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Abstract

Using techniques both of non linear analysis and geometric measure theory, we prove existence of minimizers and more generally of critical points for the Willmore functional and other L^p curvature functionals for immersions in Riemannian manifolds. More precisely, given a 3-dimensional Riemannian manifold (M, g) and an immersion of a sphere $f : \mathbb{S}^2 \hookrightarrow (M, g)$ we study the following problems.

1) The Conformal Willmore functional in a perturbative setting: consider $(M, g) = (\mathbb{R}^3, \text{eucl} + \epsilon h)$ the euclidean 3-space endowed with a perturbed metric ($h = h_{\mu\nu}$ is a smooth field of symmetric bilinear forms); we prove, under assumptions on the trace free Ricci tensor and asymptotic flatness, existence of critical points for the Conformal Willmore functional $I(f) := \frac{1}{2}\int |A^\circ|^2$ (where $A^\circ := A - \frac{1}{2}H$ is the trace free second fundamental form). The functional is conformally invariant in curved spaces. We also establish a non existence result in general Riemannian manifolds. The technique is perturbative and relies on a Lyapunov-Schmidt reduction.

2) The Willmore functional in a semi-perturbative setting: consider $(M, g) = (\mathbb{R}^3, \text{eucl} + h)$ where $h = h_{\mu\nu}$ is a $C^\infty_0(\mathbb{R}^3)$ field of symmetric bilinear forms with compact support and small C^1 norm. Under a general assumption on the scalar curvature we prove existence of a smooth immersion of $\mathbb{S}^2$ minimizing the Willmore functional $W(f) := \frac{1}{4}\int |H|^2$ (where H is the mean curvature). The technique is more global and relies on the direct method in the calculus of variations.

3) The functionals $E := \frac{1}{2}\int |A|^2$ and $W_1 := \int \left(\frac{|H|^2}{4} + 1\right)$ in compact ambient manifolds: consider (M, g) a 3-dimensional compact Riemannian manifold. We prove, under global conditions on the curvature of (M, g), existence and regularity of an immersion of a sphere minimizing the functionals E or W_1. The technique is global, uses geometric measure theory and regularity theory for higher order PDEs.

4) The functionals $E_1 := \int \left(\frac{|A|^2}{2} + 1\right)$ and $W_1 := \int \left(\frac{|H|^2}{4} + 1\right)$ in noncompact ambient manifolds: consider (M, g) a 3-dimensional asymptotically euclidean non compact Riemannian 3-manifold. We prove, under general conditions on the curvature of (M, g), existence and regularity of an immersion of a sphere minimizing the functionals E_1 or W_1. The technique relies on the direct method in the calculus of variations.

5) The supercritical functionals $\int |H|^p$ and $\int |A|^p$ in arbitrary dimension and codimension: consider (N, g) a compact n-dimensional Riemannian manifold possibly with boundary. For any $2 \le m < n$ consider the functionals $\int |H|^p$ and $\int |A|^p$ with $p > m$, defined on the m-dimensional submanifolds of N. We prove, under assumptions on (N, g), existence and partial regularity of a minimizer of such functionals in the framework of varifold theory. During the arguments we prove some new monotonicity formulas and new Isoperimetric Inequalities which are interesting by themselves.

Chapter 1

Introduction

An important problem in geometric analysis concerning the *intrinsic* geometry of manifolds sounds roughly as follows: given an n-dimensional smooth manifold find the "best metrics" on it, where with "best metric" we mean a metric whose curvature tensors satisfy special conditions (for example some traces of the Riemann curvature tensor are null or constant or prescribed, or minimize some functional; think of the Yamabe Problem, the Uniformization Theorem, etc.).

The analogous problem concerning the *extrinsic* geometry of surfaces sounds roughly as follows: given an abstract 2-dimensional surface Σ (we will always consider Σ closed: compact and without boundary) and a Riemannian 3-dimensional manifold (M, g) find the "best immersions" $f : \Sigma \hookrightarrow M$ of Σ into M. Here with "best immersion" we mean an immersion whose curvature, i.e. second fundamental form, satisfies special conditions: for example if the second fundamental form is null the immersion is totally geodesic, if the mean curvature is null the immersion is minimal, if the trace-free second fundamental form is null the immersion is totally umbilical, etc.

Before proceeding let us introduce some notation. Given an immersion $f : \Sigma \hookrightarrow (M, g)$ let us denote by $\mathring{g} = f^*g$ the pull back metric on Σ (i.e. the metric on Σ induced by the immersion f); the area form $\sqrt{\det \mathring{g}}$ is denoted with $d\mu_g$ or with $d\Sigma$; the second fundamental form is denoted with A and its trace $H := \mathring{g}^{ij} A_{ij}$ is called mean curvature (notice that we use the convention that the mean curvature is the sum of the principal curvatures and not the arithmetic mean), finally $A^\circ := A - \frac{1}{2} H \mathring{g}$ is called trace-free second fundamental form.

As written in the second paragraph, classically the "best immersions" are the ones for which the quantities A, H, A° are null or constant (i.e. parallel) but in many cases such immersions do not exist (for example if Σ is a closed surface and $(M, g) = (\mathbb{R}^3, \text{eucl})$ is the euclidean three dimensional space, by maximum principle there exist no minimal, and in particular totally geodesic, immersion of Σ into $(\mathbb{R}^3, \text{eucl})$; more references about the existence of these classical special submanifolds will be given in the following more specific introductions, here we just want to motivate the problem).
If such classical special submanifolds do not exist it is interesting to study the minimization of natural integral functionals associated to A, H, A° of the type

$$\int_\Sigma |A|^p d\mu_g, \quad \int_\Sigma |H|^p d\mu_g, \quad \int_\Sigma |A^\circ|^p d\mu_g, \quad \text{for some } p \geq 1.$$

A global minimizer, if it exists, can be seen respectively as a generalized totally geodesic, minimal, or totally umbilic immersion in a natural integral sense. The general integral functionals above have been studied, among others, by Allard [Al], Anzellotti-Serapioni-Tamanini [AST], Delladio [Del], Hutchinson [Hu1], [Hu2], [Hu3], Mantegazza [MantCVB] and Moser [Mos].

An important example of such functionals is given by the Willmore functional $\frac{1}{4} \int_\Sigma H^2 d\mu_g$. The topic is classical and goes back to the 1920-'30 when Blaschke [Bla] and Thomsen [Tho] discovered the functional and observed that it is invariant under conformal transformations of $\mathbb{R}^3$.
The functional was later rediscovered in the 60's by Willmore who proved that the standard spheres S^ρ_p are the points of strict global minimum for $\frac{1}{4} \int H^2$. The proofs of the last facts can be found in [Will] (pag. 271, 276-279).

The functional relative to immersions in $\mathbb{R}^3$ and $\mathbb{S}^3$ has been deeply studied with remarkable results (for a panoramic view up to the 80's see [Will] Chap. 7, for immersions in $\mathbb{R}^n$ see [SiL], [BK], [Chen] [KS], [Schy] and [Riv], for immersions in $\mathbb{S}^3$ and space forms see [LY], [Wei], [ZG], [GLW], [LU], [WG] [MW]). Finally, in the last years, the flow generated by the L^2-differential of the functional has been analyzed ([Sim], [KS1], [KS2]).

The Willmore functional has lots of applications in biology, general relativity, string theory and elasticity theory: in the study of lipid bilayer membranes it is called "Hellfrich energy", in general relativity it is linked with the "Hawking mass", in string theory it appears in the "Polyakov extrinsic action" and in nonlinear elasticity theory it arises as Γ-limit of some energy functionals (see [FJM]). We also mention the classical fact that in the mean curvature flow analysis one has

$$\frac{d}{dt} Vol(\mathring{M}) = -\int_{\mathring{M}} H^2 d\Sigma$$

where $\mathring{M}$ is the evolving submanifold with respect to the parameter t and $Vol(\mathring{M}) := \int_{\mathring{M}} d\Sigma$ is its area.

While, as we remarked, there is an extensive literature for immersions into $\mathbb{R}^n$ or $\mathbb{S}^n$, very little is known for general ambient manifolds (apart from the case of minimal surfaces). The aim of this thesis is to study the Willmore and other natural L^p curvature functionals in curved spaces.

Before passing to more detailed introductions let us write which problems are analyzed in the present thesis. In [Mon1], the author studied the Willmore functional in a perturbative setting: considered $\mathbb{R}^3$ with the metric $\delta_{\mu,\nu} + \epsilon h_{\mu\nu}$ which is an infinitesimal perturbation of the euclidean metric $\delta_{\mu\nu}$, under generic conditions on the scalar curvature and a fast decreasing assumption at infinity on the perturbation $h_{\mu\nu}$, existence and multiplicity of immersions which are critical point for the functional $\frac{1}{4}\int H^2 d\mu_g$ were proven (for more details see [Mon1]). The method was perturbative and the proof relied on a Lyapunov-Schmidt reduction. Using a similar technique, in Chapter 2 we study the conformal Willmore functional $\frac{1}{2}\int |A^\circ|^2 d\mu_g$, which is conformally invariant in Riemannian manifolds, in the same perturbative setting $(\mathbb{R}^3, \delta_{\mu\nu} + \epsilon h_{\mu\nu})$. Under generic conditions on the trace-free Ricci tensor $S_{\mu\nu} := Ric_{\mu\nu} - \frac{1}{3} R g_{\mu\nu}$ (where $Ric_{\mu\nu}$ is the Ricci tensor and R is the scalar curvature, see also Definition (1.5)) and a fast decreasing assumption at infinity on the perturbation $h_{\mu\nu}$, we prove existence and multiplicity of critical points; with the same technique we prove also a non existence result in general Riemannian manifolds. For more details see the corresponding Introduction 1.0.1 and Chapter 2; this work is the object of the paper [Mon2].

Using more global techniques coming from geometric measure theory, in the rest of the thesis we study the Willmore and other L^p curvature functionals in semi-perturbative and global settings. With this in mind, in Appendix 6.6 we recall some basic notions about varifold theory which will be useful in the rest of the thesis.

In Chapter 3 we study the Willmore functional $\frac{1}{4}\int |H|^2 d\mu_g$ in a semiperturbative setting: while in [Mon1] the perturbation to the euclidean metric was infinitesimal, here the perturbation is small but finite. More precisely we consider $\mathbb{R}^3$ with the metric $\delta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is a $C_0^\infty(\mathbb{R}^3)$ bilinear form with compact support and small C^1 norm. If there exist a point $\bar{p} \in \mathbb{R}^3$ where the scalar curvature of this metric is strictly positive, $R(\bar{p}) > 0$, then we prove the existence of an immersion of a sphere $f : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, \delta_{\mu,\nu} + h_{\mu\nu})$ minimizing the Willmore functional among immersed spheres. The technique is the so called direct method in the calculus of variations: we consider a minimizing sequence of immersions, we associate to it a sequence in an enlarged space where it is easier to prove compactness, then by lower semicontinuity we prove the existence of a candidate minimizer weak object and finally we prove its regularity. For more details see Introduction 1.0.2 and Chapter 3; this content is part of a joint work with J. Schygulla, see [MS1].

In the rest of the thesis we study global problems: the ambient manifold will be a Riemannian manifold under global curvature conditions. First, in Section 4.1 and Chapter 5, we study the case of a closed (compact without boundary) ambient manifold (M, g) which as always is of dimension 3. In this framework we consider immersions of 2 spheres $f : \mathbb{S}^2 \hookrightarrow M$ and the problem of the minimization of the functionals $E(f) := \frac{1}{2}\int |A|^2 d\mu_g$ and $W_1(f) := \int \left(\frac{H^2}{4} + 1\right) d\mu_g$. Notice that in $(\mathbb{R}^3, \text{eucl})$ it is equivalent to minimize the Willmore functional $W(f) := \frac{1}{4}\int |H|^2 d\mu_g$ and the functional $E(f) := \frac{1}{2}\int |A|^2 d\mu_g$ among immersions of 2-spheres, indeed by Gauss-Bonnet Theorem we have

$$E(f) := \frac{1}{2}\int |A|^2 d\mu_g = \frac{1}{2}\int |H|^2 d\mu_g - 2\pi\chi(\mathbb{S}^2) = 2W(f) - 4\pi.$$

Therefore E is a natural generalization of the Willmore functional W for immersions in manifolds. Instead, as it can be easily seen, the infimum of W_1 is not attained for immersions of spheres in $(\mathbb{R}^3, \mathrm{eucl})$. Under global conditions on the ambient manifold (M, g) we will prove the existence of a smooth immersion $f : \mathbb{S}^2 \hookrightarrow M$ minimizing E (respectively W_1) among immersions of 2-spheres. The technique is the direct method in the calculus of variations, but here the hard part is to get geometric a priori estimates on the minimizing sequences and even harder the regularity of the candidate minimizer. For more details see Introduction 1.0.3, Section 4.1 and Chapter 5; this part is the object of a joint work with E. Kuwert and J. Schygulla, see [MS2].

Next, in Section 4.2 we study the global problem of minimizing the Willmore type functionals $W_1(f) := \int \left(\frac{|H|^2}{4} + 1\right) d\mu_g$, $E_1(f) := \int \left(\frac{|A|^2}{2} + 1\right) d\mu_g$ in non compact 3-dimensional ambient Riemannian manifold (M, g) without boundary (as before $f : \mathbb{S}^2 \hookrightarrow M$ is an immersion of 2-sphere in M). Also in this case we prove, under the assumption that (M, g) is asymptotically euclidean and under curvature conditions, that there exists a smooth immersion $f : \mathbb{S}^2 \hookrightarrow M$ minimizing E_1 (respectively W_1) among immersions of spheres. As before the technique is the direct method in the calculus of variations, here the difficulty is that the surfaces in the minimizing sequence can become larger and larger or can escape to infinity so, using curvature assumptions on (M, g), we will prove a priori estimates which prevents those bad behaviors. Once the existence of a weak candidate minimizer is settled, the regularity theory is exactly the same as in the compact case (see Chapter 5); this part is contained in the joint paper [MS1] with J. Schygulla.

In the last Chapter 6 we study supercritical L^p-curvature functionals $\int |H|^p$ and $\int |A|^p$ for submanifolds of any codimension in a Riemannian manifold of arbitrary dimension. Let us be more precise. Let (N, g) be a compact (maybe with boundary) Riemannian manifold of dimension $n \in \mathbb{N}$ and consider on the m-dimensional submanifolds, $2 \leq m < n$, the L^p curvature functionals $\int |H|^p$ and $\int |A|^p$ with $p > m$. Let us stress that here the exponent $p > m$ is supercritical, in contrast with the preceding chapters where we were dealing with 2-dimensional surfaces and the exponent was 2 (so before we were dealing with the critical exponent). In this chapter we heavily use varifold theory and, using direct methods in the calculus of variations, we prove existence and partial regularity of integral rectifiable m-dimensional varifolds (the non expert reader can think at them as generalized m-dimensional submanifolds) minimizing the above functionals $\int |H|^p$ and $\int |A|^p$ in a given Riemannian n-dimensional manifold (N, g), $2 \leq m < n$ and $p > m$, under suitable assumptions on N (in the end of the chapter we give many examples of such ambient manifolds). To this aim we introduce the following new tools: some monotonicity formulas for varifolds in $\mathbb{R}^S$ involving $\int |H|^p$, to avoid degeneracy of the minimizer, and a sort of isoperimetric inequality to bound the mass (the non expert reader can think of the mass as the volume of the generalized m-dimensional submanifold) in terms of the mentioned functionals. For more details see Introduction 1.0.4 and Chapter 6; this part corresponds to the paper [MonVar].

1.0.1 Introduction and results about the Conformal Willmore Functional $\frac{1}{2}\int |A^\circ|^2$ in a perturbative setting: Chapter 2

The aim of Chapter 2 is to study a (Riemannian) conformally invariant Willmore functional. The study of Conformal Geometry was started by H. Weil and E. Cartan in the beginning of the 20^{th} century and since its foundation it has been playing ever more a central role in Riemannian Geometry; its task is to analyze how geometric quantities change under conformal transformations (i.e. diffeomorphisms which preserves angles) and possibly find out conformal invariants (i.e. quantities which remain unchanged under conformal transformations).

Let us first recall the definition of "standard" Willmore functional for immersions in $\mathbb{R}^3$ which is a topic of great interest in the contemporary research as explained before. Given a compact orientable Riemannian surface $(\mathring{M}, \mathring{g})$ isometrically immersed in $\mathbb{R}^3$ endowed with euclidean metric, the "standard"

Willmore functional of $\overset{\circ}{M}$ is defined as

$$W(\overset{\circ}{M}) = \int_{\overset{\circ}{M}} \frac{H^2}{4} d\Sigma \tag{1.1}$$

where H is the mean curvature and $d\Sigma$ is the area form of $(\overset{\circ}{M}, \mathring{g})$ (we will always adopt the convention that H is the sum of the principal curvatures: $H := k_1 + k_2$).
As written above, this functional satisfies two crucial properties:

a) W is invariant under conformal transformations of $\mathbb{R}^3$; that is, given $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ a conformal transformation, $W(\Psi(\overset{\circ}{M})) = W(\overset{\circ}{M})$.

b) W attains its strict global minimum on the standard spheres S^ρ_p of $\mathbb{R}^3$ (hence they form a critical manifold - i.e. a manifold made of critical points):

$$W(\overset{\circ}{M}) := \int_{\overset{\circ}{M}} \frac{H^2}{4} d\Sigma \geq 4\pi; \quad W(\overset{\circ}{M}) = 4\pi \Leftrightarrow \overset{\circ}{M} = S^\rho_p. \tag{1.2}$$

The proofs of the last facts can be found in [Will] (pag. 271 and pag. 276-279).

Clearly the "standard" Willmore functional W can be defined in the same way for compact oriented surfaces immersed in a *general Riemannian manifold* (M, g) of dimension three. Although this functional has several interesting applications, it turns out that W is not conformally invariant.

As proved by Bang-Yen Chen in [Chen] (see also [Wei] and for higher dimensional and codimensional analogues [PW]), the "correct" Willmore functional from the conformal point of view is defined as follows. Given a compact orientable Riemannian surface $(\overset{\circ}{M}, \mathring{g})$ isometrically immersed in the three dimensional Riemannian manifold (M, g), the *conformal Willmore functional* of $\overset{\circ}{M}$ is

$$I(\overset{\circ}{M}) := \frac{1}{2} \int_{\overset{\circ}{M}} |A^\circ|^2 d\Sigma = \int_{\overset{\circ}{M}} \left(\frac{H^2}{4} - D \right) d\Sigma \tag{1.3}$$

where $|A^\circ|^2 = \frac{1}{2}(k_1 - k_2)^2$ is the norm of the traceless second fundamental form (recall that $A^\circ = A - \frac{1}{2} H \mathring{g}$), $D := k_1 k_2$ is the product of the principal curvatures and as before H and $d\Sigma$ are respectively the mean curvature and the area form of $(\overset{\circ}{M}, \mathring{g})$. In the aforementioned papers it is proved that I is conformally invariant (i.e. given $\Psi : (M, g) \to (M, g)$ a conformal transformation, $I(\Psi(\overset{\circ}{M})) = I(\overset{\circ}{M}))$ so in this sense it is the "correct" generalization of the standard Willmore functional which, as pointed out, is conformally invariant in $\mathbb{R}^3$. We say that I generalizes W because if $\mathbb{R}^3$ is taken as ambient manifold, the quantity $D = k_1 k_2$ is nothing but the Gaussian curvature which, fixed the topology of the immersed surface, gives a constant when integrated (by the Gauss-Bonnet Theorem) hence it does not influence the variational properties of the functional.

A surface which makes the conformal Willmore functional I stationary with respect to normal variations is called *conformal Willmore surface* and it is well known (the expression of the differential in full generality is stated without proof in [PW] and the computations can be found in [HL], here we deal with a particular case which will be computed in the proof of Proposition 2.2.9) that such a surface satisfies the following PDE:

$$\frac{1}{2} \triangle_{\overset{\circ}{M}} H + H \left(\frac{H^2}{4} - D \right) + \frac{(\lambda_1 - \lambda_2)}{2} [R(\overset{\circ}{N}, e_1, \overset{\circ}{N}, e_1) - R(\overset{\circ}{N}, e_2, \overset{\circ}{N}, e_2)] + \sum_{ij} (\nabla_{e_i} R)(\overset{\circ}{N}, e_j, e_j, e_i) = 0$$

where $\triangle_{\overset{\circ}{M}}$ is the Laplace Beltrami operator on $\overset{\circ}{M}$, R is the Riemann tensor of the ambient manifold (M, g) (for details see "notations and conventions"), $\overset{\circ}{N}$ is the inward unit normal vector, λ_1 and λ_2 are the principal curvatures and e_1, e_2 are the normalized principal directions.

The goal of Chapter 1.0.1 is to study the existence of conformal Willmore surfaces.
The topic has been extensively studied in the last years: in [ZG] the author generalizes the conformal Willmore functional to arbitrary dimension and codimension and studies the existence of critical points in space forms; in [HL] the authors compute the differential of I in full generality and give examples of conformal Willmore surfaces in the sphere and in complex space forms; other existence results in spheres or in space forms are studied for instance in [GLW], [LU], [WG] and [MW].

The novelty of Chapter 1.0.1 is that the conformal Willmore functional is analyzed in an ambient manifold with non constant sectional curvature: we will give existence (resp. non existence) results for curved metrics in $\mathbb{R}^3$, close and asymptotic to the flat one (resp. in general Riemannian manifolds). More precisely, taken $h_{\mu\nu} \in C_0^\infty(\mathbb{R}^3)$ a smooth bilinear form with compact support (as we will remark later it is sufficient that $h_{\mu\nu}$ decreases fast at infinity with its derivatives) we take as ambient manifold

$$(\mathbb{R}^3, g_\epsilon) \quad \text{with} \quad g_\epsilon = \delta + \epsilon h \tag{1.4}$$

where δ is the euclidean scalar product.

The candidate critical surfaces are perturbed standard spheres (resp. perturbed geodesic spheres), let us define them. Let S_p^ρ be a standard sphere of $\mathbb{R}^3$ parametrized by

$$\Theta \in S^2 \mapsto p + \rho\Theta$$

and let $w \in C^{4,\alpha}(S^2)$ be a small function, then the *perturbed standard sphere* $S_p^\rho(w)$ is the surface parametrized as

$$\Theta \in S^2 \mapsto p + \rho(1 - w(\Theta))\Theta.$$

Analogously the *perturbed geodesic sphere* $S_{p,\rho}(w)$ is the surface parametrized by

$$\Theta \in S^2 \mapsto Exp_p[\rho(1 - w(\Theta))\Theta]$$

where S^2 is the unit sphere of T_pM, Exp_p is the exponential map centered at p and, as before, w is a small function in $C^{4,\alpha}(S^2)$.

The main results of Chapter 1.0.1 are Theorem 1.0.1 and Theorem 1.0.2 below, which will be proved in Subsection 2.3.3. Before stating them recall that given a three dimensional Riemannian manifold (M, g), the *traceless Ricci tensor* S is defined as

$$S_{\mu\nu} := R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R \tag{1.5}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the scalar curvature. Its squared norm at a point p is defined as $\|S_p\|^2 = \sum_{\mu,\nu=1}^3 S_{\mu\nu}(p)^2$ where $S_{\mu\nu}(p)$ is the matrix of S at p in an orthonormal frame. Expanding in ϵ the curvature tensors (see for example [And-Mal] pages 23-24) it is easy to see that the traceless Ricci tensor corresponding to $(\mathbb{R}^3, g_\epsilon)$ (defined in (1.4))is

$$\|S_p\|^2 = \epsilon^2 \tilde{s}_p + o(\epsilon^2) \tag{1.6}$$

where $\tilde{s}_p$ is a nonnegative quadratic function in the second derivatives of $h_{\mu\nu}$ which does not depend on ϵ. In the following Theorem, π will denote an affine plane in $\mathbb{R}^3$ and $H^1(\pi)$ will be the Sobolev space of the L^2 functions defined on π whose distributional gradient is a vector valued L^2 integrable function. $H^1(\pi)$ is equipped with the norm

$$\|f\|^2_{H^1(\pi)} := \|f\|^2_{L^2(\pi)} + \|\nabla f\|^2_{L^2(\pi)} \quad \forall f \in H^1(\pi).$$

Now we can state the Theorems.

Theorem 1.0.1. *Let $h \in C_0^\infty(\mathbb{R}^3)$ be a symmetric bilinear form with compact support and let c be such that*

$$c := \sup\{\|h_{\mu\nu}\|_{H^1(\pi)} : \pi \text{ is an affine plane in } \mathbb{R}^3,\ \mu,\nu = 1,2,3\}.$$

Then there exists a constant $A_c > 0$ depending on c with the following property: if there exists a point $\bar{p}$ such that

$$\tilde{s}_{\bar{p}} > A_c$$

then, for ϵ small enough, there exists a perturbed standard sphere $S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon)$ which is a critical point of the conformal Willmore functional I_ϵ converging to a standard sphere as $\epsilon \to 0$.

It is well-known (see Remark 1.0.5 point 3) that if a three dimensional Riemannian Manifold has non constant sectional curvature then the traceless Ricci tensor S cannot vanish everywhere. Clearly $(\mathbb{R}^3, g_\epsilon)$ has non constant sectional curvature (the metric is asymptotically flat but not flat) hence it cannot happen that $\|S\|^2 \equiv 0$; for the following existence result we ask that this non null quantity has non degenerate expansion in ϵ: we assume

$$M := \max_{p \in \mathbb{R}^3} \tilde{s}_p > 0. \tag{1.7}$$

Actually it is a maximum and not only a supremum because the metric is asymptotically flat.

The following is like a mirror Theorem to the previous existence result: in the former we bounded c and asked $\tilde{s}$ to be large enough at one point, in the latter we assume that $\tilde{s}$ is non null at one point (at least) and we ask c to be small enough.

Theorem 1.0.2. *Let h, c be as in Theorem 1.0.1 and M satisfying* (1.7)*. There exists $\delta_M > 0$ depending on M such that if $c < \delta_M$ then, for ϵ small enough, there exists a perturbed standard sphere $S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon)$ which is a critical point of the conformal Willmore functional I_ϵ converging to a standard sphere as $\epsilon \to 0$.*

Remark 1.0.3.

1. *As done in [Mon1], the assumption $h \in C_0^\infty(\mathbb{R}^3)$ in Theorem 1.0.1 and Theorem 1.0.2 can be relaxed asking that h decreases fast enough at infinity with its derivatives.*

2. *The conditions of Theorem 1.0.1*

$$\sup\{\|h_{\mu\nu}\|_{H^1(\pi)} : \pi \textit{ is an affine plane in } \mathbb{R}^3,\ \mu, \nu = 1, 2, 3\} \leq c$$

and

$$\tilde{s}_{\bar{p}} > A_c$$

are compatible. In fact the former involves only the first derivatives of h while the latter the second derivatives (see for instance [And-Mal] page 24). Of course the same fact is true for the conditions $\tilde{s}_{\bar{p}} \geq M$ and $c < \delta_M$ of Theorem 1.0.2.

3. *If the perturbation h satisfies some symmetries (invariance under reflections or rotations with respect to planes, lines or points of $\mathbb{R}^3$), it is possible to prove multiplicity results (see Subsection 5.2 of [Mon1]).*

4. *If h is C^∞ then a standard regularity argument (see the paper of Leon Simon [SiL] pag. 303 or the book by Morrey [MCB]) shows that a $C^{2,\alpha}$ conformal Willmore surface is actually C^∞. It follows that the conformal Willmore surfaces exhibited in the previous Theorems, which are $C^{4,\alpha}$ by construction, are C^∞.*

5. *The critical points $S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon)$ of I_ϵ are of (maybe degenerate) saddle type. In fact from* (1.2) *the standard spheres S_p^ρ are strict global minima in the direction of variations in $C^{4,\alpha}(S^2)^\perp = Ker[I_0''(S_p^\rho)]^\perp \cap C^{4,\alpha}(S^2)$, it is easy to see that for small ϵ the surfaces $S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon)$ are still minima in the $C^{4,\alpha}(S^2)^\perp$ direction; but, since they are obtained as maximum points of the reduced functional, in the direction of $Ker[I_0''(S_p^\rho)]$ they are (maybe degenerate) maximum points.*

As we said before, the non existence result concerns perturbed geodesic spheres of small radius. Let us state it:

Theorem 1.0.4. *Let (M, g) be a Riemannian manifold. Assume that the traceless Ricci tensor of M at the point $\bar{p}$ is not null:*

$$\|S_{\bar{p}}\| \neq 0.$$

Then there exist $\rho_0 > 0$ and $r > 0$ such that for radius $\rho < \rho_0$ and perturbation $w \in C^{4,\alpha}(S^2)$ with $\|w\|_{C^{4,\alpha}(S^2)} < r$, the surfaces $S_{\bar{p},\rho}(w)$ are not *critical points of the conformal Willmore functional I.*

Remark 1.0.5.

1. *Observe the difference with the flat case: thanks to* (1.2)*, in $\mathbb{R}^3$ the spheres of any radius are critical points of the conformal Willmore functional I (as we noticed, the term D does not influence the differential properties of the functional by Gauss-Bonnet Theorem); on the contrary, in the case of ambient metric with non null traceless Ricci curvature we have just shown that the geodesic spheres of small radius are not critical points.*

2. *The condition $\|S_p\| \neq 0$ is generic.*

3. *If (M,g) has not constant sectional curvature then there exists at least one point $\bar{p}$ such that $\|S_{\bar{p}}\| \neq 0$. In fact if $\|S\| \equiv 0$ then (M,g) is Einstein, but Einstein manifolds of dimension three have constant sectional curvature (for example see [Pet] pages 38-41).*

The abstract method employed throughout Chapter 1.0.1 is the Lyapunov-Schmidt reduction (for more details about the abstract method see Section 2.1). An analogous technique has been used in the study of constant mean curvature surfaces (see [Ye], [Ye2], [PX], [CM1], [CM2], [FMa], [FMe] and [Fe]).

We discuss next the structure of the Chapter, but first let us explain (informally) the main idea (for the details see Subsection 2.2.3 and Subsection 2.3.1).
As we remarked, (1.2) implies that the Willmore functional in the euclidean space $\mathbb{R}^3$ possesses a critical manifold Z made of standard spheres S^ρ_p. The tangent space to Z at S^ρ_p is composed of constant and affine functions on S^ρ_p so, with a pull back via the parametrization, on S^2. The second derivative of I_0 at S^ρ_p is

$$I_0''(S^\rho_p)[w] = \frac{1}{2}\triangle_{S^2}(\triangle_{S^2}+2)w$$

(for explanations and details see Remark 2.3.1) which is a Fredholm operator of index zero and whose Kernel is made of the constant and affine functions; exactly the tangent space to Z.
So, considered $C^{4,\alpha}(S^2)$ as a subspace of $L^2(S^2)$ and called

$$C^{4,\alpha}(S^2)^\perp := C^{4,\alpha}(S^2) \cap Ker[\triangle_{S^2}(\triangle_{S^2}+2)]^\perp,$$

it follows that $I_0''|_{C^{4,\alpha}(S^2)^\perp}$ is invertible on its image and one can apply the Lyapunov-Schmidt reduction. Thanks to this reduction, the critical points of I_ϵ in a neighborhood of Z are exactly the stationary points of a function (called reduced functional) $\Phi_\epsilon : Z \to \mathbb{R}$ of *finitely many* variables (we remark that in a neighborhood of Z the condition is necessary and sufficient for the existence of critical points of I_ϵ).

In order to study the function Φ_ϵ, we will compute explicit formulas and estimates of the conformal Willmore functional. More precisely for small radius ρ we will give an expansion of the functional on small perturbed geodesic spheres, for large radius we will estimate the functional on perturbed standard spheres and we will link the geodesic and standard spheres in a smooth way using a cut off function (for details see Subsection 2.3.1).

The Chapter is organized as follows: in Section 2.2 we will start in the most general setting, the conformal Willmore functional for small perturbed geodesic spheres in ambient manifold (M,g). Even in this case the reduction method can be performed, using the small radius ρ as perturbation parameter (see Lemma 2.2.10).
Employing the geometric expansions of Subsection 2.2.1 and the expression of the constrained w given in Subsection 2.2.3, in Subsection 2.2.4 we will compute the expansion of the reduced functional on small perturbed geodesic spheres of (M,g). Explicitly, in Proposition 2.2.11, we will get

$$\Phi(p,\rho) = \frac{\pi}{5}\|S_p\|^2\rho^4 + O_p(\rho^5) \tag{1.8}$$

where $\Phi(.,.)$ is the reduced functional and, as before, S_p is the traceless Ricci tensor evaluated at p. Using this formula we will show that if $\|S_{\bar{p}}\| \neq 0$ then $\Phi(\bar{p},.)$ is strictly increasing for small radius. The non existence result will follow from the necessary condition.

Section 2.3 will be devoted to the conformal Willmore functional in ambient manifold $(\mathbb{R}^3, g_\epsilon)$. In Subsection 2.3.1 we will treat the applicability of the abstract method and in the last Subsection 2.3.3 we will bound the reduced functional Φ_ϵ for large radius ρ using the computations of Subsection 2.3.2. We remark that the expansion of Φ_ϵ is degenerate in ϵ (i.e. the first term in the expansion is null and $\Phi_\epsilon = O(\epsilon^2)$), clearly this feature complicates the problem. Using the estimates on the reduced functional Φ_ϵ for large radius and the expansions for small radius (since for small radius we take geodesic spheres it will be enough to specialize (1.8) in the setting $(\mathbb{R}^3, g_\epsilon)$) we will force Φ_ϵ to have a global maximum, sufficient condition to conclude with the existence results.

1.0.2 Introduction and results about the Willmore Functional $\frac{1}{4}\int |H|^2$ in a semiperturbative setting: Chapter 3

Let $(\mathbb{R}^3, \delta_{\mu\nu} + h_{\mu\nu})$ be the Riemannian manifold associated to $\mathbb{R}^3$ with the perturbed metric $\delta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is a $C^\infty_0(\mathbb{R}^3)$ compactly supported bilinear form with small C^1 norm. The framework is semi

perturbative in the following sense: while previously the perturbation $\epsilon h_{\mu\nu}$ was infinitesimal, now the perturbation is small in C^1 norm but finite.
In this setting let us define the classical Willmore functional W_h. Let $\Sigma \subset \mathbb{R}^3$ be an embedded surface, then we define

$$W_h(\Sigma) := \frac{1}{4}\int_\Sigma |H_h|^2 d\mu_h = \frac{1}{4}\int_\Sigma |H|^2 d\mu_h$$

where $H = H_h$ is the mean curvature of the surface Σ as submanifold of $(\mathbb{R}^3, \delta_{\mu\nu} + h_{\mu\nu})$ and $d\mu_h$ the associated area form. The main result of Chapter 3 is the following theorem.

Theorem 1.0.6. *Let $(\mathbb{R}^3, \delta_{\mu\nu} + h_{\mu\nu})$ be $\mathbb{R}^3$ with the perturbed metric $\delta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is a $C_0^\infty(\mathbb{R}^3)$ compactly supported bilinear form with small C^1 norm. Assume there exists a point $\bar{p} \in \mathbb{R}^3$ where the scalar curvature is strictly positive: $R(\bar{p}) > 0$.*

Then there exists an embedded 2-sphere $\Sigma \subset \mathbb{R}^3$ which minimizes the Willmore functional W_h among embedded spheres:

$$W_h(\Sigma) := \inf\{W_h(\tilde{\Sigma}) : \tilde{\Sigma} \text{ is an embedded 2-sphere }\}.$$

The technique we adopt is the direct method in the calculus of variations. We consider a minimizing sequence and associate to each surface a Radon measure, then for having compactness we have to prove that the sequence does not shrink to a point, that there exist upper bound on the diameters and the areas of the surfaces in the minimizing sequence, and that the sequence does not escape to infinity. With this in mind we first link the euclidean and the perturbed quantities proving a monotonicity formula in a semiperturbative setting, then using these estimates we prove the desired non degeneracy of the minimizing sequence via blow up and blow down procedures which use the scale invariance of the functional and the assumption on the scalar curvature. Once we have the existence of a candidate weak minimizer then we prove C^∞ regularity following closely the theory developed by Simon in [SiL].

1.0.3 Introduction and results about the Willmore type functionals $\frac{1}{2}\int |A|^2$, $\int \left(\frac{|A|^2}{2}+1\right)$, $\int \left(\frac{|H|^2}{4}+1\right)$ in a global setting: Chapters 4 and 5

The functionals $\frac{1}{2}\int |A|^2$ and $\int \left(\frac{|H|^2}{4}+1\right)$ in COMPACT ambient manifold

Let $f : \Sigma \hookrightarrow \mathbb{R}^3$ be an immersion of a compact 2-dimensional surface Σ in $\mathbb{R}^3$. An important problem in geometric analysis is to find immersions which minimize the L^2 norm of the second fundamental form $E(f) = \frac{1}{2}\int |A|^2$. Using the Gauss Bonnet Theorem, one obtains

$$E(f) := \frac{1}{2}\int |A|^2 = \frac{1}{2}\int |H|^2 - 2\pi\chi(\Sigma) = 2W(f) - 2\pi\chi(\Sigma), \tag{1.9}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ and $W(f) = \frac{1}{4}\int |H|^2$ is the Willmore energy of f. Hence, once the topological type of Σ is fixed, it is equivalent to minimize the Willmore functional W and the functional E. If Σ is a 2-sphere then Willmore proved (Theorem 7.2.2 in [Will]) that the minimizing immersion is a round sphere, and this is actually a strict global minimum on all surfaces. If Σ is a torus then Simon, using the direct method of the calculus of variations and regularity theory for fourth order PDEs, proved in [SiL] that there exists a smooth embedding f of a torus in $\mathbb{R}^3$ minimizing the Willmore functional W among immersed tori (the Willmore conjecture asserts that the minimizing torus is actually the Clifford torus). The minimization problems for higher genuses were solved by Bauer and Kuwert in [BK]. The variational problems related to the Willmore functional (which by (1.9) are equivalent to the variational problems related to E) have become of interest for the community (see for example [KS], [LY], [Riv] and [Schy]).

Although there is a quite extensive literature about the Willmore functional in euclidean space, the analogous problems for immersions in a Riemannian manifold are almost unexplored. There are some perturbative results (see [LM], [LMS], [Mon1], [Mon2]) but the global problem has not been faced yet. In this thesis we give the first existence and full regularity results for the global problem of finding minimizers for the functional E in a compact manifold.

Before writing precisely which are the problems we study let us introduce some notation. Let (M, g) be a compact 3-dimensional Riemannian manifold without boundary; for any immersion of a 2-sphere

$f : \mathbb{S}^2 \hookrightarrow M$ we consider the functional given by the L^2 norm of the second fundamental form A of the immersion f:

$$E(f) := \frac{1}{2}\int_{\mathbb{S}^2} |A|^2 d\mu_g \tag{1.10}$$

where $d\mu_g$ is the area form induced by the pull back metric f^*g on $\mathbb{S}^2$. We consider moreover the Willmore-type functional

$$W_1(f) := \int_{\mathbb{S}^2} \left(\frac{H^2}{4} + 1\right) d\mu_g = W(f) + Area(f) \tag{1.11}$$

where H is the mean curvature (we adopt the convention that H is the sum of the principal curvatures), $W(f) := \frac{1}{4}\int_{\mathbb{S}^2} H^2 d\mu_g$ is the Willmore functional of f and $Area(f)$ is the area of $\mathbb{S}^2$ endowed with the pullback metric f^*g as above. In this thesis we study the minimization problems relative to the functionals E and W_1:

$$\inf_{f:\mathbb{S}^2 \hookrightarrow (M,g)} E(f) := \inf\{E(f) | f : \mathbb{S}^2 \hookrightarrow (M,g) \text{ is a } C^\infty \text{ immersion in } (M,g)\}, \tag{1.12}$$

$$\inf_{f:\mathbb{S}^2 \hookrightarrow (M,g)} W_1(f) := \inf\{W_1(f) | f : \mathbb{S}^2 \hookrightarrow (M,g) \text{ is a } C^\infty \text{ immersion in } (M,g)\}, \tag{1.13}$$

and we prove that the minimization problems above have a smooth solution, as follows.

Theorem 1.0.7. *Let (M,g) be a 3-dimensional Riemannian manifold whose sectional curvature $\bar{K}$ is bounded below by a positive constant:*

$$\textit{there exists a } \Lambda > 0 \quad \textit{such that} \quad \bar{K} \geq \Lambda > 0. \tag{1.14}$$

Then the minimization problem (1.12) *has a smooth solution, i.e. there exists a smooth immersion $f : \mathbb{S}^2 \hookrightarrow M$ such that*

$$E(f) = \inf\{E(h) | h : \mathbb{S}^2 \hookrightarrow (M,g) \textit{ is a } C^\infty \textit{ immersion in } (M,g)\}.$$

Notice that under the condition (1.14) the manifold M is forced to be compact. Observe moreover that the theorem is not trivial in the sense that there are examples of compact 3-manifolds satisfying the condition (1.14) which do not contain totally geodesic immersions (i.e. immersion whose second fundamental form A vanish identically, $A \equiv 0$); for instance in [ST] it is proved that the Berger Spheres $\mathbb{M}^3(k,\tau)$ with $k > 0, \tau \neq 0$ do not contain totally geodesic surfaces (note that for $k > 3\tau^2$ the space $\mathbb{M}^3(k,\tau)$ has strictly positive sectional curvature, for the computation see [Dan]).

Theorem 1.0.8. *Let (M,g) be a compact 3-dimensional Riemannian manifold which does not contain non null 2-varifolds with null second fundamental form (for the definitions see Appendix 6.6 or for more details [MonVar]) and such that the scalar curvature R is strictly positive in at least one point $\bar{p}$: $R(\bar{p}) > 0$.*

Then the minimization problem (1.12) *has a smooth solution, i.e. there exists a smooth immersion $f : \mathbb{S}^2 \hookrightarrow M$ such that*

$$E(f) = \inf\{E(h) | h : \mathbb{S}^2 \hookrightarrow (M,g) \textit{ is a } C^\infty \textit{ immersion in } (M,g)\}.$$

Notice that the condition on the non existence of non null 2-varifolds with null second fundamental form should be generic and of course implies that there exists no totally geodesic surface in (M,g).

Theorem 1.0.9. *Let (M,g) be a compact 3-dimensional Riemannian manifold whose curvature satisfies the following conditions:*

i) there exists a point $\bar{p} \in M$ where the scalar curvature is strictly greater than 6*: $R(\bar{p}) > 6$,*

ii) the sectional curvature $\bar{K}$ is bounded above by 2*: $\bar{K} \leq 2$.*

Then the minimization problem (1.13) *has a smooth solution, i.e. there exists a smooth immersion $f : \mathbb{S}^2 \hookrightarrow M$ such that*

$$W_1(f) = \inf\{W_1(h) | h : \mathbb{S}^2 \hookrightarrow (M,g) \textit{ is a } C^\infty \textit{ immersion in } (M,g)\}.$$

Notice that the conditions $i)$ and $ii)$ of Theorem 1.0.9 are compatible since 6 is the scalar curvature of the standard sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ of radius 1 (whose sectional curvature is identically equal to 1) while 2 is the sectional curvature of the sphere $\frac{1}{\sqrt{2}}\mathbb{S}^3 \subset \mathbb{R}^4$ of radius $\frac{1}{\sqrt{2}}$, so for instance $(M,g) := \frac{1}{\sqrt{2}}\mathbb{S}^3 \subset \mathbb{R}^4$ satisfies both $i)$ and $ii)$ above.

The assumption on the scalar curvature in Theorem 1.0.9 is quite natural if one thinks at surfaces which are critical points for W_1 as generalized minimal surfaces, for example in [MN] Marquez and Neves prove existence and rigidity results for min-max minimal spheres assuming that the scalar curvature of the ambient manifold is greater or equal to 6 (plus other assumptions).

Remark 1.0.10. *In the proceeding [SiProc], Simon claimed that the minimizers of $\frac{1}{4}\int |H|^2$ in a Riemannian manifold are branched $C^{1,\alpha}$ immersions but a complete proof never appeared. This chapter is the first attempt to fill in this gap in the comprehension of minimizers of integral curvature functionals in Riemannian manifolds.*

Remark 1.0.11. *Let (M,g) be as in the assumptions of Theorem 1.0.7 or Theorem 1.0.8. The two existence results imply that if (M,g) does not contain an immersed totally geodesic sphere then there is the following gap:*

$$\text{there exists } \epsilon > 0 \text{ such that for all smooth immersions } \quad h : \mathbb{S}^2 \hookrightarrow M \text{ we have } E(h) \geq \epsilon. \tag{1.15}$$

Notice moreover that if (M,g) is a simply connected 3-dimensional manifold with strictly positive sectional curvature in the sense of (1.14) *then Theorem 0.5 in [FMR] implies that any totally geodesic immersed surface is actually embedded. Hence the gap* (1.15) *is still true under the a priori weaker assumption that (M,g) does not contain* embedded *totally geodesic spheres.*

Remark 1.0.12. *In this chapter we studied the minimization problem in compact manifolds without boundary. If one studied the minimization problem in non compact manifold (respectively in compact manifold with boundary) and manages to show that a minimizing sequence is contained in a relatively compact open subset (respectively in an open subset with strictly positive distance from the boundary) then the existence and regularity theory developed in this chapter can be analogously applied.*

Now let us briefly sketch the technique of the proof and the structure of our argument.
As done in [SiL] we use the direct method of the calculus of variations: we take a minimizing sequence of immersions, associate to them weak objects (Radon measures and varifolds) for which one has a good compactness theory, prove a priori estimates which ensure compactness and non degeneracy of the minimizing sequence, by lower semicontinuity of the functional get the existence of a candidate minimizer and then prove the regularity.

We remark that in the euclidean case, from the conformal invariance of the functional, by rescaling it is trivial to have an area bound on a minimizing sequence and it is not difficult to prove that the sequence does not shrink to a point; in a Riemannian manifold the situation is different and we prove all the a priori estimates in Section 4.1.1; in this part the curvature of the ambient manifold plays a central role.

Once the needed estimates are proved, we associate to each smooth immersion a weak object (a Radon measure and a varifold) and in Section 4.1.2, using geometric measure theory, we prove compactness in the enlarged space and lower semicontinuity of the functionals, and therefore the existence of a candidate minimizer weak object. In the rest of the chapter we prove the regularity of the candidate minimizer.

We took inspiration from the work of Simon [SiL] where the regularity of the minimizers of W in euclidean setting is performed, but there are some serious modifications to be done for immersions in a Riemannian manifold. First of all, since in Euclidean setting one has an 8π bound on the Willmore functional which turns out to be very useful, using an inequality of Li and Yau [LY] and a monotonicity formula Simon manages to work with embedded surfaces; in Riemannian manifold instead we work with immersions, hence there could be multiplicity and the technique is a bit more involved. Nevertheless in Section 5.2, working locally in normal coordinates, we manage to enter into the assumptions of the Graphical Decomposition Lemma of Simon and prove that near all points (except possibly finitely many "bad points" where the curvature concentrates) of the candidate minimizer, the minimizing sequence can be written locally as union of graphs and small "pimples" with good estimates.

In Section 5.3 we prove that the candidate minimizer is locally given by graphs of $C^{1,\alpha} \cap W^{2,2}$ functions. For getting this partial regularity we first prove a local power decay on the L^2 norms of the second fundamental forms of the minimizing sequence (see Lemma 5.3.1) away from the bad points; then, still working locally away from the bad points, replacing the pimples by sort of biharmonic discs, by

Ascoli-Arzelá theorem we get existence of Lipschitz limit functions; at this point, using a generalized Poincaré inequality, the power decay of the second fundamental forms and Radon Nicodym Theorem, we show in Lemma 5.3.2 that the candidate minimizer is associated to the limit Lipschitz graphs; finally using that this candidate minimizer has weak mean curvature in L^2, together with the aforementioned power decay, a lemma of Morrey implies the $C^{1,\alpha} \cap W^{2,2}$ regularity away from the bad points. Using a topological argument involving degree theory and Gauss Bonnet theorem, in Subsection 5.3.2 we prove that actually there are no bad points and therefore the candidate minimizer is $C^{1,\alpha} \cap W^{2,2}$ everywhere. This step is quite different (and simpler) from [SiL], indeed since we work with immersed spheres we manage to exclude bad points while Simon works with surfaces of higher genus and he has to handle the bad points without excluding them.

To complete the regularity we need to show that the candidate minimizer satisfies the Euler-Lagrange equation, and for this step we need to prove that it can be parametrized on $\mathbb{S}^2$. At this point (see Subsection 5.4.1) we use the notion of generalized (r, λ)-immersions developed by Breuning in his Ph. D. Thesis [BreuTh] taking inspiration by previous work of Langer [Lan]. Once the Euler Lagrange equation is satisfied the C^∞ regularity follows (see Subsection 5.4.2)

For the reader's convenience we end Chapter 5 with some appendices about maybe non standard basic material used throughout the arguments.

The functionals $\int \left(\frac{|A|^2}{2} + 1 \right)$ and $\int \left(\frac{|H|^2}{4} + 1 \right)$ in NONCOMPACT asymptotically euclidean ambient manifold

Once the existence in compact manifolds is settled we move to noncompact ones.

Let (M, g) be a 3-dimensional non compact asymptotically euclidean Riemannian manifold without boundary and with bounded geometry. By asymptotically euclidean we mean that there exist compact subsets $K_1 \subset\subset M$ and $K_2 \subset\subset \mathbb{R}^3$ such that

$$(M \backslash K_1) \text{ is isometric to } (\mathbb{R}^3 \backslash K_2, \text{eucl} + o_1(1)) \tag{1.16}$$

where $(\mathbb{R}^3, \text{eucl} + o_1(1))$ denotes the "Riemannian manifold" $\mathbb{R}^3$ endowed with the euclidean metric $\delta_{\mu\nu} + o_1(1)_{\mu\nu}$ and $o_1(1)$ denotes a symmetric bilinear form which goes to 0 with its first derivatives at infinity:

$$\lim_{|x| \to \infty} (|o_1(1)(x)| + |\nabla o_1(1)(x)|) = 0.$$

We also assume that the Riemannian manifold (M, g) has bounded geometry: the sectional curvature is bounded and the injectivity radius is uniformly bounded below by a strictly positive constant, i.e. there exists $\Lambda \in \mathbb{R}$ such that $|\bar{K}| \leq \Lambda^2$ and $\frac{1}{\Lambda^2} \leq Inj(M)$.

For any immersion of a 2-sphere $f : \mathbb{S}^2 \hookrightarrow M$ we consider the following Willmore-type functionals:

$$W_1(f) := \int_{\mathbb{S}^2} \left(\frac{H^2}{4} + 1 \right) d\mu_g \tag{1.17}$$

$$E_1(f) := \int_{\mathbb{S}^2} \left(\frac{A^2}{2} + 1 \right) d\mu_g \tag{1.18}$$

where as before A is the second fundamental form, H is the mean curvature and $d\mu_g$ is the area form induced on $\mathbb{S}^2$ by the immersion f. We consider the minimization problems of W_1 and E_1 among smooth immersions of $\mathbb{S}^2$

$$\inf_f W_1(f) := \inf\{W_1(f) : f : \mathbb{S}^2 \hookrightarrow (M, g) \text{ is a smooth immersion in } (M, g)\}, \tag{1.19}$$

$$\inf_f E_1(f) := \inf\{E_1(f) : f : \mathbb{S}^2 \hookrightarrow (M, g) \text{ is a smooth immersion in } (M, g)\}, \tag{1.20}$$

and prove the following theorems.

Theorem 1.0.13. *Let (M, g) be a 3-dimensional non compact Riemannian manifold with bounded geometry such that:*

i) (M, g) is asymptotically euclidean in the sense of Definition (1.16),

ii) there exists a point $\bar{p}$ where the scalar curvature is strictly greater than 6, *$R(\bar{p}) > 6$,*

iii) the sectional curvature $\bar{K}$ of (M,g) is bounded above by 2: $\bar{K} \leq 2$.
Then the minimization problem (1.19) *has a smooth solution, i.e. there exists a smooth immersion $f : \mathbb{S}^2 \hookrightarrow M$ such that*

$$W_1(f) = \inf\{W_1(h) | h : \mathbb{S}^2 \hookrightarrow (M,g) \text{ is a } C^\infty \text{ immersion in } (M,g)\}.$$

Theorem 1.0.14. *Let (M,g) be a 3-dimensional non compact Riemannian manifold with bounded geometry such that:*
i) (M,g) is asymptotically euclidean in the sense of Definition (1.16),
ii) there exists a point $\bar{p}$ where the scalar curvature is strictly greater than 6, *$R(\bar{p}) > 6$.*
Then the minimization problem (1.20) *has a smooth solution, i.e. there exists a smooth immersion $f : \mathbb{S}^2 \hookrightarrow M$ such that*

$$E_1(f) = \inf\{E_1(h) | h : \mathbb{S}^2 \hookrightarrow (M,g) \text{ is a } C^\infty \text{ immersion in } (M,g)\}.$$

The technique of the proof is analogous to the one described in the introduction of the global problems in compact ambient manifolds. Here there is the major difficulty that the minimizing sequence can become larger and larger or it could escape to infinity. Using the bounded geometry condition, the scalar curvature assumption, and the asymptotic flatness we prove the geometric a priori estimates that give compactness of the minimizing sequence and hence the existence of a weak candidate minimizer. The further assumption on the sectional curvature in the case of W_1 is useful for having a small bound on $\int |A|^2$ for the minimizing sequence and this is crucial in the regularity theory in order to exclude bad points (such a bound is automatic for E_1). Once we have that the minimizing sequence is contained in a compact subset of (M,g) and we show the existence of a weak candidate minimizer then we enter into the framework of the regularity theory discussed in Chapter 5 and we conclude with the existence of a smooth minimizer.

1.0.4 Introduction and results about the supercritical functionals $\int |H|^p$ and $\int |A|^p$ in Riemannian manifolds, arbitrary dimension and codimension: Chapter 6

Given an ambient Riemannian manifold (N,g) of dimension $n \geq 3$ (with or without boundary), a classical problem in differential geometry is to find smooth immersed m-dimensional submanifolds, $2 \leq m \leq n-1$, with null mean curvature vector, $H = 0$, or with null second fundamental form, $A = 0$, namely the minimal (respectively, the totally geodesic) submanifolds of N (for more details about the existence see Example 6.5.1, Example 6.5.2, Theorem 6.5.4, Theorem 6.5.5, Remark 6.5.6 and Remark 6.5.7).

In more generality, it is interesting to study the minimization problems associated to integral functionals depending on the curvatures of the type

$$E^p_{H,m}(M) := \int_M |H|^p \quad \text{or} \quad E^p_{A,m}(M) := \int_M |A|^p, \quad p \geq 1 \tag{1.21}$$

where M is a smooth immersed m-dimensional submanifold with mean curvature H and second fundamental form A; of course the integrals are computed with respect to the m-dimensional measure of N induced on M. A global minimizer of $E^p_{H,m}$ (respectively of $E^p_{A,m}$), if it exists, can be seen as a generalized minimal (respectively totally geodesic) m-dimensional submanifold in a natural integral sense.

An important example of such functionals is given by the Willmore functional for surfaces $E^2_{H,2}$ introduced by Willmore (see [Will]) and studied in the euclidean space (see for instance the works of Simon [SiL], Kuwert and Shätzle [KS], Rivière [Riv]) or in Riemannian manifolds (see, for example, [LM], [Mon1] and [Mon2]).

The general integral functionals (1.21) depending on the curvatures of immersed submanifolds have been studied, among others, by Allard [Al], Anzellotti-Serapioni-Tamanini [AST], Delladio [Del], Hutchinson [Hu1], [Hu2], [Hu3], Mantegazza [MantCVB] and Moser [Mos].

In order to get the existence of a minimizer, the technique adopted in the present chapter (as well as in most of the aforementioned papers) is the so called direct method in the calculus of variations. As usual, it is necessary to enlarge the space where the functional is defined and to work out a compactness-lowersemicontinuity theory in the enlarged domain.

In the present chapter, the enlarged domain is made of generalized m-dimensional submanifolds of the fixed ambient Riemannian manifold (N, g): the integral rectifiable m-varifolds introduced by Almgren in [Alm] and by Allard in [Al]. Using integration by parts formulas, Allard [Al] and Hutchinson [Hu1]-Mantegazza [MantCVB] defined a weak notion of mean curvature and of second fundamental form respectively (for more details about this part see Appendix 6.6). Moreover these objects have good compactness and lower semicontinuity properties with respect to the integral functionals above.

The goal of this chapter is to prove existence and partial regularity of an m-dimensional minimizer (in the enlarged class of the rectifiable integral m-varifolds with weak mean curvature or with generalized second fundamental form in the sense explained above) of functionals of the type (1.21). Actually we will consider more general functionals modeled on this example, see Definition 6.6.2 for the expression of the considered integrand F.

More precisely, given a compact subset $N \subset\subset \bar{N}$ of an n-dimensional Riemannian manifold $(\bar{N}, g)$ (which, by Nash Embedding Theorem, can be assumed isometrically embedded in some $\mathbb{R}^S$) we will denote

$$\begin{aligned} HV_m(N) &:= \{V \text{ integral rectifiable } m\text{-varifold of } N \text{ with weak mean curvature } H^N \text{ relative to } \bar{N}\} \\ CV_m(N) &:= \{V \text{ integral rectifiable } m\text{-varifold of } N \text{ with generalized second fundamental form } A\}; \end{aligned}$$

for more details see Appendix 6.6; in any case, as written above, the non expert reader can think about the elements of $HV_m(N)$ (respectively of $CV_m(N)$) as generalized m-dimensional submanifolds with mean curvature H^N (respectively with second fundamental form A). Precisely, we consider the following two minimization problems

$$\beta^m_{N,F} := \inf \left\{ \int_{G_m(N)} F(x, P, H^N) dV : V \in HV_m(N), V \neq 0 \text{ with weak mean curvature } H^N \text{ relative to } \bar{N} \right\} \tag{1.22}$$

and

$$\alpha^m_{N,F} := \inf \left\{ \int_{G_m(N)} F(x, P, A) dV : V \in CV_m(N), V \neq 0 \text{ with generalized second fundamental form } A \right\} \tag{1.23}$$

where F is as in Definition 6.6.2 and satisfies (6.33) (respectively (6.27)). As the reader may see, the expressions $\int_{G_m(N)} F(x, P, H^N) dV$ (respectively $\int_{G_m(N)} F(x, P, A) dV$) are the natural generalizations of the functionals $E^p_{H,m}$ (respectively $E^p_{A,m}$) in (1.21) with $p > m$ in the context of varifolds.
Before stating the two main theorems, let us recall that an integral rectifiable m-varifold V on N is associated with a "generalized m-dimensional subset" spt μ_V of N together with an integer valued density function $\theta(x) \geq 0$ which carries the "multiplicity" of each point (for the precise definitions, as usual, see Appendix 6.6).

At this point we can state the two main theorems of this chapter. Let us start with the mean curvature.

Theorem 1.0.15. *Let $N \subset\subset \bar{N}$ be a compact subset with non empty interior, $int(N) \neq \emptyset$, of the n-dimensional Riemannian manifold $(\bar{N}, g)$ isometrically embedded in some $\mathbb{R}^S$ (by Nash Embedding Theorem), fix $m \leq n-1$ and consider a function $F : G_m(N) \times \mathbb{R}^S \to \mathbb{R}^+$ satisfying* (6.6.2) *and* (6.33), *namely*

$$F(x, P, H) \geq C|H|^p$$

for some $C > 0$ and $p > m$.

Then, at least one of the following two statement is true:

a) the space (N, g) contains a non zero m-varifold with null weak mean curvature H^N relative to $\bar{N}$ (in other words, N contains a stationary m-varifold; see Remark 6.6.13 for the details),

b) the minimization problem (6.33) *corresponding to F has a solution i.e. there exists a non null integral m-varifold $V \in HV_m(N)$ with weak mean curvature H^N relative to $\bar{N}$ such that*

$$\int_{G_m(N)} F(x, P, H^N) dV = \beta^m_{N,F} = \inf \left\{ \int_{G_m(N)} F(x, P, \tilde{H}^N) d\tilde{V} : \tilde{V} \in HV_m(N), \tilde{V} \neq 0 \right\}.$$

Moreover, in case b) *is true, we have* $\beta^m_{N,F} > 0$ *and the minimizer* V *has the following properties:*

b1) the support $\operatorname{spt}\mu_V$ *of the spatial measure* μ_V *associated to* V *is connected,*

b2) the diameter of $\operatorname{spt}\mu_V$ *as a subset of the Riemannian manifold* $(\bar{N}, g)$ *is strictly positive*

$$\operatorname{diam}_{\bar{N}}(\operatorname{spt}\mu_V) > 0.$$

Remark 1.0.16. *It could be interesting to study the regularity of the minimizer* V*. Notice that if* $x \in \operatorname{spt}\mu_V$*, under the hypothesis that the density in* x *satisfies* $\theta(x) = 1$ *plus other technical assumptions (see Theorem 8.19 in [Al]), Allard proved that* $\operatorname{spt}\mu_V$ *is locally around* x *a graph of a* $C^{1,1-\frac{m}{p}}$ *function since* $H \in L^p(V)$*,* $p > m$ *given by* (6.33)*. Moreover, under similar assumptions, Duggan proved local* $W^{2,p}$ *regularity in [Dug]. In the multiple density case the regularity problem is more difficult. For instance, in [Brak], is given an example of a varifold* $\tilde{V}$ *with bounded weak mean curvature whose spatial support contains a set* C *of strictly positive measure such that if* $x \in C$ *then* $\operatorname{spt}\mu_{\tilde{V}}$ *does not correspond to the graph of even a multiple-valued function in any neighborhood of* x*.*

Now let us state the second main Theorem about the second fundamental form A.

Theorem 1.0.17. *Let* $N \subset\subset \bar{N}$ *be a compact subset with non empty interior,* $int(N) \neq \emptyset$*, of the* n*-dimensional Riemannian manifold* $(\bar{N}, g)$ *isometrically embedded in some* $\mathbb{R}^S$ *(by Nash Embedding Theorem), fix* $m \leq n-1$ *and consider a function* $F : G_m(N) \times \mathbb{R}^{S^3} \to \mathbb{R}^+$ *satisfying* (6.6.2) *and* (6.27)*, namely*

$$F(x, P, A) \geq C|A|^p$$

for some $C > 0$ *and* $p > m$*.*

Then, at least one of the following two statements is true:

a) the space (N, g) *contains a non zero* m*-varifold with null generalized second fundamental form,*

b) the minimization problem (6.27) *corresponding to* F *has a solution i.e. there exists a non null curvature* m*-varifold* $V \in CV_m(N)$ *with generalized second fundamental form* A *such that*

$$\int_{G_m(N)} F(x, P, A)dV = \alpha^m_{N,F} = \inf\left\{\int_{G_m(N)} F(x, P, \tilde{A})d\tilde{V} : \tilde{V} \in CV_m(N), \tilde{V} \neq 0\right\}.$$

Moreover, in case b) *is true, we have* $\alpha^m_{N,F} > 0$ *and the minimizer* V *has the following properties:*

b1)the support $\operatorname{spt}\mu_V$ *of the spatial measure* μ_V *associated to* V *is connected,*

b2) the diameter of $\operatorname{spt}\mu_V$ *as a subset of the Riemannian manifold* $(\bar{N}, g)$ *is strictly positive*

$$\operatorname{diam}_{\bar{N}}(\operatorname{spt}\mu_V) > 0,$$

b3) For every $x \in \operatorname{spt}\mu_V$*,* V *has a unique tangent cone at* x *and this tangent cone is a finite union of* m*-dimensional subspaces* P_i *with integer multiplicities* m_i*; moreover, in some neighborhood of* x *we can express* V *has a finite union of graphs of* $C^{1,1-\frac{m}{p}}$*,* m_i*-valued functions defined on the respective affine spaces* $x + P_i$ *(*p *given in* (6.27)*).*

Remark 1.0.18. *For the precise definitions and results concerning* b3)*, the interested reader can look at the original paper [Hu2] of Hutchinson. Notice that the boundary of* N *does not create problems since, by our definitions, the minimizer* V *is a fortiori an integral* m*-varifold with generalized second fundamental form* $A \in L^p(V), p > m$*, in the* n*-dimensional Riemannian manifold* $(\bar{N}, g)$ *which has no boundary. Moreover, by Nash Embedding Theorem, we can assume* $\bar{N} \subset \mathbb{R}^S$*; therefore* V *can be seen as an integral* m*-varifold with generalized second fundamental form* $A \in L^p(V)$*,* $p > m$*, in* $\mathbb{R}^S$ *and the regularity theorem of Hutchinson can be applied.*

It could be interesting to prove higher regularity of the minimizer V*. About this point, notice that it is not trivially true that* V *is locally a union of graphs of* $W^{2,p}$ *(Sobolev) functions. Indeed in [AGP] there is an example of a curvature* m*-varifold* $\tilde{V} \in CV_m(\mathbb{R}^S)$*,* $S \geq 3$*,* $2 \leq m \leq S-1$*, with second fundamental form in* L^p*,* $p > m$*, which is not a union of graphs of* $W^{2,p}$ *functions.*

In the spirit of proving higher regularity of the minimizer of such functionals we mention the preprint of Moser [Mos] where the author proves smoothness of the minimizer of $\int |A|^2$ *in the particular case of codimension* 1 *Lipschitz graphs in* $\mathbb{R}^S$*.*

In both theorems, a delicate point is whether or not $a)$ is satisfied (fact which trivializes the result); we will study this problem in Section 6.5: we will recall two general classes of examples (given by White in [Whi]) of Riemannian manifolds with boundary where $a)$ is not satisfied in codimension 1, we will give two new examples for higher codimensions (namely Theorem 6.5.4 and Theorem 6.5.5) and we will propose a related open problem in Remark 6.5.7. Here, let us just remark that every compact subset $N \subset\subset \mathbb{R}^S$ for $s > 1$ does not satisfy $a)$ (see Theorem 6.5.5).

The idea for proving the results is to consider a minimizing sequence $\{V_k\}_{k\in\mathbb{N}}$ of varifolds, show that it is compact (i.e. there exists a varifold V and a subsequence $\{V_{k'}\}$ converging to V in an appropriate sense) and it is non degenerating: if the masses decrease to 0 the limit would be the null varifold so not a minimizer, and if the diameters decrease to 0 the limit would be a point which has no geometric relevance.

In order to perform the analysis of the minimizing sequences, in Section 6.1 we prove monotonicity formulas for integral rectifiable m-varifolds in $\mathbb{R}^S$ with weak mean curvature in L^p, $p > m$. These formulas are similar in spirit to the ones obtained by Simon in [SiL] for smooth surfaces in $\mathbb{R}^S$ involving the Willmore functional. These estimates are a fundamental tool for proving the non degeneracy of the minimizing sequences and we think they might have other applications.

To show the compactness of the minimizing sequences it is crucial to have a uniform upper bound on the masses (for the non expert reader: on the volumes of the generalized submanifolds). Inspired by the paper of White [Whi], in Section 6.2 we prove some isoperimetric inequalities involving our integral functionals which give the mass bound on the minimizing sequences in case $a)$ in the main theorems is not satisfied. The compactness follows and is proved in the same Section. Also in this case, we think that the results may have other interesting applications.

The proofs of the two main theorems is contained in Section 6.3 and 6.4. Finally, as written above, Section 6.5 is devoted to examples and remarks: we will notice that a large class of manifold with boundary can be seen as compact subset of manifold without boundary, we will give examples where the assumption for the isoperimetric inequalities are satisfied and we will end with a related open problem.

The new features of the present chapter relies, besides the main theorems, in the new tools introduced in Section 6.1 and Section 6.2, and in the new examples presented in Section 6.5.

Chapter 2

The conformal Willmore functional in a perturbative setting: existence of saddle type critical points

In this Chapter we study the conformal Willmore functional (which is conformal invariant in general Riemannian manifold (M, g)) with a perturbative method: the Lyapunov-Schmidt reduction. We show existence of critical points in ambient manifolds $(\mathbb{R}^3, g_\epsilon)$ -where g_ϵ is a metric close and asymptotic to the euclidean one. With the same technique we prove a non existence result in general Riemannian manifolds (M, g) of dimension three.

Notations and conventions

1) $\mathbb{R}^+$ denotes the set of strictly positive real numbers.

2) As mentioned before, the perturbed spheres will play a central role throughout this Chapter.
First, let us define the perturbed standard sphere $S_p^\rho(w) \subset \mathbb{R}^3$ we will use to prove the existence results. We denote by S^2 the standard unit sphere in the euclidean 3-dimensional space , $\Theta \in S^2$ is the radial versor with components Θ^μ parametrized by the polar coordinates $0 < \theta^1 < \pi$ and $0 < \theta^2 < 2\pi$ chosen in order to satisfy

$$\begin{cases} \Theta^1 = \sin\theta^1 \cos\theta^2 \\ \Theta^2 = \sin\theta^1 \sin\theta^2 \\ \Theta^3 = \cos\theta^1. \end{cases}$$

We call Θ_i the coordinate vector fields on S^2

$$\Theta_1 := \frac{\partial\Theta}{\partial\theta^1}, \quad \Theta_2 := \frac{\partial\Theta}{\partial\theta^2}$$

and $\bar{\theta}_i$ or $\bar{\Theta}_i$ the corresponding normalized ones

$$\bar{\theta}_1 = \bar{\Theta}_1 := \frac{\Theta_1}{\|\Theta_1\|}, \quad \bar{\theta}_2 = \bar{\Theta}_2 := \frac{\Theta_2}{\|\Theta_2\|}.$$

The standard sphere in $\mathbb{R}^3$ with center p and radius $\rho > 0$ is denoted by S_p^ρ; we parametrize it as $(\theta^1, \theta^2) \mapsto p + \rho\Theta(\theta^1, \theta^2)$ and call θ_i the coordinate vector fields

$$\theta_1 := \rho\frac{\partial\Theta}{\partial\theta^1}, \quad \theta_2 := \rho\frac{\partial\Theta}{\partial\theta^2}.$$

The perturbed spheres will be normal graphs on standard spheres by a function w which belongs to a suitable function space. Let us introduce the function space which has been chosen by technical reasons (to apply Schauder estimates in Lemma 2.3.3).
Denote $C^{4,\alpha}(S^2)$ (or simply $C^{4,\alpha}$) the set of the C^4 functions on S^2 whose fourth derivatives, with respect to the tangent vector fields, are α-Hölder $(0 < \alpha < 1)$. The Laplace-Beltrami operator on S^2 is denoted

by $\triangle_{S^2}$ or, if there is no confusion, as $\triangle$. The fourth order elliptic operator $\triangle(\triangle+2)$ induces a splitting of $L^2(S^2)$:

$$L^2(S^2) = Ker[\triangle(\triangle+2)] \oplus Ker[\triangle(\triangle+2)]^{\perp}$$

(the splitting makes sense because the kernel is finite dimensional, so it is closed).
If we consider $C^{4,\alpha}(S^2)$ as a subspace of $L^2(S^2)$, we can define

$$C^{4,\alpha}(S^2)^{\perp} := C^{4,\alpha}(S^2) \cap Ker[\triangle(\triangle+2)]^{\perp}.$$

Of course $C^{4,\alpha}(S^2)^{\perp}$ is a Banach space with respect to the $C^{4,\alpha}$ norm; it is the space from which we will get the perturbations w. If there is no confusion $C^{4,\alpha}(S^2)^{\perp}$ will be called simply $C^{4,\alpha^{\perp}}$.
Now we can define the perturbed spheres we will use to prove existence of critical points: fix $\rho > 0$ and a small $C^{4,\alpha^{\perp}}$ function w; the perturbed sphere $S_p^{\rho}(w)$ is the surface parametrized by

$$\Theta \in S^2 \mapsto p + \rho\big(1 - w(\Theta)\big)\Theta.$$

Now let us define the perturbed geodesic spheres $S_{p,\rho}(w)$ in the three dimensional Riemannian manifold (M, g); we will use them to prove the non-existence result.
Once a point $p \in M$ is fixed we can consider the exponential map Exp_p with center p. For $\rho > 0$ small enough, the sphere $\rho S^2 \subset T_pM$ is contained in the radius of injectivity of the exponential. We call $S_{p,\rho}$ the geodesic sphere of center p and radius ρ. This hypersurface can be parametrized by

$$\Theta \in S^2 \subset T_pM \mapsto Exp_p[\rho\Theta].$$

Analogously to the previous case, fix $p \in M$, $\rho > 0$ and a small $C^{4,\alpha}(S^2)$ function w; the perturbed geodesic sphere $S_{p,\rho}(w)$ is the surface parametrized by

$$\Theta \in S^2 \mapsto Exp_p[\rho\big(1 - w(\Theta)\big)\Theta].$$

The tangent vector fields on $S_{p,\rho}(w)$ induced by the canonical polar coordinates on S^2 are denoted by Z_i.

