};$$

To understand this result, we need to introduce the ring of modular forms. A Siegel modular form of weight k for the group $\Gamma \subseteq \Gamma_g$ is a holomorphic function $f\colon \mathbb{H}_g \to \mathbb{C}$ which satisfies

$$f(MZ) = \det(CZ + D)^k f(Z)$$

for all $M \in \Gamma$; for $g = 1$ we also require that f be holomorphic at cusps. The ring of modular forms for Γ is denoted by $A(\Gamma)$. A special type of modular forms are the thetanullwerte $\theta[\mathfrak{m}](Z)$

$$\theta[\mathfrak{m}](Z) = \sum_{n\in\mathbb{Z}^g} \exp\left(\pi i({}^{\intercal}(n + \mathfrak{m}'/2)Z(n + \mathfrak{m}'/2) + {}^{\intercal}(n + \mathfrak{m}'/2)\mathfrak{m}''\right)$$

with $\mathfrak{m} = \left(\begin{smallmatrix}\mathfrak{m}'\\ \mathfrak{m}''\end{smallmatrix}\right) \in \mathbb{F}_2^{2g}$ and $Z \in \mathbb{H}_g$, which are modular forms of weight $\frac{1}{2}$ for $\Gamma_g[4,8] \subseteq \Gamma_g$, i.e. the Igusa subgroup of matrices $M \in \Gamma_g[4]$ with $(A^{\intercal}C) : 0 \equiv (B^{\intercal}D)_0 \equiv 0 \mod 8$, where X_0 denotes the diagonal of the square matrix X. In particular, the fourth power of a thetanullwerte is a modular form of weight 2 for $\Gamma_g[2]$, and the whole $A(\Gamma_g)$ is the integral closure of the invariant ring generated by thetanullwerte; we say that $\theta[\mathfrak{m}]$ is *even* if ${}^{\intercal}\mathfrak{m}'\mathfrak{m}''$ is even. Now, the graded algebra of thetanullwerte that are modular forms with respect to $\Gamma_g[2]$ is a subring of $A(\Gamma_g[2])$, hence we can determine the hyperelliptic locus inside $\mathbb{H}_g/\Gamma_g[2]$ via vanishing of thetanullwerte. In fact we have a fundamental result by Mumford, that a component $\mathscr{H}_g^0[2]$ of the hyperelliptic locus is determined by the vanishing of $2^{g-1}(2^g+1) - \frac{1}{2}\binom{2g+2}{g+1}$ even thetanullwerte, i.e. there are exactly $\frac{1}{2}\binom{2g+2}{g+1}$ even theta functions which does not vanish.

g	Components	Vanishing thetanulls	Non Vanishing thetanulls
1	1	0	3
2	1	0	10
3	36	1	35
4	13056	10	126
5	51806208	66	462
6	2387230064640	364	1716

We have an action of the modular group Γ_g on the characteristics, and therefore on the thetanullwerte, which is given by the following formula

$$M \cdot \mathfrak{m} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \mathfrak{m} + \begin{pmatrix} (C^{\intercal}D)_0 \\ (A^{\intercal}B)_0 \end{pmatrix}$$

with $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $\mathfrak{m} \in \mathbb{F}_2^{2g}$; moreover, we have that $M \in \Gamma_g[2]$ fixes every thetanullwerte, while the whole Γ_g acts on thetanullwerte in a non trivial way. The fundamental result is that all the components are permuted under the action of $\Gamma_g/\Gamma_g[2]$. Moreover, we have that there is a homomorphism from group of braids $\mathcal{B}^{(2g+2)}$ to $\Gamma_g/\{\pm\,\mathrm{id}\}$, which is injective. Moreover, if $\mathcal{B}_0^{(2g+2)}$ is the group of pure braids, the previous homomorphism gives a surjective homomorphism onto $\Gamma_g[2]/\{\pm\,\mathrm{id}\}$. The image of $\mathcal{B}^{(2g+2)}$ in $\Gamma_g/\Gamma_g[2]$ is a finite group which is isomorphic to $S_g \times S_{g+2} \subseteq S_{2g+2}$. Hence we define a group $\Gamma_g[2] \subseteq \Gamma \subseteq \Gamma_g$ such that $\Gamma/\Gamma_g[2]$ is exactly the image of $\mathcal{B}^{(2g+2)}$ inside $\Gamma_g/\{\pm\,\mathrm{id}\}$: this group permutes the g odd characteristics of a special fundamental set and the $g+2$ even characteristics separately. Here we use the beautiful Mumford approach to associate to every characteristic an admissible 2-partition $T \cup T^c = \{1, \ldots, 2g+2\}$, with $|T| \equiv g+1 \mod 2$, of the set of the ramification points. In this way, the thetanullwerte which vanish on a component are the one associated to unbalanced partitions, i.e. $T \subseteq \{1, \ldots, 2g+2\}$ such that $|T| \neq |T^c|$. This approach is widely used in literature and we use it to give combinatorial conditions about vanishing of thetanullwerte in the boundary of hyperelliptic locus inside $\mathbb{H}_g/\Gamma_g[2]$, which is the set of hyperelliptic reducible matrices Z, i.e. there exists $M \in \Gamma_g$ such that

$$M \cdot Z = \begin{pmatrix} Z_1 & 0 & \cdots & 0 \\ 0 & Z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Z_n \end{pmatrix}$$

and Z_i hyperelliptic. We remind that if $\mathfrak{m} = {}^{\mathsf{T}}(\mathfrak{n}_1' \, \mathfrak{n}_2' \, \mathfrak{n}_1'' \, \mathfrak{n}_2'') = \left(\begin{smallmatrix} \mathfrak{n}_1' \\ \mathfrak{n}_1'' \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} \mathfrak{n}_2' \\ \mathfrak{n}_2'' \end{smallmatrix}\right)$, with $\mathfrak{n}_i \in \mathbb{F}_2^{2g_i}$, then

$$\theta[\mathfrak{m}] \left(\begin{smallmatrix} Z_1 & 0 \\ 0 & Z_2 \end{smallmatrix}\right) = \theta[\mathfrak{n}_1](Z_1)\theta[\mathfrak{n}_2](Z_2)$$

for any Z_i. If $Z_i \in \mathscr{H}_{g_i}[2]$, $Z = \left(\begin{smallmatrix} Z_1 & 0 \\ 0 & Z_2 \end{smallmatrix}\right)$ and on Z vanish all the even thetanullwerte $\theta[\mathfrak{m}]$ such that $\mathfrak{m} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ with $\mathfrak{n}_i$ odd or $\theta[\mathfrak{n}_i]$ vanishing on Z_i. Hence, there are many more thetanullwerte vanishing on points conjugate to $\left(\begin{smallmatrix} Z_1 & 0 \\ 0 & Z_2 \end{smallmatrix}\right)$, as well as on all reducible matrices. The number of vanishing functions only depends on the integer partition $\sum g_i = g$, while the actual thetanullwerte vanishing are determined by the partitions approach and depend on the chosen component. To prove these results, we need to describe the moduli space of hyperelliptic curves from another point of view.

As we already observed, every smooth hyperelliptic curve of genus g ramifies on $2g+2$ distinct points, so we can associate to the curve a polynomial equation. Let $[a_1 : b_1], \dots, [a_{2g+2} : b_{2g+2}] \in \mathbb{P}_1\mathbb{C}$ be the $2g+2$ ramification points, then the hyperelliptic curve is associated to the equation

$$y^2 = \prod_{i=1}^{2g+2} (b_i x - a_i) = f(x)$$

with $f(x)$ a polynomial of degree $2g+2$. If we homogenize $f(x)$, we get $F(X,Y) = \prod_{i=1}^{2g+2}(b_i X - a_i Y)$ which is a *binary form* of degree $2g+2$. The zeroes of the binary form are exactly the ramification points of the hyperelliptic curve. There is a natural action of $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ on the variables X, Y, i.e. $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) = \left(\begin{smallmatrix} aX+bY \\ cX+dY \end{smallmatrix}\right)$. This action induces on the variables induces an action on the coefficients of the homogeneous polynomial $F(X,Y)$. We can define invariant polynomials of degree s in the variables $\xi_1, \dots, \xi_{2g+2}$; in particular, if I is an invariant polynomial and $\xi_i = \frac{a_i}{b_i}$, assuming $[a_i : b_i] \neq \infty$, it satisfies the identity

$$I\left(\dots, \frac{a\xi_i + b}{c\xi_i + d}, \dots\right) = \left(\prod_{i=1}^{2g+2} (c\xi_i + d)^{-s}\right) I(\dots, \xi_i, \dots)$$

So, the graded ring $S(2g+2)$ of invariants of a binary form is a subring of $\mathbb{C}[\xi_i - \xi_j \mid 1 \leq i < j \leq 2g+2]$ and the homogeneous element of degree s is a polynomial of degree s with respect to each ξ_i. Probably, the most notable invariant is the discriminant of the polynomial $f(x)$, which is the product of all possible differences $(\xi_i - \xi_j)^2$.

The link between modular forms and invariants is pretty strong and it is provided by a result by Igusa [Igu67]: there is a ring homomorphism

$$\rho\colon \text{subring of } A(\Gamma_g) \to S(2, 2g+2)$$

which increases the weight by a $\frac{1}{2}g$ ratio. Moreover, Igusa points out that the domain coincides with $A(\Gamma_g)$ for every odd g and for $g = 2, 4$, and there is a sufficient condition for the domain to coincide with $A(\Gamma_g)$: we need to find an odd weight element $\psi \in A(\Gamma_g[2])_k$ which is a polynomial in the theta-constants and which satisfies $\psi(Z) \neq 0$ for at least one point Z associated with a hyperelliptic curve. We proved that this is always the case by giving explicitly the modular form of odd weight as a product of $2k$ suitable thetanullwerte, as well as the hyperelliptic point on which the modular form does not vanish. We can give an explicit version of this homomorphism by defining it on the eighth powers of thetanullwerte, i.e. let $\theta[\mathfrak{m}]^8 \in A(\Gamma_g[2])$, where $\mathfrak{m}$ is a characteristic associated to the partition $T \cup T^c = \{1, \dots, 2g+2\}$, and define

$$\rho(\theta[\mathfrak{m}]^8) = \prod_{i,j \in T} (\xi_i - \xi_j)^2 \prod_{i,j \in T^c} (\xi_i - \xi_j)^2$$

This morphism plays a central role in the understanding of the boundary structure that we will see in a while.

When we talk about invariants of binary forms, we are studying the configurations of points on a projective line. These are closely related to hyperelliptic curves, since every smooth hyperelliptic curve is an admissible double covering of a $2g+2$ pointed projective line, where the $2g+2$ are all distinct. If we take tree-like hyperelliptic curves, we need to collapse some of the branches to a *central component*, following Avritzer and Lange ideas ([AL02]); the associated binary form is a polynomial with multiple roots, which is still a stable binary form, i.e. no root is of multiplicity $m \geq g+1$; this approach let us study hyperelliptic curves from a combinatorial point of view. In fact, to any $2g+2$ pointed tree-like rational curve, we can associate its dual graph: each vertex represent a component of the curve and two vertices are connected by an edge if the corresponding components intersect. This graph is actually a tree and we can label every vertex with a subset $\hat{D}_i \subseteq \{1, \dots, 2g+2\}$ of the set of the marking points. Hence, using the combinatorial tools introduced in Section 1.2, we can establish that the curve belong to certain divisors of the moduli space, therefore to their intersection; using the generalized valuation on the ring of invariants and the ρ morphism, we can determine which thetanullwerte vanish on the period matrix of the curve. The criterion is quite simple: as usual, we associate to every characteristic a partition, and we split the partition on every $\overline{D_i}$: if the split partition is balanced on every vertex, then the corresponding thetanullwerte does not vanish. This allow us to count how many thetanullwerte vanish on a certain reducible matrix. This number was already know, since we can compute it using characteristic theory, but this new approach is easier. Moreover, a similar approach let us count how many curves share the same tree structure, and how many curves have a period matrix which is conjugate to a block period matrix, as well as the automorphism groups of every component.

Summary of the Chapters In the Chapter 1 we introduce many technical tools which are extremely useful in the rest of the thesis. We will also give the fundamental results about modular forms and projective invariants.

- In Section 1.1 we will describe the group of characteristics that parametrize the thetanullwerte.
- In Section 1.2 we will introduce the group of partitions, which is isomorphic to the group of characteristics. One isomorphism is explicitly given, and this will play a central role in our discussion. Moreover, we introduce the notion of *split* of partitions. This is a very important and technical step; it is easy to master, but difficult to describe it. This is mainly a combinatoric approach, which will be very useful also in the last chapter.
- In Section 1.3 we introduce the modular group Γ_g and its congruence subgroups. We will also recall some classical and useful results about these groups.
- In Section 1.4 we introduce the Siegel upper-half plane and its quotients with respect to the congruence subgroups. The quotient $\mathbb{H}_g/\Gamma_g = \mathscr{A}_g$ is the modular variety of the principally polarized abelian varieties, and its level 2 version $\mathscr{A}_g[2]$ will be one of the space in which the moduli space of hyperelliptic curves live. Moreover, in this Section we introduce the ring of modular forms and we sketch some fundamental properties for the thetafuntions. In particular, we prove that thetanullwerte are modular forms of weight $\frac{1}{2}$ with respect to a certain congruence subgroup $\Gamma_g[4,8]$.
- In Section 1.5 we briefly describe the braid group.
- In Section 1.6 we introduce the space $S(2g+2)$ of invariants of a binary form of degree $2g+2$. This is also useful since it is related to the space of moduli of $2g+2$-pointed curves of genus 0, which is isomorphic to the moduli space of hyperelliptic curves. The ring $S(2g+2)$ is closely related to the ring of modular forms with respect to Γ_g.

In Chapter 2 we want to develop a more geometrical point of view. We introduce various moduli spaces related to hyperelliptic curves, and we point out that the curves we are interested in are the tree-like hyperelliptic curve, which are also known as hyperelliptic curves of compact type. Moreover, we give fundamental results about the hyperelliptic locus inside $\mathscr{A}_g[2]$. In this chapter, we give two new results. The first one is Theorem 2.11, which is a generalization of a result of Igusa in [Igu67]: Igusa proved that the ρ morphism was defined on a subring of $A(\Gamma_g)$, and it was actually defined on $A(\Gamma_g)$ for odd g or $g = 2, 4$; moreover, he stated that ρ was defined on the whole ring if we can provide an odd weight element, and we proved with a combinatorial argument that this is always the case, hence ρ is always defined on the whole $A(\Gamma_g)$. The last Section of this Chapter describe an affine stratification for the moduli space of curves: this is a result of a joint work with Claudio Fontari ([FP11]). We proved that there exist suitable modular forms which determine the open sets for the covering. This proves Loojienga conjecture for $g = 4, 5$. It is still an open question if this conjecture is true also for $g \geq 6$.

In Chapter 3 we study the geometry of the curves of compact type. We generalize a work by Tsuyumine [Tsu86], and we check that we can reformulate the vanishing theorem in a more general way. In particular, we can give combinatorial criteria for a thetanullwerte vanishing on a certain divisor; this criterion will strongly rely on the notion of split of partitions that we introduced in Section 1.2

In Chapter 4 we will introduce *linegrams*, i.e. line diagrams, which are helpful in describe the combinatorial structure of the tree-like curves. In fact, it is very difficult to count the number of components of the boundary of the moduli space of hyperelliptic curves of genus g of compact type which share the same partition $\sum g_i = g$. With this construction we find a nice formula and we can give many combinatorial result in an easy way. This is a much more abstract Chapter, since we try to rely only on the combinatorial structure and not on the geometric structure.

Chapter 1

Preliminaries

1.1 The characteristic space

We first introduce *characteristics*, which we will use to parametrize characteristic theta functions. We will see that we have another useful parametrization, given by *partition*, which is equivalent to the previous one, but we have properties that are easier to see in one parametrization more than in the other. We will given an isomorphism which we will use as a *vocabulary* between the two parametrizations.

A characteristic is a vector $\mathfrak{m} \in \mathbb{F}_2^{2g}$. We can equip $\mathbb{F}_2^{2g}$ with the quadratic form q defined in the following way: if $\mathfrak{m} = \left(\begin{smallmatrix} \mathfrak{m}_1 \\ \mathfrak{m}_2 \end{smallmatrix}\right) \in \mathbb{F}_2^{2g}$, with $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathbb{F}_2^g$, then

$$q(\mathfrak{m}) = \mathfrak{m}_1^t \mathfrak{m}_2 \in \mathbb{F}_2.$$

We also have the associated bilinear form:

$$(\mathfrak{m}, \mathfrak{n}) = q(\mathfrak{m} + \mathfrak{n}) - q(\mathfrak{m}) - q(\mathfrak{n}).$$

Hence, we say that $\mathfrak{m}$ is orthogonal to $\mathfrak{n}$ if $(\mathfrak{m}, \mathfrak{n}) = 0$ and that $\mathfrak{m}$ is isotropic if $q(\mathfrak{m}) = 0$; if $\mathfrak{m}$ is isotropic we can also say that $\mathfrak{m}$ is *even*, while if $\mathfrak{m}$ is anisotropic, i.e. $q(\mathfrak{m}) = 1$, then we say it is *odd*. We can also use a multiplicative definition: let $e(\mathfrak{m}) = (-1)^{q(\mathfrak{m})}$ and $e(\mathfrak{m}, \mathfrak{n}) = (-1)^{(\mathfrak{m}, \mathfrak{n})}$: hence $\mathfrak{m}$ is even (resp. odd) if $e(\mathfrak{m}) = 1$ (resp. -1). We will denote with $\mathcal{F}_e^g$ the set of even characteristics and with $\mathcal{F}_o^g$ the set of odd characteristics. These sets are not subspaces of $\mathbb{F}_2^{2g}$ and we have the following Lemma

Lemma 1.1. *We have $2^{g-1}(2^g + 1)$ isotropic vectors in $\mathbb{F}_2^2 g$, i.e. $|\mathcal{F}_e^g| = 2^{g-1}(2^g + 1)$. The number of anisotropic vectors is $2^{g-1}(2^g - 1)$, i.e. $|\mathcal{F}_o^g| = 2^{g-1}(2^g - 1)$.*

Proof. We prove this by induction. For $g = 1$ we have three isotropic vectors $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and one anisotropic vector $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. Let $\mathfrak{m} = {}^{\mathsf{T}}(\mathfrak{m}_1 \, * \, \mathfrak{m}_2 \, *)$ be a vector in $\mathbb{F}_2^{2g}$, with $\left(\begin{smallmatrix} \mathfrak{m}_1 \\ \mathfrak{m}_2 \end{smallmatrix}\right) \in \mathbb{F}_2^{2(g-1)}$. Hence $\mathfrak{m}$ is even if and only if $\left(\begin{smallmatrix} \mathfrak{m}_1 \\ \mathfrak{m}_2 \end{smallmatrix}\right)$ is even and $\left(\begin{smallmatrix} * \\ * \end{smallmatrix}\right)$ is even of if those are both odd. Hence we have

$$\begin{aligned} |\mathcal{F}_e^g| &= 3|\mathcal{F}_e^{g-1}| + |\mathcal{F}_o^{g-1}| = 3 \cdot 2^{g-2}(2^{g-1} + 1) + 2^{g-2}(2^{g-1} - 1) = \\ &= (3+1)2^{g-1}2^{g-2} + (3-1)2^{g-2} = \\ &= 2^{g-1}(2 \cdot 2^{g-1} + 1) = 2^{g-1}(2^g + 1). \end{aligned}$$

The analogous holds for $|\mathcal{F}_o^g|$. $\square$

Definition 1.1. *If $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ are characteristics, we can define*

$$e(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3) = e(\mathfrak{m}_1)e(\mathfrak{m}_2)e(\mathfrak{m}_3)e(\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3).$$

Hence we say that a triple $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ is azygetic *if $e(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3) = -1$, while it is* sizygetic *if $e(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3) = 1$.*

Since the action of Γ_g fixes the parity of each characteristic, it also fixes the parity of the triples, i.e. it sends an azygetic sequence in another azygetic sequence.

Definition 1.2 (Azygetic Sequence)**.** *A subset of distinct characteristics $\{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ is an* azygetic sequence *if every triple $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$, with $1 \leq i < j < k \leq r$ is azygetic.*

We have that the longest azygetic sequence in $\mathbb{F}_2^{2g}$ has $2g+2$ elements, so it makes sense to give the following definition.

Definition 1.3 (Fundamental System)**.** *A* fundamental system *is an azygetic sequence with $2g+2$ elements. A* special fundamental system *is a fundamental system with g anisotropic elements and $g+2$ isotropic elements.*

We will make use of the following Lemma by Igusa ([Igu80]).

Lemma 1.2. *Let $\mathfrak{m}_1, \dots, \mathfrak{m}_{2k+1} \in \mathbb{F}_2^{2g}$, hence the following formula holds*

$$e\left(\sum_{i=1}^{2k+1} \mathfrak{m}_i\right) = \prod_{i=1}^{2k+1} e(\mathfrak{m}_i) \cdot \prod_{1<i<j} e(\mathfrak{m}_1, \mathfrak{m}_i, \mathfrak{m}_j).$$

With this Lemma we prove the following.

Lemma 1.3. *Let $\mathfrak{m}_1, \dots, \mathfrak{m}_{2g+2}$ be a special fundamental system, with $\mathfrak{m}_{2g+2} = 0$. If $\bar{\mathfrak{m}} = \sum_{i=1}^{g} \mathfrak{m}_i$, then every even characteristic is given by $\bar{\mathfrak{m}} + \sum_{i \in I} \mathfrak{m}_i$ with $I \subseteq \{1, \dots, 2g+1\}$ and $|I| \equiv g+1 \mod 4$ or $|I^c| \equiv g+1 \mod 4$; similarly, every odd characteristic is given by the same formula, but $|I| \equiv g-1 \mod 4$ or $|I^c| \equiv g-1 \mod 4$.*

Proof. We need to check that $\bar{\mathfrak{m}} + \sum_{i \in I} \mathfrak{m}_i$ with $I \subseteq \{1, \dots, 2g+1\}$ and $|I| \equiv g+1 \mod 4$ or $|I^c| \equiv g+1 \mod 4$ is an even characteristic; we note that $|I^c| \equiv g+1 \mod 4$ is equivalent to $|I| \equiv g \mod 4$. Moreover, if $I' = \{1, \dots, g\} \triangle I$,

$$\bar{m} + \sum_{i \in I} \mathfrak{m}_i = \sum_{i \in I'} \mathfrak{m}_i.$$

We eventually add $2g+2$ to I' so that $|I'| = 2k+1-g$. Hence $\bar{m} + \sum_{i \in I'} \mathfrak{m}_i$ is the sum of $2k+1$ characteristics. We have that

$$e\left(\sum_{i \in I'} \mathfrak{m}_i\right) = \underbrace{\prod_{i \in I'} e(\mathfrak{m}_i)}_{P_1} \cdot \underbrace{\prod_{1<i<j} e(\mathfrak{m}_1, \mathfrak{m}_i, \mathfrak{m}_j)}_{P_2}$$

and let $|I' \cap \{1, \dots, g\}| = t$. We compute P_1 and P_2 separately. As for P_1, we have that only $g-t$ of the first g odd characteristics survive, while the others $2k+1-g-t$ are even. Hence $P_1 = (-1)^{g-t}$. On the other side, we are using an azygetic sequence of characteristics, so $e(\mathfrak{m}_1, \mathfrak{m}_i, \mathfrak{m}_j) = -1$ for every $1 < i < j$; we have $\binom{2k+1-g-t+(g-t)-1}{2}$ such couple of i, j, hence

$$P_2 = (-1)^{\binom{2k-2t)}{2}} = (-1)^{(2k-2t)(2k-2k-1)/2} = (-1)^{t-k}.$$

So we have that

$$e\left(\sum_{i \in I'} \mathfrak{m}_i\right) = (-1)^{g-t} \cdot (-1)^{t-k} = (-1)^{g-k}$$

Since $2k + 1 - g \equiv g + 1 \mod 4$, we also have that $2k - 2g = 2(k - g) \equiv 0 \mod 4$, hence $k - g \equiv 0 \mod 2$, so $e\left(\sum_{i \in I'} \mathfrak{m}_i\right) = 1$. The same proof holds for odd characteristics. $\square$

Example 1.1. We will give a classical example of such a fundamental system. Let M be the following $2g \times 2g + 2$ matrix:

$$M = \left(\begin{array}{c|c|c|c} Id_g & Id_g & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \delta_g & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \quad \delta_g & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right),$$

where δ_g is the upper triangular unipotent matrix of degree g with $\delta_{i,j} = 1$ for all $i \leq j$ and 0 elsewhere. Hence the columns $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_{2g+2}\}$ are a special fundamental system.

We have an action of the group Γ_g, which we will describe in Section 1.3, which we need to introduce now. Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g$ and $\mathfrak{m} \in F_2^{2g}$, then

$$M \cdot \mathfrak{m} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \mathfrak{m} + \begin{pmatrix} (C^\intercal D)_0 \\ (A^\intercal B)_0 \end{pmatrix}.$$

This action preserves the quadratic form q, hence it preserves the parity of the characteristics. By the way, Γ_g acts transitively on the set of even characteristics and on the set of odd characteristics, hence $\mathbb{F}_2^{2g}$ splits in two orbits $\mathcal{F}_e^g \cup \mathcal{F}_o^g$. Moreover, the group $\Gamma_g[2] \subseteq \Gamma_g$ fixes every single characteristic, trivially. There exists a group $\Gamma_g[2] \subseteq \Gamma \subseteq \Gamma_g$ which does not fix the space of characteristics point-wise, but it fixes every special fundamental system. Hence, we have that Γ_g/Γ is a group which acts transitively on the set of special fundamental systems, while $\Gamma/\Gamma_g[2]$ permutes the characteristics inside every special fundamental system, while fixing the parity. Moreover, $\Gamma/\Gamma_g[2] \cong S_g \times S_{g+2}$.

Moreover, if $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ are azygetic, then $M\mathfrak{m}_1, M\mathfrak{m}_2, M\mathfrak{m}_3$ are still azygetic. We have the follwing Lemma.