3) Let (M, g) be a 3-dimensional Riemannian manifold.

First we make the following convention: the Greek index letters, such as $\mu, \nu, \iota, \ldots$, range from 1 to 3 while the Latin index letters, such as $i, j, k, \ldots$, will run from 1 to 2.

About the Riemann curvature tensor we adopt the convention of [Will]: denoting $\mathfrak{X}(M)$ the set of the vector fields on M, $\forall X, Y, Z \in \mathfrak{X}(M)$

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$
$$R(X,Y,Z,W) := g(R(Z,W)Y,X);$$

chosen in p an orthonormal frame E_μ, the Ricci curvature tensor is

$$\begin{aligned} Ric_p(v_1,v_2) &:= \sum_{\mu=1}^{3} R(E_\mu, v_1, E_\mu, v_2) = \sum_{1}^{3} g(R_p(E_\mu, v_2)v_1, E_\mu) \\ &= -\sum_{\mu=1}^{3} g(R_p(v_2, E_\mu)v_1, E_\mu) \quad \forall v_1, v_2 \in T_pM. \end{aligned} \tag{2.1}$$

In order to keep formulas not too long, we introduce the following notation:

$$\begin{aligned} R(0i0j) &:= g(R_p(\Theta,\Theta_i)\Theta,\Theta_j) \\ \nabla_0 R(0i0j) &:= g(\nabla_\Theta R_p(\Theta,\Theta_i)\Theta,\Theta_j) \\ \nabla_{00} R(0i0j) &:= g(\nabla_\Theta\nabla_\Theta R_p(\Theta,\Theta_i)\Theta,\Theta_j) \\ R(0i0\mu) &:= g(R_p(\Theta,\Theta_i)\Theta,E_\mu). \end{aligned}$$

In the following ambiguous cases we will mean:

$$\begin{aligned} R(0101) &:= g(R_p(\Theta,\Theta_1)\Theta,\Theta_1) \\ R(0\bar{2}0\bar{2}) &:= g(R_p(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2) \\ R(010\bar{2}) &:= g(R_p(\Theta,\Theta_1)\Theta,\bar{\Theta}_2). \end{aligned}$$

Recall the definitions of the Hessian and the Laplace-Beltrami operator on a function w:

$$Hess(w)_{\mu\nu} := \nabla_\mu \nabla_\nu w$$

$$\triangle := g^{\mu\nu}\nabla_\mu \nabla_\nu w.$$

4) Let $(\mathring{M}, \mathring{g}) \hookrightarrow (M, g)$ be an isometrically immersed surface. Recall the notion of second fundamental form $\mathring{h}$: fix a point p and an orthonormal base Z_1, Z_2 of $T_p\mathring{M}$; the (inward) normal unit vector is denoted as $\mathring{N}$. By the Weingarten equation $\mathring{h}_{ij} = -g(\nabla_{Z_i}\mathring{N}, Z_j)$.
Call k_1 and k_2 the principal curvatures (the eigenvalues of the second fundamental form with respect to the first fundamental form of $\mathring{M}$, i.e. the roots of $\det(\mathring{h}_{ij} - k\mathring{g}_{ij}) = 0$). We adopt the convention that the mean curvature is defined as $H := k_1 + k_2$.
The product of the principal curvatures will be denoted with D:

$$D := k_1 k_2 = \frac{det(\mathring{h})}{det(\mathring{g})}. \tag{2.2}$$

5) Following the notation of [PX], given $a \in \mathbb{N}$, any expression of the form $L_p^{(a)}(w)$ denotes a linear combination of the function w together with its derivatives with respect to the tangent vector fields Θ_i up to order a. The coefficients of $L_p^{(a)}$ might depend on ρ and p but, for all $k \in \mathbb{N}$, there exists a constant $C > 0$ independent of $\rho \in (0, 1)$ and $p \in M$ such that

$$\|L_p^{(a)}(w)\|_{C^{k,\alpha}(S^2)} \leq C\|w\|_{C^{k+a,\alpha}(S^2)}.$$

Similarly, given $b \in \mathbb{N}$, any expression of the form $Q_p^{(b)(a)}(w)$ denotes a nonlinear operator in the function w together with its derivatives with respect to the tangent vector fields Θ_i up to order a such that, for all $p \in M$, $Q_p^{(b)(a)}(0) = 0$. The coefficients of the Taylor expansion of $Q_p^{(b)(a)}(w)$ in powers of w and its partial derivatives might depend on ρ and p but, for all $k \in \mathbb{N}$, there exists a constant $C > 0$ independent of $\rho \in (0, 1)$ and $p \in M$ such that

$$\|Q_p^{(b)(a)}(w_2) - Q_p^{(b)(a)}(w_1)\|_{C^{k,\alpha}(S^2)} \leq c\big(\|w_2\|_{C^{k+a,\alpha}(S^2)} + \|w_1\|_{C^{k+a,\alpha}(S^2)}\big)^{b-1} \times \|w_2 - w_1\|_{C^{k+a,\alpha}(S^2)}, \tag{2.3}$$

provided $\|w_l\|_{C^a(S^2)} \leq 1$, $l = 1, 2$. If the numbers a or b are not specified, we intend that their value is 2.

We also agree that any term denoted by $O_p(\rho^d)$ is a smooth function on S^2 that might depend on p but which is bounded by a constant (independent of p) times ρ^d in C^k topology, for all $k \in N$.

6) Large positive constants are always denoted by C, and the value of C is allowed to vary from formula to formula and also within the same line. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C, as C_δ, etc.. Also constants with subscripts are allowed to vary.

2.1 A Preliminary result: the Lyapunov-Schmidt reduction

The technique used throughout this Chapter relies on an abstract perturbation method which first appeared in [AB1], [AB2] and is extensively treated with proofs and examples in [AM]. Let us briefly summarize it. Actually we present the abstract method in a form which permits to deal with degenerate expansions (as the ones we will have to handle).

Given an Hilbert space H, let $I_\epsilon : H \to \mathbb{R}$ be a C^2 functional of the form

$$I_\epsilon(u) = I_0(u) + \epsilon G_1(u) + \epsilon^2 G_2(u) + o(\epsilon^2),$$

where $I_0 \in C^2(H, \mathbb{R})$ plays the role of the unperturbed functional and $G_1, G_2 \in C^2(H, \mathbb{R})$ are the perturbations.

We first assume that there exists a finite dimensional smooth manifold Z made of critical points of I_0: $I_0'(z) = 0$ for all $z \in Z$. The set Z will be called *critical manifold* (of I_0). The critical manifold is supposed to satisfy the following non degeneracy conditions:

(ND) for all $z \in Z$, $T_z Z = Ker[I_0''(z)]$,

(Fr) for all $z \in Z$, $I_0''(z)$ is a Fredholm operator of index zero.
Under these assumptions it is known that near Z there exists a perturbed manifold Z^ϵ such that the critical points of I_ϵ constrained on Z^ϵ give rise to stationary points of I_ϵ.
More precisely, the key result is the following Theorem.

Theorem 2.1.1. *Suppose I_0 possesses a non degenerate (satisfying (ND) and (Fr)) critical manifold Z of dimension d.*
Given a compact subset Z_c of Z, there exists $\epsilon_0 > 0$ such that for all $|\epsilon| < \epsilon_0$ there is a smooth function

$$w_\epsilon(z) : Z_c \to H$$

such that
(i) for $\epsilon = 0$ it results $w_\epsilon(z) = 0$, $\forall z \in Z_c$;
(ii) $w_\epsilon(z)$ is orthogonal to $T_z Z$, $\forall z \in Z_c$;
(iii) the manifold

$$Z^\epsilon = \{z + w_\epsilon(z) : z \in Z_c\}$$

is a natural constraint for I'_ϵ. Namely, denoting

$$\Phi_\epsilon(z) = I_\epsilon(z + w_\epsilon(z)) : Z_c \to \mathbb{R}$$

the constriction of I_ϵ to Z^ϵ, if z_ϵ is a critical point of Φ_ϵ then $u_\epsilon = z_\epsilon + w_\epsilon(z_\epsilon)$ is a critical point of I_ϵ.

Thanks to this fundamental tool, in order to find critical points of I_ϵ, we can reduce ourselves to study Φ_ϵ which is a function of *finitely* many variables.

If we are slightly more accurate, it can be shown that the function $w_\epsilon(z)$ is of order $O(\epsilon)$ as $\epsilon \to 0$ uniformly in z varying in the compact Z_c. In our application, the expansion is degenerate in the sense that

$$G_1(z) = 0 \quad \forall z \in Z.$$

Using the previous facts, by a Taylor expansion it is easy to see that (we will prove it in full detail in Lemma 2.3.7)

$$\Phi_\epsilon(z) = \epsilon^2 \Big[G_2(z) - \frac{1}{2} \big(G_1'(z) | I_0''(z)^{-1} G_1'(z) \big) \Big] + o(\epsilon^2).$$

In Section 2.3 we will give sense to this formula, which will be crucial for the estimates involved in the existence result.

2.2 The conformal Willmore functional on perturbed geodesic spheres $S_{p,\rho}(w)$ of a general Riemannian Manifold (M, g)

2.2.1 Geometric expansions

In this subsection we give accurate expansions of the geometric quantities appearing in the conformal Willmore functional. First we recall and refine the well-known expansions of the first and second fundamental form and the mean curvature for the geodesic perturbed spheres $S_{p,\rho}(w)$ introduced in the previous "notations and conventions". Recall that Θ_i are the coordinate vector fields on S^2 (induced by polar coordinates) and Z_i are the corresponding coordinate vector fields on $S_{p,\rho}(w)$. The derivatives of w with respect to Θ_i are denoted by w_i.

Let $\mathring{g}$ denote the first fundamental form on $S_{p,\rho}(w)$ induced by the immersion in (M, g). The next Lemma, whose proof can be found in [PX] (Lemma 2.1), gives an expansion of the components $\mathring{g}_{ij} := g_p(Z_i, Z_j)$:

Lemma 2.2.1. *The first fundamental form on $S_{p,\rho}(w)$ has the following expansion:*

$$\begin{aligned}(1-w)^{-2}\rho^{-2}\mathring{g}_{ij} \;&=\; g(\Theta_i,\Theta_j) + (1-w)^{-2} w_i w_j + \frac{1}{3} R(0i0j)\rho^2(1-w)^2 + \frac{1}{6}\nabla_0 R(0i0j)\rho^3(1-w)^3 \\ &\quad + \Big[\frac{1}{20}\nabla_{00}R(0i0j) + \frac{2}{45} R(0i0\mu)R(0j0\mu)\Big]\rho^4(1-w)^4 + O_p(\rho^5) + \rho^5 L_p(w) + \rho^5 Q_p^{(2)}(w)\end{aligned}$$

where all curvature terms and scalar products are evaluated at p (since we are in normal coordinates, at p the metric is euclidean).

Let $\mathring{h}$ denote the second fundamental form on $S_{p,\rho}(w)$ induced by the immersion in (M,g) and $\mathring{N}$ the inward normal unit vector to $S_{p,\rho}(w)$; by the Weingarten equation $\mathring{h}_{ij} = -g(\nabla_{Z_i}\mathring{N}, Z_j)$.

Lemma 2.2.2. *The second fundamental form on $S_{p,\rho}(w)$ has the following expansion:*

$$\begin{aligned}\mathring{h}_{ij} &= \rho(1-w)g(\Theta_i,\Theta_j) + \rho(Hess_{S^2}w)_{ij} + \frac{2}{3}R(0i0j)\rho^3(1-w)^3 + \frac{5}{12}\nabla_0 R(0i0j)\rho^4(1-w)^4 \\ &\quad + \Big[\frac{3}{20}\nabla_{00}R_p(0i0j) + \frac{2}{15}R(0i0\mu)R(0j0\mu)\Big]\rho^5(1-w)^5 - \rho B^k_{ij}w_k + O_p(\rho^6) + \rho^5 L_p(w) + \rho^2 Q^{(2)}_p(w)\end{aligned}$$

where B^k_{ij} are functions on S^2 of the form $B^k_{ij} = O(\rho^2) + L_p(w) + Q^{(2)}_p(w)$ and, as usual, all curvature terms and scalar products are evaluated at p.

Proof. In [PX] the authors consider $\tilde{\mathring{N}}$ such that the normal unit vector $\mathring{N}$ has the form $\mathring{N} = \tilde{\mathring{N}}(1-\rho^2\mathring{g}^{ij}w_iw_j)^{-1/2}$. They set

$$\tilde{\mathring{h}}_{ij} = -g(\nabla_{Z_i}\tilde{\mathring{N}}, Z_j)$$

and they derive the following formula

$$\tilde{\mathring{h}}_{ij} = \frac{1}{2(1-w)}\partial_\rho \mathring{g} - \frac{1}{1-w}\rho dw \otimes dw + \rho Hess_{\mathring{g}} w.$$

Using Lemma 2.2.1 the first summand is:

$$\begin{aligned}\frac{1}{2(1-w)}\partial_\rho \mathring{g} &= g(\Theta_i,\Theta_j)\rho(1-w) + \frac{\rho}{1-w}w_iw_j + \frac{2}{3}R(0i0j)\rho^3(1-w)^3 + \frac{5}{12}\nabla_0 R(0i0j)\rho^4(1-w)^4 \\ &\quad + \frac{3}{20}\Big[\nabla_{00}R(0i0j) + \frac{2}{15}R(0i0\mu)R(0j0\mu)\Big]\rho^5(1-w)^5 + O_p(\rho^6) + \rho^6 L_p(w) + \rho^6 Q^{(2)}_p(w)\end{aligned}$$

The third summand is:

$$\rho(Hess_{\mathring{g}}w)_{ij} = \rho(w_{ij} - \mathring{\Gamma}^k_{ij}w_k).$$

With a direct computation it is easy to check that

$$\mathring{\Gamma}^k_{ij} = \Gamma^k_{ij} + B^k_{ij} \tag{2.4}$$

where Γ^k_{ij} are the Christoffel symbols of S^2 in polar coordinates and B^k_{ij} are functions on S^2 of the form

$$B^k_{ij} = O(\rho^2) + L_p(w) + Q^{(2)}_p(w).$$

Hence

$$\rho(Hess_{\mathring{g}}w)_{ij} = \rho(Hess_{S^2}w)_{ij} - \rho B^k_{ij}w_k.$$

Observing that the second summand simplifies with an adding of the first summand and that

$$\mathring{h}_{ij} = -g(\nabla_{Z_i}\mathring{N}, Z_j) = -g(\nabla_{Z_i}\tilde{\mathring{N}}(1-\rho^2\mathring{g}^{ij}w_iw_j)^{-1/2}, Z_j) = \tilde{\mathring{h}}_{ij} + \rho^2 Q^{(2)}_p(w)$$

we get the desired formula. □

Recall that the mean curvature H is the trace of $\mathring{h}$ with respect to the metric $\mathring{g}$: $H = \mathring{h}_{ij}\mathring{g}^{ij}$. Collecting the two previous Lemmas we obtain the following

Lemma 2.2.3. *The mean curvature of the hypersurface $S_{p,\rho}(w)$ can be expanded as*

$$\begin{aligned}H &= \frac{2}{\rho} + \frac{1}{\rho}(2+\triangle_{S^2})w + \frac{1}{\rho}\big[2w(w+\triangle_{S^2}w) - g^{ij}_{S^2}w_iw_j\big] - \frac{1}{\rho}g^{ij}_{S^2}B^k_{ij}w_k \\ &\quad - \frac{1}{3}\big[g^{il}_{S^2}R(0l0k)g^{kj}_{S^2}(Hess_{S^2}w)_{ij} + Ric_p(\Theta,\Theta)(1-w)\big]\rho + \frac{1}{4}g^{ij}_{S^2}\nabla_0 R(0i0j)\rho^2(1-w)^2 \\ &\quad + \Big[\frac{1}{10}g^{ij}_{S^2}\nabla_{00}R(0i0j) + \frac{4}{45}g^{ij}_{S^2}R(0i0\mu)R(0j0\mu) - \frac{1}{9}g^{il}_{S^2}R(0l0k)g^{kn}_{S^2}R(0n0i)\Big]\rho^3(1-w)^3 \\ &\quad + O_p(\rho^4) + \rho^2 L_p(w) + Q^{(2)}_p(w) + \frac{1}{\rho}L_p(w)Q^{(2)}_p(w)\end{aligned}$$

where Ric_p is the Ricci tensor computed at p.

Proof. First let us find an expansion of $\mathring{g}^{ij}$. Given an invertible matrix A,

$$(A+B\rho^2+C\rho^3+D\rho^4)^{-1} = A^{-1}-A^{-1}BA^{-1}\rho^2-A^{-1}CA^{-1}\rho^3-A^{-1}DA^{-1}\rho^4+A^{-1}BA^{-1}BA^{-1}\rho^4+O(\rho^5)$$

so we get

$$\begin{aligned}\mathring{g}^{ij} &= \frac{1}{\rho^2(1-w)^2}\Big\{g^{ij}_{S^2}-g^{il}_{S^2}(1-w)^{-2}w_lw_kg^{kj}_{S^2}-\frac{1}{3}g^{il}_{S^2}R(0l0k)g^{kj}_{S^2}\rho^2(1-w)^2-\frac{1}{6}g^{il}_{S^2}\nabla_0R(0l0k)g^{kj}_{S^2}\rho^3(1-w)^3\\ &\quad -g^{il}_{S^2}\Big[\frac{1}{20}\nabla_{00}R(0l0k)+\frac{2}{45}R(0l0\mu)R(0k0\mu)\Big]g^{kj}_{S^2}\rho^4(1-w)^4+\frac{1}{9}g^{il}_{S^2}R(0l0k)g^{kn}_{S^2}R(0n0q)g^{qj}_{S^2}\rho^4(1-w)^4\Big\}\\ &\quad +O_p(\rho^3)+\rho^3L_p(w)+\rho^2Q^{(2)}_p(w)+\frac{1}{\rho^2}(Dw)^4.\end{aligned}\tag{2.5}$$

Where $(Dw)^4$ is an homogeneous polynomial in the first derivatives w_i of order four. Putting together (2.5) and Lemma 2.2.2 it is easy to evaluate $H=\mathring{h}_{ij}\mathring{g}^{ij}$ just using the following observations:

$$\begin{aligned}\bullet\rho\mathring{g}^{ij}(Hess_{S^2}w)_{ij} &= \Big(\frac{1}{\rho}(1+2w)g^{ij}_{S^2}-\frac{\rho}{3}g^{il}_{S^2}R(0l0k)g^{kj}_{S^2}+\frac{1}{\rho}Q(w)+O(\rho^2)+\rho L(w)\Big)(Hess_{S^2}w)_{ij}\\ &= \frac{1}{\rho}(1+2w)\triangle_{S^2}w-\frac{\rho}{3}g^{il}_{S^2}R(0l0k)g^{kj}_{S^2}(Hess_{S^2}w)_{ij}+\rho^2L(w)+\rho Q(w)+\frac{1}{\rho}L(w)Q(w)\end{aligned}$$

- with a Taylor expansion

$$\frac{2}{\rho(1-w)}=\frac{2(1+w+w^2)}{\rho}+\frac{1}{\rho}wQ(w),$$
$$\frac{1}{\rho(1-w)^3}g^{ij}_{S^2}w_iw_j=\frac{1}{\rho}g^{ij}_{S^2}w_iw_j+\frac{1}{\rho}wQ(w)$$

- finally, recalling our notations, (2.1) and that $\{\Theta,\frac{\Theta_1}{\|\Theta_1\|},\frac{\Theta_2}{\|\Theta_2\|}\}$ form an orthonormal base of T_pM

$$g(\Theta_i,\Theta_j)g^{il}_{S^2}R(0l0k)g^{kj}_{S^2}=\delta^l_jg(R_p(\Theta,\Theta_l)\Theta,\Theta_k)g^{kj}_{S^2}=g(R_p(\Theta,\Theta_i)\Theta,\Theta_j)g^{ij}_{S^2}=-Ric_p(\Theta,\Theta).\tag{2.6}$$

□

Now we compute H^2:

Lemma 2.2.4. *The square of the mean curvature H^2 on $S_{p,\rho}(w)$ can be expanded as*

$$\begin{aligned}H^2 &= \frac{4}{\rho^2}+\frac{4}{\rho^2}(2+\triangle_{S^2})w+\frac{1}{\rho^2}(12w^2+12w\triangle_{S^2}w+(\triangle_{S^2}w)^2-4g^{ij}_{S^2}w_iw_j)-\frac{4}{\rho^2}g^{ij}_{S^2}B^k_{ij}w_k\\ &\quad -\frac{4}{3}g^{il}_{S^2}R(0l0k)g^{kj}_{S^2}(Hess_{S^2}w)_{ij}-\frac{2}{3}Ric_p(\Theta,\Theta)(2+\triangle_{S^2}w)+[g^{ij}_{S^2}\nabla_0R(0i0j)]\rho\\ &\quad +\Big[\frac{2}{5}g^{ij}_{S^2}\nabla_{00}R(0i0j)+\frac{16}{45}g^{ij}_{S^2}R(0i0\mu)R(0j0\mu)-\frac{4}{9}g^{il}_{S^2}R(0l0k)g^{kn}_{S^2}R(0n0i)+\frac{1}{9}Ric_p(\Theta,\Theta)Ric_p(\Theta,\Theta)\Big]\rho^2\\ &\quad +O_p(\rho^3)+\rho L_p(w)+\frac{1}{\rho}Q^{(2)}_p(w)+\frac{1}{\rho^2}L_p(w)Q^{(2)}_p(w).\end{aligned}$$

Proof. Just compute the square of H expressed as in Lemma 2.2.3. □

Lemma 2.2.5. *The determinant of the first fundamental form of $S_{p,\rho}(w)$ can be expanded as*

$$\begin{aligned}det[\mathring{g}] &= \|\Theta_2\|^2\rho^4\Big\{(1-w)^4+(g^{ij}_{S^2}w_iw_j)-\frac{1}{3}Ric_p(\Theta,\Theta)\rho^2(1-w)^6+\frac{1}{6}g^{ij}_{S^2}\nabla_0R(0i0j)\rho^3(1-w)^7\\ &\quad +\Big[\frac{1}{20}g^{ij}_{S^2}\nabla_{00}R(0i0j)+\frac{2}{45}g^{ij}_{S^2}R(0i0\mu)R(0j0\mu)+\frac{1}{9}R(0101)R(0\bar{2}0\bar{2})-R(010\bar{2})^2\Big]\rho^4(1-w)^8\Big\}\\ &\quad +O_p(\rho^9)+\rho^9L_p(w)+\rho^6Q^{(2)}_p(w)+\rho^4L_p(w)Q^{(2)}_p(w)\end{aligned}$$

where recall that $R(0101)=g(R_p(\Theta,\Theta_1)\Theta,\Theta_1)$, $R(0\bar{2}0\bar{2})=g(R_p(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)$, $R(010\bar{2})=-g(R_p(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)$ and $\bar{\Theta}_2$ is Θ_2 normalized: $\bar{\Theta}_2:=\frac{\Theta_2}{|\Theta_2|}$.

Proof. Just compute $det[\mathring{g}]$ using Lemma 2.2.1, formula (2.6) and observing that $g^{ij}_{S^2} = diag(1, 1/\|\Theta_2\|^2)$ $\square$

Lemma 2.2.6. *The determinant of the second fundamental form of $S_{p,\rho}(w)$ has the following expansion:*

$$
\begin{aligned}
det[\mathring{h}] \;=\;& \rho^2(1-w)^2\|\Theta_2\|^2 + \rho^2\|\Theta_2\|^2\triangle_{S^2}w(1-w) + \rho^2\big[(Hess_{S^2}w)_{11}(Hess_{S^2}w)_{22} - (Hess_{S^2}w)^2_{12}\big] \\
&+\frac{2}{3}\rho^4\big[R(0101)(Hess_{S^2}w)_{22} + R(0202)(Hess_{S^2}w)_{11} - 2R(0102)(Hess_{S^2}w)_{12} - Ric_p(\Theta,\Theta)(1-w)^4\|\Theta_2\|^2\big] \\
&+\|\Theta_2\|^2\Big[\frac{5}{12}g^{ij}_{S^2}\nabla_0 R(0i0j)\rho^5(1-w)^5 + \frac{3}{20}g^{ij}_{S^2}\nabla_{00}R(0i0j)\rho^6(1-w)^6 + \frac{2}{15}g^{ij}_{S^2}R(0i0\mu)R(0j0\mu)\rho^6(1-w)^6\Big] \\
&+\frac{4}{9}\rho^6(1-w)^6\|\Theta_2\|^2\big[R(0101)R(0\bar{2}0\bar{2}) - R(010\bar{2})^2\big] - \|\Theta_2\|^2\rho^2 g^{ij}_{S^2}B^k_{ij}w_k + O_p(\rho^7) + \rho^5 L_p(w) + \rho^3 Q^{(2)}_p(w).
\end{aligned}
$$

Proof. Just compute the determinant of $\mathring{h}_{ij}$ expressed as in Lemma 2.2.2 using the same tricks of the previous Lemmas. $\square$

Lemma 2.2.7. *The product of the principal curvatures of $S_{p,\rho}(w)$*

$$
D = k_1k_2 = \frac{det(\mathring{h})}{det(\mathring{g})}
$$

has the following expansion:

$$
\begin{aligned}
D \;=\;& \frac{1}{\rho^2}(1 + 2w + \triangle_{S^2}w + 3w\triangle_{S^2}w + 3w^2) - \frac{1}{\rho^2}g^{ij}_{S^2}w_iw_j + \frac{1}{\|\Theta_2\|^2\rho^2}\big[(Hess_{S^2}w)_{11}(Hess_{S^2}w)_{22} - (Hess_{S^2}w)^2_{12}\big] \\
&+\frac{2}{3\|\Theta_2\|^2}\Big[R(0101)(Hess_{S^2}w)_{22} + R(0202)(Hess_{S^2}w)_{11} - 2R(0102)(Hess_{S^2}w)_{12}\Big] \\
&+\frac{1}{3}Ric_p(\Theta,\Theta)(\triangle_{S^2}w - 1) + \frac{1}{4}g^{ij}_{S^2}\nabla_0 R(0i0j)\rho(1-w) - \frac{1}{\rho^2}g^{ij}_{S^2}B^k_{ij}w_k \\
&+\Big[\frac{1}{10}g^{ij}_{S^2}\nabla_{00}R(0i0j) + \frac{4}{45}g^{ij}_{S^2}R(0i0\mu)R(0j0\mu) + \frac{1}{3}[R(0101)R(0\bar{2}0\bar{2}) - R(010\bar{2})^2] - \frac{1}{9}Ric_p(\Theta,\Theta)^2\Big]\rho^2(1-w)^2 \\
&+O_p(\rho^3) + \rho L_p(w) + \frac{1}{\rho}Q^{(2)}_p(w) + \frac{1}{\rho^2}L_p(w)Q^{(2)}_p(w).
\end{aligned}
$$

Proof. Recalling the expansion $\frac{1}{1+x} = 1 - x + x^2 + O(x^3)$ and Lemma 2.2.5 we get

$$
\begin{aligned}
\frac{1}{det[\mathring{g}]} \;=\;& \frac{1}{\|\Theta_2\|^2(1-w)^4\rho^4}\quad\{1 - (g^{ij}_{S^2}w_iw_j) + \frac{1}{3}Ric_p(\Theta,\Theta)\rho^2(1-w)^2 - \frac{1}{6}g^{ij}_{S^2}\nabla_0 R(0i0j)\rho^3(1-w)3 \\
&-\Big[\frac{1}{20}g^{ij}_{S^2}\nabla_{00}R(0i0j) + \frac{2}{45}g^{ij}_{S^2}R(0i0\mu)R(0j0\mu) + \frac{1}{9}R(0101)R(0\bar{2}0\bar{2}) - R(010\bar{2})^2\Big]\rho^4(1-w)^8 \\
&+\frac{1}{9}Ric_p(\Theta,\Theta)^2\rho^4(1-w)^4 + O_p(\rho^5) + \rho^5L_p(w) + \rho^2Q^{(2)}_p(w) + L_p(w)Q^{(2)}_p(w)\}.
\end{aligned}
$$

Gathering together this formula and the expansion of $det(\mathring{h})$ of Lemma 2.2.6 we can conclude. $\square$

The quantity we have to integrate is $\frac{H^2}{4} - D$; collecting the previous Lemmas we finally get the following

Proposition 2.2.8. *The integrand of the conformal Willmore functional has the following expansion:*

$$
\begin{aligned}
\frac{H^2}{4} - D \;=\;& \frac{1}{\rho^2}\Big[\frac{1}{4}(\triangle_{S^2}w)^2 - \frac{1}{\|\Theta_2\|^2}(Hess_{S^2}w)_{11}(Hess_{S^2}w)_{22} + \frac{1}{\|\Theta_2\|^2}(Hess_{S^2}w)^2_{12}\Big] \\
&+\frac{1}{3\|\Theta_2\|^2}\Big[2R(0102)(Hess_{S^2}w)_{12} - R(0101)(Hess_{S^2}w)_{22} - R(0\bar{2}0\bar{2})(Hess_{S^2}w)_{11}\Big] \\
&+\frac{1}{9}\rho^2\Big[\frac{1}{4}Ric_p(\Theta,\Theta)^2 - R(0101)R(0\bar{2}0\bar{2}) + R(010\bar{2})^2\Big] - \frac{1}{6}Ric_p(\Theta,\Theta)\triangle_{S^2}w \\
&+O_p(\rho^3) + \rho L_p(w) + \frac{1}{\rho}Q^{(2)}_p(w) + \frac{1}{\rho^2}L_p(w)Q^{(2)}_p(w)
\end{aligned}
$$

Proof. Putting together the formulas of Lemma 2.2.4 and Lemma 2.2.7, we get

$$\begin{aligned}
\frac{H^2}{4}-D &= \frac{1}{4\rho^2}(\triangle_{S^2}w)^2+\frac{1}{\|\Theta_2\|^2\rho^2}\big[(Hess_{S^2}w)_{12}^2-(Hess_{S^2}w)_{11}(Hess_{S^2}w)_{22}\big]\\
&\quad -\frac{1}{3}g_{S^2}^{il}R(0l0k)g_{S^m}^{kj}(Hess_{S^2}w)_{ij}-\frac{1}{2}Ric_p(\Theta,\Theta)\triangle_{S^2}w\\
&\quad +\frac{2}{3\|\Theta_2\|^2}\big[2R(0102)(Hess_{S^2}w)_{12}-R(0101)(Hess_{S^2}w)_{22}-R(0202)(Hess_{S^2}w)_{11}\big]\\
&\quad -\frac{1}{9}\rho^2\big[g_{S^2}^{ik}g_{S^2}^{jl}R(0i0l)R(0j0k)\big]-\frac{1}{3}\rho^2\big[R(0101)R(0\bar{2}0\bar{2})-R(010\bar{2})^2\big]\\
&\quad +\frac{5}{36}Ric_p(\Theta,\Theta)^2\rho^2+O_p(\rho^3)+\rho L_p(w)+\frac{1}{\rho}Q_p^{(2)}(w)+\frac{1}{\rho^2}L_p(w)Q_p^{(2)}(w)
\end{aligned}$$

Let us simplify the second and the third lines; they can be rewritten as

$$\begin{aligned}
&-\frac{1}{3}R(0101)(Hess_{S^2}w)_{11}-\frac{1}{3}R(0\bar{2}0\bar{2})\frac{1}{\|\Theta_2\|^2}(Hess_{S^2}w)_{22}+\frac{2}{3\|\Theta_2\|^2}R(0102)(Hess_{S^2}w)_{12}\\
&-\frac{2}{3}R(0101)\frac{1}{\|\Theta_2\|^2}(Hess_{S^2}w)_{22}-\frac{2}{3}R(0\bar{2}0\bar{2})(Hess_{S^2}w)_{11}-\frac{1}{2}Ric_p(\Theta,\Theta)\triangle_{S^2}w\\
= \; & \frac{2}{3\|\Theta_2\|^2}R(0102)(Hess_{S^2}w)_{12}+\frac{1}{3}Ric_p(\Theta,\Theta)\triangle_{S^2}w\\
&-\frac{1}{3}R(0101)\frac{1}{\|\Theta_2\|^2}(Hess_{S^2}w)_{22}-\frac{1}{3}R(0\bar{2}0\bar{2})(Hess_{S^2}w)_{11}-\frac{1}{2}Ric_p(\Theta,\Theta)\triangle_{S^2}w\\
= \; & -\frac{1}{6}Ric_p(\Theta,\Theta)\triangle_{S^2}w+\frac{1}{3\|\Theta_2\|^2}\big[2R(0102)(Hess_{S^2}w)_{12}-R(0101)(Hess_{S^2}w)_{22}-R(0\bar{2}0\bar{2})(Hess_{S^2}w)_{11}\big]
\end{aligned}$$

Finally we have to simplify the fourth and the fifth lines; they can be rewritten as

$$\begin{aligned}
&\Big\{-\frac{1}{9}R(0101)^2-\frac{1}{9}R(0\bar{2}0\bar{2})^2-\frac{2}{9}R(010\bar{2})^2+\frac{5}{36}Ric_p(\Theta,\Theta)^2-\frac{1}{3}R(0101)R(0\bar{2}0\bar{2})+\frac{1}{3}R(010\bar{2})^2\Big\}\rho^2\\
= \; &\Big\{-\frac{1}{9}[R(0101)+R(0\bar{2}0\bar{2})]^2-\frac{1}{9}R(0101)R(0\bar{2}0\bar{2})+\frac{1}{9}R(010\bar{2})^2+\frac{5}{36}Ric_p(\Theta,\Theta)^2\Big\}\rho^2\\
= \; &\Big\{\frac{1}{36}Ric_p(\Theta,\Theta)^2-\frac{1}{9}R(0101)R(0\bar{2}0\bar{2})+\frac{1}{9}R(010\bar{2})^2\Big\}\rho^2
\end{aligned}$$

where, in the last equality, we used the usual identity $R(0101)+R(0\bar{2}0\bar{2})=-Ric_p(\Theta,\Theta)$.

Collecting the formulas we get the desired expansion. □

2.2.2 The differential of the conformal Willmore functional on perturbed geodesic spheres $S_{p,\rho}(w)$

Proposition 2.2.9. *On the perturbed geodesic sphere $S_{p,\rho}(w)$ the differential of the conformal Willmore functional has the following form:*

$$I'(S_{p,\rho}(w))=\frac{1}{2\rho^3}\triangle_{S^2}(\triangle_{S^2}+2)w-\frac{1}{6\rho}\triangle_{S^2}Ric_p(\Theta,\Theta)+O_p(\rho^0)+\frac{1}{\rho^2}L_p^{(4)}(w)+\frac{1}{\rho^3}Q_p^{(2)(4)}(w)$$

Proof. Let us recall the general expression of the differential of the conformal Willmore functional computed in [HL] (Theorem 3.1 plus an easy computation using Codazzi equation).

Given a compact Riemannian surface $(\mathring{M},\mathring{g})$ isometrically immersed in the three dimensional Riemannian manifold (M,g) and called $\mathring{N}$ the inward normal unit vector, the differential of the conformal Willmore functional

$$I(\mathring{M})=\int_{\mathring{M}}\Big(\frac{H^2}{4}-D\Big)d\Sigma$$

is

$$I'(\mathring{M})=\frac{1}{2}\triangle_{\mathring{M}}H+H\Big(\frac{H^2}{4}-D\Big)+\sum_{ij}R(\mathring{N},e_i,\mathring{N},e_j)\mathring{h}_{ij}-\frac{1}{2}\sum_i HR(\mathring{N},e_i,\mathring{N},e_i)+\sum_{ij}(\nabla_{e_i}R)(\mathring{N},e_j,e_j,e_i)$$

where e_1, e_2 is a local orthonormal frame of $T_p\mathring{M}$ which diagonalizes the second fundamental form $\mathring{h}_{ij}$. Since e_1, e_2 are principal directions we get

$$\sum_{ij} R(\mathring{N}, e_i, \mathring{N}, e_j)\mathring{h}_{ij} - \frac{1}{2}\sum_i HR(\mathring{N}, e_i, \mathring{N}, e_i) = \frac{(\lambda_1 - \lambda_2)}{2}[R(\mathring{N}, e_1, \mathring{N}, e_1) - R(\mathring{N}, e_2, \mathring{N}, e_2)]$$

where λ_1, λ_2 are the principal curvatures. So in this frame the differential is

$$I'(\mathring{M}) = \frac{1}{2}\triangle_{\mathring{M}} H + H\left(\frac{H^2}{4} - D\right) + \frac{(\lambda_1 - \lambda_2)}{2}[R(\mathring{N}, e_1, \mathring{N}, e_1) - R(\mathring{N}, e_2, \mathring{N}, e_2)] + \sum_{ij}(\nabla_{e_i} R)(\mathring{N}, e_j, e_j, e_i). \tag{2.7}$$

Now we want to compute the differential on the perturbed geodesic sphere $S_{p,\rho}(w)$.
Recall that

$$\begin{aligned}
\triangle_{\mathring{g}} u &= \mathring{g}^{ij}\left(u_{ij} - \mathring{\Gamma}^k_{ij} u_k\right) \\
&= \frac{1}{\rho^2} g^{ij}_{S^2}\left(u_{ij} - \Gamma^k_{ij} u_k\right) + O(\rho^0)L(u) + \frac{1}{\rho^2}L_p(w)L(u) + \frac{1}{\rho^2}Q^{(2)}_p(w)L(u) \\
&= \frac{1}{\rho^2}\triangle_{S^2} u + O(\rho^0)L(u) + \frac{1}{\rho^2}L_p(w)L(u) + \frac{1}{\rho^2}Q^{(2)}_p(w)L(u)
\end{aligned}$$

where $L(u)$ is a linear function depending on u and on its derivatives up to order two. From the above computation of H we have

$$H = \frac{2}{\rho} + \frac{1}{\rho}(2 + \triangle_{S^2})w - \frac{1}{3}Ric_p(\Theta, \Theta)\rho + O(\rho^2) + \rho L_p(w) + \frac{1}{\rho}Q^{(2)}_p(w),$$

hence

$$\triangle_{\mathring{g}} H = \frac{1}{\rho^3}\triangle_{S^2}(\triangle_{S^2} + 2)w - \frac{1}{3\rho}\triangle_{S^2} Ric_p(\Theta, \Theta) + O_p(\rho^0) + \frac{1}{\rho^2}L^{(4)}_p(w) + \frac{1}{\rho^3}Q^{(2)(4)}_p(w).$$

Now let us show that the other summands are negligible.
First we find an expansion for the principal directions λ_1 and λ_2. From the definitions, they are the roots of the polynomial equation

$$x^2 - Hx + D = 0$$

so

$$\lambda_{1,2} = \frac{H}{2} \pm \frac{\sqrt{H^2 - 4D}}{2} = \frac{1}{\rho} + O(\rho) + \frac{1}{\rho}L_p(w) + \frac{1}{\rho}Q^{(2)}_p(w)$$

and the third summand is negligible:

$$(\lambda_1 - \lambda_2)[R(\mathring{N}, e_1, \mathring{N}, e_1) - R(\mathring{N}, e_2, \mathring{N}, e_2)] = O(\rho) + \frac{1}{\rho}L_p(w) + \frac{1}{\rho}Q^{(2)}_p(w).$$

From the above computation of $\frac{H^2}{4} - D$, we have

$$\frac{H^2}{4} - D = O_p(\rho^2) + L_p(w) + \frac{1}{\rho^2}Q^{(2)}_p(w)$$

hence we get

$$H\left(\frac{H^2}{4} - D\right) = O_p(\rho) + \frac{1}{\rho}L_p(w) + \frac{1}{\rho^3}Q^{(2)}_p(w).$$

Therefore also this term is negligible and we can conclude observing that $(\nabla_{e_i} R)(\mathring{N}, e_j, e_j, e_i) = O(\rho^0)$. $\square$

2.2.3 The finite dimensional reduction

NOTATION. In this subsection, the functional space will be $C^{4,\alpha}(S^2)^\perp$: the perturbation w will be an element of $C^{4,\alpha}(S^2)^\perp$ and $B(0,r)$ will denote the ball of center 0 and radius r in $C^{4,\alpha}(S^2)^\perp$.

Lemma 2.2.10. *Fixed a compact subset $Z_c \subseteq M$, there exist $\rho_0 > 0$, $r > 0$ and a map $w_{(.,.)} : Z_c \times [0, \rho_0] \to C^{4,\alpha}(S^2)^\perp$, $(p, \rho) \mapsto w_{p,\rho}$ such that if $S_{p,\rho}(w)$ is a critical point of the conformal Willmore functional I with $(p, \rho, w) \in Z_c \times [0, \rho_0] \times B(0, r)$ then $w = w_{p,\rho}$.*
Moreover the map $w_{(.,.)}$ satisfies the following properties:
(i) the map $(p, \rho) \mapsto w_{p,\rho}$ is C^1,
(ii) $\|w_{p,\rho}\|_{C^{4,\alpha}(S^2)} = O(\rho^2)$ as $\rho \to 0$ uniformly for $p \in Z_c$,
(iii) $\|\frac{\partial}{\partial\rho} w_{p,\rho}\|_{L^2(S^2)} = O(\rho)$ as $\rho \to 0$ uniformly for $p \in Z_c$,
(iv) we have the following explicit expansion of $w_{p,\rho}$:

$$w_{p,\rho} = -\frac{1}{12}\rho^2 Ric_p(\Theta, \Theta) + \frac{1}{36}\rho^2 R(p) + O(\rho^3) \tag{2.8}$$

where the remainder $O(\rho^3)$ has to be intended in $C^{4,\alpha}(S^2)$ norm.

Proof. For the proof of $(i), (ii)$ and (iii) we refer to Lemma 4.4 of [Mon1], here we only give a sketch of the idea. Fixed a compact subset $Z_c \subseteq M$ and $p \in Z_c$, if

$$I'(S_{p,\rho}(w)) = 0 \quad \text{(equality in } L^2(S^2)),$$

then, setting $P : L^2(S^2) \to Ker[\triangle_{S^2}(\triangle_{S^2} + 2)]^\perp$ the orthogonal projection, a fortiori we have

$$PI'(S_{p,\rho}(w)) = 0;$$

that is, using the expansion of Proposition 2.2.9,

$$P\Big[\triangle_{S^2}(\triangle_{S^2} + 2)w + O_p(\rho^2) + \rho L_p^{(4)}(w) + Q_p^{(2)(4)}(w)\Big] = 0. \tag{2.9}$$

Since $\triangle_{S^2}(\triangle_{S^2}+2)$ is invertible on the space orthogonal to the Kernel and $w \in C^{4,\alpha}(S^2)^\perp := Ker[\triangle_{S^2}(\triangle_{S^2}+2)]^\perp \cap C^{4,\alpha}(S^2)$, setting

$$K := [\triangle_{S^2}(\triangle_{S^2} + 2)]^{-1} : Ker[\triangle_{S^2}(\triangle_{S^2} + 2)]^\perp \subseteq L^2(S^2) \to Ker[\triangle_{S^2}(\triangle_{S^2} + 2)]^\perp,$$

the equation (2.9) is equivalent to the fixed point problem

$$w = K[O_p(\rho^2) + \rho L_p^{(4)}(w) + Q_p^{(2)(4)}(w)] = F_{p,\rho}(w). \tag{2.10}$$

The projection in the right hand side is intended. In the aforementioned paper (using Schauder estimates) it is proved that once the compact $Z_c \subset M$ is fixed, there exist $\rho_0 > 0$ and $r > 0$ such that for all $p \in Z_c$ and $\rho < \rho_0$ the map

$$F_{p,\rho} : B(0, r) \subset C^{4,\alpha}(S^2)^\perp \to C^{4,\alpha}(S^2)^\perp$$

is a contraction. In the same paper the regularity and the decay properties are shown.

Now let us prove the expansion (iv).
Using the formula of Proposition 2.2.9, the unique solution $w \in B(0, r)$ to the fixed point problem will have to satisfy the following fourth order elliptic PDE:

$$\triangle_{S^2}(\triangle_{S^2} + 2)w = \frac{1}{3}\rho^2 \triangle_{S^2} Ric_p(\Theta, \Theta) + O_p(\rho^3) + \rho L_p^{(4)}(w) + Q_p^{(2)(4)}(w).$$

Clearly the unique solution w has the form $w = \rho^2 \bar{w} + O(\rho^3)$ where the remainder has to be intended in $C^{4,\alpha}(S^2)$ norm and $\bar{w} \in C^{4,\alpha}(S^2)$ is independent of ρ. Now we want to find an explicit formula for $\bar{w}$. Writing the radial unit vector in normal coordinates on T_pM, we have $\Theta = x^i \frac{\partial}{\partial x^i}$ and the Ricci tensor can be written as

$$Ric_p(\Theta, \Theta) = \sum_{i \neq j} R_{ij} x^i x^j + \sum_i R_{ii}(x^i)^2.$$

Recall that the eigenfunctions of $\triangle_{S^2}$ relative to the second eigenvalue $\lambda_2 = -6$ are $x^i x^j, i \neq j$ and $(x^i)^2 - (x^j)^2, i \neq j$ and notice that

$$2(x^1)^2 - 1 = (x^1)^2 - (x^2)^2 + (x^1)^2 - (x^3)^2 - (x^1)^2$$

so

$$(x^1)^2 - \frac{1}{3} = \frac{1}{3}\Big\{[(x^1)^2 - (x^2)^2] + [(x^1)^2 - (x^3)^2]\Big\}$$

is an element of the eigenspace relative to $\lambda_2 = -6$ (analogously for the others $(x^i)^2$). So

$$\begin{aligned} Ric_p(\Theta,\Theta) &= \sum_{i\neq j} R_{ij}(p)x^i x^j + \sum_i R_{ii}(p)[(x^i)^2 - \frac{1}{3}] + \frac{1}{3}\sum_i R_{ii}(p) \\ &= \sum_{i\neq j} R_{ij}(p)x^i x^j + \sum_i R_{ii}(p)[(x^i)^2 - \frac{1}{3}] + \frac{1}{3}R(p) \end{aligned}$$

and $Ric_p(\Theta,\Theta) - \frac{1}{3}R(p)$ is an element of the second eigenspace of $\triangle_{S^2}$.
Recalling that $w = \rho^2\bar{w} + O(\rho^3)$, then $\bar{w}$ has to solve the following linear elliptic PDE

$$\triangle_{S^2}(\triangle_{S^2} + 2)\bar{w} = \frac{1}{3}\triangle_{S^2}\Big[Ric_p(\Theta,\Theta) - \frac{1}{3}R(p)\Big].$$

Since the right hand side is an eigenfunction of $\triangle_{S^2}$ with eigenvalue -6 the equation is easily solved as

$$\bar{w} = -\frac{1}{12}Ric(\Theta,\Theta) + \frac{1}{36}R(p).$$

□

2.2.4 The expansion of the reduced functional $I(S_{p,\rho}(w_{p,\rho}))$

In this subsection we want to evaluate the reduced functional $I(S_{p,\rho}(w_{p,\rho}))$, that is the conformal Willmore functional on perturbed geodesic spheres with perturbation w in the constraint given by Proposition 2.2.10.

Proposition 2.2.11. *The conformal Willmore functional on perturbed geodesic spheres $S_{p,\rho}(w_{p,\rho})$ with perturbation $w_{p,\rho}$ lying in the constraint given by Proposition 2.2.10 can be expanded in ρ as follows*

$$I(S_{p,\rho}(w_{p,\rho})) = \frac{\pi}{5}\|S_p\|^2\rho^4 + O_p(\rho^5),$$

where S_p is the Traceless Ricci tensor defined in (1.5).

Proof. In the sequel we fix a point $p \in M$ and we want to evaluate $I(S_{p,\rho}(w_{p,\rho}))$ for small ρ. For simplicity of notation, let us denote $w = w_{p,\rho}$; from Proposition 2.2.10 we know that $w = \rho^2\bar{w} + O(\rho^3)$. Notice that the leading part of $H^2/4 - D$ is homogeneous of degree two in ρ, so in order to evaluate $I(S_{p,\rho}(w_{p,\rho}))$ it is sufficient to multiply $H^2/4 - D$ by the first term of $\sqrt{det[\mathring{g}]}$ (that is $\rho^2\|\Theta_2\|$). Using the expansion of Proposition 2.2.8 we get

$$\begin{aligned} I(S_{p,\rho}(w)) &= \rho^4 \int_{S^2}\Big[\frac{1}{4}(\triangle_{S^2}\bar{w})^2 - \frac{1}{\|\Theta_2\|^2}(Hess_{S^2}\bar{w})_{11}(Hess_{S^2}\bar{w})_{22} + \frac{1}{\|\Theta_2\|^2}(Hess_{S^2}\bar{w})^2_{12} - \frac{1}{6}Ric_p(\Theta,\Theta)\triangle_{S^2}\bar{w} \\ &+ \frac{2}{3\|\Theta_2\|^2}R(0102)(Hess_{S^2}\bar{w})_{12} - \frac{1}{3\|\Theta_2\|^2}R(0101)(Hess_{S^2}\bar{w})_{22} - \frac{1}{3}R(0\bar{2}0\bar{2})(Hess_{S^2}\bar{w})_{11} \\ &+ \frac{1}{9}\Big(\frac{1}{4}Ric_p(\Theta,\Theta)^2 - R(0101)R(0\bar{2}0\bar{2}) + R(010\bar{2})^2\Big)\Big]d\Sigma_0 \quad + O_p(\rho^5). \end{aligned} \tag{2.11}$$

From (iv) of Proposition 2.2.10 it follows that

$$\triangle_{S^2}\bar{w} = -6\bar{w} = \frac{1}{2}Ric_p(\Theta,\Theta) - \frac{1}{6}R(p)$$

so, after some easy computation, one can write

$$\frac{1}{4}(\triangle_{S^2}\bar{w})^2-\frac{1}{6}Ric_p(\Theta,\Theta)\triangle_{S^2}\bar{w}+\frac{1}{36}Ric_p(\Theta,\Theta)^2=\frac{1}{144}Ric_p(\Theta,\Theta)^2-\frac{1}{72}Ric_p(\Theta,\Theta)R(p)+\frac{1}{144}R(p)^2. \tag{2.12}$$

In order to simplify the other integrands of (2.11) we compute $(Hess_{S^2}\bar{w})_{ij}$. The nonvanishing Christoffel symbols of S^2 in polar coordinates θ^1,θ^2 are

$$\begin{aligned}\Gamma^2_{12}&=\Gamma^2_{21}=cotan\theta^1\\ \Gamma^1_{22}&=-\sin\theta^1\cos\theta^1.\end{aligned}$$

Hence, recalling that $(Hessw)_{ij}=w_{ij}-\Gamma^k_{ij}w_k$ and the expression of w given in (iv), we get

$$\begin{aligned}(Hess_{S^2}\bar{w})_{11}&=\bar{w}_{11}=-\frac{1}{6}\partial_{\theta^1}(Ric_p(\Theta,\Theta_1))\\ &=-\frac{1}{6}Ric_p(\Theta_1,\Theta_1)-\frac{1}{6}Ric_p(\Theta,\Theta_{11})\quad\text{but }\Theta_{11}=-\Theta\\ &=-\frac{1}{6}Ric_p(\Theta_1,\Theta_1)+\frac{1}{6}Ric_p(\Theta,\Theta)\\ (Hess_{S^2}\bar{w})_{12}&=\bar{w}_{12}-\Gamma^2_{12}\bar{w}_2=-\frac{1}{6}\partial_{\theta^1}(Ric_p(\Theta,\Theta_2))+\frac{1}{6}\Gamma^2_{12}Ric_p(\Theta,\Theta_2)\\ &=-\frac{1}{6}Ric_p(\Theta_1,\Theta_2)-\frac{1}{6}Ric_p(\Theta,\Theta_{12})+\frac{1}{6}\Gamma^2_{12}Ric_p(\Theta,\Theta_2)\text{ but }\Theta_{12}=cotan\theta^1\Theta_2\\ &=-\frac{1}{6}Ric_p(\Theta_1,\Theta_2)\end{aligned}$$

$$\begin{aligned}(Hess_{S^2}\bar{w})_{22}&=\bar{w}_{22}-\Gamma^1_{22}\bar{w}_1\\ &=-\frac{1}{6}Ric_p(\Theta_2,\Theta_2)-\frac{1}{6}Ric_p(\Theta,\Theta_{22})+\frac{1}{6}\Gamma^1_{22}Ric_p(\Theta,\Theta_1)\text{ but }\Theta_{22}=-\sin\theta^1\cos\theta^1\Theta_1-\sin^2\theta^1\Theta\\ &=-\frac{1}{6}Ric_p(\Theta_2,\Theta_2)+\frac{1}{6}\|\Theta_2\|^2Ric_p(\Theta,\Theta).\end{aligned}$$

Therefore the other part of the integrand can be written as

$$\begin{aligned}&-\frac{1}{\|\Theta_2\|^2}(Hess_{S^2}\bar{w})_{11}(Hess_{S^2}\bar{w})_{22}+\frac{1}{\|\Theta_2\|^2}(Hess_{S^2}\bar{w})^2_{12}+\frac{2}{3\|\Theta_2\|^2}R(0102)(Hess_{S^2}\bar{w})_{12}\\ &-\frac{1}{3}R(0101)\frac{1}{\|\Theta_2\|^2}(Hess_{S^2}\bar{w})_{22}-\frac{1}{3}R(0\bar{2}0\bar{2})(Hess_{S^2}\bar{w})_{11}-\frac{1}{9}R(0101)R(0\bar{2}0\bar{2})+\frac{1}{9}R(010\bar{2})^2\\ =&-\frac{1}{36}Ric(\Theta,\Theta)^2+\frac{1}{36}Ric(\Theta,\Theta)[Ric(\bar{\Theta}_2,\bar{\Theta}_2)+Ric(\Theta_1,\Theta_1)]-\frac{1}{36}Ric(\bar{\Theta}_2,\bar{\Theta}_2)Ric(\Theta_1,\Theta_1)\\ &+\frac{1}{36}Ric(\Theta_1,\bar{\Theta}_2)^2-\frac{1}{9}R(010\bar{2})Ric(\Theta_1,\bar{\Theta}_2)+\frac{1}{18}R(0101)(Ric(\bar{\Theta}_2,\bar{\Theta}_2)-Ric(\Theta,\Theta))\\ &+\frac{1}{18}R(0\bar{2}0\bar{2})(Ric(\Theta_1,\Theta_1)-Ric(\Theta,\Theta))-\frac{1}{9}R(0101)R(0\bar{2}0\bar{2})+\frac{1}{9}R(010\bar{2})^2.\end{aligned} \tag{2.13}$$

Using the following three identities (which follow from the orthogonality of $\{\Theta,\Theta_1,\bar{\Theta}_2\}$, from the definitions and the symmetries of the curvature tensors)

$$\begin{aligned}-\frac{1}{18}\Big(R(0101)+R(0\bar{2}0\bar{2})\Big)Ric_p(\Theta,\Theta)&=\frac{1}{18}Ric_p(\Theta,\Theta)^2\\ Ric_p(\Theta_1,\Theta_1)+Ric_p(\bar{\Theta}_2,\bar{\Theta}_2)&=R(p)-Ric_p(\Theta,\Theta)\\ R(010\bar{2})&=-Ric_p(\Theta_1,\bar{\Theta}_2),\end{aligned} \tag{2.14}$$

after some easy computations we can say that (2.13) equals

$$\begin{aligned}=&\frac{1}{36}Ric(\Theta,\Theta)R(p)-\frac{1}{36}Ric(\bar{\Theta}_2,\bar{\Theta}_2)Ric(\Theta_1,\Theta_1)+\frac{1}{4}Ric(\Theta_1,\bar{\Theta}_2)^2\\ &+\frac{1}{18}R(0101)Ric(\bar{\Theta}_2,\bar{\Theta}_2)+\frac{1}{18}R(0\bar{2}0\bar{2})Ric(\Theta_1,\Theta_1)-\frac{1}{9}R(0101)R(0\bar{2}0\bar{2}).\end{aligned} \tag{2.15}$$

Let us try to simplify the last line using that $R(0101) + R(\bar{2}1\bar{2}1) = -Ric(\Theta_1, \Theta_1)$ and identity (2.14):

$$\begin{aligned}
& \frac{1}{18}R(0101)Ric(\bar{\Theta}_2, \bar{\Theta}_2) + \frac{1}{18}R(0\bar{2}0\bar{2})Ric(\Theta_1, \Theta_1) - \frac{1}{9}R(0101)R(0\bar{2}0\bar{2}) \\
= & -\frac{1}{18}\big[Ric_p(\Theta_1, \Theta_1) + R(1\bar{2}1\bar{2})\big]Ric_p(\bar{\Theta}_2, \bar{\Theta}_2) - \frac{1}{18}\big[Ric(\bar{\Theta}_2, \bar{\Theta}_2) + R(1\bar{2}1\bar{2})\big]Ric_p(\Theta_1, \Theta_1) \\
& -\frac{1}{9}\big[Ric_p(\Theta_1, \Theta_1) + R(1\bar{2}1\bar{2})\big]\big[Ric(\bar{\Theta}_2, \bar{\Theta}_2) + R(1\bar{2}1\bar{2})\big] \\
= & -\frac{2}{9}Ric_p(\Theta_1, \Theta_1)Ric_p(\bar{\Theta}_2, \bar{\Theta}_2) + \frac{1}{6}R(1\bar{2}1\bar{2})Ric_p(\Theta, \Theta) - \frac{1}{6}R(1\bar{2}1\bar{2})R(p) - \frac{1}{9}R(1\bar{2}1\bar{2})^2. \quad (2.16)
\end{aligned}$$

Since $\{\Theta, \Theta_1, \bar{\Theta}_2\}$ is an orthonormal base of T_pM we have the following useful identity

$$\begin{aligned}
R(1\bar{2}1\bar{2}) &= R(1\bar{2}1\bar{2}) - Ric_p(\Theta, \Theta) + Ric_p(\Theta, \Theta) \\
&= \big[R(1\bar{2}1\bar{2}) + R(0\bar{2}0\bar{2}) + R(0101)\big] + Ric_p(\Theta, \Theta) \\
&= -\frac{1}{2}\big[Ric_p(\Theta_1, \Theta_1) + Ric_p(\bar{\Theta}_2, \bar{\Theta}_2) + Ric_p(\Theta, \Theta)\big] + Ric_p(\Theta, \Theta) \\
&= -\frac{1}{2}R(p) + Ric_p(\Theta, \Theta). \qquad (2.17)
\end{aligned}$$

Plugging the last identity (2.17) into formula (2.16), we get that (2.16) equals

$$= -\frac{2}{9}Ric_p(\Theta_1, \Theta_1)Ric_p(\bar{\Theta}_2, \bar{\Theta}_2) - \frac{5}{36}Ric_p(\Theta, \Theta)R(p) + \frac{1}{18}Ric_p(\Theta, \Theta)^2 + \frac{1}{18}R(p)^2. \qquad (2.18)$$

Therefore the last line of (2.13) equals (2.18) and the integrands (2.13) become

$$= \frac{1}{18}Ric(\Theta, \Theta)^2 - \frac{1}{9}Ric(\Theta, \Theta)R(p) + \frac{1}{18}R(p)^2 + \frac{1}{4}Ric(\Theta_1, \bar{\Theta}_2)^2 - \frac{1}{4}Ric_p(\Theta_1, \Theta_1)Ric_p(\bar{\Theta}_2, \bar{\Theta}_2);$$

hence the conformal Willmore functional expressed as in (2.11), using the last formula and (2.12), becomes

$$\begin{aligned}
I(S_{p,\rho}(w)) = \rho^4 \int_{S^2} \Big[& \frac{1}{16}Ric_p(\Theta, \Theta)^2 - \frac{1}{8}Ric_p(\Theta, \Theta)R(p) + \frac{1}{16}R(p)^2 + \frac{1}{4}Ric(\Theta_1, \bar{\Theta}_2)^2 + \\
& -\frac{1}{4}Ric_p(\Theta_1, \Theta_1)Ric_p(\bar{\Theta}_2, \bar{\Theta}_2)\Big] d\Sigma_0 + O_p(\rho^5). \qquad (2.19)
\end{aligned}$$

The integral of the first three summands is well-known (see for example the appendix of [PX]), let us compute the integral of the last two summands.

Claim.

$$\int_{S^2} \big[Ric_p(\Theta_1, \bar{\Theta}_2)^2 - Ric_p(\Theta_1, \Theta_1)Ric_p(\bar{\Theta}_2, \bar{\Theta}_2)\big] d\Sigma_0 = \frac{2\pi}{3}\big(\|Ric_p\|^2 - R(p)^2\big)$$

Proof of the Claim:
As before let us denote by $E_\mu, \mu = 1, 2, 3$ an orthonormal base of T_pM and with x^μ the induced coordinates. Under this notation the radial unit vector is

$$S^2 \ni \Theta = x^\mu E_\mu.$$

Recall that the polar coordinates $0 < \theta^1 < \pi, \quad 0 < \theta^2 < 2\pi$ have been chosen such that S^2 is parametrized as follows

$$\begin{cases} x^1 = \sin\theta^1 \cos\theta^2 \\ x^2 = \sin\theta^1 \sin\theta^2 \\ x^3 = \cos\theta^1. \end{cases}$$

The normalized tangent vectors $\bar{\Theta}_i := \frac{\Theta_i}{\|\Theta_i\|}$ have coordinates

$$\begin{aligned}
\bar{\Theta}_1 &= \Theta_1 = (\cos\theta^1 \cos\theta^2, \cos\theta^1 \sin\theta^2, -\sin\theta^1) \\
&= \left(\frac{x^1 x^3}{\sqrt{(x^1)^2 + (x^2)^2}}, \frac{x^2 x^3}{\sqrt{(x^1)^2 + (x^2)^2}}, -\sqrt{(x^1)^2 + (x^2)^2}\right) \qquad (2.20) \\
\bar{\Theta}_2 &= (-\sin\theta^2, \cos\theta^2, 0) \\
&= \left(-\frac{x^2}{\sqrt{(x^1)^2 + (x^2)^2}}, \frac{x^1}{\sqrt{(x^1)^2 + (x^2)^2}}, 0\right). \qquad (2.21)
\end{aligned}$$

Using this expressions for $\bar\Theta_i$ we get the following formulas for $Ric_p(\bar\Theta_i, \bar\Theta_j)$:

$$\begin{aligned}
Ric_p(\Theta_1,\Theta_1) &= R_{11}\frac{(x^1)^2(x^3)^2}{(x^1)^2+(x^2)^2}+2R_{12}\frac{x^1x^2(x^3)^2}{(x^1)^2+(x^2)^2}+R_{22}\frac{(x^2)^2(x^3)^2}{(x^1)^2+(x^2)^2}-2R_{13}x^1x^3-2R_{23}x^2x^3\\
&\quad +R_{33}[(x^1)^2+(x^2)^2]\\
Ric_p(\bar\Theta_2,\bar\Theta_2) &= R_{11}\frac{(x^2)^2}{(x^1)^2+(x^2)^2}-2R_{12}\frac{x^1x^2}{(x^1)^2+(x^2)^2}+R_{22}\frac{(x^1)^2}{(x^1)^2+(x^2)^2}\\
Ric_p(\Theta_1,\bar\Theta_2) &= -R_{11}\frac{x^1x^2x^3}{(x^1)^2+(x^2)^2}-2R_{12}\frac{(x^2)^2x^3}{(x^1)^2+(x^2)^2}+R_{22}\frac{x^1x^2x^3}{(x^1)^2+(x^2)^2}+R_{12}x^3+R_{13}x^2-R_{23}x^1.
\end{aligned}$$

Notice that the summands which contain a term of the type $(x^i)^{2m+1}$ $(m\in\mathbb{N})$ have vanishing integral on S^2; then, calling "Remainder" all these summands, we get

$$\begin{aligned}
Ric(\Theta_1,\bar\Theta_2)^2 &= (R_{11}^2-2R_{11}R_{22}+R_{22}^2-4R_{12}^2)\frac{(x^1)^2(x^2)^2(x^3)^2}{[(x^1)^2+(x^2)^2]^2}+R_{12}^2(x^3)^2+R_{13}^2(x^2)^2+R_{23}^2(x^1)^2\\
&\quad+\text{ Remainder},\\
Ric(\Theta_1,\Theta_1)Ric(\bar\Theta_2,\bar\Theta_2) &= (R_{11}^2-2R_{11}R_{22}+R_{22}^2-4R_{12}^2)\frac{(x^1)^2(x^2)^2(x^3)^2}{[(x^1)^2+(x^2)^2]^2}+R_{11}R_{22}(x^3)^2\\
&\quad+R_{11}R_{33}(x^2)^2+R_{22}R_{33}(x^1)^2+\text{ Remainder}.
\end{aligned}$$

Therefore the integral of the left hand side of the Claim becomes

$$=\int_{S^2}\Big[R_{12}^2(x^3)^2+R_{13}^2(x^2)^2+R_{23}^2(x^1)^2-R_{11}R_{22}(x^3)^2-R_{11}R_{33}(x^2)^2-R_{22}R_{33}(x^1)^2\Big]d\Sigma_0.$$

Recalling that $\int_{S^2}(x^\mu)^2d\Sigma_0=\frac{4\pi}{3}$,we can continue the equalities

$$\begin{aligned}
&= \frac{4\pi}{3}\left[R_{12}^2+R_{13}^2+R_{23}^2-R_{11}R_{22}-R_{11}R_{33}-R_{22}R_{33}\right]\\
&= \frac{2\pi}{3}\left[(R_{11}^2+R_{22}^2+R_{33}^2+2R_{12}^2+2R_{13}^2+2R_{23}^2)-(R_{11}^2+R_{22}^2+R_{33}^2+2R_{11}R_{22}+2R_{11}R_{33}+2R_{22}R_{33})\right]\\
&= \frac{2\pi}{3}\left(\|Ric_p\|^2-R(p)^2\right).
\end{aligned}$$

Now we are in position to conclude the computation of the integral (2.19).
It is known that $\int_{S^2}Ric_p(\Theta,\Theta)d\Sigma_0=\frac{4\pi}{3}R(p)$ and $\int_{S^2}[Ric_p(\Theta,\Theta)]^2d\Sigma_0=\frac{4\pi}{15}(2\|Ric_p\|^2+R(p)^2)$ (see the appendix of [PX]) thus, grouping together this formulas and the claim, we can say that the conformal Willmore functional on constrained small geodesic spheres can be expanded as

$$I(S_{p,\rho}(w))=\frac{\pi}{5}\Big(\|Ric_p\|^2-\frac{1}{3}R(p)^2\Big)\rho^4+O_p(\rho^5).$$

A simple computation in the orthonormal basis that diagonalizes Ric_p shows that the first term in the expansion is the squared norm of the Traceless Ricci tensor:

$$\Big(\|Ric_p\|^2-\frac{1}{3}R(p)^2\Big)=\|Ric_p-\frac{1}{3}g_pR(p)\|^2=\|S_p\|^2.$$

□

2.2.5 Proof of the non existence result

We start with a Lemma, which asserts that for small perturbation $u\in C^{4,\alpha}(S^2)$ and small radius ρ, the perturbed geodesic sphere $S_{p,\rho}(u)$ can be obtained as a normal graph on an other geodesic sphere $S_{\tilde p,\tilde\rho}$ with perturbation $\tilde w\in {C^{4,\alpha}}^\perp$: $S_{p,\rho}(u)=S_{\tilde p,\tilde\rho}(\tilde w)$; for the proof see [Mon1] Lemma 5.3.

Lemma 2.2.12. *Let (M,g) be a Riemannian manifold of dimension three and fix $\bar p\in M$. Then there exist $B(0,r_1)\subset C^{4,\alpha}(S^2)$, $\rho_1>0$, a compact neighborhood U of $\bar p$ and three continuous functions $p(.) : B(0,r_1)\to U\subset M$,*

$\rho(.,.) : (0, \rho_1) \times B(0, r_1) \to \mathbb{R}^+$,
$w(.,.) : U \times B(0, r_1) \to C^{4,\alpha}(S^2)^\perp$,
such that for all $\bar{\rho} < \rho_1$ *and* $u \in B(0, r_1)$*, all the perturbed geodesic spheres* $S_{\bar{p},\bar{\rho}}(u)$ *can be realized as*

$$S_{\bar{p},\bar{\rho}}(u) = S_{p(u),\rho(\bar{\rho},u)}[w(p(u), u)].$$

Now we are in position to prove the non existence result.

Proof of Theorem 1.0.4.
Since $\|S_{\bar{p}}\| \neq 0$, there exists $\eta > 0$ and a compact neighborhood Z_c of $\bar{p}$ such that $\|S_p\| > \eta$ for all $p \in Z_c$.

From Lemma 2.2.10 there exist $\rho_0 > 0$ and a ball $B(0, r) \subset C^{4,\alpha}(S^2)$ such that- for $w \in C^{4,\alpha\perp} \cap B(0, r)$, $p \in Z_c$ and $\rho < \rho_0$- if the perturbed geodesic sphere $S_{p,\rho}(w)$ is a critical point of I then $w = w_{p,\rho}$ with good decay properties as $\rho \to 0$. Moreover, for $p \in Z_c$ and $\rho < \rho_0$ we can consider the C^1 function

$$\Phi(p, \rho) = I(S_{p,\rho}(w_{p,\rho})).$$

Observe that if $S_{\tilde{p},\tilde{\rho}}(w_{\tilde{p},\tilde{\rho}})$ is a critical point for I then a fortiori $(\tilde{p}, \tilde{\rho})$ is a critical point of the constricted functional $\Phi(.,.)$.
Proposition 2.2.11 gives an expansion for $\Phi(p, \rho)$; differentiating it with respect to ρ and recalling (from Lemma 2.2.10) that as $\rho \to 0$ one has $\|w_{p,\rho}\|_{C^{4,\alpha}} = O(\rho^2)$ and $\|\frac{\partial}{\partial \rho} w_{p,\rho}\|_{L^2} = O(\rho)$ uniformly for $p \in Z_c$, we get

$$\frac{\partial}{\partial \rho}\Phi(p, \rho) = \frac{4\pi}{5}\|S_p\|\rho^3 + O_p(\rho^4)$$

and

$$\left|\frac{\partial}{\partial \rho}\Phi(p, \rho)\right| > \frac{4\pi}{5}\eta\rho^3 + O(\rho^4) \quad \text{for all } p \in Z_c, \tag{2.22}$$

where the remainder $O(\rho^4)$ is uniform on Z_c.
From this equation we can say that there exist $\rho_2 \in]0, \rho_0[$ such that for all $p \in Z_c$ and $\rho < \rho_2$, (p, ρ) is not a critical point of Φ.
Hence

$$\begin{aligned} &\forall w \in C^{4,\alpha}(S^2)^\perp \cap B(0, r),\ \rho < \rho_2 \text{ and } p \in Z_c \\ &\Rightarrow S_{p,\rho}(w) \text{ is NOT a critical point of } I. \end{aligned} \tag{2.23}$$

Now from Lemma 2.2.12, if $u \in B(0, r_1) \subset C^{4,\alpha}(S^2)$ and $\bar{\rho} < \rho_1$, any perturbed sphere $S_{\bar{p},\bar{\rho}}(u)$ can be realized as

$$S_{\bar{p},\bar{\rho}}(u) = S_{p(u),\rho(\bar{\rho},u)}[w(p(u), u)], \quad w(p(u), u) \in C^{4,\alpha}(S^2)^\perp.$$

From the continuity of the functions $p(.)$, $\rho(.,.)$ and $w(.,.)$, there exist $\rho_3 \in]0, \min(\rho_1, \rho_2)[$ and $r_2 \in]0, \min(r, r_1)[$ such that for all $u \in B(0, r_2) \subset C^{4,\alpha}(S^2)$ and $\bar{\rho} < \rho_3$ we have:
· $p(u) \in Z_c$,
· $\rho(\bar{\rho}, u) < \rho_2$ and
· $w(p(u), u) \in C^{4,\alpha}(S^2)^\perp \cap B(0, r)$.

It follows that if $u \in B(0, r_2)$ and $\bar{\rho} < \rho_3$, the sphere $S_{\bar{p},\bar{\rho}}(u)$ can be realized as $S_{p(u),\rho(\bar{\rho},u)}[w(p(u), u)]$ which satisfies the assumptions (2.23); so it is not a critical point of I. ■

2.3 The conformal Willmore functional on perturbed standard spheres $S_p^\rho(w)$ in $(\mathbb{R}^3, g_\epsilon)$

Throughout this section $I_\epsilon(\mathring{M}) := \int_{\mathring{M}} \left[\frac{H^2}{4} - D\right] d\Sigma_\epsilon$ will be the conformal Willmore functional of the surface $\mathring{M}$ embedded in the ambient manifold $(\mathbb{R}^3, g_\epsilon)$, where $g_\epsilon = \delta + \epsilon h$ is a perturbation of the euclidean metric (h is a bilinear form with good decay properties at infinity, for simplicity we will treat in detail the case when h has compact support but as one can see from the estimates it is enough to take h fast decreasing. See for example [Mon1] Theorem 1.1).

The problem will be studied through a perturbation method relying on the Lyapunov-Schmidt reduction: In Subsection 2.3.1 we will perform the abstract reduction, in Subsection 2.3.2 we will compute an expansion of the reduced functional and in the last Subsection 2.3.3 we will prove the main Theorems of this Chapter, that is the existence of conformal Willmore surfaces.

2.3.1 The finite dimensional reduction

We already know from Theorem 1.2 that I_0 possesses a critical manifold made up of the standard spheres S_p^ρ of $\mathbb{R}^3$, we want to study the perturbed functional I_ϵ near this critical manifold. First of all let us point out a clarification about $I_0'(S_p^\rho)$ and $I_0''(S_p^\rho)$, that are the first and second variations of the unperturbed functional on the standard spheres, which will be useful throughout this Section.

Remark 2.3.1. *In the previous paper [Mon1], (remark 3.3, notice the factor difference in the definition of the Willmore functional) we observed that*

$$I_0'(S_p^\rho(w)) = \frac{1}{2\rho^3}\triangle_{S^2}(\triangle_{S^2}+2)w + \frac{1}{\rho^3}Q_p^{(2)(4)}(w)$$

and

$$I_0''(S_p^\rho)[w] = \frac{1}{2\rho^3}\triangle_{S^2}(\triangle_{S^2}+2)[w].$$

The sense of the two formulas were the following.
By definition $S_p^\rho(w)$ is a normal graph on S_p^ρ with perturbation ρw (we chose the inward normal $\mathring{N}$ for all the computations), hence

$$I_0(S_p^\rho(w)) = I_0(S_p^\rho) + \int_{S_p^\rho}\big(I_0'(S_p^\rho)(\rho w)\big)d\Sigma_0 + \frac{1}{2}\int_{S_p^\rho}\big(I_0''(S_p^\rho)[w](\rho w)\big)d\Sigma_0 + o(|w|^2).$$

If we want to bring the expression to the standard sphere we get

$$I_0(S_p^\rho(w)) = I_0(S_p^\rho) + \int_{S^2}\big(\rho^3 I_0'(S_p^\rho)w\big)d\Sigma_0 + \frac{1}{2}\int_{S^2}\big(\rho^3 I_0''(S_p^\rho)[w]w\big)d\Sigma_0 + o(|w|^2).$$

Now we denote

$$\tilde{I}_0'(S_p^\rho(w)) = \rho^3 I_0'(S_p^\rho(w)) = \frac{1}{2}\triangle_{S^2}(\triangle_{S^2}+2)[w] + Q_p^{(2)(4)}(w)$$

and

$$\tilde{I}_0''(S_p^\rho)[w] = \rho^3 I_0''(S_p^\rho)[w] = \frac{1}{2}\triangle_{S^2}(\triangle_{S^2}+2)[w]$$

then we get the more familiar formula

$$I_0(S_p^\rho(w)) = I_0(S_p^\rho) + \int_{S^2}\big(\tilde{I}_0'(S_p^\rho)w\big) + \frac{1}{2}\int_{S^2}\big(\tilde{I}_0''(S_p^\rho)[w]w\big) + o(|w|^2).$$

This was about the functional $\int \frac{H^2}{4}$ but the same argument can be repeated for the functional $\int\big(\frac{H^2}{4}-D\big)$ (since the ambient is euclidean, $D=K$ the Gaussian curvature which by the Gauss Bonnet Theorem does not influence the differential). Since S_p^ρ are critical points for I_0 we can say that the conformal Willmore functional on perturbed standard spheres is

$$I_0(S_p^\rho(w)) = \frac{1}{4}\int_{S^2}\Big(\triangle_{S^2}(\triangle_{S^2}+2)[w]w\Big) + o(|w|^2).$$

In the following we will always denote

$$I_0'(S_p^\rho)[w] = \frac{1}{2}\triangle_{S^2}(\triangle_{S^2}+2)[w] + Q_p^{(2)(4)}(w)$$

$$I_0''(S_p^\rho)[w] = \frac{1}{2}\triangle_{S^2}(\triangle_{S^2}+2)[w]$$

since, as we saw, it is more natural.

Since from Proposition 2.2.11 we have an expansion of I_ϵ on small geodesic spheres and on the other hand the critical manifold of I_0 is made up of standard spheres, let us link the two objects. The geodesic sphere in $(\mathbb{R}^3, g_\epsilon)$ of center p and radius ρ will be denoted by $S_{p,\rho}^\epsilon$.

Lemma 2.3.2. *For small ϵ the geodesic spheres $S^\epsilon_{p,\rho}$ are normal graphs on the standard spheres S^ρ_p with a perturbation $v_\epsilon \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^3 \times S^2)$:*

$$S^\epsilon_{p,\rho} = S^\rho_p(v_\epsilon(\rho, p, .)).$$

Moreover the perturbation v_ϵ satisfies the following decreasing properties:
1) $\rho v_\epsilon = O(\epsilon)$ in C^k norm on compact subsets of $\mathbb{R}^+ \times \mathbb{R}^3 \times S^2$ for all $k \geq 0$;
2) $v_\epsilon(\rho, ., .) = O(\rho)$ as $\rho \to 0$ uniformly for $\Theta \in S^2$ and p in a compact subset of $\mathbb{R}^3$.

Proof. The geodesic spheres $S^\epsilon_{p,\rho}$ are parametrized by $\Theta \mapsto Exp_p(\rho\Theta)$. So one is interested in the solution of the geodesic equation

$$\begin{cases} \ddot{y}^i + \Gamma^i_{jk}\dot{y}^j\dot{y}^k = 0 \\ y^i(0) = p^i \\ \dot{y}^i(0) = \Theta^i \end{cases}$$

evaluated at ρ. We look for y^i of the form

$$y^i = p^i + \rho\Theta^i + \epsilon u^i + o(\epsilon)$$

where

$$u^i : \mathbb{R}^+ \times \mathbb{R}^3 \times S^2 \to \mathbb{R}, \quad (\rho, p, \Theta) \mapsto u^i(\rho, p, \Theta)$$

is $C^\infty(\mathbb{R}^+ \times \mathbb{R}^3 \times S^2)$ and have to be determined. A straightforward computation (setting $\Gamma^i_{jk} = \epsilon\tilde{\Gamma}^i_{jk}$) shows that u^i must solve the following non linear second order ODE:

$$\begin{cases} \ddot{u}^i + \tilde{\Gamma}^i_{jk}\Theta^j\Theta^k = 0 \\ u^i(0) = 0 \\ \dot{u}^i(0) = 0 \end{cases}$$

where we have denoted $\dot{u}^i = \frac{\partial}{\partial\rho}u^i$ and $\ddot{u}^i = \frac{\partial^2}{\partial\rho^2}u^i$ and the equation has to be considered at (p, Θ) fixed. Since h is compactly supported (more generally it is enough to assume that h and its first derivatives vanish at infinity), the Christoffel symbols $\tilde{\Gamma}^i_{jk}$ vanish at infinity and the ODE admits unique solution defined for all $\rho \geq 0$. From differentiable dependence on parameters, u^i is of class $C^\infty(\mathbb{R}^+ \times \mathbb{R}^3 \times S^2)$, observe also that $u^i = O(\rho^2)$ as $\rho \to 0$ uniformly for $\Theta \in S^2$ and p in a compact subset of $\mathbb{R}^3$.
It follows that the geodesic sphere $S^\epsilon_{p,\rho}$ can be obtained from the standard sphere S^ρ_p with the small variation $\epsilon u^i(\rho, p, \Theta)$. Now it is easy to see that for ϵ small enough there exists $v_\epsilon \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^3 \times S^2)$ such that

- $S^\epsilon_{p,\rho} = S^\rho_p(v_\epsilon)$
- $\rho v_\epsilon = O(\epsilon)$ in C^k norm on compact subsets of $\mathbb{R}^+ \times \mathbb{R}^3 \times S^2$ for all $k \geq 0$
- $v_\epsilon(\rho, \Theta) = O(\rho)$ as $\rho \to 0$, uniformly for $\Theta \in S^2$ and p in a compact subset of $\mathbb{R}^3$.

□

Now we define the manifold of approximate solutions that will play the role of the "critical manifold" Z. Let R_1 and R_2 be positive real numbers to be determined and χ a $C^\infty(\mathbb{R}^+)$ cut off function such that

$$\begin{cases} \chi(\rho) = 1 \text{ for } 0 \leq \rho \leq R_1 \\ 0 \leq \chi(\rho) \leq 1 \text{ for } R_1 \leq \rho \leq R_2 \\ \chi(\rho) = 0 \text{ for } \rho \geq R_2. \end{cases}$$

We denote by $\Sigma^\epsilon_{p,\rho}$ the perturbed standard sphere

$$\Sigma^\epsilon_{p,\rho} = S^\rho_p(\chi v_\epsilon) \tag{2.24}$$

and we consider it as parametrized on S^2; observe that for $\rho < R_1$ one gets the geodesic spheres $\Sigma^\epsilon_{p,\rho} = S^\epsilon_{p,\rho}$ and for $\rho > R_2$ one has the standard spheres $\Sigma^\epsilon_{p,\rho} = S^\rho_p$.
Denoted by $\mathring{N}$ the inward normal unit vector, given a function w on S^2, $\Sigma^\epsilon_{p,\rho}(w)$ will be the surface parametrized by $\Sigma^\epsilon_{p,\rho} + \rho w\mathring{N}$ (notice that we are consistent with the previous notations since Θ points outward).

At this point we can state the two Lemmas which allow us to perform the Finite Dimensional Reduction. Recall that, as always, $P : L^2(S^2) \to Ker[\triangle_{S^2}(\triangle_{S^2} + 2)]^\perp$ is the orthogonal projection.

Lemma 2.3.3. *For each compact subset $Z_c \subseteq \mathbb{R}^3 \oplus \mathbb{R}^+$, there exist $\epsilon_0 > 0$ and $r > 0$ with the following property: for all $|\epsilon| \leq \epsilon_0$ and $(p,\rho) \in Z_c$, the auxiliary equation $PI'_\epsilon(\Sigma^\epsilon_{p,\rho}(w)) = 0$ has unique solution $w = w_\epsilon(p,\rho) \in B(0,r) \subset C^{4,\alpha}(S^2)^\perp$ such that:*
1) the map $w_\epsilon(.,.) : Z_c \to C^{4,\alpha}(S^2)^\perp$ is of class C^1;
2) $\|w_\epsilon(p,\rho)\|_{C^{4,\alpha}(S^2)} \to 0$ for $\epsilon \to 0$ uniformly with respect to $(p,\rho) \in Z_c$;
3) more precisely $\|w_\epsilon(p,\rho)\|_{C^{4,\alpha}(S^2)} = O(\epsilon)$ for $\epsilon \to 0$ uniformly in $(p,\rho) \in Z_c$;
4) $\|w_\epsilon(p,\rho)\|_{C^{4,\alpha}} = O(\rho^2)$ uniformly for p in the compact set.

Proof. The proof will be rather sketchy, for more details we refer to Section 4 of [Mon1].

• $\rho \leq R_1$: Recall Lemma 2.2.10 and choose $R_1 = \rho_0$; for $\rho \leq R_1$, the surface $\Sigma^\epsilon_{p,\rho}$ coincides with the geodesic sphere $S_{p,\rho}$, so thanks to Lemma 2.2.10 there exists a unique $w_\epsilon(p,\rho) \in C^{4,\alpha}(S^2)^\perp$ which solves the auxiliary equation. During the proof of Proposition 2.2.9 we wrote I' as in equation (2.7); observing that all the curvature tensors of $(\mathbb{R}^3, g_\epsilon)$ are of order $O(\epsilon)$ (in C^k norm $\forall k \in \mathbb{N}$ on each fixed compact set of $\mathbb{R}^3$), it follows that

$$PI'_\epsilon(S^\epsilon_{p,\rho}(w_\epsilon(p,\rho))) = \frac{1}{2}\triangle_{S^\epsilon_{p,\rho}(w_\epsilon(p,\rho))}H + Q^{(2)(4)}(w_\epsilon(p,\rho)) + O(\epsilon) = 0 \quad \text{in } C^{0,\alpha}(S^2);$$

from this formula and the expansions of $\mathring{h}$, $\mathring{g}^{-1}$ and H, we have that

$$\triangle_{S^2}(\triangle_{S^2} + 2)[w_\epsilon(p,\rho)] + Q^{(2)(4)}(w_\epsilon(p,\rho)) = O(\epsilon) \quad \text{in } C^{0,\alpha}(S^2)$$

uniformly for $(p,\rho) \in Z_c$; first observe that $\|w_\epsilon\|_{C^{4,\alpha}(S^2)} \to 0$ as $\epsilon \to 0$ uniformly in Z_c so the second summand is negligible, then conclude that $\|w_\epsilon\|_{C^{4,\alpha}(S^2)} = O(\epsilon)$ uniformly on Z_c. The other properties follow from Lemma 2.2.10.

• $\rho \geq R_2$: in this case the surface $\Sigma^\epsilon_{p,\rho}$ coincides with the standard sphere S^ρ_p for which the discussion has already been done in Lemma 4.1 of [Mon1].

• $R_1 \leq \rho \leq R_2$: with a Taylor expansion the auxiliary equation becomes

$$0 = PI'_\epsilon(\Sigma^\epsilon_{p,\rho}(w_\epsilon)) = PI'_\epsilon(\Sigma^\epsilon_{p,\rho}) + PI''_\epsilon(\Sigma^\epsilon_{p,\rho})[w_\epsilon] + o(\|w_\epsilon\|_{C^{4,\alpha}(S^2)}).$$

But by definition $\Sigma^\epsilon_{p,\rho} = S^\rho_p(\chi v_\epsilon)$, so

$$I'_\epsilon(\Sigma^\epsilon_{p,\rho}) = I'_\epsilon(S^\rho_p(\chi v_\epsilon)) = I'_0(S^\rho_p) + I''_0(S^\rho_p)[\chi v_\epsilon] + O(\epsilon).$$

Since $I'_0(S^\rho_p) = 0$ and $\|v_\epsilon\|_{C^{4,\alpha}} = O(\epsilon)$ we get

$$\|I'_\epsilon(\Sigma^\epsilon_{p,\rho})\|_{C^{0,\alpha}(S^2)} = O(\epsilon).$$

Now $PI''_0(S^\rho_p) = \triangle_{S^2}(\triangle_{S^2} + 2)$ which is an invertible map $C^{4,\alpha\perp} \to C^{0,\alpha\perp}$ uniformly on Z_c; since the set of invertible operators is open, for ϵ small also $PI''_\epsilon(S^\rho_p)$ is uniformly invertible. From the fact that $\|v_\epsilon\|_{C^k(S^2)} = O(\epsilon)$ for all k it follows that also $PI''_\epsilon(\Sigma^\epsilon_{p,\rho}) = PI''_\epsilon(S^\rho_p(\chi v_\epsilon))$ is uniformly invertible on Z_c. With a fixed point argument analogous to the proof of Lemma 4.1 in [Mon1] it is possible to show that there exist $r > 0$ and a unique solution $w_\epsilon \in B(0,r) \subset C^{4,\alpha\perp}$ of

$$w_\epsilon = -PI''_\epsilon(\Sigma^\epsilon_{p,\rho})^{-1}\Big(PI'_\epsilon(\Sigma^\epsilon_{p,\rho}) + o(\|w_\epsilon\|_{C^{4,\alpha}(S^2)})\Big)$$

with the desired properties. □

Now we are in position to define the reduced functional $\Phi_\epsilon(p,\rho) = I_\epsilon(\Sigma^\epsilon_{p,\rho}(w_\epsilon(p,\rho)))$ and to state the following fundamental Lemma:

Lemma 2.3.4. *Fixed a compact set $Z_c \subseteq \mathbb{R}^3 \oplus \mathbb{R}^+$, for $|\epsilon| \leq \epsilon_0$ consider the functional $\Phi_\epsilon : Z_c \to \mathbb{R}$. Assume that, for ϵ small enough, Φ_ϵ has a critical point $(p_\epsilon, \rho_\epsilon) \in Z_c$. Then $\Sigma^\epsilon_{p_\epsilon,\rho_\epsilon}(w_\epsilon(p_\epsilon,\rho_\epsilon))$ is a critical point of I_ϵ.*

Proof. The proof is a slight modification of the proof of Lemma 4.2 in [Mon1] just using the good decay properties of v_ϵ, w_ϵ and their derivatives as $\epsilon \to 0$. □

Remark 2.3.5. *The reduced functional Φ_ϵ is defined for small ϵ once a compact $Z_c \subset \mathbb{R}^3 \oplus \mathbb{R}^+$ is fixed. In the following discussion we will study the behavior of Φ_ϵ for large ρ; this makes sense since the compact Z_c can be chosen arbitrarily large and the solution of the auxiliary equation $w_\epsilon(p,\rho)$ given in Lemma 2.3.3 is unique in a small ball of $C^{4,\alpha}(S^2)^\perp$. However the compact Z_c will be chosen in a rigorous and appropriate way in the proofs of Theorem 1.0.1 and Theorem 1.0.2.*

2.3.2 Expansion of the reduced functional $I_\epsilon(\Sigma^\epsilon_{p,\rho}(w_\epsilon(p,\rho)))$

Since Lemma 2.3.3 applies, we can perform the Finite Dimensional Reduction. In this Subsection we will study the reduced functional $\Phi_\epsilon(p,\rho) = I_\epsilon(\Sigma^\epsilon_{p,\rho}(w_\epsilon(p,\rho)))$. For $\rho < R_1$, $\Sigma^\epsilon_{p,\rho} = S^\epsilon_{p,\rho}$ so for small radius ρ we have the explicit expansion of $\Phi_\epsilon(p,\rho) = I_\epsilon(S^\epsilon_{p,\rho}(w_\epsilon(p,\rho)))$ given by Proposition 2.2.11. More generally, for all the radius we can write the conformal Willmore functional on our surfaces $\Sigma^\epsilon_{p,\rho}(w)$ as

$$I_\epsilon(\Sigma^\epsilon_{p,\rho}(w)) = I_0(\Sigma^\epsilon_{p,\rho}(w)) + \epsilon G_1(\Sigma^\epsilon_{p,\rho}(w)) + \epsilon^2 G_2(\Sigma^\epsilon_{p,\rho}(w)) + o(\epsilon^2). \tag{2.25}$$

Now let us study the case $\rho > R_2$, when $\Sigma^\epsilon_{p,\rho} = S^\rho_p$; in this circumstance we get the formula

$$I_\epsilon(S^\rho_p(w)) = I_0(S^\rho_p(w)) + \epsilon G_1(S^\rho_p(w)) + \epsilon^2 G_2(S^\rho_p(w)) + o(\epsilon^2). \tag{2.26}$$

Lemma 2.3.6. *For all standard spheres S^ρ_p one has*

$$I_0(S^\rho_p) = G_1(S^\rho_p) = 0.$$

Proof. As above, we write the functional as $I_\epsilon(S^\rho_p) = I_0(S^\rho_p) + \epsilon G_1(S^\rho_p) + o(\epsilon)$. First let us expand in ϵ the geometric quantities of interest starting from the area form $d\Sigma_\epsilon := \sqrt{E_\epsilon G_\epsilon - F_\epsilon^2}$.

$$\begin{aligned} E_\epsilon &= g_\epsilon(\theta_1,\theta_1) = (\theta_1,\theta_1) + \epsilon h(\theta_1,\theta_1) = E_0 + \epsilon h(\theta_1,\theta_1) \\ F_\epsilon &= F_0 + \epsilon h(\theta_1,\theta_2) = \epsilon h(\theta_1,\theta_2) \\ G_\epsilon &= G_0 + \epsilon h(\theta_2,\theta_2), \end{aligned}$$

where $(.,.)$ denotes the euclidean scalar product and E_0, F_0, G_0 are the coefficients of the first fundamental form in euclidean metric. The area form can be expanded as

$$\begin{aligned} d\Sigma_\epsilon &:= \sqrt{E_\epsilon G_\epsilon - F_\epsilon^2} \\ &= \sqrt{E_0G_0 + \epsilon\big(E_0 h(\theta_2,\theta_2) + G_0 h(\theta_1,\theta_1)\big) + o(\epsilon)}, \end{aligned}$$

where the remainder $o(\epsilon)$ is uniform fixed the compact set in the variables $(p,\rho), \rho > 0$.
Using the standard Taylor expansion $\sqrt{a+bx+cx^2} = \sqrt{a} + \frac{1}{2}\frac{b}{\sqrt{a}}x + o(x)$, we get

$$\sqrt{E_\epsilon G_\epsilon - F_\epsilon^2} = \sqrt{E_0G_0} + \frac{\epsilon}{2}\frac{E_0 h(\theta_2,\theta_2) + G_0 h(\theta_1,\theta_1)}{\sqrt{E_0G_0}} + o(\epsilon), \tag{2.27}$$

where the remainder $o(\epsilon)$ is uniform fixed the compact set in the variables (p,ρ).

Now let us expand the second fundamental form.
First of all we have to find an expression of the inward normal unit vector ν_ϵ on S^ρ_p in metric g_ϵ.
We look for ν_ϵ of the form

$$\nu_\epsilon = \nu_0 + \epsilon N + o(\epsilon)$$

where $\nu_0 = -\Theta$ is the inward normal unit vector on S^ρ_p in euclidean metric and the remainder is $o(\epsilon)$ uniformly fixed the compact in (p,ρ). From the orthogonality conditions $g_\epsilon(\theta_1,\nu_\epsilon) = 0$ and $g_\epsilon(\theta_2,\nu_\epsilon) = 0$, we get

$$\begin{aligned} 0 &= g_\epsilon(\theta_1,\nu_\epsilon) = (\theta_1,\nu_0) + \epsilon(\theta_1,N) + \epsilon h(\theta_1,\nu_0) + o(\epsilon) \\ 0 &= g_\epsilon(\theta_2,\nu_\epsilon) = (\theta_2,\nu_0) + \epsilon(\theta_2,N) + \epsilon h(\theta_2,\nu_0) + o(\epsilon) \end{aligned}$$

from which, being ν_0 the euclidean normal vector to S^ρ_p,

$$(N,\theta_1) = -h(\nu_0,\theta_1) \tag{2.28}$$
$$(N,\theta_2) = -h(\nu_0,\theta_2). \tag{2.29}$$

Imposing the normalization condition on ν_ϵ we obtain

$$1 = g_\epsilon(\nu_\epsilon,\nu_\epsilon) = (\nu_0,\nu_0) + 2\epsilon(\nu_0,N) + \epsilon h(\nu_0,\nu_0) + o(\epsilon)$$

from which, being $(\nu_0,\nu_0) = 1$

$$(N,\nu_0) = -\frac{1}{2}h(\nu_0,\nu_0). \tag{2.30}$$

Denote by $\bar{\theta}_i = \frac{\theta_i}{|\theta_i|}$ the normalized tangent vectors; since $(\bar{\theta}_1, \bar{\theta}_2, \nu_0)$ are an orthonormal base, the expressions (2.28),(2.29),(2.30) characterize univocally N, which can be written in this base as

$$N = -h(\nu_0, \bar{\theta}_1)\bar{\theta}_1 - h(\nu_0, \bar{\theta}_2)\bar{\theta}_2 - \frac{1}{2}h(\nu_0, \nu_0)\nu_0. \tag{2.31}$$

Knowing the normal vector we can evaluate the coefficients of the second fundamental form

$$\mathring{h}_{\epsilon ij} := -g_\epsilon(\nabla_{\theta_i}\nu_\epsilon, \theta_j),$$

where ∇ is the connection on $\mathbb{R}^3$ endowed with the metric g_ϵ. By linearity, denoting with $\frac{\partial}{\partial x^\lambda}$ the standard euclidean frame of $\mathbb{R}^3$

$$\nabla_{\theta_i}\nu_\epsilon \;=\; \theta_i^\mu \nabla_\mu(\nu_\epsilon^\lambda \frac{\partial}{\partial x^\lambda}) = \frac{\partial \nu_\epsilon}{\partial \theta^i} + \theta_i^\mu \nu_\epsilon^\lambda \Gamma_{\mu\lambda}^\nu \frac{\partial}{\partial x^\nu}$$

where $\Gamma^\nu_{\mu\lambda}$ are the Christoffel symbols of $(\mathbb{R}^3, g_\epsilon)$.
Let us find an expansion in ϵ of $\Gamma^\nu_{\mu\lambda}$. By definition

$$\Gamma^\nu_{\mu\lambda} = \frac{1}{2}g^{\nu\sigma}[D_\mu g_{\lambda\sigma} + D_\lambda g_{\sigma\mu} - D_\sigma g_{\mu\lambda}].$$

Noticing that $g^{\mu\sigma} = \delta^{\mu\sigma} - \epsilon h_{\mu\sigma} + o(\epsilon)$ and $D_\mu g_{\lambda\sigma} = \epsilon D_\mu h_{\lambda\sigma}$, we obtain

$$\begin{aligned} \Gamma^\nu_{\mu\lambda} &= \frac{1}{2}\epsilon\delta^{\nu\sigma}[D_\mu h_{\lambda\sigma} + D_\lambda h_{\sigma\mu} - D_\sigma h_{\mu\lambda}] + o(\epsilon) \\ &= \frac{1}{2}\epsilon\delta^{\nu\sigma}A_{\mu\sigma\lambda} \end{aligned} \tag{2.32}$$

where we set

$$A_{\mu\nu\lambda} := [D_\mu h_{\lambda\nu} + D_\lambda h_{\nu\mu} - D_\nu h_{\mu\lambda}]. \tag{2.33}$$

Hence

$$\nabla_{\theta_i}\nu_\epsilon = \frac{\partial \nu_\epsilon}{\partial \theta^i} + \frac{1}{2}\epsilon\theta_i^\mu \nu_0^\lambda \delta^{\nu\sigma}A_{\mu\sigma\lambda}\frac{\partial}{\partial x^\nu} + o(\epsilon)$$

and the second fundamental form becomes

$$\mathring{h}_{\epsilon ij} = -\left(\frac{\partial \nu_0}{\partial \theta^i}, \theta_j\right) - \epsilon\left[h\left(\frac{\partial \nu_0}{\partial \theta^i}, \theta_j\right) + \left(\frac{\partial N}{\partial \theta^i}, \theta_j\right)\right] - \frac{1}{2}\epsilon\theta_i^\mu\theta_j^\nu\nu_0^\lambda A_{\mu\nu\lambda}. \tag{2.34}$$

In order to simplify the expressions let us recall the values of the coefficients of the unperturbed first fundamental form

$$\begin{aligned} E_0 &= \rho^2 \\ F_0 &= 0 \\ G_0 &= \rho^2\sin^2\theta^1, \end{aligned}$$

those of the unperturbed second fundamental form (following the classical notation of the theory of surfaces, we denote by l_0, m_0, n_0 the quantities $\mathring{h}_{0_{11}}, \mathring{h}_{0_{12}}, \mathring{h}_{0_{22}}$)

$$\begin{aligned} l_0 &= \rho \\ m_0 &= 0 \\ n_0 &= \rho\sin^2\theta^1 \end{aligned}$$

and the unperturbed mean curvature and Gaussian curvature

$$H_0 = \frac{2}{\rho}$$

$$D_0 = \frac{1}{\rho^2}.$$

From formula (22) in the proof of Lemma 3.4 in [Mon1] and the above expressions of the unperturbed quantities we have immediately that

$$\begin{aligned}\int_{S_p^\rho}\frac{H_\epsilon^2}{4}d\Sigma_\epsilon &= 4\pi-\frac{1}{2}\epsilon\int_{S^2}\Big[h(\bar\theta_1,\bar\theta_1)+h(\bar\theta_2,\bar\theta_2)\Big]d\Sigma_0-\frac{1}{2}\epsilon\rho\int_{S^2}\Big[(\bar\theta_2^\mu\bar\theta_2^\nu+\bar\theta_1^\mu\bar\theta_1^\nu)\nu_0^\lambda A_{\mu\nu\lambda}\Big]d\Sigma_0\\ &-\epsilon\int_{(0,\pi)\times(0,2\pi)}\left[h\left(\frac{\partial\nu_0}{\partial\theta^2},\bar\theta_2\right)+\left(\frac{\partial N}{\partial\theta^2},\bar\theta_2\right)\right]d\theta^1d\theta^2\\ &-\epsilon\int_{S^2}\left[h\left(\frac{\partial\nu_0}{\partial\theta^1},\bar\theta_1\right)+\left(\frac{\partial N}{\partial\theta^1},\bar\theta_1\right)\right]d\Sigma_0+o(\epsilon).\end{aligned}\tag{2.35}$$

Now we have to compute $\int_{S_p^\rho}D_\epsilon d\Sigma_\epsilon$. Knowing the first and the second fundamental forms we can evaluate $D_\epsilon:=\frac{\det\mathring{h}_\epsilon}{\det\mathring{g}_\epsilon}$, in fact observing that

$$\begin{aligned}\det\mathring{h}_\epsilon &= \det\mathring{h}_0-\epsilon n_0\left[h\left(\frac{\partial\nu_0}{\partial\theta^1},\theta_1\right)+\left(\frac{\partial N}{\partial\theta^1},\theta_1\right)\right]-\frac{1}{2}\epsilon n_0\theta_1^\mu\theta_1^\nu\nu_0^\lambda A_{\mu\nu\lambda}+\\ &-\epsilon l_0\left[h\left(\frac{\partial\nu_0}{\partial\theta^2},\theta_2\right)+\left(\frac{\partial N}{\partial\theta^2},\theta_2\right)\right]-\frac{1}{2}\epsilon l_0\theta_2^\mu\theta_2^\nu\nu_0^\lambda A_{\mu\nu\lambda}+o(\epsilon)\end{aligned}\tag{2.36}$$

and that

$$\det\mathring{g}_\epsilon=\det\mathring{g}_0+\epsilon E_0h(\theta_2,\theta_2)+\epsilon G_0h(\theta_1,\theta_1),$$

using the Taylor expansion $\frac{1}{a+\epsilon b+o(\epsilon)}=\frac{1}{a}-\epsilon\frac{b}{a^2}+o(\epsilon)$, we get

$$\begin{aligned}D_\epsilon &= D_0-\epsilon\frac{n_0\left[h\left(\frac{\partial\nu_0}{\partial\theta^1},\theta_1\right)+\left(\frac{\partial N}{\partial\theta^1},\theta_1\right)\right]+\frac{1}{2}n_0\theta_1^\mu\theta_1^\nu\nu_0^\lambda A_{\mu\nu\lambda}+l_0\left[h\left(\frac{\partial\nu_0}{\partial\theta^2},\theta_2\right)+\left(\frac{\partial N}{\partial\theta^2},\theta_2\right)\right]+\frac{1}{2}l_0\theta_2^\mu\theta_2^\nu\nu_0^\lambda A_{\mu\nu\lambda}}{E_0G_0}\\ &-\epsilon\frac{\big[E_0h(\theta_2,\theta_2)+G_0h(\theta_1,\theta_1)\big]\det\mathring{h}_0}{(E_0G_0)^2}+o(\epsilon).\end{aligned}$$

Recalling (2.27) we obtain

$$\begin{aligned}\int_{S_p^\rho}D_\epsilon d\Sigma_\epsilon &= \int_{S_p^\rho}D_0d\Sigma_0+\frac{\epsilon}{2}\int_{(0,\pi)\times(0,2\pi)}D_0\left\{\frac{E_0h(\theta_2,\theta_2)+G_0h(\theta_1,\theta_1)}{\sqrt{E_0G_0}}\right\}d\theta^1d\theta^2\\ &-\epsilon\int_{(0,\pi)\times(0,2\pi)}\left\{\frac{n_0\left[h\left(\frac{\partial\nu_0}{\partial\theta^1},\theta_1\right)+\left(\frac{\partial N}{\partial\theta^1},\theta_1\right)\right]+\frac{1}{2}n_0\theta_1^\mu\theta_1^\nu\nu_0^\lambda A_{\mu\nu\lambda}}{\sqrt{E_0G_0}}\right\}d\theta^1d\theta^2\\ &-\epsilon\int_{(0,\pi)\times(0,2\pi)}\left\{\frac{l_0\left[h\left(\frac{\partial\nu_0}{\partial\theta^2},\theta_2\right)+\left(\frac{\partial N}{\partial\theta^2},\theta_2\right)\right]}{\sqrt{E_0G_0}}\right\}d\theta^1d\theta^2\\ &-\epsilon\int_{(0,\pi)\times(0,2\pi)}\left\{\frac{\frac{1}{2}l_0\theta_2^\mu\theta_2^\nu\nu_0^\lambda A_{\mu\nu\lambda}}{\sqrt{E_0G_0}}+\frac{\big[E_0h(\theta_2,\theta_2)+G_0h(\theta_1,\theta_1)\big]\det\mathring{h}_0}{(E_0G_0)^{3/2}}\right\}d\theta^1d\theta^2+o(\epsilon).\end{aligned}\tag{2.37}$$

Plugging the unperturbed quantities into (2.37), after some easy computations we get

$$\begin{aligned}\int_{S_p^\rho}D_\epsilon d\Sigma_\epsilon &= 4\pi-\frac{1}{2}\epsilon\int_{S^2}\left[h(\bar\theta_2,\bar\theta_2)+h(\bar\theta_1,\bar\theta_1)\right]d\Sigma_0-\epsilon\int_{(0,\pi)\times(0,2\pi)}\left[h\left(\frac{\partial\nu_0}{\partial\theta^2},\bar\theta_2\right)+\left(\frac{\partial N}{\partial\theta^2},\bar\theta_2\right)\right]d\theta^1d\theta^2\\ &-\epsilon\int_{S^2}\left\{\left[h\left(\frac{\partial\nu_0}{\partial\theta^1},\bar\theta_1\right)+\left(\frac{\partial N}{\partial\theta^1},\bar\theta_1\right)\right]+\frac{\rho}{2}[\bar\theta_1^\mu\bar\theta_1^\nu\nu_0^\lambda A_{\mu\nu\lambda}+\bar\theta_2^\mu\bar\theta_2^\nu\nu_0^\lambda A_{\mu\nu\lambda}]\right\}d\Sigma_0+o(\epsilon).\end{aligned}\tag{2.38}$$

Comparing the integrals (2.35) and (2.38) we see that all terms cancel out and we can conclude that

$$\int_{S_p^\rho}\Big[\frac{H_\epsilon^2}{4}-D_\epsilon\Big]d\Sigma_\epsilon=o(\epsilon).$$

□

In the following Lemma we find the expansion of the reduced functional Φ_ϵ in terms of I_0, G_1, G_2 and their derivatives. Recall the notation introduced in Remark 2.3.1 about I_0' and I_0'' and the definition of R_2 given in the Subsection 2.3.1 after Lemma 2.3.2.

Lemma 2.3.7. *For $\rho > R_2$ the reduced functional has the following expression:*

$$\Phi_\epsilon = \epsilon^2\bigg(G_2(S_p^\rho) - \frac{1}{2}\int_{S^2}\Big[G_1'(S_p^\rho)\big(I_0''(S_p^\rho)^{-1}\big[G_1'(S_p^\rho)\big]\big)\Big]d\Sigma_0\bigg) + o(\epsilon^2).$$

Proof. With a Taylor expansion in ϵ, w and recalling that $\|w\|_{C^{4,\alpha}} = O(\epsilon)$ (see Lemma 2.3.3), we have

$$\begin{aligned} I_\epsilon'(S_p^\rho(w)) &= I_0'(S_p^\rho(w)) + \epsilon G_1'(S_p^\rho(w)) + o(\epsilon) \\ &= I_0'(S_p^\rho) + I_0''(S_p^\rho)[w] + \epsilon G_1'(S_p^\rho) + o(\epsilon). \end{aligned}$$

Since $I_0'(S_p^\rho) = 0$ and w satisfies the auxiliary equation $PI_\epsilon'(S_p^\rho(w)) = 0$, we must have

$$w = -\epsilon I_0''(S_p^\rho)^{-1}[PG_1'(S_p^\rho)].$$

Observe that from $G_1(S_p^\rho) \equiv 0\ \forall p, \rho$ it follows that $G_1'(S_p^\rho) \in Ker[\triangle_{S^2}(\triangle_{S^2}+2)]^\perp$, so $PG_1'(S_p^\rho) = G_1'(S_p^\rho)$. Hence, recalling that $I_0(S_p^\rho) = 0, I_0'(S_p^\rho) = 0, G_1(S_p^\rho) = 0$ we have

$$\begin{aligned} I_\epsilon(S_p^\rho(w)) &= I_0(S_p^\rho(w)) + \epsilon G_1(S_p^\rho(w)) + \epsilon^2 G_2(S_p^\rho(w)) + o(\epsilon^2) \\ &= \frac{1}{2}\int_{S^2}\Big[I_0''(S_p^\rho)[w] \quad w\Big]d\Sigma_0 + \epsilon\int_{S^2}\big[G_1'(S_p^\rho) \quad w\big]d\Sigma_0 + \epsilon^2 G_2(S_p^\rho) + o(\epsilon^2) \\ &= -\frac{1}{2}\epsilon^2\int_{S^2}\Big[G_1'(S_p^\rho) \quad I_0''(S_p^\rho)^{-1}[G_1'(S_p^\rho)]\Big]d\Sigma_0 + \epsilon^2 G_2(S_p^\rho) + o(\epsilon^2). \end{aligned}$$

□

Now we want to estimate the quantities $G_1'(S_p^\rho)$ and $G_2(S_p^\rho)$ appearing in the expression of the reduced functional.

Lemma 2.3.8. *Writing the conformal Willmore functional on perturbed standard spheres as in* (2.26)*, we get the following expressions for the differential of G_1 and for G_2 evaluated on S_p^ρ:*

$$G_1'(S_p^\rho) = L(h) + (1+\rho)\big[L(Dh) + L(D^2h) + L(D^3h)\big]$$

$$G_2(S_p^\rho) = \int_{S_p^\rho}\Big[\frac{1}{\rho^2}L(h)L(Dh) + \frac{1}{\rho}L(h)L(Dh) + \frac{1}{\rho^2}(Q(h) + Q(Dh)) + \frac{1}{\rho}Q(Dh) + Q(Dh)\Big]$$

where $L(.)$ and $Q(.)$ denote a generic linear (respectively quadratic) function in the entries of the matrix argument with smooth coefficients on S^2 which can change from formula to formula and also in the same formula.

Proof. To get the expression of the desired quantities we compute the expansion of $I_\epsilon(S_p^\rho)$ at second order in ϵ and first order in w. In the intention of simplifying the notation, we will omit the remainder terms in the expansions. During the proof we use $L(.)$ and $Q(.)$ to denote a generic linear (respectively quadratic) in the components real, vector or matrix-valued function, with real, vector or matrix argument and with smooth coefficients on S^2. The letter a will denote a smooth real, vector or matrix-valued function on S^2. L, Q and a can change from formula to formula and also in the same formula.

Let us start with the expansion. Observe that S_p^ρ is parametrized by $p + \rho(1-w)\Theta$ so the tangent vectors are

$$Z_i = \rho(1-w)\Theta_i - \rho w_i\Theta = \rho(a + L(w) + L(Dw)).$$

The first fundamental form on S_p^ρ is

$$\mathring{g}_{ij} = g_\epsilon(Z_i, Z_j) = (Z_i, Z_j) + \epsilon h(Z_i, Z_j) = \rho^2\Big[a + L(w) + L(Dw) + \epsilon L(h)(a + L(w) + L(Dw))\Big]$$

and

$$\det\mathring{g} = \rho^4\Big[a + L(w) + L(Dw) + \epsilon L(h)(a + L(w) + L(Dw)) + \epsilon^2 Q(h)\Big]$$

$$\sqrt{\det \mathring{g}} = \rho^2\Big[a + L(w) + L(Dw) + \epsilon L(h)(a + L(w) + L(Dw)) + \epsilon^2 Q(h)\Big];$$

it is easy to see that the inverse of metric is

$$\mathring{g}^{ij} = \frac{1}{\rho^2}\Big[a + L(w) + L(Dw) + \epsilon L(h)(a + L(w) + L(Dw)) + \epsilon^2 Q(h)\Big].$$

The normal versor ν_ϵ has to satisfy the three following equations:

$$0 = g_\epsilon(\nu_\epsilon, Z_i) = (\nu_\epsilon, Z_i) + \epsilon h(\nu_\epsilon, Z_i) = \nu_\epsilon(1 + \epsilon L(h))\Big(a + L(w) + L(Dw)\Big)$$

$$1 = g_\epsilon(\nu_\epsilon, \nu_\epsilon).$$

Hence, just solving the linear system given by the first two conditions and plugging in the third one, we realize that

$$\nu_\epsilon = a + L(w) + L(Dw) + \epsilon L(h)(a + L(w) + L(Dw)) + \epsilon^2 Q(h).$$

In order to compute the second fundamental form $\mathring{h}_\epsilon = -g_\epsilon(\nabla_{Z_i}\nu_\epsilon, Z_j)$ recall that

$$\nabla_{Z_i}\nu_\epsilon = \frac{\partial \nu_\epsilon}{\partial \theta^i} + Z_i^\mu \nu_\epsilon^\lambda \Gamma^\nu_{\mu\lambda}\frac{\partial}{\partial x^\nu}$$

and that

$$\Gamma^\nu_{\mu\lambda} = \frac{1}{2}\epsilon\delta^{\nu\sigma}[D_\mu h_{\lambda\sigma} + D_\lambda h_{\sigma\mu} - D_\sigma h_{\mu\lambda}] = \epsilon L(Dh),$$

so the covariant derivative of ν_ϵ can be written as

$$\begin{aligned}\nabla_{Z_i}\nu_\epsilon &= a + L(w) + L(Dw) + L(D^2w) + \epsilon L(Dh)(a + L(w) + L(Dw)) + \epsilon L(h)(a + L(w) + L(Dw) + L(D^2w))\\ &\quad + \epsilon\rho L(Dh)\big(a + L(w) + L(Dw)\big) + \epsilon^2(1+\rho)L(h)L(Dh)\end{aligned}$$

and the second fundamental form becomes

$$\begin{aligned}\mathring{h}_\epsilon &= \rho\Big[a + L(w) + L(Dw) + L(D^2w) + \epsilon L(Dh)(a + L(w) + L(Dw)) + \epsilon L(h)(a + L(w) + L(Dw) + L(D^2w))\Big]\\ &\quad + \epsilon\rho^2 L(Dh)\big(a + L(w) + L(Dw)\big) + \epsilon^2\rho(1+\rho)L(h)L(Dh) + \epsilon^2\rho Q(h).\end{aligned}$$

Using the previous formulas now we are in position to estimate H, H^2 and D. With some easy computations one gets

$$\begin{aligned}H &= \frac{1}{\rho}\Big[a + L(w) + L(Dw) + L(D^2w) + \epsilon L(Dh)(a + L(w) + L(Dw)) + \epsilon L(h)(a + L(w) + L(Dw) + L(D^2w))\Big]\\ &\quad + \epsilon L(Dh)\big(a + L(w) + L(Dw)\big) + \epsilon^2\frac{1}{\rho}(1+\rho)L(h)L(Dh) + \epsilon^2\frac{1}{\rho}Q(h).\\ H^2 &= \frac{1}{\rho^2}\Big[a + L(w) + L(Dw) + L(D^2w) + \epsilon(L(h) + L(Dh) + \rho L(Dh))(a + L(w) + L(Dw) + L(D^2w))\Big]\\ &\quad + \epsilon^2\frac{1}{\rho^2}(1+\rho)L(h)L(Dh) + \frac{\epsilon^2}{\rho^2}(Q(h) + Q(Dh)) + \frac{\epsilon^2}{\rho}L(Dh)(L(h) + L(Dh)) + \epsilon^2 Q(Dh)\\ \det\mathring{h} &= \rho^2\Big[a + L(w) + L(Dw) + L(D^2w) + \epsilon(L(h) + L(Dh) + \rho L(Dh))(a + L(w) + L(Dw) + L(D^2w))\Big]\\ &\quad + \epsilon^2\rho^2(1+\rho)L(h)L(Dh) + \epsilon^2\rho^2(Q(h) + Q(Dh)) + \epsilon^2\rho^3(1+\rho)Q(Dh)\\ D &= \frac{\det\mathring{h}}{\det\mathring{g}} = \frac{1}{\rho^2}\Big[a + L(w) + L(Dw) + L(D^2w) + \epsilon(L(h) + L(Dh) + \rho L(Dh))(a + L(w) + L(Dw) + L(D^2w))\Big]\\ &\quad + \frac{\epsilon^2}{\rho^2}(1+\rho)L(h)L(Dh) + \frac{\epsilon^2}{\rho^2}(Q(h) + Q(Dh)) + \epsilon^2\frac{1}{\rho}(1+\rho)Q(Dh)\end{aligned}$$

Now we can compute $I_\epsilon(S^\rho_p(w)) = I_0(S^\rho_p(w)) + \epsilon G_1(S^\rho_p(w)) + \epsilon^2 G_2(S^\rho_p(w))$ at the second order in ϵ and first order in w:

$$\begin{aligned}I_\epsilon(S^\rho_p)) &= \int_{S^\rho_p}\left[\frac{H^2}{4} - D\right]d\Sigma_0 = \int_{S^2}\big[a + L(w) + L(Dw) + L(D^2w)\big]d\Sigma_0\\ &\quad + \epsilon\int_{S^2}\Big[\big(L(h) + L(Dh) + \rho L(Dh)\big)\big(a + L(w) + L(Dw) + L(D^2w)\big)\Big]d\Sigma_0\\ &\quad + \epsilon^2\int_{S^2}\Big[(1+\rho)L(h)L(Dh) + \rho L(Dh)(L(h) + L(Dh)) + Q(h) + Q(Dh) + \rho Q(Dh) + \rho^2 Q(Dh)\Big]d\Sigma_0.\end{aligned}$$

So $G_1(S_p^\rho(w)) = \int_{S^2}[(L(h)+L(Dh)+\rho L(Dh))\big(a+L(w)+L(Dw)+L(D^2w)\big)]$, but also

$$G_1(S_p^\rho(w)) = G_1(S_p^\rho) + \int_{S^2} G_1'(S_p^\rho) w d\Sigma_0;$$

with an integration by parts we get the first variation

$$\int_{S^2} G_1'(S_p^\rho)w = \int_{S^2}\Big[\Big(L(h)+(1+\rho)\big(L(Dh)+L(D^2h)+L(D^3h)\big)\Big)w\Big]d\Sigma_0,$$

then the differential of G_1 at S_p^ρ is

$$G_1'(S_p^\rho) = L(h)+(1+\rho)\big[L(Dh)+L(D^2h)+L(D^3h)\big].$$

Finally observe that

$$G_2(S_p^\rho) = \int_{S_p^\rho}\Big[\frac{1+\rho}{\rho^2}L(h)L(Dh)+\frac{1}{\rho}L(Dh)(L(h)+L(Dh))+\frac{1}{\rho^2}(Q(h)+Q(Dh))+\frac{1}{\rho}Q(Dh)+Q(Dh)\Big]d\Sigma_0$$

$\square$

2.3.3 Proof of the existence Theorems

In order to get existence of critical points we study the reduced functional $\Phi_\epsilon : \mathbb{R}^3 \oplus \mathbb{R}^+ \to \mathbb{R}$. Since for small radius ρ, the reduced functional coincides with the conformal Willmore functional evaluated on the perturbed geodesic spheres $S^\epsilon_{p,\rho}(w_\epsilon(p,\rho))$ obtained in Lemma 2.2.10, then we know the expansion of Φ_ϵ for small radius from Proposition 2.2.11. Now, using the expression of the reduced functional for large radius given in Lemma 2.3.7 and the estimates of Lemma 2.3.8, we are able to bound $\Phi_\epsilon(p,\rho)$ for large radius. This is done in the following Lemma:

Lemma 2.3.9. *Let $h_{\mu\nu} \in C_0^\infty(\mathbb{R}^3)$ a symmetric bilinear form with compact support (it is enough that h and its first derivatives decrease fast at infinity) and let $c \in \mathbb{R}$ such that*

$$c := \sup\{\|h_{\mu\nu}\|_{H^1(\pi)} : \pi \text{ is an affine plane in } \mathbb{R}^3,\ \mu,\nu = 1,2,3\}.$$

Then there exists a constant $C_c > 0$ depending on c and $R_3 > 0$ such that for all $\rho > R_3$

$$|\Phi_\epsilon(p,\rho)| < \epsilon^2 C_c.$$

Moreover one has that $\forall \eta > 0$ there exist $\delta > 0$ small enough and $R_4 \geq 0$ large enough such that for $c < \delta$ and $\rho > R_4$

$$|\Phi_\epsilon(p,\rho)| < \eta\epsilon^2.$$

Proof. For simplicity the proof of the Lemma is done in the case $h \in C_0^\infty$. Using the notations established in Remark 2.3.1, from Lemma 2.3.7 and Lemma 2.3.8 we can write the reduced functional as

$$\begin{aligned}
\Phi_\epsilon(p,\rho) &= \epsilon^2\Bigg(G_2(S_p^\rho) - \frac{1}{2}\int_{S^2}\Big[G_1'(S_p^\rho)((I_0''(S_p^\rho))^{-1}\big[G_1'(S_p^\rho)\big])\Big]\Bigg) + o(\epsilon^2) \\
&= \epsilon^2\int_{S_p^\rho}\Big[\frac{1}{\rho^2}L(h)L(Dh)+\frac{1}{\rho}L(h)L(Dh)+\frac{1}{\rho^2}(Q(h)+Q(Dh))+\frac{1}{\rho}Q(Dh)+Q(Dh)\Big]d\Sigma_0 \\
&\quad +\epsilon^2\int_{S_p^\rho}\frac{1}{\rho^2}\Big[L(h)+(1+\rho)(L(Dh)+L(D^2h)+L(D^3h))\times \\
&\qquad \times(\triangle_{S^2}(\triangle_{S^2}+2))^{-1}\big[L(h)+(1+\rho)(L(Dh)+L(D^2h)+L(D^3h))\big]\Big]d\Sigma_0.
\end{aligned}$$

Now denote $K = supp(h)$ which is a compact subset of $\mathbb{R}^3$; of course in the formula above the domain of integration can be replaced with $S_p^\rho \cap K$.

Observe that for all $\sigma > 0$ there exists $R > 0$ with the following property:
for all standard spheres S_p^ρ with radius $\rho > R$ there exists an affine plane $\pi \subset \mathbb{R}^3$ such that

$$\|h\|^2_{H^1(S_p^\rho\cap K)} < \|h\|^2_{H^1(\pi\cap K)} + \sigma. \tag{2.39}$$

This is simply because one can approximate (in C^k norm for all $k \in \mathbb{N}$) the portion of standard sphere $S^\rho_p \cap K$ with a portion of an affine plane π provided that the radius ρ is large enough.

So the first integral can be bounded by a constant times $\|h\|^2_{H^1(\pi\cap K)} + \sigma$. Using the standard elliptic regularity estimates and integration by parts also the second integral can be bounded with a constant times $\|h\|^2_{H^1(\pi\cap K)} + \sigma$.

Hence for all $\sigma > 0$ there exists $R > 0$ and $\tilde{C} > 0$ such that for all (p, ρ) with $\rho > R$, there exists an affine plane π such that

$$|\Phi_\epsilon(p, \rho)| < \epsilon^2 \tilde{C}(\|h\|^2_{H^1(\pi\cap K)} + \sigma).$$

Notice that $\tilde{C}$ depends on the structure of the functions $L(.)$ and $Q(.)$ but is uniform in (p, ρ), R and σ as above. Recalling the definition of c we get:

For all $\sigma > 0$ there exists $R > 0$ such that for all (p, ρ) with $\rho > R$,

$$|\Phi_\epsilon(p, \rho)| < \epsilon^2 \tilde{C}(c^2 + \sigma).$$

Clearly setting $\sigma = 1$, $R_3 = R$ and $C_c = \tilde{C}(c^2 + 1)$ one obtains the first part of the thesis. For the second part we have to show that for all $\eta > 0$ there exist $\delta > 0$ and $R_4 > 0$ such that if $c < \delta$ then for all (p, ρ) with $\rho > R_4$ one has $\Phi_\epsilon(p, \rho) < \epsilon^2 \eta$; but this is true setting above $\delta^2 = \sigma = \frac{\eta}{2\tilde{C}}$ and $R_4 = R$ associated to σ as before (observe that the estimate is uniform in p). $\square$

Now we are in position to prove the main results of the Chapter.

Proof of Theorem 1.0.1 In order to show the Theorem, by Lemma 2.3.4, it is enough to prove that Φ_ϵ has a critical point.

Observe that for $\rho < R_1$

$$\Phi_\epsilon(p, \rho) = I_\epsilon(S^\epsilon_{p,\rho}(w_\epsilon)) = O(\rho^4),$$

so Φ_ϵ can be extended to a C^1 function up to $\rho = 0$ just putting $\Phi_\epsilon(p, 0) = 0$ for all $p \in \mathbb{R}^3$.

Let R_3 and C_c be as in Lemma 2.3.9. Since h has compact support, there exists a $R > 0$ such that for $|p| \geq R$ and $\rho \leq R_3$, $S^\rho_p \cap supp(h) = \emptyset$.

In order to apply the Finite Dimensional Reduction, we have to fix a compact $Z_c \subset \mathbb{R}^3 \oplus \mathbb{R}^+$. Let us choose it as

$$Z_c := \{(p, \rho) : |p| \leq R, 0 \leq \rho \leq R_3\}.$$

Apply Lemma 2.3.3 to the compact Z_c and observe that on the boundary ∂Z_c we have:

- $\rho = 0$: $\Phi_\epsilon = 0$.
- $|p| = R$: $\Phi_\epsilon = 0$. In fact for $|p| = R$ the standard sphere S^ρ_p does not intersect the support of h, so $\Sigma^\epsilon_{p,\rho} = S^\rho_p$ for all the radius $0 \leq \rho \leq R_3$; since the solution of the auxiliary equation $PI'_\epsilon(\Sigma^\epsilon_{p,\rho}(w_\epsilon)) = 0$ is unique for w_ϵ small enough and since S^ρ_p is already a critical point for $I_\epsilon (= I_0$ since $S^\rho_p \cap supp(h) = \emptyset)$ it follows that $\Sigma^\epsilon_{p,\rho}(w_\epsilon) = S^\rho_p$, hence

$$\Phi_\epsilon(p, \rho) = I_\epsilon(\Sigma^\epsilon_{p,\rho}(w_\epsilon)) = I_\epsilon(S^\rho_p) = I_0(S^\rho_p) = 0.$$

-$\rho = R_3$: from Lemma 2.3.9 we have that $|\Phi_\epsilon| < \epsilon^2 C_c$.

Now observe that $\Phi_\epsilon = O(\epsilon^2)$ uniformly on Z_c:
from the definition of reduced functional, with a Taylor expansion one gets

$$\Phi_\epsilon(p, \rho) = I_\epsilon(\Sigma^\epsilon_{p,\rho}(w_\epsilon)) = I'_\epsilon(\Sigma^\epsilon_{p,\rho})[w_\epsilon] + O(\|w_\epsilon\|^2),$$

but $\|w_\epsilon\|_{C^{4,\alpha}(S^2)} = O(\epsilon)$ and $\|v_\epsilon\|_{C^{4,\alpha}(S^2)} = O(\epsilon)$ uniformly for $(p, \rho) \in Z_c$, so

$$I'_\epsilon(\Sigma^\epsilon_{p,\rho}) = I'_\epsilon(S^\rho_p(v_\epsilon)) = I''_0(S^\rho_p)[v_\epsilon] + \epsilon G'_1(S^\rho_p) + o(\epsilon) = O(\epsilon)$$

hence $\Phi_\epsilon = O(\epsilon^2)$ uniformly on Z_c.

At this moment we know that Φ_ϵ is of order $O(\epsilon^2)$ uniformly on Z_c and we know its behavior on the boundary ∂Z_c.

Now we are going to use the expansion for small radius computed in Proposition 2.2.11. Recall that for $\rho < R_1$, $\Phi_\epsilon(p, \rho) = I_\epsilon(S^\epsilon_{p,\rho}(w_\epsilon(p, \rho)))$ and from Proposition 2.2.11 we have the expansion:

$$\Phi_\epsilon(p, \rho) = \frac{\pi}{5}\|S_p\|^2\rho^4 + O(\epsilon^2)O_p(\rho^5).$$

Recalling (1.6), the first term can be written as $\|S_p\|^2 = \epsilon^2 \tilde{s}_p + o(\epsilon^2)$, so

$$\Phi_\epsilon(p,\rho) = \frac{\pi}{5}\epsilon^2 \tilde{s}_p \rho^4 + \rho^4 o(\epsilon^2) + O(\epsilon^2) O_p(\rho^5).$$

Choose $\bar{\rho} < R_1$ such that for small ϵ the remainder $|\bar{\rho}^4 o(\epsilon^2) + O(\bar{\rho}^5) O(\epsilon^2)| < \epsilon^2$ and choose $A_c > \frac{5}{\pi}\frac{C_c+1}{\bar{\rho}^4}$. If there exists a point $\bar{p}$ such that $\tilde{s}_{\bar{p}} > A_c$ then

$$\Phi_\epsilon(\bar{p},\bar{\rho}) > \epsilon^2 C_c$$

so Φ_ϵ attains its global maximum on Z_c at an interior point $(p_\epsilon, \rho_\epsilon)$ for all ϵ small enough and applying Lemma 2.3.4 we can say that $\Sigma^\epsilon_{p,\rho}(w_\epsilon(p,\rho))$ is a critical point of I_ϵ for ϵ small enough.
Since for $\epsilon \to 0$ we have $\|v_\epsilon\|_{C^{4,\alpha}(S^2)} \to 0$ and $\|w_\epsilon\|_{C^{4,\alpha}(S^2)} \to 0$ (see Lemma 2.3.2 and Lemma 2.3.3), then the critical point $\Sigma^\epsilon_{p,\rho}(w_\epsilon(p,\rho))$, for small ϵ, can be realized as normal graph on a standard sphere and it converges to a standard sphere as $\epsilon \to 0$. ∎

Proof of Theorem 1.0.2 Recall (1.6) and let $\bar{p} \in \mathbb{R}^3$ be a maximum point of the first term in the expansion of the squared norm of the Traceless Ricci tensor: $\tilde{s}_{\bar{p}} = M$. Observe that from Proposition 2.2.11 and from the proof of the last Theorem, for small radius ρ the reduced functional $\Phi_\epsilon(\bar{p},\rho)$ expands as

$$\Phi_\epsilon(\bar{p},\rho) = \frac{\pi}{5}\epsilon^2 \tilde{s}_{\bar{p}} \rho^4 + \rho^4 o(\epsilon^2) + O(\epsilon^2) O_{\bar{p}}(\rho^5).$$

Let $\bar{\rho}$ and ϵ small enough such that the remainder $|\bar{\rho}^4 o(\epsilon^2) + O(\epsilon^2) O_{\bar{p}}(\bar{\rho}^5)| < \frac{\pi}{10} M \epsilon^2 \bar{\rho}^4$; in this way

$$\Phi_\epsilon(\bar{p},\bar{\rho}) > \frac{\pi}{10} M \epsilon^2 \bar{\rho}^4.$$

From the second part of Lemma 2.3.9 there exist $\delta_M > 0$ and $R_4 > 0$ such that, if $c < \delta_M$

$$|\Phi_\epsilon(p,\rho)| < \frac{\pi}{11} M \epsilon^2 \bar{\rho}^4 \quad \forall (p,\rho) : \rho \geq R_4.$$

(Recall that h has compact support and if $\Sigma^\epsilon_{p,\rho}(w_\epsilon(p,\rho))$ does not intersect $supp(h)$ then $\Phi_\epsilon(p,\rho) = 0$.)

As in the proof of Theorem 1.0.1, let $R > 0$ be such that for $|p| \geq R$ and $\rho \leq R_4$, $S^\rho_p \cap supp(h) = \emptyset$; now we apply the Finite Dimensional Reduction to the compact subset $Z_c \subset \mathbb{R}^3 \oplus \mathbb{R}^+$ defined as

$$Z_c := \{(p,\rho) : |p| \leq R, 0 \leq \rho \leq R_4\}.$$

If we apply Lemma 2.3.3 to the compact Z_c, from the previous discussion and from the proof of Theorem 1.0.1, on the boundary ∂Z_c we have:
- $\rho = 0$: $\Phi_\epsilon = 0$.
- $|p| = R$: $\Phi_\epsilon = 0$.
-$\rho = R_4$: $|\Phi_\epsilon(p,\rho)| < \frac{\pi}{11} M \epsilon^2 \bar{\rho}^4$.

Observe that $(\bar{p},\bar{\rho})$ is an interior point of ∂Z_c and that

$$\Phi_\epsilon(\bar{p},\bar{\rho}) > \frac{\pi}{10} M \epsilon^2 \bar{\rho}^4 > \sup_{(p,\rho)\in \partial Z_c} |\Phi_\epsilon(p,\rho)|$$

so Φ_ϵ attains its global maximum on Z_c at an interior point $(p_\epsilon,\rho_\epsilon)$ for all ϵ small enough. Applying Lemma 2.3.4 we can say that $\Sigma^\epsilon_{p,\rho}(w_\epsilon(p,\rho))$ is a critical point of I_ϵ for ϵ small enough and we conclude as in the previous Theorem. ∎

Chapter 3

Existence of a smooth embedded sphere minimizing the Willmore functional in a semi perturbative setting

3.1 Preliminaries and notations of the chapter

1) We use the following notation: Greek index letters, such as $\mu, \nu, \iota, \dots$, range from 1 to 3 while Latin index letters, such as $i, j, k, \dots$, will run from 1 to 2.

2) Let $h = h_{\mu\nu}(x)$ be a symmetric bilinear form with compact support in $\mathbb{R}^3$ and small C^1 norm; the support of h will be called $\operatorname{spt} h$. With C^0 and C^1 norm we mean

$$\|h\|_{C^0} := \sup_{x\in\mathbb{R}^3} \sup_{u,v\in S^2} |h(x)(u,v)|$$

$$\|Dh\|_{C^0} := \sup_{x\in\mathbb{R}^3} \sup_{u,v,w\in S^2} |D_w(h(x)(u,v))|$$

where D_w is just the directional derivative and of course $\|h\|_{C^1} = \|h\|_{C^0} + \|Dh\|_{C^0}$.
In all the chapter $\mathbb{R}^3$ will be just the "manifold" without metric, $(\mathbb{R}^3, \delta)$ will be the "Riemannian manifold" of the three dimensional euclidean space and $(\mathbb{R}^3, \delta + h)$ will denote the manifold $\mathbb{R}^3$ endowed with the Riemannian metric $\delta_{\mu\nu} + h_{\mu\nu}(x)$.
We will call $B^e_\rho(x)$ (and $B^h_\rho(x)$) the euclidean ball (respectively the geodesic ball in $(\mathbb{R}^3, \delta + h)$) of center x and radius ρ.

3) We will denote by $\Sigma \hookrightarrow \mathbb{R}^3$ an immersed smooth closed (i.e. compact without boundary) orientable surface of genus g (for simplicity we will assume $g = 0$ but most of the results remain true only with a uniform bound on the genus).
The surface Σ can be seen as immersed in two different Riemannian manifolds: $(\mathbb{R}^3, \delta)$ and $(\mathbb{R}^3, \delta + h)$. It follows that all the geometric quantities can be computed with respect the two different spaces and will have two values: the euclidean and the perturbed ones. We use the convention that all the quantities computed with respect to the euclidean metric will have a subscript "e" and the corresponding ones evaluated in perturbed metric will have a subscript "h":

$$|\Sigma|_e, (A_e)_{ij}, H_e, W_e(\Sigma) \dots$$

are the euclidean area of Σ, euclidean second fundamental form, euclidean mean curvature, euclidean Willmore functional while

$$|\Sigma|_h, (A_h)_{ij}, H_h, W_h(\Sigma) \dots$$

are the corresponding quantities in metric $\delta + h$. The first fundamental form induced on Σ by the two different metrics will be denoted respectively by $\mathring{\delta}_{ij}$ and $(\delta \mathring{+} h)_{ij}$ or simply by $\mathring{\delta}$ and $\delta \mathring{+} h$.

We will call $\Sigma_{x,\rho} := \Sigma \cap B^e_\rho(x)$.

4) Let Σ be as above. Recall that the euclidean Willmore functional of Σ is defined as

$$W_e(\Sigma) := \frac{1}{4}\int_\Sigma |H_e|^2 \sqrt{\det \mathring{\delta}}$$

where $H_e = k_1 + k_2$ is the sum of the principal curvatures and $\sqrt{\det \mathring{\delta}}$ is the area form induced by the euclidean metric. Analogously, just taking the corresponding quantities in metric $\delta + h$, one defines W_h. Let us denote

$$\alpha^g_h := \inf\{W_h(\Sigma) : \Sigma \hookrightarrow \mathbb{R}^3 \text{is an immersed smooth closed orientable surface of genus } \leq g\} \tag{3.1}$$

and when g is not written we mean genus= 0:

$$\alpha_h := \alpha^0_h.$$

3.2 Geometric estimates and a monotonicity formula in a perturbed setting

The goal of this Section is to prove monotonicity formulas which link the area, the diameter and the Willmore functional of a surface $\Sigma \hookrightarrow (\mathbb{R}^3, \delta + h)$; in order to obtain it we get estimates from above and below of the perturbed geometric quantities in terms of the corresponding euclidean ones.
Let us start with a straightforward but useful Lemma.

Lemma 3.2.1. *Let $(\mathbb{R}^3, \delta+h)$ be the euclidean space with compactly supported perturbation h and assume that $\|h\|_{C^0(\mathbb{R}^3)} \leq \eta < 1$.*
Then
i) $(\mathbb{R}^3, \delta + h)$ is a complete Riemannian manifold
ii) for every pair of points $p_1, p_2 \in \mathbb{R}^3$ we have

$$\frac{1}{\sqrt{1+\eta}} d_h(p_1, p_2) \leq d_e(p_1, p_2) \leq \frac{1}{\sqrt{1-\eta}} d_h(p_1, p_2)$$

where $d_e(p_1, p_2)$ (respectively $d_h(p_1, p_2)$) is the distance in $(\mathbb{R}^3, \delta)$ (respectively in $(\mathbb{R}^3, \delta+h)$) between the points p_1, p_2.

Proof. To get i) it is sufficient to prove that all the geodesics of $(\mathbb{R}^3, \delta + h)$ are defined globally; but this is a simple exercise of ODE just considering the geodesic differential equation $\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu \dot{x}^\lambda = 0$ and observing that the Christoffel symbols $\Gamma^\mu_{\nu\lambda}$ of $(\mathbb{R}^3, \delta + h)$ are bounded. Indeed, since the the geodesics of $(\mathbb{R}^3, \delta + h)$ can be parametrized by arclength, the geodesic differential equation can be interpreted as a dynamical system on the Spherical bundle $S(\mathbb{R}^3, \delta + h)$ of $(\mathbb{R}^3, \delta + h)$ (the bundle of the unit tangent vectors) generated by the vector field $X_h(x^\mu, y^\mu) := (y^\mu, -\Gamma^\mu_{\nu\lambda} y^\nu y^\lambda)$ where $x \in \mathbb{R}^3, y \in T_x\mathbb{R}^3$ with $|y|_h = 1$. But X_h is a bounded vector field on $S(\mathbb{R}^3, \delta + h)$ which implies by standard and simple ODE arguments (see for instance Lemma 7.2 and Lemma 7.3 of [AMNonLin]) that the integral curves are defined on the whole $\mathbb{R}$.

ii) Consider the segment of straight line $[p_1, p_2]$ connecting p_1 and p_2. Then by definition of distance as inf of the lengths of the curves connecting p_1 and p_2

$$d_h(p_1, p_2) \leq length_h([p_1, p_2]) := \int_0^1 \sqrt{(\delta + h)(p_2 - p_1, p_2 - p_1)} \leq \sqrt{1+\eta}\, d_e(p_1, p_2)$$

where of course $length_h([p_1, p_2])$ is the length of the segment $[p_1, p_2]$ in the metric $\delta + h$.

Let us prove the other inequality; let $\gamma_h : [0, 1] \to \mathbb{R}^3$ be the minimizing geodesic in $(\mathbb{R}^3, \delta + h)$ connecting p_1 and p_2 (it exists since $(\mathbb{R}^3, \delta + h)$ is complete by part i)). Then

$$d_h(p_1, p_2) = \int_0^1 \sqrt{(\delta + h)(\dot{\gamma_h}, \dot{\gamma_h})} \geq \int_0^1 \sqrt{(1 - \eta)\, \delta(\dot{\gamma_h}, \dot{\gamma_h})} = \sqrt{1-\eta}\, length_e(\gamma_h) \geq \sqrt{1-\eta}\, d_e(p_1, p_2)$$

where of course $length_e(\gamma_h)$ is the length of the curve γ_h in euclidean metric. ■

Lemma 3.2.2. *Let $\Sigma \hookrightarrow \mathbb{R}^3$ be an immersed smooth closed orientable surface and let $||h||_{C^0} \leq \eta < 1/4$. Then the following pointwise estimate on the area form holds:*

$$(1-4\eta)\sqrt{(\det \mathring{\delta})} \leq \sqrt{\det(\delta \mathring{+} h)} \leq (1+4\eta)\sqrt{(\det \mathring{\delta})}. \tag{3.2}$$

Denoted with $B^e_\rho(x)$ the euclidean ball of radius ρ and center x, we will call $\Sigma_{x,\rho} := \Sigma \cap B^e_\rho(x)$. Then just integrating one gets

$$(1-4\eta)|\Sigma_{x,\rho}|_e \leq |\Sigma_{x,\rho}|_h \leq (1+4\eta)|\Sigma_{x,\rho}|_e$$

for all $x \in \mathbb{R}^3$ and $\rho > 0$.

PROOF. Let us call $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ a coordinate patch for the surface Σ (Ω is a regular open subset of $\mathbb{R}^2$); of course it is enough to do all the computation for a general patch, moreover we can assume that the patch is conformal with respect to the euclidean metric (i.e. we are using isothermal coordinates w.r.t. the euclidean structure). Recall that

$$|\Sigma_{x,\rho}|_e := \int_{\Sigma_{x,\rho}} \sqrt{\det(\mathring{\delta})}$$

and

$$|\Sigma_{x,\rho}|_h := \int_{\Sigma_{x,\rho}} \sqrt{\det(\delta \mathring{+} h)}$$

where $\mathring{\delta}$ and $\delta \mathring{+} h$ are the first fundamental forms induced by euclidean and perturbed metric.
Let $f_i, i = 1, 2$ be the derivatives of f with respect to the two coordinates (i.e. the two tangent vectors of the coordinate frame), then by definition:

$$(\delta \mathring{+} h)_{ij} = (\delta + h)(f_i, f_j) = \mathring{\delta}_{ij} + h(f_i, f_j)$$

where $\mathring{\delta}_{ij}$ is diagonal. We can evaluate the determinant:

$$\det(\delta \mathring{+} h) = \det(\mathring{\delta}) + \mathring{\delta}_{11} h(f_2, f_2) + \mathring{\delta}_{22} h(f_1, f_1) + \det(h(f_i, f_j)). \tag{3.3}$$

From the assumptions we have

$$|h(f_i, f_i)| \leq \eta \mathring{\delta}_{ii}$$

$$h(f_1, f_2)^2 \leq \eta^2 \mathring{\delta}_{11} \mathring{\delta}_{22} \leq \eta^2 \det \mathring{\delta}.$$

Putting the last two estimates in (3.3) and observing that $\eta^2 < \eta$ we get

$$(1-4\eta)(\det \mathring{\delta}) \leq \det(\delta \mathring{+} h) \leq (1+4\eta)(\det \mathring{\delta}). \tag{3.4}$$

Since in our range $1 - 4\eta \leq \sqrt{1-4\eta}$ and $\sqrt{1+4\eta} \leq 1 + 4\eta$, we have the thesis just taking the square root of (3.4) and integrating on the desired domain. ■

In the following Lemma we derive a pointwise estimate from above and below of the mean curvature squared in a perturbed setting in terms of the corresponding euclidean quantities.

Lemma 3.2.3. *Let Σ be as in Lemma 3.2.2 and assume that $\|h\|_{C^0} \leq \eta$, $\|Dh\|_{C^0} \leq \theta$ (η is supposed to be small while no assumption is made on θ). Then the following pointwise estimate holds:*

$$(1 - C\eta - \gamma)|H_e|^2 - (C\eta + \gamma)|A_e|^2 - C_\gamma \theta^2 \leq |H|^2_h \leq (1 + C\eta + \gamma)|H_e|^2 + (C\eta + \gamma)|A_e|^2 + C_\gamma \theta^2$$

where $\gamma > 0$ can be chosen arbitrarily small and C_γ is a constant depending on γ such that $C_\gamma \to \infty$ if $\gamma \to 0$ but which can be bounded by $C_\gamma \leq C(1 + \frac{1}{\gamma})$ for C large enough independent of γ.