Lemma 1.4. *Let $(\mathfrak{m}_1, \ldots, \mathfrak{m}_k)$ and $(\mathfrak{n}_1, \ldots, \mathfrak{n}_k)$ be two ordered k-tuples of characteristics. Then, there exists $M \in \Gamma_g$ such that $M\mathfrak{m}_i = \mathfrak{n}_i$ for each $i = 1, \ldots, n$ if and only if the following holds:*

1. $e(\mathfrak{m}_i) = e(\mathfrak{n}_i)$ *for each* $i = 1, \dots, k$,
2. $e(\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_l) = e(\mathfrak{n}_i, \mathfrak{n}_j, \mathfrak{n}_l)$ *for each* $1 \leq i < j < l \leq k$,
3. *whenever* $\{\mathfrak{m}_1, \dots, \mathfrak{m}_{2l}\} \subseteq \{\mathfrak{m}_1, \dots, \mathfrak{m}_k\}$ *is such that* $\mathfrak{m}_1 + \dots + \mathfrak{m}_{2l} \neq 0$, *then* $\mathfrak{n}_1 + \dots + \mathfrak{n}_{2l} \neq 0$.

Hence, we have the following important Corollary

Corollary 1.4.1. *Let* $\mathfrak{m}_1, \dots, \mathfrak{m}_g$ *and* $\mathfrak{n}_1, \dots, \mathfrak{n}_g$ *be two different orderings of a* g*-tuple of even characteristics, such that*

$$e(\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k) = e(\mathfrak{n}_i, \mathfrak{n}_j, \mathfrak{n}_k);$$

hence, there exists an elementi $M \in \Gamma_g/\Gamma_g[2]$ *such that* $M\mathfrak{m}_i = \mathfrak{n}_i$ *for each* $i = 1, \dots, g$.

Let $k_1 + k_2 = g$. We can decompose $\mathbb{F}_2^{2g} = \mathbb{F}_2^{2k_1} \oplus \mathbb{F}_2^{2k_2}$ in the following way: if $\mathfrak{m} \in \mathbb{F}_2^{2g}$, then

$$\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{n}' = \begin{pmatrix} \mathfrak{n}_1 \\ \mathfrak{n}'_1 \\ \mathfrak{n}_2 \\ \mathfrak{n}'_2 \end{pmatrix}, \tag{1.1}$$

where $\mathfrak{n} = \left(\begin{smallmatrix} \mathfrak{n}_1 \\ \mathfrak{n}_2 \end{smallmatrix}\right) \in \mathbb{F}_2^{2k_1}$ and $\mathfrak{n}' = \begin{pmatrix} \mathfrak{n}'_1 \\ \mathfrak{n}'_2 \end{pmatrix} \in \mathbb{F}_2^{2k_2}$.

Remark 1.1. If $\mathfrak{n} \in \mathcal{F}_e^{k_1}$ and $\mathfrak{n}' \in \mathcal{F}_e^{k_2}$, then $\mathfrak{n} \oplus \mathfrak{n}' \in \mathcal{F}_e^g$. By the way, if $\mathfrak{n} \in \mathcal{F}_o^{k_1}$ and $\mathfrak{n}' \in \mathcal{F}_o^{k_2}$ then $\mathfrak{n} \oplus \mathfrak{n}' \in \mathcal{F}_e^g$. On the contrary, if $\mathfrak{n} \in \mathcal{F}_e^{k_1}$ and $\mathfrak{n}' \in \mathcal{F}_o^{k_2}$ (or viceversa) then $\mathfrak{n} \oplus \mathfrak{n}' \in \mathcal{F}_o^g$. Hence we can write

$$\mathcal{F}_e^g = \mathcal{F}_e^{k_1} \oplus \mathcal{F}_e^{k_2} \cup \mathcal{F}_o^{k_1} \oplus \mathcal{F}_o^{k_2}$$
$$\mathcal{F}_o^g = \mathcal{F}_e^{k_1} \oplus \mathcal{F}_o^{k_2} \cup \mathcal{F}_o^{k_1} \oplus \mathcal{F}_e^{k_2}$$

Moreover, we can iterate this decomposition for every integer partition $\sum k_i = g$, and get $\bigoplus \mathbb{F}_2^{2k_i} = \mathbb{F}_2^{2g}$. This is only a canonical split. More generally, any isomorphism

$$\psi_M \colon \mathbb{F}_2^{g_1} \oplus \dots \mathbb{F}_2^{g_n} \to \mathbb{F}_2^g$$

with $\sum g_i = g$, such that $e(\mathfrak{m}_1 \oplus \dots \oplus \mathfrak{m}_n) = e(\mathfrak{m}_1) \cdot \dots \cdot e(\mathfrak{m}_n)$, is given fixing a matrix $M \in \Gamma_g$ which acts on characteristics via 1.8; hence we have $\psi_M(\mathfrak{m}_1, \dots, \mathfrak{m}_n) = M \cdot (\mathfrak{m}_1 \oplus \dots \oplus \mathfrak{m}_n)$. Indeed, every matrix $M \in \Gamma_g$ fixes the quadratic form q (see Section 1.8) hence the parity is preserved. Note that there are more choices for M which define the same ψ_M. At any rate, let Ψ_g be the set of all the maps ψ_M.

1.2 The partitions space

Let $S = \{1, \dots, 2g+2\}$ and $T \subseteq S$, with $|T| = k$; we can consider the partition $T \cup T^c = S$ and we say that this is an *admissible partition* if $k \equiv g+1 \mod 2$. Let

$$\mathcal{P}^g = \{T \subseteq S \mid |T| \equiv g+1 \mod 2\} \big/ \sim,$$

where $T \sim T'$ if $T' = T$ or $T' = T^c$ and this is an equivalence relation; it's easy to check that $|\mathcal{P}^g| = 2^{2g}$. From now on we will say simply *partition* instead of *admissible partition*.

Definition 1.4. *A partition $T \in \mathcal{P}^g$ is* even *if* $|T| \equiv g+1 \mod 4\}$*; A partition* $T \in \mathcal{P}^g$ *is* odd *if* $|T| \equiv g-1 \mod 4\}$.

We need to check that this is a good definition, i.e. that $|T| \equiv |T^c| \mod 4$, which is straight forward. We will denote by $\mathcal{P}^g_e \subseteq \mathcal{P}^g$ the subset of even partitions and by $\mathcal{P}^g_o \subseteq \mathcal{P}^g$ the subset of odd partitions. We can also introduce a notation which mimics the one we used for characteristics.

Notation. *Let $T \in \mathcal{P}^g$. We will denote by $e(T) = (-1)^{\frac{g+1-|T|}{2}}$, hence we have that $T \in \mathcal{P}^g_e$ if $e(T) = 1$ and $T \in \mathcal{P}^g_o$ if $e(T) = -1$. Note that $\frac{g+1-|T|}{2} \in \mathbb{Z}$, hence $e(T)$ is either 1 or -1.*

Definition 1.5 (Balanced Admissible Partition)**.** *A partition $T \in \mathcal{P}^g$ is* balanced *if $|T| = g+1$, i.e. $|T| = |T^c|$. We will denote by $\mathcal{P}^g_b$ the set of balanced partitions.*

Obviously, every balanced partition is also an even partition. We want to endow $\mathcal{P}^g$ with an abelian group structure.

Definition 1.6 (Partitions Group Law)**.** *Let $U \in \mathcal{P}^g_b$, which will serve as identity element and let $\triangle$ denote the usual symmetric difference between sets. We define $\triangledown \colon \mathcal{P}^g \times \mathcal{P}^g \to \mathcal{P}^g$ in the following way: if $T_1, T_2 \in \mathcal{P}^g$, then*

$$T_1 \triangledown T_2 = (T_1 \triangle T_2) \triangle U$$

So $(\mathcal{P}^g, \triangledown, U)$ is an abelian group.

We need to check that the operation is well defined, i.e. that if $T_1, T_2 \in \mathcal{P}^g$, then $T_1 \triangledown T_2 \in \mathcal{P}^g$. We note that

$$|A \triangle B| \equiv |A| + |B| - 2|A \cup B| \equiv |A| + |B| \mod 2;$$

hence, if $|T_1| \equiv |T_2| \equiv g+1 \mod 2$ and $|U| = g+1$, we have that

$$\begin{aligned} |T_1 \triangledown T_2| &\equiv |(T_1 \triangle T_2) \triangle U| \mod 2 \\ &\equiv |T_1 \triangle T_2| + |U| \mod 2 \\ &\equiv |T_1| + |T_2| + |U| \mod 2 \\ &\equiv 3(g+1) \equiv g+1 \mod 2; \end{aligned}$$

by the way, since the same equivalences does not hold $\mod 4$, i.e. it is not true that $\mathcal{P}^g_e$ or $\mathcal{P}^g_o$ are closed under $\triangledown$.

Remark 1.2. It is easy to check that $T \triangledown T = U$ for every $T \in \mathcal{P}^g$, hence $\mathcal{P}^g$ is an abelian group of order 2^{2g} and every element have order 2, so it is isomorphic to $\mathbb{Z}_2^{2g}$, which is the additive group underlying $\mathbb{F}_2^{2g}$. We will give an explicit, and useful, isomorphism in Theorem 1.6.

Definition 1.7 (Fundamental System of Partitions)**.** *We say that a triple of partitions $T_1, T_2, T_3 \in \mathcal{P}^g$ is* azygetic *if $e(T_1)e(T_2)e(T_3)e(T_1 \triangledown T_2 \triangledown T_3) = -1$, and $\{T_1, \dots, T_r\}$ is an azygetic sequence if every triple is azygetic. Hence, a* fundamental system of partitions *is a maximal azygetic sequence of partitions.*

Remark 1.3. We defined $\triangledown$ using a chosen partition $U \in \mathcal{P}_b^g$ to be the identity, but $T_1 \triangledown T_2 \triangledown T_3$ does not depend on the choice of U, in fact

$$\begin{aligned}(T_1 \triangledown T_2) \triangledown T_3 &= ((T_1 \triangle T_2) \triangle U) \triangledown T_3 = \\ &= T_1 \triangle T_2 \triangle T_3 \triangle U \triangle U = \\ &= T_1 \triangle T_2 \triangle T_3,\end{aligned}$$

and $\triangle$ does not depend on the choice of U. Hence the fundamental systems are defined a priori.

Example 1.2. We will give an example of such a fundamental system of partitions. The following $2g+2$ partitions form a fundamental system:

$$\begin{aligned}E_i &= \{1, \ldots, g, g+1+i\} \text{ for } i = 0, \ldots, g+1; \\ O_i &= \{1, \ldots, \hat{\imath}, \ldots, g\} \text{ for } i = 1, \ldots, g.\end{aligned}$$

Note that $E_i \in \mathcal{P}_e^g$ and $O_i \in \mathcal{P}_o^g$; moreover, we have that

$$\begin{aligned}(O_i \triangledown O_j) \triangledown O_k &= \{1, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, \hat{k}, \ldots, g\} \in \mathcal{P}_o^g; \\ (E_i \triangledown E_j) \triangledown E_k &= \{1, \ldots, g, g+1+i, g+1+j, g+1+k\} \in \mathcal{P}_e^g;\end{aligned}$$

hence E_i, E_j, E_k and O_i, O_j, O_k are azygetic; further computations show that $F = \{O_1, \ldots, O_g, E_0, \ldots, E_{g+1}\}$ is an azygetic sequence. Moreover, if we fix $U \in \mathcal{P}_b^g$, we have that

$$E_0 \triangledown \ldots \triangledown E_{g+1} \triangledown O_1 \triangledown \ldots \triangledown O_g = U,$$

We want to give an explicit way to write every $T \in \mathcal{P}^g$ as a combination of partitions of the fundamental system.

Lemma 1.5. *Let $g+1 \notin T \in \mathcal{P}^g$, $T_o = T \cap \{1, \ldots, g\}$ and $T_e = T \cap \{g+2, \ldots, 2g+2\}$. Let $\bar{O} = \triangledown_{i=1}^g O_i$. We have that*

$$T = \bar{O} \triangledown \mathop{\triangledown}_{i \in T_o} O_i \triangledown \mathop{\triangledown}_{i \in T_e} E_{i-(g+1)}.$$

The proof is an easy check. Note that the hypothesis $g+1 \notin T$ is just a technical hypothesis which can be dropped since we can always switch between T and T^c.

Theorem 1.6. *Let M as in Example 1.1 and let M_i denote the i-th column of the matrix. Let F be the fundamental system of partitions defined in Example 1.2. Let $\varphi\colon \{M_1, \ldots, M_{2g+2}\} \to F$ such that*

$$\varphi(M_i) = \begin{cases} O_i & \text{if } i = 1, \ldots, g \\ E_{2g+2-i} & \text{if } i = 0, \ldots, g+1, \end{cases}$$

Then φ defines an abelian groups morphism, which we will call φ again, $\varphi\colon \mathbb{F}_2^{2g} \to \mathcal{P}^g$ such that

(i) $\varphi(0) = E_0$;

(ii) if $\mathfrak{m} \in \mathcal{F}_e^g$ (resp. $\mathcal{F}_o^g$), then $\varphi(\mathfrak{m}) \in \mathcal{P}_e^g$ (resp. $\mathcal{P}_o^g$);

(iii) if $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ are azygetic, then $\varphi(\mathfrak{m}_1), \varphi(\mathfrak{m}_2), \varphi(\mathfrak{m}_3)$ are azygetic.

Proof. It's easy to check that $\operatorname{rk} M = 2g$, hence $\{M_1, \dots, M_{2g+2}\}$ generate $\mathbb{F}_2^{2g}$. Moreover, we can check that $\{M_1, \dots, M_{2g}\}$ generate $\mathbb{F}_2^{2g}$, since $M_{2g+2} = 0$ and $M_{2g+1} = \sum_{i=1}^{2g} M_i$.

Moreover, E_0 is the identity and

$$E_1 = O_1 \triangledown \dots \triangledown O_g \triangledown E_2 \triangledown \dots \triangledown E_{g+1},$$

hence $\{O_1, \dots, O_g, E_2, \dots, E_{g+1}\}$ generate $\mathcal{P}^g$. Hence φ takes a set of generators in a set of generators, hence it extends on the whole group; it is obvious that $\varphi(0) = \varphi(M_{2g+2}) = E_0$.

Let $\mathfrak{m} \in \mathcal{F}_e^g$; by Lemma 1.3 we have $\mathfrak{m} = \bar{\mathfrak{m}} + \sum_{i \in I} \mathfrak{m}_i$, with $I \subseteq \{1, \dots, 2g+1\}$, $|I| \equiv g+1 \mod 4$ or $|I^c| \equiv g+1 \mod 4$ and $\bar{\mathfrak{m}} = \sum_{i=1}^{g+1} \mathfrak{m}_i$; We want to use Lemma 1.5, so we have to distinguish four cases, since we would like to choose between I and I^c and to check if $g+1$ belongs to I; We will use the notation $I_o = I \cup \{1, \dots, g\}$ and $I_e = I \cup \{g+2, \dots, 2g+1\}$.

1. If $|I| \equiv g+1 \mod 4$ and $g+1 \notin I$, then we simply apply φ.

$$\begin{aligned}
\varphi(\mathfrak{m}) &= \varphi\left(\bar{\mathfrak{m}} + \sum_{i \in I} \mathfrak{m}_i\right) = \varphi(\bar{\mathfrak{m}}) \triangledown \varphi\left(\sum_{i \in I} \mathfrak{m}_i\right) = \\
&= \varphi\left(\sum_{i=1}^{g} \mathfrak{m}_i\right) \triangledown \mathop{\nabla}_{i \in I} \varphi(\mathfrak{m}_i) = \mathop{\nabla}_{i=1}^{g} \varphi(\mathfrak{m}_i) \triangledown \mathop{\nabla}_{i \in I} \varphi(\mathfrak{m}_i) = \\
&= \mathop{\nabla}_{i=1}^{g} O_i \triangledown \mathop{\nabla}_{i \in I_o} O_i \triangledown \mathop{\nabla}_{i \in I_e} E_{2g+2-i} = \\
&= \bar{O} \triangledown \mathop{\nabla}_{i \in I_o} O_i \triangledown \mathop{\nabla}_{i \in I_e} E_{2g+2-i}.
\end{aligned}$$

   Hence, by Lemma 1.5, $|\varphi(\mathfrak{m})| = |I_o| + |I_e| = |I| \equiv g+1 \mod 4$.

2. If $|I^c| \equiv g+1 \mod 4$ and $g+1 \notin I$, then we let $I' = I \cup \{2g+2\}$ and observe that

$$\mathfrak{m} = \bar{\mathfrak{m}} + \sum_{i \in I} \mathfrak{m}_i = \bar{\mathfrak{m}} + \sum_{i \in I} \mathfrak{m}_i + 0 = \bar{\mathfrak{m}} + \sum_{i \in I'} \mathfrak{m}_i$$

   Since $|I^c| \equiv g+1 \mod 4$ a simple computation shows that $|I'| = |I|+1 \equiv g+1 \mod 4$, then we apply φ.

$$\varphi(\mathfrak{m}) = \bar{O} \triangledown \mathop{\nabla}_{i \in I'_o} O_i \triangledown \mathop{\nabla}_{i \in I'_e} E_{2g+2-i}.$$

   Hence, by Lemma 1.5, $|\varphi(\mathfrak{m})| = |I'_o| + |I'_e| = |I'| \equiv g+1 \mod 4$.

3. If $|I^c| \equiv g+1 \mod 4$ and $g+1 \notin I$, then we observe that

$$\mathfrak{m} = \bar{\mathfrak{m}} + \sum_{i \in I} \mathfrak{m}_i = \bar{\mathfrak{m}} + \sum_{i \in I^c} \mathfrak{m}_i.$$

   Since $g+1 \notin I$, we apply φ.

$$\varphi(\mathfrak{m}) = \bar{O} \triangledown \mathop{\nabla}_{i \in I^c_o} O_i \triangledown \mathop{\nabla}_{i \in I^c_e} E_{2g+2-i}.$$

   Hence, by Lemma 1.5, $|\varphi(\mathfrak{m})| = |I^c| \equiv g+1 \mod 4$.

4. If $|I| \equiv g+1 \mod 4$ and $g+1 \in I$, then we observe that
$$\mathfrak{m} = \bar{\mathfrak{m}} + \sum_{i\in I} \mathfrak{m}_i = \bar{\mathfrak{m}} + \sum_{i\in I^c} \mathfrak{m}_i + 0 = \bar{\mathfrak{m}} + \sum_{i\in I^c\cup\{2g+2\}} \mathfrak{m}_i.$$
Since $|I| \equiv g+1 \mod 4$ then $|I^c \cup \{2g+2\}| = |I_c| + 1 \equiv g+1 \mod 4$, then we apply φ.
$$\varphi(\mathfrak{m}) = \bar{O} \triangledown \mathop{\triangledown}_{i\in I_o^c} O_i \triangledown \mathop{\triangledown}_{i\in I_e^c\cup\{2g+2\}} E_i.$$
Hence, by Lemma 1.5, $|\varphi(\mathfrak{m})| = |I_o^c| + |I_e^c \cup \{2g+2\}| = |I^c \cup \{2g+2\}| \equiv g+1 \mod 4$.

The same holds for odd characteristics. The third point follows trivially. □

Remark 1.4. For every g we have the morphism $\varphi = \varphi_g$ but we will drop g if there is no need to distinguish.

We will discuss *balanced admissible partitions* in a more geometrical context later on (see Chapter 2).

What we will need in the second part of this thesis is a notion of *split*. As we saw in Section 1.1, there is a canonic decomposition for the characteristics. When we talk about partitions, there is no evident choice. We will proceed as follows.

Let $D \subseteq S$, $|D| = 2g_1 + 1 \geq 3$; hence we have that $S = D \cup D^c$ and we can complete both D and D^c to sets with an even number of elements, adding a point to each set: let $\overline{D} = D \cup \{\star\}$ and $\overline{D^c} = D^c \cup \{\star\}$, hence $|\overline{D}| = 2g_1 + 2$ and $|\overline{D^c}| = 2g_2 + 2$, where $g_1 + g_2 = g$. The $\star$ element will be a place holder for the glueing; this symbolize the same element in the geometric context as we will see later. For any partition $T \in \mathcal{P}^g$, we want to associate a partition T' on $\overline{D}$ and a partition T'' on $\overline{D^c}$. We are asking the following properties:

- T' is an admissible partition in $\overline{D}$, i.e. upon a fixed isomorphism $\overline{D} \to \{1, \dots, 2g_1 + 2\} = S'$, $T' \in \mathcal{P}^{g_1}$;
- T'' is an admissible partition in $\overline{D^c}$, i.e. upon a fised isomorphism $\overline{D^c} \to \{1, \dots, 2g_2 + 2\} = S''$, $T'' \in \mathcal{P}^{g_2}$;
- $T' \cup T'' = T \cup \{\star\}$;
- $\star$ belongs either to T' or T''.

This is fairly easy to realize, but we need to take care of the fact that a partition is an equivalence class and not only a set. Briefly, what we need to check is $|T \cap D|$. If $|T \cap D| \equiv g_1 + 1 \mod 2$, then $|T \cap D^c| \equiv g_2 \mod 2$; if $|T \cap D| \equiv g_1 \mod 2$, then $|T \cap D^c| \equiv g_2 + 1 \mod 2$. Hence we have
$$\begin{cases} T' = T \cap D, \quad T'' = (T \cap D^c) \cup \{\star\} & \text{if } |T \cap D| \equiv g_1 + 1 \mod 2 \\ T' = (T \cap D) \cup \{\star\}, \quad T'' = T \cap D^c & \text{if } |T \cap D| \equiv g_1 \mod 2; \end{cases} \tag{1.2}$$
It is easy to check that (1.2) satisfies the properties that we require.

Moreover, if we have two partitions $T' \in \mathcal{P}^{g_1}$ in $\overline{D}$ and $T'' \in \mathcal{P}^{g_2}$ in $\overline{D^c}$, we want to glue them together to get a partition $T \in \mathcal{P}^g$ in S. If $\star$ is in T' and not in T'', then we just set $T = T' \cup T'' \setminus \{\star\}$; otherwise, either $\star$ belongs to both T' and T''

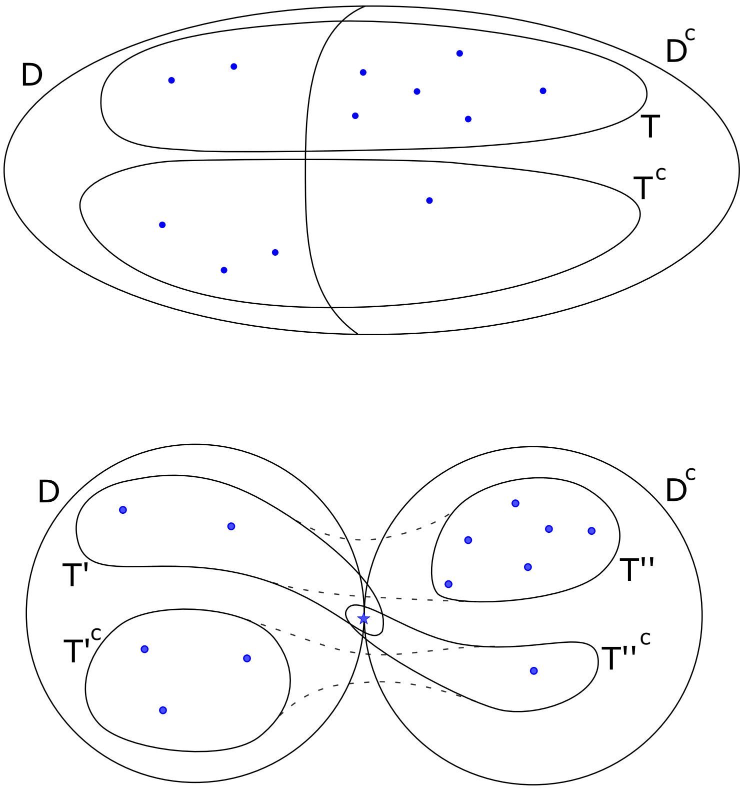


Figure 1.1. Check

or it does not belong to any of them: in both cases, we switch T' and T'^c, which define the same partition, and we set $T = (T')^c \cup T'' \setminus \{\star\}$. Note that we could have switched T'' with $(T'')^c$: in this case, $\tilde{T} = T' \cup (T'')^c \setminus \{\star\}$ and $T \sim \tilde{T}$, since $\tilde{T} = T^c$.
Let $D_1, D_2 \subseteq S$; we define $D_1 \circledast D_2$ if one of the following holds:

$$D_1 \subseteq D_2, \quad D_2 \subseteq D_1, \quad D_1 \cap D_2 = \emptyset, \quad D_1 \cup D_2 = S; \tag{1.3}$$

Remark 1.5. The relation $\circledast$ is an equivalence relation. More over, it is compatible with the relation $\sim$; in fact we have that

$$D_1 \subseteq D_2 \Leftrightarrow D_2^c \subseteq D_1^c \Leftrightarrow D_1 \cap D_2^c = \emptyset \Leftrightarrow D_1^c \cup D_2 = S$$

for any $D_1, D_2 \subseteq S$; the same holds for the other three cases.

Definition 1.8. *A partition $T \in \mathcal{P}^g$ is D-balanced if T' is balanced in $\overline{D}$ and T'' is balanced in $\overline{D^c}$.*

Remark 1.6. Let $T \in \mathcal{P}^g$ and $D \subseteq S$, $|D|$ odd. If T is D-balanced, then it is balanced. Moreover, if $|D| = 2g_1 + 1$, T is D-balanced if T is balanced and $|T \cap D| = g_1$ or $|T \cap D| = g_1 + 1$.

This Remark will be really useful but it does not generalize in an easy way. We need some more.