PROOF. Let us fix a point $p \in \Sigma$ and use the same notation of Lemma 3.2.2; all the computations will be done at the point p. Choose the parametrization f given by the normal coordinates at p (with respect to the metric $\mathring{\delta}$) such that the coordinate vectors f_i are euclidean-orthonormal and diagonalize the euclidean second fundamental form A_e at p (the first condition is trivial, the second can be achieved by a rotation). With this choice of coordinates, the euclidean-Christoffel symbols $\tilde{\Gamma}^k_{ij}$ of Σ vanish at p and one can say that

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = (A_e)_{ij}\nu_e + \tilde{\Gamma}^k_{ij} f_k = (A_e)_{ij}\nu_e \tag{3.5}$$

at the point p; this will be useful later. In the formula above and in what follows, ν_e denotes the euclidean normal vector to Σ:

$$\nu_e := f_1 \times f_2.$$

The normal vector to Σ in the perturbed metric is denoted with ν_h and it has the form

$$\nu_h = \nu_e + N$$

where the correction N is small since $\|h\|_{C^0}$ is small. More precisely from the orthogonality conditions $(\delta + h)(f_1, \nu_h) = 0$ and $(\delta + h)(f_2, \nu_h) = 0$ we get

$$\begin{aligned} \delta(N, f_1) &= -h(\nu_e, f_1) + \text{higher order terms} \\ \delta(N, f_2) &= -h(\nu_e, f_2) + \text{higher order terms.} \end{aligned}$$

Imposing the normalization condition $(\delta + h)(\nu_h, \nu_h) = 1$ we obtain

$$\delta(N, \nu_e) = -\frac{1}{2} h(\nu_e, \nu_e) + \text{higher order terms.}$$

Collecting the formulas above, being (f_1, f_2, ν_e) an orthonormal frame in euclidean metric, we can represent N as

$$N = -h(\nu_e, f_1) f_1 - h(\nu_e, f_2) f_2 - \frac{1}{2} h(\nu_e, \nu_e)\nu_e + \text{higher order terms.} \tag{3.6}$$

Observe that the higher order terms can be computed in an inductive way using the orthonormality conditions above and that for η small

$$|N|_e := \sqrt{\delta(N, N)} \leq C\eta. \tag{3.7}$$

Now let us compute the perturbed second fundamental form

$$(A_h)_{ij} := (\delta + h)(\nu_h, {}^{\delta+h}\nabla_{f_i} f_j)$$

where ${}^{\delta+h}\nabla$ is the covariant derivative in $(\mathbb{R}^3, \delta + h)$; by definition

$${}^{\delta+h}\nabla_{f_i} f_j = \frac{\partial^2 f}{\partial x^i \partial x^j} + {}^{\delta+h}\,\Gamma f_i f_j$$

where ${}^{\delta+h}\Gamma$ are the Christoffel symbols of $(\mathbb{R}^3, \delta+h)$ and ${}^{\delta+h}\Gamma f_i f_j := {}^{\delta+h}\Gamma^{\mu}_{\nu\lambda} f_i^{\nu} f_j^{\lambda} e_{\mu}$ ($\{e_\mu\}$ is the standard euclidean orthonormal basis of $(\mathbb{R}^3, \delta)$ and $f_i = f_i^{\mu} e_{\mu}$).
Using (3.5), the perturbed second fundamental form becomes

$$(A_h)_{ij} = (\delta + h)(\nu_e + N, (A_e)_{ij}\nu_e + {}^{\delta+h}\Gamma f_i f_j).$$

Observing that $|{}^{\delta+h}\Gamma| \leq C\theta$ and recalling (3.7) one gets

$$(A_e)_{ij} - C\eta (A_e)_{ij} - C\theta \leq (A_h)_{ij} \leq (A_e)_{ij} + C\eta (A_e)_{ij} + C\theta. \tag{3.8}$$

Just squaring and using γ-Cauchy inequality we get that for any small $\gamma > 0$ there exists a $C_\gamma > 0$ such that the following estimate holds

$$(1 - 2\gamma - C\eta)|A_e|^2 - C_\gamma \theta^2 (1 + \eta^2) \leq |A_h|^2 \leq (1 + 2\gamma + C\eta)|A_e|^2 + C_\gamma \theta^2 (1 + \eta^2). \tag{3.9}$$

Taking the trace of (3.8) with respect to $\delta \overset{\circ}{+} h$,

$$H_h := (\delta \overset{\circ}{+} h)^{ij} (A_h)_{ij};$$

and since

$$(\mathring{\delta})^{ij} - C\eta \leq (\delta \mathring{+} h)^{ij} \leq (\mathring{\delta})^{ij} + C\eta$$

we get

$$H_e - C\eta|A_e|_e - C\theta \leq H_h \leq H_e + C\eta|A_e|_e + C\theta \tag{3.10}$$

where $|A_e|_e$ (in the sequel called just $|A_e|$) is the euclidean norm of the euclidean second fundamental form. At this point we can compute the estimate of $|H_h|^2$, let us do the one from above (the other one is analogous). Using the Cauchy inequality we can write

$$\begin{aligned} |H_h|^2 &\leq |H_e|^2 + C\eta|H_e||A_e| + C\theta|H_e| + C\eta^2|A_e|^2 + C\theta|A_e| + C\theta^2 \\ &\leq (1 + C\eta + \gamma)|H_e|^2 + (C\eta + \gamma)|A_e|^2 + C_\gamma\theta^2 \end{aligned}$$

where $C_\gamma \to \infty$ as $\gamma \to 0$ but can be bounded by $C_\gamma \leq C(1 + \frac{1}{\gamma})$ for C large enough. ■

Lemma 3.2.4. *Let* $\operatorname{spt} h \subseteq B^e_{r_0}(x_0)$ *for some* $x_0 \in \mathbb{R}^3$ *and* $r_0 > 0$. *As before* $\|h\|_{C^0} \leq \eta$, $\|Dh\|_{C^0} \leq \theta$ *(η is supposed to be small while no assumption is made on θ) and* $\Sigma \hookrightarrow \mathbb{R}^3$ *is a closed smooth orientable surface of genus* 0 *(it is enough to ask the uniform bound* $\operatorname{genus}(\Sigma_k) \leq g$*) immersed in* $\mathbb{R}^3$. *Then*

$$(1 - C\eta - C\gamma - C_\gamma r_0^2\theta^2)W_e(\Sigma) - C_g(\eta + \gamma) \leq W_h(\Sigma) \tag{3.11}$$

where $C_g \to \infty$ *as* $g \to \infty$, $\gamma > 0$ *can be chosen arbitrarily small and* C_γ *is a constant depending on* γ *such that* $C_\gamma \to \infty$ *if* $\gamma \to 0$ *but which can be bounded by* $C_\gamma \leq C(1 + \frac{1}{\gamma})$ *for* C *large enough independent of* γ. *It follows that for* γ, η *and* $r_0\theta$ *small enough*

$$W_e(\Sigma) \leq \frac{3}{2}W_h(\Sigma) + 1. \tag{3.12}$$

Proof. Recalling the estimate of the area form (3.2), just integrating the formula of Lemma 3.2.3 one gets

$$W_h(\Sigma) := \frac{1}{4}\int_\Sigma |H_h|^2\sqrt{\det(\delta \mathring{+} h)} \geq \int_\Sigma \Big[\Big(\frac{1}{4} - C\eta - \gamma\Big)|H_e|^2 - (C\eta + \gamma)|A_e|^2 - C_\gamma\theta^2\chi_h\Big](1 - 4\eta)\sqrt{\det\mathring{\delta}}$$

where χ_h is the characteristic function of $\operatorname{spt} h$ (i.e $\chi_h(x) = 1$ if $x \in \operatorname{spt} h$ and $\chi_h(x) = 0$ otherwise). From the Gauss-Bonnet Theorem

$$\int_\Sigma |A_e|^2\sqrt{\det\mathring{\delta}} = \int_\Sigma |H_e|^2\sqrt{\det\mathring{\delta}} - 4\pi\chi_E(\Sigma)$$

where $\chi_E(\Sigma) = 2 - 2\operatorname{genus}(\Sigma)$ is the Euler Characteristic of Σ. In the case $\operatorname{genus}(\Sigma) = 0$ of course $\chi_E(\Sigma) = 2$ but more generally if $\operatorname{genus}(\Sigma)$ is uniformly bounded also $-4\pi\chi_E(\Sigma)$ will be uniformly bounded from above. Hence

$$W_h(\Sigma) \geq (1 - C\eta - \gamma)W_e(\Sigma) - C_g(\eta + \gamma) - C_\gamma\theta^2|\Sigma \cap \operatorname{spt} h|_e$$

where $C_g \to \infty$ as $g \to \infty$. From formula (1.3) in [SiL],

$$|\Sigma \cap \operatorname{spt} h|_e \leq |\Sigma \cap B^e_{r_0}(x_0)|_e \leq Cr_0^2 W_e(\Sigma). \tag{3.13}$$

We can conclude that

$$W_h(\Sigma) \geq (1 - C\eta - \gamma - C_\gamma\theta^2 r_0^2)W_e(\Sigma) - C_g(\eta + \gamma).$$

We get the thesis by first fixing γ small enough and then choosing sufficiently small η, θ. ■

Using the estimates of the previous Lemmas, we get the desired monotonicity formulas in the following proposition.

Proposition 3.2.5. *As before let* $\operatorname{spt} h \subseteq B^e_{r_0}(x_0)$ *for some* $x_0 \in \mathbb{R}^3$, $r_0 > 0$ *and* $\|h\|_{C^0} \leq \eta$, $\|Dh\|_{C^0} \leq \theta$*; recall that* $\Sigma \hookrightarrow \mathbb{R}^3$ *is a closed smooth orientable surface of genus* $g \geq 0$ *immersed in* $\mathbb{R}^3$*. Then for* η *and* $r_0\theta$ *small enough the following inequality holds*

$$\sigma^{-2}|\Sigma_{x,\sigma}|_h \leq C\Big[\rho^{-2}|\Sigma_{x,\rho}|_h + W_h(\Sigma_{x,\rho}) + [C_g(\eta+\gamma) + C_\gamma r_0^2\theta^2](W_h(\Sigma)+1)\Big] \quad 0 < \sigma \leq \rho < \infty$$

where $\gamma > 0$ *can be chosen arbitrarily small and* C_γ, C_g *are constants depending on* γ *(respectively on* g*) such that* $C_\gamma \to \infty$ *if* $\gamma \to 0$ *(respectively* $C_g \to \infty$ *if* $g \to \infty$*). It follows the more simple estimate*

$$\sigma^{-2}|\Sigma_{x,\sigma}|_h \leq C_g\Big[\rho^{-2}|\Sigma_{x,\rho}|_h + W_h(\Sigma) + 1\Big] \quad 0 < \sigma \leq \rho < \infty \tag{3.14}$$

and just taking the limit $\rho \to \infty$*,*

$$|\Sigma|_h \leq C_g(W_h(\Sigma)+1)(\operatorname{diam}_e \Sigma)^2 \tag{3.15}$$

where $\operatorname{diam}_e \Sigma$ *is the euclidean diameter of* Σ*.*

Proof. Let us recall the euclidean monotonicity formula proved by Simon (formula (1.3) in [SiL]):

$$\sigma^{-2}|\Sigma_{x,\sigma}|_e \leq C(\rho^{-2}|\Sigma_{x,\rho}|_e + W_e(\Sigma_{x,\rho})). \tag{3.16}$$

We just have to estimate from above and below the area part and from above the Willmore term. From Lemma 3.2.2

$$\frac{1}{1+4\eta}|\Sigma_{x,\sigma}|_h \leq |\Sigma_{x,\sigma}|_e,$$

$$|\Sigma_{x,\rho}|_e \leq \frac{1}{1-4\eta}|\Sigma_{x,\rho}|_h$$

and integrating the formula of Lemma 3.2.3 one gets

$$W_h(\Sigma_{x,\rho}) \geq \int_{\Sigma_{x,\rho}} \Big[\Big(\frac{1}{4} - C\eta - \gamma\Big)|H_e|^2 - (C\eta+\gamma)|A_e|^2\Big](1-4\eta)\sqrt{\det \mathring{\delta}} - C_\gamma\theta^2 \int_{\Sigma_{x,\rho}} \chi_h \sqrt{\det(\delta \mathring{+} h)}$$

where χ_h is the characteristic function of $\operatorname{spt} h$.
From the Gauss Bonnet Theorem and the estimate (3.12),

$$\int_{\Sigma_{x,\rho}} |A_e|^2\sqrt{\det \mathring{\delta}} \leq \int_\Sigma |A_e|^2\sqrt{\det \mathring{\delta}} \leq C_g(W_e(\Sigma)+1) \leq C_g(W_h(\Sigma)+1)$$

where C_g is a constant depending on genus(Σ) such that $C_g \to \infty$ if genus(Σ) $\to \infty$. Hence

$$W_h(\Sigma_{x,\rho}) \geq (1 - C\eta - C\gamma)W_e(\Sigma_{x,\rho}) - C_g(\eta+\gamma)(W_h(\Sigma)+1) - C_\gamma\theta^2|\Sigma_{x,\rho} \cap \operatorname{spt} h|_h$$

and for η, γ small enough (γ will be small but fixed while η can vary and be arbitrarily closed to 0)

$$W_e(\Sigma_{x,\rho}) \leq CW_h(\Sigma_{x,\rho}) + C_g(\eta+\gamma)(W_h(\Sigma)+1) + C_\gamma\theta^2|\Sigma_{x,\rho} \cap \operatorname{spt} h|_h.$$

From the previous inequalities (3.2)-(3.13) and (3.12)

$$|\Sigma_{x,\rho} \cap \operatorname{spt} h|_h \leq C|\Sigma \cap B^e_{r_0}(x_0)|_e \leq Cr_0^2 W_e(\Sigma) \leq Cr_0^2(W_h(\Sigma)+1),$$

hence

$$W_e(\Sigma_{x,\rho}) \leq CW_h(\Sigma_{x,\rho}) + C_g(\eta+\gamma)(W_h(\Sigma)+1) + C_\gamma r_0^2\theta^2(W_h(\Sigma)+1)$$

and we can conclude that

$$\sigma^{-2}|\Sigma_{x,\sigma}|_h \leq C\Big[\rho^{-2}|\Sigma_{x,\rho}|_h + W_h(\Sigma_{x,\rho}) + [C_g(\eta+\gamma) + C_\gamma r_0^2\theta^2](W_h(\Sigma)+1)\Big].$$

■

3.3 Global a priori estimates on the minimizing sequence Σ_k

Under a very general assumption on the metric (we just ask that the scalar curvature of the ambient manifold is strictly positive at one point) we will show global a priori estimates on the minimizing sequence of the Willmore functional: more precisely we will get a uniform upper area bound and uniform upper and lower bounds on the diameters.

Proposition 3.3.1. *Following the previous notation, let $(\mathbb{R}^3, \delta + h)$ be the ambient manifold and $\alpha_h^g = \inf(W_h)$ over the surfaces of genus less or equal than g (for the precise definition see equation (3.1)). If there exists a point $\bar{p} \in \mathbb{R}^3$ such that the scalar curvature R_h of $(\mathbb{R}^3, \delta + h)$ is strictly positive*

$$R_h(\bar{p}) > 0$$

then there exists $\epsilon > 0$ such that

$$\alpha_h^g < 4\pi - 2\epsilon.$$

Proof. From Proposition 3.1 of [Mon1], on geodesic spheres $S_{\bar{p},\rho}$ of center $\bar{p}$ and small radius ρ one has

$$W_h(S_{\bar{p},\rho}) = 4\pi - \frac{2\pi}{3} R_h(\bar{p})\rho^2 + O(\rho^3).$$

Since the genus of these surfaces is 0 and $R_h(\bar{p}) > 0$ the conclusion follows. ■

Corollary 3.3.2. *Let Σ_k, with* $\operatorname{genus}\Sigma_k \leq g$, *be a minimizing sequence for W_h of bounded genus*

$$W_h(\Sigma_k) \downarrow \alpha_h^g$$

and assume there exists a point $\bar{p} \in \mathbb{R}^3$ such that the scalar curvature $R_h(\bar{p}) > 0$.
Then there exists $\epsilon > 0$ such that for large k

$$W_h(\Sigma_k) < 4\pi - \epsilon.$$

Now let us state and prove uniform a priori upper bounds on the minimizing sequence Σ_k. The idea is to use just that $W_h(\Sigma_k) < 4\pi - \epsilon$ and then perform a blow down procedure making use of the rescaling invariance of the Willmore functional (see equation (3.24) below).

Proposition 3.3.3. *Let $(\mathbb{R}^3, \delta + h)$ be as before with small $\|h\|_{C^1}$ and let $\Sigma_k \hookrightarrow \mathbb{R}^3$ be a sequence of immersed smooth closed orientable surfaces of genus 0 (more generally one can ask the uniform bound* $\text{genus}(\Sigma_k) \leq g$, *but in this case the required smallness of $\|h\|_{C^1}$ depends on g and goes to 0 as $g \to \infty$). Assume that*

$$\limsup_k W_h(\Sigma_k) < 4\pi,$$

then
i)there exists a compact subset $K \subset \mathbb{R}^3$ such that

$$\Sigma_k \subseteq K \quad \forall k \in \mathbb{N},$$

ii) there exists a uniform area bound

$$|\Sigma_k|_h \leq C.$$

for some large $C > 0$.

Proof. We can assume that each surface Σ_k is connected, otherwise just replace the sequence $\{\Sigma_k\}_{k\in\mathbb{N}}$ with the sequence of the connected components and observe that it satisfies the same assumptions.
As before call $\eta := \|h\|_{C^0}$ and $\theta := \|Dh\|_{C^0}$. Since h has compact support then $\operatorname{spt} h \subseteq B^e_{r_0}(0)$ for some $r_0 > 0$ and from $W_h(\Sigma_k) < 4\pi$ it follows that

$$\Sigma_k \cap B^e_{r_0}(0) \neq \emptyset \quad \text{for k large.}$$

In fact if $\Sigma_k \cap B^e_{r_0}(0) = \emptyset$ then $W_h(\Sigma_k) = W_e(\Sigma_k)$ and $W_e(\Sigma_k) \geq 4\pi$ from Theorem 7.2.2 in [Will].
If we prove that $\limsup_k(\operatorname{diam}_e \Sigma_k) < \infty$ then of course we get *i)* and statement *ii)* follows in virtue of estimate (3.15). Let us prove it by contradiction and assume that up to subsequences

$$\operatorname{diam}_e \Sigma_k \uparrow \infty.$$

For each k rescale both, Σ_k and the perturbation h, by $1/\operatorname{diam}_e \Sigma_k$ in the following sense:

$$\tilde{\Sigma}_k := \frac{1}{\operatorname{diam}_e \Sigma_k}\Sigma_k \tag{3.17}$$

$$(h_k)_{\mu\nu}(x) := h_{\mu\nu}\big((\operatorname{diam}_e \Sigma_k)\, x\big) \tag{3.18}$$

where the right hand side of (3.17) denotes the multiplication in $\mathbb{R}^3$ (as vector space) of each point of Σ_k by $1/(\operatorname{diam}_e \Sigma_k)$. It follows that

$$\operatorname{diam}_e \tilde{\Sigma}_k = 1 \tag{3.19}$$

$$\operatorname{spt} h_k = \frac{1}{\operatorname{diam}_e \Sigma_k}\operatorname{spt} h \subseteq B^e_{r_k}(0) \tag{3.20}$$

where

$$r_k := \frac{1}{\operatorname{diam}_e \Sigma_k} r_0 \downarrow 0. \tag{3.21}$$

Called $\eta_k := \|h_k\|_{C^0}$ and $\theta_k := \|Dh_k\|_{C^0}$, observe that

$$\eta_k = \eta := \|h\|_{C^0} \tag{3.22}$$

$$r_k\theta_k = \frac{1}{\operatorname{diam}_e \Sigma_k} r_0 \cdot \operatorname{diam}_e \Sigma_k\ \theta = r_0\theta. \tag{3.23}$$

The second equality follows simply from the chain rule

$$\frac{\partial}{\partial x^\lambda}(h_k)_{\mu\nu}|_x = \frac{\partial}{\partial x^\lambda}[h_{\mu\nu}(\operatorname{diam}_e \Sigma_k\ .)]|_x = \operatorname{diam}_e \Sigma_k\ \frac{\partial}{\partial x^\lambda} h_{\mu\nu}|_{\operatorname{diam}_e \Sigma_k\ x}.$$

Moreover, just from the definitions, it is easy to check the scale invariance of the Willmore functional

$$W_{h_k}(\tilde{\Sigma}_k) = W_h(\Sigma_k). \tag{3.24}$$

Now for each k consider $\tilde{\Sigma}_k \hookrightarrow (\mathbb{R}^3, \delta + h_k)$. Since $\eta_k = \eta$ and $r_k\theta_k = r_0\theta$, for small $\|h\|_{C^1}$, we can apply equation (3.15) at each step k to get

$$|\tilde{\Sigma}_k|_{h_k} \leq C \tag{3.25}$$

where we used the uniform diameter and Willmore bound on the $\tilde{\Sigma}_k$.
Applying for each k the inequality (3.12) and Lemma 3.2.2, we have a bound on the global euclidean quantities

$$|\tilde{\Sigma}_k|_e \leq C \tag{3.26}$$

$$W_e(\tilde{\Sigma}_k) \leq C. \tag{3.27}$$

Let us denote by $V^e_{\tilde{\Sigma}_k}$ the associated Allard varifold in $(\mathbb{R}^3, \delta)$ (i.e. integral varifold with finite first variation $\delta V^e_{\tilde{\Sigma}_k} := \int_{\tilde{\Sigma}_k} |H_e| \sqrt{\det(\mathring{\delta})}$; for the definition and properties see the book of Simon [SiGMT], the original paper of Allard [Al], or the thesis of Mantegazza [Mant]). Observe that $V^e_{\tilde{\Sigma}_k}$ are Allard varifolds without boundary in $(\mathbb{R}^3, \delta)$ which have uniform bound on the mass (inequality (3.26)) and on the first variation (from Schwartz inequality and equation (3.27))

$$|\delta V^e_{\tilde{\Sigma}_k}|_e := \int_{\tilde{\Sigma}_k} |H_e| \sqrt{\det \mathring{\delta}} \leq C\sqrt{W_e(\tilde{\Sigma}_k)}\ \sqrt{|\tilde{\Sigma}_k|_e} \leq C.$$

Then, from Allard Compactness Theorem (see for example [SiGMT] Remark 42.8 or the original paper of Allard [Al]), the varifolds $V^e_{\tilde{\Sigma}_k}$ converge (in the sense of Allard varifolds) up to subsequences to a limit Allard varifold V in $(\mathbb{R}^3, \delta)$. Moreover, from lower semicontinuity of the Willmore functional under Allard-varifold convergence, we have

$$W_e(V) \leq \liminf_k W_e(\tilde{\Sigma}_k) \leq C.$$

Of course the Willmore functional of the integral 2-varifold V

$$W_e(V) := \frac{1}{4}\int_{\mathbb{R}^3} |H_V|^2 d\|V\|$$

is just, up to a factor, the L^2 norm of the weak mean curvature H_V with respect to the mass measure $\|V\|$ of V. In particular the limit integral 2-varifold V has square integrable mean curvature $H_V \in L^2(V)$ in the whole $(\mathbb{R}^3, \delta)$. Now let us prove that actually $V \neq 0$ is not the null varifold and it does not shrink to 0.

Claim. *The mass of V in $(\mathbb{R}^3, \delta)$ is strictly positive and, the spatial support of V,* $\operatorname{spt}\|V\| \neq \{0\}$.

Proof of the Claim: we will prove that there exists $\beta > 0$ such that

$$|\tilde{\Sigma}_k \cap (\mathbb{R}^3 \backslash B^e_{1/2}(0))|_e > \beta \quad \text{for large } k. \tag{3.28}$$

Since the varifold convergence implies the weak convergence of measures of the associated mass measures, we will have for the converging subsequence

$$\|V\|(\mathbb{R}^3 \backslash B^e_{1/2-\varepsilon}(0)) = \lim_k |\tilde{\Sigma}_k \cap (\mathbb{R}^3 \backslash B^e_{1/2-\varepsilon}(0))|_e \geq \limsup_k |\tilde{\Sigma}_k \cap (\mathbb{R}^3 \backslash B^e_{1/2}(0))|_e > \beta \tag{3.29}$$

for some $\varepsilon \geq 0$ such that $\|V\|(\partial B^e_{1/2-\varepsilon}(0)) = 0$ (it exists since $\|V\|$ is a finite measure); observe we denoted with $\|V\|(\mathbb{R}^3 \backslash B^e_{1/2}(0))$ the measure of $\mathbb{R}^3 \backslash B^e_{1/2}(0)$ with respect to the mass measure $\|V\|$ of the varifold V. This will prove the Claim.

Now let us prove (3.28). Since $\tilde{\Sigma}_k$ is connected, $\operatorname{diam}_e \tilde{\Sigma}_k = 1$, $\tilde{\Sigma}_k \cap B^e_{r_k}(0) \neq \emptyset$ and $r_k \to 0$ it follows that

$$\operatorname{spt} h_k \subset B_{1/2} \quad \text{for } k \text{ large and} \tag{3.30}$$

$$\tilde{\Sigma}_k \cap S_{3/4} \neq \emptyset \quad \text{for } k \text{ large}$$

where $S_{3/4} := \{x \in \mathbb{R}^3 : |x|_e = 3/4\}$ is the euclidean sphere of center 0 and radius 3/4.
Let us consider a partition of $B^e_{3/4}(0) \backslash B^e_{1/2}(0)$ with N annuli at distance $\frac{1}{4N}$ one each other, i.e. the i^{th} annulus is of the type

$$A_i := B^e_{1/2+\frac{i}{4N}}(0) \backslash B^e_{1/2+\frac{(i-1)}{4N}}(0) \quad i = 1, \ldots, N.$$

We can assume that each $\tilde{\Sigma}_k \cap (\mathbb{R}^3 \backslash B^e_{1/2}(0))$ is connected, otherwise just take a connected component of $\tilde{\Sigma}_k \cap (\mathbb{R}^3 \backslash B^e_{1/2}(0))$ which intersects $S_{3/4}$.
From the connection property, for each annulus and for each $\tilde{\Sigma}_k$ there is a point $x^k_i \in \tilde{\Sigma}_k \cap A_i$ such that $B^e_{1/(8N)}(x^k_i) \subset A_i$. From Simon's monotonicity formula (formula (1.4) page 285 of [SiL]),

$$\pi \leq C\big(64N^2 |\tilde{\Sigma}_k \cap B^e_{1/(8N)}(x^k_i)|_e + W_e(\tilde{\Sigma}_k \cap B^e_{1/(8N)}(x^k_i)\big).$$

It follows that

$$|\tilde{\Sigma}_k \cap B^e_{1/(8N)}(x^k_i)|_e \geq \frac{1}{64N^2}\Big(\frac{\pi}{C} - W_e(\tilde{\Sigma}_k \cap B^e_{1/(8N)}(x^k_i)\Big). \tag{3.31}$$

Now it is enough to prove that $\exists N$ large enough: $\forall k$ large $\exists x^k_i$ (notation above) such that

$$W_e(\tilde{\Sigma}_k \cap B^e_{1/(8N)}(x^k_i)) < \frac{\pi}{2C}.$$

If it is not true, $\forall N > 0$ there exists a large k such that $\forall x^k_i$ $i = 1, \ldots, N$,

$$W_e(\tilde{\Sigma}_k \cap B^e_{1/(8N)}(x^k_i)) \geq \frac{\pi}{2C}.$$

But, for k fixed, the balls $B^e_{1/(8N)}(x^k_i)$ $i = 1, \ldots, N$ are disjoint; hence

$$W_e(\tilde{\Sigma}_k \cap (\mathbb{R}^3 \backslash B_{1/2})) \geq \sum_{i=1}^{N} W_e(\tilde{\Sigma}_k \cap B^e_{1/(8N)}(x^k_i)) \geq N\frac{\pi}{2C}.$$

Since N is arbitrarily large, this contradicts the boundness of $W_e(\tilde{\Sigma}_k) \geq W_e(\tilde{\Sigma}_k \cap (\mathbb{R}^3 \backslash B_{1/2}))$.
This concludes the proof of the claim.

From the Claim and the discussion above, we can say that V is a non null integral 2-varifold with square integrable weak mean curvature $H \in L^2(V)$ in the whole $(\mathbb{R}^3, \delta)$ such that $\operatorname{spt}\|V\| \neq \{0\}$. Kuwert and Shätzle proved that the euclidean monotonicity formula of Simon generalizes to non null integral 2-varifolds with square integrable weak mean curvature (see the Appendix A of [KS]; in particular we use formula (A.18) at page 355) so

$$W_e(V) \geq 4\pi. \tag{3.32}$$

In order to reach a contradiction we want to prove that from the assumptions it also follows $W_e(V) < 4\pi$.

Let us denote by $V^{h_k}_{\tilde{\Sigma}_k}$ the Allard varifold in the Riemannian manifold $(\mathbb{R}^3, \delta + h_k)$ associated to $\tilde{\Sigma}_k$ (i.e. integral varifold with finite first variation $\delta V_{\tilde{\Sigma}_k} := \int_{\tilde{\Sigma}_k} |H_{h_k}| \sqrt{\det(\delta + \mathring{h}_k)}$). Consider the sequence of closed shrinking balls $B_{1/n} := \bar{B}^e_{\frac{1}{n}}(0)$ and the restriction of the varifolds to the open subsets $\mathbb{R}^3 \backslash B_{1/n}$

$$V^{h_k}_{\tilde{\Sigma}_k \lfloor (\mathbb{R}^3 \backslash B_{1/n})}.$$

Actually, to be precise, $V^{h_k}_{\tilde{\Sigma}_k \lfloor (\mathbb{R}^3 \backslash B_{1/n})}$ denotes the restriction of V to the Grassmannian of the 2-planes based on the points of $\mathbb{R}^3 \backslash B_{1/n}$ (by definition of varifold, V is a measure on the Grassmannian). Observe that for each fixed n and for varying large k, $\operatorname{spt} h_k \subseteq B_{1/n}$ then

$$V^{h_k}_{\tilde{\Sigma}_k \lfloor (\mathbb{R}^3 \backslash B_{1/n})} = V^{e}_{\tilde{\Sigma}_k \lfloor (\mathbb{R}^3 \backslash B_{1/n})} \quad \text{for large } k. \tag{3.33}$$

Recall that up to subsequences $V^e_{\tilde{\Sigma}_k} \to V$ in $(\mathbb{R}^3, \delta)$ in varifold sense; since varifold convergence is a local property (one perform tests with C^0_c functions) it follows that $V^{e}_{\tilde{\Sigma}_k \lfloor (\mathbb{R}^3 \backslash B_{1/n})} \to V_{\lfloor (\mathbb{R}^3 \backslash B_{1/n})}$ and using equation (3.33)

$$V^{h_k}_{\tilde{\Sigma}_k \lfloor (\mathbb{R}^3 \backslash B_{1/n})} \overset{k \to \infty}{\to} V_{\lfloor (\mathbb{R}^3 \backslash B_{1/n})} \quad \text{in varifold sense.}$$

Now from the lower semicontinuity of Willmore functional under varifold convergence

$$W_e(V_{\lfloor (\mathbb{R}^3 \backslash B_{1/n})}) \leq \liminf_k W_e(V^{e}_{\tilde{\Sigma}_k \lfloor (\mathbb{R}^3 \backslash B_{1/n})}) = \liminf_k W_{h_k}(V^{h_k}_{\tilde{\Sigma}_k \lfloor (\mathbb{R}^3 \backslash B_{1/n})}) \leq \liminf_k W_{h_k}(\tilde{\Sigma}_k) \leq 4\pi - \epsilon \tag{3.34}$$

for some $\epsilon > 0$ independent of n (the last inequality comes directly from the statement assumption on $W_h(\Sigma_k)$ and the invariance under rescaling (3.24)). Using Simon's euclidean monotonicity formula (formula (1.3) in [SiL] recalled before in equation (3.16)) $|\tilde{\Sigma}_k \cap B^e_\rho(0)|_e \leq C\rho^2$, so we have the local area bound

$$\|V\|(B^e_\rho(0)) \leq C\rho^2. \tag{3.35}$$

Since $H_V \in L^2(V)$, the local area bound (3.35) and the inequality (3.34) imply

$$W_e(V) = \lim_{n \to \infty} W_e(V_{\lfloor (\mathbb{R}^3 \backslash B_{1/n})}) \leq 4\pi - \epsilon$$

which contradicts (3.32).

As remarked in the beginning of the proof, the contradiction proves a uniform bound on $\operatorname{diam}_e \Sigma_k$ and the existence of the compact set K; the uniform area bound follows from equation (3.15).

■

We also would like to say that the minimizing sequence does not shrink to a point. This is proved in the following proposition in the more general framework of a Riemannian manifold as ambient space. The idea is the following: if a sequence of surfaces shrinks to a point and we use normal coordinates in that point, we are reduced to the previous framework of perturbed metric. Hence we can use all the estimates computed in Section 6.1.

Proposition 3.3.4. *Let (M, g) be a (maybe non compact) 3 dimensional Riemannian manifold without boundary: $\partial M = \emptyset$. Let $\{\Sigma_k\}_{k \in \mathbb{N}}$ be a sequence of immersed smooth closed oriented surfaces of genus $\leq G$. Assume that*
i) there exists a compact subset $K \subset\subset M$ such that

$$\Sigma_k \subset K \quad \textit{for all } k \in \mathbb{N},$$

ii)

$$\limsup_k W_g(\Sigma_k) \leq 4\pi - \epsilon$$

for some $\epsilon > 0$*; where* $W_g(\Sigma_k) := \frac{1}{4}\int_{\Sigma_k} |H_g|^2 \sqrt{\det \mathring{g}}$ *is the Willmore functional of* $\Sigma_k \hookrightarrow (M, g)$*.*

Then

$$\liminf_k (\operatorname{diam}_g \Sigma_k) > 0$$

where $\operatorname{diam}_g \Sigma_k$ *is the diameter of* Σ_k *in the Riemannian manifold* (M, g)*.*

Proof. Let us prove it by contradiction and assume that up to subsequences $\operatorname{diam}_g \Sigma_k \downarrow 0$.

For each surface take a point $y_k \in \Sigma_k$; from assumption $i)$, $(y_k)_{k\in\mathbb{N}}$ is a sequence in the compact subset $K \subset\subset M$ and up to subsequences

$$y_k \to \bar{x}$$

for some $\bar{x} \in K$. Since $\operatorname{diam}_g \Sigma_k \downarrow 0$ then

$$\Sigma_k \to \bar{x} \quad \text{in Hausdorff distance sense.}$$

Consider geodesic normal coordinates centered at $\bar{x}$ (the coordinates of $\bar{x}$ are 0); in these coordinates the metric can be written as (see for example [LP] formula (5.4) page 61)

$$\begin{aligned} g_{\mu\nu}(x) &= \delta_{\mu\nu} + \frac{1}{3} R_{\mu\sigma\lambda\nu} x^\sigma x^\lambda + O(|x|^3) \qquad (3.36) \\ &= \delta_{\mu\nu} + h_{\mu\nu}(x) \qquad (3.37) \end{aligned}$$

where

$$h_{\mu\nu}(0) = 0 \quad \text{and} \quad D_\lambda h_{\mu\nu}(0) = 0 \quad \forall \lambda, \mu, \nu = 1, 2, 3. \qquad (3.38)$$

Call $inj(\bar{x}) > 0$ the injectivity radius at $\bar{x}$; for k large, up to subsequences, $\Sigma_k \subset B^g_{inj(\bar{x})}(\bar{x})$ (the geodesic ball of center $\bar{x}$ and radius $inj(\bar{x})$). In this ball we have the geodesic normal coordinates so, using (5.4) and (5.5), we can argue as if the ambient manifold was a euclidean ball centered at 0 in the perturbed manifold $(\mathbb{R}^3, \delta + h)$ as before (since the Σ_k are inside a euclidean ball, one can use cutoff functions to make $h_{\mu\nu}$ with compact support in $\mathbb{R}^3$).

Now let us perform the following

Blow up procedure: Recall that in our coordinates $\Sigma_k \to 0$ in Hausdorff distance sense (in $(\mathbb{R}^3, \delta + h)$). Since the Hausdorff convergence is a topological notion and of course the topology of $(\mathbb{R}^3, \delta)$ and of $(\mathbb{R}^3, \delta + h)$ is the same, we have that $\Sigma_k \to 0$ in Hausdorff distance sense in $(\mathbb{R}^3, \delta)$ so

$$\Sigma_k \subset B^e_{r_k}(0) \quad \text{and} \quad r_k \downarrow 0 \qquad (3.39)$$

where, as in the previous notation, $B^e_{r_k}(0)$ denotes the ball of center 0 and radius r_k in $(\mathbb{R}^3, \delta)$. As done in Proposition 3.3.3 using the vector space structure of $\mathbb{R}^3$, let us rescale everything (the surfaces and the metric) by $1/r_k$ getting

$$\begin{aligned} \tilde{\Sigma}_k &:= \frac{1}{r_k} \Sigma_k \qquad (3.40) \\ (h_k)_{\mu\nu}(x) &:= h_{\mu\nu}(r_k x). \qquad (3.41) \end{aligned}$$

It follows that

$$\begin{aligned} \tilde{\Sigma}_k &\subset B^e_1(0) \qquad (3.42) \\ \|h_k\|_{C^1(B^e_2(0))} &\downarrow 0 \qquad (3.43) \end{aligned}$$

where equation (3.43) just expresses the uniform convergence in $B^e_2(0)$ of h_k and Dh_k given by the continuity of h, Dh and by the property (5.5). Observe that we can assume that $\operatorname{spt} h_k \subseteq B^e_2(0)$; indeed if we multiply each h_k by a fixed cutoff function identically 1 in $B^e_1(0)$, 0 outside $B^2_e(0)$ and with bounded gradient, nothing changes on $\tilde{\Sigma}_k$ and (3.43) remains true. So summarizing we can assume that

$$\operatorname{spt} h_k \subseteq B^e_2(0) \quad \text{and} \quad \|h_k\|_{C^1(B^e_2(0))} \downarrow 0. \qquad (3.44)$$

As in the Proof of Proposition 3.3.3, from the definitions it is easy to check the scale invariance of the Willmore functional given in (3.24); the fact implies that $W_g(\Sigma_k) =: W_h(\Sigma_k) = W_{h_k}(\tilde{\Sigma}_k)$, so from assumption ii)

$$\limsup_k W_{h_k}(\tilde{\Sigma}_k) = \limsup_k W_g(\Sigma_k) \leq 4\pi - \epsilon. \tag{3.45}$$

Now, from the properties of h_k stated in (3.44), for large k we are in position to apply the inequality (3.11) of Lemma 3.2.4 to $\tilde{\Sigma}_k \hookrightarrow (\mathbb{R}^3, \delta + h_k)$. In the present case we get

$$(1 - C\|h_k\|_{C^1(B_2^c(0))} - C\gamma - C_\gamma \|h_k\|^2_{C^1(B_2^c(0))}) W_e(\tilde{\Sigma}_k) - C_G(\|h_k\|_{C^1(B_2^c(0))} + \gamma) \leq W_{h_k}(\tilde{\Sigma}_k) \tag{3.46}$$

where all the constants are independent of k, $\gamma > 0$ can be chosen arbitrarily small and C_γ, C_G are constants depending on γ (respectively on the genus bound G) such that $C_\gamma \to \infty$ if $\gamma \to 0$ (respectively $C_G \to \infty$ if $G \to \infty$) but which can be bounded by $C_\gamma \leq C(1 + \frac{1}{\gamma})$ for C large enough independent of γ. Now for each k choose $\gamma_k := \|h_k\|_{C^1(B_2^c(0))}$; then using the bound on C_γ and the boundness of $W_e(\tilde{\Sigma}_k)$ given by (3.12), taking the lim sup of both sides we obtain

$$\limsup_{k\to\infty} W_{h_k}(\tilde{\Sigma}_k) \geq \limsup_{k\to\infty} W_e(\tilde{\Sigma}_k) \geq 4\pi$$

where the last inequality comes from Theorem 7.2.2 in [Will]. The inequality clearly contradicts (3.45). ■

Putting together Corollary 3.3.2, Proposition 3.3.3 and Proposition 3.3.4 we have the following useful Corollary:

Corollary 3.3.5. *Let $(\mathbb{R}^3, \delta + h)$ be, as before, the euclidean space with a small in C^1 norm compactly supported perturbation h and assume that there exists a point $\bar{p}$ where the scalar curvature R_h is positive:*

$$R_h(\bar{p}) > 0.$$

Let $\{\Sigma_k\}_{k\in\mathbb{N}}$ be a sequence of immersed, smooth, closed, oriented surfaces of genus $\leq g$ and assume they are minimizing for α_h^g (see (3.1) for the definition).

Then, for $\|h\|_{C^1(\mathbb{R}^3)}$ small enough,

i)there exists a compact subset $K \subset \mathbb{R}^3$ such that

$$\Sigma_k \subseteq K \quad \forall k \in \mathbb{N},$$

ii) there exists a uniform area bound

$$|\Sigma_k|_h \leq C.$$

iii) there exists a lower diameter bound

$$\liminf_k (\operatorname{diam}_e \Sigma_k) > 0$$

where $\operatorname{diam}_e \Sigma_k$ *is the diameter of Σ_k in* $(\mathbb{R}^3, \delta)$.

iv) the 8π bound on the euclidean Willmore functional of the minimizing sequence holds asymptotically:

$$\limsup_k W_e(\Sigma_k) < 8\pi.$$

It follows by Theorem 6 in [LY] that the surfaces Σ_k are embedded for large k.

Proof. i) and ii) follow directly from Corollary 3.3.2 and Proposition 3.3.3. From Corollary 3.3.2 and Proposition 3.3.4 it follows that $\liminf_k(\operatorname{diam}_h \Sigma_k) > 0$ where $\operatorname{diam}_h \Sigma_k$ is the diameter of Σ_k in the Riemannian manifold $(\mathbb{R}^3, \delta + h)$. We can end the proof of iii) using part ii) of Lemma 3.2.1. Finally, putting together Corollary 3.3.2 and estimate (3.12) we obtain iv). ■

3.4 C^∞ regularity of an embedded 2-sphere minimizing W_h

In this section we work with a minimizing sequence Σ_k of embedded surfaces of genus 0. Using the a priori estimates on the minimizing sequence Σ_k of Section 4.1.1, adapting the regularity theory developed in [SiL], we will get the existence of a minimizer for the Willmore functional among embedded 2-spheres.

3.4.1 Existence of a minimizer, definitions of good/bad points and graphical decomposition Lemma

Thanks to the Allard compactness Theorem and the a priori estimates on the minimizing sequence of Section 4.1.1 we can state and prove the following compactness and lower semi continuity result.

Proposition 3.4.1. *Let $(\mathbb{R}^3, \delta + h)$ be as above, $\|h\|_{C^1(\mathbb{R}^3)}$ be small enough and assume there exists a point $\bar{p}$ such that the scalar curvature*

$$R_h(\bar{p}) > 0.$$

Consider $\{\Sigma_k\}_{k\in\mathbb{N}}$ a minimizing sequence for W_h of smooth embedded 2-spheres:

$$\lim_k W_h(\Sigma_k) \downarrow \alpha_h$$

where α_h is defined in (3.1).

Then there exists a non null integral 2-varifold V^h (associated to a 2-rectifiable set of $\mathbb{R}^3$) with square integrable weak mean curvature $H_h \in L^2(V^h)$ such that
i) up to subsequences

$$V^h_{\Sigma_k} \to V^h \quad \textit{in varifold sense}$$

where $V^h_{\Sigma_k}$ is the Allard varifold associated to $\Sigma_k \hookrightarrow (\mathbb{R}^3, \delta + h)$.
ii) from the lower semi continuity of the Willmore functional under varifold convergence

$$W_h(V^h) := \frac{1}{4}\int_{\mathbb{R}^3} |H_h|^2 d\mu^h \leq \liminf_k W_h(\Sigma_k) = \alpha_h$$

where $\mu^h := \|V^h\|$ is the mass measure associated to the varifold V^h.
iii) up to subsequences

$$\mu^e_k \to \mu^e \quad \textit{weak convergence of Radon measures}$$

where, for every $B \subseteq \mathbb{R}^3$ Borel set, $\mu^e_k(B) := |\Sigma_k \cap B|_e$ are the euclidean Radon measures naturally associated to Σ_k and μ^e is a Radon measure on $\mathbb{R}^3$ which satisfies $\operatorname{spt}\mu^e = \operatorname{spt}\mu^h$.
iv) the Radon measures converging subsequence also converges in Hausdorff distance sense:

$$\operatorname{spt}\mu^h_k = \operatorname{spt}\mu^e_k \to \operatorname{spt}\mu^e = \operatorname{spt}\mu^h =: \Sigma \quad \textit{in Hausdorff distance sense in } \mathbb{R}^3.$$

In particular from the lower diameter bound on the minimizing sequence (see iii) of Corollary 3.3.5) we have $\operatorname{diam}_e(\operatorname{spt}\mu^e) > 0$.

Proof. From Corollary 3.3.5 there exists a compact subset $K \subset\subset \mathbb{R}^3$ such that $\Sigma_k \subseteq K$ for all k and there exists C such that $|\Sigma_k|_h \leq C$. Since the surfaces have no boundary, using Schwartz inequality, we have the uniform bound on the first variation

$$|\delta V^h_{\Sigma_k}| = \int_{\Sigma_k} |H_h|\sqrt{\det(\delta \mathring{+} h)} \leq 2\sqrt{W_h(\Sigma_k)}\sqrt{|\Sigma_k|_h} \leq C.$$

From Allard compactness Theorem (see [Al], [SiGMT] or [Mant]), there exists an integral 2-varifold V^h (associated to a 2-rectifiable set of $\mathbb{R}^3$) with finite first variation (i.e. V^h_Σ has integrable weak mean curvature H_h) such that, up to subsequences,

$$V^h_{\Sigma_k} \to V^h \quad \text{in varifold sense.}$$

Since $\|h\|_{C^1(\mathbb{R}^3)}$ is small, from Lemma 3.2.4, we have

$$W_e(\Sigma_k) \leq C \quad \forall k \in \mathbb{N}$$

for some $C > 0$; using that (see Corollary 3.3.5)

$$0 < \liminf_k \operatorname{diam}_e(\Sigma_k) \leq \sup_k \operatorname{diam}_e(\Sigma_k) \leq C$$

and Lemma 1.1 of [SiL] we have

$$\frac{1}{C} \leq |\Sigma_k|_e \leq C \quad \text{for large } k \tag{3.47}$$

and, thanks to Lemma 3.2.2,

$$\frac{1}{C} \leq |\Sigma_k|_h \leq C \quad \text{for large } k. \tag{3.48}$$

Since the varifold convergence implies the weak convergence of the mass measures, on the converging subsequence we have

$$\|V^h\|(\mathbb{R}^3) = \lim_k \|V^h_{\Sigma_k}\|(\mathbb{R}^3) := \lim_k |\Sigma_k|_h \geq \frac{1}{C},$$

hence V^h is non null.

From the lower semi continuity of the Willmore functional under varifold convergence it follows that the weak mean curvature H_h of the limit varifold V^h_Σ is square integrable and, denoted $\mu^h := \|V^h\|$,

$$W_h(V^h) := \frac{1}{4}\int_{\mathbb{R}^3} |H_h|^2 d\mu^h \leq \liminf_k W_h(\Sigma_k) = \alpha_h.$$

To get iii) observe that (from Corollary 3.3.5) all the Radon measures μ^e_k have support in the same compact subset $K \subset\subset \mathbb{R}^3$ and (from inequality (3.47)) $\mu^e_k(K) := |\Sigma_k|_e \leq C$. From Banach-Alaoglu Theorem (see for example [SiGMT] page 22), μ^e_k converge up to subsequences to a limit Radon measure μ^e with $\operatorname{spt}\mu^e \subseteq K$. But recalling that the mass measures of $V^h_{\Sigma_k}$ weak converge to the mass measure of V^h, namely μ^h, and using Lemma 3.2.2 it is easy to see that $\operatorname{spt}\mu^e = \operatorname{spt}\mu^h$ previously defined.

In order to prove iv) recall that, from iii), $\mu^e_k \to \mu^e$ weak as Radon measures; moreover we have the uniform bound on the euclidean Willmore functional $W_e(\Sigma_k) \leq C$. These conditions imply the Hausdorff distance convergence of $\operatorname{spt}\mu^e_k \to \operatorname{spt}\mu^e$ (see [SiL] page 311).

■

Remark 3.4.2. *From Proposition 3.4.1 we have existence of a candidate minimizer V^h with spatial support Σ in metric $\delta + h$. Observe that up to now V^h is not a minimizer since it could be not smooth and a priori it may happen that $W_h(V^h) < \alpha_h$. Hence we have to study the regularity of $\Sigma := \operatorname{spt}\mu^h$; to this aim it is useful to consider both the euclidean geometric quantities of the minimizing sequence and the perturbed ones. The perturbed ones have been analyzed in Corollary 3.3.5, Proposition 3.4.1 and estimate* (3.48)*.*

About the euclidean quantities we have estimate (3.47)*; from Corollary 3.3.5 we know that for small $\|h\|_{C^1(\mathbb{R}^3)}$ we have $W_e(\Sigma_k) < 8\pi$ for large k. Moreover from the Gauss Bonnet Theorem*

$$\int_{\Sigma_k} |A_e|^2 \sqrt{\det \mathring{\delta}} = 4W_e(\Sigma_k) - 4\pi\chi_E(\Sigma_k) \leq C$$

Now we define the so called bad points with respect to a given $\varepsilon > 0$ in the following way: define the Radon measures α_k on $\mathbb{R}^3$ by

$$\alpha_k = \mu^h_k \llcorner |A_k|^2.$$

From Remark 3.4.2 we know $\alpha_k(\mathbb{R}^3) \leq C$, by compactness there exists a Radon measure α on $\mathbb{R}^3$ such that (after passing to a subsequence) $\alpha_k \to \alpha$ weak as Radon measures. It follows that $\operatorname{spt}\alpha \subset \Sigma = \operatorname{spt}\mu^h$ and $\alpha(\mathbb{R}^3) \leq C$. Now we define the bad points with respect to $\varepsilon > 0$ by

$$\mathcal{B}_\varepsilon = \left\{ \xi \in \Sigma \,\middle|\, \alpha(\{\xi\}) > \varepsilon^2 \right\}. \tag{3.49}$$

Since $\alpha(\mathbb{R}^3) \leq c$, there exist only finitely many bad points. Moreover for $\xi_0 \in \Sigma \setminus \mathcal{B}_\varepsilon$ there exists a $0 < \rho_0 = \rho_0(\xi_0, \varepsilon) \leq 1$ such that $\alpha(B_{\rho_0}(\xi_0)) < \frac{3}{2}\varepsilon^2$, and since $\alpha_k \to \alpha$ weakly as measures we get

$$\int_{\Sigma_k \cap B^e_{\rho_0}(\xi_0)} |A^h_k|^2 \, d\mathcal{H}^2_h \leq \frac{3}{2}\varepsilon^2 \quad \text{for } k \text{ sufficiently large} \tag{3.50}$$

where, as before, A^h_k and $\mathcal{H}^2_h$ denote the second fundamental form of Σ_k and the 2-dimensional Hausdorff measure in $(\mathbb{R}^3, \delta + h)$. Consider geodesic normal coordinates of the Riemannian manifold $(\mathbb{R}^3, \delta + h)$ centered at ξ_0 (the coordinates of ξ_0 are 0); in these coordinates the metric can be written as (see for example [LP] formula (5.4) page 61)

$$(\delta + h)_{\mu\nu}(x) = \delta_{\mu\nu} + \frac{1}{3}R_{\mu\sigma\lambda\nu}x^\sigma x^\lambda + O(|x|^3) = \delta_{\mu\nu} + o_1(1)(x)_{\mu\nu}$$

where, as before if $x \to 0$ we have $|o_1(1)(x)| + |Do_1(1)(x)| \to 0$. Called $inj(\xi_0) > 0$ the injectivity radius at ξ_0, for $\rho_0 < inj(\xi_0)$ we can put on $B_{\rho_0}(\xi_0)$ the normal coordinates just introduced and work on $\Sigma_k \cap B_{\rho_0}(\xi_0)$ as it was immersed in the manifold $(\mathbb{R}^3, \delta + \tilde{h})$ where $\|\tilde{h}\|_{C^1}$ can be taken arbitrarily small (for ρ_0 small enough). Then taking $\gamma > 0$ sufficiently small in estimate (3.9), using (3.2) and (3.14) we conclude that for ρ_0 small enough

$$\int_{\Sigma_k \cap B^e_{\rho_0}(\xi_0)} |A^e_k|^2 \, d\mathcal{H}^2_e \leq 2\varepsilon^2 \quad \text{for } k \text{ sufficiently large.} \tag{3.51}$$

Now fix $\xi_0 \in \Sigma \setminus \mathcal{B}_\varepsilon$ and let ρ_0 as in (5.2). Let $\xi \in \Sigma \cap B_{\frac{\rho_0}{2}}(\xi_0)$. We want to apply Simon's graphical decomposition lemma to show that the surfaces Σ_k can be written as a graph with small Lipschitz norm together with some "pimples" with small diameter in a neighborhood around the point ξ. This is done in exactly the same way Simon did in [SiL]. We just sketch this procedure. By the Hausdorff convergence there exists a sequence $\xi_k \in \Sigma_k$ such that $\xi_k \to \xi$. In view of (5.2) and the Monotonicity formula applied to Σ_k and ξ_k the assumptions of Simon's graphical decomposition lemma are satisfied for $\rho \leq \frac{\rho_0}{4}$ and infinitely many $k \in \mathbb{N}$. Since $W_e(\Sigma_k) \leq 8\pi - \delta_0$, we can apply Lemma 1.4 in [SiL] to deduce that for $\theta \in \left(0, \frac{1}{2}\right)$ small enough, $\tau \in \left(\frac{\rho}{4}, \frac{\rho}{2}\right)$ and infinitely many $k \in \mathbb{N}$ only one of the discs $D^k_{\tau,l}$ appearing in the graphical decomposition lemma can intersect the ball $B_{\theta\frac{\rho}{4}}(\xi_k)$. Moreover, by a slight perturbation from ξ_k to ξ, we may assume that $\xi \in L_k$ for all $k \in \mathbb{N}$. Now $L_k \to L$ in $\xi + G_2(\mathbb{R}^3)$, and therefore we may furthermore assume that the planes, on which the graph functions are defined, do not depend on $k \in \mathbb{N}$. After all we get a graphical decomposition in the following way.

Lemma 3.4.3. *For $\varepsilon \leq \varepsilon_0$, $\rho \leq \frac{\rho_0}{4}$ and infinitely many $k \in \mathbb{N}$ there exist pairwise disjoint closed subsets $P^k_1, \dots, P^k_{N_k}$ of Σ_k such that*

$$\Sigma_k \cap \overline{B_{\theta\frac{\rho}{8}}(\xi)} = D_k \cap \overline{B_{\theta\frac{\rho}{8}}(\xi)} = \left(\operatorname{graph} u_k \cup \bigcup_n P^k_n \right) \cap \overline{B_{\theta\frac{\rho}{8}}(\xi)},$$

where D_k is a topological disc and where the following holds:

1. *The sets P^k_n are topological discs disjoint from* $\operatorname{graph} u_k$.
2. *$u_k \in C^\infty(\overline{\Omega_k}, L^\perp)$, where $L \subset \mathbb{R}^3$ is a 2-dim. plane such that $\xi \in L$, and $\Omega_k = (B_{\lambda_k}(\xi) \cap L) \setminus \bigcup_m d_{k,m}$. Here $\lambda_k > \frac{\rho}{4}$ and the sets $d_{k,m} \subset L$ are pairwise disjoint closed discs.*
3. *The following inequalities hold:*

$$\sum_m \operatorname{diam} d_{k,m} + \sum_n \operatorname{diam} P^k_n \leq c \left(\int_{\Sigma_k \cap B^e_{2\rho}(\xi)} |A^e_k|^2 \, d\mathcal{H}^2_e \right)^{\frac{1}{4}} \rho \leq c\varepsilon^{\frac{1}{2}} \rho, \tag{3.52}$$

$$||u_k||_{L^\infty(\Omega_k)} \leq c\varepsilon^{\frac{1}{6}} \rho + \delta_k \quad \textit{where } \delta_k \to 0, \tag{3.53}$$

$$||Du_k||_{L^\infty(\Omega_k)} \leq c\varepsilon^{\frac{1}{6}} + \delta_k \quad \textit{where } \delta_k \to 0. \tag{3.54}$$

3.4.2 C^∞ regularity of Σ

Since this semiperturbative setting is closely related with the setting in [SiL], we just sketch the procedure for proving regularity pointing out the main differences with [SiL] and referring to the mentioned paper for more details (for details see also Chapter 5 or [Schy]).

Now we prove a power decay for the L^2-norm of the second fundamental form on small balls around the good points $\xi \in \Sigma \setminus \mathcal{B}_\varepsilon$. This will help us to show that Σ is actually $C^{1,\alpha} \cap W^{2,2}$ away from the bad points.

Lemma 3.4.4. *Let $\xi_0 \in \Sigma \setminus \mathcal{B}_\varepsilon$. There exists a $\rho_0 = \rho_0(\xi_0, \varepsilon) > 0$ such that for all $\xi \in \Sigma \cap B_{\frac{\rho_0}{2}}(\xi_0)$ and all $\rho \leq \frac{\rho_0}{4}$ we have*

$$\liminf_{k \to \infty} \int_{\Sigma_k \cap B_{\theta\frac{\rho}{8}}(\xi)} |A^h_k|^2 \, d\mathcal{H}^2_h \leq c\rho^\alpha,$$

where $\alpha \in (0,1)$ and $c < \infty$ are universal constants.

The proof of this Lemma is the same as in [SiL] (pages 299-300) noticing that in view of the expansion of the metric in normal coordinates as above one can pass from the setting $(\mathbb{R}^3, \delta + h)$ to the standard euclidean setting up to an error bounded by $c\rho^2$ (for more details see also the proof Lemma 5.3.1 where the proof is carried in an general Riemannian manifold).

Next we show that our candidate minimizer limit measure μ^h is given locally by a Lipschitz graph with small Lipschitz norm away from the bad points. Again we briefly sketch the construction referring to the aforementioned references for more details.

First of all one can replace the pimples of the Graphical Decomposition Lemma 3.4.3 with appropriate graph extensions with small C^1 norm, thus they converge to a Lipschitz function with small Lipschitz norm. Then, using a generalized Poincaré inequality proved in Lemma A.1 in [SiL], together with the previous Lemma 3.4.4 one proves that for all $\xi \in \Sigma \cap B^e_{\frac{\rho_0}{2}}(\xi_0)$ and all sufficiently small ρ

$$\mu^h \llcorner B^e_\rho(\xi) = \mathcal{H}^2_h \llcorner \big(\operatorname{graph} u \cap B^e_\rho(\xi)\big)\,, \tag{3.55}$$

where $u \in C^{0,1}(B^e_{\frac{\rho}{4}}(\xi) \cap L, L^\perp)$. For more details see the proof of Lemma 5.3.2 carried in general Riemannian manifold.

Since the limit measure μ^h has weak mean curvature in L^2, it follows that $u \in W^{2,2}$; moreover using Lemma 3.4.4 one has that the L^2 norm of the Hessian of u satisfies the following power decay

$$\int_{B_\rho \cap L} |D^2 u|^2 \leq c\rho^\alpha. \tag{3.56}$$

From Morrey's lemma (see [GT], Theorem 7.19) it follows that $u \in C^{1,\alpha} \cap W^{2,2}$. Thus our limit varifold V^h can be written as a $C^{1,\alpha} \cap W^{2,2}$-graph away from the bad points.

Now one excludes the bad points $\mathcal{B}_\varepsilon$ by proving a similar power decay as in Lemma 3.4.4 for balls around the bad points (notice that at this point we use that we are minimizing among spheres, because for higher bad points might appear), for details see Subsection 5.3.2. Therefore our candidate minimizer limit varifold V^h is given locally by a $C^{1,\alpha} \cap W^{2,2}$-graph everywhere.

Let us point out that by [SiL], $\operatorname{genus}(\Sigma) \leq \liminf_k \operatorname{genus}(\Sigma_k) = 0$ (for a different proof see Lemma 5.4.2). Via a standard approximation argument one can check that

$$\inf\{W_h(\Sigma) | \Sigma \text{ is a smooth embedded 2-sphere}\} = \inf\{W_h(\Sigma) | \Sigma \text{ is a } C^1 \cap W^{2,2} \text{ embedded 2-sphere}\}$$

Then by lower semicontinuity (Proposition 3.4.1) the limit embedded surface Σ is an embedded 2-sphere which minimizes W_h among $C^1 \cap W^{2,2}$-embedded 2-spheres, in particular it satisfies the Euler Lagrange equation

$$W'_h(\Sigma) = \frac{1}{2}\triangle H - \frac{1}{4} H(H^2 - 2|A|^2 - 2Ric_h(\nu,\nu))$$

where $\triangle$ is the Laplace Beltrami of the surface Σ and $Ric_h(\nu,\nu)$ is the Ricci tensor of $(\mathbb{R}^3, \delta+h)$ evaluated on the unit normal ν to Σ. It is a long and tedious computation but it is possible to check that the Euler Lagrange equation of W_h fits in Lemma 3.2 in [SiL].

It follows that the function u locally representing μ^h is actually $C^{2,\alpha} \cap W^{3,2}$ and the L^2 norm of the 3^{rd} derivatives satisfies the power decay $\int_{B_\rho} |D^3 u|^2 \leq c\rho^\alpha$. Now using the difference quotients method one proves that the function u is actually $C^{3,\alpha} \cap W^{4,2}$ and the L^2 norm of the 4^{th} derivatives satisfies the power decay $\int_{B_\rho} |D^4 u|^2 \leq c\rho^\alpha$; continuing this bootstrap argument one shows the smoothness of u and thus of Σ.

Chapter 4

Willmore type functionals in global setting: existence of a weak minimizing surface

4.1 Existence of a weak minimizer for $\frac{1}{2}\int |A|^2$ and $\int \left(\frac{|H|^2}{4}+1\right)$ in COMPACT Riemannian manifolds under curvature assumptions

Throughout this section (M,g) will be a closed (compact without boundary) Riemannian 3-dimensional manifold.

4.1.1 Global a priori estimates on the minimizing sequence

Under geometric assumptions on the ambient manifold we will show global a priori estimates on the minimizing sequences of the functionals E and W_1: more precisely we will get uniform upper area bounds, uniform upper and lower bounds on the diameters and lower 2-density bounds.

Upper Area bounds

Proposition 4.1.1. *Let (M,g) be a closed 3-dimensional manifold with positive sectional curvature $\bar{K}$:*

$$\exists \lambda \text{ such that } \bar{K} > \lambda^2 > 0. \tag{4.1}$$

Then, for every smooth immersion $f : \mathbb{S}^2 \hookrightarrow (M,g)$, the following area estimate holds:

$$|f(\mathbb{S}^2)|_g \leq \frac{1}{\lambda^2}\Big(4\pi + E(f)\Big) \tag{4.2}$$

where $|f(\mathbb{S}^2)|_g := \int_{\mathbb{S}^2} d\mu_g$ is the area of $\mathbb{S}^2$ equipped with the pull back metric f^*g given by the immersion.

Proof. Recall that by the Gauss equation

$$\bar{K}(T_x f) = K_G - k_1 k_2 = K_G - \frac{1}{4}H^2 + \frac{1}{2}|A^\circ|^2$$

where $\bar{K}(T_x f)$ is the sectional curvature of the ambient manifold evaluated on the plane $T_x f \subset T_x M$ with $x \in f(\mathbb{S}^2)$, K_G is the Gaussian curvature of $(\mathbb{S}^2, f^*g)$ (also called sectional curvature of the surface) and k_1, k_2 are the principal curvatures.
Integrating the assumption (4.1) and using Gauss Bonnet theorem we get

$$\begin{aligned} \lambda^2 |f(\mathbb{S}^2)|_g &\leq \int_{\mathbb{S}^2} \bar{K} d\mu_g = 2\pi\chi_E(\mathbb{S}^2) - W(f) + W_c(f) \\ &\leq 4\pi + W_c(f) \\ &\leq 4\pi + E(f) \end{aligned} \tag{4.3}$$

where, in the last two inequalities, we used that $W(f) > 0$ and $E(\Sigma) \leq W_c(f)$. ■

The next condition for the area bound is a very special case of the Isoperimetric Inequality given in Theorem 6.2.1 which holds for more general functionals in the contest of varifolds. For the basic concepts about varifolds see Appendix 6.6 (for more material the interested reader is referred to the paper of Hutchinson [Hu1] or to the book of Simon [SiGMT]).

Proposition 4.1.2. *Let (M, g) be a closed (compact, without boundary) Riemannian 3-manifold and assume that there are no nonzero 2-varifolds with null generalized second fundamental form.*

Then there exists a constant C such that for every smooth immersion $f : \mathbb{S}^2 \hookrightarrow (M, g)$, the following area estimate is true

$$|f(\mathbb{S}^2)|_g \leq C \int_{\mathbb{S}^2} |A|^2 d\mu_g. \tag{4.4}$$

Proof. In Section 5 of [Hu1] is proved that to an immersed closed smooth k-submanifold one can associate a k-varifold with generalized curvature; moreover the function $F(x, P, q) := |q|^2, q \in \mathbb{R}^{p^3}$ trivially satisfies the condition (6.6.2). Hence it is enough to apply Theorem 6.2.1 to the 2-varifold with curvature associated to $f(\mathbb{S}^2)$ and the specified function F. ■

Remark 4.1.3. *Let (M, g) be a Riemannian 3-manifold and $f_k : \mathbb{S}^2 \hookrightarrow (M, g)$ be a sequence of smooth immersions such that*

$$W_1(f_k) := \int_{\mathbb{S}^2} \left(\frac{|H|^2}{4} + 1 \right) d\mu_g < C$$

for some $C > 0$ independent of k. Then, of course, the area $|f_k(\mathbb{S}^2)|_g$ is uniformly bounded.

Diameter bounds on the minimizing sequences

First of all, since the ambient manifold is closed, we have a trivial upper bound on the diameter of the immersed surfaces. In this section we want to establish a lower bound on the diameters.

Lower diameter bounds for minimizing sequences of E

First let us prove that under quite general assumptions on the ambient manifold (namely we assume that the scalar curvature is strictly positive at a point) the infimum of the functional E is strictly less then the one of the corresponding functional in the euclidean space $\mathbb{R}^3$.

Lemma 4.1.4. *Let (M, g) be a (not necessarily compact) Riemannian 3-manifold and assume that there exists a point $\bar{p} \in M$ where the scalar curvature is strictly positive*

$$R_g(\bar{p}) > 0.$$

Then there exist $\epsilon > 0$ and small $\rho > 0$ such that the geodesic sphere $S_{\bar{p},\rho}$ of center $\bar{p}$ and radius ρ satisfies

$$E(S_{\bar{p},\rho}) := \frac{1}{4} \int_{S_{\bar{p},\rho}} |H|^2 d\mu_g + \frac{1}{2} \int_{S_{\bar{p},\rho}} |A^\circ|^2 d\mu_g < 4\pi - 2\epsilon.$$

Proof. From Proposition 3.1 of [Mon1], on geodesic spheres $S_{\bar{p},\rho}$ of center $\bar{p}$ and small radius ρ one has

$$W(S_{\bar{p},\rho}) := \frac{1}{4} \int_{S_{\bar{p},\rho}} |H|^2 d\mu_g = 4\pi - \frac{2\pi}{3} R_g(\bar{p}) \rho^2 + O(\rho^3).$$

Moreover, the quantity

$$\frac{1}{2} \int_{S_{\bar{p},\rho}} |A^\circ|^2 d\mu_g = \frac{1}{4} \int_{S_{\bar{p},\rho}} (k_1 - k_2)^2 d\mu_g = \int_{S_{\bar{p},\rho}} \Big(\frac{|H|^2}{4} - k_1 k_2 \Big) d\mu_g$$

is what we called Conformal Willmore functional and studied in Chapter 2. In that chapter the expansion of the functional on geodesic spheres of small radius is computed. Considering $w = 0$ (w is the perturbation of the geodesic sphere) in Lemma 2.2.5 and in Proposition 2.2.8, it is easy to check that

$$\frac{1}{2} \int_{S_{\bar{p},\rho}} |A^\circ|^2 d\mu_g = \int_{S_{\bar{p},\rho}} \Big(\frac{|H|^2}{4} - k_1 k_2 \Big) d\mu_g = O(\rho^4).$$

It follows that, since $R_g(\bar{p}) > 0$,

$$E(S_{\bar{p},\rho}) = 4\pi - \frac{2\pi}{3} R_g(\bar{p})\rho^2 + O(\rho^3) < 4\pi - 2\epsilon$$

for $\rho > 0$ and $\epsilon > 0$ small enough. ■

Remark 4.1.5. *Observe that if the ambient manifold (M, g) is the euclidean space $(\mathbb{R}^3, \delta)$, for every smooth closed immersed surface Σ, the functional $E(\Sigma) \geq W(\Sigma) \geq 4\pi$ (the last inequality is a famous Theorem of Willmore which can be found in [Will], Theorem 7.2.2); moreover the equality $E(\Sigma) = 4\pi$ is reached if and only if Σ is a round sphere.*

Thanks to Lemma 4.1.4 and Remark 4.1.5, using a blow up procedure it is possible to prove a lower diameter bound on the minimizing sequences of E:

Proposition 4.1.6. *Let (M, g) be a closed Riemannian 3-manifold whose scalar curvature is strictly positive at a point:*

$$\exists \bar{p} \in M : \quad R_g(\bar{p}) > 0.$$

Consider a sequence of immersions $\{f_k : \mathbb{S}^2 \hookrightarrow M\}_{k\in\mathbb{N}}$ minimizing E (i.e a minimizing sequence of the problem (1.12))

Then

$$\liminf_k (\operatorname{diam}_g f_k(\mathbb{S}^2)) > 0$$

where $\operatorname{diam}_g f_k(\mathbb{S}^2)$ is the diameter of $f_k(\mathbb{S}^2)$ in the Riemannian manifold (M, g).

Proof. From the assumption on the scalar curvature and since $\{f_k\}_{k\in\mathbb{N}}$ is a minimizing sequence of E, Lemma 4.1.4 implies that

$$\lim_k E(f_k) \leq 4\pi - 2\epsilon.$$

Since M is compact and $W(f_k) \leq E(f_k)$, the minimizing sequence satisfies the assumptions of Proposition 3.3.4 and the conclusion follows. ■

Lower diameter and E bounds for the minimizing sequences of W_1

In this subsection we want to prove a bound from below on the diameters of minimizing sequences of W_1. In order to get this bound it is useful to prove that the $\inf W_1$ in the manifold is less then the corresponding inf in euclidean space; so let us compute the expansion of W_1 on small geodesic spheres.

Lemma 4.1.7. *Let (M, g) be a (not necessarily compact) Riemannian 3-manifold and assume that there exists a point $\bar{p} \in M$ where the scalar curvature is greater than 6*

$$R_g(\bar{p}) > 6$$

Then there exists $\epsilon > 0$ and small $\rho > 0$ such that the geodesic sphere $S_{\bar{p},\rho}$ of center $\bar{p}$ and radius ρ satisfies

$$W_1(S_{\bar{p},\rho}) := \int_{S_{\bar{p},\rho}} \frac{H^2}{4} d\mu_g + |S_{\bar{p},\rho}|_g < 4\pi - 2\theta.$$

Proof. From Proposition 3.1 of [Mon1], on geodesic spheres $S_{\bar{p},\rho}$ of center $\bar{p}$ and small radius ρ one has

$$W(S_{\bar{p},\rho}) := \frac{1}{4}\int_{S_{\bar{p},\rho}} |H|^2 d\mu_g = 4\pi - \frac{2\pi}{3} R_g(\bar{p})\rho^2 + O(\rho^3).$$

From equation (8) in the Proof of Proposition 3.1 in [Mon1], we have the following expansion of the area of small geodesic spheres:

$$|S_{\bar{p},\rho}|_g = 4\pi\rho^2 + O(\rho^4).$$

Hence the expansion of W_1 on small geodesic spheres is

$$W_1(S_{\bar{p},\rho}) = 4\pi - \Big(\frac{2}{3}R_g(\bar{p}) - 4\Big)\pi\rho^2 + O(\rho^3).$$

We can conclude that, if $R_g(\bar{p}) > 6$, for $\rho > 0$ and $\epsilon > 0$ small enough we have the thesis. ■

Remark 4.1.8. *Observe that if the ambient manifold (M, g) is the euclidean space $(\mathbb{R}^3, \delta)$, for every smooth closed immersed surface Σ, the functional $W_1(\Sigma) > W(\Sigma) \geq 4\pi$. Moreover taking the sequence of round spheres $S_p^{1/n}$ of center p and radius $1/n$, $W_1(S_p^{1/n}) = 4\pi + \frac{\pi\epsilon}{n^2} \downarrow 4\pi$. Therefore in the euclidean space the infimum of W_1 is 4π and it is never achieved, so the condition on the scalar curvature is not purely technical but is somehow necessary to prevent the shrinking of the minimizing sequences.*

Using Lemma 4.2.1 one can repeat the proof of Proposition 4.1.6 and obtain the desired lower bound on the diameters:

Proposition 4.1.9. *Let (M, g) be a closed Riemannian 3-manifold whose scalar curvature is greater than 6 at one point:*

$$\exists \bar{p} \in M : \quad R_g(\bar{p}) > 6.$$

Consider a sequence of immersions $\{f_k : \mathbb{S}^2 \hookrightarrow M\}_{k \in \mathbb{N}}$ minimizing W_1 (i.e a minimizing sequence of the problem (1.13))

Then

$$\limsup_k W_1(f_k) < 4\pi \quad \text{and} \quad \liminf_k (\operatorname{diam}_g f_k(\mathbb{S}^2)) > 0$$

where $\operatorname{diam}_g f_k(\mathbb{S}^2)$ is the diameter of $f_k(\mathbb{S}^2)$ in the Riemannian manifold (M, g).

A local area bound for surfaces in Riemannian manifolds

In this subsection we will prove a quadratic area decay for immersions with equibounded area and Willmore functional, and lying in a fixed compact subset of a Riemannian 3-manifold (M, g).

Lemma 4.1.10. *Let (M, g) be a (maybe non compact) Riemannian 3-manifold and $K \subset\subset M$ a compact subset. Consider a smooth immersion $f : \mathbb{S}^2 \hookrightarrow K \subset M$ with bounded area:*

$$|f(\mathbb{S}^2)|_g \leq c_1$$

and whose Willmore functional is bounded by a constant c_2:

$$W(f) := \frac{1}{4} \int_{\mathbb{S}^2} |H_g|^2 d\mu_g \leq c_2.$$

Then there exists a constant $C_{K,c_1,c_2} > 0$ depending only on K, c_1 and c_2 such that for every $\xi \in M$ and every $\rho > 0$

$$\mu_f(B^g_\rho(\xi)) \leq C_{K,c_1,c_2}\, \rho^2,$$

where $\mu_f(B^g_\rho(\xi)) = \int_{f^{-1}(B^g_\rho(\xi))} d\mu_g$ is the area of the intersection $f(\mathbb{S}^2) \cap B^g_\rho(\xi)$.

Proof. By Nash Theorem we can assume that $M \hookrightarrow \mathbb{R}^S$ is isometrically embedded for some $p \geq 4$; hence $f(\mathbb{S}^2) \subset M \subset \mathbb{R}^S$ can be seen both as a surface in M and as a surfaces in $\mathbb{R}^S$. We call $H_{\mathbb{S}^2 \hookrightarrow \mathbb{R}^S}$ and $H_{\mathbb{S}^2 \hookrightarrow M}$ the mean curvature of $f(\mathbb{S}^2)$ as immersed surface in $\mathbb{R}^S$ (respectively in M); $A_{M \hookrightarrow \mathbb{R}^S}$ denotes the second fundamental form of M as submanifold of $\mathbb{R}^S$. The following estimate holds:

$$|H_{\mathbb{S}^2 \hookrightarrow \mathbb{R}^S}|^2 \leq |H_{\mathbb{S}^2 \hookrightarrow M}|^2 + C|A_{M \hookrightarrow \mathbb{R}^S}|^2$$

where C is universal constant depending only on the dimensions. Integrating over $\mathbb{S}^2$ we obtain

$$\int_{\mathbb{S}^2} |H_{\mathbb{S}^2 \hookrightarrow \mathbb{R}^S}|^2 d\mathcal{H}^2_{\mathbb{R}^S} \leq 4W(f) + C \int_{\mathbb{S}^2} |A_{M \hookrightarrow \mathbb{R}^S}|^2 d\mathcal{H}^2_{\mathbb{R}^S} \tag{4.5}$$

where $\mathcal{H}^2_{\mathbb{R}^S}$ is the area form induced by the immersion in $\mathbb{R}^S$ (observe that the area measure $d\mu_g$ of $\mathbb{S}^2$ as surface in M is the same as $d\mathcal{H}^2_{\mathbb{R}^S}$ from the Nash isometric embedding). Since $f(\mathbb{S}^2) \subset K$ and K is a compact set, then $\int_{\mathbb{S}^2} |A_{M \hookrightarrow \mathbb{R}^S}|^2 d\mathcal{H}^2_{\mathbb{R}^S} \leq \max_K |A_{M \hookrightarrow \mathbb{R}^S}|^2 |f(\mathbb{S}^2)|_g \leq C_{K,c_1}$. Using the assumed estimate on the Willmore functional we obtain

$$\int_{\mathbb{S}^2} |H_{\mathbb{S}^2 \hookrightarrow \mathbb{R}^S}|^2 d\mathcal{H}^2_{\mathbb{R}^S} \leq C_{K,c_1,c_2}. \tag{4.6}$$

Let us denote by $B^{\mathbb{R}^S}_\rho(\xi)$ the ball in $\mathbb{R}^S$ of center ξ and radius ρ. From Simon's Monotonicity formula for immersed surfaces in $\mathbb{R}^S$ (see formula (1.3) in [SiL])

$$|f(\mathbb{S}^2) \cap B^{\mathbb{R}^S}_\rho(\xi)|_{\mathbb{R}^S} \leq C\rho^2 \int_{\mathbb{S}^2} |H_{\Sigma \hookrightarrow \mathbb{R}^S}|^2 d\mathcal{H}^2_{\mathbb{R}^S} \leq C_{K,c_1,c_2}\, \rho^2,$$

where $|f(\mathbb{S}^2) \cap B^{\mathbb{R}^S}_\rho(\xi)|_{\mathbb{R}^S}$ is the area of $f^{-1}(B^{\mathbb{R}^S}_\rho(\xi))$ with respect to the area form induced by the immersion in $\mathbb{R}^S$ and the constant C in the first inequality depends only on the dimensions.

Now observe that the metric ball $B^g_\rho(\xi)$ in the Riemannian manifold (M,g) with center ξ and radius ρ is always contained in $B^{\mathbb{R}^S}_\rho(\xi)$: $B^g_\rho(\xi) \subset B^{\mathbb{R}^S}_\rho(\xi)$; moreover, as remarked above, $d\mu_g$ coincide with $d\mathcal{H}^2_{\mathbb{R}^S}$. We can conclude that

$$\mu_f(B^g_\rho(\xi)) = |f(\mathbb{S}^2) \cap B^g_\rho(\xi)|_{\mathbb{R}^S} \leq |f(\mathbb{S}^2) \cap B^{\mathbb{R}^S}_\rho(\xi)|_{\mathbb{R}^S} \leq C_{K,c_1,c_2}\, \rho^2.$$

■

4.1.2 Properties of the functionals and compactness of the minimizing sequences

Compactness and lower semicontinuity of the functional E

Before stating the fundamental theorem, let us recall some notation. If $f_k : \mathbb{S}^2 \hookrightarrow M$ is a sequence of smooth immersions, as explained in Appendix 6.6, to each f_k we can associate a 2-varifold with curvature (V_{f_k}, A_k) also denoted for simplicity with (V_k, A_k). The spatial measure $\|V_k\|$ will be also called μ_k and is simply $\mu_k(B) := \int_{f_k^{-1}(B)} d\mu_g$, $\forall B \subset M$ Borel set, i.e. the area of $f_k(S^2) \cap B$ counted with multiplicity.

Theorem 4.1.11. *Let (M,g) be a closed Riemannian 3-manifold which satisfies at least one of the two conditions below:*
- (M,g) has uniformly strictly positive sectional curvature in the sense of (4.1)*; or*
- there is no nonzero 2-varifold of M with null generalized second fundamental form and there is a point $\bar{p}$ where the scalar curvature is strictly positive: $R_g(\bar{p}) > 0$.

Let $\{f_k\}_{k\in\mathbb{N}}$ be a minimizing sequence of smooth immersions $f_k : \mathbb{S}^2 \hookrightarrow M$ for E, i.e. a minimizing sequence for problem (1.12)*. Then the following properties hold:*
i) there exists an integral 2-varifold V of M such that, up to subsequences, $V_k \to V$ in varifold sense,
ii) V is an integral varifold with generalized second fundamental form A (so we write $(V,A) \in CV_2(M)$),
iii) called $\{(V_k, A_k)\}_{k\in\mathbb{N}}$ the measure-function pairs associated to the immersions f_k with second fundamental forms A_k,

$$(V_k, A_k) \rightharpoonup (V, A) \quad \textit{weak converge of measure-function pairs,}$$

iv) called $\mu := \|V\|$ the mass measure of the limit varifold V then up to subsequences

$$\mu_k \rightharpoonup \mu \quad \textit{weak as Radon measures and}$$

$$\operatorname{spt}\mu_k \to \operatorname{spt}\mu \quad \textit{in Hausdorff distance sense,}$$

v) $E(\mu) := E(V) := \frac{1}{2}\int_{G_2(M)} |A(x,P)|^2 dV \leq \liminf_k E(f_k) < 4\pi$,
vi) $\operatorname{spt}\mu$ *is compact, connected and*

$$\operatorname{diam}_g(\operatorname{spt}\mu) \geq \liminf_k(\operatorname{diam}_g \operatorname{spt}\mu_k) > 0.$$

Proof.

i): First, by Lemma 4.1.4, since $\{f_k\}_{k\in\mathbb{N}}$ is a minimizing sequence of E, we have

$$\liminf_k E(f_k) = \lim_k E(f_k) < 4\pi.$$

From Proposition 4.1.1 and Corollary 4.1.2 it follows a uniform bound on the areas of $f_k(\mathbb{S}^2)$:

$$\exists C > 0 : \forall k \in \mathbb{N} \quad |f_k(\mathbb{S}^2)|_g < C.$$

Using Cauchy-Schwartz we can estimate the first variation of the varifolds V_k associated to the immersions f_k:

$$\|\delta V_k\| := \int_{\mathbb{S}^2} |H_k| d\mu_g \leq C \int_{\mathbb{S}^2} |A_k| d\mu_g \leq C \Big(\int_{\mathbb{S}^2} |A_k|^2 d\mu_g \Big)^{\frac{1}{2}} \sqrt{|f_k(\mathbb{S}^2)|_g} \leq C.$$

Now applying Allard compactness Theorem (Theorem 6.4 in [Al]) to the sequence of varifolds V_k, we can say that there exists an integral 2-varifold V such that

$$V_k \to V \quad \text{in varifold sense.}$$

ii), iii) and v): Of course the integrand of the functional E satisfies the condition (6.6.2). Using the previous point i), we can apply the Theorem 6.6.7 of Hutchinson and say that $V \in CV_2(M)$ with generalized second fundamental form A. Hence we can define the functional E also on V:

$$E(V) := \int_{G_2(M)} |A(x,P)|^2 dV.$$

Properties iii) and v) follow again from Theorem 6.6.7.

iv): Of course, since the Grassmannian of the 2-planes in $\mathbb{R}^3$ is compact, the varifold convergence of a sequence of varifolds implies the measure theoretic convergence of the spatial supports; so up to subsequences

$$\mu_k \rightharpoonup \mu \quad \text{weak as Radon measures.}$$

In order to get the Hausdorff convergence recall that $M \subset \mathbb{R}^S$ is isometrically embedded by Nash Theorem, so we can see the surfaces $f_k(\mathbb{S}^2)$ as immersed in $\mathbb{R}^S$.

Exactly as in the proof of Lemma 4.1.10, it is possible to prove that $f_k(\mathbb{S}^2) \subset \mathbb{R}^S$ is a sequence of surfaces with uniformly bounded Willmore functional. Moreover we know that the associated measures $\mu_k \rightharpoonup \mu$; under this conditions Leon Simon proved (see [SiL] pages 310-311) that actually

$$\operatorname{spt} \mu_k \to \operatorname{spt} \mu \quad \text{in Hausdorff distance sense}$$

as subsets of $\mathbb{R}^S$; but since M is isometrically embedded in $\mathbb{R}^S$ it clearly implies that $\operatorname{spt} \mu_k \to \operatorname{spt} \mu$ in Hausdorff distance as subsets of M.

vi): The inequality $\liminf_k (\operatorname{diam}_g \operatorname{spt} \mu_k) > 0$ follows from Proposition 4.1.6. Called k' the subsequence converging in Hausdorff distance sense, from the definition of Hausdorff convergence it is easy to see that $\operatorname{diam}_g \operatorname{spt} \mu = \lim_{k'} (\operatorname{diam}_g \operatorname{spt} \mu_{k'})$. Hence

$$\operatorname{diam}_g \operatorname{spt} \mu = \lim_{k'} (\operatorname{diam}_g \operatorname{spt} \mu_{k'}) \geq \liminf_k (\operatorname{diam}_g \operatorname{spt} \mu_k) > 0.$$

About the topological properties of $\operatorname{spt} \mu$ observe that by definitions it is a closed subset of the compact manifold M so it is compact. Moreover $f_k(\mathbb{S}^2)$ is connected and the Hausdorff distance limit of a sequence of connected subsets must be connected. ■

Remark 4.1.12. *The limit varifold with curvature (V, A) is the candidate to be the minimizer of E among the immersions of $\mathbb{S}^2$. We remark that V, up to now, is only a candidate minimizer since it may not be smooth (so the value of the functional could be a priori strictly less than the inf on the smooth immersions) and it may even vanish as measure since we have not yet proved a lower bound on the areas of $f_k(\mathbb{S}^2)$. In the following we will prove the desired lower bound and the regularity.*

Compactness and lower semicontinuity of the functional W_1

In this subsection we prove a counterpart of the compactness lower semicontinuity Theorem 4.2.6 for the functional W_1. Before stating it let us recall that the theory of the varifolds with weak mean curvature can be seen as a part of the theory of the measure-function pairs of Hutchinson [Hu1] (see Appendix 6.6). We mean that a varifold with weak mean curvature can be seen as a measure function pair (V, H) with H vector valued $L^1_{loc}(V)$ function which satisfies an integration by parts formula (the corresponding of the tangential divergence theorem for smooth surfaces $\int_\Sigma div_\Sigma X d\mu_g = -\int_\Sigma H \cdot X d\mu_g$ where X is any $C^1_c(M)$ vector field tangent to the ambient manifold M and div_Σ is the tangential divergence on Σ).

Theorem 4.1.13. *Let (M,g) be a compact Riemannian 3-manifold with scalar curvature strictly greater than 6 at a point:*

$$\exists \bar{p} \in M : \quad R_g(\bar{p}) > 6.$$

Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of smooth immersions $f_k : \mathbb{S}^2 \hookrightarrow M$ minimizing W_1, i.e. a minimizing sequence for problem (1.13).

Then the following compactness and lower semicontinuity properties hold:
i) there exists an integral 2-varifold V of M such that, up to subsequences, $V_k \to V$ in varifold sense,
ii) V is an integral varifold with weak mean curvature $H \in L^2(V)$,
iii) called $\{(V_k, H_k)\}_{k\in\mathbb{N}}$ the measure-function pairs associated to the immersions f_k with mean curvatures H_k,

$$(V_k, H_k) \rightharpoonup (V, H) \quad \textit{weak converge of measure-function pairs,}$$

iv) called $\mu := \|V\|$ the mass measure of the limit varifold V then up to subsequences

$$\mu_k \rightharpoonup \mu \quad \textit{weak as Radon measures and}$$

$$\operatorname{spt} \mu_k \to \operatorname{spt} \mu \quad \textit{in Hausdorff distance sense,}$$

v) We can define

$$W_1(\mu) := W_1(V) := \int_{G_2(M)} \left(\frac{|H(x,P)|^2}{4} + 1 \right) dV \leq \liminf_k W_1(f_k) < 4\pi,$$

vi) $\operatorname{spt} \mu$ *is compact, connected and*

$$\operatorname{diam}_g(\operatorname{spt} \mu) \geq \liminf_k (\operatorname{diam}_g \operatorname{spt} \mu_k) > 0.$$

Proof. i): since $W_1(f_k)$ is uniformly bounded on k, by the very definition on W_1 it follows a uniform bound on the areas and on the Willmore functionals of f_k:

$$\exists C > 0 \quad : \quad \forall k \in \mathbb{N} \quad |f_k(\mathbb{S}^2)|_g < C, \tag{4.7}$$

$$\exists C > 0 \quad : \quad \forall k \in \mathbb{N} \quad \int_{\mathbb{S}^2} |H_k|^2 d\mu_g < C. \tag{4.8}$$

Using Schwartz inequality we can estimate the first variation of the varifold V_k associated to the immersion f_k:

$$\|\delta V_k\| := \int_{\mathbb{S}^2} |H_k| d\mu_g \leq \Big(\int_{\mathbb{S}^2} |H_k|^2 d\mu_g \Big)^{\frac{1}{2}} \sqrt{|f_k(\mathbb{S}^2)|_g} \leq C.$$

Now applying Allard compactness Theorem (Theorem 6.4 in [Al]) to the sequence of varifolds V_k, we can say that there exists an integral 2-varifold V such that

$$V_k \to V \quad \text{ in varifold sense.}$$

iv) and vi): the proof is analogous to the corresponding statements in Theorem 4.2.6.

ii), iii) and v): first of all let us observe that, since from *i)* up to subsequences $V_k \to V$ in varifold sense, then we have the convergence of the masses

$$\|V_k\|(M) = |f_k(\mathbb{S}^2)|_g \to \|V\|(M).$$

Hence the second adding of the functional W_1 is continuous under varifold convergence and we are left to consider the Willmore functional $\frac{1}{4}\int_{G_2(M)} |H_k|^2 dV_k$. Recall the uniform bound on the Willmore functionals (4.8) and observe that the function $F(x,p,q) := |q|^2$ satisfies the condition 6.6.2. Then we can apply the compactness-lower semicontinuity Theorem 6.2.7 for integral varifolds with weak mean curvatures (observe that now we don't need the hypothesis on the non existence of a stationary varifold since we already have the uniform mass bound (4.7)). ii), iii) and v) follow. ■

For the regularity theory it will be useful the following Lemma.

Lemma 4.1.14. *Let (M,g) be a (maybe non compact) manifold with sectional curvature bounded above by 2, $\bar{K} \leq 2$, and let $f : \mathbb{S}^2 \hookrightarrow M$ be a smooth immersion. Then one has the following bound:*

$$E(f) \leq 2W_1(f) - 4\pi$$

PROOF. For a general immersion $f : \mathbb{S}^2 \hookrightarrow M$, by Gauss equation we can write

$$\frac{1}{2}|A|^2 = \frac{1}{2}H^2 - k_1 k_2 = \frac{1}{2}H^2 - K_G + \bar{K}(T_x f) = \left(\frac{H^2}{4} + 1\right) + \left(\frac{H^2}{4} + \bar{K}(T_x f) - 1\right) - K_G$$

where $\bar{K}(T_x f)$ is the sectional curvature of the ambient manifold evaluated on the plane $T_x f \subset T_x M$ with $x \in f(\mathbb{S}^2)$, K_G is the Gaussian curvature of $(\mathbb{S}^2, f^*g)$ and k_1, k_2 are the principal curvatures. Integrating, by Gauss Bonnet theorem and the bound $\bar{K} \leq 2$, we get

$$E(f) \leq 2W_1(f) - 2\pi\chi_E(\mathbb{S}^2) = 2W_1(f) - 4\pi$$

■

Remark 4.1.15. *The limit varifold V is the candidate minimizer of W_1 among smooth immersions of $\mathbb{S}^2$ in M but it is not trivially a minimizer since, up to now, it may not be smooth and its measure may even vanish since we have not yet proved a lower bound on the areas of $f_k(\mathbb{S}^2)$.*

4.2 Existence of a weak minimizer for $\int \left(\frac{|H|^2}{4} + 1\right)$ and $\int \left(\frac{|A|^2}{2} + 1\right)$ in NONCOMPACT asymptotically euclidean Riemannian manifolds under curvature assumptions

4.2.1 A priori bounds on the minimizing sequences of W_1 and E_1

In this subsection we want to prove a bound from below on the diameters of minimizing sequences of W_1 and E_1. In order to get this bound it is useful to prove that the infimum of W_1 in the manifold is less than the corresponding infimum in euclidean space; so let us compute the expansion of W_1 on small geodesic spheres.

Lemma 4.2.1. *Let (M,g) be a (maybe non compact) Riemannian 3-manifold and assume that there exists a point $\bar{p} \in M$ where the scalar curvature is greater than 6*

$$R_g(\bar{p}) > 6.$$

Then there exists $\epsilon > 0$ and small $\rho > 0$ such that the geodesic sphere $S_{\bar{p},\rho}$ of center $\bar{p}$ and radius ρ satisfies

$$W_1(S_{\bar{p},\rho}) := \int_{S_{\bar{p},\rho}} \frac{H^2}{4} d\mu_g + |S_{\bar{p},\rho}|_g < 4\pi - 2\epsilon,$$

$$E_1(S_{\bar{p},\rho}) := \int_{S_{\bar{p},\rho}} \frac{|A|^2}{2} d\mu_g + |S_{\bar{p},\rho}|_g < 4\pi - 2\epsilon.$$

PROOF. From Proposition 3.1 of [Mon1], on geodesic spheres $S_{\bar{p},\rho}$ of center $\bar{p}$ and small radius ρ one has

$$W(S_{\bar{p},\rho}) := \frac{1}{4}\int_{S_{\bar{p},\rho}} |H|^2 d\mu_g = 4\pi - \frac{2\pi}{3} R_g(\bar{p})\rho^2 + O(\rho^3).$$

From equation (8) in the Proof of Proposition 3.1 in [Mon1], we have the following expansion of the area of small geodesic spheres:

$$|S_{\bar{p},\rho}|_g = 4\pi\rho^2 + O(\rho^4).$$

Hence the expansion of W_1 on small geodesic spheres is

$$W_1(S_{\bar{p},\rho}) = 4\pi - \left(\frac{2}{3}R_g(\bar{p}) - 4\right)\pi\rho^2 + O(\rho^3).$$

We conclude that, if $R_g(\bar{p}) > 6$, for $\rho > 0$ and $\epsilon > 0$ small enough we have the first inequality.

For the second inequality, observe that $\frac{1}{2}|A^2| = \frac{1}{4}H^2 + \frac{1}{2}|A^\circ|^2$. Moreover, the quantity

$$\frac{1}{2}\int_{S_{\bar{p},\rho}} |A^\circ|^2 d\mu_g = \frac{1}{4}\int_{S_{\bar{p},\rho}} (k_1 - k_2)^2 d\mu_g = \int_{S_{\bar{p},\rho}} \Big(\frac{|H|^2}{4} - k_1 k_2\Big) d\mu_g$$

is what we called Conformal Willmore functional and studied in Chapter 2. In that chapter the expansion of the functional on geodesic spheres of small radius is computed. Considering $w = 0$ (w is the perturbation of the geodesic sphere) in Lemma 2.2.5 and in Proposition 2.2.8, it is easy to check that

$$\frac{1}{2}\int_{S_{\bar{p},\rho}} |A^\circ|^2 d\mu_g = \int_{S_{\bar{p},\rho}} \Big(\frac{|H|^2}{4} - k_1 k_2\Big) d\mu_g = O(\rho^4).$$

It follows that $E_1(S_{\bar{p},\rho}) = W_1(S_{\bar{p},\rho}) + O(\rho^4)$ and we conclude as above. ■

Remark 4.2.2. *Observe that if the ambient manifold (M, g) is the euclidean space $(\mathbb{R}^3, \delta)$, for every smooth closed immersed surface Σ, the functional $W_1(\Sigma) > W(\Sigma) \geq 4\pi$. Moreover taking the sequence of round spheres $S_p^{1/n}$ of center p and radius $1/n$, $W_1(S_p^{1/n}) = 4\pi + \frac{4\pi}{n^2} \downarrow 4\pi$. So in the euclidean space the infimum of W_1 is 4π and it is never achieved.*

Proposition 4.2.3. *Let (M, g) be a Riemannian 3-manifold whose scalar curvature is strictly greater than 6 at one point:*

$$\exists \bar{p} \in M : \quad R_g(\bar{p}) > 6.$$

Consider a sequence of immersions $\{f_k : \mathbb{S}^2 \hookrightarrow M\}_{k\in\mathbb{N}}$ minimizing W_1 ,respectively E_1 (i.e a minimizing sequence of the problem (1.19), respectively (1.20))

Then

$$\liminf_k (\mathrm{diam}_g\, f_k(\mathbb{S}^2)) > 0$$

where $\mathrm{diam}_g\, f_k(\mathbb{S}^2)$ is the diameter of $f_k(\mathbb{S}^2)$ in the Riemannian manifold (M, g).

Proof. From the assumption on the scalar curvature, if $\{f_k\}_{k\in\mathbb{N}}$ is a minimizing sequence of W_1, Lemma 4.2.1 implies that

$$\lim_k W_1(f_k) \leq 4\pi - 2\epsilon. \tag{4.9}$$

Observe that if instead $\{f_k\}_{k\in\mathbb{N}}$ is a minimizing sequence of E_1, Lemma 4.2.1 implies $\lim_k E_1(f_k) \leq 4\pi - 2\epsilon$, but as shown in the proof of the previously cited Lemma, $E_1(f_k) \geq W_1(f_k)$, so inequality (4.9) also holds for minimizing sequences of E_1.

Let us assume by contradiction that $\liminf_k(\mathrm{diam}_g\, f_k(\mathbb{S}^2)) = 0$, so up to subsequences $\mathrm{diam}_g\, f_k(\mathbb{S}^2) \to 0$.

First, using inequality (4.9) and that (M, g) is asymptotically euclidean let us show that there exists a compact subset $K \subset\subset M$ such that $f_k(\mathbb{S}^2) \subset K$. If it in not the case than, up to subsequence, for every $k \in \mathbb{N}$ we can take a point $\xi_k \in f_k(\mathbb{S}^2) \subset \mathbb{R}^3$ (recall that outside a compact subset (M, g) is isometric to $(\mathbb{R}^3, \delta + o_1(1))$ such that $|\xi_k| \to \infty$. Since we are assuming that $\mathrm{diam}_g\, f_k(\mathbb{S}^2) \to 0$, for k large enough all of the surface $f_k(\mathbb{S}^2)$ is contained in a region where $o_1(1)$ is arbitrarily small in C^1 norm:

$$\liminf_k \|o_1(1)\|_{C^1(f_k(\mathbb{S}^2))} = 0.$$

Now for every fixed $k \in \mathbb{N}$ we can put $\gamma = r_0 = \mathrm{diam}_e(f_k(\mathbb{S}^2)) \overset{k\to\infty}{\to} 0$ in estimate (3.11) (repeat the proof of Lemma 3.2.4 with such quantities), since for k large also η and θ are arbitrarily small, passing to the liminf in estimate (3.11) we can conclude that

$$\liminf W_1(f_k) > \liminf W(f_k) \geq \liminf W_e(f_k) \geq 4\pi$$

contradicting (4.9); so there exists a compact subset $K \subset\subset M$ such that $f_k(\mathbb{S}^2) \subset K$ for all $k \in \mathbb{N}$.

Since $W_1(f_k) > W(f_k)$, the estimate (4.9) and the argument above put us in position to apply Proposition 3.3.4 and the conclusion follows. ■

Since the ambient manifold is non compact, it is not trivial a priori that the minimizing sequence of surfaces has a uniform upper diameter bound. But this is true, using a monotonicity formula of Link (see his Ph.D. Thesis [FL]).

Proposition 4.2.4. *Let (M, g) be a (maybe non compact) Riemannian 3-manifold such that*
i) the sectional curvature is uniformly bounded

$$|\bar{K}| \leq \Lambda^2 \quad \textit{for some } \Lambda \in \mathbb{R},$$

ii) the injectivity radius is uniformly bounded below away from 0

$$Inj(M, g) \geq \bar{\rho} > 0.$$

Then there exists a constant $C = C(\bar{\rho}, \Lambda) > 0$ such that for every $\Sigma \hookrightarrow (M, g)$ connected smooth closed immersed oriented surface we have

$$\operatorname{diam}_g \Sigma \leq \max\{1, C(|\Sigma|_g + W(\Sigma))\}.$$

Proof. If $\operatorname{diam}_g \Sigma \leq 1$ we have finished, so we can assume that $\operatorname{diam}_g \Sigma \geq 1$.
Under the assumptions i) and ii), Link proved (see [FL]) that there exists a constant $C = C(\bar{\rho}, \Lambda)$ such that for $0 < \sigma \leq \rho < \rho_0 = c \min(Inj(M), \frac{1}{\Lambda})$

$$\frac{|\Sigma \cap B_\sigma(x)|_g}{\sigma^2} \leq C\Big(\frac{|\Sigma \cap B_\rho(x)|_g}{\rho^2} + W(\Sigma \cap B_\rho(x))\Big).$$

From the smoothness and the compactness of Σ, sending $\sigma \to 0$ in the formula above, for every $\rho \leq \rho_0$ and $x \in \Sigma$ one has

$$1 \leq C\Big(\frac{|\Sigma \cap B_\rho(x)|_g}{\rho^2} + W(\Sigma \cap B_\rho(x))\Big). \tag{4.10}$$

Since Σ is compact there exists a pair of points $x, y \in \Sigma$ such that $d_g(x, y) = \operatorname{diam}_g \Sigma$. Let us divide the interval $[0, \operatorname{diam}_g \Sigma]$ in $N \geq 1$ ($N \in \mathbb{N}$ to be determined from ρ) subintervals, of the same length ρ with

$$\frac{1}{2}\min(1, \rho_0) < \rho \leq \min(1, \rho_0).$$

Consider the corresponding partition of the metric ball $B_{\operatorname{diam}_g \Sigma}(x)$ into N spherical (metric) annuli at distance ρ one to each other

$$A_i = B_{i\rho}(x) \backslash B_{(i-1)\rho}(x) \quad i = 1, \ldots, N$$

where $B_{i\rho}(x)$ is the metric ball.

Since the surface Σ is connected, for each annulus A_i there exists a metric ball $B_{\frac{\rho}{3}}(x_i) \subset A_i$ all contained in the annulus with center $x_i \in \Sigma$ and radius $\frac{\rho}{3}$. For each ball $B_{\frac{\rho}{3}}(x_i)$ we can apply the estimate (5.11) and summing on i we get

$$\begin{aligned} N &\leq C \sum_{i=1}^{N} \Big(\frac{|\Sigma \cap B_{\frac{\rho}{3}}(x_i)|_g}{\rho^2} + W(\Sigma \cap B_{\frac{\rho}{3}}(x_i))\Big) \\ &\leq C \left(\frac{|\Sigma|_g}{\rho^2} + W(\Sigma)\right) \end{aligned} \tag{4.11}$$

where the last inequality comes from the disjointness of the balls $B_{\frac{\rho}{3}}(x_i)$. Now multiplying both sides by ρ^2 we get

$$\rho \operatorname{diam}_g \Sigma = N\rho^2 \leq C(|\Sigma|_g + \rho^2 W(\Sigma)) \leq C(|\Sigma|_g + W(\Sigma)),$$

where the last inequality comes from the condition $\rho \leq 1$. Now, from the estimate $\frac{1}{2}\min(1, \rho_0) < \rho$ we have the bound $\frac{1}{\rho} < 2\max(1, 1/\rho_0) \leq C_{\bar{\rho},\Lambda}$ where $C_{\bar{\rho},\Lambda}$ is a constant depending on $\bar{\rho} \leq Inj(M)$ and Λ. We can conclude that

$$\operatorname{diam}_g \Sigma \leq C_{\bar{\rho},\Lambda}\,(|\Sigma|_g + W(\Sigma)).$$

■

Called $\{f_k\}_{k \in \mathbb{N}}$ a minimizing sequence of W_1 (respectively E_1), thanks to Proposition 4.2.4, Lemma 4.2.1 and Remark 4.2.2, it is possible to prove that the immersions $\{f_k\}_{k \in \mathbb{N}}$ are all valued in a compact subset of M.

Proposition 4.2.5. *Let (M,g) be a non compact asymptotically euclidean Riemannian 3-manifold (in the sense of (1.16)) with bounded geometry (i.e. with bounded sectional curvature $|\bar{K}| \leq \Lambda^2$ for some $\Lambda \in \mathbb{R}$, and strictly positive injectivity radius $Inj(M,g) \geq \bar{\rho} > 0$) whose scalar curvature is strictly greater than 6 at a point:*

$$\exists \bar{p} \in M : \quad R_g(\bar{p}) > 6.$$

Consider a minimizing sequence $\{f_k : \mathbb{S}^2 \hookrightarrow M\}_{k\in\mathbb{N}}$ of smooth immersions of W_1 (respectively of E_1) among immersions of the same kind.
Then there exists a compact subset $K \subset\subset M$ such that $f_k(\mathbb{S}^2) \subset K$ for all $k \in \mathbb{N}$.

Proof. As in the proof of Lemma 4.2.3, since $\{f_k\}_{k\in\mathbb{N}}$ is a minimizing sequence of W_1 (respectively E_1) we know that estimate (4.9) holds, namely $W_1(f_k) \leq 4\pi - 2\epsilon$.

From (4.9), we have a uniform bound on $W(f_k)$ and on $|f_k(\mathbb{S}^2)|_g$; since the ambient manifold (M,g) is of bounded geometry, the conditions i) and ii) of Proposition 4.2.4 are satisfied and we can say that

$$\mathrm{diam}_g(f_k(\mathbb{S}^2)) \leq C(W(f_k) + |f_k(\mathbb{S}^2)|_g) \leq C$$

for some $C > 0$ independent of k.

If by contradiction there exists no compact subset $K \subset\subset M$ such that $f_k(\mathbb{S}^2) \subset K$ then, up to subsequences, for every $k \in \mathbb{N}$ we can take a point $\xi_k \in f_k(\mathbb{S}^2) \subset \mathbb{R}^3$ (recall that outside a compact subset (M,g) is isometric to $(\mathbb{R}^3, \delta + o_1(1))$ such that $|\xi_k| \to \infty$. Since $\mathrm{diam}_g\, f_k(\mathbb{S}^2) \leq C$, for k large enough all the surface $f_k(\mathbb{S}^2)$ is contained in a region where $o_1(1)$ is arbitrarily small in C^1 norm:

$$\liminf_k \|o_1(1)\|_{C^1(f_k(\mathbb{S}^2))} = 0.$$

Now consider estimate (3.11) and apply it to $f_k(\mathbb{S}^2)$ for k large; observe that in proof of that estimate one can consider $r_0 = \mathrm{diam}_g\, f_k(\mathbb{S}^2) \leq C$, moreover the quantities η and θ converge to 0 as $k \to \infty$. Choosing γ small enough (depending on the ϵ of (4.9)) it follows that for k sufficiently large we have the Euclidean Willmore functional $W_e(f_k) < 4\pi$ contradicting Theorem 7.2.2 in [Will]. ■

4.2.2 Existence of a smooth immersion of $\mathbb{S}^2$ minimizing W_1, respectively E_1

Let us start by summarizing the estimates and properties of a minimizing sequence $f_k : \mathbb{S}^2 \hookrightarrow M$ for W_1 (respectively E_1).

Theorem 4.2.6. *Let (M,g) be an asymptotically flat Riemannian 3-manifold (in the sense of (1.16)) of bounded geometry which satisfies:*
- For the minimization problems of W_1 and E_1: there exists a point $\bar{p} \in M$ such that $R(\bar{p}) > 6$,
- For the minimization problems of W_1: the sectional curvature $\bar{K}$ is bounded by 2, $\bar{K} \leq 2$.

Let $\{f_k\}_{k\in\mathbb{N}}$ be a minimizing sequence of smooth immersions $f_k : \mathbb{S}^2 \hookrightarrow M$ for W_1 (respectively E_1), i.e. a minimizing sequence for problem (1.19) (resp. problem (1.20)).

Then, called V_k the varifolds associated to f_k and $\mu_k := \|V_k\|$ the associated spatial measures, the following holds:
i) there exists a compact subset $K \subset\subset M$ such that $f_k(\mathbb{S}^2) \subset K$,
ii) there exists a constant C such that $\frac{1}{C} \leq \mathrm{diam}_g(f_k(\mathbb{S}^2)) \leq C$,
iii) $\limsup_k \frac{1}{2}\int |A_k|^2 d\mu_k < 4\pi$, where A_k is the second fundamental form of f_k,
iv) there exists an integral 2-varifold V of M such that, up to subsequences, $V_k \to V$ in varifold sense,
v) V is an integral varifold with weak mean curvature H (resp. generalized second fundamental form A),
vi) called $\mu := \|V\|$ the mass measure of the limit varifold V then $\mathrm{spt}\,\mu$ is compact, connected and up to subsequences

$$\mu_k \rightharpoonup \mu \quad \textit{weak as Radon measures and}$$

$$\mathrm{spt}\,\mu_k \to \mathrm{spt}\,\mu \quad \textit{in Hausdorff distance sense,}$$

vii)

$$W_1(\mu) := W_1(V) := \int_{G_2(M)} \left(\frac{|H(x,P)|^2}{4} + 1 \right) dV \leq \liminf_k W_1(f_k) < 4\pi, \ \textit{and respectively}$$

$$E_1(\mu) := E_1(V) := \int_{G_2(M)} \left(\frac{|A(x,P)|^2}{2} + 1 \right) dV \leq \liminf_k E_1(f_k) < 4\pi,$$

PROOF. i) follows from Proposition 4.2.5, ii) follows from Proposition 4.2.3 and Proposition 4.2.4.

iii): by Lemma 4.2.1 we know that if f_k is a minimizing sequence for E_1 (respectively W_1) then for large k we have $\frac{1}{2}\int |A_k|^2 d\mu_k \leq E_1(f_k) \leq 4\pi - \epsilon$ for some $\epsilon > 0$ (respectively $W_1(f_k) \leq 4\pi - \epsilon$). In the first case we conclude; in the second case by Gauss equation, for an immersion $f : \mathbb{S}^2 \hookrightarrow M$, we get

$$\frac{1}{2}|A|^2 = \frac{1}{2}H^2 - k_1 k_2 = \frac{1}{2}H^2 - K_G + \bar{K}(T_x f) = \Big(\frac{H^2}{4} + 1\Big) + \Big(\frac{H^2}{4} + \bar{K}(T_x f) - 1\Big) - K_G$$

where $\bar{K}(T_x f)$ is the sectional curvature of the ambient manifold evaluated on the plane $T_x f \subset T_x M$ with $x \in f(\mathbb{S}^2)$, K_G is the Gaussian curvature of $(\mathbb{S}^2, f^* g)$ and k_1, k_2 are the principal curvatures. Integrating, by Gauss Bonnet theorem and the bound $\bar{K} \leq 2$, we get

$$\frac{1}{2}\int |A|^2) \leq 2W_1(f) - 2\pi\chi_E(\mathbb{S}^2) = 2W_1(f) - 4\pi.$$

Since for k large we know that $W_1(f_k) < 4\pi - \epsilon$ we conclude that $\frac{1}{2}\int |A_k|^2 d\mu_k \leq 4\pi - \epsilon$.

iv) For the minimizing sequences f_k of both E_1 and W_1, from part iii) we have the uniform bound on the L^2 norms of the second fundamental forms $\frac{1}{2}\int |A_k|^2 d\mu_k \leq 4\pi - \epsilon$; moreover by definition of the functionals it is clear the uniform bound on the areas of $f_k(\mathbb{S}^2)$: $\exists C > 0$ such that $\forall k \in \mathbb{N} \quad |f_k(\mathbb{S}^2)|_g < C$.

Using the Cauchy-Schwartz inequality we can estimate the first variation of the varifolds V_k associated to the immersions f_k:

$$\|\delta V_k\| := \int |H_k| d\mu_k \leq C \int |A_k| d\mu_k \leq C\Big(\int |A_k|^2 d\mu_k\Big)^{\frac{1}{2}} \sqrt{|f_k(\mathbb{S}^2)|_g} \leq C.$$

Now applying Allard compactness Theorem (Theorem 6.4 in [Al]) to the sequence of varifolds V_k, we can say that there exists an integral 2-varifold V such that

$$V_k \to V \quad \text{in varifold sense.}$$

v) and vii): from the uniform bound $\frac{1}{2}\int |A_k|^2 d\mu_k \leq 4\pi - \epsilon$, using the previous point i), we can apply Theorem 5.3.2 in [Hu1] and say that $V \in CV_2(M)$ is a curvature 2-varifold with generalized second fundamental form A (hence in particular with weak mean curvature H, see Remark 5.2.3 in [Hu1]). By the lower semicontinuity of the functionals proved in the aforementioned paper by Hutchinson (notice that the functionals are sum of a lower semicontinuous and a continuous part under varifold convergence), we can define W_1 and E_1 on the limit varifold V and vii) follows.

vi): Of course, since the Grassmannian of the 2-planes is compact, the varifold convergence of a sequence of varifolds implies the measure theoretic convergence of the spatial supports; so up to subsequences

$$\mu_k \rightharpoonup \mu \quad \text{weak as Radon measures.}$$

In order to get the Hausdorff convergence recall that $K \subset\subset M \subset \mathbb{R}^S$ is isometrically embedded by Nash Theorem, so we can see the surfaces $f_k(\mathbb{S}^2)$ as immersed in $\mathbb{R}^S$. Since f_k as immersions in K have uniformly bounded Willmore energy and area, and since K is compact and isometrically embedded in $\mathbb{R}^S$ it follows that f_k as immersions in $\mathbb{R}^S$ have uniformly bounded Willmore energy (for more details see the proof of Lemma 4.1.10). Moreover we know that the associated measures $\mu_k \rightharpoonup \mu$; under this conditions Leon Simon proved (see [SiL] pages 310-311) that actually

$$\operatorname{spt}\mu_k \to \operatorname{spt}\mu \quad \text{in Hausdorff distance sense}$$

as subsets of $\mathbb{R}^S$; but since M is isometrically embedded in $\mathbb{R}^S$ it clearly implies that $\operatorname{spt}\mu_k \to \operatorname{spt}\mu$ in Hausdorff distance as subsets of M. ∎

At this point we proved the existence of a candidate minimizer varifold V for the functional E_1, respectively W_1; moreover we showed that the minimizing sequence is contained in a compact subset $K \subset\subset M$. Henceforth we are in the setting of Chapter 5, for convenience the regularity theory in that chapter is stated for closed ambient manifold and for the functionals analyzed in Section 4.1 but can be repeated analogously for the functionals E_1 and W_1 since the we just proved that the minimizing sequences stay in a compact subset (see Remark 1.0.12). It follows that the candidate minimizer V is a non null 2-varifold associated to a smooth immersion $f : \mathbb{S}^2 \hookrightarrow M$ which therefore is a minimizer for E_1, respectively W_1.

Chapter 5

Regularity theory for minimizers of Willmore type functionals in Riemannian manifolds

In this chapter we prove the regularity of the candidate minimizer given in Chapter 4, the content of the chapter is a joint work with E. Kuwert and J. Schygulla from Freiburg (see [MS2])

5.1 Introduction to the chapter

For the regularity theory we took inspiration from the work of Simon [SiL] where the regularity of the minimizers of W in euclidean setting is performed, but there are some non trivial modifications to be done for immersions in a Riemannian manifold. First of all, since in Euclidean setting one has an 8π bound on the Willmore functional which turns out to be very useful, using an inequality of Li and Yau [LY] and a monotonicity formula Simon manages to work with embedded surfaces; in Riemannian manifold instead we work with immersions, hence there could be multiplicity and the technique is a bit more involved. Nevertheless in Section 5.2, working locally in normal coordinates, we manage to enter into the assumptions of the *Graphical Decomposition Lemma* of Simon and prove that near all the points (except possibly finitely many "bad points" where the curvature concentrates) of the candidate minimizer, the minimizing sequence can be written locally as union of graphs and small "pimples" with good estimates.

In Section 5.3 we prove that the candidate minimizer is locally given by graphs of $C^{1,\alpha} \cap W^{2,2}$ functions. For getting this partial regularity we first prove a local power decay on the L^2 norms of the second fundamental forms of the minimizing sequence (see Lemma 5.3.1) away from the bad points; then, still working locally away from the bad points, replacing the pimples by sort of biharmonic discs, by Ascoli-Arzelá theorem we get existence of Lipschitz limit functions; at this point, using a generalized Poincaré inequality, the power decay of the second fundamental forms and Radon Nicodym Theorem, we show in Lemma 5.3.2 that the candidate minimizer is associated to the limit Lipschitz graphs; finally using that this candidate minimizer has weak mean curvature in L^2, together with the aforementioned power decay, a lemma of Morrey implies the $C^{1,\alpha} \cap W^{2,2}$ regularity away from the bad points. Using a topological argument involving degree theory and Gauss Bonnet theorem, in Subsection 5.3.2 we prove that actually there are no bad points and therefore the candidate minimizer is of class $C^{1,\alpha} \cap W^{2,2}$ everywhere. This step is quite different (and simpler) from [SiL], indeed since we work with immersed spheres we manage to exclude bad points while Simon works with surfaces of higher genus and he has to handle the bad points without excluding them.

To complete the regularity we need to show that the candidate minimizer satisfies the Euler-Lagrange equation, and for this step we need to prove that it can be parametrized on $\mathbb{S}^2$. At this point (see Subsection 5.4.1) we use the notion of generalized (r, λ)-immersions developed by Breuning in his Ph. D. Thesis [BreuTh] taking inspiration by previous work of Langer [Lan]. Once the Euler Lagrange equation is satisfied the C^∞ regularity follows (see Subsection 5.4.2)

5.2 Good/Bad Points and the Graphical Decomposition Lemma

In the present section we will define the good and bad points, we will state the Graphical Decomposition Lemma of Leon Simon and we will show that it can be applied in our settings. This is the starting point for the regularity theory of the candidate minimizer varifold V.

5.2.1 Definition and first properties of the good/bad points

Let us start introducing some notation: since we will work in normal coordinates we will see our surfaces immersed locally either in $\mathbb{R}^3$ with euclidean metric $\delta_{\mu\nu}$ or with the Riemannian metric $g_{\mu\nu}$. The quantities in "euclidean" setting will be denoted with an "e" (ex. μ_k^e, H_k^e, A_k^e, ...) and the Riemannian quantities will be denoted with a "g" (ex. μ_k^g, H_k^g, A_k^g, ...).

This subsection is common to the two functionals E and W_1 since we use properties that both the functionals satisfy. First we define the so called bad points with respect to a given $\varepsilon > 0$ in the following way: define the Radon measures α_k on M by

$$\alpha_k = \mu_k^g \llcorner |A_k^g|^2.$$

From the definition of E and for Lemma 4.1.14 we know that $\alpha_k(M) \leq C$. By compactness there exists a Radon measure α on M such that (after passing to a subsequence) $\alpha_k \rightharpoonup \alpha$ weak as Radon measures. It follows that $\operatorname{spt}\alpha \subset \operatorname{spt}\mu$ and $\alpha(M) \leq C$.

Definition 5.2.1. *We define the bad points with respect to $\varepsilon > 0$ by*

$$\mathcal{B}_\varepsilon = \left\{ \xi \in \operatorname{spt}\mu \,\middle|\, \alpha(\{\xi\}) > \varepsilon^2 \right\}. \tag{5.1}$$

The points of $\operatorname{spt}\mu \setminus \mathcal{B}_\varepsilon$ *are called* ε-good points.

Remark 5.2.2. *Since $\alpha(M) \leq C$, there exist only finitely many bad points. Moreover if ξ_0 is an ε-good point there exists a $0 < \rho_0 = \rho_0(\xi_0, \varepsilon) \leq 1$ such that $\alpha(B^g_{\rho_0}(\xi_0)) < 2\varepsilon^2$, and since $\alpha_k \rightharpoonup \alpha$ weakly as measures we get*

$$\int_{B^g_{\rho_0}(\xi_0)} |A_k^g|^2 \, d\mu_k^g \leq 2\varepsilon^2 \quad \textit{for } k \textit{ sufficiently large,} \tag{5.2}$$

where $B^g_{\rho_0}(\xi_0)$ is the metric ball in (M,g) of center ξ_0 and radius ρ_0.

5.2.2 Some geometric estimates in normal coordinates

Throughout this subsection, let (M,g) be our closed Riemannian 3-manifold and $\{f_k : \mathbb{S}^2 \hookrightarrow M\}_{k\in\mathbb{N}}$ a sequence of smooth immersions.

Fix $\xi \in \operatorname{spt}\mu$ an $\varepsilon-$good point for some $\varepsilon > 0$ and consider $x^\mu, \mu = 1,2,3$, normal coordinates of (M,g) centered at ξ (i.e the coordinates of ξ are 0). Recall that, in this coordinates, the metric $g_{\mu\nu}$ takes the following shape (see for example [LP] formula (5.4) page 61):

$$\begin{aligned} g_{\mu\nu}(x) &= \delta_{\mu\nu} + \frac{1}{3} R_{\mu\sigma\lambda\nu} x^\sigma x^\lambda + O(|x|^3) && (5.3)\\ &= \delta_{\mu\nu} + h_{\mu\nu}(x) && (5.4)\end{aligned}$$

where

$$h_{\mu\nu}(0) = 0 \quad \text{and} \quad D_\lambda h_{\mu\nu}(0) = 0 \quad \forall \lambda, \mu, \nu = 1,2,3. \tag{5.5}$$

Called $inj(\xi) > 0$ the injectivity radius at ξ, observe that inside the geodesic ball $B^g_{inj(\xi)}(\xi)$ we have two metrics: the metric $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ of the Riemannian manifold (M,g) and the euclidean metric $\delta_{\mu\nu}$. Denoted with $|x|$ the norm of x as a vector in $\mathbb{R}^3$, observe that (5.3) (or (5.5)) implies

$$|h_{\mu\nu}(x)| = O(|x|^2) \quad |D_\lambda h_{\mu\nu}(x)| = O(|x|) \quad \text{for small } |x|, \tag{5.6}$$

where the notation $O(t), t > 0$ of course means that there exists $C > 0$ such that $\lim_{t\to 0^+} \frac{O(t)}{t} \leq C$. Moreover, since M is compact, all the curvatures of g are bounded on M and the remainders $O(|x|^2)$, $O(|x|)$ (which clearly depend on the base point ξ) are uniform in M; i.e. there exist a constant C_M

depending on M which can be used in the definition of O for the remainders in (5.6) for all the base points $\xi \in M$.

Since we have two metrics, all the geometric quantities associated to a surface have two values: the euclidean and the Riemannian one. The euclidean quantities are labeled with an "e" ($d_e(x,y)$ is the euclidean distance between x and y,$d\mu^e$ is the euclidean area form, $|.|_e$ is the euclidean area, H^e is the euclidean mean curvature, A^e is the euclidean second fundamental form, …) and the Riemannian ones with a "g" ($d_g(x,y)$,$d\mu^g$, $|.|_g$, H^g, A^g … are the corresponding Riemannian ones). We also adopt the convention that $|A^e|^2$ (respectively $|A^g|^2$) is the squared euclidean (respectively Riemannian) norm of the euclidean (respectively Riemannian) second fundamental form; analogous notation is used for $|H^e|^2$ and $|H^g|^2$.

For the regularity theory it is important to relate the euclidean and the Riemannian quantities; let us do it.

Proposition 5.2.3. *Let (M,g) be a Riemannian 3-manifold, consider a point $\xi \in M$ and normal coordinates $x^\mu, \mu = 1,2,3$, centered in ξ. Let $\Sigma \hookrightarrow M$ be an immersed smooth surface, as explained above we have couples of geometric quantities: the euclidean and the Riemannian ones.*

Then the following relations between the two of them hold:

$$
\begin{array}{rl}
i) & d_e(x,y) \approx (1+O(\rho^2))d_g(x,y) \quad \text{for } |x|,|y| \le \rho \\
ii) & d\mu^e(x) \approx (1+O(|x|^2))d\mu^g(x) \\
iii) & \mu^e(B^e_\rho(\xi)) := |\Sigma \cap B^e_\rho(\xi)|_e \approx (1+O(\rho^2))|\Sigma \cap B^g_{\rho+O(\rho^3)}(\xi)|_g =: (1+O(\rho^2))\mu^g(B^g_{\rho+O(\rho^3)}(\xi)) \\
iv) & (A^g)_{ij}(x) \approx [1+O(|x|^2)](A^e)_{ij}(x)+O(|x|) \\
v) & |A^g|^2 \approx (1+O(|x|^{2-2\alpha}))|A^e|^2(x)+O(|x|^{2\alpha}) \quad \forall\, 0 \le \alpha \le 1 \\
vi) & H^g(x) \approx H^e(x)+O(|x|^2)|A^e|(x)+O(|x|) \\
vii) & |H^g|^2(x) \approx (1+O(|x|^{2-2\alpha}))|H^e|^2(x)+O(|x|^2)|A^e|^2+O(|x^{2\alpha}|) \quad \forall\, 0 \le \alpha \le 1
\end{array}
$$

where $x \in \Sigma$ is a small vector of $\mathbb{R}^3$ and with the symbol $\approx$ we mean that we have an upper and lower bound of the left hand side with the right hand side.

Proof. In Chapter 3, we considered the "manifold" $\mathbb{R}^3$ with two metrics: the standard euclidean one and a perturbed one. We denoted the euclidean scalar product by $\delta_{\mu\nu}$ and the perturbed metric as $\delta_{\mu\nu}+h_{\mu\nu}(x)$ where $h_{\mu\nu}(.)$ was a compactly supported field of smooth symmetric bilinear forms. We called $\eta := \|h\|_{C^0(\mathbb{R}^3)}$ and $\theta := \|Dh\|_{C^0(\mathbb{R}^3)}$ and we worked out estimates of the geometric quantities (distance, area, second fundamental form, mean curvature,…) in perturbed metric in terms of the euclidean ones and the remainders depended on η and θ.

Now, as remarked above, near the point ξ the Riemannian metric in normal coordinates is a perturbation of the euclidean metric, moreover $|h_{\mu\nu}(x)| = O(|x^2|)$ and $|D_\lambda h_{\mu\nu}(x)| = O(|x|)$. Since the estimates of Chapter 3 are punctual then, in their proof adapted to the present contest, it does not matter if h has not compact support; moreover one can estimate $|h_{\mu\nu}(x)|$ with $O(|x|^2)$ instead of η and $D_\lambda h_{\mu\nu}(x)$ with $O(|x|)$ instead of θ.

i) follows from statement *ii)* of Lemma 3.2.1;

ii) follows from Lemma 3.2.2; about the notation observe that now $d\mu_e$ is what we called $\sqrt{\det(\overset{\circ}{\delta})}$ and $d\mu_g$ is what we called $\sqrt{\det(\delta \overset{\circ}{+} h)}$;

iii) from statement *i)* above we have that $B^e_\rho(\xi) \approx B^g_{\rho(1+O(\rho^2))}(\xi)$ where we mean that the left hand side is contained in and contains a set in the form of the right hand side. Now we apply statement *ii)* above to get

$$|\Sigma \cap B^e_\rho(\xi)|_e \approx |\Sigma \cap B^g_{\rho+O(\rho^3)}(\xi)|_e \approx (1+O(\rho^2))|\Sigma \cap B^g_{\rho+O(\rho^3)}(\xi)|_g;$$

iv) follows from estimate (3.8) in the proof of Lemma 3.2.3;

v) at the point $x \in \Sigma$ take a g-orthonormal base of the tangent space $T_x\Sigma$ which diagonalizes A_g. Then $|A_g|^2 = (A_g)^2_{11} + (A_g)^2_{22}$. Plugging statement *iv)* above into the last equality we get

$$
\begin{aligned}
|A_g|^2 &\approx [1+O(|x|^2)][(A_e)^2_{11}+(A_e)^2_{22}]+O(|x|)H_e(x)+O(|x|^2) \\
&\approx [1+O(|x|^2)][(A_e)^2_{11}+(A_e)^2_{22}]+O(|x|)|A_e|(x)+O(|x|^2)
\end{aligned} \tag{5.7}
$$

Notice that the chosen g-orthonormal frame of $T_x\Sigma$ may not be euclidean-orthonormal and it can also happen that in this base A_e is not diagonal. Nevertheless, using statement $iv)$ we have $(A_e)_{12}(x) = O(|x|)$, moreover the inverse of the euclidean first fundamental form is $\delta_{ij} + O(|x|^2)$, then

$$|A_e|^2(x) = [\delta_{ik} + O(|x|^2)][\delta_{jl} + O(|x|^2)](A_e)_{ij}(x)(A_e)_{kl}(x) \approx [1 + O(|x|^2)][(A_e)^2_{11} + (A_e)^2_{22}] + O(|x|^2)$$

it follows that $[(A_e)^2_{11}(x) + (A_e)^2_{22}(x)] \approx [1 + O(|x|^2)]|A_e|^2(x) + O(|x|^2)$ and plugging into (5.7) we obtain

$$|A_g|^2 \approx (1 + O(|x|^2))|A_e|^2(x) + O(|x|)|A_e|(x) + O(|x|^2).$$

Using the estimate $2ab \leq a^2 + b^2$ observe that, for any $0 \leq \alpha \leq 1$,

$$O(|x|)|A_e|(x) = O(|x|^\alpha)O(|x|^{1-\alpha})|A_e|(x) \approx O(|x|^{2\alpha}) + O(|x|^{2-2\alpha})|A_e|^2(x)$$

hence we can conclude that

$$|A_g|^2 \approx (1 + O(|x|^{2-2\alpha}))|A_e|^2(x) + O(|x|^{2\alpha}) \quad \forall\, 0 \leq \alpha \leq 1;$$

$vi)$ follows from estimate (3.10) in the proof of Lemma 3.2.3;

$vii)$ from statement $vi)$, just taking the norm with respect with the two metrics, recalling that $g_{\mu\nu}(x) = \delta_{\mu\nu} + O(|x|^2)$ and that $|H_e| \leq C|A_2|$we have

$$|H_g|^2(x) \approx (1 + O(|x|^2))|H_e|^2(x) + O(|x|^2)|A_e|^2(x) + O(|x|)|H_e| + O(|x|^2).$$

With the same trick of statement $v)$, $O(|x|)|H_e| = O(|x|^\alpha)O(|x|^{1-\alpha})|H_e|(x) \approx O(|x|^{2\alpha}) + O(|x|^{2-2\alpha})|H_e|^2(x)$ for all $0 \leq \alpha \leq 1$; we can conclude that

$$|H_g|^2(x) \approx (1 + O(|x|^{2-2\alpha}))|H_e|^2(x) + O(|x|^2)|A_e|^2(x) + O(|x|^{2\alpha}) \quad \forall\, 0 \leq \alpha \leq 1.$$

■

Using Proposition 5.2.3, in the next Lemma we will get easy but fundamental estimates in order to apply the Graphical Decomposition Lemma of Leon Simon.

Lemma 5.2.4. *Let (M, g), f_k and μ as before and assume a uniform bound on the L^2 norms of the second fundamental forms of f_k*

$$\exists\, C > 0 \text{ such that } \int |A^g_k|^2 d\mu^g_k \leq C \;\forall\, k \in \mathbb{N}$$

and on the areas

$$\exists\, C > 0 \text{ such that } |f_k(\mathbb{S}^2)|_g \leq C.$$

Fix $\varepsilon > 0$, take a good point $\xi_0 \in \operatorname{spt}\mu \backslash \mathcal{B}_\varepsilon$ (by Remark 5.2.2, see also Remark 5.2.6, we know that the set of the good points $\operatorname{spt}\mu \backslash \mathcal{B}_\varepsilon$ is non empty).

Then there exist $\rho_0 = \rho_0(\xi_0, \varepsilon) > 0$ (maybe smaller than the ρ_0 of Remark 5.2.2), $\beta > 0$ (β depending only on M and on the assumed two uniform bounds) and infinitely many k such that the following is true:

for all $\xi \in \operatorname{spt}\mu \cap B^e_{\frac{\rho_0}{2}}(\xi_0)$ there exist $\xi_k \in f_k(\mathbb{S}^2)$ such that $\xi_k \to \xi$ and for all $0 < \rho \leq \frac{\rho_0}{4}$ we have that

$$\begin{aligned} &i) \quad \mu^e_k(\overline{B^e_\rho(\xi_k)}) = |f_k(\mathbb{S}^2) \cap \overline{B^e_\rho(\xi_k)}|_e \leq \beta\rho^2, \\ &ii) \quad (\partial f_k(\mathbb{S}^2)) \cap \overline{B^e_\rho(\xi_k)} = \emptyset, \\ &iii) \quad \int_{B^e_\rho(\xi_k)} |A^e_k|^2 d\mu^e_k \leq 3\varepsilon^2. \end{aligned}$$

PROOF. Let us call $\Sigma := \operatorname{spt}\mu$ and $\Sigma_k := \operatorname{spt}\mu_k$. Let $\xi \in \Sigma \cap B^e_{\frac{\rho_0}{2}}(\xi_0)$ and $\Sigma_k \ni \xi_k \to \xi \in \Sigma$, with ρ_0 and ξ_k to be determined during the proof (of course since $\Sigma_k \to \Sigma$ in Hausdorff distance sense, then there exists a sequence $\xi_k \in \Sigma_k$ such that $\xi_k \to \xi$).

i) Statement *i)* of Proposition 5.2.3, tells us that for small $\rho > 0$ we have $B^e_\rho(\xi_k) \approx B^g_{\rho+O(\rho^3)}(\xi_k)$ (for the notation see the proof of Proposition 5.2.3); then, using the estimate on the area form (statement *ii)* of Proposition 5.2.3)

$$|\Sigma_k \cap \overline{B^e_\rho(\xi_k)}|_e \approx |\Sigma_k \cap \overline{B^g_{\rho+O(\rho^3)}(\xi_k)}|_e \approx (1+O(\rho^2))|\Sigma_k \cap \overline{B^g_{\rho+O(\rho^3)}(\xi_k)}|_g.$$

Since we can estimate $|H_g|^2 \leq 2|A_g|^2$, the assumed uniform bounds permit us to apply Lemma 4.1.10 and say that $|\Sigma_k \cap \overline{B^g_\rho(\xi_k)}|_g \leq C\rho^2$ (we get the estimate with the closed ball with a limit process on decreasing open balls containing it). Thus

$$|\Sigma_k \cap \overline{B^e_\rho(\xi_k)}|_e \approx (1+O(\rho^2))|\Sigma_k \cap \overline{B^g_{\rho+O(\rho^3)}(\xi_k)}|_g \leq C[1+O(\rho^2)][\rho+O(\rho^3)]^2 \leq \beta\rho^2$$

for small ρ; let us say $\forall\ 0 < \rho \leq \rho_0$, for some $\rho_0 > 0$ (notice that we used that the remainders $O(.)$ are uniform in the compact set K, moreover the last estimate holds with the same β for every choice of $\xi_k \in \Sigma_k$).

ii) is trivial since by assumption the surfaces have no boundary

iii) Let $\rho_0 = \rho_0(\xi_0, \varepsilon)$ be as in Remark 5.2.2 (or smaller in a way that statement *i)* above is satisfied); then

$$\int_{B^g_{\rho_0}(\xi_0)} |A^g_k|^2 \, d\mu^g_k \leq 2\varepsilon^2 \quad \text{for infinitely many } k. \tag{5.8}$$

Using statements $i), ii)$ and $v)$ of Proposition 5.2.3 we get

$$\begin{aligned}
\int_{B^e_\rho(\xi_0)} |A^e_k|^2 \, d\mu^e_k &\approx \int_{B^g_{\rho+O(\rho^3)}(\xi_0)} \left[(1+O(\rho))|A^g_k|^2 + O(\rho)\right][1+O(\rho^2)] \, d\mu^g_k \\
&\approx [1+O(\rho)] \int_{B^g_{\rho+O(\rho^3)}(\xi_0)} |A^g_k|^2 \, d\mu^g_k + O(\rho)|\Sigma_k \cap B^e_\rho(\xi_0)|_e \\
&\leq [1+O(\rho)]2\varepsilon^2 + O(\rho^3) \\
&\leq 3\varepsilon^2 \quad \text{for infinitely many } k
\end{aligned} \tag{5.9}$$

for small ρ; let us say $\forall\ 0 < \rho \leq \rho_0$, for an even smaller $\rho_0 > 0$. Notice that we used the local area estimate we got in statement *i)* and the property (5.8). Then, for $\xi \in B^e_{\frac{\rho_0}{2}}(\xi_0)$ we have that $B^e_{\frac{\rho_0}{2}}(\xi) \subset B^e_{\rho_0}(\xi_0)$. Since $\Sigma_k \to \Sigma$ in Hausdorff distance sense, for k large enough (uniformly on ξ) we can choose $\xi_k \in B^e_{\frac{\rho_0}{4}}(\xi) \cap \Sigma_k$; it follows that $B^e_{\frac{\rho_0}{4}}(\xi_k) \subset B^e_{\frac{\rho_0}{2}}(\xi) \subset B^e_{\rho_0}(\xi_0)$ and, using (5.9),

$$\int_{B^e_\rho(\xi_k)} |A^e_k|^2 \, d\mu^e_k \leq 3\varepsilon^2$$

for $0 < \rho \leq \frac{\rho_0}{4}$ as desired. ■

5.2.3 A lower 2-density bound near the good points

In this subsection we prove that for both the functionals E and W_1 we have a lower 2-density bound on the minimizing sequence of immersions f_k near the good points. The result is crucial since it avoids the trivial case when the candidate minimizer limit measure is null.

Proposition 5.2.5. *Let (M,g), f_k and μ be as before. Then there exists $\varepsilon_0 > 0$, $\rho_0 > 0$ small enough and $C > 0$ such that the following is true: fix a ε_0-good point ξ_0 and take $\xi \in B^e_{\rho_0}(\xi_0)$; thus there exists a sequence $\xi_{k'} \in f_{k'}(\mathbb{S}^2)$ (where k' is a subsequence of the ks) which satisfies the following two properties*

i) $\xi_{k'} \to \xi$

ii)

$$\frac{\mu^g_{k'}(B^g_\rho(\xi_{k'}))}{\rho^2} = \frac{|\Sigma_{k'} \cap B^g_\rho(\xi_{k'})|_g}{\rho^2} \geq C > 0, \quad \forall\, 0 < \rho \leq \rho_0.$$

It follows a 2-density lower bound on the limit measure μ at the point $\xi \in B^e_{\rho_0}(\xi_0)$:

$$\frac{\mu(B^e_\rho(\xi))}{\rho^2} \geq C \quad \forall\, 0 < \rho \leq \rho_0.$$

PROOF. As before let us call $\Sigma_k := \operatorname{spt}\mu_k$ and $\Sigma := \operatorname{spt}\mu$. Since by assumption $\Sigma_k \to \Sigma$ in Hausdorff distance sense, for each $\xi \in \Sigma$ there exists a sequence $\xi_k \in \Sigma_k$ such that $\xi_k \to \xi$.
The numbers $\varepsilon_0 > 0$ and ρ_0 will be chosen later in the proof. Since ξ_0 is a good point, then by Remark 5.2.2, for $\rho_0 > 0$ small enough

$$\int_{B^g_{\rho_0}(\xi_0)} |A^g_k|^2 \, d\mu^g_k \leq 2\varepsilon_0^2 \quad \text{for infinitely many } k.$$

From statement $i)$ of Proposition 5.2.3 and from the assumption $\xi \in B^e_{\frac{\bar\rho}{2}}(\xi_0)$ ($\bar\rho$ small to be determined), it follows that $B^e_{\frac{\bar\rho}{2}}(\xi) \subset B^e_{\bar\rho}(\xi_0) \subset B^g_{\bar\rho+O(\bar\rho^3)}(\xi_0) \subset B^g_{\rho_0}(\xi_0)$ for $\bar\rho$ small enough. Analogously, since $\xi_k \to \xi$, for k large enough and ρ small enough we have that $B^g_\rho(\xi_k) \subset B^g_{2\rho}(\xi) \subset B^e_{\frac{\bar\rho}{2}}(\xi) \subset B^g_{\rho_0}(\xi_0)$. Recalling that the norm of the mean curvature can be estimated with the norm of the second fundamental form, $|H^g_k|^2 \leq 2|A^g_k|^2$, we get for infinitely many k

$$W(\Sigma_k \cap B^g_\rho(\xi_k)) = \frac{1}{4}\int_{B^g_\rho(\xi_k)} |H^g_k|^2 \, d\mu^g_k \leq \frac{1}{2}\int_{B^g_\rho(\xi_k)} |A^g_k|^2 \, d\mu^g_k \leq \frac{1}{2}\int_{B^g_{\rho_0}(\xi_0)} |A^g_k|^2 \, d\mu^g_k \leq \varepsilon_0^2. \tag{5.10}$$

Let us recall a monotonicity formula proved by Florian Link in his Ph. D. Thesis [FL]. Under the assumptions on the ambient manifold (M, g) of strictly positive lower bound on the injectivity radius and bounded sectional curvature (which now are of course satisfied since our M is compact) he proves that there exists a constant $C = C(M)$ such that for $0 < \sigma \leq \rho < \rho_0 = c(M)$ and every smooth immersed surface $\tilde\Sigma$

$$\frac{|\tilde\Sigma \cap B^g_\sigma(x)|_g}{\sigma^2} \leq C\Big(\frac{|\tilde\Sigma \cap B^g_\rho(x)|_g}{\rho^2} + W(\tilde\Sigma \cap B^g_\rho(x))\Big).$$

From the smoothness of $\tilde\Sigma$, sending $\sigma \to 0$ in the formula above, for every $\rho \leq \rho_0$ and $x \in \tilde\Sigma$ one has

$$1 \leq C\Big(\frac{|\tilde\Sigma \cap B^g_\rho(x)|_g}{\rho^2} + W(\tilde\Sigma \cap B^g_\rho(x))\Big). \tag{5.11}$$

Using estimate (5.11) for the subsequence k' for which the inequality (5.10) holds, we obtain that for every $0 < \rho \leq \rho_0$ (taking ρ_0 even smaller in a way that Link's monotonicity formula can be applied)

$$1 \leq C\left(\frac{|\Sigma_{k'} \cap B^g_\rho(\xi_{k'})|_g}{\rho^2} + W\big(\Sigma_{k'} \cap B^g_\rho(\xi_{k'})\big)\right) \leq C\left(\frac{|\Sigma_{k'} \cap B^g_\rho(\xi_{k'})|_g}{\rho^2} + \varepsilon_0^2\right).$$

Chosen $\varepsilon_0^2 \leq \frac{1}{2C}$ we get

$$\frac{|\Sigma_{k'} \cap B^g_\rho(\xi_{k'})|_g}{\rho^2} \geq C > 0 \tag{5.12}$$

for the subsequence of the k', for some $C > 0$ and for $0 < \rho < \rho_0$, ρ_0 maybe smaller. Now let us show the lower 2-density bound on μ, the limit of the measures μ_k associated to Σ_k.
Since μ is a finite Radon measure, for almost every $0 < \rho \leq \rho_0$ we have $\mu(\partial B^g_{2\rho}(\xi)) = 0$ then the weak convergence of measures implies

$$\mu(B^g_{2\rho}(\xi)) = \lim_k[\mu_k(B^g_{2\rho}(\xi))] = \lim_{k'}[\mu_{k'}(B^g_{2\rho}(\xi))].$$

Since $\xi_{k'} \to \xi$, for k' large enough $\xi_{k'} \in B^g_\rho(\xi)$ and $B^g_\rho(\xi_{k'}) \subset B^g_{2\rho}(\xi)$; it follows that

$$\lim_{k'}[\mu_{k'}(B^g_{2\rho}(\xi))] \geq \limsup_{k'}[\mu_{k'}(B^g_\rho(\xi_{k'}))] = \limsup_{k'} |\Sigma_{k'} \cap B^g_\rho(\xi_{k'})|_g \geq C\rho^2 > 0$$

where in the last step we used inequality (5.12). Collecting the last two chains of inequalities we get

$$\frac{\mu(B^g_{2\rho}(\xi))}{4\rho^2} \geq C > 0$$

for almost every $0 < \rho \leq \rho_0$. Now fix an arbitrary $\rho \in (0, 2\rho_0)$, then there exists a sequence $\rho_n \uparrow \rho$ such that the last inequality is satisfied: $\mu(B^g_{\rho_n}(\xi)) \geq C\rho_n^2$. Passing to the limit in n we get

$$\frac{\mu(B^g_\rho(\xi))}{\rho^2} \geq C > 0 \quad \forall\, \rho \in (0, 2\rho_0).$$

We can conclude using statement $i)$ of Proposition 5.2.3 and the smallness of ρ_0; indeed, for small ρ we have $B^e_\rho(\xi) \supset B^g_{\rho+O(\rho^3)}(\xi)$ then

$$\mu(B^e_\rho(\xi)) \geq \mu(B^g_{\rho+O(\rho^3)}(\xi)) \geq C[\rho^2 + O(\rho^6)] \geq C\rho^2$$

for all $\rho \in (0, \rho_0)$, ρ_0 small enough. ■

Remark 5.2.6. *Proposition 5.2.5 is crucial for our minimization problems since it avoids the trivial case when the candidate minimizer limit measure μ is null. Indeed in the case $\{f_k\}_{k\in\mathbb{N}}$ is a minimizing sequence for either E or W_1, we know from Theorem 4.2.6 (respectively Theorem 4.1.13) that the support* spt μ *of the limit measure μ is compact, connected and with positive diameter; hence it contains infinitely many points. Since for both the functionals the L^2 norms of the second fundamental forms of f_k are uniformly bounded (for E it is trivial, see Lemma 4.1.14 for W_1), by Remark 5.2.2, for every $\varepsilon > 0$ there are infinitely many ε-good points. Thus, applying Proposition 5.2.5, we have that there exists a small $\rho_0 > 0$ such that $\mu(B^e_{\rho_0}(\xi_0)) > C\rho_0^2 > 0$.*

5.2.4 The Graphical Decomposition Lemma

Thanks to Lemma 5.2.4 we are in position to apply the Graphical Decomposition Lemma of Leon Simon (Lemma 2.1 in [SiL]).