We can carry on with this method in the following general way. We will need some simple terminology from graph theory. Let $k_1 + \ldots + k_n = g$ an integer partition of g, with $k_i \geq 1$; obviously $n \leq g$. Let $\mathcal{T} = (V, E)$ be a weighted tree with $|V| = n$, where V is the set of vertices and E is the set of edges and the weight of $v_i \in V$ is k_i; since $\mathcal{T}$ is a tree, it is connected and $n = |V| = |E| + 1$. Let $d(v)$ the degree of a vertex v, i.e. the number of edges which have an endpoint in v. We ask an additional condition

$$d(v_i) \leq 2k_i + 2. \tag{$\star$}$$

For every vertex v_i, we choose a subset $\hat{D}_i \subseteq S = \{1, \ldots, 2g+2\}$ such that

$$|\hat{D}_i| = 2k_i + 2 - d(v_i) \text{ and } \bigcup_{i=1}^{n} \hat{D}_i = S$$

The condition ($\star$) tell us that $\{\hat{D}_1, \ldots, \hat{D}_n\}$ is a partition of the set S. Indeed

$$\sum_{i=1}^{n} |\hat{D}_i| = \sum_{i=1}^{n} (2k_i + 2 - d(v_i)) = 2g + 2n - 2|E| = 2g + 2 = |S|$$

hence $\hat{D}_i \cap \hat{D}_j = \emptyset$ for every $i \neq j$. Since $\mathcal{T}$ is a tree, when we remove and edge $e_i \in E$ we are disconnecting the tree in two subtrees $\mathcal{T}_i = (V_i, E_i)$ and $\mathcal{T}_i'(V_i', E_i')$, where $V_i \cup V_i' = V$ and $E_i \cup E_i' \cup \{e_i\} = E$.

$$D_i = \bigcup_{j\colon v_j \in V_i} \hat{D}_j \text{ and } D_i' = \bigcup_{j\colon v_j \in V_i'} \hat{D}_j.$$

Hence we have $D_i' = D_i^c$ and so D_i define a 2-partition of S. Moreover, it is easy to check that $D_i \circledast D_j$ for every $i, j \in \{1, \ldots, n-1\}$.

Hence, to every tree $\mathcal{T} = (V, E)$ we associate a $(|E|)$-ple $\{D_e\}_{e \in E}$ of 2-partitions of S. To simplify the notation, we can fix an enumeration of the edges such that for every leaf v_i the only edge which has an endpoint in v_i is e_i. We have the following further properties.

- If $v_i \in V$ is a leaf, i.e. $d(v_i) = 1$ and $e_i \in E$ is the only edge which have an endpoint in v_i, then $|D_i| = 2k_i + 1$;
- If $v_i \in V$ is a leaf, then we can define $\mathcal{T}^i = (V \setminus \{v_i\}, E \setminus \{e_i\})$.

We will use this construction in a while. Before that, we need to define when $T \in \mathcal{P}^g$ is a $\mathcal{T}$-admissile partition.

Let $\{\hat{D}_1, \ldots, \hat{D}_n\}$ as before. Let $\overline{D_i} = D_i \cup \bigcup_{e\colon v_i \in e} \{\star_e\}$, where $\star_e$ is a place holder for the glueing, as before; if $e = \{v_i, v_j\} \in E$, then we will use the following notation $\star_e = \star_{i,j}$. Hence

$$|\overline{D_i}| = 2k_i + 2;$$

Let $T \in \mathcal{P}^g$; we want to define $\overline{T_1}, \ldots, \overline{T_n}$ such that

- $\overline{T_i} \cap S = T \cap \hat{D}_i$;
- $\overline{T_i}$ is an admissible partition on $\overline{D_i}$, i.e. upon a fixed isomorphism $\overline{D_i} \to \{1, \ldots, 2k_i + 2\}$, $T_i \in \mathcal{P}^{k_i}$;
- $\bigcup_{i=1}^{n} \overline{T_i} = T \cup \{\star_e\}_{e \in E}$;

- $\star_{i,j}$ belongs either to $\overline{T_i}$ or to $\overline{T_j}$ but not to both.

There is a unique way to achieve this. Let $v_i \in V$ be a leaf of $\mathcal{T}$; hence, e_i is the only edge which has an endpoint on v_i. Let $\overline{D_i} = D_i \cup \{\star_{i,j}\}$, and define $\overline{T_i}$ by

$$\overline{T_i} = \begin{cases} T_i & \text{if } |T_i| \equiv k_i + 1 \mod 2 \\ T_i \cup \{\star_{i,j}\} & \text{if } |T_i| \equiv k_i \mod 2 \end{cases};$$

therefore $\overline{T_i}$ is an admissible partition on $\overline{D_i}$. Moreover, if $e_i = \{v_i, v_j\}$, we define T'_j by

$$T'_j = \begin{cases} T_j & \text{if } |T_i| \equiv k_i \mod 2 \Leftrightarrow \star_{i,j} \in \overline{T_i} \\ T_j \cup \{\star_{i,j}\} & \text{if } |T_i| \equiv k_i + 1 \mod 2 \Leftrightarrow \star_{i,j} \notin \overline{T_i} \end{cases};$$

note that, in general, T'_j will be different from $\overline{T_j}$, in fact it is just a partial update. Hence

$$\{T_1, \dots, T_n\} \longrightarrow \{T_1, \dots, \overline{T_i}, \dots, T'_j, \dots, T_n\}$$

We proceed with this method inductively on $\mathcal{T}^i$; at every step we will get $\overline{T_{i'}}$ from $T_{i'}$ corresponding to a leaf $v_{i'} \in V \setminus \{v_i\}$ and update the $T_{j'}$ corresponding to the other node $v_{j'}$ connected to $v_{i'}$. In $n-2$ steps we are left with a tree with two leaves v_l and v'_l and one edge e_l, so we can repeat this procedure one more time starting from any of the two leaves.

Example 1.3. Let $g = 4$, $S = \{1, \dots, 10\}$, $k_i = 1$ for all $i = 1,2,3,4$. Let $\hat{D}_1 = \{1,2,3\}$, $\hat{D}_2 = \{4,5,6\}$, $\hat{D}_3 = \{7,8,9\}$ and $\hat{D}_4 = \{10\}$. Hence the tree is $\mathcal{T} = (\{v_1, v_2, v_3, v_4\}, \{(v_1, v_4), (v_2, v_4), (v_3, v_4)\})$. By definition, we have $\overline{D_1} = \{1,2,3,\star_{1,4}\}$, $\overline{D_2} = \{4,5,6,\star_{2,4}\}$, $\overline{D_3} = \{7,8,9,\star_{3,4}\}$, $\overline{D_4} = \{10, \star_{1,4}, \star_{2,4}, \star_{3,4}\}$. Moreover, $D_i = \hat{D}_i$ for $i = 1,2,3$.

Let $T = \{1,2,5\} \in \mathcal{P}^g$; $T_1 = \{1,2\}$, $T_2 = \{5\}$, $T_3 = T_4 = \emptyset$. Let start with v_1: T_1 is admissible in $\overline{D_1}$, hence $\overline{T_1} = T_1$ and $T'_4 = \{\star_{1,4}\}$ and we prune the leaf v_1. Hence $\mathcal{T}^1 = (\{v_2, v_3, v_4\}, \{(v_2, v_4), (v_3, v_4)\})$. Next is the leaf v_2: T_2 is not an admissible partition in $\overline{D_2}$, hence $\overline{T_2} = T_2 \cup \{\star_{2,4}\}$ and we prune the leaf v_2; note that we cannot use $\star_{2,4}$ again for T_4. Last step is removing the leaf v_3: T_3 is an admissible partition in $\overline{D_3}$, hence $\overline{T_3} = T_3$ and $\overline{T_4} = T''_4 = T'_4 \cup \{\star_{3,4}\}$. Everything is fine, indeed

- $\overline{T_1} \cap S = \{1,2\} = T \cap \hat{D}_1$, and so on;
- $\overline{T_1}, \overline{T_2}, \overline{T_3}$ are admissible partitions by construction; $\overline{T_4}$ is admissible on $\overline{D_4}$;
- $\overline{T_1} \cup \overline{T_2} \cup \overline{T_3} \cup \overline{T_4} = \{1,2,5,\star_{1,4},\star_{2,4},\star_{3,4}\} = T \cup \{\star_e\}_{e \in E}$;
- $\star_{1,4} \in \overline{T_4}$, $\star_{2,4} \in \overline{T_2}$, $\star_{3,4} \in \overline{T_3}$.

Note that T is an odd partition and it splits in $\overline{T_1}$, which is even, $\overline{T_2}$, which is even, $\overline{T_3}$, which is odd, and $\overline{T_4}$, which is even.

On the other hand, if $\{\overline{T_i}\}$ are admissible partitions on $\{\overline{D_i}\}$, we want to glue them together to get an admissible partition $T \in \mathcal{P}^g$. The first guess is to take $\bigcup \overline{T_i} \setminus \{\star_i\}$, but this is only right if each $\star$-point belongs to a unique $\overline{T_i}$; by the way, we can switch between $\overline{T_i}$ and $\overline{T_i}^c$, where the complementary is inside $\overline{D_i}$, hence we can always arrange $\overline{T_i}$ in a unique way such that our condition holds. We choose any $\overline{T_i}$, and we look at every $\overline{T_j}$ such that $\{i,j\} \in E$; this means that $\overline{D_i} \cap \overline{D_j} = \star_{i,j}$. Hence we have the following

$$\tilde{T}_j = \begin{cases} \overline{T_j} & \text{if } \star_{i,j} \in \overline{T_i} \text{ and } \star_{i,j} \notin \overline{T_j} \\ & \quad \text{or } \star_{i,j} \in \overline{T_i} \text{ and } \star_{i,j} \in \overline{T_i} \\ \overline{T_j}^c & \text{if } \star_{i,j} \in \overline{T_i} \cap \overline{T_j} \\ & \quad \text{or } \star_{i,j} \notin \overline{T_i} \text{ and } \star_{i,j} \notin \overline{T_j} \end{cases};$$

by induction we get $\{\tilde{T}_i\}$ which is $\sim$-equivalent to $\{\overline{T_i}\}$ but now our first guess is right and

$$T = \bigcup_{i=1}^{n} \tilde{T}_i \setminus \{\star_{i,j}\}_{(i,j)\in E}$$

We need to check that $T \in \mathcal{P}^g$, but this is straightforward since

$$|T| = \sum_{i=1}^{n} |\overline{T_i}| - |\{\star_e\}_{e\in E}| \equiv \sum_{i=1}^{n}(k_i+1) - |E| \equiv g+n-(n-1) \equiv g+1 \mod 2. \tag{1.4}$$

Moreover, if we start with $\tilde{T}_i = \overline{T_i}^c$ in $\overline{D_i}$ and we proceed as before, we will end up with T^c, which is $\sim$-equivalent to T. We will simply write $T = \bigoplus T_i = \bigcup_{i=1}^{n} \tilde{T}_i \setminus \{\star_e\}_{e\in E}$ since now this makes sense.

Lemma 1.7. *If $\overline{T_i}$ is a balanced partition on $\overline{D_i}$, then $T = \bigoplus T_i$ is a balanced partition; if T is a D_i-balanced partition, then $T = \bigoplus T_i$ is a balanced partition.*

Proof. The first part is easy to prove, since $\overline{T_i} = k_i + 1$ and all the equivalences in (1.4) are, in fact, equalities:

$$|T| = \sum_{i=1}^{n} |\overline{T_i}| - |\{\star_e\}_{e\in E}| = \sum_{i=1}^{n}(k_i + 1) - |E| = g + n - (n-1) = g + 1;$$

The last part follows trivially from the first part since we can consider a tree with two vertices with associated sets D_i and D_i^c, i.e. we are contracting the subtrees we obtain when we remove the edge e_i. Hence the result follows. □

Theorem 1.8. *Let $T \in \mathcal{P}^g$ and $\mathcal{T} = (V, E)$ be a tree, with associated sets $\hat{D}_v$, for $v \in V$. Then $\overline{T_v}$ is a balanced partition on $\overline{D_v}$ for every $v \in V$ if and only if T_e is a D_e-balanced partition $\forall e \in E$.*

Proof. By Lemma 1.7, we can assume that T is a balanced partition, i.e. $|T| = g + 1$.

- $\Rightarrow$ If we remove the edge $e \in E$, we get two subtrees $\mathcal{T}_1 = (V_1, E_1)$ and $\mathcal{T}_2 = (V_2, E_2)$. Let $D_e = D_1 = \bigcup_{i\in V_1} \hat{D}_i$ and $D_2 = \bigcup_{i\in V_2} \hat{D}_i$; we have that $D_2 = D_e^c$ and we complete both sets adding an extra point $\star_e$ corresponding to the edge we removed. We have $\overline{D_i} = D_i \cup \{\star_e\}$ and

  $$|\overline{D_i}| = \sum_{i\in V} |\hat{D}_i| + 1 = \sum_{i\in V}(2k_i + 2 - d(i)) + 1 = 2(\sum k_i) + 2|V_i| - 2|E_i|.$$

  Let $T' = \bigcup_{i\in V_1} T_i$ and $T'' = \bigcup_{i\in V_2} T_i$. Since $|\overline{T_i}| = k_i + 1$, we have that

  $$|T'| = \sum_{i\in V} |\overline{T_i}| - |E_i| = \sum k_i + |V_i| - |E_i|,$$

  and the same goes for T''. Hence, T is D_e balanced.

$\Leftarrow$ If v is a leaf and $e \in E$ is the only edge which contains v, then $\overline{T_v}$ is balanced on $\overline{D_i}$ since it is equivalent to the fact that T is D_e-balanced. When we remove a vertex $v \in V$ which is not a leaf, we disconnect the tree in $d(v) = n$ subtrees $\mathcal{T}_1, \dots, \mathcal{T}_n$. As we did in the previous step, to every subtree we can associate a set $D_e = \bigcup_{i \in V_e} \hat{D}_i$ which is the 2-partition associated to the edge e. Hence, we know that T is D_e-balanced by hypothesis, i.e. $|\overline{T'_e}| = \frac{|D_e|}{2}$. Hence we have

$$\sum_{e \ni i} |\overline{D_e}| + \overline{D_i} = 2g + 2d(i) + 2$$

and, since T is balanced

$$\sum_{e \ni i} |\overline{T'_e}| + \overline{T_i} = |\bigcup T_i| + d(g) = g + 1 + d(i)$$

Rearranging the terms in both equations we get

$$\begin{aligned} \overline{D_i} &= 2g + 2d(i) + 2 - \sum_{e \ni i} |\overline{D_e}| = 2k_i + 2 \\ \overline{T_i} &= g + 1 + d(i) - \sum_{e \ni i} |\overline{T'_e}| = k_i + 1, \end{aligned}$$

which implies that $\overline{T_i}$ is balanced on $\overline{D_i}$.

$\square$

Hence, we have a way to split partitions. Since the way of splitting partition depends on the tree structure and not only on $|\overline{D_i}|$, the usual split for characteristics is not enough to describe what is happening. Hence, we will use that split as the canonical one and describe all other splits in terms of the tree. We start checking what the canonical split of characteristics looks like in terms of partitions.

Example 1.4. Let $g = 4$ and $k_i = 1$ for $i = 1, \dots, 4$. We want to determine how the partition $T_i = \{i\}$ splits, and therefore the shape of the tree.

$$\left.\begin{aligned} \varphi^{-1}(\{1\}) &= {}^{\mathsf{T}}(0\,1\,1\,1\,1\,1\,0\,1) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \\ \varphi^{-1}(\{2\}) &= {}^{\mathsf{T}}(1\,0\,1\,1\,1\,0\,0\,1) = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \\ \varphi^{-1}(\{3\}) &= {}^{\mathsf{T}}(1\,1\,0\,1\,1\,0\,1\,1) = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \\ \varphi^{-1}(\{4\}) &= {}^{\mathsf{T}}(1\,1\,1\,0\,1\,0\,1\,0) = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \\ \varphi^{-1}(\{5\}) &= {}^{\mathsf{T}}(1\,1\,1\,1\,0\,1\,0\,1) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \\ \varphi^{-1}(\{6\}) &= {}^{\mathsf{T}}(1\,1\,1\,1\,1\,0\,1\,0) = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \\ \varphi^{-1}(\{7\}) &= {}^{\mathsf{T}}(1\,1\,1\,0\,1\,0\,1\,1) = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \\ \varphi^{-1}(\{8\}) &= {}^{\mathsf{T}}(1\,1\,0\,1\,1\,0\,0\,1) = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \\ \varphi^{-1}(\{9\}) &= {}^{\mathsf{T}}(1\,0\,1\,1\,1\,1\,0\,1) = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \\ \varphi^{-1}(\{10\}) &= {}^{\mathsf{T}}(0\,1\,1\,1\,0\,1\,0\,1) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \end{aligned}\right. ;$$

as we can spot, for example, both $\varphi^{-1}(\{1\})$, $\varphi^{-1}(\{5\})$ and $\varphi^{-1}(\{10\})$ share the last three elements in the decomposition, i.e. the three characteristics are even $\oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$. We will denote $\{i\} = \overline{T_{i,1}} \oplus \overline{T_{i,2}} \oplus \overline{T_{i,3}} \oplus \overline{T_{i,4}}$, hence $\overline{T_{i,j}}$ give us information about $\overline{D_i}$, and therefore about $\{D_i\}$ and the tree structure. Note that $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ identifies via φ the empty partition. Hence we have the following relations.

$$\overline{T_{i,1}} = \emptyset \qquad \text{for } i \neq 1,5,10 \tag{C_1}$$
$$\overline{T_{i,2}} = \emptyset \qquad \text{for } i = 1,5,10 \tag{C_2}$$
$$\overline{T_{i,3}} = \emptyset \qquad \text{for } i = 4,6,7 \tag{C_3}$$
$$\overline{T_{i,4}} = \emptyset \qquad \text{for } i \neq 4,6,7 \tag{C_4}$$
$$\overline{T_{i,2}} = \overline{T_{j,2}} \qquad \text{for } i,j = 3,4,6,7,8 \tag{C_5}$$
$$\overline{T_{i,3}} = \overline{T_{j,3}} \qquad \text{for } i,j = 1,2,5,9,10 \tag{C_6}$$

The conditions (C_1) and (C_4) tell us that $\hat{D}_1 = \{1,5,10\}$ and $\hat{D}_4 = \{4,6,7\}$, since $\{i\} \cap \overline{D_1}$ is non empty only for $i = 1,5,10$ and the same goes for D_4. The conditions (C_5) and (C_6) tell us that $\hat{D}_2 = \{2,9\}$ and $\hat{D}_3 = \{3,8\}$. Moreover, conditions (C_2) and (C_3), together with other conditions, tell us that $\overline{D_1} \cap \overline{D_2}$, $\overline{D_2} \cap \overline{D_3}$ and $\overline{D_3} \cap \overline{D_4}$ are non-empty, hence we have

$$\overline{D_1} = \{1,5,10,\star_{1,2}\} \qquad \overline{T_{i,1}} = \begin{cases} \{i,\star_{1,2}\} & \text{if } i = 1,5,10 \\ \emptyset & \text{if } i \neq 1,5,10 \end{cases}$$

$$\overline{D_2} = \{2,9,\star_{1,2},\star_{2,3}\} \qquad \overline{T_{i,2}} = \begin{cases} \{i,\star_{1,2}\} & \text{if } i = 2,9 \\ \{\star_{1,2},\star_{2,3}\} & \text{if } i = 4,6,7 \\ \emptyset & \text{if } i = 1,3,5,8,10 \end{cases}$$

$$\overline{D_3} = \{3,8,\star_{2,3},\star_{3,4}\} \qquad \overline{T_{i,3}} = \begin{cases} \{i,\star_{3,4}\} & \text{if } i = 3,8 \\ \{\star_{2,3},\star_{3,4}\} & \text{if } i = 1,5,10 \\ \emptyset & \text{if } i = 2,4,6,7,9 \end{cases}$$

$$\overline{D_4} = \{4,6,7,\star_{3,4}\} \qquad \overline{T_{i,4}} = \begin{cases} \{i,\star_{3,4}\} & \text{if } i = 4,6,7 \\ \emptyset & \text{if } i \neq 4,6,7 \end{cases}$$

And we can summarize this with a picture.

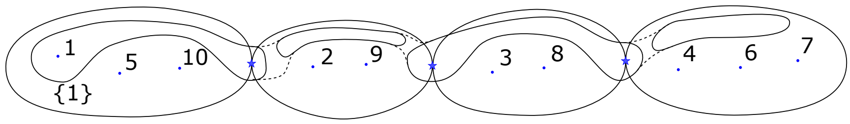


Figure 1.2. The split $\bigoplus \overline{D_i}$ and the partition $\{1\} = \{1\star_{1,2}\} \oplus \emptyset \oplus \{\star_{2,3},\star_{3,4}\} \oplus \emptyset$

Hence, the tree structure is the following:

This very example apply every time we are giving an explicit way, i.e. $\psi = M \circ \varphi$, with $M \in \Gamma_g$, to identify $\bigoplus \mathfrak{m}_i = \mathfrak{m}$ which may differ from the canonical one. What we want to do is to find a matrix M associated to the partition and the tree structure. We will do it in the case where every vertex of the tree is of weight 1, i.e. $|\overline{D_i}| = 4$ for every $i = 1,\dots,g$. We fix[1] isomorphisms $\overline{D_i} \to \{1,2,3,4\}$ for every i, then we check how a partition T splits on the $\overline{D_i}$'s.

[1] this is arbitrary for our purposes, since it only implies a choice of a very special permutation of the characteristics (see Chapter 2)

Let $\mathcal{D} = (\{\hat{D}_1, \dots, \hat{D}_g\}, \mathcal{T})$ be the data of partition and tree. Let $e_0 = {}^\intercal(0 \dots 0)$ and $e_1, \dots, e_{2g}$ be the canonical generators for $\mathbb{F}_2^{2g}$. We want to compute φ^{-1} and ψ^{-1}, where ψ is the map related to the data $\mathcal{D}$, of this characteristics. We can see how these characteristics split in the canonical way and associate to each component a characteristic $\overline{T_{i,j}}$ on $\overline{D_i}$:

$$\begin{aligned}
e_0 &= \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \oplus \dots \oplus \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) && = \overline{T_{1,0}} \oplus \dots \oplus \overline{T_{g,0}}\\
e_1 &= \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \oplus \dots \oplus \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) && = \overline{T_{1,1}} \oplus \dots \oplus \overline{T_{g,1}}\\
e_2 &= \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \oplus \dots \oplus \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) && = \overline{T_{1,2}} \oplus \dots \oplus \overline{T_{g,2}}\\
&\vdots && \vdots\\
e_g &= \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \oplus \dots \oplus \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) && = \overline{T_{1,g}} \oplus \dots \oplus \overline{T_{g,g}}\\
&\vdots && \vdots\\
e_{2g} &= \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \oplus \dots \oplus \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) && = \overline{T_{1,2g}} \oplus \dots \oplus \overline{T_{g,2g}};
\end{aligned}$$

hence, by the glueing algorithm we saw before, we get $\overline{T_0}, \overline{T_1}, \dots, \overline{T_{2g}}$. Now, we can apply φ^{-1} to these partitions to check the corresponding characteristics. Hence we have

$$M \cdot e_i = \varphi^{-1}(\overline{T_i}),$$

and we can determine M from these relations.

We have that φ is a $\Gamma_g[2]$-equivariant isomorphism and that choosing a partition is equivalent to choosing a fundamental system.

1.3 The congruence group $\Gamma_g[2]$

We recall that $\mathrm{Sp}_g(\mathbb{R})$ is the group of $2g \times 2g$ matrices $M = \left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ which preserves the symplectic form $J_g = \left(\begin{smallmatrix}0 & \mathrm{Id}_g\\ -\mathrm{Id}_g & 0\end{smallmatrix}\right)$, i.e.

$${}^\intercal M \cdot J_g \cdot M = J_g.$$

The same condition can be written in terms of A, B, C, D:

$$A^\intercal B = B^\intercal A, \quad C^\intercal D = D^\intercal C, \quad A^\intercal D - B^\intercal C = \mathrm{Id}_g.$$

We will denote by $\Gamma_g = \Gamma_g[1]$ the group $\mathrm{Sp}_g(\mathbb{Z})$.

Definition 1.9. *We will define the* principal congruence subgroup of level *n:*

$$\Gamma_g[n] := \{M \in \Gamma_g \mid M \equiv \mathrm{Id}_{2g} \mod n\},$$

and the Igusa subgroup of level *n:*

$$\Gamma_g[n, 2n] := \{\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \Gamma_g[n] \mid (A^\intercal C)_0 \equiv (B^\intercal D)_0 \equiv 0 \mod 2n\} \subseteq \Gamma_g[n],$$

where $(M)_0$ denotes the diagonal of a square matrix M.

We will briefly give some result about these subgroups. We have that $\Gamma_g[n]$ is a normal subgroup of Γ_g and the index is

$$[\Gamma_g : \Gamma_g[n]] = n^{g(2g+1)} \prod_{\substack{p|n \\ 1\leq k\leq g}} \left(1 - p^{-2k}\right).$$

Moreover let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g[n, 2n]$; if n is an even number, then $(A^{\intercal}B)_0 \equiv (B)_0 \mod 2$ and $(C^{\intercal}D)_0 \equiv (C)_0 \mod 2$. We will state the following Lemma, which proof can be found in [Igu64]

Lemma 1.9 ([Igu64])**.** *If n is even, then the following holds.*

(i) $\Gamma_g[n, 2n]$ *is a normal subgroup of* Γ_g *and its index in* $\Gamma_g[n]$ *is* 2^{2g}*;*

(ii) $\Gamma_g[n]/\Gamma_g[2n, 4n]$ *is an abelian group;*

(iii) $\Gamma_g[n, 2n]/\Gamma_g[2n, 4n]$ *is a vector space over* $\mathbb{F}_2$ *of dimension* $g(2g + 1)$*.*

We will now give an explicit set of the generators for $\Gamma_g[2]$.

(Type A) Let $1 \leq i, j \leq g$ and $i \neq j$; we set A_{ij} to be the $2g \times 2g$ identity matrix with the (i, j)-coefficient replaced by 2 and the $(g + j, g + i)$ coefficient replaced by -2; if $1 \leq i = j \leq g$, we set A_{ii} to be the $2g \times 2g$ identity matrix with the (i, i) and the $(g + i, g + 2)$ coefficients replaced by -1;

(Type B) let $1 \leq i < j \leq g$; we set B_{ij} to be the $2g \times 2g$ identity matrix with the $(g+i, j)$ and the $(g + j, i)$ coefficients replaced by 2; if $1 \leq i = j \leq g$, we set B_{ii} to be the $2g \times 2g$ identity matrix with the $(g + i, i)$ coefficient replaced by 2;

(Type C) we simply set $C_{ij} = {}^{\intercal}B_{ij}$.

All these matrices are in $\Gamma_g[2]$, and we have $g^2 + 2\frac{g(g+1)}{2} = g(2g + 1)$ such matrices, which are clearly independent.