Lemma 5.2.7. *Let (M, g),f_k and μ be as in the assumptions of Lemma 5.2.4. Let β be given by Lemma 5.2.4 and $\varepsilon_0 = \varepsilon_0(\beta)$ the associated one by Lemma 2.1 in [SiL]. Let $\varepsilon < \varepsilon_0$, fix a good point point ξ_0 with respect to ε and consider $\rho_0 = \rho_0(\xi_0, \varepsilon_0)$ given by Lemma 5.2.4.*

Then for any $\xi \in \operatorname{spt}\mu \cap B^e_{\frac{\rho_0}{2}}(\xi_0)$, for all $\rho \leq \frac{\rho_0}{4}$ and for infinitely many $k \in \mathbb{N}$ the following holds:

There exist 2-dimensional planes L_l containing ξ and functions $u^l_k \in C^\infty\left(\overline{\Omega^l_k}, L^\perp_l\right)$ such that

$$f_k(\mathbb{S}^2) \cap \overline{B^e_{\frac{\rho}{4}}(\xi)} = \bigcup_{l=1}^{M_k} D^l_k \cap \overline{B^e_{\frac{\rho}{4}}(\xi)} = \left(\bigcup_{l=1}^{M_k} \operatorname{graph} u^l_k \cup \bigcup_{j=1}^{N_k} P^k_j\right) \cap \overline{B^e_{\frac{\rho}{4}}(\xi)}$$

where

$$\Omega^l_k = (B^e_\lambda(\xi) \cap L^l) \setminus \bigcup_m d^l_{k,m} \qquad \left(\lambda \in \left(\frac{67}{128}\frac{\rho}{4}, \frac{67}{128}\frac{\rho}{2}\right)\right)$$

and where the $d^l_{k,m} \subset L^l$ are pairwise disjoint closed discs disjoint from $\partial B^e_\lambda(\xi)$.

Furthermore each D^l_k is a topological disc with $\operatorname{graph} u^l_k \cap \overline{B^e_{\frac{\rho}{4}}(\xi)} \subset D^l_k$ *and* $D^l_k \setminus \operatorname{graph} u^l_k$ *is a union of a subcollection of the $P^k_j \subset f_k(\mathbb{S}^2)$, and each P^k_j is diffeomorphically a closed disc.*

We have the following estimates:

$$\begin{aligned}
M_k &\leq c\beta, \quad (M_k = \textit{the number of slices for a fixed } k) \\
\sum_{m=1}^{M^l_k} \operatorname{diam} d^l_{k,m} &\leq c\left(\int_{B^e_{2\rho}(\xi_k)} |A^e_k|^2 \, d\mu^e_k\right)^{\frac{1}{4}} \rho \leq c\varepsilon^{\frac{1}{2}}\rho, &(5.13)\\
\sum_{j=1}^{N_k} \operatorname{diam} P^k_j &\leq c\left(\int_{B^e_{2\rho}(\xi_k)} |A^e_k|^2 \, d\mu^e_k\right)^{\frac{1}{4}} \rho \leq c\varepsilon^{\frac{1}{2}}\rho, &(5.14)\\
\frac{1}{\rho}||u^l_k||_{L^\infty(\Omega_k)} &\leq c\varepsilon^{\frac{1}{6}} + \frac{\delta_k}{\rho} \quad \textit{where } \delta_k \to 0, &(5.15)\\
||Du^l_k||_{L^\infty(\Omega_k)} &\leq c\varepsilon^{\frac{1}{6}} + \delta_k \quad \textit{where } \delta_k \to 0. &(5.16)
\end{aligned}$$

5.3 $C^{1,\alpha} \cap W^{2,2}$ regularity of the limit measure μ

5.3.1 Regularity in the good points

In the next step we estimate the squared integral of the second fundamental form on small balls around the "good points". This estimate will help us to show that the candidate minimizer (for W_1 or E) μ is actually the measure associated to $C^{1,\alpha} \cap W^{2,2}$ graphs in a neighborhood around the good points.

Lemma 5.3.1. *Consider the following two cases:*

i)Let (M,g) be a closed Riemannian 3-manifold which satisfies at least one of the two conditions below:
- (M,g) has uniformly strictly positive sectional curvature in the sense of (4.1); or
- there is no nonzero 2-varifold of M with null generalized second fundamental form and there is a point $\bar{p}$ where the scalar curvature is strictly positive: $R_g(\bar{p}) > 0$.
Let $\{f_k : \mathbb{S}^2 \hookrightarrow M\}_{k\in\mathbb{N}}$ be a minimizing sequence of smooth immersions for E among the immersions of the same type.

ii) Let (M,g) be a closed Riemannian 3-manifold whose scalar curvature is strictly greater than 6 at a point:

$$\exists \bar{p} \in M : \quad R_g(\bar{p}) > 6.$$

Let $\{f_k : \mathbb{S}^2 \hookrightarrow M\}_{k\in\mathbb{N}}$ be a minimizing sequence of smooth immersions for W_1 among the immersions of the same type.

Let μ be the limit measure given by the corresponding compactness Theorems (Theorem 4.2.6 and Theorem 4.1.13) and consider $\varepsilon_0 > 0$, $\xi_0 \in \operatorname{spt}\mu \backslash \mathcal{B}_\varepsilon$ for $\varepsilon \le \varepsilon_0$, $\rho_0 = \rho_0(\xi_0,\varepsilon_0)$ as in Lemma 5.2.7 (actually $\varepsilon_0 > 0$ can be chosen smaller during the proof).

Then we have for all $\xi \in \operatorname{spt}\mu \cap B^e_{\frac{\rho_0}{2}}(\xi_0)$ and all $\rho \le \frac{\rho_0}{4}$ that

$$\liminf_{k\to\infty} \int_{B^e_{\frac{\rho}{8}}(\xi)} |A^e_k|^2 \, d\mu^e_k \le c\rho^\alpha \quad \text{where } c = c(\rho_0) \text{ and } \alpha \in (0,1).$$

PROOF. First of all observe that from Proposition 4.1.1, Corollary 4.1.2, Remark 4.1.3 (which give the area bounds), Lemma 4.1.14 (which give the L^2 bound on A), and the compactness results of Theorem 4.2.6 and Theorem 4.1.13, then we are in the assumptions of Definition 5.2.1, Lemma 5.2.4 and Lemma 5.2.7.

For infinitely many $k \in \mathbb{N}$ apply the Graphical Decomposition given by Lemma 5.2.7 and for those $k \in \mathbb{N}$ (surface index), $l \in \{1,\dots,M_k\}$ (slice index) and $\gamma \in \left(\frac{\rho}{16}, \frac{3\rho}{32}\right)$ define the set

$$C^l_\gamma(\xi) = \left\{ x + y \,\middle|\, x \in B^e_\gamma(\xi) \cap L_l, y \in L_l^\perp \right\}.$$

From the estimates on the diameters of the pimples and the C^1 estimates on the graph functions u^l_k, it follows that

$$D^l_k \cap C^l_\gamma(\xi) = D^l_k \cap C^l_\gamma(\xi) \cap \overline{B^e_{\frac{\rho}{4}}(\xi)} \quad \text{for } \varepsilon \le \varepsilon_0 \text{ and } \delta_k \le \frac{\rho}{8}. \tag{5.17}$$

To see this, let $z \in D^l_k \cap C^l_\gamma(\xi)$, then $z = x_1 + y_1$ with $x_1 \in B_\gamma(\xi) \cap L_l$, $y_1 \in L_l^\perp$. Since D^l_k is disjoint union of a graph and a pimple part, there are two possible cases:

1) $z \in \operatorname{graph} u^l_k \cap C^l_\gamma(\xi)$: thus $|y_1| = |u^l_k(x_1)| \le c\varepsilon^{\frac{1}{6}}\rho + \delta_k$ and

$$\begin{aligned} |z - \xi| &\le |x_1 - \xi| + |y_1| \quad \le \gamma + c\varepsilon^{\frac{1}{6}}\rho + \delta_k \quad \le \frac{3\rho}{32} + c\varepsilon^{\frac{1}{6}}\rho + \delta_k \\ &\le \frac{\rho}{8} + \delta_k \quad \text{for } \varepsilon \le \varepsilon_0,\ \varepsilon_0 \text{ maybe smaller} \\ &\le \frac{\rho}{4} \quad \text{for } \delta_k \le \frac{\rho}{8}. \end{aligned} \tag{5.18}$$

2) $z \in D^l_k \cap P^k_j \cap C^l_\gamma(\xi)$ for some $j \in \mathbb{N}$: Since $\operatorname{diam} P^k_j \le c\varepsilon^{\frac{1}{2}}\rho$ it follows that $|y_1| \le c\varepsilon^{\frac{1}{6}}\rho + \delta_k + \operatorname{diam} P^k_j \le c\varepsilon^{\frac{1}{6}}\rho + \delta_k$. Now the claim follows in the same way as above in 1).

Next define the set A^l_k by

$$A^l_k(\xi) = \left\{ \gamma \in \left(\frac{\rho}{16}, \frac{3\rho}{32}\right) \,\middle|\, \partial C^l_\gamma(\xi) \cap \bigcup_j P^k_j = \emptyset \right\}.$$

For $\varepsilon \le \varepsilon_0$ (ε_0 maybe smaller) it follows that

$$\mathcal{L}^1(A^l_k(\xi)) \ge \frac{\rho}{32} - \sum_j \operatorname{diam} P^k_j \ge \frac{\rho}{32} - c\varepsilon^{\frac{1}{2}}\rho \ge \frac{\rho}{64}.$$

From Lemma 5.5.2 it follows that there exists a set $T_l \subset \left(\frac{\rho}{16}, \frac{3\rho}{32}\right)$ with $\mathcal{L}^1(T_l) \geq \frac{\rho}{64}$ such that for all $\gamma \in T_l$,

$$\partial C^l_\gamma(\xi) \cap \bigcup_j P^k_j = \emptyset \quad \text{for infinitely many } k \in \mathbb{N}.$$

Now let $\gamma \in T_l$ be arbitrary (it will be chosen later); we apply the Extension Lemma given in the Appendix (see Lemma 5.5.1, for the proof see [Schy]) to get a function $w^l_k \in C^\infty\left(\overline{B^e_\gamma(\xi)} \cap L_l, L_l^\perp\right)$ for infinitely many k such that

$$\begin{aligned} w^l_k = u^l_k \quad &, \quad \frac{\partial w^l_k}{\partial \nu} = \frac{\partial u^l_k}{\partial \nu} \quad \text{on } \partial B^e_\gamma(\xi) \cap L_l, \\ \frac{1}{\gamma}||w^l_k||_{L^\infty(B^e_\gamma(\xi)\cap L_l)} &\leq c\varepsilon^{\frac{1}{6}} + \frac{\delta_k}{\gamma} \quad \text{where } \delta_k \to 0, \\ ||Dw^l_k||_{L^\infty(\partial B^e_\gamma(\xi)\cap L_l)} &\leq c\varepsilon^{\frac{1}{6}} + \delta_k \quad \text{where } \delta_k \to 0, \\ \int_{B^e_\gamma(\xi)\cap L_l} |D^2 w^l_k|^2 &\leq c\gamma \int_{\operatorname{graph} u^l_k|_{\partial B^e_\gamma(\xi)\cap L_l}} |A^e_k|^2 \, d\mathcal{H}^1_e. \end{aligned}$$

where $d\mathcal{H}^1_e$ is the 1 dimensional euclidean Hausdorff measure.

Observe that, with an analogous argument as above using the estimates on w^l_k, we get

$$\operatorname{graph} w^l_k \subset \overline{B^e_{\frac{\rho}{4}}(\xi)} \quad \text{for } \varepsilon \leq \varepsilon_0 \ (\varepsilon_0 \text{ maybe smaller) and } \delta_k \leq \frac{\rho}{8}. \tag{5.19}$$

Now we consider the immersed surfaces

$$\tilde{\Sigma}_k = \left(f_k(\mathbb{S}^2) \setminus \left(\bigcup_l D^l_k \cap C^l_\gamma(\xi)\right)\right) \cup \bigcup_l \operatorname{graph} w^l_k. \tag{5.20}$$

Let us check that $\tilde{\Sigma}_k$ can be parametrized on $\mathbb{S}^2$ by a $C^{1,1}$ immersion $\tilde{f}_k : \mathbb{S}^2 \hookrightarrow M$: Since the pimples are diffeomorphic to discs and since we have chosen a good radius γ for the cylinder $C^l_\gamma(\xi)$, it is possible to show that $D^l_k \cap C^l_\gamma(\xi)$ is diffeomorphic to $\operatorname{graph} w^l_k$ for all k, l. By the boundary properties of w^l_k one can define a $C^{1,1}$ immersion $\tilde{f}_k : \mathbb{S}^2 \hookrightarrow M$ which parametrizes $\tilde{\Sigma}_k$.

From the definition of γ we have that

$$\int_{\operatorname{graph} w^l_k} |A_e|^2 \, d\mathcal{H}^2_e \leq c \int_{B^e_\gamma(\xi)\cap L_l} |D^2 w^l_k|^2 \leq c\gamma \int_{\operatorname{graph} u^l_k|_{\partial B^e_\gamma(\xi)\cap L_l}} |A^e_k|^2 d\mathcal{H}^1_e = c\gamma \int_{\partial C^l_\gamma(\xi)\cap D^l_k} |A^e_k|^2 d\mathcal{H}^1_e.$$

Until now, $\gamma \in T_l \subset \left(\frac{\rho}{16}, \frac{3\rho}{32}\right)$ was arbitrary and $\mathcal{L}^1(T_l) \geq \frac{\rho}{64}$. Therefore, with a Fubini-type argument, we get that the set

$$S^l_k = \left\{ \gamma \in T_l \,\Big|\, \int_{\partial C^l_\gamma(\xi)\cap D^l_k} |\mathrm{A}_e|^2 d\mu_e \leq \frac{128}{\rho} \int_{\left(D^l_k \cap C^l_{k,\frac{3\rho}{32}}(\xi) \setminus C^l_{k,\frac{\rho}{16}}(\xi)\right) \setminus \bigcup_j P^k_j} |\mathrm{A}_e|^2 \, d\mu_e \right\}$$

has measure $\mathcal{L}^1(S^l_k) \geq \frac{\rho}{128}$. Indeed otherwise we would have that

$$\begin{aligned} \int_{\left(D^l_k \cap C^l_{k,\frac{3\rho}{32}}(\xi) \setminus C^l_{k,\frac{\rho}{16}}(\xi)\right) \setminus \bigcup_j P^k_j} |\mathrm{A}_e|^2 \, d\mu_e \quad &\geq \quad \int_{T_l \setminus S^l_k} \int_{\partial C^l_\gamma(\xi)\cap D^l_k} |\mathrm{A}_e|^2 \\ &\geq \quad \mathcal{L}^1\left(T_l \setminus S^l_k\right) \frac{128}{\rho} \int_{\left(D^l_k \cap C^l_{k,\frac{3\rho}{32}}(\xi) \setminus C^l_{k,\frac{\rho}{16}}(\xi)\right) \setminus \bigcup_j P^k_j} |\mathrm{A}_e|^2 \, d\mu_e \\ &> \quad \left(\frac{\rho}{64} - \frac{\rho}{128}\right) \frac{128}{\rho} \int_{\left(D^l_k \cap C^l_{k,\frac{3\rho}{32}}(\xi) \setminus C^l_{k,\frac{\rho}{16}}(\xi)\right) \setminus \bigcup_j P^k_j} |\mathrm{A}_e|^2 \, d\mu_e \\ &= \quad \int_{\left(D^l_k \cap C^l_{k,\frac{3\rho}{32}}(\xi) \setminus C^l_{k,\frac{\rho}{16}}(\xi)\right) \setminus \bigcup_j P^k_j} |\mathrm{A}_e|^2 \, d\mu_e, \end{aligned}$$

a contradiction.

Until now, $\gamma \in T_l \subset \left(\frac{\rho}{16}, \frac{3\rho}{32}\right)$ was arbitrary and $\mathcal{L}^1(T_l) \geq \frac{\rho}{64}$. Therefore, with a simple Fubini-type argument as done in [SiL], it is easy to see that we can choose γ such that for every l, k (fixed)

$$\int_{\mathrm{graph}\, w_k^l} |A_e|^2 \, d\mathcal{H}_e^2 \leq c \int_{\left(D_k^l \cap C^l_{\frac{3\rho}{32}}(\xi) \setminus C^l_{\frac{\rho}{16}}(\xi)\right) \setminus \bigcup_j P_j^k} |A_k^e|^2 \, d\mathcal{H}_e^2 .$$

Now notice that (this follows from the estimates on u_k^l and Du_k^l for $\varepsilon \leq \varepsilon_0$, ε_0 maybe smaller)

$$\begin{aligned} B^e_{\frac{\rho}{16}}(\xi) \quad &\subset \quad C^l_{\frac{\rho}{16}}(\xi), \\ \left(D_k^l \cap C^l_{\frac{3\rho}{32}}(\xi)\right) \setminus \bigcup_j P_j^k \quad &\subset \quad \left(D_k^l \cap B^e_{\frac{\rho}{8}}(\xi)\right) \setminus \bigcup_j P_j^k . \end{aligned}$$

We get that

$$\int_{\mathrm{graph}\, w_k^l} |A_e|^2 \, d\mathcal{H}_e^2 \leq c \int_{D_k^l \cap B^e_{\frac{\rho}{8}}(\xi) \setminus B^e_{\frac{\rho}{16}}(\xi)} |A_k^e|^2 \, d\mathcal{H}_e^2 .$$

It follows that (using the uniform bound on M_k),

$$\sum_{l=1}^{M_k} \int_{\mathrm{graph}\, w_k^l} |A_e|^2 \, d\mathcal{H}_e^2 \leq c \sum_{l=1}^{M_k} \int_{D_k^l \cap B^e_{\frac{\rho}{8}}(\xi) \setminus B^e_{\frac{\rho}{16}}(\xi)} |A_k^e|^2 \, d\mathcal{H}_e^2 = c \int_{B^e_{\frac{\rho}{8}}(\xi) \setminus B^e_{\frac{\rho}{16}}(\xi)} |A_k^e|^2 \, d\mu_k^e . \tag{5.21}$$

Under the assumptions of the present Lemma, there are two cases:

i) f_k is a minimizing sequence for the functional E, then

$$E(\tilde{f}_k) \geq E(f_k) - \varepsilon_k \quad \text{where } \varepsilon_k \to 0,$$

which implies, since $B^e_{\frac{\rho}{16}}(\xi) \subset C^l_\gamma(\xi)$,

$$\sum_{l=1}^{M_k} \int_{\mathrm{graph}\, w_k^l} |A_g|^2 \, d\mathcal{H}_g^2 \geq \int_{B^e_{\frac{\rho}{16}}(\xi)} |A_k^g|^2 \, d\mu_k^g - \varepsilon_k . \tag{5.22}$$

Using statements *ii)* and *v)* of Proposition 5.2.3,

$$|A_e|^2 \quad \approx \quad (1 + O(\rho^2))|A_g|^2 + C \tag{5.23}$$

$$d\mu_e \quad \approx \quad [1 + O(\rho)^2] d\mu_g , \tag{5.24}$$

we can compare the L^2-norm of the second fundamental form in metric g and in euclidean metric.
On the one hand we have

$$\begin{aligned} \int_{\mathrm{graph}\, w_k^l} |A_g|^2 \, d\mathcal{H}_g^2 \quad &\approx \quad \int_{\mathrm{graph}\, w_k^l} \left((1 + O(\rho^2))|A_e|^2 + C\right) \left(1 + O\left(\rho^2\right)\right) d\mathcal{H}_e^2 \\ &\approx \quad \int_{\mathrm{graph}\, w_k^l} |A_e|^2 \, d\mathcal{H}_e^2 + c\mathcal{H}_e^2(\mathrm{graph}\, w_k^l) + O\left(\rho^2\right) \int_{\mathrm{graph}\, w_k^l} |A_e|^2 \, d\mathcal{H}_e^2 . \end{aligned}$$

The bounds on the gradient of w_k^l imply $\mathcal{H}_e^2(\mathrm{graph}\, w_k^l) \leq c\rho^2$. Using (5.21) we also have $\sum_{l=1}^{M_k} \int_{\mathrm{graph}\, w_k^l} |A_e|^2 \, d\mathcal{H}_e^2 \leq c$. It follows that

$$\sum_{l=1}^{M_k} \int_{\mathrm{graph}\, w_k^l} |A_g|^2 \, d\mathcal{H}_g^2 \leq \sum_{l=1}^{M_k} \int_{\mathrm{graph}\, w_k^l} |A_e|^2 \, d\mathcal{H}_e^2 + c\rho^2 . \tag{5.25}$$

On the other hand, with analogous estimates,

$$\int_{B^e_{\frac{\rho}{16}}(\xi)} |A_k^g|^2 \, d\mu_k^g \approx \int_{B^e_{\frac{\rho}{16}}(\xi)} |A_k^e|^2 \, d\mu_k^e + C\mu_k^e(B^e_{\frac{\rho}{16}}(\xi)) + O\left(\rho^2\right) \int_{B^e_{\frac{\rho}{16}}(\xi)} |A_k^e|^2 \, d\mu_k^e .$$

From statement $i)$ in Lemma 5.2.4, we know that $\mu_k^e(B^e_{\frac{\rho}{16}}(\xi)) \leq \beta\rho^2$; since as above $\int_{B^e_{\frac{\rho}{16}}(\xi)} |A_k^e|^2 \, d\mu_k^e \leq c$, it follows that

$$\int_{B^e_{\frac{\rho}{16}}(\xi)} |A_k^g|^2 \, d\mu_k^g \geq \int_{B^e_{\frac{\rho}{16}}(\xi)} |A_k^e|^2 \, d\mu_k^e - c\rho^2. \tag{5.26}$$

Therefore, putting estimates (5.25) and (5.26) into the inequality (5.22), we get

$$\sum_{l=1}^{M_k} \int_{\operatorname{graph} w_k^l} |A_e|^2 d\mathcal{H}_e^2 \geq c \int_{B^e_{\frac{\rho}{16}}(\xi)} |A_k^e|^2 \, d\mu_k^e - \varepsilon_k - c\rho^2. \tag{5.27}$$

$ii)$ f_k is a minimizing sequence for the functional W_1, then

$$W_1(\tilde{f}_k) \geq W_1(f_k) - \varepsilon_k \quad \text{where } \varepsilon_k \to 0. \tag{5.28}$$

Integrating the Gauss equation

$$\frac{1}{4}|H_g|^2 = \frac{1}{4}|A_g|^2 + \frac{1}{2}K_G - \frac{1}{2}\bar{K}(Tf_k)$$

(K_G is the Gauss curvature and $\bar{K}(Tf_k)$ is the sectional curvature of the ambient manifold evaluated on the tangent space to f_k) and applying Gauss-Bonnet Theorem, we get

$$W_1(f_k) := \int \left(\frac{|H_k^g|^2}{4} + 1 \right) d\mu_k^g = \int \left(\frac{|A_k^g|^2}{4} + 1 \right) d\mu_k^g + \pi\chi_E(\mathbb{S}^2) - \frac{1}{2}\int \bar{K}(Tf_k) d\mu_k^g.$$

Since both f_k and $\tilde{f}_k$ are immersions of a sphere,the last inequality and (5.28) imply

$$\int \left(\frac{|\tilde{A}_k^g|^2}{4} + 1 \right) d\tilde{\mu}_k^g - \frac{1}{2}\int \bar{K}(T\tilde{f}_k) d\tilde{\mu}_k^g \geq \int \left(\frac{|A_k^g|^2}{4} + 1 \right) d\mu_k^g - \frac{1}{2}\int \bar{K}(Tf_k) d\mu_k^g - \varepsilon_k$$

where, of course, $\tilde{A}_k^g$ and $\tilde{\mu}_k^g$ are respectively the second fundamental form and the area measure associate to the immersion $\tilde{f}_k$. Since from the definition of $\tilde{\Sigma}_k$ and inclusions (5.17), (5.19) outside the ball $\overline{B^e_{\frac{\rho}{4}}(\xi)}$ the surfaces $f_k(\mathbb{S}^2)$ and $\tilde{\Sigma}_k$ coincide at k fixed, then

$$\begin{aligned} &\sum_{l=1}^{M_k} \int_{\operatorname{graph} w_k^l} |A_g|^2 \, d\mathcal{H}_g^2 + 4\sum_{l=1}^{M_k} \mathcal{H}_g(\operatorname{graph} w_k^l) - 2\sum_{l=1}^{M_k} \int_{\operatorname{graph} w_k^l} \bar{K} \, d\mathcal{H}_g^2 \\ &\geq \int_{B^e_{\frac{\rho}{16}}(\xi)} |A_k^g|^2 \, d\mu_k^g + 4\mu_k^g(B^e_{\frac{\rho}{16}}(\xi)) - 2\int_{B^e_{\frac{\rho}{4}}(\xi)} \bar{K} \, d\mu_k^g - \varepsilon_k. \end{aligned} \tag{5.29}$$

Using that the sectional curvature $\bar{K}$ is bounded since M is compact, the local area bounds written for case $i)$, and estimates (5.25), (5.26), we get

$$\sum_{l=1}^{M_k} \int_{\operatorname{graph} w_k^l} |A_e|^2 d\mathcal{H}_e^2 \geq \int_{B^e_{\frac{\rho}{16}}(\xi)} |A_k^e|^2 \, d\mu_k^e - \varepsilon_k - c\rho^2. \tag{5.30}$$

In both cases $i)$ and $ii)$, plugging (5.21) into (5.27) or in (5.30) we obtain

$$\int_{B^e_{\frac{\rho}{16}}(\xi)} |A_k^e|^2 \, d\mu_k^e \leq c \int_{B^e_{\frac{\rho}{8}}(\xi) \setminus B^e_{\frac{\rho}{16}}(\xi)} |A_k^e|^2 \, d\mu_k^e + \varepsilon_k + c\rho^2.$$

By adding c times the left hand side of this inequality to both sides ("hole filling") we deduce the following: for all $\rho \leq \frac{\rho_0}{4}$ we have for infinitely many $k \in \mathbb{N}$ that

$$\int_{B^e_{\frac{\rho}{16}}(\xi)} |A_k^e|^2 \, d\mu_k^e \leq \theta \int_{B^e_{\frac{\rho}{8}}(\xi)} |A_k^e|^2 \, d\mu_k^e + \varepsilon_k + c\rho^2,$$

where $\theta = \frac{c}{c+1} \in (0,1)$ is a fixed universal constant. Now if we let $g(\rho) = \liminf_{k\to\infty} \int_{B^e_\rho(\xi)} |A^e_k|^2 \, d\mu^e_k$ we get that

$$g(\rho) \le \theta g(2\rho) + c\rho^2 \quad \text{for all } \rho \le \frac{\rho_0}{64}.$$

In view of Lemma 5.5.3 in the Appendix, the present Lemma is proved.

■

Now we are able to show that, in a neighborhood of the good points, the limit measure μ is the Radon measure associated to $C^{1,\alpha} \cap W^{2,2}$-graphs. First we recall the setting shortly: let $0 < \varepsilon \le \varepsilon_0$, $\xi_0 \in \operatorname{spt}\mu \backslash \mathcal{B}_\varepsilon$ be an ε-good point and let $\rho_0 = \rho_0(\xi_0, \varepsilon) > 0$ be as in Lemma 5.2.4. Let $\xi \in \operatorname{spt}\mu \cap B^e_{\frac{\rho_0}{2}}(\xi_0)$, $\rho \le \frac{\rho_0}{4}$ and recall Lemma 5.2.7.

We had that $u^l_k : \Omega^l_k \to L^\perp_l$ where the set Ω^l_k was given by

$$\Omega^l_k = (B^e_\lambda(\xi) \cap L_l) \setminus \bigcup_m d^l_{k,m}$$

where $\lambda \in \left(\frac{\rho}{4}, \frac{\rho}{2}\right)$ and where the sets $d^l_{k,m} \subset L_l$ are pairwise disjoint closed discs which do not intersect $\partial B^e_\lambda(\xi)$.

Define the quantity $\alpha_k(\rho)$ by

$$\alpha_k(\rho) = \int_{B^e_{4\rho}(\xi)} |A^e_k|^2 \, d\mu^e_k$$

and notice that by Lemma 5.3.1 we have

$$\liminf_{k\to\infty} \alpha_k(\rho) \le c\rho^\alpha \quad \text{for all } \rho \le \frac{\rho_0}{128}. \tag{5.31}$$

Since $\xi_k \to \xi$ and therefore $B^e_{2\rho}(\xi_k) \subset B^e_{4\rho}(\xi)$ for k sufficiently large we have that

$$\sum_m \operatorname{diam} d^l_{k,m} \le c\alpha_k(\rho)^{\frac{1}{4}} \rho \le c\varepsilon^{\frac{1}{2}} \rho. \tag{5.32}$$

Therefore for $\varepsilon \le \varepsilon_0$ we may apply Lemma 5.5.4 to the functions $f^l_j = D_j u^l_k$ and $\delta = c\alpha_k(\rho)^{\frac{1}{4}}\rho$ in order to get a constant vector η^l_k, with $|\eta^l_k| \le c\varepsilon^{\frac{1}{6}} + \delta_k \le c$ and $\delta_k \to 0$, such that

$$\int_{\Omega^l_k} \left|Du^l_k - \eta^l_k\right|^2 \le c\rho^2 \int_{\Omega^l_k} \left|D^2 u^l_k\right|^2 + c\alpha_k(\rho)^{\frac{1}{4}} \rho^2 \sup_{\Omega^l_k} \left|Du^l_k\right|^2.$$

Now we have that

$$\int_{\Omega^l_k} \left|D^2 u^l_k\right|^2 \le c \int_{\operatorname{graph} u^l_k} |A^e_k|^2 \, d\mathcal{H}^2_e \le c \int_{B^e_{2\rho}(\xi)} |A^e_k|^2 \, d\mu^e_k \le c\alpha_k(\rho).$$

Since $|Du^l_k| \le c$ and $\alpha_k(\rho) \le 1$ for $\varepsilon \le \varepsilon_0$, it follows that

$$\int_{\Omega^l_k} \left|Du^l_k - \eta^l_k\right|^2 \le c\alpha_k(\rho)^{\frac{1}{4}} \rho^2. \tag{5.33}$$

Now let $\overline{u}^l_k \in C^{1,1}(B^e_\lambda(\xi) \cap L_l, L^\perp_l)$ be an extension of u^l_k to all of $B^e_\lambda(\xi) \cap L_l$ as in Lemma 5.5.1, i.e.

$$\begin{aligned}
\overline{u}^l_k \quad &= \quad u^l_k \quad \text{in } B^e_\lambda(\xi) \cap L_l \setminus \bigcup_m d^l_{k,m}, \\
\overline{u}^l_k = u^l_k \quad &, \quad \frac{\partial \overline{u}^l_k}{\partial \nu} = \frac{\partial u^l_k}{\partial \nu} \quad \text{on } \bigcup_m \partial d^l_{k,m}, \\
\|\overline{u}^l_k\|_{L^\infty(d^l_{k,m})} \quad &\le \quad c\varepsilon^{\frac{1}{6}} \rho + \delta_k \quad \text{where } \delta_k \to 0, \\
\|D\overline{u}^l_k\|_{L^\infty(d^l_{k,m})} \quad &\le \quad c\varepsilon^{\frac{1}{6}} + \delta_k, \quad \text{where } \delta_k \to 0.
\end{aligned}$$

It follows that $||\overline{u}_k^l||_{L^\infty(B^e_\lambda(\xi)\cap L_l)}+||D\overline{u}_k^l||_{L^\infty(B^e_\lambda(\xi)\cap L_l)}\leq c$ where c is independent of k. From the gradient estimates for the function $\overline{u}_k^l$, since $|\eta_k^l|\leq c$ and because of (5.32) we get that

$$
\begin{aligned}
\int_{B^e_\lambda(\xi)\cap L_l}\left|D\overline{u}_k^l-\eta_k^l\right|^2 &= \int_{\Omega_k^l}\left|Du_k^l-\eta_k^l\right|^2+\sum_m\int_{d^l_{k,m}}\left|D\overline{u}_k^l-\eta_k^l\right|^2\\
&\leq c\alpha_k(\rho)^{\frac{1}{4}}\rho^2+c\sum_m\int_{d^l_{k,m}}\left|D\overline{u}_k^l\right|^2+c\sum_m\int_{d^l_{k,m}}\left|\eta_k^l\right|^2\\
&\leq c\alpha_k(\rho)^{\frac{1}{4}}\rho^2+c\sum_m\mathcal{L}^2(d^l_{k,m})\leq c\alpha_k(\rho)^{\frac{1}{4}}\rho^2+c\left(\sum_m\operatorname{diam}d^l_{k,m}\right)^2\\
&\leq c\alpha_k(\rho)^{\frac{1}{4}}\rho^2+c\alpha_k(\rho)^{\frac{1}{2}}\rho^2\leq c\alpha_k(\rho)^{\frac{1}{4}}\rho^2,
\end{aligned}
$$

so

$$\int_{B^e_\lambda(\xi)\cap L_l}\left|D\overline{u}_k^l-\eta_k^l\right|^2\leq c\alpha_k(\rho)^{\frac{1}{4}}\rho^2.$$

Thus, in view of (5.31), we conclude that

$$\liminf_{k\to\infty}\int_{B^e_\lambda(\xi)\cap L_l}\left|D\overline{u}_k^l-\eta_k^l\right|^2\leq c\rho^{2+\alpha}\quad\text{for all }\rho\leq\frac{\rho_0}{128}. \tag{5.34}$$

Moreover, it trivially follows that $\left\|\overline{u}_k^l\right\|_{W^{1,2}(B^e_\lambda(\xi)\cap L_l)}\leq c\rho^2\leq c$. Therefore it follows that the sequence $\overline{u}_k^l$ is equicontinuous and uniformly bounded in $C^1(B^e_\lambda(\xi)\cap L_l,L_l^\perp)$ and $W^{1,2}(B^e_\lambda(\xi)\cap L_l,L_l^\perp)$ and we get the existence of a function $u^l_\xi\in C^{0,1}(B^e_\lambda(\xi)\cap L_l,L_l^\perp)\cap W^{1,2}(B^e_\lambda(\xi)\cap L_l,L_l^\perp)$ such that (after passing to a subsequence)

$$
\begin{aligned}
\overline{u}_k^l &\to u^l_\xi \quad\text{in } C^0(B^e_\lambda(\xi)\cap L_l,L_l^\perp),\\
\overline{u}_k^l &\rightharpoonup u^l_\xi \quad\text{weakly in } W^{1,2}(B^e_\lambda(\xi)\cap L_l,L_l^\perp)
\end{aligned}
$$

and such that the following estimates hold for the function u^l_ξ:

$$\frac{1}{\rho}||u^l_\xi||_{L^\infty(B^e_\lambda(\xi)\cap L_l)}+||Du^l_\xi||_{L^\infty(B^e_\lambda(\xi)\cap L_l)}\leq c\varepsilon^{\frac{1}{6}}.$$

We notice that, a priori, the limit function might depend on the point ξ; indeed, the sequence u_k^l depends on ξ since it comes from the graphical decomposition lemma which is a local statement.

Observe that, up to subsequences, $\eta_k^l\to\eta^l$ with $|\eta^l|\leq c\varepsilon^{\frac{1}{6}}$. Since $D\overline{u}_k^l\rightharpoonup Du^l_\xi$ weakly in $L^2(B^e_\lambda(\xi)\cap L_l)$, then $D\overline{u}_k^l-\eta_k^l\rightharpoonup Du^l_\xi-\eta^l$ weakly in $L^2(B^e_\lambda(\xi)\cap L_l)$; therefore, by lower-semicontinuity, estimate (5.34) implies that

$$\int_{B^e_\lambda(\xi)\cap L_l}\left|Du^l_\xi-\eta^l\right|^2\leq c\rho^{2+\alpha}\quad\text{for all }\rho\leq\frac{\rho_0}{128}. \tag{5.35}$$

Lemma 5.3.2. *Let f_k and μ be as in Lemma 5.3.1 (μ is the limit candidate minimizer measure). Thus there exists $\varepsilon_0>0$ such that for every $0<\varepsilon\leq\varepsilon_0$ and every good point $\xi_0\in\Sigma\setminus\mathcal{B}_\varepsilon$ there exists $\rho_0=\rho_0(\xi_0,\varepsilon_0,M)$ such that the following is true:*

For all $\xi\in\operatorname{spt}\mu\cap B^e_{\frac{\rho_0}{2}}(\xi_0)$ and all $\rho\leq\rho_0$ such that

$$\mu\llcorner B^e_\rho(\xi)=\sum_{l=1}^M\mathcal{H}^2_g\llcorner\left(\operatorname{graph}u^l_\xi\cap B^e_\rho(\xi)\right),$$

where each $u^l_\xi\in C^{0,1}\left(B^e_{2\rho}(\xi)\cap L_l,L_l^\perp\right)$ such that

$$\frac{1}{\rho}||u^l_\xi||_{L^\infty(B^e_{2\rho}(\xi)\cap L_l)}+\left\|Du^l_\xi\right\|_{L^\infty(B^e_{2\rho}(\xi)\cap L_l)}\leq c\varepsilon^{\frac{1}{6}}$$

and where $\mathcal{H}^2_g$ denotes the 2 dimensional Hausdorff measure of the Riemannian manifold (M,g).

PROOF. First we claim that for all $\rho \leq \frac{\rho_0}{128}$ the following equation holds:

$$\mu_k^g \llcorner B_\rho^e(\xi) = \sum_{l=1}^{M} \mathcal{H}_g^2 \llcorner \big(\operatorname{graph} \overline{u}_k^l \cap B_\rho^e(\xi)\big) + \theta_k \tag{5.36}$$

where θ_k is a signed measure with $\liminf_{k\to\infty}$ of the total mass is smaller than $c\rho^{2+\alpha}$, i.e. there exist two radon measures θ_k^1 and θ_k^2 such that $\theta_k = \theta_k^1 - \theta_k^2$ and such that $\liminf_{k\to\infty}\big(\theta_k^1(M) + \theta_k^2(M)\big) \leq c\rho^{2+\alpha}$.

To prove the claim, recall that we have (from the diameter estimates in Lemma 5.2.7 and from the quadratic area decay) $\sum_{m,l} \mathcal{L}^2\left(d_{k,m}^l\right) + \sum_j \mathcal{H}_g^2\left(P_j^k\right) \leq c\alpha_k(\rho)^{\frac{1}{2}}\rho^2$; thus for $\rho \leq \frac{\rho_0}{128}$ Lemma 5.3.1 yields $\liminf_{k\to\infty}\sum_{m,l}\mathcal{L}^2\left(d_{k,m}^l\right) + \liminf_{k\to\infty}\sum_j \mathcal{H}_g^2\left(P_j^k\right) \leq c\rho^{2+\alpha}$.

It follows that $\mu_k^g \llcorner B_\rho^e(\xi) = \sum_{l=1}^M \mathcal{H}_g^2 \llcorner \big(\operatorname{graph}\overline{u}_k^l \cap B_\rho^e(\xi)\big) + \theta_k$ where

$$\theta_k = \sum_{l=1}^{M} \mathcal{H}_g^2 \llcorner \big((D_k^l \setminus \operatorname{graph}\overline{u}_k^l) \cap B_\rho^e(\xi)\big) - \sum_{l=1}^{M} \mathcal{H}_g^2 \llcorner \big((\operatorname{graph}\overline{u}_k^l \setminus D_k^l) \cap B_\rho^e(\xi)\big) = \theta_k^1 - \theta_k^2.$$

We have that $\theta_k^1(M) \leq \sum_j \mathcal{H}_g^2\left(P_j^k\right)$ and that $\theta_k^2(M) \leq c\sum_{m,l}\mathcal{L}^2\left(d_{k,m}^l\right)$, and (5.36) follows.

Now by taking limits in the measure theoretic sense we claim that for all $\xi \in \operatorname{spt}\mu \cap B_{\frac{\rho_0}{2}}^e(\xi_0)$:

$$\mu \llcorner B_\rho^e(\xi) = \sum_{l=1}^{M} \mathcal{H}_g^2 \llcorner \big(\operatorname{graph} u_\xi^l \cap B_\rho^e(\xi)\big) + \theta \tag{5.37}$$

where θ is a signed measure with total mass smaller than $c\rho^{2+\alpha}$. This equation holds for all $\rho \leq \frac{\rho_0}{128}$ such that

$$\mu\left(\partial B_\rho^e(\xi)\right) = \mathcal{H}_g^2 \llcorner \operatorname{graph} u_\xi^l\left(\partial B_\rho^e(\xi)\right) = 0 \quad \text{for all } l.$$

Notice that the last sequence of equations holds for a.e. ρ (since we are dealing with radon measures with finite mass).

To prove (5.37) let $U \subset M$ be an open subset. We have that

1) Let $\rho \leq \frac{\rho_0}{128}$ be such that $\mu\left(\partial B_\rho^e(\xi)\right) = 0$. Moreover assume that $\mu \llcorner B_\rho^e(\xi)\,(\partial U) = 0$. It follows that $\mu\left(\partial\left(U \cap B_\rho^e(\xi)\right)\right) = 0$ and therefore $\mu_k^g\left(U \cap B_\rho^e(\xi)\right) \to \mu\left(U \cap B_\rho^e(\xi)\right)$.

2) Let $\rho \leq \frac{\rho_0}{128}$ be such that $\mathcal{H}_g^2 \llcorner \operatorname{graph} u_\xi^l\left(\partial B_\rho^e(\xi)\right) = 0$. Moreover assume that $\mathcal{H}_g^2 \llcorner \Big(\operatorname{graph} u_\xi^l \cap B_\rho^e(\xi)\Big)(\partial U) = 0$. It follows from $ii)$ of Proposition 5.2.3 that

$$\mathcal{H}_g^2 \llcorner \big(\operatorname{graph}\overline{u}_k^l \cap B_\rho^e(\xi)\big)(U) = \int_{L_l} \chi_{U\cap B_\rho^e(\xi)}(x + \overline{u}_k^l(x))\sqrt{1 + |D\overline{u}_k^l(x)|^2} + O(\rho^4).$$

Now we have that

$$\left|\int_{L_l} \chi_{U\cap B_\rho^e(\xi)}(x + \overline{u}_k^l(x))\sqrt{1 + |D\overline{u}_k^l(x)|^2} - \int_{L_l} \chi_{U\cap B_\rho^e(\xi)}(x + u_\xi^l(x))\sqrt{1 + |Du_\xi^l(x)|^2}\right|$$

$$\leq c\int_{L_l}\left|\chi_{U\cap B_\rho^e(\xi)}(x + \overline{u}_k^l(x)) - \chi_{U\cap B_\rho^e(\xi)}(x + u_\xi^l(x))\right| + \int_{L_l} \chi_{U\cap B_\rho^e(\xi)}(x + u_\xi^l(x))\left|\sqrt{1 + |D\overline{u}_k^l(x)|^2} - \sqrt{1 + |Du_\xi^l(x)|^2}\right|.$$

Since $\overline{u}_k^l \to u_\xi^l$ uniformly and since $\mathcal{H}_g^2 \llcorner \operatorname{graph} u_\xi^l\left(\partial B_\rho^e(\xi)\right) = 0$ it follows that

$$\chi_{U\cap B_\rho^e(\xi)}(x + \overline{u}_k^l(x)) \to \chi_{U\cap B_\rho^e(\xi)}(x + u_\xi^l(x)) \quad \text{for a.e. } x \in L_l.$$

To see this we have to consider two distinct cases:

(i) $x \in L_l$ such that $x + u_\xi^l(x) \in U \cap B_\rho^e(\xi)$: Then $\chi_{U\cap B_\rho^e(\xi)}(x + u_\xi^l(x)) = 1$ and since $U \cap B_\rho^e(\xi)$ is open and $\overline{u}_k^l \to u_\xi^l$ uniformly it follows that $x + \overline{u}_k^l(x) \in U \cap B_\rho^e(\xi)$ for k sufficiently large so that $\chi_{U\cap B_\rho^e(\xi)}(x + \overline{u}_k^l(x)) = 1$.

(ii) $x \in L_l$ such that $x + u^l_\xi(x) \in \overline{U \cap B^e_\rho(\xi)}^c$: Now $\chi_{U\cap B^e_\rho(\xi)}(x+u^l_\xi(x)) = 0$ and since $\overline{U \cap B^e_\rho(\xi)}^c$ is open it again follows that $x + \overline{u}^l_k(x) \in \overline{U \cap B^e_\rho(\xi)}^c$ for k sufficiently large and $\chi_{U\cap B^e_\rho(\xi)}(x + \overline{u}^l_k(x)) = 0$.

Now notice that

$$\mathcal{L}^2\left(\left\{x \in L_l \,|\, x + u^l_\xi(x) \in \partial\left(U \cap B^e_\rho(\xi)\right)\right\}\right) = 0$$

since $\mathcal{H}^2_g \llcorner \text{graph}\, u^l_\xi\left(\partial\left(U \cap B^e_\rho(\xi)\right)\right) = 0$ by the choice of the set U and

$$\begin{aligned}\mathcal{H}^2_g \llcorner \text{graph}\, u^l_\xi\left(\partial\left(U \cap B^e_\rho(\xi)\right)\right) &= (1 + O(\rho^2)) \int_{\left\{x \in L_l \,|\, x+u^l_\xi(x) \in \partial\left(U \cap B^e_\rho(\xi)\right)\right\}} \sqrt{1 + |Du^l_\xi|^2} \\ &\geq \frac{1}{C}\, \mathcal{L}^2\left(\left\{x \in L_l \,|\, x + u^l_\xi(x) \in \partial\left(U \cap B^e_\rho(\xi)\right)\right\}\right).\end{aligned}$$

Therefore

$$\chi_{U\cap B^e_\rho(\xi)}(x + \overline{u}^l_k(x)) \to \chi_{U\cap B^e_\rho(\xi)}(x + u^l_\xi(x)) \quad \text{for a.e. } x \in L_l.$$

By the dominated convergence theorem we get that $\int_{L_l} \left|\chi_{U\cap B^e_\rho(\xi)}(x + \overline{u}^l_k(x)) - \chi_{U\cap B^e_\rho(\xi)}(x + u^l_\xi(x))\right| \to 0$. Because of the elementary inequality $\left|\sqrt{1+a^2} - \sqrt{1+b^2}\right| \leq |a - b|$ we get on the other hand that

$$\int_{L_l} \chi_{U\cap B^e_\rho(\xi)}(x + u^l_\xi(x)) \left|\sqrt{1 + |D\overline{u}^l_k(x)|^2} - \sqrt{1 + |Du^l_\xi(x)|^2}\right|$$

$$\leq \int_{L_l} \chi_{U\cap B^e_\rho(\xi)}(x+u^l_\xi(x)) \left|D\overline{u}^l_k(x) - \eta^l_k\right| + \int_{L_l} \chi_{U\cap B^e_\rho(\xi)}(x+u^l_\xi(x)) \left|\eta^l_k - \eta^l\right| + \int_{L_l} \chi_{U\cap B^e_\rho(\xi)}(x+u^l_\xi(x)) \left|\eta^l - Du^l_\xi(x)\right|.$$

Now we have that $\chi_{U\cap B^e_\rho(\xi)}(x + u^l_\xi(x)) = 0$ if $x \notin B^e_{\left(1-c\varepsilon^{\frac{1}{6}}\right)\rho}(\xi) \cap L_l$. This follows from the L^∞-bound for the function u^l_ξ. Therefore we get that $\left(\int_L \chi_{U\cap B^e_\rho(\xi)}(x + u^l_\xi(x))\right)^{\frac{1}{2}} \leq \mathcal{L}^2\left(B^e_{\left(1-c\varepsilon^{\frac{1}{6}}\right)\rho}(\xi) \cap L_l\right)^{\frac{1}{2}} \leq c\rho$. In view of (5.34) the lim inf of the first term can now be estimated by

$$\liminf_{k\to\infty} \int_{L_l} \chi_{U\cap B^e_\rho(\xi)}(x + u^l_\xi(x)) \left|D\overline{u}^l_k(x) - \eta^l_k\right| \leq c\rho^{2+\alpha}.$$

With (5.35) we get in the same way that $\int_{L_l} \chi_{U\cap B^e_\rho(\xi)}(x+u^l_\xi(x)) \left|\eta^l - Du^l_\xi(x)\right| \leq c\rho^{2+\alpha}$. Now since $\eta^l_k \to \eta^l$ (strongly) we have that $\lim_{k\to\infty} \int_{L_l} \chi_{U\cap B^e_\rho(\xi)}(x + u^l_\xi(x)) \left|\eta^l_k - \eta^l\right| = 0$. Therefore after all we get that

$$\mathcal{H}^2_g \llcorner \left(\text{graph}\, \overline{u}^l_k \cap B^e_\rho(\xi)\right)(U) = \mathcal{H}^2_g \llcorner \left(\text{graph}\, u^l_\xi \cap B^e_\rho(\xi)\right)(U) + \tilde{\theta}_k(U) + O(\rho^4)$$

where $\tilde{\theta}_k$ is a signed measure with $\liminf_{k\to\infty}$ total mass $\leq c\rho^{2+\alpha}$.

Since $\tilde{\theta}_k$ is a signed measure it converges weakly (after passing to a subsequence) to some signed measure $\tilde{\theta}$ with total mass smaller than $c\rho^{2+\alpha}$. Assume that $\tilde{\theta}(\partial U) = 0$. Then it follows that $\tilde{\theta}_k(U) \to \tilde{\theta}(U)$ and therefore we get that

$$\lim_{k\to\infty} \mathcal{H}^2_g \llcorner \left(\text{graph}\, \overline{u}^l_k \cap B^e_\rho(\xi)\right)(U) = \mathcal{H}^2_g \llcorner \left(\text{graph}\, u^l_\xi \cap B^e_\rho(\xi)\right)(U) + \tilde{\theta}(U) + O(\rho^4). \tag{5.38}$$

3) Since the θ_k's were signed measures such that the lim inf of the total mass is smaller than $c\rho^{2+\alpha}$ they converge in the weak sense (after passing to a subsequence) to a signed measure $\overline{\theta}$ with total mass smaller than $c\rho^{2+\alpha}, c\varepsilon^{\frac{1}{4}}\rho^2\}$. Assuming $\overline{\theta}(\partial U) = 0$ it follows that

$$\theta_k(U) \to \overline{\theta}(U). \tag{5.39}$$

Now by taking limits in (5.36) we get that

$$\mu \llcorner B^e_\rho(\xi)(U) = \sum_{l=1}^M \mathcal{H}^2_g \llcorner \left(\text{graph}\, u^l_\xi \cap B^e_\rho(\xi)\right)(U) + \theta(U)$$

where $\theta = \overline{\theta} + \tilde{\theta} + O(\rho^4)$ is a signed measure with total mass smaller than $c\rho^{2+\alpha}$. Notice that this equation holds for every $U \subset M$ open such that $\mu \llcorner B^e_\rho(\xi)(\partial U) = \mathcal{H}^2_g \llcorner \Big(\operatorname{graph} u^l_\xi \cap B^e_\rho(\xi)\Big)(\partial U) = \overline{\theta}(\partial U) = \tilde{\theta}(\partial U) = 0$. By choosing an appropriate exhaustion this equation holds for arbitrary open sets $U \subset M$ and (5.37) is shown.

Next we claim that $\operatorname{spt}\mu$ is locally given by the union of the graphs of the functions u^l_ξ, i.e. there exists a $\rho_0 = \rho_0(\xi_0, \varepsilon_0, M)$ such that for all $\xi \in \operatorname{spt}\mu \cap B^e_{\frac{\rho_0}{2}}(\xi_0)$ and all $\rho \leq \rho_0$ it follows that

$$\operatorname{spt}\mu \cap B^e_\rho(\xi) = \bigcup_{l=1}^M \operatorname{graph} u^l_\xi \cap B^e_\rho(\xi). \tag{5.40}$$

To prove the claim (5.40) let ρ_0 be such that Proposition 5.2.5 holds. Let $\xi \in \operatorname{spt}\mu \cap B^e_{\frac{\rho_0}{2}}(\xi_0)$ and choose $\rho \in \left(\frac{\rho_0}{256}, \frac{\rho_0}{128}\right)$ according to (5.37) such that $\mu \llcorner B^e_\rho(\xi) = \sum_{l=1}^M \mathcal{H}^2_g \llcorner \Big(\operatorname{graph} u^l_\xi \cap B^e_\rho(\xi)\Big) + \theta$.

-″ $\subseteq$ ″: let $x \in \operatorname{spt}\mu \cap B^e_{\frac{\rho}{2}}(\xi)$. Since ξ_0 is a good point and ξ, x are near ξ_0, the lower density bound given in Proposition 5.2.5 holds and we get $\mu \llcorner B^e_\rho(\xi)\left(B^e_{\frac{\rho}{2}}(x)\right) = \mu\left(B^e_{\frac{\rho}{2}}(x)\right) \geq \hat{c}\rho^2$. We get

$$\hat{c}\rho^2 \leq \sum_{l=1}^M \mathcal{H}^2_g\left(\operatorname{graph} u^l_\xi \cap B^e_{\frac{\rho}{2}}(x)\right) + c\rho^{2+\alpha}.$$

For $\rho \leq \rho_0$, where $\rho_0 = \rho_0(\xi_0, \varepsilon_0, M)$, we conclude that $\sum_{l=1}^M \mathcal{H}^2_g\left(\operatorname{graph} u^l_\xi \cap B^e_{\frac{\rho}{2}}(x)\right) > 0$ and therefore $x \in \bigcup_{l=1}^M \operatorname{graph} u^l_\xi$.

-″ $\supseteq$ ″: let $z \in \bigcup_{l=1}^M \operatorname{graph} u^l_\xi \cap B^e_{\frac{\rho}{2}}(\xi)$. Write $z = x + u^l_\xi(x)$ for some $l \in \{1, \dots, M\}$ and some $x \in L_l$. If $y \in B^e_{\frac{\rho}{4}}(x) \cap L_l$ we claim that $y + u^l_\xi(y) \in B^e_{\frac{\rho}{2}}(z)$, indeed for $\varepsilon \leq \varepsilon_0$ (ε_0 maybe smaller) we get

$$|z - y - u^l_\xi(y)| \leq |x-y| + |u^l_\xi(x) - u^l_\xi(y)| \leq \left(1 + c\varepsilon^{\frac{1}{6}}\right)|x-y| \leq \left(1 + c\varepsilon^{\frac{1}{6}}\right)\frac{\rho}{4} \leq \frac{\rho}{2}.$$

Therefore

$$\mathcal{H}^2_g \llcorner \operatorname{graph} u^l_\xi\left(B^e_{\frac{\rho}{2}}(z)\right) \geq \int_{B^e_{\frac{\rho}{4}}(x) \cap L_l} \chi_{B^e_{\frac{\rho}{2}}(z)}(y + u^l_\xi(y))\, d\mu_g \geq c\,\mathcal{L}^2\left(B^e_{\frac{\rho}{4}}(x) \cap L_l\right) = \hat{c}\rho^2.$$

As above we obtain $\mu\left(B^e_{\frac{\rho}{2}}(z)\right) \geq \hat{c}\rho^2 - c\rho^{2+\alpha} > 0$ for $\rho \leq \rho_0$, where $\rho_0 = \rho_0(\xi_0, \varepsilon_0, M)$, and conclude that $z \in \operatorname{spt}\mu$.

This shows (5.40). Now the claim (5.40) implies that the functions u^l_ξ do not depend on the point ξ in the following sense: let $\eta \in \Sigma \cap B^e_{\frac{\rho_0}{2}}(\xi_0)$. Then we have for all $\rho \leq \rho_0$ that

$$\bigcup_{l=1}^M \operatorname{graph} u^l_\xi \cap \left(B^e_\rho(\xi) \cap B^e_\rho(\eta)\right) = \bigcup_{l=1}^N \operatorname{graph} u^l_\eta \cap \left(B^e_\rho(\xi) \cap B^e_\rho(\eta)\right). \tag{5.41}$$

In the next step choose $\rho \leq \rho_0$ such that $\mu\left(\partial B^e_\rho(\xi)\right) = \mathcal{H}^2_g \llcorner \operatorname{graph} u^l_\xi\left(\partial B^e_\rho(\xi)\right) = 0$ for all l, therefore, from (5.37),

$$\mu \llcorner B^e_\rho(\xi) = \sum_{l=1}^M \mathcal{H}^2_g \llcorner \big(\operatorname{graph} u^l_\xi \cap B^e_\rho(\xi)\big) + \theta. \tag{5.42}$$

Let $z \in \operatorname{spt}\mu \cap B^e_\rho(\xi) = \bigcup_{l=1}^M \operatorname{graph} u^l_\xi \cap B^e_\rho(\xi)$ and let $\sigma > 0$ be such that $B^e_\sigma(z) \subset B^e_\rho(\xi)$ and such that (due to (5.37) for the point z) $\mu \llcorner B^e_\sigma(z) = \sum_{l=1}^N \mathcal{H}^2_g \llcorner \big(\operatorname{graph} u^l_z \cap B^e_\sigma(z)\big) + \theta_z$, where the total mass of θ_z is smaller than $c\sigma^{2+\alpha}$.

From (5.41) it follows that $\theta\left(B^e_\sigma(z)\right) = \theta_z\left(B^e_\sigma(z)\right)$, hence we get a nice decay for the signed measure θ:

$$\lim_{\sigma \to 0} \frac{\theta\left(B^e_\sigma(z)\right)}{\sigma^2} = 0 \quad \text{for all } z \in \operatorname{spt}\mu \cap B^e_\rho(\xi) = \bigcup_{l=1}^M \operatorname{graph} u^l_\xi \cap B^e_\rho(\xi). \tag{5.43}$$

Next we claim that for all $z \in \operatorname{spt}\mu \cap B^e_\rho(\xi) = \bigcup_{l=1}^M \operatorname{graph} u^l_\xi \cap B^e_\rho(\xi)$,

$$\liminf_{\sigma\to 0} \frac{\sum_{l=1}^M \mathcal{H}^2_g \llcorner \Big(\operatorname{graph} u^l_\xi \cap B^e_\rho(\xi)\Big)\,(B^e_\sigma(z))}{\pi\sigma^2} \geq C > 0. \tag{5.44}$$

To prove the claim (5.44), let $z \in \operatorname{spt}\mu \cap B^e_\rho(\xi) = \bigcup_{l=1}^M \operatorname{graph} u^l_\xi \cap B^e_\rho(\xi)$ and let $\sigma > 0$ be such that $B^e_\sigma(z) \subset B^e_\rho(\xi)$. Let $z \in \operatorname{graph} u^l_\xi$ for some l, then

$$\mathcal{H}^2_g \llcorner \big(\operatorname{graph} u^l_\xi \cap B^e_\rho(\xi)\big)\,(B^e_\sigma(z)) \;\; = \;\; \mathcal{H}^2_g \left(\operatorname{graph} u^l_\xi \cap B^e_\sigma(z)\right) = \int_{L_l} \chi_{\,B^e_\sigma(z)}(y + u^l_\xi(y))\, d\mu_g.$$

Now let $z = x + u^l_\xi(x)$ with $x \in B^e_\rho(\xi) \cap L_l$. We have that

$$|z - y - u^l_\xi(y)| \leq |x-y| + |u^l_\xi(x) - u^l_\xi(y)| \leq \left(1 + c\varepsilon^{\frac{1}{6}}\right)|x-y|,$$

therefore

$$\chi_{\,B^e_\sigma(z)}(y + u^l_\xi(y)) = 1 \quad \text{if } |x-y| \leq \frac{1}{1 + c\varepsilon^{\frac{1}{6}}}\sigma.$$

Estimating as before (5.44) follows (with ε_0 maybe smaller).

Now for $z \in \operatorname{spt}\mu \cap B^e_\rho(\xi) = \bigcup_{l=1}^M \operatorname{graph} u^l_\xi \cap B^e_\rho(\xi)$ and $\sigma > 0$ such that $B^e_\sigma(z) \subset B^e_\rho(\xi)$, it follows from (5.42), (5.43) and (5.44) that

$$\frac{\mu\llcorner B^e_\rho(\xi)\,(B^e_\sigma(z))}{\sum_{l=1}^M \mathcal{H}^2_g \llcorner \Big(\operatorname{graph} u^l_\xi \cap B^e_\rho(\xi)\Big)\,(B^e_\sigma(z))} = 1 + \frac{\theta\,(B^e_\sigma(z))}{\sum_{l=1}^M \mathcal{H}^2_g \llcorner \Big(\operatorname{graph} u^l_\xi \cap B^e_\rho(\xi)\Big)\,(B^e_\sigma(z))}.$$

Since the right hand side converges to 1 this shows that $D_{\left(\sum_{l=1}^M \mathcal{H}^2_g \llcorner (\operatorname{graph} u^l_\xi \cap B^e_\rho(\xi))\right)} \left(\mu \llcorner B^e_\rho(\xi)\right)(z) = 1$ for all $z \in \operatorname{spt}\mu \cap B^e_\rho(\xi) = \bigcup_{l=1}^M \operatorname{graph} u^l_\xi \cap B^e_\rho(\xi)$ and the lemma follows from the Theorem of Radon-Nikodym. ■

Proposition 5.3.3. *Let (M,g) and f_k be as in i) or ii) of Lemma 5.3.1. Let μ be the candidate minimizer limit measure given by Theorem 4.2.6 or Theorem 4.1.13. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the following is true: for every ε-good point ξ_0 there exist*
- $\rho_0 = \rho_0(\xi_0, \varepsilon_0)$,
- 2-dimensional subspaces $L_l \subset T_{\xi_0}M, l = 1, \ldots, M_{\xi_0}$ (by estimates in Lemma 5.2.7, Lemma 5.2.4 and the above discussion $M_{\xi_0} \leq c\beta$ is uniformly bounded with respect to ξ_0) and
- functions $u^l_{\xi_0} : L_l \cap B^e_{\rho_0}(\xi_0) \to L_l^\perp$, with $u^l_{\xi_0} \in C^{1,\alpha}(L_l \cap B^e_{\rho_0}(\xi_0)) \cap W^{2,2}(L_l \cap B^e_{\rho_0}(\xi_0))$ which satisfy the following power decay

$$\int_{B^e_\sigma(x)\cap L_l} |D^2 u^l_{\xi_0}|^2 \leq C\sigma^\alpha \tag{5.45}$$

for all $x \in B^e_{\rho_0}(\xi_0) \cap L_l$ and all $\sigma > 0$ sufficiently small,

such that for all $\rho \leq \rho_0$ the following equation holds

$$\mu \llcorner B^e_\rho(\xi_0) = \sum_{l=1}^{M_{\xi_0}} \mathcal{H}^2_g \llcorner \big(\operatorname{graph} u^l_{\xi_0} \cap B^e_\rho(\xi_0)\big)\,.$$

Proof. In Lemma 5.3.2 we showed that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the following is true: for every ε-good point ξ_0 there exist $\rho_0 = \rho_0(\xi_0, \varepsilon_0)$, 2-dimensional subspaces $L_l \subset T_{\xi_0}M, l = 1, \ldots, M_{\xi_0}$ and functions $u^l_{\xi_0} : L_l \cap B^e_{\rho_0}(\xi_0) \to L_l^\perp$, with $u^l_{\xi_0} \in C^{0,1}(L_l \cap B^e_{\rho_0}(\xi_0))$ such that for all $\rho \leq \rho_0$, $\mu \llcorner B^e_\rho(\xi_0) = \sum_{l=1}^{M_{\xi_0}} \mathcal{H}^2_g \llcorner \Big(\operatorname{graph} u^l_{\xi_0} \cap B^e_\rho(\xi_0)\Big)$.

Here we have to prove that the functions $u^l_{\xi_0}$ are actually $C^{1,\alpha} \cap W^{2,2}$ regular. Observe that, by the uniform bounds on the area and on the Willmore functional of the immersions f_k in metric g, using Proposition 5.2.3 it is easy to see that for ρ_0 maybe smaller we have $\mu^e_k(B^e_{\rho_0}(\xi_0)) \leq C$ and $\int_{B^e_{\rho_0}(\xi_0)} |H^e_k|^2 d\mu^e_k \leq C$.

Called $V^e_k \llcorner B^e_{\rho_0}(\xi_0)$ the varifold associated to $f_k(\mathbb{S}^2) \cap B^e_{\rho_0}(\xi_0)$, where $B^e_{\rho_0}(\xi_0)$ is endowed with euclidean metric, with a Schwartz inequality we get the uniform bound on the first variation $|\delta V^e_k| :=$

$\int_{B^e_{\rho_0}(\xi_0)} |H^e_k| d\mu^e_k \leq C$. By Allard Compactness Theorem there exists a 2-varifold V^e such that, up to subsequences $V^e_k \to V^e$ in varifold sense; moreover the varifold V^e has finite first variation and, applying the theory of the measure-function pairs of Hutchinson (see Theorem 4.4.2 in [Hu1]) to the measure-function pairs (V^e_k, H^e_k) (for more explanations see Appendix 6.6) we get that the limit varifold V^e has weak mean curvature H^e. Moreover, called μ^e its spatial measure, repeating the proof of Lemma 5.3.2 by replacing everywhere the Hausdorff measure $\mathcal{H}^2_g$ of the manifold with the euclidean Hausdorff measure $\mathcal{H}^2_e$ we obtain

$$\mu^e \llcorner B^e_\rho(\xi_0) = \sum_{l=1}^{M_{\xi_0}} \mathcal{H}^2_e \llcorner \big(\operatorname{graph} u^l_{\xi_0} \cap B^e_\rho(\xi_0)\big)$$

for all $\rho \leq \rho_0$ maybe smaller. Notice that by the lower semicontinuity of the Willmore functional under varifold convergence we have

$$\int_{B^e_\rho(\xi_0)} |H^e|^2 d\mu^e \leq \liminf_k \int_{B^e_\rho(\xi_0)} |H^e_k|^2 d\mu^e_k \leq 2 \liminf_k \int_{B^e_\rho(\xi_0)} |A^e_k|^2 d\mu^e_k \leq c\rho^\alpha$$

for $0 < \rho \leq \rho_0$, ρ_0 maybe smaller, where we used the inequality $|H^e_k|^2 \leq 2|A^e_k|^2$ and Lemma 5.3.1. Observe that for every $\xi \in B^e_{\rho_0}(\xi_0)$, ρ_0 maybe smaller, choosing ρ_ξ in a way that $B^e_{\rho_\xi}(\xi) \subset B^e_{\rho_0}(\xi_0)$ and repeating the arguments above just replacing ξ_0 with ξ and ρ_0 with ρ_ξ it is easy to check that the following conditions are satisfied:

- there exist 2-dimensional affine subspaces $L^\xi_l \subset T_{\xi_0}M, l = 1, \ldots, M_{\xi_0}$ passing through ξ
- there exist functions $u^l_\xi : L^\xi_l \cap B^e_{\rho_\xi}(\xi) \to L^{\xi\perp}_l$, with $u^l_\xi \in C^{0,1}(L_l \cap B^e_{\rho_\xi}(\xi))$
- $\mu^e \llcorner B^e_{\rho_\xi}(\xi) = \sum_{l=1}^{M_{\xi_0}} \mathcal{H}^2_e \llcorner \Big(\operatorname{graph} u^l_\xi \cap B^e_{\rho_\xi}(\xi)\Big)$
- the radius power decay of the Willmore functional $\int_{B^e_\rho(\xi)} |H^e|^2 d\mu^e \leq c\rho^\alpha$ for all $\rho \leq \rho_\xi$.

It is not difficult to check that the graph functions u^l_ξ are weak solutions to the following equation:

$$\sum_{i,j=1}^{2} \partial_j \left(\sqrt{\det g_l}\; g^{ij}_l \partial_i F_l\right) = \sqrt{\det g_l}\; H^e \circ F \quad \text{in } B_{\rho_\xi}(\xi) \cap L_l$$

where $F(x) = x + u^l_\xi(x)$ and $g_{ij} = \delta_{ij} + \partial_i u^l_\xi \cdot \partial_j u^l_\xi$; this follows directly from the definition of the weak mean curvature vector and the graph representation of Lemma 5.3.2.

Using the bounds on Du^l_ξ and the power decay of the Willmore functional above one gets for all $\sigma \leq \rho_\xi$ sufficiently small that (choosing $\varepsilon \leq \varepsilon_0$, ε_0 small enough)

$$\int_{B^e_{\frac{\sigma}{2}}(\xi) \cap L^\xi_l} |D^2 u^l_\xi|^2 \leq c \int_{B^e_\sigma(\xi) \setminus B^e_{\frac{\sigma}{2}}(\xi) \cap L^\xi_l} |D^2 u^l_\xi|^2 + c\sigma^\alpha.$$

Now again by "hole-filling" we get $\int_{B^e_{\frac{\sigma}{2}}(\xi) \cap L^\xi_l} |D^2 u^l_\xi|^2 \leq \theta \int_{B^e_\sigma(\xi) \cap L^\xi_l} |D^2 u^l_\xi|^2 + c\sigma^\alpha$ for some $\theta \in (0,1)$, for all $\xi \in B^e_{\rho_0}(\xi_0) \cap \Sigma$ and all $\sigma \leq \rho_\xi$ sufficiently small.
Applying Lemma 5.5.3 we obtain $\int_{B^e_\sigma(\xi) \cap L^\xi_l} |D^2 u^l_\xi|^2 \leq c\sigma^\alpha$. It is important to observe that c and α do not depend on $\xi \in B^e_{\rho_0}(\xi_0)$. Finally, with a Schwartz inequality we get

$$\int_{B^e_\sigma(\xi) \cap L^\xi_l} |D^2 u^l_\xi| \leq c\sigma \left(\int_{B^e_\sigma(\xi) \cap L^\xi_l} |D^2 u^l_\xi|^2 \right)^{\frac{1}{2}} \leq c\sigma^{1+\alpha}$$

for all $\xi \in B^e_{\rho_0}(\xi_0) \cap \operatorname{spt} \mu$ and all $\sigma \leq \rho_\xi$ sufficiently small.

By the bound $\|Du^l_\xi\|_{L^\infty(B^e_{\rho_\xi}(\xi))} \leq c\varepsilon^{\frac{1}{6}}$, varying ξ we observe that the planes L^ξ_l are obtained from L^l by rotations of order $\varepsilon^{\frac{1}{6}}$ and translations. Since $\bigcup_{l=1}^{M_{\xi_0}} \operatorname{graph} u^l_{\xi_0} \cap B^e_{\rho_\xi}(\xi) = \bigcup_{l=1}^{M_{\xi_0}} \operatorname{graph} u^l_\xi \cap B^e_{\rho_\xi}(\xi)$ we get that $|D^2 u^l_{\xi_0}|^2 \leq c|D^2 u^l_\xi|^2 + c$ for ε_0 small enough; moreover, taken $x \in L^l \cap B^e_{\rho_0}(\xi_0)$ and called ξ its projection on $\operatorname{graph} u^l_{\xi_0}$, it is easy to see that the projection of $B^e_\sigma(x) \cap L_l$ into L^ξ_l is contained in the ball $B^e_{c\sigma}(\xi) \cap L^\xi_l$ for some $c > 1$ uniform on x.

Hence, there exist $C > 0$ and $0 < \alpha < 1$ such that for all $x \in L^l \cap B^e_{\rho_0}(\xi_0)$

$$\int_{B^e_\sigma(x)\cap L_l} |D^2 u^l_{\xi_0}| \leq c \int_{B^e_{c\sigma}(\xi)\cap L^\xi_l} |D^2 u^l_\xi| + c\sigma^2 \leq C\sigma^{1+\alpha}.$$

Since $u^l_{\xi_0} \in W^{2,2}\left(B^e_{\rho_0}(\xi_0) \cap L_l, L_l^\perp\right)$ and therefore $Du^l_\xi \in W^{1,1}\left(B^e_{\rho_0}(\xi_0) \cap L_l, L_l^\perp\right)$, it follows from Morrey's lemma (see [GT], Theorem 7.19) that

$$Du^l_{\xi_0} \in C^{0,\alpha}\left(B^e_{\rho_0}(\xi_0) \cap L_l, L_l^\perp\right)$$

and therefore that each graph function $u^l_{\xi_0} \in C^{1,\alpha}\left(B^e_{\rho_0}(\xi_0) \cap L_l, L_l^\perp\right)$. This finally shows that for each ε-good point ξ_0 there exists a neighborhood of ξ_0 in which our limit varifold is given as a union of $C^{1,\alpha} \cap W^{2,2}$-graphs $u^l_{\xi_0} : B^e_{\rho_0}(\xi_0) \cap L_l \to L_l^\perp$ with small gradient bounded by $c\varepsilon^{\frac{1}{6}}$ and such that

$$\int_{B^e_\sigma(x)\cap L_l} |D^2 u^l_{\xi_0}|^2 \leq C\sigma^\alpha$$

for all $x \in B^e_{\rho_0}(\xi_0) \cap L_l$ and all $\sigma > 0$ sufficiently small. ■

5.3.2 Non existence of bad points

Let us start with a technical but useful Lemma.

Lemma 5.3.4. *Let (M, g), $\{f_k\}_{k\in\mathbb{N}}$ and μ be as in i) or ii) of Lemma 5.3.1. Fix an arbitrary $\xi \in \operatorname{spt}\mu$ (good or bad point) and consider normal coordinates centered in ξ on a neighborhood $U \subset M$. Given $x \in U$ take $p \in f_k^{-1}(\{x\})$ a preimage and consider the tangent space $T_p f_k$; we denote by $(T_p f_k)^{\perp_e}$ the orthogonal complement in the normal coordinates and with $\perp_e$ the projection on $(T_p f_k)^{\perp_e}$.*

Then for every $\epsilon > 0$ there exists $\rho_0 > 0$ small enough such that, up to subsequences in $\{f_k\}$,

$$\frac{|(x-\xi)^{\perp_e}|_e}{|x-\xi|_e} \leq \epsilon \quad \textit{for all } x \in \left(f_k(\mathbb{S}^2) \cap B^e_\rho(\xi) \setminus B^e_{\frac{\rho}{2}}(\xi)\right) \setminus \mathcal{B}_k \tag{5.46}$$

where $\mathcal{B}_k \subset f_k(\mathbb{S}^2) \cap B^e_{\rho_0}(\xi_j)$ with $\mathcal{H}^2_e\left(f_k(\mathbb{S}^2) \cap B^e_\rho(\xi) \setminus B^e_{\frac{\rho}{2}}(\xi) \cap \mathcal{B}_k\right) \leq c\epsilon\rho^2$.

Proof. First of all, as before, by Nash Theorem we can assume that $M \subset \mathbb{R}^S$ is isometrically embedded for some p; therefore the sequence $\{f_k\}_{k\in\mathbb{N}}$ can be seen also as a sequence of immersions in $\mathbb{R}^S$. Recall the uniform area bounds given by Proposition 4.1.1, Proposition 4.1.2 and Remark 4.1.3, and the uniform bound on the Willmore functionals $W(f_k)$; then the proof of Lemma 4.1.10 gives (4.6) namely $\int |H_{\mathbb{S}^2\hookrightarrow\mathbb{R}^S}|^2 d\mathcal{H}^2_{\mathbb{R}^S} \leq C$.

From (3.32) in [SiL] (for more details see also (3.54) in [Schy]) there exists ρ_0 such that for $\rho < \frac{\rho_0}{4}$, up to subsequences in $\{f_k\}_{k\in\mathbb{N}}$, for k large enough (the subsequence does not depend on ρ while the largeness of k does)

$$\frac{|(x-\xi)^{\perp_{\mathbb{R}^S}}|_{\mathbb{R}^S}}{|x-\xi|_{\mathbb{R}^S}} \leq \frac{\varepsilon}{2} \quad \text{on } \left(f_k(\mathbb{S}^2) \cap B^{\mathbb{R}^S}_{2\rho}(\xi) \setminus B^{\mathbb{R}^S}_{\frac{\rho}{4}}(\xi)\right) \setminus \mathcal{B}_k$$

where $\mathcal{B}_k \subset f_k(\mathbb{S}^2) \cap B^{\mathbb{R}^S}_{\rho_0/2}(\xi)$ with $\mathcal{H}^2{}_{\mathbb{R}^S}\left(f_k(\mathbb{S}^2) \cap B^{\mathbb{R}^S}_{2\rho}(\xi) \setminus B^{\mathbb{R}^S}_{\frac{\rho}{4}}(\xi) \cap \mathcal{B}_k\right) \leq c\varepsilon\rho^2$. Now it is easy to see that

$$\frac{|(x-\xi)^{\perp_e}|_e}{|x-\xi|_e} \leq \frac{|(x-\xi)^{\perp_{\mathbb{R}^S}}|_{\mathbb{R}^S}}{|x-\xi|_{\mathbb{R}^S}} + remainder(\rho)$$

where $remainder(\rho) \downarrow 0$ as $\rho \downarrow 0$. Therefore, choosing ρ_0 small enough such that for $\rho < \rho_0$ we have $remainder(\rho) < \varepsilon/2$, $M \cap (B^e_\rho(\xi) \setminus B^e_{\frac{\rho}{2}}(\xi)) \subseteq M \cap (B^{\mathbb{R}^S}_{2\rho}(\xi) \setminus B^{\mathbb{R}^S}_{\frac{\rho}{4}}(\xi))$ and $B^{\mathbb{R}^S}_{\rho_0/2}(\xi) \cap M \subseteq B^e_{\rho_0}(\xi) \cap M$, we obtain for such ρ

$$\frac{|(x-\xi)^{\perp_e}|_e}{|x-\xi|_e} \leq \epsilon \quad \text{for all } x \in \left(f_k(\mathbb{S}^2) \cap B^e_\rho(\xi) \setminus B^e_{\frac{\rho}{2}}(\xi)\right) \setminus \mathcal{B}_k$$

where $\mathcal{B}_k \subset f_k(\mathbb{S}^2) \cap B^e_{\rho_0}(\xi)$ with $\mathcal{H}^2{}_e\left(f_k(\mathbb{S}^2) \cap B^e_\rho(\xi) \setminus B^e_{\frac{\rho}{2}}(\xi) \cap \mathcal{B}_k\right) \leq c\epsilon\rho^2$. $\blacksquare$

Now we will handle the bad points and prove a similar power decay as in Lemma 5.3.1, but now for balls around the bad points. Since the bad points are discrete and since we want to prove a local decay, we can assume here that there is only one bad point ξ. As in the Definition 5.2.1, there exists $\rho_0 > 0$ such that for $\rho < \rho_0$ and k sufficiently large

$$\int_{B^g_{\frac{3}{2}\rho}(\xi) \setminus B^g_{\frac{\rho}{4}}(\xi)} |A^g_k|^2 \, d\mu^g_k < \frac{\varepsilon^2}{2}.$$

Now the statements $i)$, $ii)$ and $v)$ of Proposition 5.2.3 and Lemma 4.1.10, for a maybe smaller ρ_0 we have

$$\int_{B^e_\rho(\xi) \setminus B^e_{\frac{\rho}{2}}(\xi)} |A^e_k|^2 \, d\mu^e_k \leq \varepsilon^2. \tag{5.47}$$

Now let us show that for $\rho < \rho_0$ and k sufficiently large

$$f_k(\mathbb{S}^2) \cap \partial B^e_{\frac{3}{4}\rho}(\xi) \neq \emptyset. \tag{5.48}$$

Let $\xi_k \in f_k(\mathbb{S}^2)$ be such that $\xi_k \to \xi$. Thus $f_k(\mathbb{S}^2) \cap B^e_{\frac{3}{4}\rho}(\xi) \neq \emptyset$ for k sufficiently large. Now suppose that $f_k(\mathbb{S}^2) \cap \partial B^e_{\frac{3}{4}\rho}(\xi) = \emptyset$. Since $f_k(\mathbb{S}^2)$ is connected, we get that $f_k(\mathbb{S}^2) \subset B^e_{\frac{3}{4}\rho}(\xi)$. Due to Proposition 5.2.3 it follows that

$$\operatorname{diam}_g(f_k(\mathbb{S}^2)) \approx (1 + O(\rho^2)) \operatorname{diam}_e(f_k(\mathbb{S}^2)) \leq c\rho < c\rho_0,$$

and therefore, by choosing ρ_0 smaller, we get a contradiction with the lower diameter bound given by Proposition 4.1.6 (or Proposition 4.2.3).

Let $z \in \Sigma_k \cap \partial B^e_{\frac{3}{4}\rho}(\xi)$; recalling Lemma 4.1.10, we may apply the graphical decomposition lemma to f_k, z and infinitely many k to get that there exist pairwise disjoint closed subsets $P^k_1, \ldots, P^k_{N_k} \subset \Sigma_k$ such that

$$\Sigma_k \cap \overline{B^e_{\frac{\rho}{32}}(z)} = \left(\bigcup_{l=1}^{M_k(z)} \operatorname{graph} u^l_k \cup \bigcup_n P^k_n \right) \cap \overline{B^e_{\frac{\rho}{32}}(z)},$$

where the following holds:

1. The sets P^k_n are diffeomorphic to discs and disjoint from $\operatorname{graph} u^l_k$.
2. $u^l_k \in C^\infty(\overline{\Omega^l_k}, (L^l_k)^\perp)$, where L^l_k is a 2-dim. plane and $\Omega^l_k = \left(B_{\lambda^l_k}(\pi_{L^l_k}(z)) \cap L^l_k\right) \setminus \bigcup_m d^l_{k,m}$, where $\lambda^l_k > \frac{\rho}{16}$ and where the sets $d^l_{k,m}$ are pairwise disjoint closed discs in L^l_k.
3. The following inequalities hold:

$$M_k(z) \leq c, \quad \text{where } c < \infty \text{ does not depend on } z, k \text{ and } \rho, \tag{5.49}$$

$$\sum_m \operatorname{diam} d^l_{k,m} + \sum_n \operatorname{diam} P^k_n \leq c\varepsilon^{\frac{1}{2}}\rho, \tag{5.50}$$

$$\frac{1}{\rho}\|u^l_k\|_{L^\infty(\Omega^l_k)} + \|Du^l_k\|_{L^\infty(\Omega^l_k)} \leq c\varepsilon^{\frac{1}{6}}. \tag{5.51}$$

Remark 5.3.5. *Notice that $z \in f_k(\mathbb{S}^2) \cap \partial B^e_{\frac{3}{4}\rho}(\xi)$ was arbitrary. Now cover $B^e_{(\frac{3}{4}+\frac{1}{128})\rho}(\xi) \setminus B^e_{(\frac{3}{4}-\frac{1}{128})\rho}(\xi)$ by finitely many balls $B^e_{\frac{\rho}{64}}$ with center in $\partial B^e_{\frac{3}{4}\rho}(\xi)$, where the number does not depend on ρ, namely*

$$B^e_{(\frac{3}{4}+\frac{1}{128})\rho}(\xi) \setminus B^e_{(\frac{3}{4}-\frac{1}{128})\rho}(\xi) \subset \bigcup_{i=1}^{I} B^e_{\frac{\rho}{64}}(y_i),$$

where $y_i \in \partial B^e_{\frac{3}{4}\rho}(\xi)$ and I is an universal constant. From this it follows that there exist points $\{z_k^1, \dots, z_k^{J_k}\} \subset f_k(\mathbb{S}^2) \cap \partial B^e_{\frac{3}{4}\rho}(\xi)$ with $J_k \le I$ such that

$$f_k(\mathbb{S}^2) \cap B^e_{(\frac{3}{4}+\frac{1}{128})\rho}(\xi) \setminus B^e_{(\frac{3}{4}-\frac{1}{128})\rho}(\xi) \subset \bigcup_{i=1}^{J_k} B^e_{\frac{\rho}{32}}(z_i^k). \tag{5.52}$$

Now denote by

$$\left\{\Sigma_k^p \middle| \, 1 \le p \le P_k\right\} \tag{5.53}$$

the images via f_k of the connected components of $f_k^{-1}(B^e_{(\frac{3}{4}+\frac{1}{128})\rho}(\xi) \setminus B^e_{(\frac{3}{4}-\frac{1}{128})\rho}(\xi))$. From the above inclusion, the universal bound on J_k, the graphical decomposition from above and the universal bound on $M_k(z_i^k)$ we get that

$$P_k \le c, \quad \textit{where } c \textit{ is an universal constant independent of } k \textit{ and } \rho. \tag{5.54}$$

In the next step we show that

$$\operatorname{dist}\left(\xi, L_k^l\right) \le c\varepsilon^{\frac{1}{6}}\rho \quad \text{for all } l \in \{1, \dots, M_k(z)\}. \tag{5.55}$$

To prove this notice that Proposition 5.2.5 and Proposition 5.2.3 imply

$$\mu_k^e(B^e_{\frac{\rho}{32}}(z)) \ge c\rho^2. \tag{5.56}$$

Now to prove (5.55) notice that

$$\left(\operatorname{graph} u_k^l \cap B^e_{\frac{\rho}{32}}(z)\right) \setminus \mathcal{B}_k \neq \emptyset,$$

where $\mathcal{B}_k$ was defined in Lemma 5.3.4. This follows from the graphical decomposition above, the diameter estimates for the sets P_n^k, the area estimate concerning the set $\mathcal{B}_k$ and (5.56).

Let $y \in \left(\operatorname{graph} u_k^l \cap B^e_{\frac{\rho}{32}}(z)\right) \setminus \mathcal{B}_k \subset \left(f_k(\mathbb{S}^2) \cap B^e_\rho(\xi) \setminus B^e_{\frac{\rho}{2}}(\xi)\right) \setminus \mathcal{B}_k$. It follows that

$$|\xi - \pi_{T_y f_k}(\xi)| \le \varepsilon|y-\xi| \le \varepsilon\left(|y-z| + |z-\xi|\right) \le c\varepsilon\rho.$$

Define the perturbed plane $\tilde{L}_k^l$ by $\tilde{L}_k^l = L_k^l + (y - \pi_{L_k^l}(y))$. Thus $\operatorname{dist}(\tilde{L}_k^l, L_k^l) = |y - \pi_{L_k^l}(y)| \le c\varepsilon^{\frac{1}{6}}\rho$ (since $y \in \operatorname{graph} u_k^l \cap B^e_{\frac{\rho}{32}}(z)$). Thus clearly, by Pythagoras Theorem, $|y - \pi_{\tilde{L}_k^l}(\pi_{T_y f_k}(\xi))|^2 \le |y - \pi_{T_y f_k}(\xi)|^2 \le |y-\xi|^2 \le c\rho^2$. Since $T_y f_k$ can be parametrized in terms of $Du_k^l(y)$ over $\tilde{L}_k^l$ we get that

$$|\pi_{T_y f_k}(\xi) - \pi_{\tilde{L}_k^l}(\pi_{T_y f_k}(\xi))| \le \|Du_k^l\|_{L^\infty}|y - \pi_{\tilde{L}_k^l}(\pi_{T_y f_k}(\xi))| \le c\varepsilon^{\frac{1}{6}}\rho.$$

Therefore by triangle inequality we finally get (5.55).

$$\begin{aligned}
dist(\xi, L_k^l) &= \left|\xi - \pi_{L_k^l}(\xi)\right| \\
&\le \left|\xi - \pi_{L_k^l}(\pi_{T_y\Sigma_k}(\xi))\right| \\
&\le \left|\xi - \pi_{T_y\Sigma_k}(\xi)\right| + \left|\pi_{T_y\Sigma_k}(\xi) - \pi_{\tilde{L}_k^l}(\pi_{T_y\Sigma_k}(\xi))\right| \\
&\quad + \left|\pi_{\tilde{L}_k^l}(\pi_{T_y\Sigma_k}(\xi)) - \pi_{L_k^l}(\pi_{T_y\Sigma_k}(\xi))\right| \\
&\le c\varepsilon^{\frac{1}{6}}\rho,
\end{aligned}$$

Since $\operatorname{dist}(\xi, L_k^l) \le c\varepsilon^{\frac{1}{6}}\rho$, we may assume (after translation) that $\xi \in L_k^l$ for all $l \in \{1, \dots, M_k(z)\}$ and k without changing the estimates for the functions u_k^l. Moreover we again have that $L_k^l \to L^l$ with $\xi \in L^l$. Therefore for k sufficiently large we may assume that L_k^l is a fixed 2-dim. plane L^l.

Now we have that either the point z lies in one of the graphs or can be connected to one of the graphs. Without loss of generality we may assume that this graph corresponds to the function u_k^1. Subsequently we will work only with this function u_k^1, which is defined on some part of the plane L_1 with some discs $d_{k,m}^1$ removed. We will therefore drop the index 1. Define the set

$$T_k(z) = \left\{ \tau \in \left(\frac{\rho}{64}, \frac{\rho}{\sqrt{2} \cdot 32} \right) \middle| \partial B_\tau^e(\pi_L(z)) \cap \bigcup_m d_{k,m} = \emptyset \right\}.$$

It follows from the diameter estimates and the selection principle in [SiL] that for $\varepsilon \leq \varepsilon_0$ there exists a $\tau \in \left(\frac{\rho}{64}, \frac{\rho}{\sqrt{2} \cdot 32} \right)$ such that $\tau \in T_k(z)$ for infinitely many k.

Since $\xi \in L$, it follows from the choice of τ that for $\varepsilon \leq \varepsilon_0$

$$\partial B_{\frac{3}{4}\rho}^e(\xi) \cap \partial B_\tau^e(\pi_L(z)) \cap L = \{p_{1,k}, p_{2,k}\},$$

where $p_{1,k}, p_{2,k} \in \left(B^e_{\frac{\rho}{\sqrt{2} \cdot 32}}(\pi_L(z)) \cap L \right) \setminus \bigcup_m d_{k,m}$ are distinct.

Define the image points $z_{i,k} \in \operatorname{graph} u_k$ by

$$z_{i,k} = p_{i,k} + u_k(p_{i,k}).$$

Using the L^∞-estimates for u_k we get for $\varepsilon \leq \varepsilon_0$ that $\frac{5}{8}\rho < |z_{i,k} - \xi| < \frac{7}{8}\rho$ and thus $\int_{B^e_{\frac{\rho}{8}}(z_{i,k})} |A_k^e|^2 \, d\mu_k^e \leq \varepsilon^2$. Therefore we can again apply the graphical decomposition lemma to the points $z_{i,k}$. Thus there exist pairwise disjoint subsets $P_1^{i,k}, \ldots, P_{N_{i,k}}^{i,k} \subset f_k(\mathbb{S}^2)$ such that

$$f_k(\mathbb{S}^2) \cap \overline{B^e_{\frac{\rho}{32}}(z_{i,k})} = \left(\bigcup_{l=1}^{M_{i,k}(z_{i,k})} \operatorname{graph} u_{i,k}^l \cup \bigcup_n P_n^{i,k} \right) \cap \overline{B^e_{\frac{\rho}{32}}(z_{i,k})},$$

where the usual properties and estimates holds.

1. The sets $P_n^{i,k}$ are closed topological discs disjoint from $\operatorname{graph} u_{i,k}$.
2. $u_{i,k}^l \in C^\infty(\overline{\Omega_{i,k}^l}, (L_{i,k}^l)^\perp)$, where $L_{i,k}^l$ is a 2-dim. plane and $\Omega_{i,k}^l = \left(B_{\lambda_{i,k}^l}(\pi_{L_{i,k}^l}(z_{i,k})) \cap L_{i,k}^l \right) \setminus \bigcup_m d_{i,k,m}^l$, where $\lambda_{i,k}^l > \frac{\rho}{16}$ and where the sets $d_{i,k,m}^l$ are pairwise disjoint closed discs in $L_{i,k}^l$.
3. The following inequalities hold:

$$M_{i,k}(z_{i,k}) \leq c, \quad \text{where } c < \infty \text{ does not depend on } z_{i,k}, k \text{ and } \rho, \tag{5.57}$$

$$\sum_m \operatorname{diam} d_{i,k,m}^l + \sum_n \operatorname{diam} P_n^{i,k} \leq c\varepsilon^{\frac{1}{2}} \rho, \tag{5.58}$$

$$\frac{1}{\rho} \|u_{i,k}^l\|_{L^\infty(\Omega_{i,k}^l)} + \|Du_{i,k}^l\|_{L^\infty(\Omega_{i,k}^l)} \leq c\varepsilon^{\frac{1}{6}}. \tag{5.59}$$

Now we have again that the points $z_{i,k}$ either lie in one of the graphs $u_{i,k}^l$ or can be connected to one of them. Without loss of generality let this be the graph corresponding to $u_{i,k}^1$. We will again drop the upper index. Since $z_{i,k} \in \operatorname{graph} u_k$ it follows that $\operatorname{dist}(z_{i,k}, L) \leq c\varepsilon^{\frac{1}{6}} \rho$ and that $\operatorname{graph} u_{i,k}$ is connected to $\operatorname{graph} u_k$. Since the L^∞-norms of u_k and $u_{i,k}$ and their derivatives are small, we may assume (after translation and rotation as done before) that $L_{i,k} = L$.

By continuing with this procedure we get after a finite number of steps, depending not on ρ and k, an open cover of $\partial B^e_{\frac{3}{4}\rho}(\xi) \cap L$ which also covers the set

$$A(L) = \left\{ x + y \,\middle|\, x \in L, \operatorname{dist}\left(x, \partial B^e_{\frac{3}{4}\rho}(\xi) \cap L\right) < \frac{\rho}{\sqrt{2} \cdot 64}, y \in L^\perp, |y| < \frac{\rho}{\sqrt{2} \cdot 64} \right\}.$$

Now it can happen that after one "walk-around" we do not end up in the same disc of $f_k(\mathbb{S}^2) \cap B^e_{\frac{\rho}{32}}(z)$ which contains the point z. But then we can proceed in a similar way and do another "walk-around". Now by construction, the "flatness" of the involved graph functions and the diameter bounds for the discs, every "walk-around" corresponds to a part of $f_k(\mathbb{S}^2)$ with an area that is bounded from below by $c\rho^2$, where c is a universal constant independent of k and ρ. On the other hand we have that $\mu_k^e(B_\rho^e(\xi)) \leq c\rho^2$. It follows that after a finite number of "walk-arounds" (which is bounded by a universal constant) we have to get back to the disc of $f_k(\mathbb{S}^2) \cap B^e_{\frac{\rho}{32}}(z)$ which contains the point z.

We summarize the above procedure and the resulting properties in the following remark.

Remark 5.3.6. *If $\varepsilon \leq \varepsilon_0$, for each component $\tilde{\Sigma}_k^p$ there exist pairwise disjoint subsets $P_1^{k,p}, \ldots, P_{N_{k,p}}^{k,p} \subset f_k(\mathbb{S}^2)$, a natural number $k_p \in \mathbb{N}$ and a smooth function u_k^p defined on the rectangular set*

$$B_k^p = \left[\left(\left(\frac{3}{4} - \frac{1}{\sqrt{2} \cdot 64} \right) \rho, \left(\frac{3}{4} + \frac{1}{\sqrt{2} \cdot 64} \right) \rho \right) \times [0, 2\pi k_p) \right] \setminus \bigcup d_{k,m}^p,$$

where the $d_{k,m}^p$ are closed discs in $\left(\left(\frac{3}{4} - \frac{1}{\sqrt{2} \cdot 64} \right) \rho, \left(\frac{3}{4} + \frac{1}{\sqrt{2} \cdot 64} \right) \rho \right) \times [0, 2\pi k_p)$, such that

$$\tilde{\Sigma}_k^p = \left(R_p \left(\operatorname{graph} U_k^p \right) \cup \bigcup_j P_j^{k,p} \right) \cap B^e_{\left(\frac{3}{4}+\frac{1}{128}\right)\rho}(\xi) \setminus B^e_{\left(\frac{3}{4}-\frac{1}{128}\right)\rho}(\xi),$$

where $\operatorname{graph} U_k^p = \left\{ \left(re^{i\theta}, u_k^p(r,\theta)\right) \,\middle|\, (r,\theta) \in B_k^p \right\}$ *and R_p denotes a rotation such that $R_p(\mathbb{R}^2) = L_p$, where L_p is the 2-dim. plane from before. Moreover we have*

$$\sum_m \operatorname{diam} d_{k,m}^p + \sum_j \operatorname{diam} P_j^{k,p} \leq c\varepsilon^{\frac{1}{2}} \rho, \quad \frac{1}{\rho} \|u_k^p\|_{L^\infty(B_k^p)} + \|Du_k^p\|_{L^\infty(B_k^p)} \leq c\varepsilon^{\frac{1}{6}}.$$

We may assume without loss of generality that the discs $d_{k,m}^p$ are pairwise disjoint, since otherwise we can exchange two intersecting discs by one disc whose diameter is smaller than the sum of the diameters of the intersecting discs.

Now let $\rho \leq \rho_0$ and define the set

$$C_k(\xi) = \left\{ \sigma \in \left(\left(\frac{3}{4} - \frac{1}{256} \right) \rho, \left(\frac{3}{4} + \frac{1}{256} \right) \rho \right) \,\middle|\, \partial B_\sigma^e(\xi) \cap \bigcup_{p,j} P_j^{k,p} = \emptyset, \int_{\partial B_\sigma^e(\xi)} |A_k^e|^2 \, ds_k^e \leq \frac{512}{\rho} \varepsilon^2 \right\}.$$

It follows that

$$\mathcal{L}^1(C_k(\xi)) \geq \frac{1}{512} \rho,$$

since by the diameter estimates for the "pimples" we have for $\varepsilon \leq \varepsilon_0$ that

$$\mathcal{L}^1 \left(\left\{ \sigma \in \left(\left(\frac{3}{4} - \frac{1}{256} \right) \rho, \left(\frac{3}{4} + \frac{1}{256} \right) \rho \right) \,\middle|\, \partial B_\sigma^e(\xi) \cap \bigcup_{p,j} P_j^{k,p} = \emptyset \right\} \right) \geq \frac{1}{256} \rho,$$

and therefore we would get, assuming that $\mathcal{L}^1(C_k(\xi)) < \frac{1}{512}\rho$,

$$\begin{aligned} \varepsilon^2 &\geq \int_{\Sigma_k \cap B_\rho^e(\xi) \setminus B^e_{\frac{\rho}{2}}(\xi)} |A_e|^2 \, d\mu_e \\ &\geq \int_{\left\{ \sigma \in \left(\left(\frac{3}{4} - \frac{1}{256} \right) \rho, \left(\frac{3}{4} + \frac{1}{256} \right) \rho \right) \,\middle|\, \partial B_\sigma^e(\xi) \cap \bigcup_{p,j} P_j^{k,p} = \emptyset \right\} \setminus C_k(\xi)} \int_{\Sigma_k \cap \partial B_\sigma^e(\xi)} |A_e|^2 \, ds_e \, d\sigma \\ &> \varepsilon^2. \end{aligned}$$

Again it follows from the diameter bounds, a simple Fubini argument and Lemma 5.5.2 that there exists a $\sigma \in \left(\left(\frac{3}{4} - \frac{1}{256} \right) \rho, \left(\frac{3}{4} + \frac{1}{256} \right) \rho \right)$ such that $\sigma \in C_k(\xi)$ for infinitely many $k \in \mathbb{N}$. Denote by

$$\left\{ \tilde{\Sigma}_k^q \,|\, 1 \leq q \leq Q_k \right\} \tag{5.60}$$

the images of the components of $f_k^{-1}(B_\sigma^e(\xi))$. By Remark 5.3.5, we get that Q_k is bounded by an universal constant which is independent of k and ρ.

Lemma 5.3.7. *Suppose that*

$$\frac{1}{2}\int |A_k^g|^2\, d\mu_k^g \le 4\pi - \delta$$

for some $\delta > 0$ (this holds in both our cases by Lemma 4.1.4, and Lemma 4.1.14 together with Proposition 4.2.3). Then for $\varepsilon \le \varepsilon_0$ each $\tilde{\Sigma}_k^q$ is a topological disc. Moreover we have that $k_p = 1$ for all p (for the definition of k_p see Remark 5.3.6).

Proof. Fix $k \in \mathbb{N}$. First of all we construct a new surface $\bar{\Sigma}_k$ such that, called $\bar{\mu}_k$ the associated Radon measure, we have

$$\begin{aligned}
&(i) \quad \bar{\mu}_k \llcorner B_\sigma^e(\xi) = \mu_k^e \llcorner B_\sigma^e(\xi),\\
&(ii) \quad \left| \int_{B^e_{\left(\frac{3}{4}+\frac{1}{128}\right)\rho}(\xi)\setminus B^e_\sigma(\xi)} K_G\, d\bar{\mu}_k \right| \le c\varepsilon^{\frac{1}{3}} \quad (K_G = \text{Gauss curvature of } \bar{\Sigma}_k),\\
&(iii) \quad \int_{\bar{\Sigma}_k\setminus B^e_{\left(\frac{3}{4}+\frac{1}{128}\right)\rho}(\xi)} K_G\, d\bar{\mu}_k = 0.
\end{aligned}$$

To define $\bar{\Sigma}_k$ recall Remark 5.3.6 and notice that $\sum_{p,m} \operatorname{diam} d^p_{k,m} \le c\varepsilon^{\frac{1}{2}}\rho$. Now denote by M_k the number of all discs $d^p_{k,m}$. Because of the diameter estimate it follows for $\varepsilon \le \varepsilon_0$ that there exists an interval $I_k^p \subset \left(\left(\frac{3}{4}-\frac{1}{256}\right)\rho, \left(\frac{3}{4}+\frac{1}{128}\right)\rho\right)$ with $\mathcal{L}^1(I_k^p) \ge \frac{1}{512 M_k}\rho$ such that $(I_k^p \times [0, 2\pi k_p)) \cap \bigcup_m d^p_{k,m} = \emptyset$.

Let $I_k^p = (a_k^p, b_k^p)$ and $\varphi_p \in C^\infty((0,\infty)\times[0,2\pi k_p))$ with $0 \le \varphi_p \le 1$ such that

$$\varphi_p = 1 \text{ on } (0, a_k^p)\times[0,2\pi k_p), \quad \varphi_p = 0 \text{ on } (b_k^p,\infty)\times[0,2\pi k_p), \quad |D\varphi_p| \le \frac{c}{\rho} \text{ and } |D^2\varphi_p| \le \frac{c}{\rho^2}.$$

Now define new "components" $\bar{\Sigma}_k^p$ by

$$\bar{\Sigma}_k^p = \left(R_p\left(\operatorname{graph} \bar{U}_k^p\right) \cup \bigcup_j P_j^{k,p} \right) \cap B^e_{\left(\frac{3}{4}+\frac{1}{128}\right)\rho}(\xi) \setminus B^e_{\left(\frac{3}{4}-\frac{1}{128}\right)\rho}(\xi),$$

where $\operatorname{graph} \bar{U}_k^p$ is given by

$$\operatorname{graph} \bar{U}_k^p = \left\{ \left(re^{i\theta}, \varphi_p(r,\theta) u_k^p(r,\theta)\right) \,\middle|\, (r,\theta) \in B_k^p \right\},$$

and R_p denotes a rotation such that $R_p(\mathbb{R}^2) = L_p$, where L_p is the 2-dim. plane from before. Namely we just "flattened out" the components Σ_k^p.

Now define the new surface $\bar{\Sigma}_k$ by

$$\bar{\Sigma}_k = \left(\left(f_k(\mathbb{S}^2) \cap B^e_{\left(\frac{3}{4}+\frac{1}{128}\right)\rho}(\xi) \right) \setminus \bigcup_p \Sigma_k^p \right) \cup \bigcup_p \left(\bar{\Sigma}_k^p \cup L_p \right).$$

By definition i) follows immediately. Since $\bar{\Sigma}_k \setminus B^e_{\left(\frac{3}{4}+\frac{1}{128}\right)\rho}(\xi) = \bigcup_p L_p \setminus B^e_{\left(\frac{3}{4}+\frac{1}{128}\right)\rho}(\xi)$, we moreover have $\int_{\bar{\Sigma}_k\setminus B^e_{\left(\frac{3}{4}+\frac{1}{128}\right)\rho}(\xi)} K_G\, d\bar{\mu}_k = 0$. To prove property (ii) above notice that

$$\int_{\bar{\Sigma}_k\cap B^e_{\left(\frac{3}{4}+\frac{1}{128}\right)\rho}(\xi)\setminus B^e_\sigma(\xi)} |K_G|\, d\bar{\mu}_k \quad \le \quad \int_{f_k(\mathbb{S}^2)\cap B^e_\rho(\xi)\setminus B^e_{\frac{\rho}{2}}(\xi)} |K_G|\, d\mu_k^e + \sum_p \int_{R_p\left(\operatorname{graph} \bar{U}_k^p\right)} |K_G|\, d\bar{\mu}_k.$$

Now the first integral on the right hand side can be estimated by

$$\int_{f_k(\mathbb{S}^2)\cap B^e_\rho(\xi)\setminus B^e_{\frac{\rho}{2}}(\xi)} |K_G|\, d\mu_k^e \le \frac{1}{2}\int_{B^e_\rho(\xi)\setminus B^e_{\frac{\rho}{2}}(\xi)} |A_k^e|^2\, d\mu_k^e \le \varepsilon^2.$$

The second integral can be estimated by

$$\int_{R_p\left(\operatorname{graph} \bar{U}_k^p\right)} |K_G|\, d\bar{\mu}_k \le \frac{1}{2}\int_{\operatorname{graph} \bar{U}_k^p} |A_e|^2\, d\bar{\mu}_k \le c\int_{B_k^p} |D^2(\varphi_p u_k^p)|^2.$$

Because of the properties of the functions u_k^p and φ_p we have

$$|D^2(\varphi_p u_k^p)|^2 \le c\left(|u_k^p|^2|D^2\varphi_p|^2 + |Du_k^p|^2|D\varphi_p|^2 + |\varphi_p|^2|D^2u_k^p|^2\right) \le c\frac{\varepsilon^{\frac{1}{3}}}{\rho^2} + |D^2u_k^p|^2$$

and therefore we get

$$\int_{B_k^p}|D^2(\varphi_p u_k^p)|^2 \le c\varepsilon^{\frac{1}{3}} + c\int_{\operatorname{graph} U_k^p}|A_k^e|^2\,d\mu_k^e \le c\varepsilon^{\frac{1}{3}} + c\int_{B_\rho^e(\xi)\setminus B_{\frac{\rho}{2}}^e(\xi)}|A_k^e|^2\,d\mu_k^e \le c\varepsilon^{\frac{1}{3}}.$$

Thus the desired property (ii) is shown (here we also use that $1 \le p \le P_k \le c$) and the construction of $\bar{\Sigma}_k$ is concluded.

Now denote by $N : \bar{\Sigma}_k \to \mathbb{S}^2$ the Gauss-map. Since the degree of the Gauss-map $\deg(N)$ is half the Euler characteristic, it follows from Gauss-Bonnet that

$$\deg(N) = \frac{1}{4\pi}\int_{\bar{\Sigma}_k} K_G\,d\bar{\mu}_k = \frac{1}{4\pi}\int_{\bar{\Sigma}_k\cap B^e_{\left(\frac{3}{4}+\frac{1}{128}\right)\rho}(\xi)\setminus B_\sigma^e(\xi)} K_G\,d\bar{\mu}_k + \frac{1}{4\pi}\int_{\bar{\Sigma}_k\cap B_\sigma^e(\xi)} K_G\,d\bar{\mu}_k,$$

and therefore we get using (ii) above that

$$\left|\int_{\bar{\Sigma}_k\cap B_\sigma^e(\xi)} K_G\,d\bar{\mu}_k - 4\pi\deg(N)\right| \le c\varepsilon^{\frac{1}{3}}.$$

On the other hand it follows from the assumptions and Proposition 5.2.3 that

$$\left|\int_{\bar{\Sigma}_k\cap B_\sigma^e(\xi)} K_G\,d\bar{\mu}_k\right| = \left|\int_{f_k(\mathbb{S}^2)\cap B_\sigma^e(\xi)} K_G\,d\mu_k^e\right| \le \frac{1}{2}\int_{B_\sigma^e(\xi)}|A_k^e|^2\,d\mu_k^e \le 4\pi - \frac{\delta}{2}$$

by choosing ρ_0 smaller if necessary (remember: $\rho \le \rho_0$). Since $\deg(N) \in \mathbb{Z}$ it follows for $\varepsilon \le \varepsilon_0$ that

$$\left|\int_{f_k(\mathbb{S}^2)\cap B_\sigma^e(\xi)} K_G\,d\mu_k^e\right| = \left|\int_{\bar{\Sigma}_k\cap B_\sigma^e(\xi)} K_G\,d\bar{\mu}_k\right| \le c\varepsilon^{\frac{1}{3}}. \tag{5.61}$$

Now by the choice of σ we have for all $p = 1,\dots,P_k$ that

$$\Sigma_k^p \cap \partial B_\sigma^e(\xi) = \gamma_p,$$

where each γ_p is a closed, immersed smooth curve and where P_k is bounded by a universal constant. By construction and the choice of σ we have that $\gamma_p \cap \bigcup_j P_j^{k,p} = \emptyset$, therefore (see the almost graph representation of Σ_k^p above) γ_p is almost a flat circle of radius σ which can be parametrized on the interval $[0, 2\pi k_p)$. After some computations it follows from the choice of σ that (called κ the geodesic curvature)

$$\left|\int_{\gamma_p}\kappa\,ds_k^e - 2\pi k_p\right| \le c\varepsilon^{\frac{1}{6}} + c\int_{\gamma_p}|A_k^e|\,ds_k^e \le c\varepsilon^{\frac{1}{6}} + c\sigma^{\frac{1}{2}}\left(\int_{\partial B_\sigma^e(\xi)}|A_k^e|^2\,ds_k^e\right)^{\frac{1}{2}} \le c\varepsilon^{\frac{1}{6}} + c\left(\frac{\sigma}{\rho}\right)^{\frac{1}{2}}\varepsilon \le c\varepsilon^{\frac{1}{6}},$$

and therefore it follows from the bound on P_k that

$$\left|\int_{\partial B_\sigma^e(\xi)}\kappa\,ds_k^e - 2\pi\sum_{p=1}^{P_k}k_p\right| \le c\varepsilon^{\frac{1}{6}}. \tag{5.62}$$

On the other hand we have that $f_k(\mathbb{S}^2)\cap B_\sigma^e(\xi) = \bigcup_{q=1}^{Q_k}\tilde{\Sigma}_k^q$. Now the Euler characteristic of the components is given by

$$\chi(\tilde{\Sigma}_k^q) = 2(1-g_q) - b_q,$$

where b_q is the number of boundary components of $\tilde{\Sigma}_k^q$ and g_q is the genus of the closed surface which arises by gluing b_q topological discs. Especially we have that

$$b_q \ge 1 \quad \text{and} \quad \sum_{q=1}^{Q_k} b_q = P_k.$$

By summing over q we get that the Euler characteristic of $\bigcup_{q=1}^{Q_k}\tilde{\Sigma}_k^q$ is

$$\chi_E\left(\bigcup_{q=1}^{Q_k}\tilde{\Sigma}_k^q\right) = 2(Q_k-g)-P_k, \quad \text{with } g=\sum_{q=1}^{Q_k} g_q \geq 0.$$

Since $Q_k \leq P_k$ we finally get that

$$P_k \geq 2(Q_k-g)-P_k = \frac{1}{2\pi}\int_{f_k(\mathbb{S}^2)\cap B_\sigma^e(\xi)} K_G\, d\mu_k^e + \frac{1}{2\pi}\int_{f_k(\mathbb{S}^2)\cap\partial B_\sigma^e(\xi)} \kappa\, ds_k^e \geq \sum_{p=1}^{P_k} k_p - c\varepsilon^{\frac{1}{6}} \geq P_k - c\varepsilon^{\frac{1}{6}}.$$

Since $2(Q_k-g)-P_k \in \mathbb{N}$ it follows for $\varepsilon \leq \varepsilon_0$ that $P_k = 2(Q_k-g)-P_k$ and therefore (since $Q_k \leq P_k$) that $Q_k = P_k$ and $g=0$. It follows that $g_q = 0$ and $b_q = 1$ for all q. This yields that the Euler characteristic of $\tilde{\Sigma}_k^q$ is 1 for all q and therefore each $\tilde{\Sigma}_k^q$ is a topological disc. Moreover the estimate above yields that $k_p = 1$ for all p. ■

Remark 5.3.8. Notice that it follows from the last Lemma and Remark 5.3.6 that for each component $\tilde{\Sigma}_k^p$ of $\Sigma_k \cap B^e_{(\frac{3}{4}+\frac{1}{128})\rho}(\xi)\setminus B^e_{(\frac{3}{4}-\frac{1}{128})\rho}(\xi)$ there exist pairwise disjoint subsets $P_1^{k,p},\dots,P_{N_{k,p}}^{k,p}\subset\Sigma_k$ and a smooth function u_k^p defined on the rectangular set

$$B_k^p = \left[\left(\left(\frac{3}{4}-\frac{1}{\sqrt{2}\cdot 64}\right)\rho, \left(\frac{3}{4}+\frac{1}{\sqrt{2}\cdot 64}\right)\rho\right)\times[0,2\pi)\right]\setminus\bigcup d_{k,m}^p,$$

where the $d_{k,m}^p$ are pairwise disjoint, closed discs in $\left(\left(\frac{3}{4}-\frac{1}{\sqrt{2}\cdot 64}\right)\rho, \left(\frac{3}{4}+\frac{1}{\sqrt{2}\cdot 64}\right)\rho\right)\times[0,2\pi)$, such that

$$\tilde{\Sigma}_k^p = \left(R_p\left(\operatorname{graph} U_k^p\right)\cup\bigcup_j P_j^{k,p}\right)\cap B^e_{(\frac{3}{4}+\frac{1}{128})\rho}(\xi)\setminus B^e_{(\frac{3}{4}-\frac{1}{128})\rho}(\xi),$$

where $\operatorname{graph} U_k^p$ is given by

$$\operatorname{graph} U_k^p = \left\{\left(re^{i\theta}, u_k^p(r,\theta)\right)\,\middle|\,(r,\theta)\in B_k^p\right\}$$

and R_p denotes a rotation such that $R_p(\mathbb{R}^2) = L_p$, where L_p is the 2-dim. plane. Moreover we have

$$\sum_m \operatorname{diam} d_{k,m}^p + \sum_j \operatorname{diam} P_j^{k,p} \leq c\varepsilon^{\frac{1}{2}}\rho,$$

$$\frac{1}{\rho}\|u_k^p\|_{L^\infty(B_k^p)} + \|Du_k^p\|_{L^\infty(B_k^p)} \leq c\varepsilon^{\frac{1}{6}}.$$

Moreover it follows from the last Lemma that the number of components of $\Sigma_k\cap B_{(\frac{3}{4}+\frac{1}{128})\rho}(\xi)$ equals the number of components of $\Sigma_k\cap B^e_{(\frac{3}{4}+\frac{1}{128})\rho}(\xi)\setminus B^e_{(\frac{3}{4}-\frac{1}{128})\rho}(\xi)$. ■

Define the set

$$C_k^p = \left\{s\in\left(0,\frac{\rho}{128}\right)\,\middle|\,\left(\left(\frac{3}{4}\rho+s\right)\times[0,2\pi)\right)\cap\bigcup_m d_{k,m}^p = \emptyset\right\}.$$

By the diameter estimates for the discs $d_{k,m}^p$ and Lemma 5.5.2 there exists a $s\in\left(0,\frac{\rho}{128}\right)$ such that $s\in C_k^p$ for infinitely many k.

Now define the set

$$D_k^p = \left\{s\in C^p\,\middle|\,\int_{R_p\left(\operatorname{graph} U^p_{k|\left(\left(\frac{3}{4}\rho+s\right)\times[0,2\pi)\right)}\right)} |A_k^e|^2\, ds_k^e \leq \frac{512}{\rho}\int_{\Sigma_k^p}|A_k^e|^2\, d\mu_k^e\right\}.$$

As before there exists $s \in \left(0, \frac{\rho}{128}\right)$ such that $s \in D_k^p$ for infinitely many k and u_k^p is defined on the line $\left(\frac{3}{4}\rho + s\right) \times [0, 2\pi)$. Now it follows from the last Remark that $R_p(\operatorname{graph} U_k^p|_{\left(\left(\frac{3}{4}\rho+s\right)\times[0,2\pi)\right)})$ divides $f_k(\mathbb{S}^2)$ into two topological discs $\Sigma_1^{k,p}, \Sigma_2^{k,p}$, one of them, w.l.o.g. $\Sigma_1^{k,p}$, intersecting $B^e_{\frac{3}{4}\rho}(\xi)$.
From the estimates for the function u_k^p and the choice of s we get that $\Sigma_1^{k,p} \subset B^e_{\left(\frac{3}{4}+\frac{1}{128}\right)\rho}(\xi)$ and Lemma 4.1.10 yields $\mu_k^e(\Sigma_1^{k,p}) \leq c\rho^2$.

According to the Lemma 5.5.1, let $w_k^p \in C^\infty\left(B^e_{\frac{3}{4}\rho+s}(\xi) \cap L_p, L_p^\perp\right)$ be an extension of $R_p(U_k^p)$ restricted to $\partial B^e_{\frac{3}{4}\rho+s}(\xi) \cap L_p$. In view of the estimates for u_k^p and therefore for w_k^p we get that $\operatorname{graph} w_k^p \subset B^e_{\left(\frac{3}{4}+\frac{1}{128}\right)\rho}(\xi)$.

Now, we can define the surface $\tilde{\Sigma}_k$ by

$$\tilde{\Sigma}_k = \left(f_k(\mathbb{S}^2) \setminus \bigcup_p \Sigma_1^{k,p}\right) \cup \bigcup_p \operatorname{graph} w_k^p$$

and we can do exactly the same as in the proof of Lemma 5.3.1 to get the same power decay as for the good points, but now for balls around the bad points. But by definition the bad points do not allow a decay like this, and therefore we have proved that there are no bad points.

5.4 C^∞ regularity of the minimizer

At this point we proved that the candidate minimizer limit measure μ is locally the measure associated to $C^{1,\alpha} \cap W^{2,2}$ graphs. To prove higher regularity we have to show that such graphs satisfy the Euler Lagrange equation of the functional. Since we started from a minimizing sequence among immersions of 2-spheres, we first have to show that μ is the measure associated to an immersion of a 2-sphere, then we will prove that the graphs satisfy the equation and finally, using a PDE Lemma of Simon and a bootstrap argument, we conclude with the smoothness of μ.

5.4.1 $C^{1,\alpha} \cap W^{2,2}$ parametrization on $\mathbb{S}^2$ of the limit measure μ

Up to now we have shown that the limit measure μ is everywhere the sum of the Radon measures associated to the graphs of some $C^{1,\alpha} \cap W^{2,2}$ functions; now we want to show that there exists a $C^{1,\alpha} \cap W^{2,2}$ immersion $f : \mathbb{S}^2 \hookrightarrow M$ such that μ is the Radon measure on M associated to f. In order prove this, let us show that we can modify the immersions $\{f_k\}_{k\in\mathbb{N}}$ of the minimizing sequence such that the new immersions $\tilde{f}_k$ are generalized (r,λ)-immersions for some $\lambda < \frac{1}{4}$ and $r > 0$ (see Definition 5.5.5) and the associated measures $\tilde{\mu}_k$ converge to μ weakly as measures.

Lemma 5.4.1. *Let f_k and μ as before. Then it is possible to modify the smooth immersions $\{f_k : \mathbb{S}^2 \hookrightarrow M\}_{k\in\mathbb{N}}$ into the new $C^{1,1}$-immersions $\{\tilde{f}_k : \mathbb{S}^2 \hookrightarrow M\}_{k\in\mathbb{N}}$ which are generalized (r,λ)-immersions in the sense of Definition 5.5.5 with $\lambda < \frac{1}{4}$, some $r > 0$ and $|\tilde{f}_k(\mathbb{S}^2)| \leq C$, namely $\{\tilde{f}_k\}_{k\in\mathbb{N}} \subset \mathcal{F}^1_C(r,\lambda)$.*

Proof. Recall that both μ and the f_k's are locally representable as union of graphs and pimples in the following way: for every $\xi \in \operatorname{spt}\mu$ there exists $r_\xi > 0$ and there exists K_ξ such that

i) for $k \geq K_\xi$ we have $\mu_k \llcorner B^e_{r_\xi}(\xi) = \sum_{l=1}^{M_\xi} \mathcal{H}^2_g \llcorner \Big(\operatorname{graph} u_l^k \cup \bigcup_i P_i^k \cap B^e_{r_\xi}(\xi)\Big)$ where u_l^k are smooth functions defined on appropriate planes L_l with the usual properties and estimates (see Lemma 5.2.7),

ii) $\mu \llcorner B^e_{r_\xi}(\xi) = \sum_{l=1}^{M_\xi} \mathcal{H}^2_g \llcorner \Big(\operatorname{graph} u_l \cap B^e_{r_\xi}(\xi)\Big)$, where u_l are $C^{1,\alpha} \cap W^{2,2}$ functions defined on the planes L_l (see Lemma 5.3.2).

For $\xi \in \operatorname{spt}\mu$ denote $\rho_\xi := \sup\{r_\xi$ which satisfies i) and ii) $\}$.

Claim. *We have that $\rho := \inf\{\rho_\xi : \xi \in \operatorname{spt}\mu\} > 0$.*

Proof. If by contradiction $\rho = 0$ then there exists a sequence $\{\xi_i\}_{i\in\mathbb{N}}$ of points in $\operatorname{spt}\mu$ such that $\rho_{\xi_i} \to 0$. By compactness of $\operatorname{spt}\mu$, up to subsequences $\xi_i \to \xi \in \operatorname{spt}\mu$; let $r_\xi > 0$ be as in i) and ii) above. Since $\xi_i \to \xi$, we have $B^e_{\frac{r_\xi}{2}}(\xi_i) \subset B^e_{r_\xi}(\xi)$ for i large therefore on the ball $B^e_{\frac{r_\xi}{2}}(\xi_i)$ we have the desired graphical decomposition i),ii). It follows that $\rho_{\xi_i} \geq \frac{r_\xi}{2} > 0$, contradiction. $\square$

By compactness of $\operatorname{spt}\mu$ there exist $\{\xi_1,\dots,\xi_J\}\subset\operatorname{spt}\mu$ such that $\operatorname{spt}\mu\subset\bigcup_{j=1}^J B^e_{\frac{\rho}{4}}(\xi_j)$. Since $f_k(\mathbb{S}^2)$ converges to $\operatorname{spt}\mu$ in Hausdorff distance sense, for k large we also have $f_k(\mathbb{S}^2)\subset\bigcup_{j=1}^J B^e_{\frac{\rho}{4}}(\xi_j)$.

From i) above recall that $\mu_k\llcorner B^e_\rho(\xi_j)=\sum_{l=1}^{M_{\xi_j}}\mathcal{H}^2_g\llcorner\Big(\operatorname{graph} u_l^{k,j}\cup\bigcup_i P_i^{k,j}\cap B^e_\rho(\xi_j)\Big)$.
By the diameter estimates on the pimples $P_i^{k,j}$ and the selection principle 5.5.2 there exists $\frac{\bar\rho}{2}\in(\frac{\rho}{4},\frac{\rho}{2})$ such that $\partial B_{\bar\rho}(\xi_l)\cap\bigcup_{i,j}P_i^{k,j}=\emptyset$ for all $l\in\{1,\dots,J\}$ and for infinitely many k.

Of course we still have that $\bigcup_{k\geq K}f_k(\mathbb{S}^2)\cup\operatorname{spt}\mu\subset\bigcup_{j=1}^J B^e_{\frac{\bar\rho}{2}}(\xi_j)$ and the graphical decomposition as in i), ii) still holds in $B^e_{\bar\rho}(\xi_j)$.

First consider $f_k(\mathbb{S}^2)\cap B^e_{\bar\rho}(\xi_1)$. We replace the pimples $\{P_i^{k,1}\}_{i,k}$ of $f_k(\mathbb{S}^2)\cap B^e_{\bar\rho}(\xi_1)$ with the extension Lemma 5.5.1 as done in the proof of Lemma 5.3.1 (see (5.20) by graphs of functions with small C^1-norms defined on the discs $d_i^{k,1}$. It follows that the area of these graphs is bounded by $c(\operatorname{diam} d_i^{k,1})^2$, thus the sum of the areas of all the graphs is bounded by $c\sum_i(\operatorname{diam} d_i^{k,1})^2\leq c\varepsilon\bar\rho$, which followed from the graphical decomposition lemma. Notice that by the choice of $\bar\rho$ no pimple intersects $\partial B^e_{\bar\rho}(\xi_1)$ and we obtain a new $C^{1,1}$ immersion $f_k^1:\mathbb{S}^2\hookrightarrow M$ such that

$$f_k(\mathbb{S}^2)\setminus B^e_{\bar\rho}(\xi_1)=f_k^1(\mathbb{S}^2)\setminus B^e_{\bar\rho}(\xi_1),\quad f_k^1(\mathbb{S}^2)\cap B^e_{\bar\rho}(\xi_1)=\bigcup_{l=1}^{M_1}\operatorname{graph} w^k_{l,1}. \tag{5.63}$$

Moreover by the above area estimate we get $|f_k^1(\mathbb{S}^2)|\leq|f_k(\mathbb{S}^2)|+c\varepsilon\bar\rho$. Observe that $w^k_{l,1}:L_1^l\cap B^e_{\bar\rho}(\xi_1)\to(L_1^l)^\perp$ are $C^{1,1}$ functions which satisfy: $\frac{1}{\bar\rho}\|w^k_{l,1}\|_{L^\infty}+\|Dw^k_{l,1}\|_{L^\infty}\leq c\varepsilon^{\frac16}+\delta_k$ with $\delta_k\to0$. Moreover we have that $\mu\llcorner B^e_{\bar\rho}(\xi_1)=\sum_{l=1}^{M_1}\mathcal{H}^2_g\llcorner\big(\operatorname{graph} u_{l,1}\cap B^e_{\bar\rho}(\xi_1)\big)$ and by the construction after Lemma 5.3.1 we have that $w^k_{l,1}\to u_{l,1}$ uniformly.

Now consider some ξ_j such that $B^e_{\frac{\bar\rho}{2}}(\xi_1)\cap B^e_{\frac{\bar\rho}{2}}(\xi_j)\neq\emptyset$, without loss of generality we can take $j=2$. Recall that $\mu_k\llcorner B^e_{\bar\rho}(\xi_2)=\sum_{l=1}^{M_2}\mathcal{H}^2_g\llcorner\Big(\operatorname{graph} u^k_{l,2}\cup\bigcup_i P_i^{k,2}\cap B^e_{\bar\rho}(\xi_2)\Big)$ where $u_l^{k,2}$ are smooth functions defined on appropriate planes $L_{l,2}$.

Observe that on the intersection of the doubled balls, $f_k^1(\mathbb{S}^2)\cap B^e_{\bar\rho}(\xi_1)\cap B^e_{\bar\rho}(\xi_2)=\bigcup_{l=1}^{M_1}\operatorname{graph} w^k_{l,1}$ and because of the C^1 estimates for $w^k_{l,1}$ and $u_l^{k,2}$ and the diameter estimate on the pimples, these functions can be written as graphs on the planes $L_{l,2}$ satisfying analogous estimates. We conclude that $f_k^1(\mathbb{S}^2)\cap B^e_{\bar\rho}(\xi_1)\cap B^e_{\bar\rho}(\xi_2)=\bigcup_{l=1}^{M_1}\operatorname{graph} w^k_{l,2}$ where now the functions $w^k_{l,2}$ are defined on the planes L_2^l. From (5.63), the graphical representation of $f_k(\mathbb{S}^2)\cap B^e_{\bar\rho}(\xi_2)\setminus B^e_{\bar\rho}(\xi_1)$ and the choice of $\bar\rho$, we can replace the pimples inside $B^e_{\bar\rho}(\xi_2)\setminus B^e_{\bar\rho}(\xi_1)$ with new graphs as done before obtaining a new $C^{1,1}$ immersion $f_k^2:\mathbb{S}^2\hookrightarrow M$ which is union of graphs (without pimples) in both balls such that the corresponding graph functions converge uniformly to the graph functions representing μ, and such that $|f_k^2(\mathbb{S}^2)|\leq|f_k(\mathbb{S}^2)|+2c\varepsilon\bar\rho$.

Repeating the above procedure a finite number of times we obtain the desired $C^{1,1}$ immersion $\tilde f_k:=f_k^J:\mathbb{S}^2\hookrightarrow M$ with $|\tilde f_k(\mathbb{S}^2)|\leq|f_k(\mathbb{S}^2)|+cJ\varepsilon\bar\rho\leq C$ because of the uniform area estimate given in Proposition 4.1.1, Proposition 4.1.2 and Remark 4.1.3.

Now let us show that the $C^{1,1}$ immersion $\tilde f_k$ are actually generalized (r,λ)-immersions. Recall that $\operatorname{spt}\mu\subset\bigcup_{j=1}^J B^e_{\bar\rho}(\xi_j)$ is an open cover of $\operatorname{spt}\mu$, then by Lebesgue Lemma there exists the Lebesgue number $\tilde\rho>0$ with the following property: for every $\xi\in\operatorname{spt}\mu$ we have that $B^e_{\tilde\rho}(\xi)\subset B^e_{\bar\rho}(\xi_j)$ for some $j\in\{1,\dots,J\}$.

Now observe that also $\tilde f_k(\mathbb{S}^2)$ converges to $\operatorname{spt}\mu$ in Hausdorff distance sense (this follows by the uniform convergence of the corresponding graphs), then $B_{\frac{\tilde\rho}{2}}(\tilde f_k(\mathbb{S}^2))\subset\bigcup_{j=1}^J B^e_{\bar\rho}(\xi_j)$ for k large. Let us take a point $p\in\mathbb{S}^2$ and observe that $B^e_{\frac{\tilde\rho}{2}}(\tilde f_k(p))\subset B^e_{\bar\rho}(\xi_j)$ for some j; therefore by construction of $\tilde f_k$ we have $\tilde f_k(\mathbb{S}^2)\cap B^e_{\frac{\tilde\rho}{2}}(\tilde f_k(p))=\bigcup_{l=1}^{M_j}\operatorname{graph} w^k_{l,j}$ where $w^k_{l,j}:L_j^l\cap B^e_{\frac{\tilde\rho}{2}}(\pi_j^l(\tilde f_k(p)))\to(L_j^l)^\perp$ are $C^{1,1}$ functions which satisfy $\|Dw^k_{l,j}\|_{L^\infty}\leq c\varepsilon^{\frac16}+\delta_k$ with $\delta_k\to0$ (where π_j^l denotes the orthogonal projection onto L_j^l). Now let us recall that by Nash embedding theorem we can assume that our ambient manifold is isometrically embedded in some $\mathbb{R}^S$; let us denote by $A^k_{p,L_j^l}:\mathbb{R}^S\to\mathbb{R}^S$ an Euclidean isometry which maps the origin to $\tilde f_k(p)$ and the subspace $\mathbb{R}^2\times\{0\}$ onto $\tilde f_k(p)+(L_j^l-\pi_j^l(\tilde f_k(p)))$. We get that $\tilde f_k(\mathbb{S}^2)\cap B^e_{\frac{\tilde\rho}{2}}(\tilde f_k(p))=\bigcup_{l=1}^{M_j}\Big(A^k_{p,L_j^l}(\operatorname{graph}\tilde w^k_{l,j})\Big)$ where $\tilde w^k_{l,j}:\mathbb{R}^2\cap B^e_{\frac{\tilde\rho}{2}}(0)\to(\mathbb{R}^2)^\perp$ is given by $\tilde w^k_{l,j}=(A^k_{p,L_j^l})^{-1}\circ w^k_{l,j}\circ A^k_{p,L_j^l}-(\tilde f_k(p)-\pi_j^l(\tilde f_k(p)))$ are $C^{1,1}$ functions which satisfy $\|D\tilde w^k_{l,j}\|_{L^\infty}\leq c\varepsilon^{\frac16}+\delta_k$

with $\delta_k \to 0$. Now call $U^k_{\frac{\tilde\rho}{2},p} \subset \mathbb{S}^2$ the p-component of the set $(\pi \circ (A^k_{p,L^l_j})^{-1} \circ \tilde f_k)^{-1}(\mathbb{R}^2 \cap B^e_{\frac{\tilde\rho}{2}}(0))$ where $\pi : \mathbb{R}^S \to \mathbb{R}^2$ is the projection on the first two coordinates. By construction we have that $(A^k_{p,L^l_j})^{-1} \circ \tilde f_k(U^k_{\frac{\tilde\rho}{2},p}) = \operatorname{graph} \tilde w^k_{l,j}$, where $\tilde w^k_{l,j} : \mathbb{R}^2 \cap B^e_{\frac{\tilde\rho}{2}}(0) \to (\mathbb{R}^2)^\perp$ is a $C^{1,1}$ function which satisfies $\|D\tilde w^k_{l,j}\|_{L^\infty} \le c\varepsilon^{\frac{1}{6}} + \delta_k$ with $\delta_k \to 0$.

For any $\lambda < \frac{1}{4}$, choosing in the beginning of the construction ε small enough, for k large enough we have that $\tilde f_k : \mathbb{S}^2 \hookrightarrow M$ is a generalized $(\frac{\tilde\rho}{2}, \lambda)$-immersion. ■

By the compactness Theorem 5.5.6 for generalized (r,λ)-immersions of Breuning, we know that there exists a generalized $(\frac{\tilde\rho}{2}, \lambda)$-function $f : \mathbb{S}^2 \hookrightarrow M$ (see Definition 5.5.5) and diffeomorphisms $\phi_k : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\tilde f_k \circ \phi_k \to f$ uniformly. Let us briefly recall Breuning's construction of the limit f (see page 57 of [BreuTh]). Let q be in $\mathbb{S}^2$ and $q_k = \phi_k(q)$; by the uniform convergence of the $\tilde f_k \circ \phi_k$ we have that for k large $B^e_{\frac{\tilde\rho}{2}}(\tilde f_k(q_k)) \subset B^e_{\tilde\rho}(\xi_j)$ for some j. By the construction carried in the proof of Lemma 5.4.1 we know that for each large k

$$(A^k_{q_k,L^l_j})^{-1} \circ \tilde f_k(U^k_{\frac{\tilde\rho}{2},q_k}) = \operatorname{graph} \tilde w^k_{l,j}.$$

As Breuning proves, there exist λ-Lipschitz functions $\tilde u_{l,j}$ such that

$$\tilde w^k_{l,j} \overset{k\to\infty}{\longrightarrow} \tilde u_{l,j} \quad \text{and} \quad (A_{q,L^l_j})^{-1} \circ f(U_{\frac{\tilde\rho}{2},q}) = \operatorname{graph} \tilde u_{l,j}.$$

On the other hand we know from the representation of the limit measure μ we have

$$\mu \llcorner B^e_{\tilde\rho}(\xi_j) = \sum_{l=1}^{M_j} \mathcal{H}^2_g \llcorner \left(\operatorname{graph} u_{l,j} \cap B^e_{\tilde\rho}(\xi_j)\right),$$

where $u_{l,j}$ are $C^{1,\alpha} \cap W^{2,2}$ functions defined on the planes L^j_l (see Lemma 5.3.2) and $A_{q,L^l_j} \circ \tilde w^k_{l,j} \circ (A_{q,L^l_j})^{-1} \overset{k\to\infty}{\longrightarrow} u_{l,j}$. By uniqueness of the limit it follows that $\tilde u_{l,j} = (A_{q,L^l_j})^{-1} \circ u_{l,j} \circ A_{q,L^l_j}$ is actually $C^{1,\alpha} \cap W^{2,2}$ and $A_{q,L^l_j}(\operatorname{graph} \tilde u_{l,j}) = \operatorname{graph} u_{l,j}$. Thus

$$f(U_{\frac{\tilde\rho}{2},q}) = A_{q,L^l_j}(\operatorname{graph} \tilde u_{l,j}) = \operatorname{graph} u_{l,j}.$$

We have therefore shown that the generalized $(\frac{\tilde\rho}{2}, \lambda)$-function $f : \mathbb{S}^2 \hookrightarrow M$ is actually a $C^{1,\alpha} \cap W^{2,2}$ immersion and μ is the Radon measure on M associated to the immersion f. Henceforth we have just proven the following Lemma.

Lemma 5.4.2. *Let f_k and μ as before. Then there exists a $C^{1,\alpha} \cap W^{2,2}$ immersion $f : \mathbb{S}^2 \hookrightarrow M$ such that μ is the Radon measure associated to the immersion f.*

5.4.2 Smoothness of the immersion f parametrizing μ

First of all let us point out that via a standard approximation argument one can check that

$$\inf\{E(h) | h : \mathbb{S}^2 \hookrightarrow M \text{ smooth immersion }\} = \inf\{E(h) | h : \mathbb{S}^2 \hookrightarrow M \ C^1 \cap W^{2,2} \text{ immersion }\}$$

(analogous equality for W_1 replacing E). Then by lower semicontinuity (Theorem 4.2.6 and Theorem 4.1.13) the limit immersion f constructed in Lemma 5.4.2 minimizes E (respectively W_1) among $C^1 \cap W^{2,2}$ immersions of $\mathbb{S}^2$ into M, in particular it satisfies the Euler Lagrange equation (at the point x)

$$E'(f)[x] = \triangle H - \frac{1}{2} H(H^2 - 2|A|^2 - R_g(f(x))) + (\nabla_\nu \bar K)(T_x f)$$

where $\triangle$ is the Laplace Beltrami of the immersion f and as before R_g and $\nabla_\nu \bar K$ are respectively the scalar curvature and the covariant derivative of the sectional curvature of the ambient manifold (M,g) (instead, the Euler equation for W_1 is $W_1(f)' = \frac{1}{2}\triangle H - \frac{1}{4} H(H^2 - 2|A|^2 - 2Ric_g(\nu,\nu) + 4)$ where, of course, $Ric_g(\nu,\nu)$ is the Ricci tensor of (M,g) evaluated on the unit normal ν to f). It is a long and tedious computation but it is possible to check that the Euler Lagrange equation of E (resp. of W_1) fits in Lemma 3.2 in [SiL].

It follows that the functions u^l representing μ are actually $C^{2,\alpha}\cap W^{3,2}$ and the L^2 norm of the 3^{rd} derivatives satisfies the power decay $\int_{B_\rho}|D^3u^l|^2\leq c\rho^\alpha$. Now using the difference quotients method one proves that the functions u^l are actually $C^{3,\alpha}\cap W^{4,2}$ and the L^2 norm of the 4^{th} derivatives satisfies the power decay $\int_{B_\rho}|D^4u^l|^2\leq c\rho^\alpha$; continuing this bootstrap argument one shows the smoothness of u^l and thus that of f.

5.5 Appendix to regularity theory

5.5.1 Some useful lemmas

The following Lemmas have been used in the regularity theory and their proof is essentially due to Simon (see [SiL]). Simon's formulation of the next lemma a is bit different, here we use a little modification given by Schygulla in [Schy] which has the advantage of having a more constructive proof.

Lemma 5.5.1. *Let L be a 2-dimensional plane in $\mathbb{R}^n$, $x_0\in L$ and $u\in C^\infty\left(U,L^\perp\right)$ where $U\subset L$ is an open neighborhood of $L\cap\partial B_\rho(x_0)$. Moreover let $|Du|\leq c$ in U. Then there exists a function $w\in C^\infty\left(\overline{B_\rho(x_0)},L^\perp\right)$ with the following properties:*

1.) $w=u$ *on* $\partial B_\rho(x_0)$

2.) $\dfrac{\partial w}{\partial\nu}=\dfrac{\partial u}{\partial\nu}$ *on* $\partial B_\rho(x_0)$ ($\nu=$ *outer unit normal to* $\partial B_\rho(x_0)$)

3.) $\dfrac{1}{\rho}||w||_{L^\infty(B_\rho(x_0))}\leq c(n)\left(\dfrac{1}{\rho}||u||_{L^\infty(\partial B_\rho(x_0))}+||Du||_{L^\infty(\partial B_\rho(x_0))}\right)$

4.) $||Dw||_{L^\infty(B_\rho(x_0))}\leq c(n)||Du||_{L^\infty(\partial B_\rho(x_0))}$

5.) $\displaystyle\int_{B_\rho(x_0)}|D^2w(x)|^2\,dx\leq c(n)\rho\int_{\operatorname{graph}u_{|\partial B_\rho(x_0)}}|A(x)|^2\,d\mu$

The next Lemma can be found with proof in the appendix of [SiL]

Lemma 5.5.2. *Let $\delta>0, I\subset\mathbb{R}$ a bounded interval and $A_k\subset I, k\in\mathbb{N}$, measurable sets with $\mathcal{L}^1(A_k)\geq\delta$ for all k. Then there exists a set $A\subset I$ with $\mathcal{L}^1(A)\geq\delta$ such that each point $x\in A$ lies in A_k for infinitely many k.*

The Lemma below is a little modification of what Simon uses in [SiL], hence we give a proof.

Lemma 5.5.3. *Let $g:(0,b)\to[0,+\infty[$ be a real valued bounded function such that*

$$g\left(x\right)\leq\gamma g(2x)+Cx^\alpha\quad\text{for all } x\in\left(0,\frac{b}{2}\right)$$

where $\alpha>0$, $\gamma\in(0,1)$ and C some positive constant. It follows that there exists $\beta\in(0,1)$ and a constant $C=C\left(b,||g||_{L^\infty(0,b)}\right)$ such that

$$g(x)\leq Cx^\beta\quad\text{for all } x\in(0,b)\,.$$

Proof. First of all observe that since g is non negative we can choose $\gamma\in(0,1)$ maybe a little larger such that $\gamma\neq\left(\frac{1}{2}\right)^\alpha$. Next choose $\beta\in(0,\min(1,\alpha))$ such that $\gamma\leq\left(\frac{1}{2}\right)^\beta$. Now let $x\in\left(\frac{b}{2},b\right)$ and $m\in\mathbb{N}$. It follows that

$$g(2^{-m}x)\leq\gamma^m g(x)+\sum_{j=0}^{m-1}C\gamma^j\left(2^{j-m}x\right)^\alpha.$$

Let us estimate the second adding term in the right hand side:

$$\begin{aligned}\sum_{j=0}^{m-1} C\gamma^j \left(2^{j-m}x\right)^\alpha &= \frac{C}{2^{\alpha m}}x^\alpha \sum_{j=0}^{m-1} (2^\alpha\gamma)^j = \frac{C}{2^{\alpha m}}x^\alpha \frac{1-2^{\alpha m}\gamma^m}{1-2^\alpha\gamma} \\ &\le C_b\gamma^m + \frac{C}{2^{\alpha m}}x^\alpha = C_b\gamma^m + C\left(2^{-m}x\right)^\alpha = C_b\gamma^m + C\left(2^{-m}b\frac{x}{b}\right)^\alpha \\ &\le C_b\gamma^m + C_b\left(2^{-m}x\right)^\beta,\end{aligned}$$

therefore it follows that (since g is bounded)

$$g(2^{-m}x) \le C_{b,\|g\|_{L^\infty(0,b)}}\gamma^m + C_b\left(2^{-m}x\right)^\beta \quad \text{for all } x \in \left(\frac{b}{2}, b\right) \text{ and } m \in \mathbb{N}.$$

Since g is bounded, for $m = 0$ and $x \in \left(\frac{b}{2}, b\right)$ we have

$$g(x) \le C(=\|g\|_{L^\infty(0,b)}) \le C_{\|g\|_{L^\infty(0,b)}}\frac{2x}{b} \le C_{\|g\|_{L^\infty(0,b)}}\frac{x}{b} \le C_{\|g\|_{L^\infty(0,b)}}\left(\frac{x}{b}\right)^\beta \le C_{b,\|g\|_{L^\infty(0,b)}}x^\beta.$$

Now for $m \ge 1$ let $I_m = \left(2^{-(m+1)}b, 2^{-m}b\right)$. For $y \in I_m$ there exists $x \in \left(\frac{b}{2}, b\right)$ such that $y = 2^{-m}x$ and therefore we get (notice that $\gamma^m \le \left(\frac{2}{b}\right)^\beta y^\beta$)

$$g(y) \le C_{b,\|g\|_{L^\infty(0,b)}}\gamma^m + C_b y^\beta \le C_{b,\|g\|_{L^\infty(0,b)}}y^\beta.$$

Therefore the lemma is proved. ■

Lemma 5.5.4. *Let $\mu > 0$, $\delta \in \left(0, \frac{\mu}{2}\right)$ and $\Omega = B_\mu^{\mathbb{R}^2}(0)\backslash E$ where $E \subset \mathbb{R}^2$ is measurable with $\mathcal{L}^1(p_1(E)) \le \frac{\mu}{2}$ and $\mathcal{L}^1(p_2(E)) \le \delta$ where p_1 is the projection onto the x-axis and p_2 is the projection onto the y-axis. Then for any $f \in C^1(\Omega)$ there exists a point $(x_0, y_0) \in \Omega$ such that*

$$\int_\Omega |f - f(x_0,y_0)|^2 \le C\mu^2 \int_\Omega |Df|^2 + C\delta\mu \sup_\Omega |f|^2$$

where C is a absolute constant.