1.4 The Siegel upper-half plane and characteristic Theta functions

Let $\mathbb{H}_g$ be the Siegel space of degree g:

$$\mathbb{H}_g := \{Z \in M_g(\mathbb{C}) \mid {}^{\intercal}Z = Z, \operatorname{Im} Z > 0\},$$

and z_{ij} denotes the (i, j)-component of the matrix $Z \in \mathbb{H}_g$, as usual. We associate to a matrix $Z \in \mathbb{H}_g$ an abelian variety $\mathbb{C}^g/\langle \mathrm{Id}_g, Z\rangle_{\mathbb{Z}}$, where $\langle \mathrm{Id}_g, Z\rangle_{\mathbb{Z}}$ denotes the lattice generated by column vectors of the matrix (Id_g, Z), which is a lattice of full rank, i.e. $2g$, because $\operatorname{Im} Z > 0$.

Let Γ_g denote the Siegel modular group $\mathrm{Sp}_g(\mathbb{Z})$; see Section 1.3 for details. It acts on the Siegel space by usual modular transformation: let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g$,

$$M \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

The quotient space $\mathscr{A}_g$ of $\mathbb{H}_g$ by Γ_g is the *moduli space* of principally polarized abelian varieties over $\mathbb{C}$. If n is a positive integer, then $\mathscr{A}_g[n]$ denotes the quotient of $\mathbb{H}_g$ by $\Gamma_g[n]$ which is called a *principally congruence subgroup of level* n, i.e.

$$\Gamma_g[n] := \{M \in \Gamma_g \mid M \equiv \mathrm{Id}_{2g} \mod n\}.$$

So $\mathscr{A}_g[n]$ is the moduli of pairs of principally polarized abelian varieties together with their n-division points and a symplectic structure. We may define also $\mathscr{A}_g[n, 2n]$ which is the quotient of $\mathbb{H}_g$ by $\Gamma_g[n, 2n] \subseteq \Gamma_g[n]$, which is defined by

$$\Gamma_g[n, 2n] := \left\{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g[n] \mid (A^\intercal C)_0 \equiv (B^\intercal D)_0 \equiv 0 \mod 2n\right\},$$

where $(M)_0$ denotes the diagonal of a square matrix M.

Since $\Gamma_g[n, 2n] \subseteq \Gamma_g[n]$, then there is a natural covering map of $\mathscr{A}_g[n]$ onto $\mathscr{A}_g[n, 2n]$ and similar natural maps exists for similar inclusions.

Definition 1.10 (Automorphic Factor). *Let $\Gamma \subseteq \Gamma_g$ a subgroup of finite index. A function $\rho\colon \Gamma \times \mathbb{H}_g \to \mathbb{C} \setminus \{0\}$ is called* automorphic factor *if*

(i) $\rho(M, Z)$ *is holomorphic in* Z*;*

(ii) $\rho(M_1M_2, Z) = \rho(M_1, M_2Z) \cdot \rho(M_2, Z)$ *for* $M_1, M_2 \in \Gamma$*;*

(iii) $\rho(-M, Z) = \rho(M, Z)$ *if* $\pm M \in \Gamma$*;*

Since every automorphic factor ρ can be written in the form

$$\rho(M, Z) = \chi(M) \det(CZ + D)^w$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$, w a positive rational number and χ a root of unity, then we can attach can define w as the *weight* of ρ. Now we can define *modular forms.*

Definition 1.11 (Siegel Modular Form). *A holomorphic function f on $\mathbb{H}_g$ is called a* Siegel modular form *of weight ρ for Γ if f satisfies*

$$f(MZ) = \rho(M, Z)f(Z) \quad \textit{for all } M \in \Gamma.$$

Moreover, if $g = 1$ we need the additional condition that f be holomorphic at cusps.

In this thesis, we will restrict to modular forms which satisfies the functional equation

$$f(MZ) = \det(CZ + D)^k f(Z),$$

i.e. we want $\chi(M) = 1$ and weight k.

Remark 1.7. With this definition, a modular form f for Γ is a global section of the coherent sheaf determined by the automorphy factor ρ, which is automatically invertible except, at least, at the fixed point set of Γ.

Now, we will define special modular forms which will play a big role in this thesis.

Definition 1.12 (Characteristic Theta function)**.** *Let* $\mathfrak{m} = \left(\begin{smallmatrix}\mathfrak{m}'\\ \mathfrak{m}''\end{smallmatrix}\right) \in \mathbb{Z}^{2g}$, *with* $\mathfrak{m}', \mathfrak{m}'' \in \mathbb{Z}^g$*. A* theta function *with characteristic* $\mathfrak{m}$ *is a function* $\theta[\mathfrak{m}](Z,x)\colon \mathbb{H}_g \times \mathbb{C}^g \to \mathbb{C}$ *such that*

$$\theta[\mathfrak{m}](Z,x) = \sum_{r\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}{}^{\mathsf{T}}\!\left(r+\frac{\mathfrak{m}'}{2}\right) Z \left(r+\frac{\mathfrak{m}'}{2}\right) + {}^{\mathsf{T}}\!\left(r+\frac{\mathfrak{m}'}{2}\right)\left(x+\frac{\mathfrak{m}''}{2}\right)\right),$$

where $\mathbf{e}(\bullet) = \exp(2\pi i\bullet)$.

Following Igusa [Igu64], we will prove that the evaluation at $x=0$ of Theta functions are, in a sense, modular functions of weight $\frac{1}{2}$; this is not trivial and we will just give a sketch of the proof. We first observe that, as an analytic function of x, the theta function $\theta[\mathfrak{m}](Z,x)$ is characterized, up to a constant factor, by the functional equation

$$\theta[\mathfrak{m}](Z, x+(Z,\mathrm{Id}_g)\mathfrak{n}) = \mathbf{e}(\mathfrak{m}\cdot\mathfrak{n})\,\mathbf{e}\left(-{}^{\mathsf{T}}\mathfrak{n}'x - \frac{1}{2}{}^{\mathsf{T}}\mathfrak{n}'Z\mathfrak{n}'\right)\theta[\mathfrak{m}](Z,x) \tag{1.5}$$

satisfied for every $\mathfrak{n}\in\mathbb{Z}^{2g}$; the scalar product $\mathfrak{m}\cdot\mathfrak{n}$ is given by the symplectic matrix $\left(\begin{smallmatrix}0 & \mathrm{Id}_g\\ -\mathrm{Id}_g & 0\end{smallmatrix}\right)$, hence

$$\mathfrak{m}\cdot\mathfrak{n} = {}^{\mathsf{T}}\mathfrak{m}\begin{pmatrix}0 & \mathrm{Id}_g\\ -\mathrm{Id}_g & 0\end{pmatrix}\mathfrak{n} = \begin{pmatrix}{}^{\mathsf{T}}\mathfrak{m}' & {}^{\mathsf{T}}\mathfrak{m}''\end{pmatrix}\begin{pmatrix}0 & \mathrm{Id}_g\\ -\mathrm{Id}_g & 0\end{pmatrix}\begin{pmatrix}\mathfrak{n}'\\ \mathfrak{n}''\end{pmatrix} = {}^{\mathsf{T}}\mathfrak{m}'\mathfrak{n}'' - {}^{\mathsf{T}}\mathfrak{m}''\mathfrak{n}'.$$

Since every theta function satisfies (1.5), we should check that if an analytic function $f(x)$ satisfies (1.5), then $\mathbf{e}(-\frac{1}{2}{}^{\mathsf{T}}\mathfrak{m}'x)f(x)$ is invariant under the translation $x\mapsto x+\mathfrak{n}''$ and, by using Fourier expansion, we get $f(x) = \mathrm{const}\cdot\theta[\mathfrak{m}](Z,x)$.

The next step is to observe that, as an analytic function of both Z and x, the theta function is characterized, up to a constant factor, by the following *heat equation*:

$$\sum_{1\le j,k\le g}\gamma_{jk}\frac{\partial^2\theta[\mathfrak{m}]}{\partial z_j\partial z_k} = 4\pi i\sum_{1\le j\le k\le g}\gamma_{jk}\frac{\partial\theta[\mathfrak{m}]}{\partial Z_{j,k}} \tag{1.6}$$

which is satisfied for every complex coefficients $\gamma_{jk} = \gamma_{kj}$.

Now we can state some transformation formulas which involve theta functions; this Lemma also allow us to introduce the action of the modular group on the characteristic space.

Lemma 1.10. *Let* λ *be a complex non-singular matrix and* μ *a complex symmetric matrix, both of degree* g*. Then*

$$\theta[\mathfrak{n}](Z',x') = K\,\mathbf{e}\left(\frac{1}{2}{}^{\mathsf{T}}x\mu x\right)\theta[\mathfrak{m}](Z,x) \tag{1.7}$$

with $K = K(M,Z,\mathfrak{m}) =$ *constant, as functions of* $x' = \lambda x$ *if and only if there exists an element* $M = \left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in\Gamma_g[1]$ *such that*

- $\lambda = {}^{\mathsf{T}}(CZ+D)^{-1}$
- $\mu = (CZ+D)^{-1}C$
- $Z' = M\cdot Z \mod 2$

- $\mathfrak{n} = M \cdot \mathfrak{m} \mod 2$

where $M \cdot \mathfrak{m}$ *is defined in the following way*

$$M \cdot \mathfrak{m} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \mathfrak{m} + \begin{pmatrix} (C^{\intercal}D)_0 \\ (A^{\intercal}B)_0 \end{pmatrix}. \tag{1.8}$$

This can be proved showing that $\theta[\mathfrak{n}](Z', x')$ satisfies the same functional equation as $\mathbf{e}(\frac{1}{2}{}^{\intercal}x\mu x)\theta[\mathfrak{m}](Z, x)$, hence they only differ by a constant factor.

We will come back to (1.8) in Section 1.1, but we need now the notion of parity of a characteristic: we will say that $\mathfrak{m}$ is *even* if $\mathbf{e}(\mathfrak{m}) := \mathbf{e}({}^{\intercal}\mathfrak{m}'\mathfrak{m}'') = 1$, *odd* otherwise. These characteristics are called even or odd because we have

$$\theta[\mathfrak{m}](Z, -x) = \mathbf{e}(\mathfrak{m})\theta[\mathfrak{m}](Z, x). \tag{1.9}$$

We will now introduce the Thetanullwerte, which are special theta functions specialized in $x = 0$, which have a special role in our work.

Definition 1.13 (Thetanullwerte). *Let* $\mathfrak{m} \in \mathbb{Z}^{2g}$. *A* thetanullwert *is a function* $\theta[\mathfrak{m}](Z) \colon \mathbb{H}_g \to \mathbb{C}$ *such that*

$$\theta[\mathfrak{m}](Z) := \theta[\mathfrak{m}](Z, 0)$$

Hence, by (1.9) we have clearly that odd thetnullwerte are identically 0 as a function of Z. Moreover, we can state (1.7) for the thetanullwerte:

$$\theta[M\mathfrak{m}](MZ) = K\theta[\mathfrak{m}](Z), \tag{1.10}$$

and our next target is to determine K more explicitly. Using the heat equation (1.6) and the fact that

$$dZ' = {}^{\intercal}(XZ + D)^{-1}dZ(CZ + D)^{-1}$$

holds between differentials, we get, for any even characteristic,

$$\sum_{1 \le j \le k \le g} \gamma_{jk} \frac{\partial \log K}{\partial Z_{j,k}} = \frac{1}{2} \sum_{1 \le j,k \le g} \gamma_{jk}\mu_{j,k}.$$

It is non trivial to check that $\det(CZ + D)^{\frac{1}{2}}$ satisfies the same differential equation as K, hence

$$K = K_1 \cdot \det(CZ + D)^{\frac{1}{2}},$$

and $K_1 = K_1(M, \mathfrak{m})$ no longer depends on Z. Hence we have the important *transformation formula* for thetanullwerte

$$\theta[M\mathfrak{m}](MZ) = K_1 \det(CZ + D)^{\frac{1}{2}}\theta[\mathfrak{m}](Z). \tag{1.11}$$

We can express the transformation formula in a even more explicit way.

$$\theta[M\mathfrak{m}](MZ) = \kappa(M)\,\mathbf{e}(\varphi_{\mathfrak{m}}(M))\det(CZ + D)^{\frac{1}{2}}\theta[\mathfrak{m}](Z),$$

where

$$-\frac{1}{8}({}^{\intercal}\mathfrak{m}'{}^{\intercal}BD\mathfrak{m}' + {}^{\intercal}\mathfrak{m}''{}^{\intercal}AC\mathfrak{m}'' - 2{}^{\intercal}\mathfrak{m}'{}^{\intercal}BC\mathfrak{m}'' - 2({}^{\intercal}AB)_0(D\mathfrak{m}' - C\mathfrak{m}'')$$

and $\kappa(M)$ is a constant which only depends on M and we can give an explicit formula for $\kappa(M)^2$.

Lemma 1.11 ([Igu64]). *We choose* $\det(CZ+D)^{\frac{1}{2}}$ *to be positive if* Z *is a pure imaginary diagonal matrix. Hence we have*

$$\begin{aligned}\kappa(A_{ij}) &= 1 \quad (i \neq j)\\ \kappa(A_{ii}) &= \mathbf{e}\left(-\frac{1}{4}\right)\\ \kappa(B_{ij}) &= \kappa(C_{ij}) = 1 \quad (i \leq j)\end{aligned}$$

This allow us to compute $\kappa(M)$ for every M, since those matrices generate the whole $\Gamma_g[2]$. By the way, we can make all this easier to compute: let $(-4 \mid n)$ be the Kronecker symbol for the prime discriminant -4. This means that $(-4 \mid n) \equiv 0 \mod 4$ if n is even and ± 1 if $n \equiv \pm 1 \mod 4$. If $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g[2]$ as usual, we can define

$$(-4 \mid M) = \prod_{1 \leq i,j \leq g} (-4 \mid d_{i,j}),$$

and $M \mapsto (-4 \mid M)$ is a character of $\Gamma_g[2]$. We want to use this information to write the transformation formula for $\psi = \theta[\mathfrak{m}]\theta[\mathfrak{n}]$; there exists a character χ of $\Gamma_g[2]$ such that

$$\psi(M \cdot Z) = (-4 \mid M)\chi(M)\det(CZ+D)\psi(Z)$$

In particular, we have that for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g[4,8]$ the following holds

$$\begin{aligned}(-4 \mid M) &= 1\\ A &\equiv 0 \mod 2\\ B &\equiv 0 \mod 4\end{aligned}$$

so $\chi(M) = 1$; moreover, if $M \in \Gamma_g$, $\kappa(M)\,\mathbf{e}(\varphi_{\mathfrak{m}}(M))$ is an eight root of 1, while if $M \in \Gamma_g[2]$, $\kappa(M)\,\mathbf{e}(\varphi_{\mathfrak{m}}(M))$ is a fourth root of 1. Hence we have the following result.

Lemma 1.12. *Let* $\psi(Z) = \theta[\mathfrak{m}](Z)\theta[\mathfrak{n}](Z)$*; then, for every* $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g[4,8]$ *we have*

$$\psi(MZ) = \det(CZ+D)\psi(Z)$$

This shows that $\psi[\mathfrak{m}]$ is a modular form for $\Gamma_g[4,8]$, with weight $w = 1$. Moreover,

$$\frac{\theta[\mathfrak{m}](Z)}{\theta[\mathfrak{n}](Z)}$$

is a meromorphic function on $\mathscr{A}_g[4,8]$ and on every covering of $\mathscr{A}_g[4,8]$.

Let us go back to modular forms. It's easy to check that the sum of two modular forms of the same weight is a modular form of the same weight, while the product of two modular forms is a modular form of weight the sum of the weights. Hence we can define the graded algebra

$$A(\Gamma_g) := \oplus A_k(\Gamma_g),$$

where $A_k(\Gamma)$ is the ring of modular forms of weight k. We will give some explicit examples of modular forms.

Example 1.5. The thetanullwerte $\theta[\mathfrak{m}]^4$ is a modular form relative to $\Gamma_g[2]$. This holds since

$$\theta[\mathfrak{m}](M \cdot Z)^4 = \kappa(M)^4 \, \mathbf{e}(4\varphi_{\mathfrak{m}}(M)) \det(CZ+D)^2 \theta[\mathfrak{m}](Z)$$

and $\kappa(M)^4 = \mathbf{e}(4\varphi_{\mathfrak{m}}(M)) = 1$.

Example 1.6. The polynomial $\sum_{\mathfrak{m}\in\mathcal{F}_e^g} \theta[\mathfrak{m}]^8$, in the thetanullwerte, is a modular form relative to Γ_g. In fact, $\kappa(m)^8 = \mathbf{e}(8\varphi_m(M)) = 1$, and this means that Γ_g permutes the $\theta[\mathfrak{m}]$, fixing the parity. Hence, if we act with $\gamma \in \Gamma_g$ on $\sum_{\text{even}} \theta[\mathfrak{m}]^8$ we are just permuting terms. Note that, if $M \in \Gamma_g$, $\kappa(M)^4$ and $\mathbf{e}(4\varphi_{\mathfrak{m}}(M))$ are either ± 1, hence

$$\sum_{\mathfrak{m}\in\mathcal{P}_e^g} \theta[\mathfrak{m}]^4$$

is modular with respect to $\Gamma_g[2]$ but it is identically 0 with respect to Γ_g, hence it is not modular.

Example 1.7. If $\sigma \in \Gamma_g/\Gamma$, we denote by

$$\sigma \cdot \{\mathfrak{m}_1, \dots, \mathfrak{m}_k\} = \{\sigma\mathfrak{m}_1, \dots, \sigma\mathfrak{m}_k\}$$

the action of σ on a set of characteristics. Let $P(\{\mathfrak{m}_1, \dots, \mathfrak{m}_k\}) = \prod_{i=1}^k \theta[\mathfrak{m}_i]$. Moreover, let $V = \{T \in \mathcal{P}_e^g \mid |T| = |g+1|\}$ and $\mathfrak{V} = \varphi^{-1}(V)$; then

$$F_H = \sum_{\sigma\in\Gamma_g/\Gamma} P(\sigma \cdot \mathfrak{V})^8$$

is a modular for relative to Γ_g. This modular form is very important since $P(V)^8$ is a product of all the theta functions which does not vanish on a chosen component, while it vanishes on every other component of the hyperelliptic locus. Hence we have that F_H never vanishes on the hyperelliptic locus, indeed on every component there is precisely one term which is not 0, while all other terms are 0. We will make use of this modular form in Section 2.5. This modular form also proves that the *slope estimate* (see 2.16.1) is sharp, in fact it has slope $8 + \frac{4}{g}$. This modular form was first introduced by Salvati Manni [SM00].

The last two examples belong to a more general picture: every symmetrization with respect to Γ_g/Γ of a form which is modular with respect to Γ is a modular form with respect to Γ_g. Another important class of modular forms is the class of *Eisenstein Series.*

Example 1.8. Let Λ be a lattice in $\mathbb{C}^g$, i.e. a $2g$-rank discrete $\mathbb{Z}$-module inside $\mathbb{C}^g$. Define, for $k > 2$,

$$E_k(\Lambda) = \sum_{\lambda\in\Lambda\setminus\{0\}} \lambda^{-k}.$$

The condition $k > 2$ is needed for the convergence of the series. This is a modular form since every lattice $\Lambda = \Lambda(Z)$ is represented as a point $Z \in \mathbb{H}_g$, i.e. $\Lambda = \langle \mathrm{Id}_g, Z\rangle_{\mathbb{Z}}$ and two lattices are the same if they differ by an element of Γ_g, i.e. the group of automorphisms of the lattice, hence the function is defined on $\mathbb{H}_g/\Gamma_g$. Moreover, E_k is a modular form of weight k, hence it is identically 0 if k is odd.

$$\mathfrak{m} = {}^{\intercal}(\mathfrak{n}'_1\, \mathfrak{n}'_2\, \mathfrak{n}''_1\, \mathfrak{n}''_2) = \left(\begin{smallmatrix} \mathfrak{n}'_1 \\ \mathfrak{n}''_1 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} \mathfrak{n}'_2 \\ \mathfrak{n}''_2 \end{smallmatrix}\right), \text{ with } \mathfrak{n}_i \in \mathbb{F}_2^{2g_i}$$

Remark 1.8. Let $Z \in \mathbb{H}_g$ be a period matrix of the form $Z = \left(\begin{smallmatrix} Z_1 & 0 \\ 0 & Z_2 \end{smallmatrix}\right)$, with $Z_1 \in \mathbb{H}_{g_1}$ and $Z_2 \in \mathbb{H}_{g_2}$, $g_1 + g_2 = g$. We split $\mathfrak{m} = {}^{\intercal}(\mathfrak{n}'_1\, \mathfrak{n}'_2\, \mathfrak{n}''_1\, \mathfrak{n}''_2) \in \mathbb{F}^{2g}$ as $\mathfrak{m}_1 \oplus m_2$, where $\mathfrak{m}_1 = \left(\begin{smallmatrix} \mathfrak{n}'_1 \\ \mathfrak{n}''_1 \end{smallmatrix}\right) \in \mathbb{F}^{2g_1}$, $\mathfrak{m}_2 = \left(\begin{smallmatrix} \mathfrak{n}'_2 \\ \mathfrak{n}''_2 \end{smallmatrix}\right) \in \mathbb{F}^{2g_2}$; then we have

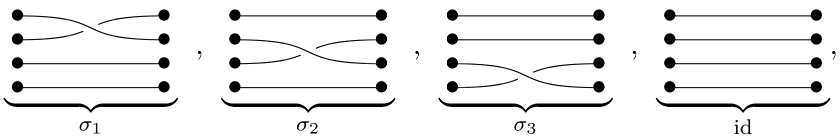
$$\theta\mathfrak{m}(Z) = \theta[\mathfrak{m}_1](Z_1) \cdot \theta[\mathfrak{m}_2](Z_2).$$

.

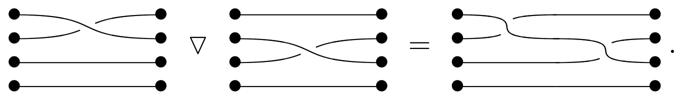
1.5 The Braid Group

In this section we will briefly introduce the Braid Group $\mathcal{B}^{(n)}$. Intuitively, if we have 2 sets of n ordered points each, we can tie together each point on the first set with a point on the second set: we call this tying a *braid*; by topological meaning, this is more than a simple permutation, since some braids which are associated to the same permutation might not be topologically equivalent. We will give a description of the group by generators and relations for the group $\mathcal{B}^{(4)}$.

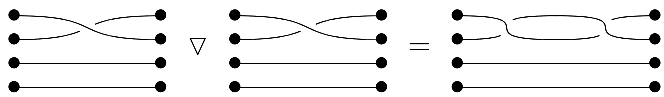
The generators and the identity for $\mathcal{B}^{(4)}$ are the following:



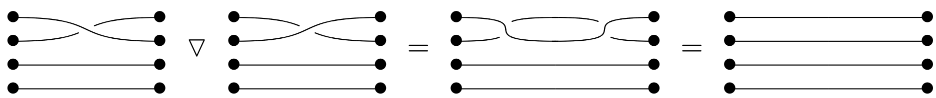
and the operation $\triangledown$ composes two braids identifying the objects in the middle; e.g. $\sigma_1 \triangledown \sigma_2$ is the following:



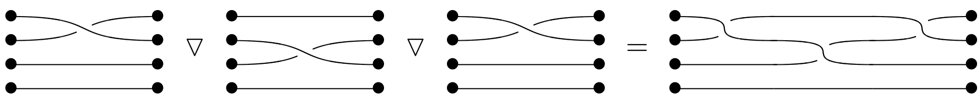
As we pointed out before, every braid induces a permutation, but there is some more. In the following example, $\sigma_1 \triangledown \sigma_1$ is the identity permutation, but the braid is not the identity braid id:



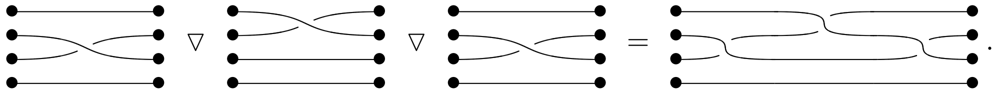
In particular, each generator in $\mathcal{B}^{(n)}$ has infinite order; the inverse for a generator is the same braid, but with the crossing inverted:



We also have the following relation $\sigma_1 \triangledown \sigma_2 \triangledown \sigma_1 = \sigma_2 \triangledown \sigma_1 \triangledown \sigma_2$, in fact



is the same as



This kind of relation works for every σ_i and σ_j where $|i-j|=1$, while σ_i and σ_j commute if $|i-j|\geq 2$. So we can define

$$\mathcal{B}^{(n)} = \left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, {}^{\sigma_i\sigma_j\sigma_i=\sigma_j\sigma_i\sigma_j \text{ for } |i-j|=1}_{\sigma_i\sigma_j=\sigma_j\sigma_i \text{ for } |i-j|\geq 2} \right\rangle,$$

where the $\triangledown$ is omitted for notation sake. We already noted that $\mathcal{B}^{(n)}$ is an infinite group and we have a homomorphism $\mathcal{B}^{(n)} \to S_n$ which associate each braid to the permutation it induces, i.e. we do not care any longer about topological differences. Hence, we can define

$$\mathcal{B}_0^{(n)} := \ker(\mathcal{B}^{(n)} \to S_n)$$

which is the group of *pure braids*. This group will come in handle soon.

1.6 Invariants and covariants of a binary form

A *binary form* of degree r is polynomial in two variables which is homogeneous of degree r. Let

$$f(u,\bar{x}) = \sum_{i=0}^{r} \binom{r}{i} u_i x_1^{r-i} x_2^i,$$

be a special binary form in x_1, x_2 of degree r, where $u = \{u_0, \ldots, u_r\}$ and $\bar{x} = \{x_1, x_2\}$.

We have an action of $\mathrm{SL}_2(\mathbb{C})$ over $\bar{x}$ in an obvious way: if $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$, then

$$M \cdot \bar{x} = \{ax_1 + bx_2, cx_1 + dx_2\}.$$

This action induces an action on u in the following way: we define $M \cdot u$ to be $\tilde{u}$, where

$$f(M\bar{x}, u) = f(\bar{x}, \tilde{u})$$

for any binary form f.

Example 1.9. Let $f(\bar{x}, u)$ be a binary form of degree 2. Let us see how the matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts on $u = \{u_0, u_1, u_2\}$. We have that

$$f(\bar{x}, u) = u_0 x_1^2 + 2u_1 x_1 x_2 + u_2 x_2^2,$$

hence

$$\begin{aligned} f(M\bar{x}, u) =& f(\{ax_1 + bx_2, cx_1 + dx_2\}, u) = \\ =& u_0(ax_1 + bx_2)^2 + 2u_1(ax_1 + bx_2)(cx_1 + dx_2) + u_2(cx_1 + dx_2)^2 = \\ =& (a^2u_0 + 2acu_1 + c^2u_2)x_1^2 + 2(abu_0 + (bc + ad)u_1 + cdu_2)x_1x_2 + \\ &+ (b^2u_0 + 2bdu_1 + d^2u_2)x_2^2. \end{aligned}$$

Hence we have the explicit action:

$$M \cdot u = \{a^2u_0 + 2acu_1 + c^2u_2, abu_0 + (bc + ad)u_1 + cdu_2, b^2u_0 + 2bdu_1 + d^2u_2\}.$$

Example 1.10. Let $f(\bar{x}, u)$ be a binary form of degree r. Let us see how the matrix $M = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, with $\lambda \in \mathbb{C}^\star$, acts on u. We have that

$$\begin{aligned} f(M\bar{x}, u) &= f(\{\lambda x_1, \lambda^{-1} x_2\}, u) = \\ &= \sum_{i=0}^{r} \binom{r}{i} u_i (\lambda x_1)^{r-i} (\lambda^{-1} x_2)^i = \\ &= \sum_{i=0}^{r} \binom{r}{i} \lambda^{r-2i} u_i x_1^{r-i} x_2^i. \end{aligned}$$

Hence we have the explicit action:

$$M \cdot u = \{\lambda^r u_0, \lambda^{r-2} u_1, \ldots, \lambda^{-r} u_r\}.$$

Definition 1.14. *A polynomial $g(\bar{x}, u) \in \mathbb{C}[\bar{x}, u] = \mathbb{C}[x_1, x_2, u_0, \ldots, u_r]$ which is homogeneous in x_1, x_2 is called* covariant *if it is invariant under the action of* $\mathrm{SL}_2(\mathbb{C})$*, i.e for any $M \in \mathrm{SL}_2(\mathbb{C})$ holds*

$$g(M\bar{x}, u) = g(\bar{x}, Mu),$$

or equivalently

$$g(M^{-1}\bar{x}, Mu) = g(\bar{x}, u).$$

We will denote by $C(r) \subseteq \mathbb{C}[\bar{x}, u]$ the ring of covariants; note that we used the binary form f to define the action of M on u, hence $f \in C(r)$, but a general element of $C(r)$ is not necessarily a binary form of degree r, since r is only related to the number of variables u_i, which is $r+1$. A covariant polynomial $g(\bar{x}, u)$ can be written in the following way

$$g(\bar{x}, u) = \sum_i g_i(\bar{x}, u)$$

where each g_i is a polynomial which is homogeneous also in u of degree i; since the action of $\mathrm{SL}_2(\mathbb{C})$ is homogeneous on u, each g_i is a covariant polynomial.

Let us set the weight of u_i to be i; we can define the *weighted degree* of a monomial $u_0^{a_0} \cdots u_r^{a_r}$:

$$\operatorname{wdeg} u_0^{a_0} \cdots u_r^{a_r} = \sum_{i=0}^{r} i a_i,$$

and we say that a polynomial in $u_0, \ldots, u_r$ is *isobaric* of weight w if the weighted degree of each monomial is w. A homogeneous polynomial is not necessarily isobaric and an isobaric polynomial is not necessarily homogeneous.

Example 1.11. The polynomial $u_0 u_1 + u_2^2$ is homogeneous and not isobaric; the polynomial $u_1^2 + u_2$ is isobaric and not homogeneous. Moreover, an isobaric polynomial of weight 0 and degree d is a scalar multiple of u_0^d; an isobaric polynomial of weight 1 and degree $d > 0$ is scalar multiple of $u_0^{d-1} u_1$; an isobaric polynomial of weight 2 and degree $d > 1$ is a linear combination of the two polynomials $u_0^{d-1} u_2$ and $u_0^{d-2} u_1^2$.

Lemma 1.13. *Let $g \in C(r)$, homogeneous of degree k in $\bar{x}$ and of degree s' in u. The coefficient for $x_1^{k-i} x_2^i$ is an isobaric polynomial in u of weight $s+i$, where s satisfies $rs' = 2s + k$.*

Proof. We can write

$$g(\bar{x}, u) = \sum_{i=0}^{k} g^{(i)}(u) x_1^{k-i} x_2^i,$$

where $g^{(i)}(u)$ is the coefficient of $x_1^{k-i}x_2^i$; we already know that $g^{(i)}$ is homogeneous in u and we want to prove that it is also isobaric. Let $M = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$; so, by covariance and by the Example 1.10, we have that

$$\begin{aligned} g(\bar{x}, u) &= g(M^{-1}\bar{x}, Mu) = \\ &= \sum_{i=0}^{k} g^{(i)}(\lambda^r u_0, \dots, \lambda^{-r} u_r)(\lambda^{-1} x_1)^{k-i} (\lambda x_2)^i = \\ &= \sum_{i=0}^{k} g^{(i)}(\lambda^r u_0, \dots, \lambda^{-r} u_r) \lambda^{2i-k} x_1^{k-i} x_2^i \end{aligned}$$

Hence, for every $i = 0, \dots, k$ we have

$$\begin{aligned} g^{(i)}(u) &= \lambda^{2i-k} g^{(i)}(\lambda^r u_0, \dots, \lambda^{-r} u_r) = (\text{since } \deg_u g^{(i)} = s') \\ &= \lambda^{2i-k+rs'} g^{(i)}(u_0, \lambda^{-2} u_1, \dots, \lambda^{-2r} u_r). \end{aligned} \tag{1.12}$$

Let $\alpha u_0^{a_0} \cdots u_r^{a_r}$ be a generic monomial of $g^{(i)}$; let $n = \operatorname{wdeg} \alpha u_0^{a_0} \cdots u_r^{a_r} = \sum i a_i$. Since M acts diagonally on u and by (1.12) we have that

$$\begin{aligned} \alpha u_0^{a_0} \cdots u_r^{a_r} &= \alpha \lambda^{2i-k-rs'} \lambda^{\Sigma - 2 i a_i} u_0^{a_0} \cdots u_r^{a_r} = \\ &= \alpha \lambda^{2i-k+rs'-2n} u_0^{a_0} \cdots u_r^{a_r} \end{aligned}$$

Hence $2i - k + rs' - 2n = 0$, so $2n = rs' - k + 2i$ and n does not depend on the monomial, since r, s', k, i which are determined a priori. So $g^{(i)}$ is isobaric and if s satisfies $rs' = 2s - k$, then $n = s + i$. $\square$

Definition 1.15. *Let g be a polynomial as in the hypothesis of Lemma 1.13; we say that g is a (k, s)-covariant*

Hence, we have a bigraded decomposition

$$C(r) = \bigoplus_{k,s \geq 0} C_{k,s}(r)$$

where $C_{k,s}(r)$ denotes the vector space of (k, s)-covariants.

Let $y = x_1/x_2$; we have an action of $\mathrm{SL}_2(\mathbb{C})$ on y which comes from the action on $\{x_1, x_2\}$:

$$M \cdot y = \frac{ay + b}{cy + d}.$$

In this case, the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acts trivially. Moreover, if $f(\bar{x}, u)$ as above, we can consider

$$x_2^{-r} f(\bar{x}, u) = \hat{f}(y, u)$$

which is still a polynomial in the variable y.

Lemma 1.14. *If $f(\bar{x}, u)$ as above and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$, then*

$$\hat{f}(My, u) = (cy + d)^{-r} \hat{f}(y, Mu)$$

Proof. This is a one-line computation:

$$\begin{aligned} \hat{f}(My, u) &= (cx_1 + dx_2)^{-r} f(M\bar{x}, u) = (cx_1 + dx_2)^{-r} f(\bar{x}, Mu) = \\ &= (cx_1 + dx_2)^{-r} x_2^r \hat{f}(y, Mu) = (cy + d)^{-r} \hat{f}(y, Mu). \end{aligned}$$

□

Let $\xi_1, \ldots, \xi_r$ be the roots of the polynomial $\hat{f}$ in the variable y; we can write

$$\frac{1}{u_0} \hat{f}(y, u) = \sum_{i=0}^{r} \binom{r}{i} \frac{u_i}{u_0} y^{r-i} = \prod_{i=1}^{r} (y - \xi_i);$$

if $\sigma_i(\bar{\xi})$ denotes the i-th symmetrical function in the variables $\{\xi_1, \ldots, \xi_r\} = \bar{\xi}$, then

$$\binom{r}{i} \frac{u_i}{u_0} = (-1)^{r-i} \sigma_i(\bar{\xi}). \tag{1.13}$$

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$; then the action of M on u induces an action of M on $\bar{\xi}$:

$$M \cdot \bar{\xi} = \left\{ \ldots, \frac{d\xi_i - b}{-c\xi_i + a}, \ldots \right\};$$

to prove this, let

$$\frac{1}{Mu_0} \hat{f}(y, Mu) = \prod_{i=1}^{r} (y - \zeta_i)$$

where the ζ_i are the roots of the polynomial $\hat{f}(y, Mu)$; hence $\bar{\zeta} = M\bar{\xi}$. If Mu_0 denotes the first component of Mu, we have

$$\begin{aligned} \prod_{i=1}^{r} (y - \zeta_i) &= \frac{1}{Mu_0} \hat{f}(y, Mu) = \frac{(c + yd)^r}{Mu_0} \hat{f}(My, u) = \\ &= \frac{(cy + d)^r}{Mu_0/u_0} \prod_{i=1}^{r} (My - \xi_i) = \frac{(cy + d)^r}{Mu_0/u_0} \prod_{i=1}^{r} \left(\frac{ay + b}{cy + d} - \xi_i \right) \end{aligned}$$

Since the first and the last polynomial are equals they share the same roots; up to a permutation, we can say that ζ_i is the root of $My - \xi_i$:

$$My - \xi_i = 0 \Rightarrow \frac{ay + b}{cy + d} - \xi_i = 0 \Rightarrow y - \frac{d\xi_i - b}{-c\xi_i + a} = 0,$$

hence $\zeta_i = \frac{d\xi_i - b}{-c\xi_i + a}$. Moreover, we can explicitly compute Mu_0, since it is the coefficient of x_1^r in $f(M\bar{x}, u)$: if M as usual, then

$$\begin{aligned} f(M\bar{x}, u) &= f(\{ax_1 + bx_2, cx_1 + dx_2\}, u) \\ &= \sum_{i=0}^{r} \binom{r}{i} u_i (ax_1 + bx_2)^{r-i} (cx_1 + dx_2)^i; \end{aligned}$$

hence

$$Mu_0 = \sum_{i=0}^{r} \binom{r}{i} u_i a^{r-i} c^i = u_0 \prod_{i=1}^{r} (a - \xi_i c)$$

Definition 1.16. *Let* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ *as usual. A polynomial* $h(y, \bar{\xi}) \in \mathbb{C}[y, \xi_1, \ldots, \xi_r]$ *is a* (k, s)-covariant *if*

(i) $h\left(\frac{ay+b}{cy+d}, \frac{a\xi_1+b}{c\xi_1+d}, \ldots, \frac{a\xi_r+b}{c\xi_r+d}\right) = (cy+d)^{-k} \prod_{i=1}^r (c\xi_i + d)^{-s} h(y, \bar{\xi})$.

(ii) $h(y, \bar{\xi})$ *is symmetric in* $\xi_1, \ldots, \xi_r$.

Since we have two notions of (k, s)-covariance, we should check that this definition is coherent with the notion we used before. In fact, let $g(\bar{x}, u) \in C_{k,s}(r)$; let us put $\hat{g}(y, u) = x_2^{-k} g(\bar{x}, \bar{u})$ and, using (1.13) and specializing $u_0 = 1$, we can express $\hat{g}(y, u)$ as a polynomial in the ξ_i's, i.e. $\tilde{g}(y, \bar{\xi}) = \hat{g}(y, \bar{u})$. By the transformation formulas we checked before, we get that $\tilde{g}$ is a (k, s)-covariant polynomial in the sense of the Definition 1.16. On the other hand, let $h(y, \bar{\xi})$ be a covariant polynomial in the sense of the Definition 1.16; by condition (ii), we can express h as a polynomial in $u_1, \ldots, u_r$ and we can homogenize it with u_0; if we multiply by x_2^k, we get a polynomial in $\bar{x}, u$ which is (k, s)-covariant.

A $(0, s)$-covariant is called an *invariant* of degree s. We will denote by $S(2, r)$ the graded ring of invariants of a binary form, i.e.

$$S(2, r) = \bigoplus_s C_{0,s}(r).$$

If we drop the condition (ii), then we can define the space

$$S(r) = \oplus_s S(r)_s \subseteq \mathbb{C}[\xi_i - \xi_j \mid 1 \leq i < j \leq r],$$

where the homogeneous element $I \in S(r)_s$ is a polynomial of degree s with respect to each ξ_i and satisfies the identity

$$I\left(\ldots, \frac{a\xi_i + b}{c\xi_i + d}, \ldots\right) = \prod_{i=1}^r (c\xi_i + d)^{-s} I(\ldots, \xi_i, \ldots),$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$. Then, we have that $S(2, r)$ is just the invariant subring of $S(r)$ under the natural action of the symmetric group S_r, which assures the condition (ii) again.

Chapter 2

The moduli space of hyperelliptic curves

Curve: The loveliest distance
between two points

Mae West

2.1 Jacobian Map and moduli spaces

Let M be a compact Riemann surface of genus $g \geq 1$. Let $A_1, \dots, A_g$, $B_1, \dots, B_g$ be canonical generators for $\pi_1(M, p_o)$; we also take $[A_i]$ and $[B_i]$ as a basis for $H_1(M, \mathbb{Z})$ so that the intersection pairing in this basis is

$$\mathrm{J}_g = \begin{pmatrix} 0 & \mathrm{Id}_g \\ -\mathrm{Id}_g & 0 \end{pmatrix}.$$

Lemma 2.1. *Given a marked Riemann surface M of genus $g \geq 1$, there exists a unique basis $\omega_1, \dots, \omega_g$ of abelian differentials such that $\int_{A_j} \omega_i = \delta_{i,j}$.*

Hence, let $\omega_1, \dots, \omega_g$ as in Lemma 2.1 and we can form the matrix

$$\begin{pmatrix} \int_{A_1} \omega_1 & \cdots & \int_{A_g} \omega_1 & \int_{B_1} \omega_1 & \cdots & \int_{B_g} \omega_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \int_{A_g} \omega_1 & \cdots & \int_{A_g} \omega_g & \int_{B_1} \omega_g & \cdots & \int_{B_g} \omega_g \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_g & \Omega \end{pmatrix}.$$

If $\pi \colon \hat{M} \to M$ is the universal covering and $\pi(z_0) = p_0$, we have the Abel-Jacobi map $w \colon \hat{M} \to \mathbb{C}^g$ given by

$$w(z) = {}^{\mathsf{T}}\left(\int_{z_0}^{z} \omega_1 \cdots \int_{z_0}^{z} \omega_g \right).$$

Let $\Lambda_M = \mathbb{Z} + \mathbb{Z}\Omega$ be a lattice associated to M; if we act on $\hat{M}$ via deck transformation, the result via Abel-Jacobi map is a translation via an element the lattice Λ. Hence we have a map from M to the torus $\mathbb{C}^g/\Lambda_M$ and we can associate to every Riemann surface M the torus $\mathrm{Jac}(M) = \mathbb{C}^g/\Lambda_M$.

Proposition 2.2. *Let M be a marked Riemann surface, then $\Omega \in \mathbb{H}_g$ and* $\mathrm{Jac}(M)$ *is a principally polarized abelian variety.*

If the marking is changed, we have a different matrix Ω. By the way, the matrix Ω only depends on the homology classes $[A_i]$ and $[B_i]$; a different homology basis which preserves the intersection matrix J will be related to the previous one by an element of Γ_g. Hence the equivalence class of $\mathrm{Jac}(M)$ depends only upon M, regardless of the marking.

Since $\mathrm{Jac}(M)$ is an abelian group, we can extend the Abel-Jacobi map to a homomorphism $w\colon \mathrm{Div}(M) \to \mathrm{Jac}(M)$ where $\mathrm{Div}(M)$ is the free abelian group of points of M. The map restricted to $\mathrm{Div}^0(M)$ is independent of the base point and the kernel of the restriction is exactly the set of principal divisors and we define

$$\mathrm{Pic}(M) = {}^{\mathrm{Div}(M)}\!\big/_{\text{(principal divisors)}}.$$

Hence, the Abel-Jacobi map is also a group isomorphism between $\mathrm{Pic}^0(M)$ and $\mathrm{Jac}(M)$.

We want to give some general vanishing results about functions on M. If $D \in \mathrm{Div}(M)$, let $\mathcal{L}(D) = \{f \text{ meromorphic on } M \mid \mathrm{div}(f) + D \geq 0\}$.

Theorem 2.3 (Riemann Vanishing)**.** *Let M be a marked compact Riemann surface of genus $g \geq 1$ and let* $\mathrm{Jac}(M)$ *be its Jacobian. There exists $r \in \mathbb{C}^g$ such that $[2r] = w(K_M)$ in* $\mathrm{Jac}(M)$ *and*

$$\mathrm{ord}_0\, \theta(r - w(D) + z, \Omega) = \dim_{\mathbb{C}} \mathcal{L}(D),$$

for every $D \in \mathrm{Div}^{g-1}(M)$.

Definition 2.1. *A divisor class $\chi \in \mathrm{Pic}^{g-1}(M)$ such that $2\chi = K_M$ in $Pic^{2g-2}(M)$ is called a* theta characteristic*.*

We will make use of this notions in the context of hyperelliptic curves in the next section. Before that, we want to make a picture of the moduli spaces we are working with.

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g and $\mathscr{A}_g = \mathbb{H}_g/\Gamma_g$ the moduli space of principally polarized abelian varieties. Via the Jacobian map, we have $\mathrm{Jac}\colon \mathcal{M}_g \to \mathscr{A}_g$ which takes every smooth curve of genus g to the class of his period matrix; the image of $\mathcal{M}_g$ via Jac is the *Jacobian locus.* The moduli space $\mathscr{A}_g$ also contains abelian varieties which are associate to matrices which are *reducible.* By an abuse of notations, we denote the corresponding point in $\mathscr{A}_g$ by the same symbol Z.

Definition 2.2. *We say that Z is* reducible *if there exist $Z_i \in \mathbb{H}_{g_i}$, $\sum g_i = g$, and $M \in \Gamma_g$ such that*

$$M \cdot Z = \begin{pmatrix} Z_1 & 0 & \cdots & 0 \\ 0 & Z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Z_n \end{pmatrix} \tag{2.1}$$

Hence, we define $\overline{\mathrm{Jac}(\mathcal{M}_g)}$ as the closure of the Jacobian locus inside $\mathscr{A}_g$; a typical element of the closure is a matrix Z which is either the jacobian of a smooth curve, or a reducible matrix Z such that there exist $M \in \Gamma_g$ and $M \cdot Z$ is like in (2.1), and every Z_i is a jacobian of a smooth curve of genus g_i (cfr. [Hoy63]). Recall that the *Deligne-Mumford compactification* of $\mathcal{M}_g$ is the moduli space of stable genus g curves. An element of $\overline{\mathcal{M}_g}$ is a curve of genus g which is connected, projective and has at worst nodal singularities and a finite automorphism group. This is a smooth compactification with well-known boundary components. Moreover, there exists a morphism from $\overline{\mathcal{M}_g}$ to the closure of $\overline{\mathrm{Jac}(\mathcal{M}_g)}$ inside the Satake compactification $\mathscr{A}_g^*$ of $\mathscr{A}_g$ (see 2.4). We define $\mathcal{M}_g^{\mathrm{comp}}$ inside $\overline{\mathcal{M}_g}$ to be the pre-image of $\overline{\mathrm{Jac}(\mathcal{M}_g)}$; this is the moduli space of curves of compact type. The fiber of this map over a product of curves is at least 2-dimensional, except for the case $(g-1,1)$, i.e. the case in which we have a curve of genus $g-1$ and an elliptic curve, where the fiber is 1-dimensional. Anyway, we are interested in the hyperelliptic case, where $\mathcal{H}_g^{\mathrm{comp}}$ maps 1-1 on the closure of $\mathscr{H}_g$ inside $\mathscr{A}_g$.

We can consider on $\mathscr{A}_g$ a level 2 structure: let $\mathscr{A}_g[2] = \mathbb{H}_g/\Gamma_g[2]$ which is a covering of $\mathscr{A}_g[2]$ since $\Gamma_g[2] \subseteq \Gamma_g$. Hence we can also determine define $\overline{\mathrm{Jac}\,\mathcal{M}_g}[2]$ via pull back.

$$
\begin{tikzcd}
\overline{\mathrm{Jac}\,\mathcal{M}_g} \arrow[r, "Jac"] & \mathscr{A}_g \\
\overline{\mathrm{Jac}\,\mathcal{M}_g}[2] \arrow[u] \arrow[r, "\mathrm{Jac}[2]"] & \mathscr{A}_g[2] \arrow[u, "\pi"']
\end{tikzcd}
\tag{2.2}
$$

Let $\mathscr{C}_i$ be a smooth curve of genus g_i, for $i = 1, \ldots, n$, with $\sum g_i = g$, and let $Z_i = \mathrm{Jac}(\mathscr{C}_i)$. Any disjoint union of the smooth curves $\mathscr{C}_1, \ldots, \mathscr{C}_n$, with some points identified according to a tree structure, will result in a stable nodal curve in $\overline{\mathrm{Jac}\,\mathcal{M}_g}$ which is mapped, via Jac, onto a reducible matrix Z which is conjugate to a matrix as in (2.1). We remark that whenever the map Jac, or Jac[2], is injective on $\mathcal{M}_g$ it has positive dimensional fiber. Indeed, for example, when the curve $\mathscr{C}$ splits in two components, we need to choose an arbitrary point on each component, and we glue them together: all this possibilities maps on the same period matrix. On the other side, if we talk about hyperelliptic curves we have that the fibre is discrete since we can only glue two components via ramification points, which are a finite set in both components.

2.2 Meet a Hyperelliptic Curve

Let $g \geq 1$ and let $\xi_1, \ldots, \xi_{2g+2}$ be $2g+2$ distinct points of $\mathbb{P}^1\mathbb{C}$. If none of the ξ_i is ∞, then we construct the Riemann surface M associated to the plane curve $y^2 = \prod_{i=1}^{2g+2}(x - \xi_i)$. If one of the ξ_i is the point at infinity, we permute them so that $\xi_{2g+2} = \infty$ and let M be given by $y^2 = \prod_{i=1}^{2g+1}(x - \xi_i)$. The Riemann surface M is hyperelliptic because x has degree 2 as a function from M to $\mathbb{P}^1\mathbb{C}$. All hyperelliptic Riemann surfaces are known to be given in this way.

The $(\xi_i, 0)$ are the ramification point of x on M and we denote these by their images in $\mathbb{P}^1\mathbb{C}$, ξ_i, as is traditional. We denote by L_∞ the divisor at infinity: it is of the form $\inf_1 + \inf_2$ if inf is not a ramification point, i.e. $\xi_i \neq \infty$ for all i, and of the form 2∞ if ∞ is a ramification point, i.e. $\xi_{2g+2} = \infty$.

Let $S = \{1, \dots, 2g+2\}$ be the set of indexes of the branching points. We will use notation and results from Section 1.2: let $\mathcal{P}^g$ be the set of admissible partitions of S.

Definition 2.3. *Let $T \in \mathcal{P}^g$, define $f_T \in \operatorname{Pic}^{g-1}(M)$ by*

$$f_T := \sum_{i \in T} \xi_i - \frac{g-1-|T|}{2} L.$$

Moreover, if $U = \{1, \dots, g+1\}$ we have that $|T \triangle U| \equiv 0 \mod 2$; let $T' = T \triangle U$ and define $e_{T'} \in \operatorname{Pic}^0(M)$ by

$$e_{T'} := \sum_{i \in T'} \xi_i - \frac{|T'|}{2} L.$$

Lemma 2.4. *Let $T, T_1, T_2 \in \mathcal{P}^g$. We have that*

- $2e_{T'} = 0$ *in* $\operatorname{Pic}^0(M)$ *and* $2f_T = (g-1)L = K_M$ *in* $\operatorname{Pic}^{2g-2}(M)$, *where* K_M *is the canonical divisor;*
- $e_{T_1'} + e_{T_2'} = e_{T_1' \triangle T_2'} = e_{(T_1 \triangle T_2)'}$ *and* $f_{T_1} + f_{T_2} = f_{T_1 \triangledown T_2}$; *moreover,* $f_{T_1} + e_{T_2'} = f_{T_1 \triangle T_2}$;
- $e_{T_1'} = e_{T_2'}$ *if and only if* $T_1 \sim T_2$; *moreover* $f_{T_1} = f_{T_2}$ *if and only if* $T_1 \sim T_2$;
- *The set of* f_T *in* $\operatorname{Pic}^{g-1}(M)$ *gives all theta characteristics for* M.
- $\dim_{\mathbb{C}} \mathcal{L}(f_T) = \left| \frac{g+1-|T|}{2} \right|$

Recall, from Section 1.2 that $\varphi \colon \mathbb{F}_2^{2g} \to \mathcal{P}^g$ is an isomorphism and let $\mathfrak{m} = \mathfrak{m}_T \in \mathbb{F}_2^{2g}$ if $\varphi(\mathfrak{m}) = T$ and $T = T_{\mathfrak{m}}$ if $T = \varphi(\mathfrak{m})$ for some $\mathfrak{m} \in \mathbb{F}_2^{2g}$. Hence $\mathfrak{m}_T$ satisfies the same properties as f_T

Proposition 2.5. *Let M be a marked hyperelliptic curve given by $y^2 = \prod_{i \in S}(x - \xi_i)$, with $\xi_{2g+2} \in \hat{M}$ as the base point. For the map $\varphi \colon \mathbb{F}_2^{2g} \to \mathcal{P}^g$ the following conditions hold:*

- $\varphi \colon \mathbb{F}_2^{2g} \to \mathcal{P}^g$ *is an isomorphism;*
- $e(\mathfrak{m}_T) = (-1)^{\frac{g+1-|T|}{2}}\ \forall T \in \mathcal{P}^g$ *(recall that $e(\bullet)$ is the multiplicative form $q(\bullet)$ defined in (1.1));*
- $e(\mathfrak{m}_{T_1}, \mathfrak{m}_{T_2}) = (-1)^{|T_1 \cap T_2|}\ \forall T_1, T_2 \in \mathcal{P}^g$;
- $w(e_T) = w(e_{T'}) = (\Omega \ \mathrm{Id})\mathfrak{m}_T$ *in* $\operatorname{Jac}(M)$, $\forall T \in \mathcal{P}^g$;
- $w(e_U) = 0$;

Proof. We already proved that φ is an isomorphism; the second and the third property follow from the fact that φ preserves parity and azygeticity. For the fourth property, we have that

$$w(e_{T'}) = w\left(\sum_{i \in T'} \xi_i - \frac{|T'|}{2} L\right) = \sum_{i \in T'} w(\xi_i) - \frac{|T'|}{2}(2\xi_{2g+2}) = \sum_{i \in T'} (\Omega \ \mathrm{Id})\eta_i = (\Omega \ \mathrm{Id})\eta_{T'}.$$

The last point is just a corollary of the previous one, since $\varphi^{-1}(U) = 0$. $\square$

The following fundamental Proposition settles the question about which theta-nullwerte vanish and non-vanish on a hyperelliptic point. This result was first due to Thomae, in 1880's and then improved by Mumford [Mum84]. Mumford proved that if a jacobian point satisfies the same vanishing conditions, then it is a hyperelliptic point. We are only going to show the first part of this result, about the vanishing and non-vanishing.

Proposition 2.6 (Vanishing and non-vanishing)**.** *Let Ω be the period matrix of a marked hyperelliptic Riemann surface M with ξ_{2g+2} as the base point. Let $w(\xi_{2g+2}) = (\Omega\, I)\eta$. We have that $\theta[\mathfrak{m}](\Omega)$ does not vanish if and only if $\mathfrak{m}$ has hyperelliptic η-order zero, i.e.*

$$\theta[\mathfrak{m}](\Omega) \neq 0 \Leftrightarrow \exists T \in \mathcal{P}^g, \varphi(\mathfrak{m}) = T \text{ and } |T| = g+1.$$

Corollary 2.6.1. *There are exactly $\frac{1}{2}\binom{2g+2}{g+1}$ even theta functions which does not vanish on an hyperelliptic point Ω.*

Proof. We just need to count the number of T's with the desired property, i.e. $T \in \mathcal{P}^g$ and $|T| = g+1$. We need to choose $g+1$ points among $2g+2$, then we divide by 2 since $|T| = |T^c|$. $\square$

Let $\mathcal{H}_g$ denote the moduli space of hyperelliptic curves inside $\mathcal{M}_g$. As we did for $\mathcal{M}_g$, we can extend $\mathcal{H}_g$ to $\mathcal{H}_g^{\text{cpt}}$ which is the moduli space of tree-like hyperelliptic curves. We can complete diagram (2.2) with the moduli space of hyperelliptic curves:

$$\begin{tikzcd}
\mathcal{H}_g^{\text{cpt}} \arrow[r] & \overline{\operatorname{Jac}\mathcal{M}_g} \arrow[r, "\text{Jac}"] & \mathscr{A}_g \\
\mathcal{H}_g^{\text{cpt}}[2] \arrow[r] \arrow[u] & \overline{\operatorname{Jac}\mathcal{M}_g}[2] \arrow[r, "\text{Jac}"] \arrow[u] & \mathscr{A}_g[2] \arrow[u, "\pi"']
\end{tikzcd}$$

We will denote by $\mathscr{H}_g[2]$ the image of $\mathcal{H}_g^{\text{cpt}}[2]$ inside $\mathscr{A}_g[2]$. We want to recall some results about $\mathscr{H}_g[2]$.

The moduli space $\mathscr{H}_g[2]$ is a variety of dimension $2g-1$; for $g \leq 2$ it is irreducible, since it coincides with $\overline{\operatorname{Jac}\mathcal{M}_g}[2]$, i.e. all curves of genus 1 and 2 are hyperelliptic. However, if $g \geq 3$, $\mathscr{H}_g[2]$ breaks into irreducible components isomorphic to each other. Tsuyumine, in [Tsu90], shows that there is a natural homomorphism of the pure braid group $\mathcal{B}_0^{(2g+2)}$ to $\Gamma_g[2]$, while the image of the whole braid group $\mathcal{B}^{(2g+2)}$ in $\Gamma_g/\Gamma_g[2]$ is isomorphic to the symmetric group S_{2g+2}, which acts naturally on $\xi_1, \dots, \xi_{2g+2}$. Hence we have the following result by Tsuyumine [Tsu90]

Proposition 2.7. *Let $g \geq 2$. Then the number of components of hyperelliptic loci in $\overline{\operatorname{Jac}\mathcal{M}_g}[2]$ is*

$$\frac{[\Gamma_g : \Gamma_g[2]]}{(2g+2)!} = \frac{2^{g(2g+1)}}{(2g+2)!} \prod_{k=1}^{g} \frac{2^{2k}-1}{2^{2k}}.$$

Let $W \subseteq \operatorname{Spec}\mathbb{C}[\xi_1, \dots, \xi_{2g+2}]$; to $\xi \in W$ we associate the Riemann surface M as before, which maps to a point $Z = Z(\xi) \in \mathscr{A}_g[2]$. Hence we have a map $h\colon W \to \mathscr{A}_g[2]$, and the image of W via h is just one component of $\mathscr{H}_g[2]$, which we will denote by $\mathscr{H}_g^0[2]$. Hence, it is sufficient to study a single component, and then act with $\Gamma_g/\Gamma_g[2]$.

Moreover, the locus

$$\mathrm{RH}(g_1, g_2) := \left\{ Z \in \mathscr{A}_g[2] \mid M \cdot Z = \left(\begin{smallmatrix} Z_1 & 0 \\ 0 & Z_2 \end{smallmatrix}\right), M \in \Gamma_g, Z_i \in \mathscr{A}_{g_i}[2], g_1 + g_2 = g \right\}$$

is the set of reducible period matrices which are jacobians of hyperelliptic curves, and we can take the hyperelliptic points inside $\mathrm{RH}(g_1, g_2)$, which is a divisor in $\mathcal{H}_g^{\mathrm{cpt}}$ since $\dim \mathrm{RH}(g_1, g_2) = (2g_1 - 1) + (2g_2 - 1) = 2g - 2$, indeed every matrix which is conjugated to a 2-block diagonal matrix with every block hyperelliptic is still a hyperelliptic curve.

Before going to the fundamental vanishing property, we need the moduli space of *admissible double coverings*

2.3 Stable n-pointed curves and admissible double coverings

A stable (resp. semistable) n-pointed curve is, by definition, a complete connected curve B that has only nodes as singularities, together with an ordered collection $p_1, \ldots, p_n \in B$ of distinct smooth points such that every smooth rational component of the normalization of B has at least 3 (resp. 2) points lying over singular points or points among $p_1 \ldots, p_n$. Note that any stable n-pointed curve may admit non-trivial automorphisms, but its group of automorphisms is always finite. Since we are interested in n-pointed curves of genus 0, it's useful to remember that the group of automorphisms is a subgroup of the symmetric group S_n. According to [Knu83], the moduli space $\mathcal{M}_{0,n}$ of stable n-pointed curves of genus zero exists and it is a *fine* moduli space. By the way, we want to define $\mathcal{M}_{0,2g+2}^c$, which will be the moduli space of $2g+2$ pointed curves of genus zero $(B, p_1, \ldots, p_{2g+2})$ such that every smooth rational component of the normalization of B has an even number, greater than 2, of points lying over singular points or points among $p_1, \ldots, p_{2g+2}$.

Let $(B, p_1, \ldots, p_n) \in \mathcal{M}_{0,n}$ be a stable n-pointed curve of genus 0 and let $q_1, \ldots, q_k$ be the nodes of the curve B. An *admissible d-fold cover* of the curve $(B, p_1, \ldots, p_n)$ is a connected nodal curve C together with a regolar map $\pi \colon C \to B$ such that:

(i) $\pi^{-1}(B_{\mathrm{ns}}) = C_{\mathrm{ns}}$ and the restriction of π to this open set is a d-fold covering simply branched over the points p_i and otherwise unramified;

(ii) $\pi^{-1}(B_{\mathrm{sing}}) = C_{\mathrm{sing}}$ and for every node q of B and every node r of C lying over q, the d branches of C near r map to the branches of B near q with the same ramification index.

Here B_{ns} (resp. C_{ns}) is the set of smooth points of B (resp. C) and B_{sing} (resp. C_{sing}) is the set of nodes of B (resp. C). According to [HM82], the moduli space $\mathbf{H}_{d,g}$ of admissible d-coverings of stable $(2(g + d) - 2)$-pointed curves of genus 0 exists.

We want to show how to get an admissible 2-sheeted cover of a curve $(B, p_1, \ldots, p_{2g+2}) \in \mathcal{M}_{0,2g+2}^c$ in a unique way. We will use an algorithm which is very similar to the one we use to define the split of partitions (see Section 1.2). Write $B = L_1 \cup \ldots \cup L_n$, where L_i are the irreducible components of B. Let $p(i)$ be the set of points of the marking belonging to L_i and $q(i)$ the set of the nodes, i.e. the extra points that give the glueing of the rational components; hence we have that $|p(i)| + |q(i)| = 2g_i + 2$ and $\sum |p(i)| = 2g + 2$ and $\sum |q(i)| = 2n - 2$. Since every component $(L_i, p(i), q(i)$ is a smooth $2g_i + 2$-pointed curve of genus g_i, we have a unique double cover C_i of L_i branched over $p(i)$ and $q(i)$. The curve $C = C_1 \cup \dots C_n$ is an admissible double cover of B and it is unique since it is unique on every smooth component. On the contrary, if $\pi\colon C \to B$ is an admissible double cover, then B is a stable $(2g+2)$-pointed curve of genus 0. Hence we have the following Proposition ([AL02]).

Proposition 2.8. *There is a canonical isomorphism $\mathbf{H}_{2,g} \cong \mathcal{M}_{0,2g+2}$ of the moduli space $\mathbf{H}_{2,g}$ of admissible double covers of stable $(2g+2)$-pointed curves of genus 0 onto $\mathcal{M}_{0,2g+2}$.*

Moreover, every admissible double cover of a stable $(2g+2)$-pointed curve of genus 0 is a limit of a family of smooth double covers of a rational curve, which are all hyperelliptic. The limit is a tree of smooth hyperelliptic curves. On the other side, every tree of hyperelliptic curves is a limit of a family of smooth hyperelliptic curves, hence the canonical isomorphism extend to the boundary. Hence we have the following Proposition ([AL02]).

Proposition 2.9. *There is a canonical isomorphism $\mathbf{H}_{2,g}^{cpt} \cong \mathcal{H}_g^{cpt}[2]$ of the moduli space $\mathbf{H}_{2,g}$ of admissible double covers of stable $(2g+2)$-pointed curves of genus 0 onto $\mathcal{H}_g^{cpt}[2]$.*

Hence, as an immediate corollary, we have the following

Corollary 2.9.1. *There is a canonical isomorphism $\mathcal{M}_{0,2g+2}^{c} \cong \mathcal{H}_g^{cpt}[2]$.*

We will see in the next Chapter that the boundary of $\mathcal{M}_{0,2g+2}^{\mathrm{cpt}} \setminus \mathcal{M}_{0,2g+2}$ is a divisor consisting in all possible irreducible divisors Δ_D, $D \subseteq \{1, \dots, 2g+2\}$ and $|D|$ odd, where a general point of Δ_D represent a curve $B = B_1 \cup B_2$ with B_1 containing all the points p_i, $i \in D$ and B_2 containing all the points p_i, $i \in D^c$.

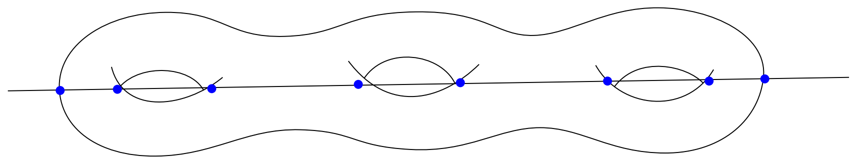


Figure 2.1. How we can try to visualize an admissible double covering of a 8-pointed curve of genus 0, resulting in a hyperelliptic curve of genus 3

This approach let us use the theory of invariants of a binary form the we introduced in Section 1.6. Indeed, we have a morphism $\mathcal{M}_{0,2g+2} \to \operatorname{Proj} S(2g+2)$: if $(B, p_1, \dots, p_{2g+2}) \in \mathcal{M}_{0,2g+2}$, we can consider the binary form which has exactly p_i's as roots. In a more explicity way: let $B = \mathbb{P}^1$ and $p_i = [a_i : b_i]$. Hence

$$f(x, y) = \prod_{i=1}^{2g+2} (b_i x - x_i y)$$

is the binary form associated to $(B, p_1, \dots, p_{2g+2})$. We want to extend this morphism to product of curves. We will make use of some of the notation introduced in Section 1.2. Recall that to any product of curves C, we can associate its dual graph

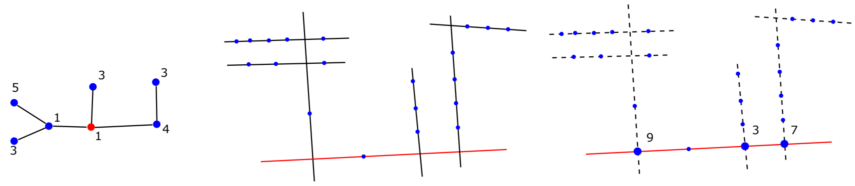


Figure 2.2. The tree associated to a product of smooth genus 0 curves, with $|p(i)|$ on every vertex and the central vertex, and how we are going to contract it onto the central component; note that $\sum |p(i)| = 20$ which is greater than 9, 3 and 7.

$\mathcal{T} = (V, E)$: each vertex $v_i \in V$ represent a component C_i of C and two vertices v_i and v_j are connected by an edge $e_{i,j} \in E$ if the corresponding components intersect. Thus, a connected tree all of whose vertices represent smooth curves of genus 0 correspond to a product of smooth curves of genus 0. To every vertex v_i we want to associate a weight $w(v_i) = k_i$ with $|p(i)| + |q(i)| = 2k_i + 2$, $p(i)$, $q(i)$ as before. If we remove a vertex $v(i)$, together with all the edges containing $v(i)$, we obtain $d(v_i)$ subtrees $\mathcal{T}_{i,j} = (v_{i,j}, E_{i,j})$, $j = 1, \dots d(v_i)$. We can associate to every such subtree the set $p(\mathcal{T}_{i,j}) = \cup_{l \in v_{i,j} p(l)}$. We say that v_i is a *central vertex* if $|p(\mathcal{T}_{i,j})| < g + 1$ for every $j = 1, \dots, d(v_i)$. We can do the same when we remove an edge $e \in E$: this disconnect the tree in 2 subtrees $\mathcal{T}_{e,1}$ and $\mathcal{T}_{e,2}$ and we have $p(\mathcal{T}_{e,1})$ and $p(\mathcal{T}_{e,2})$, which is, in particular, a 2-partition of $\{1, \dots, 2g + 2\}$

Lemma 2.10. *Let $\mathcal{T} = (V, E)$ denote a weighted tree as above and with the further property $(*)$ there is no edge $e \in E$ such that $|p(\mathcal{T}_{e,1})| = |p(\mathcal{T}_{e,2})| = g + 1$. Then $\mathcal{T}$ admits a unique central vertex.*

This is a trivial consequence of a more general result by Avritzer and Lange ([AL02, Lemma 3.2]). We will make use of this result to describe the image of $\mathcal{M}^{\text{cpt}}_{0,2g+2}$ inside Proj $\overline{S(2g + 2)}$, which is the space of stable (resp. semistable) binary forms, i.e. binary forms with multiple roots with no root of multiplicity $\geq g+1$ (resp. $> g + 1$). If we are in the hypothesis of Lemma 2.10 and $(*)$ holds, then we have a central vertex $\bar{v} = v_i$. The corresponding curve C_i will be the component on which we are going to contract all the others component. We can see an example in Figure 2.2. Let $q_{i,j} \in q(i)$ be point where C_i which intersects the curve corresponding to the subtree $\mathcal{T}_{i,j}$. Our binary form will be the one whose zeros are $[a_i : b_i] \in p(i)$, with multiplicity 1, and $[c_j : d_j] = q_{i,j} \in q(i)$, with multiplicity $m_j = |p(\mathcal{T}_{i,j})|$. Hence

$$f(x, y) = \prod_{l=1}^{|p(i)|} (b_l x - a_l) \prod_{l=1}^{|q(i)|} (d_l x - c_l y)^{m_l}$$

We map all other curves for which condition $(*)$ does not hold to the semistable point of $\overline{S(2g + 2)}$.

2.4 The ρ homomorphism

In this section we are going to define a ring homomorphism ρ between the graded ring of modular forms and the graded ring of projective invariants of a binary form.

Let $W = W^g$ as in the previous section. Let $a \in W$ and consider the point $Z \in \mathbb{H}_g$ associated with a. Then, exactly $\frac{1}{2}\binom{2g+2}{g+1}$ of the $\theta[\mathfrak{m}](Z)$ are different from zero.

Theorem 2.11. *Let $A(\Gamma_g)$ denote the graded ring of Siegel modular forms of degree g and of level one, and let $S = S(2, 2g+2)$. Then, there exists a ring homomorphism*

$$\rho\colon A(\Gamma_g) \to S$$

which increase the weight by a $\frac{1}{2}g$ ratio. The homomorphism ρ is uniquely defined except for the freedom $\rho \to i^{\alpha k}\rho$ on the homogeneous part of $A(\Gamma_g)_k$ of weight k for $\alpha = 0, 1, 2, 3$. An element ψ of $A(\Gamma_g)$ belongs to the kernel of ρ if and only if ψ vanishes at every point of $\mathbb{H}_g$ associated with a hyperelliptic curve.

Igusa showed that the theorem is actually true for ρ defined on a subring of $A(\Gamma_g)$, which coincides with $A(\Gamma_g)$ for every odd g. Moreover, he proves that a sufficient condition for the domain of ρ to be the whole $A(\Gamma_g)$, even if g is even, is that there exists an element $\psi \in A(\Gamma_g[2])_k$ for an odd k which is a polynomial in the thetanullwerte and which satisfies $\psi(Z) \neq 0$ for at least one point Z associated with a hyperelliptic curve. Hence if we prove the following Claim we have the result.

Claim 2.12. *For every even g there exist some thetanullwerte whose product is a modular form with respect to $\Gamma_g[2]$ of odd weight which doesn't vanish everywhere on the hyperelliptic locus.*

We prove this in two steps.

Lemma 2.13. *For $g = 2^n$, the product of all even thetanullwerte which does not vanish on a component of the hyperelliptic locus is a modular form for $\Gamma_g[2]$ of odd weight.*

Proof. Since $\Gamma_g[2]$ fixes the parity of every thetanyllwerte and the single components, the product of all the even thetanullwerte which does not vanish on a component of the hyperelliptic locus is a modular form for the whole $\Gamma_g[2]$. We only need to show that the weight is odd. The number of even thetanullwerte which does not vanish on a chosen component is given by $\frac{1}{2}\binom{2g+2}{g+1}$; hence, for $g = 2^n$

$$\begin{aligned}
\frac{1}{2}\binom{2\cdot 2^n+2}{2^n+1} &= \frac{1}{2}\frac{(2\cdot 2^n+2)\cdot\ldots\cdot(2^n+2)}{(2^n+1)\cdot\ldots\cdot 1} = \\
&= \frac{1}{2}\frac{(2\cdot 2^n+2)\cdot(2\cdot 2^n)\cdot\ldots\cdot(2^n+2)}{2^n\cdot(2^n-2)\cdot\ldots\cdot 2}\,\frac{(2\cdot 2^n+1)\cdot\ldots\cdot(\cdot 2^n+3)}{(2^n+1)\cdot(2^n-1)\cdot\ldots\cdot 1} = \\
&= \frac{1}{2}\frac{2^{2^{n-1}+1}}{2^{2^{n-1}-1}}\frac{\text{odd}}{\text{odd}} = 2\frac{\text{odd}}{\text{odd}},
\end{aligned}$$

so there are $2\cdot\text{odd}$ thetanullwerte which does not vanish on a component and their product is a modular form of weight $\frac{2\cdot\text{odd}}{2} = \text{odd}$. □

Since we chose the thetanullwerte which does not vanish on a component, there exists a point $\bar{Z}$ in that component such that the product of the thetanullwerte does not vanish on $\bar{Z}$; we will use this in the next step. We will need the following Lemma (see [SM00], [Igu80]):

Lemma 2.14. *If M is a matrix whose columns are even characteristics, and*

$$M\cdot{}^{\mathsf{T}}M \equiv \begin{pmatrix} 0 & 1_g \\ 1_g & 0 \end{pmatrix} \mod 2 \tag{2.3}$$

$$\operatorname{diag}(M\cdot{}^{\mathsf{T}}M) \equiv 0 \mod 4, \tag{2.4}$$

then the product of all the thetanullwerte related to the columns of M is a modular form with respect to $\Gamma_g[2]$. The converse is also true.

Lemma 2.15. *For g even, there exists a modular form for $\Gamma_g[2]$ of odd weight.*

Proof. Since g even, there exist n, positive integer, and k, odd, such that $g = 2^n \cdot k$. Hence, let M be a matrix which columns are the characteristics associated to the thetanullwerte which does not vanish on a component of the hyperelliptic locus of genus 2^n. If $\mathfrak{m} = \left(\begin{smallmatrix}\mathfrak{m}_1\\ \mathfrak{m}_2\end{smallmatrix}\right)$ is such a characteristic, then we can write $M = \left(\begin{smallmatrix}M_1\\ M_2\end{smallmatrix}\right)$, where M_i is the matrix which columns are the $\mathfrak{m}_i$. With this notation, we can define the following matrix

$$\tilde{M} = \begin{pmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & M_1 \\ M_2 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & M_2 \end{pmatrix},$$

where M_1 and M_2 appear k times each. The columns of this matrix are even characteristics and the corresponding thetanullwerte do not vanish on a point of the type

$$\begin{pmatrix} \bar{Z} & 0 & \dots & 0 \\ 0 & \bar{Z} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{Z} \end{pmatrix}$$

with $\bar{Z}$ as before. Since this point is on the boundary of the hyperelliptic locus, there there is a whole open set of the hyperelliptic locus which contains this point on which the thetanullwerte does not vanish, so there is at least one hyperelliptic point on which it does not vanish. We only need to show that the product of all these thetanullwerte is a modular form for $\Gamma_g[2]$ of odd weight. For the weight, it's easy to see that it is odd, by Lemma 2.13, since we have a product of $k \cdots 2 \cdot \text{odd}$ thetanullwerte of weight $\frac{1}{2}$, hence the weight is $k \cdot \text{odd}$ which is odd. As for the modularity, we use Lemma 2.14, together with Lemma 2.13. In fact, we have that

$$\tilde{M}^{\intercal}\tilde{M} \equiv \begin{pmatrix} 0 & 1_g \\ 1_g & 0 \end{pmatrix} \mod 2$$

and

$$\operatorname{diag}(\tilde{M}^{\intercal}\tilde{M}) \equiv 0 \mod 4$$

because the same equivalences holds for the matrix M, by Lemma 2.13, and the two previous equations follows blockwise. □

Hence, for every even g we have found a modular form for $\Gamma_g[2]$ of odd weight and the Theorem 2.11 is proved.

Now we want to describe the ρ homomorphism in a more explicit way. We have that $A(\Gamma_g[2])$ is the integral closure of the ring generated by the fourth power of the thetaconstants; by the way, we will define the morphism on the eighth powers to make the definition easier. Let $\theta[\mathfrak{m}_T] \in A(\Gamma_g[2])$ and let

$$\rho(\theta[\mathfrak{m}_T]^8) = \prod_{i \neq j \in T} (x_i - x_j)^2 \cdot \prod_{i \neq j \notin T} (x_i - x_j)^2$$

Moreover, if $w\colon W \to \mathbb{H}_g/\Gamma_g[2]$, then we have that for any $f, g \in \Gamma_g[2]$ of the same weight

$$\frac{f(w(\xi))}{g(w(\xi))} = \frac{\rho(f)}{\rho(g)}$$

as functions of ξ.

We will now give another criterion for the vanishing and non-vanishing of a thetanullwerte on the hyperelliptic locus; Proposition **??** was first stated by Weissauer, unpublished, but we will give our own proof. Let $f \in A(\Gamma_g)$ be a cuspidal form, i.e. a form which vanishes at the boundary of $\mathscr{A}_g$.

For any modular form $f \in A(\Gamma_g)$, we have that the usual Fourier expansion

$$f(Z) = \sum a(n_{ij})\, \mathbf{e}\left(2\pi i \sum_{1 \leq i \leq j \leq n} n_{ij} z_{ij}\right)$$

can be written in the form

$$f(Z) = \sum_T a(T)\, \mathbf{e}(\pi i \operatorname{tr}(TZ) \tag{2.5}$$

Hence, a Lemma by Freitag [Fre10, Proposition 3.1] tell us that any holomorphic function with a Fourier expansion like in 2.5, where the sum is taken over even, symmetrical matrices T, then $a(T) \neq 0$ implies $T \geq 0$. Hence, we can define the vanishing order of f at the boundary, i.e.

$$f(Z) = \sum_{T \geq 0} a(T)\, \mathbf{e}\left(\frac{1}{2} \operatorname{tr} TZ\right)$$

and

$$k = \frac{1}{2} \min_{x \in \mathbb{Z}^g \setminus \{0\}} \{T[x] \mid a(T) \neq 0\}$$

where $T[x]$ is the quadratic form ${}^{\intercal}xTx$, which represent only even numbers, and k is $\frac{1}{2}$ of the minimum integer representable.

Proposition 2.16. *weiss The image of a cusp for with vanishing order k is divisible by the k-th power of the discriminant $\Delta = \prod_{i,j}(x_i - x_j)^2$.*

Proof. Form Igusa, we know that

$$\Delta \mid \rho(f).$$

Now, let us consider the homomorphism

$$A(\Gamma_g[2]) \to S(2g + 2);$$

we know that
$$\rho(\theta[\mathfrak{m}_T]^8) = \prod_{i\neq j\in T}(x_i - x_j)^2 \cdot \prod_{i\neq j\in T^c}(x_i - x_j)^2$$
and we let $Z \mapsto \left(\begin{smallmatrix} Z_{g-1} & 0 \\ 0 & i\infty \end{smallmatrix}\right)$. Since this is a cusp and we want $\theta[\mathfrak{m}_T]$ to be a cusp form, we have that
$$\theta[\mathfrak{m}_T]\left(\left(\begin{smallmatrix} Z_{g-1} & 0 \\ 0 & i\infty \end{smallmatrix}\right)\right) = 0.$$
This correspond to $x_{2g+1}, x_{2g+2} \mapsto \infty$, hence
$$(x_{2g+1} - x_{2g+2})^2 \mid \rho(\theta[\mathfrak{m}_T]^8)$$
Since this is true for every pair x_i, x_j, it follows that
$$\Delta \mid \rho(\theta[\mathfrak{m}_T]^8).$$
□

We are interested in the following Corollary.

Corollary 2.16.1. *Let f be a cusp form with weight λ and vanishing order k. Let*
$$\frac{\lambda}{k} > 8 + \frac{4}{g}. \tag{2.6}$$
Then $\rho(f) \equiv 0$, i.e. $f \equiv 0$ on the hyperelliptic locus. The quantity $\frac{\lambda}{k}$ is the slope *of the cusp form.*

Proof. By the property of ρ, $\rho(f)$ has weight $\frac{\lambda g}{2}$. If $\rho(f) \neq 0$, then $\Delta^k \mid \rho f$. But Δ is an invariant of weight $4g+4$, hence we have
$$(4g+2)k \leq \lambda \frac{g}{2}$$
and $\frac{\lambda}{k} \geq 8 + \frac{4}{g}$. This is a contradiction, so $\rho(f) \equiv 0$. Another equivalent way to prove this is to check that if the slope is small enough, then Δ^k cannot divide $\rho(f)$. □

Remark 2.1. The previous estimate 2.6 is sharp, in fact it is possible to find a cusp form with slope exactly $8 + \frac{4}{g}$ which does not vanish on the hyperelliptic locus. An example is given by the cusp form of the Example 1.7; a proof for this fact can be find in [SM00], indeed we have an upper bound estimate for the slope of F_H which combined with Corollary 2.16.1 gives the exact slope.

2.5 An affine stratification for the moduli space of curves

In this section, we will give one of the fundamental results of this Thesis. The following section is the contento of a joint paper with Fontanari [FP11] which was accepted by Geometria Dedicata and which is already available on-line.

A purely algebro-geometric approach to the cohomology of the moduli spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}_{g,n}}$ of smooth (respectively, stable) curves of genus g with n marked points has been recently developed by Enrico Arbarello and Maurizio Cornalba in the two papers [AC98], [AC08], where the only essential result borrowed from geometric topology is a vanishing theorem due to John Harer. Namely, the fact that $H_k(\mathcal{M}_{g,n})$ vanishes for $k > 4g-4+n$ if $n > 0$ and for $k > 4g-5$ if $n = 0$ was deduced in [Har86] from the construction of a $(4g-4+n)$-dimensional spine for $\mathcal{M}_{g,n}$ by means of Strebel differentials. On the other hand, it is conceivable that Harer's vanishing is only the tip of an iceberg of deeper geometrical properties (see [HL98], Problem (6.5)). For instance, a conjecture of Eduard Looijenga says that $\mathcal{M}_g$ is a union of $g-1$ open subsets (see [FL99], Conjecture 11.3), but (as far as we know) until now there are no advances in this direction. Notice that Looijenga's conjecture trivially holds for $g = 2, 3$: indeed, it is well-known that $\mathcal{M}_2$ is affine and that non-hyperelliptic curves of genus 3 can be canonically embedded as quartic plane curves, hence $\mathcal{M}_3 = \mathcal{M}_3 \setminus \{\text{the hyperelliptic locus}\} \cup \mathcal{M}_3 \setminus \{\text{the locus of plane quartics with at least one hyperflex}\}$ is the union of two affine open subsets.

Theorem 2.17. *For every g with $2 \leq g \leq 5$ the moduli space $\mathcal{M}_g$ is the union of $g-1$ affine open subsets.*

We point out that, under the same assumptions on g, from the properties of the linear system of quadrics passing through the canonical image of a smooth projective genus g curve it follws that $\mathcal{M}_g$ admits an affine stratification of depth $g-2$ (see [FL99]). Our approach in the present note relies instead on the theory of modular forms.

We need four special modular forms which will give the affine open covering we are looking for. We denote by $\mathcal{F}_e^g$ the subset of $\mathbb{F}^{2g}$ of even characteristics.

$$F_{\text{null}} = P(\mathcal{F}_e^g) = \prod_{m \in \mathcal{F}_e^g} \theta_m; \tag{2.7}$$

$$F_1 = \sum_{m \in \mathcal{F}_e^g} \left(P(\mathcal{F}_e^g)/\theta_m\right)^8; \tag{2.8}$$

$$F_H = \sum_{A \subseteq \mathbb{F}^{2g}} \left(P(\mathcal{F}_e^g \setminus A)\right)^8; \tag{2.9}$$

$$F_T = 2^g \sum_{m \in \mathcal{F}_e^g} \theta_m^{16} - \left(\sum_{m \in \mathcal{F}_e^g} \theta_m^8 \right)^2. \tag{2.10}$$

In (2.9), A varies among all suitable sets of $v(g)$ characteristics corresponding to the irreducible components of $\mathcal{H}_g[2]$ as in Proposition 2.6. We need to take the eighth power of the theta constants in order to obtain the modularity of the above forms with respect to the modular group.

Remark 2.2. The modular form F_{null} has weight $2^{g-2}(2^g+1)$. Its vanishing locus is the divisor Θ_{null} on $\mathscr{A}_g$.

Remark 2.3. The modular form F_1 has weight $2^{g+1}(2^g+1)-4$. It defines a divisor D_1 on $\mathscr{A}_g$.

Remark 2.4. The modular form F_H has weight $2\binom{2g+2}{g+1}$ and it coincides with F_{null}^8 for $g = 2$, since no theta constant vanishes on the hyperelliptic locus. In the case $g = 3$, the modular form F_H coincides with F_1 since every component of the hyperelliptic locus is characterized by the vanishing of a single theta constant. Let D_H be the divisor defined by $\{F_H = 0\}$ on $\mathscr{A}_g$.

Note that the slope of the modular form F_H is exactly $8 + \frac{4}{g}$.

Remark 2.5. The modular form F_T has weight 8 and it is not identically 0 only for $g \geq 4$. Indeed, for $g = 2, 3$ it vanishes identically on $\mathbb{H}_g$, while for $g = 4$ it vanishes on the of $\mathcal{M}_4$ in $\mathbb{H}_4$. When $g \geq 4$, it defines a divisor D_T on $\mathscr{A}_g$, which coincides with the closure of $\mathcal{M}_4$ when $g = 4$ and with the closure of the trigonal locus when $g = 5$ (see [GS09]).

We will give the following classical definition of the *Satake compactification* of $\mathscr{A}_g$.

Definition 2.4. *For $g = 1$,the Satake compactification $\mathscr{A}_1^*$ is $\mathscr{A}_1 \cup \{i\infty\}$, which is the only cusp. For $g > 1$, we define $\mathscr{A}_g^*$ as the union of $\mathscr{A}_r$ for $1 \leq r \leq g$, namely*

$$\mathscr{A}_g^* = \bigcup_{1 \leq r \leq g} \mathscr{A}_r,$$

and the closure of each $\mathscr{A}_r$ inside $\mathscr{A}_g^*$ is homeomorphic to $\mathscr{A}_r^*$

In order to handle the divisors defined by modular forms on $\mathscr{A}_g^*$, we will make use of the following fact.

Lemma 2.18. *For $g \geq 3$ all modular forms define ample divisors on $\mathscr{A}_g^*$.*

Proof. For $g \geq 3$ the group of the Weil divisors of $\mathscr{A}_g^*$ modulo principal divisors is isomorphic to $\mathbb{Z}$, hence a suitable multiple of any effective divisor D in $\mathscr{A}_g^*$ is very ample. By [Deb92], Lemma 2.1, D is ample. Since every divisor defined by a modular form is effective, our claim follows. $\square$

Let D be a divisor defined by a modular form. Since $\mathcal{M}_g$ contains complete curves when $g \geq 3$, we have that $D \cap \mathcal{M}_g \neq \emptyset$ (indeed, the Satake compactification is projective and has boundary of codimension 2 for $g \geq 3$). Hence each of the previously described divisors either contain $\mathcal{M}_g$ or define a divisor in $\mathcal{M}_g$. In the latter case, we use the same notation for the induced divisor inside $\mathcal{M}_g$.

Next, we prove a fundamental result about F_H.

Lemma 2.19. *The modular form F_H never vanishes on $\mathcal{H}_g$.*

Proof. Let τ be the period matrix of a hyperelliptic curve. By Proposition 2.6, there is a suitable set A of $v(g)$ theta constants which vanish at τ. Hence all summands but $P(\mathcal{F}_e^g \setminus A)^8$ contain at least one of the vanishing theta constants and $F_H(\tau) = P(\mathcal{F}_e^g \setminus A)^8(\tau) \neq 0$. It follows that F_H never vanishes on $\mathcal{H}_g$. $\square$

Now we can exhibit an explicit open covering of $\mathcal{M}_g$ for every $2 \leq g \leq 5$. We have already recalled the description for $g = 2, 3$ in the Introduction, hence here we focus on the cases $g = 4$ and $g = 5$. Namely, we are going to prove that

$$\mathcal{M}_4 = \mathcal{M}_4 \setminus \Theta_{\text{null}} \cup \mathcal{M}_4 \setminus D_1 \cup \mathcal{M}_4 \setminus D_H, , \tag{2.11}$$

$$\mathcal{M}_5 = \mathcal{M}_5 \setminus \Theta_{\text{null}} \cup \mathcal{M}_5 \setminus D_1 \cup \mathcal{M}_5 \setminus D_H \cup \mathcal{M}_5 \setminus D_T. \tag{2.12}$$

Our proof relies on two key ideas. The first one is a straightforward application of Cornalba-Harris ampleness criterion (see [CH88]):

Proposition 2.20. *Let D be an effective divisor on $\mathcal{M}_g$ and let $\overline{D}$ be its closure in the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$. If $[\overline{D}] = a\lambda - \sum b_i\delta_i$ with $a, b_i > 0$, then $\overline{\mathcal{M}_g} \setminus D$ is an affine open subset.*

Proof. Just notice that $E = a\lambda - \sum b_i\delta_i + \sum(b_i - \varepsilon)\delta_i = a\lambda - \varepsilon\delta$ with $\varepsilon > 0$ small enough is an effective divisor on $\overline{\mathcal{M}_g}$ such that $\overline{\mathcal{M}_g} \setminus E = \mathcal{M}_g \setminus D$ and E is ample by Cornalba-Harris ampleness criterion [CH88]. □

We will exploit in particular the following consequence of Proposition 2.20:

Corollary 2.20.1. *Let D be an effective divisor on $\mathcal{M}_g$ and let $\tilde{D}$ be its closure in the Satake compactification $\mathscr{A}_g^*$. If $\tilde{D}$ contains the product of periods of smooth curves and periods of nodal curves, then $\mathcal{M}_g \setminus D$ is affine.*

Proof. There is a standard map from the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ to the Satake compactification $\mathcal{M}_g^*$ (i.e. the closure of $\mathcal{M}_g$ in the Satake compactification) which takes boundary divisors of $\overline{\mathcal{M}_g}$ to products of periods of smooth curves and periods of nodal curves. Thus in the notation of Proposition 2.20 we have $b_i > 0$, since $\tilde{D}$ intersects the image of the boundary, and our claim follows. □

Remark 2.6. We could avoid the Cornalba-Harris ampleness criterion observing that a modular form always induces an ample divisor $\tilde{D}$ on $\mathcal{M}_g^*$ and it defines a divisor D on $\mathcal{M}_g$. Obviously $\mathcal{M}_g^* \setminus \tilde{D}$ is affine. Now, whenever $\tilde{D} = D \cup (\mathcal{M}_g^* \setminus \mathcal{M}_g)$ we have that $\mathcal{M}_g \setminus D = \mathcal{M}_g^* \setminus \tilde{D}$ is affine.

Next we present a nice criterion to check whether F_H vanishes on τ.

Lemma 2.21. *Let $\tau \in \mathbb{H}_g$. If there exist more than $v(g)$ even theta constants vanishing on τ, then $F_H(\tau) = 0$.*

Proof. This is just a trivial application of the pigeonhole principle. Indeed, each summand of F_H is the product of $\frac{1}{2}\binom{2g+2}{g+1}$ even theta constants out of $2^{g-1}(2^g + 1)$ total even theta constants, since there are exactly $v(g)$ theta constants left out of the product. If there are more than $v(g)$ theta constants vanishing at τ, then every summand contains at least one of them, hence it vanishes at τ. □

We shall also apply the following auxiliary result, which is essentially Lemma 3 in [Acc74]. Recall that g^r_d stands for a linear system of dimension r and degree d; in particular, a hyperelliptic curve has a g^1_2.

Lemma 2.22. *If a curve C of genus 5 with a base point free g^1_3 carries two half-canonical g^1_4, then C is hyperelliptic.*

Proof. We claim (see [Acc74], Lemma 2) that if C is a curve of genus 5 with a base point free g^1_3, then every half-canonical g^1_4 has a fixed point. Indeed, let $x+y+z$ be a divisor in the g^1_3 with three distint points and notice that $h^0(C, K_C - x - y - z) = h^0(C, K_C - x - y) = h^0(C, K_C - x - z)$. If D_x and D_y are two divisors in the half-canonical g^1_4 containing x and y, respectively, it follows that z is contained in the canonical divisor $D_x + D_y$, say in D_x. Now, if y is not a base point of the g^1_4, then there is a divisor D in the g^1_4 which does not contain y. On the other hand,

y has to be contained in the canonical divisor $D_x + D$ containing x and z, hence $D_x = x + y + z + w$ and $g^1_4 = g^1_3 + w$, as claimed. By the claim, both half-canonical g^1_4 have a fixed point, namely, the first one is $g^1_3 + x$ and the second one is $g^1_3 + y$ with $x \neq y$. We have $2g^1_3 + 2x = |K_C| = 2g^1_3 + 2y$, hence $2x \sim 2y$ with $x \neq y$ and C turns out to be hyperelliptic. □

Proof of Theorem 2.17. For $g = 2, 3$ the statement is trivial, as recalled in the Introduction, hence we need to check it only for $g = 4, 5$. Let $g = 4$; according to (2.11), our three open sets are the following:

$$\mathcal{M}_4 = (\mathcal{M}_4 \setminus \Theta_{\mathrm{null}}) \cup (\mathcal{M}_4 \setminus D_1) \cup (\mathcal{M}_4 \setminus D_H).$$

By our previous discussion it is enough to prove that the above divisors satisfy the condition of Corollary 2.20.1. For D_1 and Θ_{null} it is obvious that Lemma 2.21 holds. Hence we just need to check that the closure $\tilde{D}_H$ of D_H in $\mathscr{A}^*_4$ contains the product of periods of smooth curves and periods of nodal curves, in order to apply Corollary 2.20.1. If $\tau \in \tilde{D}_H$ is a product of periods, i.e. $\tau = \left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)$, with $\tau_1 \in \mathcal{M}_{g_1}$ and $\tau_2 \in \mathcal{M}_{g_2}$, $g_1 + g_2 = 4$, then by Remark 1.8 we have $\theta_m(\tau) = 0$ if $m = m_1 \oplus m_2$ with $m_1 \in \mathbb{F}^{2g_1}$ and $m_2 \in \mathbb{F}^{2g_2}$ odd characteristics. There are two possible cases. If $g_1 = g_2 = 2$, then we have $6 \cdot 6 = 36 > 10 = v(4)$ even characteristics which split as odd$\oplus$odd in the notation of Remark 1.8; by Lemma 2.21, F_H vanishes on τ. If $g_1 = 1$ and $g_2 = 3$, then we have $1 \cdot 28 = 28 > 10 = v(4)$ even characteristics and F_H vanish on τ. The analogue result holds for nodal curves. Hence $\mathcal{M}_4 \setminus D_H$ is an affine open set. Moreover, $\Theta_{\mathrm{null}} \cap D_1$ is the hyperelliptic locus in $\mathcal{M}_4$. Indeed, set-theoretically,

$$\Theta_{\mathrm{null}} \cap D_1 = \bigcup_{m_1 \neq m_2} \{\theta_{m_1} = \theta_{m_2} = 0\} \tag{2.13}$$

and the intersection of this locus with $\mathcal{M}_4$ gives the hyperelliptic locus (see [Igu81]). By Lemma 2.19 we have $\mathcal{M}_4 \supset \Theta_{\mathrm{null}} \cap D_1 \cap D_H = \emptyset$, hence (2.11) holds.

Let now $g = 5$; according to (2.12), our four open sets are the following:

$$\mathcal{M}_5 = (\mathcal{M}_5 \setminus \Theta_{\mathrm{null}}) \cup (\mathcal{M}_5 \setminus D_1) \cup (\mathcal{M}_5 \setminus D_H) \cup (\mathcal{M}_5 \setminus D_T).$$

Again, we just need to check that the divisors satisfies the condition of Corollary 2.20.1. For D_1 and Θ_{null} this is obvious. We need to check that the closure $\tilde{D}_H$ of D_H in $\mathscr{A}^*_5$ contains the product of periods of smooth curves and periods of nodal curves. As before, it is sufficient to prove that if $\tau = \left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right) \in \mathcal{M}_{g_1} \times \mathcal{M}_{g_2}$, with $g_1 + g_2 = 5$, then more than $v(5) = 66$ theta constants vanish on τ. If $g_1 = 1$ and $g_2 = 4$, then we have $1 \cdot 120 = 120 > 66$ even theta constants vanishing on τ. If $g_1 = 2$ and $g_2 = 3$, then we have $6 \cdot 28 = 168 > 66$ even theta constants vanishing on τ. By Lemma 2.21, F_H vanish on every product of smooth curves. The analogue result holds for nodal curves. Hence $\mathcal{M}_4 \setminus D_H$ is an affine open set. Again, we have that (2.13) holds and together with Lemma 2.22 we conclude that $\Theta_{\mathrm{null}} \cap D_1 \cap D_T$ is exactly the hyperelliptic locus. By Lemma 2.19 we have $\mathcal{M}_5 \supset \Theta_{\mathrm{null}} \cap D_1 \cap D_T \cap D_H = \emptyset$, hence (2.12) holds. □

Chapter 3

Divisors on Moduli Space of Hyperelliptic Curves

3.1 Degeneration of Hyperelliptic Curves

Following Tsuyumine and Keel, we want to describe the boundary of the hyperelliptic locus in the various compactifications.

3.2 Geometrical description of the divisors

Let $S = \{1, \ldots, 2g+2\}$ be the set of indexes of the $2g+2$ points on a curve of genus 0 in $\mathcal{M}^c_{0,2g+2}$. Let $D \subseteq \{1, \ldots, 2g+2\}$, with $|D|, |D^c| \geq 2$, we let Δ_D be the divisor whose generic point is a curve with two components, the points of D on one branch, the points of D^c on the other. This also implies that Δ_D and Δ_{D^c} are the same divisor.

Lemma 3.1 ([Kee92]). *Let $D, D' \subseteq S$, then $\Delta_D \cap \Delta_{D'} \neq \emptyset$, if and only if $D \circledast D'$.*

We want to study case where $|D|$ is odd.

Let $D \subseteq S$, $|D| = 2g_1 + 1$ and $\xi = \{\xi_1, \ldots, \xi_{2g+2}\} \in W$. We define $\xi_D = \{\xi_i\}_{i \in D} \in W^*_D$ and $\xi_{D^c} = \{\xi_i\}_{i \in D^c} \in W^*_{D^c}$. Hence, we have two smooth hyperelliptic curves $\mathscr{C}_1$ and $\mathscr{C}_2$, which have equation

$$\mathscr{C}_1 \colon y^2 = \prod_{i \in D} (x - \xi_i)$$

and

$$\mathscr{C}_2 \colon y^2 = \prod_{i \in D^c} (x - \xi_i).$$

The lower part of Figure 3.1 might be rather misleading, since the two smooth Riemann surfaces should meet transversely in one point and this cannot happen in the three dimensional space; however, it makes sense in a four dimensional space. The upper part, which is the 6-pointed nodal curve of genus 0 underlying the double admissible covering, is a much more faithful representation.

We have that the intersection point is the point ∞_1 on $\mathscr{C}_1$ and ∞_2 on $\mathscr{C}_2$ and for both curves it is a ramification point. With a suitable homology basis on

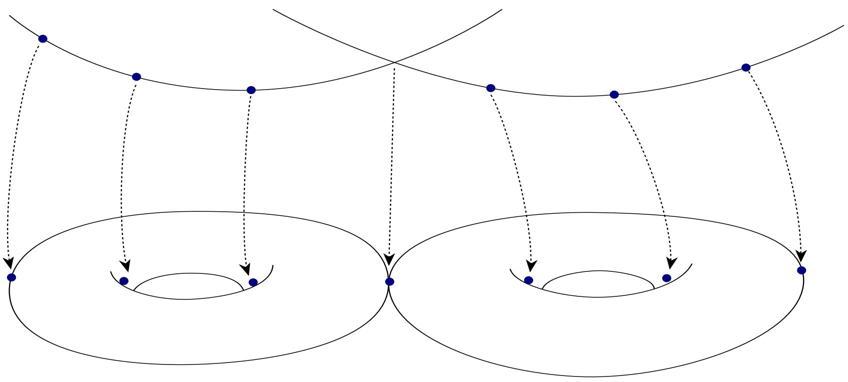


Figure 3.1. A generic point of Δ_D when $D \subset \{1,\dots,6\}$ is odd

the curves, i.e. we take the canonical generators for $\pi_1(\mathscr{C}_1, \infty_1)$ on $\mathscr{C}_1$ and the canonical generators of $\pi_1(\mathscr{C}_2, \infty_2)$ on $\mathscr{C}_2$, we have two maps $w_1 \colon \mathscr{C}_1 \mapsto \Omega_1 \in \mathbb{H}_{g_1}$ and $w_2 \colon \mathscr{C}_2 \mapsto \Omega_2 \in \mathbb{H}_{g_2}$.

Now, let $\mathscr{C}(t)$ denote the curve defined by

$$\mathscr{C}(t) \colon y^2 = \prod_{i\in D}(x-\xi_i)\cdot \prod_{i \in D^c}(x-(t+\xi_i));$$

for $t = 0$, it is a smooth curve, while for $t \to \infty$, we are collapsing all the ξ_i for $i \in D$ to a single point ∞. If $\Omega(t) = w(\mathscr{C}(t))$, then we have the following result

Lemma 3.2. *Let $\Omega = \lim_{t\to\infty}\Omega(t)$. Then Ω is conjugate to $\left(\begin{smallmatrix}\Omega_1 & 0\\ 0 & \Omega_2\end{smallmatrix}\right)$ via Γ_g.*

Proof. Via a base point change in $\pi_1(\mathscr{C}, p_0)$. □

Since it is not easy to find the matrix which gives the explicit conjugate, we want to find a simple criterion to determine which theta functions vanish on $\Omega = \lim_{t\to\infty}\Omega(t)$.

Let $S(r)$ be the ring of invariants of a binary form of degree r as in Section 1.6. We introduce a valuation on $S(r)$ associated to a divisor $\Delta = \Delta_D$:

$$\nu_\Delta(I) = -\deg_t I(\xi + \chi_D t) + \frac{s(|D||D^c|-1)}{r-2}.$$

for any $0 \neq I \in S(r)$, and $\nu_\Delta(0) = +\infty$. It is easy to check that this is a valuation and it generalizes the valuation given by Tsuyumine ([Tsu86]). Indeed, if $D = \{r-2, r-1, r\}$ and $I \in S(r)$, we have

$$\nu_\Delta(I) = -\deg_t I(\xi_1, \dots, \xi_{r-3}, \xi_{r-2}+t, \xi_{r-1}+t, \xi_r + t) + \frac{s(3r-10)}{r-2}$$

Let

$S(r)_{\Delta,0} = \{$graded subring of $S(r)$ generated by homogeneous elements I, with $\nu_\Delta(I) \geq 0\}$

We define the operator Ψ_Δ by

$$\Psi_\Delta(I) = \lim_{t\to\infty} t^{-\frac{-s(|D||D^c|-1)}{r-2}} I(\xi + \chi_D t).$$

If $I \in S(r)_{\Delta,0}$, then $\Psi_\Delta(I)$ is a well defined element in the ring $\mathbb{C}[\xi_i - \xi_j, \xi_i, \xi_j \in W_D^*] \otimes \mathbb{C}[\xi_i - \xi_j, \xi_i, \xi_j \in W_{D^c}^*]$. Moreover $\Psi_\Delta(I)$ is symmetric in ξ_D and ξ_{D^c}.

Lemma 3.3. *Suppose that $\theta[\mathfrak{m}_T]$ is a nonzero function on $h(W)$ and $\Delta = \Delta_D$. Then $\Psi_\Delta\theta[\mathfrak{m}_T] \neq 0$ if and only if T is D-balanced. Moreover, $\Psi_\Delta\theta[\mathfrak{m}_T]$ does not vanish if and only if $\nu_\Delta(\rho(\theta[\mathfrak{m}_T]^8)) = 0$.*

Proof. Since $\theta[\mathfrak{m}_T]$ does not vanish on $h(W)$, we have that T must be a balanced partition, hence $|T| = g+1$. This implies that $\rho(\theta[\mathfrak{m}_T]^8(\xi))$ is an element of $S(2g+2)$ and every ξ_i appears with degree $2g$. We need to compute the degree of t in $\rho(\theta[\mathfrak{m}_T]^8)(\xi + \chi_D t)$.

$$\begin{aligned}\rho(\theta[\mathfrak{m}_T]^8)(\xi + \chi_D t) = &\prod_{i\neq j\in T} ((\xi_i + \chi_D(i)t) - (\xi_j + \chi_D(j)t))^2\cdot\\ &\cdot\prod_{i\neq j\in T^c} ((\xi_i + \chi_D(i)t) - (\xi_j + \chi_D(j)t))^2\end{aligned}$$

which is essentially

$$\begin{aligned}\deg_t(\rho(\theta[\mathfrak{m}_T]^8)(\xi + \chi_D t)) =&2(\deg_t(\prod_{i\neq j\in T}((\xi_i - \xi_j) - (\chi_D(i) - \chi_D(j))t))+\\ &+\deg_t(\prod_{i\neq j\in T^c}((\xi_i - \xi_j) - (\chi_D(i) - \chi_D(j))t))))\end{aligned}$$

Hence we have that $\chi_D(i) - \chi_D(j) \neq 0$ in the first factor if and only if $i \in D\cap T$ and $j \in D^c \cap T$. For the second factor, we have the mirror condition: $\chi_D(i) - \chi_D(j) \neq 0$ if and only if $i \in D \cap T^c$ and $j \in D^c \cap T^c$. Hence

$$\deg_t(\rho(\theta[\mathfrak{m}_T]^8)(\xi + \chi_D t)) = 2(|D\cap T||D^c\cap T| + |D\cap T^c||D^c\cap T^c|) \tag{3.1}$$

Since we want to simplify the computations, we set $2g + 2 = 2s$, $|T| = s$ (this is because we already know that if T is unbalanced, the corresponding thetanullwerte vanish on the component) $|D| = \delta$ and $|D\cap T| = h$. Hence we have $|D\cap T^c| = \delta - h$, $|D^c \cap T| = s - h$ and $|D^c \cap T^c| = s + h - \delta$. Putting this in equation (3.1) we have:

$$\begin{aligned}\deg_t(\rho(\theta[\mathfrak{m}_T]^8)(\xi + \chi_D t)) &= 2(h(s-h) + (\delta - h)(s + h - \delta)) =\\ &= 2(-h^2 + sh + \delta s + \delta h - \delta^2 - hs - h^2 + h\delta) =\\ &= -4h^2 + 4h\delta + 2\delta(s-\delta)\\ &= -4(h^2 - \delta) + 2s\delta - 2\delta^2 =: f(h).\end{aligned}$$

As a function of the real variable h, $f(h)$ is a concave parabola, with his maximum for $h = \frac{\delta}{2}$. Since δ is odd and we are looking for integer solution, we have that the admissible maximum is for $h = \frac{\delta\pm 1}{2}$ and

$$\begin{aligned}f\left(\frac{\delta\pm1}{2}\right) &= -4\left(\frac{\delta\pm1}{2}\right)^2 + 4\frac{\delta\pm1}{2}\delta + 2s\delta - 2\delta^2\\ &= -\delta^2 \mp 2\delta - 1 + 2\delta^2 \pm 2\delta + 2s\delta - 2\delta^2\\ &= 2s\delta - \delta^2 - 1 = \delta(2s-\delta) - 1 = |D||D^c| - 1.\end{aligned}$$

Hence

$$\deg_t(\rho(\theta[\mathfrak{m}_T]^8)(\xi + \chi_D t)) \leq |D||D^c| - 1,$$

where the equality holds if $|T \cap D| = h = \frac{\delta \pm 1}{2} = k$ or $k+1$. Hence

$$\Psi_\Delta(\rho(\theta[\mathfrak{m}_T])) = \lim_{t\to\infty} t^{-|D||D^c|+1}\rho(\theta[\mathfrak{m}_T])(\xi + \chi_D t)$$

since $r = 2g+2$ and $s = 2g$. Therefore $\Psi_\Delta(\rho(\theta[\mathfrak{m}_T])) \equiv 0$ if $\nu_\Delta(\rho(\theta[\mathfrak{m}_T]^8)(\xi)) > 0$. Moreover, there is a theta constant $\theta[\mathfrak{m}]$ which does not vanish on $h(W)$, and the order of the ρ-image of $\theta[\mathfrak{m}]^8$ must be 0. Then any theta constant with $\nu_\Delta(\rho(\theta[\mathfrak{m}]^8))) = 0$ does not vanish by Ψ. Indeed the values of the ratio

$$\frac{\theta[\mathfrak{m}]^8}{\theta[\mathfrak{m}_T]^8}$$

on $h(W)$ only depends on the ratio of their ρ-images. □

Note that this Lemma generalizes a Lemma in [Tsu86, Lemma 4]. Indeed, let $D = \{2g, 2g+1, 2g+2\}$: T is not D-balanced if $D \cap T = 3, 0$, i.e. $D \in T$ or $D \in T^c$. In those cases, we have $\Psi_D(\theta[\mathfrak{m}_T]) = 0$.

Theorem 3.4. *Let h_g, h_D, h_{D^c}, with $|D| = 2g_1 + 1$, be the morphisms determined above and let $\rho_g, \rho_D, \rho_{D^c}$ be their associated homomorphisms; let $g_2 = g - g_1$. Then we have the commutative diagram*

$$\begin{array}{ccc} A(\Gamma_g[2]) & \xrightarrow{\rho_g} & S(2g+2)_0 \\ \downarrow{\scriptstyle \Psi_D} & & \downarrow{\scriptstyle \Psi_D} \\ A(\Gamma_{g_1}[2]) \otimes A(\Gamma_{g_2}[2]) & \xrightarrow{\rho_D^* \otimes \rho_{D^c}^*} & S(2g_1+2)^* \otimes S(2g_2+2)^* \end{array}$$

Proof. Since the order of the ρ_D-image of eight power of theta constants are greater or equal than 0, the ρ_D image of any modular form is contained in $S(2g+2)_0$. □

Now, an element of the boundary of $\mathcal{M}^c_{0,2g+2}$ is in the intersection of some divisors. If we associate to a curve $\mathscr{C} \in \mathcal{M}^c_{0,2g+2} \setminus \mathcal{M}_{0,2g+2}$ a tree as we did in Section 2.3, we have a family of divisors which contain $\mathscr{C}$. Using the notation of Section 1.2, we have that $\hat{D}_i \subseteq S = \{1, \dots, 2g+2\}$ is the subset of points belonging to the smooth component $\mathscr{C}_i$. Hence, we can construct the D_i's and we have that $\mathscr{C} \in \bigcap_{i=1}^{n-1} \Delta_{D_i}$. We already showed that $D_i \circledast D_j$, hence that intersection is non empty. We want to know if $\theta[\mathfrak{m}]$ vanish on the hyperelliptic point Ω associated to $\mathscr{C}$. Since $\mathscr{C}$ is in the intersection of Δ_{D_i}'s, we want to know if $\theta[\mathfrak{m}_T]$ vanishes identically on those divisors. By Lemma 3.3, we have that $\theta[\mathfrak{m}_T]$ vanishes on Δ_{D_i} if T is not D-balanced. Hence, T should be D_i-balanced for every $i = 1, \dots, n$. This is equivalent to ask that $\overline{T_i}$ is balanced on $\overline{D_i}$. This will come in handle when we will look for combinatorics.

Proposition 3.5. *Let $Z \in \mathscr{H}^c_g[2]^0$ be the point associated to a $\mathscr{C}$ and let C be its model inside $\mathcal{M}^c_{0,2g+2}$; let $\mathcal{T} = (V, E)$ be the tree dual graph associated to C, which induces the partition $\cup \hat{D}_i = S$ of the $2g+2$ marking points. Hence the following are equivalent:*

(i) T *is* D_e *balanced* $\forall e \in E$;

(ii) $\overline{T_i}$ *is balanced over* D_i $\forall i \in V$;

(iii) $\theta[\mathfrak{m}_T](Z) \neq 0$.

Chapter 4

Combinatorics of the boundary

The supreme accomplishment is to
blur the line between work and play

Arnold Toynbee

The period matrix Z of a curve $\mathbb{C} \in \mathcal{M}^c_{0,2g+2}$ does not take care of the topology given by the glueing of the components. By the way, the thetanullwerte vanishing on Z only depend on the balancedness of the partition related to the characteristic. Hence, even if we permute points on each component we are fixing the same set of vanishing thetanullwerte. We want to count how many different components have the same set of vanishing functions and how many thetanullwerte vanish on a fixed Z.

This last chapter will be a little more descriptive, since the tools we are going to define are just combinatorial tools. We may give them a geometrical meaning, and we will try to, but we are interested in the combinatorial setting.

We need to introduce *linegrams*, which name is a contraction for *line diagrams*. A linegram is a the data of n lines, an integer partition $\sum_{i=1}^n g_i = g$, $g_i \geq 1$, and a set of coloured and distinct points on each line, satisfying a basic set of rules.

- There is a unique *red* point R_i on every line; this also gives an ordering to lines, i.e. the i-th line is the line with R_i on it;
- there are $n-2$ distinct and numbered *green* points G_i;
- there are $2g+2$ distinct and numbered *black*points P_i;
- there are exactly $2g_i+2$ point on the i-th line, i.e. one red point R_i and $2g_i+1$ points among blacks or greens.

An Example for $n=6$ and $g=9$ is given in Figure 4.1 We want to associate to every linegram a labeled tree $(\mathcal{T}, \{\hat{D}_1, \dots, \hat{D}_n\})$; we use the same notation of Section 1.2. The algorithm consist in matching green points and red points in a unique way, such that this application is a bijection and it only depends on the green and black points on each component. Then, we have a vertex for every line and an edge if two lines share a matching pair. Finally, we label each vertex with a set $\hat{D}_i$ which is the set of black points on the corresponding line.

We will describe the algorithm step by step.

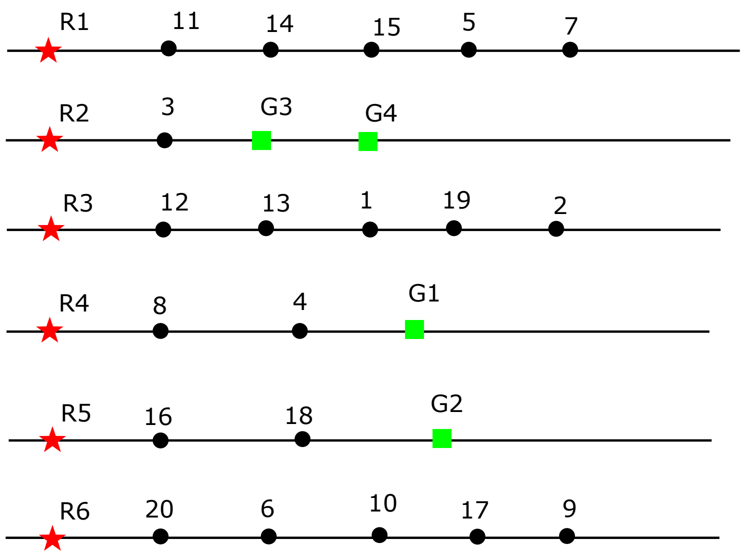


Figure 4.1. A linegram with $n = 6$ and $g = 9$.

1. If $n = 2$, we just join together the two red points. This same step will be done when we are out of green points.

2. Since there are n red points and $n - 2$ green points, there is a line with no green points[1]. Let $\mu = \min\{i \mid \text{there are no green points on } i\text{-th line}\}$. Hence, we match G_1 and R_μ. We say that G_1 and R_μ are no more *free* points. Now we still have $n - 1$ red points and $n - 3$ green points.

3. If we have no more free green points, we join the two remaining red points. Otherwise, we still have free green points to match. We will say that a line i is *available* if there are no free green points and R_i is free. Let $\bar{G}$ be the first free green point and

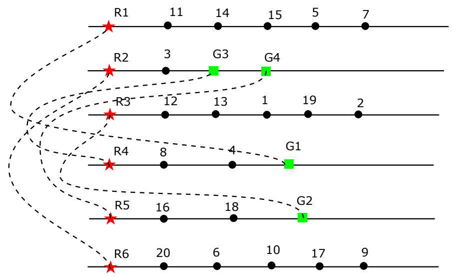
$$\bar{\mu} = \min\{i \mid i\text{-th line is available}\}.$$

Hence, we match $\bar{G}$ and $R_{\bar{\mu}}$ and $\bar{G}$ is no more a free point. We repeat this step until we have no more green points. It will require $n - 2$ steps, since we match a single green point at every step.

Example 4.1. We want to apply algorithm to the linegram in Figure 4.1. We have $n = 6 > 2$, hence we start with step 3. The lines with no green points are 1, 3, 6, so we match G_1 and R_1, which are no more free. We repeat step 2. The lines with not free green points and the red point available are 3, 4, 6, so we match G_2 and R_3. Next step: lines available are 4, 5, 6, so we match G_3 with R_4. Last step, lines available are 5 and 6, so we match G_4 with R_5. The two remaining red points are R_2 and R_6 and we match them together. We can see in Figure 4.2 the matching and the tree associate to linegram in Figure 4.1.

Proposition 4.1. *The above algorithm is well defined and it defines an explicit bijection between the set of labelled trees coming from curves of compact type and the set of linegrams.*

[1] actually, there are at least *two* lines with no green points, but this would be misleading for the algorithm



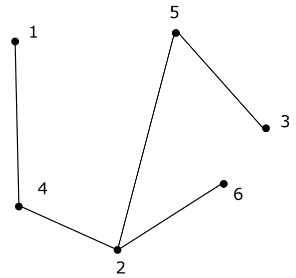


Figure 4.2. The matching on linegram in Figure 4.1 and the associated tree.

Proof. We need to verify that the algorithm is well defined, i.e. if we apply the algorithm to any linegram, we get a tree. Since we have exactly $n-1$ edges, we just need to check that the graph has no simple cycles. This is true and we prove it by induction. If $n = 2$, then we apply step 1 and it we get trivially a tree. The same is also trivial for $n = 3$. Suppose it is true for $n-1$, i.e. if we have a linegram with $n-1$ lines, $n-1$ red points and $n-3$ green points we apply algorithm to get a tree. Now, if we have a linegram with n lines, we can remove the μ-th line and turn G_1 in a black point P_0. We are in the previous case and we have a tree with $n-1$ vertices and n edge. We need to attach to this tree the point associated to the line we removed. This is a leaf since the μ-th line has no green points and it is attached only to the vertex associated to the line with P_0. Hence, we still have a tree.

We also have an inverse algorithm, which create a linegram starting from a labelled tree. Starting with the tree $(\mathcal{T}, \{\hat{D}_i, \ldots, \hat{D}_n\})$, we will have a linegram with n lines. The i-th line is the one related to the vertex $v_i \in V$; on this line, we mark a black point P_l for every $l \in \hat{D}_i$ and the red point R_i. We need to determine the position of the green points. If v_i is a leaf, then there are no green points on the i-th line. If v_i is not a leaf, we have $\deg v_i \geq 2$ and we mark $\deg v_i - 1$ green points on the i-th line. We only need to determine the ordering of the green points. Let $\mu = \min\{i \mid v_i \text{ is a leaf }\}$ and let λ such that $\{v_\mu, v_\lambda\} \in E$ is the only edge which contains v_μ; the λ-th line has at least one green point, which we will label with G_1. Then, we remove the leaf v_μ and proceed by induction on the pruned tree. This is exactly the inverse algorithm, so we have a bijection between labelled trees and linegrams. □

We want to count how many different linegrams are there, with a fixed partition $\sum g_i = g$.

The red points are fixed points, while we can permute all other points. In particular, if we fix the green points, we are only changing the enumeration of the black points, fixing the tree shape. If we permute green points, we may change the tree shape. By the way, if we change all the points which belong to the same line i, with all the points in another line j, and this implies that $g_i = g_j$, we still get the same linegram. The number of linegrams depends on the different g_i's counted with multiplicity. If m_i is the multiplicity of g_i and we have k different values for the g_i's, i.e. $\sum_{i=1}^{k} m_i g_i = g$, the total number of linegrams is given by

$$\frac{(2g+n)!}{\prod_{i=1}^{k}(2g_i+1)!^{m_i} \cdot m_i!}, \tag{4.1}$$

where

- $(2g+n)!$ is the number of all permutations, disregarding the distinction between black and green points;
- $(2g_i+1)!$ is the number of permutations of all free point (both green and black ones) on the i-th component;
- $m_i!$ is the number of permutations of all equivalent components, i.e. such that $g_i = g_j$.

Hence we have the following proposition.

Proposition 4.2. *The number of all $(g_1, \dots, g_k)$-boundaries of a component $\mathscr{H}_g^0[2]$ of $\mathscr{H}_g[2]$ is*

$$\frac{(2g+n)!}{\prod_{i=1}^{k}(2g_i+1)!^{m_i} \cdot m_i!}$$

Proof. A point of a $(g_1, \dots, g_k)$-bounday of $\mathscr{H}_g^0[2]$ is associated with a product of k abelian varieties of dimension g_i. Hence, we have a tree like curve with k components and the i-th component is a curve of genus g. We can associate to this curve a labeled tree, and hence a linegram. Hence the total number of linegrams counts the total number of $(g_1, \dots, g_k)$-boundaries. □

We can explicitly compute the number of labelled trees for n small enough.

Example 4.2. We want to compute how many different trees with $g_i \neq g_j$, for $1 \leq i \neq j \leq 3$ and $\sum g_i = g$ we can find. From (4.1) we expect $\frac{(2g+3)!}{(2g_1+1)!(2g_2+1)!(2g_3+1)!}$. Indeed, we have three possibilities, which we can see in Figure 4.3 Hence, we can

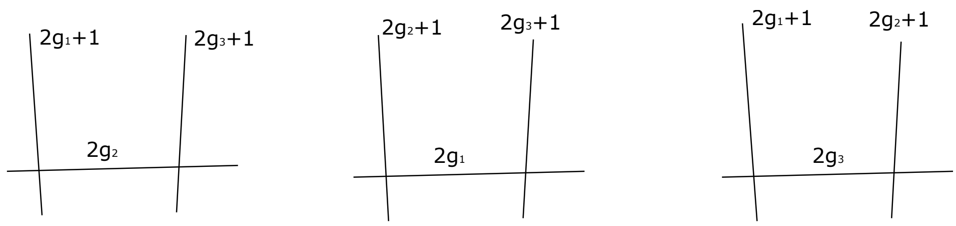


Figure 4.3. The three possible trees. The central component is different every time; we write down the number of black points on each component.

count how many permutations of the black points we have which does not fix the set on a single component. We do this computation for the first tree, then we generalize to the other two trees.

$$\binom{2g+2}{2g_1+1}\binom{(2g+2)-(2g+1)}{2g_2}\binom{(2g+2)-(2g_1+1)-2g_2}{2g_3+1} =$$
$$= \frac{(2g+2)!}{(2g_1+1)!(2g_2)!(2g_3+1)!}$$

Hence, we have

$$\frac{(2g+2)!}{(2g_1)!(2g_2+1)!(2g_3+1)!}+\frac{(2g+2)!}{(2g_1+1)!(2g_2)!(2g_3+1)!}+\frac{(2g+2)!}{(2g_1+1)!(2g_2+1)!(2g_3)!}=$$
$$=\frac{(2g+2)!((2g_1+1)+(2g_2+1)+(2g_3+1))}{(2g_1+1)!(2g_2+1)!(2g_3+1)!}=\frac{(2g+3)!}{(2g_1+1)!(2g_2+1)!(2g_3+1)!}.$$

For $n=4$ we have two different trees and it's much harder to do the computation directly.

We want to count how many thetanullwerte vanishes on a component associated to a labelled tree $(\mathcal{T},\{\hat{D}_1,\dots,\hat{D}_n\})$. If we try a direct approach to this problem, we need to use the fact that if $\theta[\mathfrak{m}]$ vanishes on the boundary of a component, then either $\theta[\mathfrak{m}]$ vanishes on the whole component, i.e. $|\varphi(\mathfrak{m})|\neq g+1$, or $\mathfrak{m}=\oplus_{i=1}^{n}\mathfrak{m}_i$, $\mathfrak{m}_i\in\mathbb{F}_2^{g_i}$, $|\overline{D_i}|=2g_i+2$ and $\sum g_i=g$, and for at least one index i we have that $|\varphi(\mathfrak{m}_i)|\neq g_i$. This computation is pretty hard since it involves the integer partitions of g and also some complicated combinatorics. We want to use another approach.

We already know from Lemma 1.7, that $T=\oplus T_i$ is balanced if $\overline{T_i}$ is balanced for every i. On the other hand, we have that if $\overline{T_i}$ is not balanced on $\overline{D_i}$ for some i, then the corresponding thetanullwerte vanishes on the corresponding block diagonal matrix. Hence we need to count how many partitions $T=\oplus\mathcal{T}_i$ split in balanced partitions.

Proposition 4.3. *Let $Z\in\mathscr{H}_g^c[2]^0$ be the point associated to a $\mathscr{C}$ and let C be its model inside $\mathcal{M}_{0,2g+2}^c$; let $\mathcal{T}=(V,E)$ be the tree dual graph associated to C, which induces the partition $\cup\hat{D}_i=S$ of the $2g+2$ marking points. Let $|\overline{D_i}|=2g_i+2$. Then there are exactly*

$$2^{g-1}(2^g+1)-\frac{1}{2^k}\prod_{i=1}^{k}\binom{2g_i+2}{g_i+1}$$

thetanullwerte which vanish on Z.

Proof. It is sufficient to check that we have

$$\frac{1}{2}\binom{2g_i+2}{g_i+1}$$

balanced partitions on $\overline{D_i}$, hence we have

$$\prod_{i=1}^{k}\frac{1}{2}\binom{2g_i+2}{g_i+1}=\frac{1}{2^k}\prod_{i=1}^{k}\binom{2g_i+2}{g_i+1}$$

partitions of S which are D_i-balanced partitions for every i. Any other partition is not D_i-balanced for some i, hence, by Proposition 3.5 the corresponding thetanullwerte vanish on Z. $\square$

Bibliography

[AC98] E. Arbarello and M. Cornalba *Calculating cohomology groups of moduli spaces of curves via algebraic geometry*, 97–127, Inst. Hautes Études Sci. Publ. Math. **88** (1998)

[AC08] E. Arbarello and M. Cornalba, *Divisors in the moduli space of curves*, Pre-print arXiv:0810.5373, to appear in Surveys in Differential Geometry.

[Acc74] R. D. Accola *Some loci of Teichmuller space for genus five defined by vanishing theta nulls*, 11–18, Contributions to analysis (Lipman Bers), Academic Press, New York, (1974).

[AL02] D. Avitzer, H. Lange, *The moduli space of hyperelliptic curves and binary forms*, 615–632, Math. Z. **242** (2002)

[CH88] M. Cornalba, J. Harris, *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Sc. ENS, **21** (1988)

[Deb92] O. Debarre, *Le lieu des variétés abéliennes dont le diviseur thêta est singulier a deux composantes*, 687–708, Ann. Sc. Ec. Norm. Sup. **25** (1992)

[Fay79] J. Fay, *On the Riemann-Jacobi Formula*, 61–73, Nachr. Akad. Wiss. Göttingen Math.Phys. Kl. **5** (1979)

[FL99] C. Faber, E. Looijenga, *Remarks on moduli of curves. Moduli of curves and abelian varieties*, 23–45, Aspects Math., **33** (1999)

[Fre10] E. Freitag, *Complex Analysis 2*, Springer - Universitext (2010)

[FSM05] E. Freitag, R. Salvati Manni, *Modular forms for the even unimodular lattice of signature* $(2, 10)$, Pre-print arXiv:0511.433v1

[FSM07] E. Freitag, R. Salvati Manni, *The modular variety of hyperelliptic curves of genus three*, Pre-print arXiv:0710.5920v1

[FP11] C. Fontanari, S. Pascolutti, *An affine open covering of $\mathcal{M}_g$ for $g \leq 5$*, Pre-print arXiv:1103.2646 (to appear on Geometriae Dedicata)

[Gla80] J.P. Glass, *Theta constants of genus three*, 123–137, Comp. Math., **40** (1980)

[GS09] S. Grushevsky, R. Salvati Manni *The superstring cosmological constant and the Schottky form in genus* 5, arXiv:0809.1391 (2008).

[Har86] J. Harer *The virtual cohomological dimension of the mapping class group of an orientable surface*, 157–176, Inv. Math. **84** (1986)

[HL98] R. Hain, E. Looijenga *Mapping class groups and moduli spaces of curves*, 97–142, Proc. Symp. Pure Math. **62** (1998)

[HM82] J. Harris, D. Mumford *On the Kodaira dimension of the moduli space of curves*, 23–88, Invent. Math. **67** (1982)

[Hoy63] W.L. Hoyt, *On products and Algebraic Families of Jacobian Varieties*, 415–423, Ann. Math. **77** (1963)

[Igu62] J. Igusa, *On Siegel Modular Forms of genus 2*, 175–200, Am. J. Math, **84** (1962)

[Igu64] J. Igusa, *On the graded ring of Theta Constants*, 219–246, Am. J. Math, **86** (1964)

[Igu67] J. Igusa, *Modular Forms and Projective Invariants*, 817–855, Am. J. Math, **89** (1967)

[Igu80] J. Igusa, *On Jacobi's derivative formula and its generalizations*, 409–446, Am. J. Math, **102** (1980)

[Igu81] J. Igusa, *On the irreducibility of Schottky's divisor*, 531–545, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981)

[Kee92] S. Keel, *Intersection theory of moduli spaces of stable n-pointed curves of genus 0*, 545–574, Trans.Am.Math.Soc **330** (1992)

[Knu83] F. F. Knudsen, *The projectivity of the moduli space of stable curves II*, 161–199, Math. Scand. **52** (1983)

[Kon05] S. Kondo, *The Moduli Space of 8 points on $\mathbb{P}^1$ and Automorphic Forms*, Pre-print arXiv:0504.233v1

[KW22] A. Krazer, W. Wirtinger, *Abelsche Funktionen und allgemeine Theta-funktionen*, 604–873, Encykl. der Math. Wissenschaften, **2** (1922)

[Mum84] D. Mumford, *Tata Lectures on Theta II*, Birkhäuser **43** (1984)

[PY09] C. Poor, D. Yuen, *Binary Forms and the Hyperelliptic superstring ansatz*, Preprint arXiv:0911.4545v1

[Poo93] C. Poor, *The hyperelliptic locus*, 809–884, Duke Math. J. **76** (1993)

[Run95] B. Runge, *On Siegel Modular Forms, Part II*, 179–197, Nag. Math. J. **138** (1995)

[SM00] R. Salvati Manni, *Slope cusp forms and Theta Series*, 282–296, J. Num. Th. **83** (2000)

[Tsu86] S. Tsuyumine, *On Siegel modular forms of degree three*, 755–862, Am. J. Math. **108** (1986)

[Tsu90] S. Tsuyumine, *Thetanullwerte on a moduli space of curves and hyperelliptic loci*, 539–568, Math. Z. **207** (1991)