Proof. First consider the case $\mu = 1$.

Let $f \in C^1(\Omega)$ and define the set S by

$$S := \left\{ x \in \left(-\frac{3}{4}, \frac{3}{4}\right) \,\middle|\, x \notin p_1(E) \right\}.$$

It follows by assumption that

$$\mathcal{L}^1(S) \ge \frac{6}{4} - \mathcal{L}^1(p_1(E)) \ge 1.$$

We also have that

$$l_x \cap E = \emptyset \qquad \text{for all } x \in S$$

where $l_x = \{(x,y) \,|\, y \in \mathbb{R}\}$. Now let $T \subset S$ be given by

$$T := \left\{ x \in S \,\middle|\, \int_{l_x\cap\Omega} |Df(x,y)|^2\, dy \le 4\int_\Omega |Df|^2 \right\}.$$

It follows that

$$\mathcal{L}^1(T) \ge \frac{3}{4}$$

since otherwise we would have that

$$\begin{aligned}\int_\Omega |Df|^2 &\ge \int_{S\backslash T}\int_{l_x\cap\Omega} |Df(x,y)|^2\, dy\, dx > \int_{S\backslash T} 4\int_\Omega |Df|^2 = 4\left(\mathcal{L}^1(S) - \mathcal{L}^1(T)\right)\int_\Omega |Df|^2 \\ &> 4\left(1 - \frac{3}{4}\right)\int_\Omega |Df|^2 = \int_\Omega |Df|^2,\end{aligned}$$

a contradiction.

Since $\mathcal{L}^1(l_x \cap \Omega) \le 2$ for all $x \in T$ and $|f(x,y) - f(x,0)| \le \int_{l_x \cap \Omega} |Df(x,y)|\, dy$ for all $y \in l_x \cap \Omega$ we therefore get, by using the Cauchy-Schwartz-Inequality, that

$$\sup_{l_x \cap \Omega} |f - f(x,0)|^2 \le 8 \int_\Omega |Df|^2 \quad \text{for all } x \in T.$$

On the other hand the inequality

$$\int_a^b h^2 \le (b-a)^2 \int_a^b (h')^2$$

holds for every function $h \in C^1((a,b))$ such that $h = 0$ at some point of (a,b).
For $y \in (-1,1) \backslash p_2(E)$ define the set $L_y = \{(x,y) \,|\, x \in \mathbb{R}\}$. Since $\mathcal{L}^1(L_y \cap \Omega) \le 2$ we get for all $y \in (-1,1) \backslash p_2(E)$ and all $x_0 \in T$ that

$$\int_{L_y \cap \Omega} |f(x,y) - f(x_0,y)|^2\, dx \le 4 \int_{L_y \cap \Omega} |Df|^2.$$

By the above estimates, we conclude that for all $y \in (-1,1) \backslash p_2(E)$ and all $x_0 \in T$

$$\int_{L_y \cap \Omega} |f(x,y) - f(x_0,0)|^2\, dx \le 4 \int_{L_y \cap \Omega} |Df|^2 + 16 \int_\Omega |Df|^2.$$

By integration over all $y \in (-1,1) \backslash p_2(E)$ we get that

$$\int_{\Omega \backslash p_2^{-1}(E)} |f(x,y) - f(x_0,0)|^2 \le C \int_\Omega |Df|^2 \quad \text{for all } x_0 \in T.$$

Since $|f(x,y) - f(x_0,0)|^2 \le 4 \sup_\Omega |f|^2$ and $\mathcal{L}^2(\Omega \cap p_2^{-1}(E)) \le 2\delta$ by assumption, we also get that

$$\int_{\Omega \cap p_2^{-1}(E)} |f - f(x_0,0)|^2 \le 8\delta \sup_\Omega |f|^2 \quad \text{for all } x_0 \in T.$$

Therefore we conclude that

$$\int_\Omega |f - f(x_0,0)|^2 \le C \int_\Omega |Df|^2 + C\delta \sup_\Omega |f|^2 \quad \text{for all } x_0 \in T.$$

Since $T \neq \emptyset$ the Lemma is proven for $\mu = 1$.

Now let $\mu > 0$ be arbitrary and $f \in C^1(\Omega)$. Define the set

$$\tilde{\Omega} = \frac{1}{\mu}\Omega = B_1(0) \setminus \tilde{E}$$

where the set $\tilde{E}$ is given by $\tilde{E} = \frac{1}{\mu}E$. By the assumptions it follows that

$$\mathcal{L}^1(p_1(\tilde{E})) \le \frac{1}{2} \quad \text{and} \quad \mathcal{L}^1(p_2(\tilde{E})) \le \frac{\delta}{\mu} \le \frac{1}{2}.$$

Define also the function $\tilde{f} \in C^1(\tilde{\Omega})$ by $\tilde{f}(x) = f(\mu x)$. From the $\mu = 1$-case it follows that

$$\int_{\tilde{\Omega}} \left| \tilde{f} - \tilde{f}(\tilde{x}_0, \tilde{y}_0) \right|^2 \le C \int_{\tilde{\Omega}} \left| D\tilde{f} \right|^2 + C\frac{\delta}{\mu} \sup_{\tilde{\Omega}} |\tilde{f}|^2 \tag{5.64}$$

for some point $(\tilde{x}_0, \tilde{y}_0) \in \tilde{\Omega}$.
Elementary calculations using the transformation formula show that

$$\begin{aligned} \int_{\tilde{\Omega}} \left| \tilde{f} - \tilde{f}(\tilde{x}_0, \tilde{y}_0) \right|^2 &= \mu^{-2} \int_\Omega |f - f(x_0, y_0)|^2 \quad \text{where } (x_0, y_0) = \mu(\tilde{x}_0, \tilde{y}_0) \in \Omega, \\ \int_{\tilde{\Omega}} \left| D\tilde{f} \right|^2 &= \int_\Omega |Df|^2 \quad \text{and} \quad \sup_{\tilde{\Omega}} |\tilde{f}|^2 = \sup_\Omega |f|^2. \end{aligned}$$

Plugging the last formulas in (5.64), the desired estimate follows for arbitrary $\mu > 0$. ■

5.5.2 Definitions and properties of generalized (r, λ)-immersions

In this subsection we briefly recall the definitions and properties of generalized (r, λ)-immersions of $f : \mathbb{S}^2 \hookrightarrow M \subset \mathbb{R}^S$ appearing in [BreuTh].

We call a mapping $A : \mathbb{R}^S \to \mathbb{R}^S$ a Euclidean isometry, if there is a rotation $R \in SO(p)$ and a translation $T \in \mathbb{R}^S$, such that $A(x) = Rx + T$ for all $x \in \mathbb{R}^S$.

For a given point $q \in \mathbb{S}^2$ and a given 2-plane $E \in G(p, 2)$ let $A_{q,E} : \mathbb{R}^S \to \mathbb{R}^S$ be a Euclidean isometry, which maps the origin to $f(q)$ and the subspace $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^S$ onto $f(q) + E$.

Let $U^E_{r,q} \subset \mathbb{S}^2$ be the q-component of the set $(\pi \circ A^{-1}_{q,E} \circ f)^{-1}(B_r)$, where $\pi : \mathbb{R}^S \to \mathbb{R}^2$ is the projection on the first two coordinates.

Definition 5.5.5. *An immersion $f : \mathbb{S}^2 \hookrightarrow (M, g) \subset \mathbb{R}^S$ is called a generalized (r, λ)-immersion, if for each point $q \in \mathbb{S}^2$ there is an $E = E(q) \in G(p, 2)$, such that $A^{-1}_{q,E} \circ f(U^E_{r,q})$ is the graph of a differentiable function $u : B_r \to (\mathbb{R}^2)^\perp$ with $\|Du\|_{C^0(B_r)} \leq \lambda$.*

The set of generalized (r, λ)-immersions is denoted by $\mathcal{F}^1(r, \lambda)$. Moreover let $\mathcal{F}^1_V(r, \lambda)$ be the set of all immersions $f \in \mathcal{F}^1(r, \lambda)$ such that $|f(\mathbb{S}^2)| \leq V$ (of course $|f(\mathbb{S}^2)|$ is the area of $\mathbb{S}^2$ with respect to the pullback metric f^*g).

A continuous function $f : \mathbb{S}^2 \hookrightarrow (M, g) \subset \mathbb{R}^S$ is called an (r, λ)-function, if for each point $q \in \mathbb{S}^2$ there is an $E = E(q) \in G(p, 2)$, such that $A^{-1}_{q,E} \circ f(U^E_{r,q})$ is the graph of a Lipschitz function $u : B_r \to (\mathbb{R}^2)^\perp$ with with Lipschitz constant λ.

The set of (r, λ)-functions is denoted by $\mathcal{F}^0(r, \lambda)$.

Now we recall the compactness theorem Theorem 0.5 in [BreuTh].

Theorem 5.5.6. *Let $\lambda \leq \frac{1}{4}$. Then $\mathcal{F}^1_V(r, \lambda)$ is relatively compact in $\mathcal{F}^0(r, \lambda)$ in the following sense: Let $f_k : \mathbb{S}^2 \hookrightarrow (M, g) \subset \mathbb{R}^S$ be a sequence in $\mathcal{F}^1_V(r, \lambda)$. Then, after passing to a subsequence, there exists an $f \in \mathcal{F}^0(r, \lambda)$ and a sequence of diffeomorphisms $\phi_k : \mathbb{S}^2 \to \mathbb{S}^2$, such that $f_k \circ \phi_k$ is uniformly Lipschitz bounded and converges uniformly to f.*

Chapter 6

Existence and partial regularity of minimizers for supercritical L^p curvature integral functionals in Riemannian manifolds, arbitrary dimension and codimension

In this chapter we prove existence and partial regularity of integral rectifiable m-dimensional varifolds minimizing functionals of the type $\int |H|^p$ and $\int |A|^p$ in a given Riemannian n-dimensional manifold (N,g), $2 \leq m < n$ and $p > m$, under suitable assumptions on N (in the end of the chapter we give many examples of such ambient manifolds). To this aim we introduce the following new tools: some monotonicity formulas for varifolds in $\mathbb{R}^S$ involving $\int |H|^p$, to avoid degeneracy of the minimizer, and a sort of isoperimetric inequality to bound the mass in terms of the mentioned functionals. The content of the chapter corresponds to the paper [MonVar].

6.1 Monotonicity formulas for integral m-varifolds with weak mean curvature in L^p, $p > m$

Let $V = V(M,\theta)$ be an integral varifold of $\mathbb{R}^S$ (associated to the rectifiable set $M \subset \mathbb{R}^S$ and with integer multiplicity function θ) with weak mean curvature H (since throughout this section we consider only varifolds in $\mathbb{R}^S$ and there is no ambiguity, we adopt the easier notation H for $H^{\mathbb{R}^S}$). Let us write μ for $\mu_V := \pi_\sharp(V)$ the push forward of the varifold measure V on $G_m(N)$ to N via the standard projection $\pi : G_m(N) \to N, \pi(x,P) = x$ (see Appendix 6.6 for more details); of course μ_V can also be seen as $\mu_V = \mathcal{H}^m \lfloor \theta$, the restriction of the m-dimensional Hausdorff measure to the multiplicity function θ.

The first Lemma is a known fact (see for example the book of Leon Simon [SiGMT] at page 82) of which we report also the proof for completeness.

Lemma 6.1.1. *Let $V = V(M,\theta) \in IV_m(\mathbb{R}^S)$ be with weak mean curvature H as above and fix a point $x_0 \in M$. For μ-a.e. $x \in M$ call $r(x) := |x - x_0|$ and $D^\perp r$ the orthogonal projection of the gradient vector Dr onto $(T_x M)^\perp$. Consider a nonnegative function $\phi \in C^1(\mathbb{R})$ such that*

$$\phi'(t) \leq 0 \; \forall t \in \mathbb{R}, \quad \phi(t) = 1 \; \textit{for} \; t \leq \frac{1}{2}, \quad \phi(t) = 0 \; \textit{for} \; t \geq 1.$$

For all $\rho > 0$ let us denote

$$\begin{aligned} I(\rho) &:= \int_M \phi(r/\rho)d\mu, \\ L(\rho) &:= \int_M \phi(r/\rho)(x-x_0)\cdot H d\mu, \\ J(\rho) &:= \int_M \phi(r/\rho)|D^\perp r|^2 d\mu; \end{aligned}$$

then

$$\frac{d}{d\rho}[\rho^{-m}I(\rho)] = \rho^{-m}J'(\rho) + \rho^{-m-1}L(\rho). \tag{6.1}$$

Proof. The idea is to use formula (6.44) and chose the vector field X in an appropriate way in order to get informations about V. First of all let us recall that for any function $f \in C^1(\mathbb{R}^S)$ and any $x \in M$ where the approximate tangent space T_xM exists (it exists for μ-a.e. $x \in M$ see [SiGMT] 11.4-11.6) one can define the tangential gradient as the projection of the gradient in $\mathbb{R}^S$ onto T_xM:

$$\nabla^M f := \sum_{j,l=1}^{S} P^{jl} D_l f(x) e_j$$

where $D_l f$ denotes the partial derivative $\frac{\partial f}{\partial x^l}$ of f, P^{jl} is the matrix of the orthogonal projection of $\mathbb{R}^S$ onto T_xM and $\{e_j\}_{j=1,\dots,S}$ is an orthonormal basis of $\mathbb{R}^S$. Denoted $\nabla^M_j := e_j \cdot \nabla^M$, recall that the tangential divergence is defined as

$$div_M X := \sum_{j=1}^{S} \nabla^M_j X^j;$$

moreover it is easy to check the Leibniz formula

$$div_M fX := \nabla^M f \cdot X + f\, div_M X \quad \forall f \in C^1(\mathbb{R}^S) \text{ and } \forall X \in C^1(\mathbb{R}^S) \text{ vector field.}$$

Now let us choose the vector field. Fix $\rho > 0$ and consider the function $\gamma \in C^1(\mathbb{R})$ defined as

$$\gamma(t) := \phi(t/\rho);$$

then of course we have the following properties:

$$\gamma'(t) \le 0\ \forall t \in \mathbb{R}, \quad \gamma(t) = 1 \text{ for } t \le \frac{\rho}{2}, \quad \gamma(t) = 0 \text{ for } t \ge \rho.$$

Call $r(x) := |x - x_0|$ and choose the vector field

$$X(x) := \gamma(r(x))(x - x_0).$$

Using the Leibniz formula we get

$$\begin{aligned} div_M X &= \nabla^M \gamma(r)\cdot(x-x_0) + \gamma(r) div_M(x-x_0) \\ &= r\gamma'(r)\frac{(x-x_0)^T}{|x-x_0|}\frac{(x-x_0)^T}{|x-x_0|} + m\gamma(r) \\ &= r\gamma'(r)(1-|D^\perp r|^2) + m\gamma(r), \end{aligned} \tag{6.2}$$

where u^T is the projection of the vector $u \in \mathbb{R}^S$ onto T_pM and $D^\perp r = \frac{(x-x_0)^\perp}{|x-x_0|}$ is the orthogonal projection of the gradient vector Dr onto $(T_xM)^\perp$. The equation (6.44) of the weak mean curvature thus yields

$$m\int_M \gamma(r)d\mu + \int_M r\gamma'(r)d\mu = \int_M r\gamma'(r)|D^\perp r|^2 d\mu - \int_M H\cdot(x-x_0)\gamma(r)d\mu. \tag{6.3}$$

Now recall that $\gamma(r) = \phi(r/\rho)$, so $r\gamma'(r) = \frac{r}{\rho}\phi'(r/\rho) = -\rho\frac{\partial}{\partial\rho}[\phi(r/\rho)]$. Thus, combining (6.3) and the definitions of $I(\rho), J(\rho)$ and $L(\rho)$ one gets

$$mI(\rho) - \rho I'(\rho) = -\rho J'(\rho) - L(\rho).$$

Thus, multiplying both sides by ρ^{-m-1} and rearranging we obtain

$$\frac{d}{d\rho}[\rho^{-m}I(\rho)] = \rho^{-m}J'(\rho) + \rho^{-m-1}L(\rho).$$

This concludes the proof ■

Estimating from below the right hand side of (6.1) and integrating, we get the following useful inequalities.

Proposition 6.1.2. *Let $V = V(M,\theta) \in IV_m(\mathbb{R}^S)$ be with weak mean curvature $H \in L^p(V)$, $p > m$ (we mean that $\int_{G_m(\mathbb{R}^S)} |H|^p dV < \infty$ or equivalently, denoted with an abuse of notation $H(x) = H(x, T_xM)$, $\int_M |H|^p d\mu < \infty$). Fixed a point $x_0 \in M$ and $0 < \sigma < \rho < \infty$, then*

$$[\sigma^{-m}\mu(B_\sigma(x_0))]^{\frac{1}{p}} \leq [\rho^{-m}\mu(B_\rho(x_0))]^{\frac{1}{p}} + \frac{p^2}{p-m}\rho^{1-\frac{m}{p}}\Big(\int_{B_\rho(x_0)} |H|^p d\mu\Big)^{\frac{1}{p}} - \frac{p^2}{p-m}\sigma^{1-\frac{m}{p}}\Big(\int_{B_\sigma(x_0)} |H|^p d\mu\Big)^{\frac{1}{p}}. \tag{6.4}$$

Proof. Let us estimate from below the right hand side of equation (6.1). Observe that

$$J'(\rho) = \frac{d}{d\rho}\int_M \phi(r/\rho)|D^\perp r|^2 d\mu = -\rho^{-2}\int_M r\phi'(r/\rho)|D^\perp r|^2 d\mu \geq 0$$

since $\phi'(t) \leq 0$ for all $t \in \mathbb{R}$. Thus we can say that

$$\frac{d}{d\rho}[\rho^{-m}I(\rho)] \geq \rho^{-m-1}L(\rho). \tag{6.5}$$

Let us estimate from below the right hand side by the Schwartz inequality:

$$\begin{aligned}
\rho^{-m-1}L(\rho) &= \rho^{-m-1}\int_M \phi(r/\rho)(x-x_0)\cdot H d\mu \\
&\geq -\rho^{-m-1}\int_M \big(\phi(r/\rho)^{\frac{1}{p}}|H|\big)\,|x-x_0|\phi(r/\rho)^{\frac{p-1}{p}} d\mu.
\end{aligned}$$

Now recalling that $\phi(t) = 0$ for $t \geq 1$ we get that $\phi(r/\rho) = 0$ for $r \geq \rho$ so $|x - x_0|$ in the integral can be estimated from above by ρ and we can say that

$$\rho^{-m-1}L(\rho) \geq -\rho^{-m}\int_M \big(\phi(r/\rho)^{\frac{1}{p}}|H|\big)\,\phi(r/\rho)^{\frac{p-1}{p}} d\mu;$$

thus, by Holder inequality, for all $p > 1$

$$\begin{aligned}
\rho^{-m-1}L(\rho) &\geq -\rho^{-m}\Big(\int_M \phi(r/\rho)|H|^p d\mu\Big)^{\frac{1}{p}}\Big(\int_M \phi(r/\rho) d\mu\Big)^{\frac{p-1}{p}} \\
&= -\rho^{-m}\Big(\int_M \phi(r/\rho)|H|^p d\mu\Big)^{\frac{1}{p}}\; I(\rho)^{\frac{p-1}{p}}.
\end{aligned} \tag{6.6}$$

Putting together inequalities (6.5) and (6.6) we get

$$\frac{d}{d\rho}[\rho^{-m}I(\rho)] \geq -\rho^{-m}\Big(\int_M \phi(r/\rho)|H|^p d\mu\Big)^{\frac{1}{p}}\; I(\rho)^{\frac{p-1}{p}};$$

multiplying both sides by $\rho^{m-\frac{m}{p}}I(\rho)^{\frac{1}{p}-1}$ and rearranging we get

$$\frac{d}{d\rho}[\rho^{-m}I(\rho)]^{\frac{1}{p}} \geq -p\,\rho^{-\frac{m}{p}}\Big(\int_M \phi(r/\rho)|H|^p d\mu\Big)^{\frac{1}{p}}.$$

Now, after choosing $p > m$, integrate the last inequality from σ to ρ (the same ρ chosen in the statement of the Proposition) and get with an integration by parts of the right hand side

$$\begin{aligned}
\rho^{-\frac{m}{p}}I(\rho)^{\frac{1}{p}} - \sigma^{-\frac{m}{p}}I(\sigma)^{\frac{1}{p}} &\geq -p\int_\sigma^\rho \Big[\Big(t^{-\frac{m}{p}}\Big)\Big(\int_M \phi(r/t)|H|^p d\mu\Big)^{\frac{1}{p}}\Big]dt \\
&= -p\Big[\Big(1-\frac{m}{p}\Big)^{-1}\Big(\rho^{1-\frac{m}{p}}\Big(\int_M \phi(r/\rho)|H|^p d\mu\Big)^{\frac{1}{p}} - \sigma^{1-\frac{m}{p}}\Big(\int_M \phi(r/\sigma)|H|^p d\mu\Big)^{\frac{1}{p}}\Big)\Big] \\
&\quad + p\int_\sigma^\rho \Big[\Big(1-\frac{m}{p}\Big)^{-1} t^{1-\frac{m}{p}}\Big(\frac{d}{dt}\int_M \phi(r/t)|H|^p d\mu\Big)\Big]dt
\end{aligned} \tag{6.7}$$

Observe that, as before for $J'(\rho)$, since $\phi'(t) \leq 0$ for all t it follows

$$\frac{d}{dt}\int_M \phi(r/t)|H|^p d\mu = -t^{-2}\int_M r\phi'(r/t)|H|^p d\mu \geq 0$$

so the second integral in equation (6.7) is non negative and, recalling the definition of I, we can write

$$\begin{aligned}\Big(\rho^{-m}\int_M \phi(r/\rho)d\mu\Big)^{\frac{1}{p}} - \Big(\sigma^{-m}\int_M \phi(r/\sigma)d\mu\Big)^{\frac{1}{p}} \quad &\geq \quad \frac{p^2}{p-m}\Big[-\rho^{1-\frac{m}{p}}\Big(\int_M \phi(r/\rho)|H|^p d\mu\Big)^{\frac{1}{p}} \\ &\qquad + \sigma^{1-\frac{m}{p}}\Big(\int_M \phi(r/\sigma)|H|^p d\mu\Big)^{\frac{1}{p}}\Big]. \quad (6.8)\end{aligned}$$

Now observe that during all this proof and during all the proof of Lemma 6.1.1 the only used properties of ϕ have been

$$\phi \in C^1(\mathbb{R}), \quad \phi'(t) \leq 0\ \forall t \in \mathbb{R}, \quad \phi(t) \leq 1\ \forall t \in \mathbb{R}, \quad \phi(t) = 0\ \forall t \geq 1;$$

thus, for all such ϕ, the inequality (6.8) holds. Now taking a sequence ϕ_k of such functions pointwise converging to the characteristic function of $]-\infty, 1]$ and, using the Dominated Convergence Theorem, passing to the limit on k in (6.8) we get

$$\big[\rho^{-m}\mu(B_\rho(x_0))\big]^{\frac{1}{p}} - \big[\sigma^{-m}\mu(B_\sigma(x_0))\big]^{\frac{1}{p}} \geq \frac{p^2}{p-m}\Big[-\rho^{1-\frac{m}{p}}\Big(\int_{B_\rho(x_0)}|H|^p d\mu\Big)^{\frac{1}{p}} + \sigma^{1-\frac{m}{p}}\Big(\int_{B_\sigma(x_0)}|H|^p d\mu\Big)^{\frac{1}{p}}\Big].$$

Rearranging we can conclude that

$$\big[\sigma^{-m}\mu(B_\sigma(x_0))\big]^{\frac{1}{p}} \leq \big[\rho^{-m}\mu(B_\rho(x_0))\big]^{\frac{1}{p}} + \frac{p^2}{p-m}\rho^{1-\frac{m}{p}}\Big(\int_{B_\rho(x_0)}|H|^p d\mu\Big)^{\frac{1}{p}} - \frac{p^2}{p-m}\sigma^{1-\frac{m}{p}}\Big(\int_{B_\sigma(x_0)}|H|^p d\mu\Big)^{\frac{1}{p}}.$$

■

From Corollary 17.8 page 86 of [SiGMT], if $H \in L^p(V)$ for some $p > m$, then the density $\theta(x) = \lim_{\rho\downarrow 0}\frac{\mu(\bar{B}_\rho(x))}{w_m\rho^m}$ exists at every point $x \in \mathbb{R}^S$ and θ is an upper semicontinuous function. Hence, letting $\sigma \to 0$, one has

$$[\omega_m\theta(x_0)]^{\frac{1}{p}} \leq \Big[\frac{\mu(B_\rho(x_0))}{\rho^m}\Big]^{\frac{1}{p}} + \frac{p^2}{p-m}\Big[\rho^{p-m}\int_{B_\rho(x_0)}|H|^p d\mu\Big]^{\frac{1}{p}}.$$

Using the inequality $a^{\frac{1}{p}} + b^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}}(a+b)^{\frac{1}{p}}$ given by the concavity of the function $t \mapsto t^{\frac{1}{p}}$ with $p > 1$ and $t > 0$, we get

$$\omega_m\theta(x_0) \leq 2^{p-1}\Big[\frac{\mu(B_\rho(x_0))}{\rho^m} + \Big(\frac{p^2}{p-m}\Big)^p\rho^{p-m}\int_{B_\rho(x_0)}|H|^p d\mu\Big].$$

Since $V \in IV_m(\mathbb{R}^S)$, then θ is integer valued and by definition $\theta \geq 1$ μ-a.e. From the upper semicontinuity of θ it follows that $\theta(x) \geq 1$ for all $x \in \operatorname{spt}\mu$ (where, as before, μ is the spatial measure associated to V). Then the last formula can be written more simply getting the **fundamental inequality**

$$1 \leq C_{p,m}\Big[\frac{\mu(B_\rho(x_0))}{\rho^m} + \rho^{p-m}\int_{B_\rho(x_0)}|H|^p d\mu\Big] \quad \forall x_0 \in \operatorname{spt}\mu, \tag{6.9}$$

where $C_{p,m} > 0$ is a positive constant depending on p, m and such that $C_{p,m} \to \infty$ if $p \downarrow m$.

Using the fundamental inequality now we can link through inequalities the mass of V, the diameter of M and the L^p norm of the weak mean curvature H.

Lemma 6.1.3. *Let $V = V(M,\theta) \in IV_m(\mathbb{R}^S)$ be a non null integral m-varifold with compact spatial support $\operatorname{spt}\mu \subset \mathbb{R}^S$ and weak mean curvature $H \in L^p(V)$ for some $p > m$. Then, called $d = \operatorname{diam}_{\mathbb{R}^S}(\operatorname{spt}\mu)$ the diameter of $\operatorname{spt}\mu$ as a subset of $\mathbb{R}^S$,*

$$|V| \leq \Big(\frac{d}{m}\Big)^p\int_M |H|^p d\mu. \tag{6.10}$$

PROOF. In the same spirit of the proof of Lemma 6.1.1 we choose a suitable vector field X to plug in the mean curvature equation (6.44)

$$\int_M div_M X d\mu = -\int_M X \cdot H d\mu$$

in order to get informations about the varifold $V = V(M, \theta)$. Now fix a point $x_0 \in \operatorname{spt} \mu$ and simply let $X(x) = x - x_0$. Since $div_M X = m$ μ-a.e. (for more details see the proof of Lemma 6.1.1), observing that $|X| \leq d$ μ-a.e. and estimating the right hand side by Holder inequality we get

$$m|V| \leq d\Big(\int_M |H|^p d\mu\Big)^{\frac{1}{p}} |V|^{\frac{p-1}{p}}.$$

Now multiply both sides by $|V|^{\frac{1}{p}-1}$ and raise to the power p in order to get the thesis.
■

Lemma 6.1.4. *Let $V = V(M, \theta) \in IV_m(\mathbb{R}^S)$ be a non null integral m-varifold with compact connected spatial support $\operatorname{spt} \mu \subset \mathbb{R}^S$ and weak mean curvature $H \in L^p(V)$ for some $p > m$. Then, called $d = \operatorname{diam}_{\mathbb{R}^S}(\operatorname{spt} \mu)$,*

$$d \leq C_{p,m}\Big(\int_M |H|^p d\mu\Big)^{\frac{m-1}{p}} |V|^{1-\frac{m-1}{p}} \tag{6.11}$$

where $C_{p,m} > 0$ is a positive constant depending on p, m and such that $C_{p,m} \to \infty$ if $p \downarrow m$.

PROOF. Since $\operatorname{spt} \mu \subset \mathbb{R}^S$ is compact, then there exist $x_0, y_0 \in \operatorname{spt} \mu$ such that

$$d = |x_0 - y_0|.$$

Let $\rho \in]0, d/2]$ and call $N := \lfloor d/\rho \rfloor$ the integer part of d/ρ. For $j = 1, \ldots, N-1$ take

$$y_j \in \partial B_{(j+\frac{1}{2})\rho}(y) \cap \operatorname{spt} \mu$$

(observe that, since $\operatorname{spt} \mu$ is connected, $\partial B_{(j+\frac{1}{2})\rho}(y_0) \cap \operatorname{spt} \mu \neq \emptyset$ for $j = 1, \ldots, N-1$). For each ball $B_{\rho/2}(y_j)$, $j = 0, \ldots, N-1$ we have the fundamental inequality (6.9); since the balls $B_{\rho/2}(y_j)$, $j = 0, \ldots, N-1$ are pairwise disjoint, summing up over j we get

$$N \leq C_{p,m}\left(\frac{|V|}{\rho^m} + \rho^{p-m}\int_M |H|^p d\mu\right).$$

Moreover, since $N = \lfloor d/\rho \rfloor \geq \frac{d}{2\rho}$, we have

$$d \leq 2\rho N \leq C_{p,m}\left(\frac{|V|}{\rho^{m-1}} + \rho^{p-m+1}\int_M |H|^p d\mu\right). \tag{6.12}$$

Now let us choose ρ in an appropriate way; observe that taken

$$\rho = \frac{m}{2}\left(\frac{|V|}{\int_M |H|^p d\mu}\right)^{\frac{1}{p}},$$

in force of the estimate (6.10), the condition $\rho \leq d/2$ is satisfied. Finally, plugging this value of ρ into equation (6.12), after some trivial computation we conclude that

$$d \leq C_{p,m}\,|V|^{\frac{p-m+1}{p}}\left(\int_M |H|^p d\mu\right)^{\frac{m-1}{p}}.$$

■

Combining the Fundamental Inequality with the previous lemmas we are in position to prove a lower diameter and mass bound.

Lemma 6.1.5. *Let $V = V(M,\theta) \in IV_m(\mathbb{R}^S)$ be a non null integral m-varifold with spatial support* $\operatorname{spt}\mu \subset \mathbb{R}^S$ *and weak mean curvature $H \in L^p(V)$ for some $p > m$. Then, called $d := \operatorname{diam}_{\mathbb{R}^S}(\operatorname{spt}\mu)$*

$$d \geq \frac{1}{C_{p,m}\Big(\int_M |H|^p d\mu\Big)^{\frac{1}{p-m}}} \tag{6.13}$$

where $C_{p,m} > 0$ is a positive constant depending on p, m and such that $C_{p,m} \to \infty$ if $p \downarrow m$.

Proof. If $d = \infty$, the inequality (6.13) is trivially satisfied; hence we can assume that $\operatorname{spt}\mu \subset \mathbb{R}^S$ is compact. It follows that there exist $x_0, y_0 \in \operatorname{spt}\mu$ such that

$$d = |x_0 - y_0|.$$

Recall the Fundamental Inequality (6.9) and choose $\rho = d$ obtaining

$$1 \leq C_{p,m}\Big(\frac{|V|}{d^m} + d^{p-m}\int_M |H|^p d\mu\Big). \tag{6.14}$$

From Lemma 6.1.3,

$$|V| \leq \frac{1}{m^p} d^p \int_M |H^p| d\mu,$$

hence the inequality (6.14) becomes

$$1 \leq C_{p,m}\, d^{p-m} \int_M |H|^p d\mu$$

and we can conclude.

■

Lemma 6.1.6. *Let $V = V(M,\theta) \in IV_m(\mathbb{R}^S)$ be a non null integral m-varifold with compact spatial support* $\operatorname{spt}\mu \subset \mathbb{R}^S$ *and weak mean curvature $H \in L^p(V)$ for some $p > m$. Then*

$$|V| \geq \frac{1}{C_{p,m}\Big(\int_M |H|^p d\mu\Big)^{\frac{m}{p-m}}} \tag{6.15}$$

where $C_{p,m} > 0$ is a positive constant depending on p, m and such that $C_{p,m} \to \infty$ if $p \downarrow m$.

Proof. First of all we remark that each connected component of $\operatorname{spt}\mu$ is the support of an integral varifold with weak mean curvature in L^p. Hence can assume that $\operatorname{spt}\mu \subset \mathbb{R}^S$ is connected, otherwise just argue on a non null connected component of $\operatorname{spt}\mu$ and observe that the inequality (6.15) is well behaved for bigger subsets.

Call as before $d := \operatorname{diam}_{\mathbb{R}^S}(\operatorname{spt}\mu)$; from the inequality (6.11),

$$|V| \geq \frac{d^{\frac{p}{p-m+1}}}{\big(\int_M |H|^p d\mu\big)^{\frac{m-1}{p-m+1}}}.$$

But from the last inequality (6.13),

$$d^{\frac{p}{p-m+1}} \geq \frac{1}{C_{p,m}\Big(\int_M |H|^p d\mu\Big)^{\frac{p}{(p-m)(p-m+1)}}}.$$

Combining the two estimates, with an easy computation we get the conclusion.

■

Proposition 6.1.7. *Let $\{V_k = V_k(M_k,\theta_k)\}_{k\in\mathbb{N}} \subset IV_m(\mathbb{R}^S)$ be a sequence of integral varifolds with weak mean curvature $H_k \in L^p(V_k)$ for some $p > m$ and associated spatial measures μ_k. Assume a uniform bound on the L^p norms of H_k:*

$$\exists C > 0 :\ \forall k \in \mathbb{N} \quad \int_{M_k} |H_k|^p d\mu_k = \int_{G_m(\mathbb{R}^S)} |H_k|^p dV_k \leq C,$$

and assume a uniform bound on the spatial supports $\operatorname{spt}\mu_k$*:*

$$\exists R>0 : \operatorname{spt}\mu_k \subset B_R^{\mathbb{R}^S}$$

where $B_R^{\mathbb{R}^S}$ *is the ball of radius* R *centered in the origin in* $\mathbb{R}^S$*.*

It follows that if there exists a Radon measure μ *on* $\mathbb{R}^S$ *such that*

$$\mu_k \to \mu \quad \textit{weak as Radon measures,}$$

then

$$\operatorname{spt}\mu_k \to \operatorname{spt}\mu \quad \textit{in Hausdorff distance sense.}$$

Proof. First of all observe that the uniform bound on the spatial supports $\operatorname{spt}\mu_k$ implies that $\operatorname{spt}\mu$ is compact. Since $\operatorname{spt}\mu$ is compact, recall that $\operatorname{spt}\mu_k \to \operatorname{spt}\mu$ if and only if the set of the all possible limit points of all possible sequences $\{x_k\}_{k\in\mathbb{N}}$ with $x_k \in \operatorname{spt}\mu_k$ coincides with $\operatorname{spt}\mu$. Let us prove it by double inclusion.

i) since $\mu_k \to \mu$ weak as Radon measures of course $\forall x \in \mathit{spt}\,\mu$ there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ with $x_k \in \operatorname{spt}\mu_k$ such that $x_k \to x$. Otherwise there would exist $\epsilon>0$ such that for infinitely many k'

$$B_\epsilon(x)\cap \operatorname{spt}\mu_{k'} = \emptyset.$$

This would imply that $\mu_{k'}(B_\epsilon(x))=0$, but $x\in\operatorname{spt}\mu$ so we reach the contradiction

$$0<\mu(B_\epsilon(x)) = \lim_{k'} \mu_{k'} B_\epsilon(x) = 0.$$

ii) Let $\{x_k\}_{k\in\mathbb{N}}$ with $x_k\in\operatorname{spt}\mu_k$ be such that $x_k\to x$. We have to show that $x\in\operatorname{spt}\mu$. Let us argue by contradiction:
if $x\notin\operatorname{spt}\mu$ then there exists $\epsilon_0>0$ such that

$$0=\mu(B_{\epsilon_0}(x)) = \lim_k \mu_k(B_{\epsilon_0}(x)). \tag{6.16}$$

Since $\operatorname{spt}\mu_k \ni x_k \to x$, then for every $\epsilon\in(0,\epsilon_0/2)$ there exists $K_\epsilon>0$ large enough such that

$$x_k \in (\operatorname{spt}\mu_k \cap B_\epsilon(x)) \quad \forall k>K_\epsilon.$$

Now consider the balls $B_\epsilon(x_k)$ for $k>K_\epsilon$: by the triangle inequality $B_\epsilon(x_k)\subset B_{\epsilon_0}(x)$, moreover, since by construction $x_k\in\operatorname{spt}\mu_k$, we can apply the fundamental inequality (6.9) to each $B_\epsilon(x_k)$ and obtain

$$\begin{aligned} 1 \quad &\leq \quad C_{p,m}\Big[\frac{\mu_k(B_\epsilon(x_k))}{\epsilon^m} + \epsilon^{p-m}\int_{B_\epsilon(x_k)} |H_k|^p d\mu_k\Big] \\ &\leq \quad C_{p,m}\Big[\frac{\mu_k(B_{\epsilon_0}(x))}{\epsilon^m} + \epsilon^{p-m}\int_{M_k} |H_k|^p d\mu_k\Big] \quad \forall k>K_\epsilon. \end{aligned} \tag{6.17}$$

Keeping in mind (6.16), for every fixed $\epsilon\in(0,\epsilon_0/2)$ we can pass to the limit on k in inequality (6.17) and get

$$\liminf_k \int_{M_k} |H_k|^p d\mu_k \geq \frac{1}{C_{p,m}\,\epsilon^{p-m}}.$$

But $\epsilon>0$ can be arbitrarily small, contradicting the uniform bound $\int_{M_k}|H_k|^p d\mu_k \leq C$ of the assumptions.
■

6.2 Isoperimetric inequalities and compactness results

6.2.1 An isoperimetric inequality involving the generalized second fundamental form

The following Isoperimetric Inequality involving the generalized second fundamental form is inspired by the paper of White [Whi] and uses the concept of varifold with second fundamental form introduced by Hutchinson [Hu1]. Actually we need a slight generalization of the definition of curvature varifold given by Hutchinson: in Definition 5.2.1 of [Hu1], the author considers only integral varifolds but, as a matter of fact, a similar definition makes sense for a general varifold. In Appendix 6.6 we recalled the needed concepts.

Theorem 6.2.1. *Let $N \subset\subset \bar{N}$ be a compact subset of a (maybe non compact) n-dimensional Riemannian manifold $(\bar{N}, g)$ (which, by Nash Embedding Theorem we can assume isometrically embedded in some $\mathbb{R}^S$) and let $m \leq n-1$. Then the following conditions are equivalent:*

i) N contains no nonzero m-varifold with null generalized second fundamental form

ii) There is an increasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Phi(0) = 0$ and a function $F : G_m(N) \times \mathbb{R}^{S^3} \to \mathbb{R}^+$ satisfying (6.6.2) *(see Appendix 6.6) such that for every m-varifold V in N with generalized second fundamental form A*

$$|V| \leq \Phi\Big(\int_{G_m(N)} F(x, P, A(x, P)) dV \Big).$$

iii) for every function $F : G_m(N) \times \mathbb{R}^{S^3} \to \mathbb{R}^+$ satisfying (6.6.2) *(see Appendix 6.6) there exists a constant $C_F > 0$ such that for every m-varifold V in N with generalized second fundamental form A*

$$|V| \leq C_F \int_{G_m(N)} F(x, P, A(x, P)) dV.$$

Proof. Of course iii) $\Rightarrow$ ii) $\Rightarrow$ i). It remains to prove that i) $\Rightarrow$ iii). Let us argue by contradiction: assume that iii) is not satisfied and prove that also i) cannot be satisfied.
First fix the function F. If iii) is not satisfied then there exists a sequence $\{(V_k, A_k)\}_{k \in \mathbb{N}}$ of m-varifolds in N with generalized second fundamental form (see Definition 6.6.5) such that

$$|V_k| \geq k \int_{G_m(N)} F(x, P, A_k(x, P)) dV_k.$$

We can assume that $|V_k| = 1$ otherwise replace V_k with the normalized varifold $\tilde{V}_k := \frac{1}{|V_k|} V_k$ (observe that the second fundamental form is invariant under this rescaling of the measure and that $\int_{G_m(N)} F(x, P, A_k) dV_k = |V_k| \int_{G_m(N)} F(x, P, A_k) d\tilde{V}_k$). Hence

$$\int_{G_m(N)} F(x, P, A_k(x, P)) dV_k \leq \frac{1}{k}.$$

Recall that $|V_k| = 1$ so, from Banach-Alaoglu and Riesz Theorems, there exists a varifold V such that, up to subsequences, $V_k \to V$ in varifold sense (i.e weak convergence of Radon measures on $G_m(N)$). Of course $|V| = \lim_k |V_k| = 1$.

Using the notation of [Hu1] (see the Appendix 6.6) we have that the measure-function pairs (V_k, A_k) over $G_m(N)$, up to subsequences, satisfy the assumptions of Theorem 6.6.4. From (i) of the mentioned Theorem 6.6.4, it follows that there exists a measure-function pair $(V, \tilde{A})$ with values in $\mathbb{R}^{S^3}$ (i.e a Radon measure V on $G_m(N)$ and a matrix valued function $\tilde{A} \in L^1_{loc}(V)$) such that $(V_k, A_k) \rightharpoonup (V, \tilde{A})$ (i.e $V_k \lfloor A_k \to V \lfloor \tilde{A}$ weak convergence of Radon vector valued measures).

From Remark 6.6.6 we can express the generalized curvatures B_k of the varifolds V_k in terms of the second fundamental forms A_k. Moreover, calling B the corresponding quantity to $\tilde{A}$, from the explicit expression (6.42) it is clear that the weak convergence $(V_k, A_k) \rightharpoonup (V, \tilde{A})$ implies the weak convergence $(V_k, B_k) \rightharpoonup (V, B)$.
Passing to the limit in k in (6.40) we see that (V, B) satisfies the equation, so V is an m-varifold with generalized curvature B.

Now let us check that the corresponding generalized second fundamental form (in sense of equation (6.41)) to B is $\tilde{A}$.
Call

$$\Lambda^l_{ij}(x, P) := P_{pj} B_{ilp}(x, P) - P_{pj} P_{iq} \frac{\partial Q_{lp}}{\partial x_q}(x)$$

the corresponding second fundamental form to B and $\Lambda_k = A_k$ the corresponding to B_k (in a varifold with generalized curvature, B_{ijl} is uniquely determined by the integration by parts formula (6.40) and, by definition, $A^l_{ij} = \Lambda^l_{ij}$; but for our limit varifold it is not a priori clear that $\tilde{A} = \Lambda$).

Since $(V_k, B_k) \rightharpoonup (V, B)$, from the definitions it is clear that $(V_k, \Lambda_k) \rightharpoonup (V, \Lambda)$; but, from the definition of $\tilde{A}$, $(V_k, \Lambda_k) = (V_k, A_k) \rightharpoonup (V, \tilde{A})$. It follows that $\Lambda = \tilde{A}$ V-almost everywhere and that $\tilde{A}$ is the generalized second fundamental form of V.

Finally, the lower semicontinuity of the functional (sentence (ii) of Theorem 6.6.4) implies

$$\int_{G_m(N)} F(x,P,A)dV \leq \liminf_k \int_{G_m(N)} F(x,P,A_k)dV_k = 0.$$

From the assumption ii) of condition (6.6.2) on F it follows that $A = 0$ V-almost everywhere; henceforth we constructed a non null m-varifold V in N with null second fundamental form and this concludes the proof. ■

Remark 6.2.2. *A trivial but fundamental example of $F : G_m(N) \times \mathbb{R}^{S^3} \to \mathbb{R}$ satisfying the assumptions of Theorem 6.2.1 is $F(x,P,A) = |A|^p$ for any $p > 1$. Hence the Theorem implies that if a compact subset N of a Riemannian n-dimensional manifold $(\bar{N}, g)$ does not contain any non null k-varifold $(k \leq n-1)$ with null generalized second fundamental form then for every $p > 1$ there exists a constant $C_p > 0$ such that*

$$|V| \leq C_p \int_{G_m(N)} |A|^p dV$$

for every k-varifold V in N with generalized second fundamental form A.

Putting together the fundamental compactness and lower semicontinuity Theorem 6.6.7 of Hutchinson and the Isoperimetric Theorem 6.2.1 we get the following useful compactness-lower semicontinuity result.

Theorem 6.2.3. *Let $N \subset\subset \bar{N}$ be a compact subset of a (maybe non compact) n-dimensional Riemannian manifold $(\bar{N}, g)$ (which, by Nash Embedding Theorem we can assume isometrically embedded in some $\mathbb{R}^S$), fix $m \leq n-1$ and let $F : G_m(N) \times \mathbb{R}^{S^3} \to \mathbb{R}^+$ be a function satisfying* (6.6.2).

Assume that, for some $m \leq n-1$, the space (N, g) does not contain any non zero m-varifold with null generalized second fundamental form.

Consider a sequence $\{V_k\}_{k\in\mathbb{N}} \subset CV_m(N)$ of curvature varifolds with generalized second fundamental forms $\{A_k\}_{k\in\mathbb{N}}$ such that

$$\int_{G_m(N)} F(x,P,A_k)dV_k \leq C$$

for some $C > 0$ independent of k.

Then there exists $V \in CV_m(N)$ with generalized second fundamental form A such that, up to subsequences,

i) $(V_k, A_k) \rightharpoonup (V, A)$ in the weak sense of measure-function pairs,

ii) $\int_{G_m(N)} F(x,P,A)dV \leq \liminf_k \int_{G_m(N)} F(x,P,A_k)dV_k$.

Proof. From Theorem 6.2.1 there exists a constant $C_F > 0$ depending on the function F such that $|V_k| \leq C_F \int_{G_m(N)} F(x,P,A_k(x,P))dV_k$, thus from the boudness of $\int_{G_m(N)} F(x,P,A_k)dV_k$ we have the uniform mass bound

$$|V_k| \leq C \tag{6.18}$$

for some $C > 0$ independent of k. This mass bound, together with Banach Alaoglu and Riesz Theorems, implies that there exists an m-varifold V on N such that, up to subsequences, $V_k \to V$ in varifold sense.

In order to apply Hutchinson compactness Theorem 6.6.7 we have to prove that V actually is an integral m-varifold.

From assumption *iv*) on F of Definition 6.6.2, there exists a continuous function $\phi : G_m(N) \times [0,\infty) \to [0,\infty)$, with $0 \leq \phi(x,P,s) \leq \phi(x,P,t)$ for $0 \leq s \leq t$ and $(x,P) \in G_m(N)$, $\phi(x,P,t) \to \infty$ locally uniformly in (x,P) as $t \to \infty$, such that

$$\phi(x,P,|A|)|A| \leq F(x,P,A) \tag{6.19}$$

for all $(x,P,A) \in G_m(N) \times \mathbb{R}^{S^3}$. Since N is compact, also $G_m(N)$ is so and from the properties of ϕ there exists $C > 0$ such that $\phi(x,P,|A|) \geq 1$ for $|A| > C$ and any $(x,P) \in G_m(N)$. Thus for every k we can split the computation of the $L^1(V_k)$ norm of A_k as

$$\int_{G_m(N)} |A_k|dV_k = \int_{G_m(N)\cap\{|A_k|\leq C\}} |A_k|dV_k + \int_{G_m(N)\cap\{|A_k|>C\}} |A_k|dV_k.$$

The first term is bounded above by the mass bound (6.18). About the second term observe that, for $|A| > C$ the inequality (6.19) implies that $|A| \leq F(x, P, A)$; then also the second term is bounded in virtue on the assumption that $\int_{G_m(N)} F(x, P, A_k) dV_k$ is uniformly bounded.

We have proved that there exists a constant C such that, for all $k \in \mathbb{N}$,

$$\int_{G_m(N)} |A_k| dV_k \leq C. \tag{6.20}$$

Now, change point of view and look at the varifolds V_k as curvature varifolds in $\mathbb{R}^S$. Recall (see Remark 6.6.6) that the curvature function B can be written in terms of the generalized second fundamental form A relative to $\bar{N}$ and of the extrinsic curvature of the manifold $\bar{N}$ (as submanifold of $\mathbb{R}^S$) which is uniformly bounded on N from the compactness assumption. Using the triangle inequality together with estimate (6.20) and the mass bound (6.18) we obtain the uniform estimate of the $L^1(V_k)$ norms of the curvature functions B_k

$$\int_{G_m(\mathbb{R}^S)} |B_k| dV_k \leq C \tag{6.21}$$

for some $C > 0$ independent of k.

Estimate (6.21) and Remark 6.6.10 tell us that the integral varifolds V_k of $\mathbb{R}^S$ have uniformly bounded first variation: there exists a $C > 0$ independent of k such that

$$|\delta V_k(X)| \leq C \sup_{\mathbb{R}^S} |X|, \quad \forall X \in C^1_c(\mathbb{R}^S) \text{ vector field.}$$

The uniform bound on the first variations and on the masses of the integral varifolds V_k allow us to apply Allard's integral compactness Theorem (see for example [SiGMT] Remark 42.8 or the original paper of Allard [Al]) and say that the limit varifold V is actually integral.

The conclusions of the Theorem then follow from Hutchinson Theorem 6.6.7. ■

Corollary 6.2.4. *Let $N \subset\subset \bar{N}$ be a compact subset with non empty interior, $int(N) \neq \emptyset$, of a (maybe non compact) n-dimensional Riemannian manifold $(\bar{N}, g)$ (which, by Nash Embedding Theorem can be assumed isometrically embedded in some $\mathbb{R}^S$) and let $F : G_m(N) \times \mathbb{R}^{S^3} \to \mathbb{R}^+$ be a function satisfying (6.6.2).*

Assume that, for some $m \leq n-1$, the space (N, g) does not contain any non zero m-varifold with null generalized second fundamental form.

Call

$$\alpha^m_{N,F} := \inf \left\{ \int_{G_m(N)} F(x, P, A) dV : V \in CV_m(N), V \neq 0 \textit{ with generalized second fundamental form } A \right\} \tag{6.22}$$

and consider a minimizing sequence $\{V_k\}_{k \in \mathbb{N}} \subset CV_m(N)$ of curvature varifolds with generalized second fundamental forms $\{A_k\}_{k \in \mathbb{N}}$ such that

$$\int_{G_m(N)} F(x, P, A_k) dV_k \downarrow \alpha^m_{N,F}.$$

Then there exists $V \in CV_m(N)$ with generalized second fundamental form A such that, up to subsequences,

i) $(V_k, A_k) \rightharpoonup (V, A)$ in the weak sense of measure-function pairs,

ii) $\int_{G_m(N)} F(x, P, A) dV \leq \alpha^m_F$.

Proof. We only have to check that $\alpha^m_{N,F} < \infty$, then the conclusion follows from Theorem 6.2.3. But the fact is trivial since $int(N) \neq \emptyset$, indeed we can always construct a smooth compact m-dimensional embedded submanifold of N, which of course is a curvature m-varifold with finite energy. ■

Remark 6.2.5. *Notice that, a priori, Corollary 6.2.4 does not ensure the existence of a minimizer since it can happen that the limit m-varifold V is null. In the next Section 6.3 we will see that, if $F(x, P, A) \geq C|A|^p$ for some $C > 0$ and $p > m$, then this is not the case and we have a non trivial minimizer.*

6.2.2 An isoperimetric inequality involving the weak mean curvature

In this Subsection we adapt to the context of varifolds with weak mean curvature the results of the previous Subsection 6.2.1 about varifolds with generalized second fundamental form (for the basic definitions and properties see Appendix 6.6). The following Isoperimetric Inequality involving the weak mean curvature can be seen as a variant of Theorem 2.3 in [Whi].

Theorem 6.2.6. *Let $N \subset\subset \bar{N}$ be a compact subset of a (maybe non compact) n-dimensional Riemannian manifold $(\bar{N}, g)$ (which, by Nash Embedding Theorem we can assume isometrically embedded in some $\mathbb{R}^S$) and let $m \leq n-1$. Then the following conditions are equivalent:*

i) N contains no nonzero m-varifold with null weak mean curvature relative to $\bar{N}$ (i.e N contains no nonzero stationary m-varifold; see Remark 6.6.13).

ii) There is an increasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Phi(0) = 0$ and a function $F : G_m(N) \times \mathbb{R}^S \to \mathbb{R}^+$ satisfying (6.6.2) (see Appendix 6.6) such that for every m-varifold V in N with weak mean curvature H^N relative to $\bar{N}$

$$|V| \leq \Phi\Big(\int_{G_m(N)} F(x, P, H^N(x, P))dV\Big).$$

iii) for every function $F : G_m(N) \times \mathbb{R}^S \to \mathbb{R}^+$ satisfying (6.6.2) (see Appendix 6.6) there exists a constant $C_F > 0$ such that for every m-varifold V in N with weak mean curvature H^N relative to $\bar{N}$

$$|V| \leq C_F \int_{G_m(N)} F(x, P, H^N(x, P))dV.$$

Proof. The proof is similar to the proof of Theorem 6.2.1. Of course iii) $\Rightarrow$ ii) $\Rightarrow$ i). We prove by contradiction that i) $\Rightarrow$ iii): assume that iii) is not satisfied and show that also i) cannot be satisfied. First fix the function F. If iii) is not satisfied then there exists a sequence $\{V_k\}_{k\in\mathbb{N}}$ of m-varifolds in N with weak mean curvatures H_k^N relative to $\bar{N}$ (see Definition 6.6.11) such that

$$|V_k| \geq k \int_{G_m(N)} F(x, P, H_k^N(x, P))dV_k.$$

We can assume that $|V_k| = 1$ otherwise replace V_k with the normalized varifold $\tilde{V}_k := \frac{1}{|V_k|}V_k$ (observe that the weak mean curvature is invariant under this rescaling of the measure and that $\int_{G_m(N)} F(x, P, H_k^N)dV_k = |V_k| \int_{G_m(N)} F(x, P, H_k^N)d\tilde{V}_k$). Hence

$$\int_{G_m(N)} F(x, P, H_k^N(x, P))dV_k \leq \frac{1}{k}.$$

Recall that $|V_k| = 1$ so, from Banach-Alaoglu and Riesz Theorems, there exists a varifold V such that, up to subsequences, $V_k \to V$ in varifold sense (i.e weak convergence of Radon measures on $G_m(N)$). Of course $|V| = \lim_k |V_k| = 1$.

Now the measure-function pairs (V_k, H_k^N) over $G_m(N)$, up to subsequences, satisfy the assumptions of Theorem 6.6.4 and (i) (of the mentioned Theorem 6.6.4) implies that there exists a measure-function pair $(V, \tilde{H}^N)$ with values in $\mathbb{R}^S$ such that $(V_k, H_k^N) \rightharpoonup (V, \tilde{H}^N)$ weak convergence of measure-function pairs (see Definition 6.6.1).

At this point we have to check that V is an m-varifold of N with weak mean curvature $\tilde{H}^N$ relative to $\bar{N}$. Recall that $N \hookrightarrow \mathbb{R}^S$, so the varifolds V_k can be seen as varifolds with weak mean curvatures $H_k^{\mathbb{R}^S}$ in $\mathbb{R}^S$; from equation (6.45), the measure-function pair convergence $(V_k, H_k^N) \rightharpoonup (V, \tilde{H}^N)$ implies the measure-function pair convergence $(V_k, H_k^{\mathbb{R}^S}) \rightharpoonup (V, \tilde{H}^N + P_{jk}\frac{\partial Q_{ij}}{\partial x^k})$ which says (pass to the limit in Definition 6.6.9) that V is an m-varifold in $\mathbb{R}^S$ with weak mean curvature $\tilde{H}^N + P_{jk}\frac{\partial Q_{ij}}{\partial x^k}$. Thus, by Definition 6.6.11, V is an m-varifold of N with weak mean curvature $H^N := \tilde{H}^N$ relative to $\bar{N}$.

Finally, the lower semicontinuity of the functional (sentence (ii) of Theorem 6.6.4) implies

$$\int_{G_m(N)} F(x, P, H^N)dV \leq \liminf_k \int_{G_m(N)} F(x, P, H_k^N)dV_k = 0.$$

From the assumption ii) of condition (6.6.2) on F it follows that $H^N = 0$ V-almost everywhere; henceforth we constructed a non null m-varifold V in N with null weak mean curvature relative to $\bar{N}$ and this concludes the proof. ■

We also have a counterpart of Theorem 6.2.3 concerning the weak mean curvature:

Theorem 6.2.7. *Let $N \subset\subset \bar{N}$ be a compact subset of a (maybe non compact) n-dimensional Riemannian manifold $(\bar{N}, g)$ (which, by Nash Embedding Theorem we can assume isometrically embedded in some $\mathbb{R}^S$), fix $m \leq n-1$ and let $F : G_m(N) \times \mathbb{R}^S \to \mathbb{R}^+$ be a function satisfying* (6.6.2).

Assume that, for some $m \leq n-1$, the space (N, g) does not contain any non zero m-varifold with null weak mean curvature relative to $\bar{N}$.

Consider a sequence $\{V_k\}_{k\in\mathbb{N}} \subset HV_m(N)$ of integral m-varifolds with weak mean curvatures $\{H_k^N\}_{k\in\mathbb{N}}$ relative to $\bar{N}$ such that

$$\int_{G_m(N)} F(x, P, H_k^N) dV_k \leq C$$

for some $C > 0$ independent of k.

Then there exists $V \in HV_m(N)$ integral varifold with weak mean curvature H^N relative to $\bar{N}$ such that, up to subsequences,

i) $(V_k, H_k^N) \rightharpoonup (V, H^N)$ in the weak sense of measure-function pairs,

ii) $\int_{G_m(N)} F(x, P, H^N) dV \leq \liminf_k \int_{G_m(N)} F(x, P, H_k^N) dV_k$.

Proof. The proof is analogous to the proof of Theorem 6.2.3. From Theorem 6.2.6 there exists a constant $C_F > 0$ depending on the function F such that $|V_k| \leq C_F \int_{G_m(N)} F(x, P, H_k^N(x, P)) dV_k$, thus from the boudness of $\int_{G_m(N)} F(x, P, H_k^N) dV_k$ we have the uniform mass bound

$$|V_k| \leq C \tag{6.23}$$

for some $C > 0$ independent of k. This mass bound, together with Banach Alaoglu and Riesz Theorems, implies that there exists an m-varifold V on N such that, up to subsequences, $V_k \to V$ in varifold sense.

The proof that V actually is an integral m-varifold is completely analogous to the same statement in the proof of Theorem 6.2.3: formally substituting H_k^N to A_k in the mentioned proof we arrive to

$$\int_{G_m(N)} |H_k^N| dV_k \leq C. \tag{6.24}$$

Now, change point of view and look at the varifolds V_k as integral varifolds in $\mathbb{R}^S$. From Definition 6.6.11 the weak mean curvature $H_k^{\mathbb{R}^S}$ in $\mathbb{R}^S$ can be written in terms of H_k^N and of the extrinsic curvature of the manifold $\bar{N}$ (as submanifold of $\mathbb{R}^S$) which is uniformly bounded on N from the compactness assumption. Using the triangle inequality together with estimate (6.24) and the mass bound (6.23) we obtain the uniform estimate of the $L^1(V_k)$ norms of the weak mean curvatures $H_k^{\mathbb{R}^S}$

$$\int_{G_m(\mathbb{R}^S)} |H_k^{\mathbb{R}^S}| dV_k \leq C \tag{6.25}$$

for some $C > 0$ independent of k. It follows (see Definition 6.6.9) that the integral varifolds V_k of $\mathbb{R}^S$ have uniformly bounded first variation: there exists a constant $C > 0$ independent of k such that

$$|\delta V_k(X)| \leq C \sup_{\mathbb{R}^S} |X|, \quad \forall X \in C_c^1(\mathbb{R}^S) \text{ vector field.}$$

The uniform bound on the first variations and on the masses of the integral varifolds V_k allow us to apply Allard's integral compactness Theorem (see for example [SiGMT] Remark 42.8 or the original paper of Allard [Al]) and say that the limit varifold V is actually integral.

With the same arguments in the end of the proof of Theorem 6.2.6, one can show that the varifold convergence of a subsequence $V_k \to V$ and the uniform energy bound $\int_{G_m(N)} F(x, P, H_k^N) dV_k < C$ implies the existence of a measure-function pair converging subsequence $(V_k, H_k^N) \rightharpoonup (V, H^N)$ for some $\mathbb{R}^S$ -valued function $H^N \in L^1_{loc}(V)$ which actually is the weak mean curvature of V relative to $\bar{N}$.

We conclude that $V \in HV_m(N)$ is an integral m-varifold of N with weak mean curvature H^N relative to $\bar{N}$ and i) holds; property ii) follows from the general Theorem 6.6.7 about measure-function pair convergence (specifically see sentence ii) of the mentioned Theorem). ∎

Finally we have a counterpart of Corollary 6.2.4

Corollary 6.2.8. *Let $N \subset\subset \bar{N}$ be a compact subset with non empty interior, $int(N) \neq \emptyset$, of a (possibly non compact) n-dimensional Riemannian manifold $(\bar{N}, g)$ (which, by Nash Embedding Theorem can be assumed isometrically embedded in some $\mathbb{R}^S$) and let $F : G_m(N) \times \mathbb{R}^S \to \mathbb{R}^+$ be a function satisfying* (6.6.2).

Assume that, for some $m \leq n-1$, the space (N, g) does not contain any non zero m-varifold with null weak mean curvature relative to $\bar{N}$.

Call

$$\beta^m_{N,F} := \inf \left\{ \int_{G_m(N)} F(x, P, H^N) dV : V \in HV_m(N), V \neq 0 \text{ with weak wean curvature } H^N \text{ relative to } \bar{N} \right\} \tag{6.26}$$

and consider a minimizing sequence $\{V_k\}_{k\in\mathbb{N}} \subset HV_m(N)$ of integral varifolds with weak mean curvatures $\{H^N_k\}_{k\in\mathbb{N}}$ such that

$$\int_{G_m(N)} F(x, P, H^N_k) dV_k \downarrow \beta^m_{N,F}.$$

Then there exists an integral m-varifold $V \in HV_m(N)$ with weak mean curvature H^N relative to $\bar{N}$ such that, up to subsequences,

i) $(V_k, H^N_k) \rightharpoonup (V, H^N)$ in the weak sense of measure-function pairs,

ii) $\int_{G_m(N)} F(x, P, H^N) dV \leq \beta^m_{N,F}$.

Proof. As in Corollary 6.2.4 we have that $\beta^m_{N,F} < \infty$, then the conclusion follows from Theorem 6.2.7. ■

Remark 6.2.9. *As for the generalized second fundamental form, a priori, Corollary 6.2.4 does not ensure the existence of a minimizer since it can happen that the limit m-varifold V is null. In Section 6.4 we will see that, if $F(x, P, H^N) \geq C|H^N|^p$ for some $C > 0$ and $p > m$, then this is not the case and we have a non trivial minimizer.*

6.3 Case $F(x, P, A) \geq C|A|^p$ with $p > m$: non degeneracy of the minimizing sequence and existence of a $C^{1,\alpha}$ minimizer

Throughout this Section, $(\bar{N}, g)$ stands for a compact n-dimensional Riemannian manifold isometrically embedded in some $\mathbb{R}^S$ (by Nash Embedding Theorem) and $N \subset\subset \bar{N}$ is a compact subset with non empty interior (as subset of N). Fix $m \leq n-1$; we will focus our attention and specialize the previous techniques to the case

$$\begin{aligned} F &: G_m(N) \times \mathbb{R}^{S^3} \to \mathbb{R}^+ \text{ is a function satisfying (6.6.2)} \\ F(x, P, A) &\geq C|A|^p \text{ for some } p > m \text{ and } C > 0. \end{aligned} \tag{6.27}$$

Recall that we are considering the minimization problem

$$\alpha^m_{N,F} := \inf \left\{ \int_{G_m(N)} F(x, P, A) dV : V \in CV_m(N), V \neq 0 \text{ with generalized second fundamental form } A \right\}.$$

Our goal is to prove the existence of a minimizer for $\alpha^m_{N,F}$, F as in (6.27).

Let $\{V_k\}_{k\in\mathbb{N}} \subset CV_m(N)$ be a minimizing sequence of curvature varifolds with generalized second fundamental forms $\{A_k\}_{k\in\mathbb{N}}$ such that

$$\int_{G_m(N)} F(x, P, A_k) dV_k \downarrow \alpha^m_{N,F};$$

from Corollary 6.2.4 we already know that there exists $V \in CV_m(N)$ with generalized second fundamental form A such that, up to subsequences,

i) $(V_k, A_k) \rightharpoonup (V, A)$ in the weak sense of measure-function pairs,

ii) $\int_{G_m(N)} F(x, P, A) dV \leq \alpha^m_{N,F}$.

In order to have the existence of a minimizer we only have to check that V is not the zero varifold; this will be done in the next Subsection 6.3.1 using the estimates of Section 6.1.

6.3.1 Non degeneracy properties of the minimizing sequence

First of all, since $N \subset \mathbb{R}^S$, a curvature m-varifold V of N can be seen as a curvature varifold in $\mathbb{R}^S$ (for the precise value of the curvature function B in $\mathbb{R}^S$ see Remark 6.6.6); as before we write $V = V(M,\theta)$ where M is a rectifiable set and θ is the integer multiplicity function. Let us call $H^{\mathbb{R}^s}$ the weak mean curvature of V as integral m-varifold in $\mathbb{R}^S$ and, as in Section 6.1, let us denote by $\mu = \mu_V = \mathcal{H}^m \lfloor \theta = \pi_\sharp V$ the spatial measure associated to V and with $\operatorname{spt}\mu$ its support.

Lemma 6.3.1. *Let $N \subset\subset \bar{N}$ be a compact subset of the n-dimensional Riemannian manifold (N,g) isometrically embedded in some $\mathbb{R}^S$ (by Nash Embedding Theorem) and fix $p > 1$. Then there exists a constant $C_{N,p} > 0$ depending only on p and N such that for every $V = V(M,\theta) \in CV_m(N)$ curvature m-varifold of N*

$$\int_M \left| H^{\mathbb{R}^S} \right|^p d\mu \leq C_{N,p} \left(|V| + \int_{G_m(N)} |A|^p dV \right).$$

PROOF. Recall (see Remark 6.6.6) that it is possible to write the curvature function B of V seen as curvature m-varifold of $\mathbb{R}^S$ in terms of the second fundamental form A relative to $\bar{N}$ and the curvature of the manifold $\bar{N}$ seen as submanifold of $\mathbb{R}^S$ (the terms involving derivatives of Q):

$$B_{ijk} = A_{ij}^k + A_{ik}^j + P_{jl}P_{iq}\frac{\partial Q_{lk}}{\partial x_q}(x) + P_{kl}P_{iq}\frac{\partial Q_{lj}}{\partial x_q}(x).$$

From Remark 6.6.10 the weak mean curvature $H^{\mathbb{R}^S}$, which is a vector of $\mathbb{R}^S$, can be written in terms of B as

$$\left(H^{\mathbb{R}^S}\right)_i = \sum_{j=1}^S B_{jij} = \sum_{j=1}^S \left(A_{ji}^j + A_{jj}^i + P_{il}P_{jq}\frac{\partial Q_{lj}}{\partial x_q}(x) + P_{jl}P_{jq}\frac{\partial Q_{li}}{\partial x_q}(x) \right) \qquad i = 1\ldots,S.$$

Notice that, since $N \subset\subset \bar{N}$ is a compact subset of the manifold $\bar{N}$ smoothly embedded in $\mathbb{R}^S$, the functions $\frac{\partial Q_{lj}}{\partial x_m}$ are uniformly bounded by a constant C_N depending on the embedding $N \hookrightarrow \mathbb{R}^S$; moreover the P_{jm} are projection matrices so they are also uniformly bounded and we can say that

$$\left| \left(\sum_{j,l,m=1}^S P_{il}P_{jq}\frac{\partial Q_{lj}}{\partial x_q} + P_{jl}P_{jq}\frac{\partial Q_{li}}{\partial x_q} \right)_{i=1,\ldots,S} \right| \leq C_N$$

as vector of $\mathbb{R}^S$.
About the first term observe that, from the triangle inequality applied to the $\mathbb{R}^S$-vectors $(A_{ji}^j)_{i=1,\ldots,S}$ (j is fixed for each single vector),

$$\left| \left(\sum_{j=1}^S A_{ji}^j \right)_{i=1,\ldots,S} \right| \leq \sum_{j=1}^S \left| (A_{ji}^j)_{i=1,\ldots,S} \right| \leq S|A|$$

where, of course $|A| := \sqrt{\sum_{i,j,k=1}^S (A_{jk}^i)^2} \geq |(A_{ji}^j)_{i=1,\ldots,S}|$ for all $j = 1,\ldots,S$. The second adding term is analogous.
Putting together the two last estimates, by a triangle inequality, we have

$$\left| H^{\mathbb{R}^S} \right| \leq 2S|A| + C_N.$$

Using the standard inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for $a,b \geq 0$ and $p > 1$ given by the convexity of the function $t \mapsto t^p$ for $t \geq 0, p > 1$ we can write

$$\left| H^{\mathbb{R}^S} \right|^p \leq C_{N,p}\left(|A|^p + 1\right). \tag{6.28}$$

With an integration we get the conclusion. ■

Using the estimates of Section 6.1 and the last Lemma we have uniform lower bounds on the mass and on the diameter of the spatial support of a curvature m-varifold $V \in CV_m(N)$ of N with bounded $\int_{G_m(N)} |A|^p dV$, $p > m$.

Theorem 6.3.2. *Let $N \subset\subset \bar{N}$ be a compact subset of the n-dimensional Riemannian manifold (N,g) isometrically embedded in some $\mathbb{R}^S$ (by Nash Embedding Theorem) and fix $m \leq n-1$, $p > m$.*

Then there exists a constant $C_{N,p,m} > 0$ depending only on p, m and on the embedding of N into $\mathbb{R}^S$ such that $C_{N,p,m} \uparrow +\infty$ as $p \downarrow m$ and such that for every $V = V(M,\theta) \in CV_m(N)$ curvature m-varifold of N with spatial measure μ

$$i) \qquad \operatorname{diam}_{\bar{N}}(\operatorname{spt}\mu) \geq \frac{1}{C_{N,p,m}\left(|V| + \int_{G_m(N)} |A|^p dV\right)^{\frac{1}{p-m}}} \tag{6.29}$$

where $\operatorname{diam}_{\bar{N}}(\operatorname{spt}\mu)$ is the diameter of $\operatorname{spt}\mu$ as a subset of the Riemannian manifold $\bar{N}$;

$$ii) \qquad C_{N,p,m}\,|V| \left(|V| + \int_{G_m(N)} |A|^p dV\right)^{\frac{m}{p-m}} \geq 1. \tag{6.30}$$

Notice that $ii)$ implies the existence of a constant $a_{N,m,p,\int|A|^p} > 0$ depending only on p, m, on $\int_{G_m(N)} |A|^p dV$ and on the embedding of N into $\mathbb{R}^S$, with $a_{N,p,m,\int|A|^p} \downarrow 0$ if $p \downarrow m$ or if $\int_{G_m(N)} |A|^p dV \uparrow +\infty$ such that

$$|V| \geq a_{N,p,m,\int|A|^p} > 0.$$

Proof.

$i)$ From Lemma 6.1.5

$$diam_{\bar{N}}(\operatorname{spt}\mu) \geq diam_{\mathbb{R}^S}(\operatorname{spt}\mu) \geq \frac{1}{C_{p,m}\left(\int_M |H|^p d\mu\right)^{\frac{1}{p-m}}}$$

where $C_{p,m} > 0$ is a positive constant depending on p, m and such that $C_{p,m} \to \infty$ if $p \downarrow m$. The conclusion follows plugging into the last inequality the estimate of Lemma 6.3.1.

$ii)$ From Lemma 6.1.6,

$$|V| \geq \frac{1}{C_{p,m}\left(\int_M |H|^p d\mu\right)^{\frac{m}{p-m}}}$$

with $C_{p,m} > 0$ as above. The conclusion, again, follows plugging into the last inequality the estimate of Lemma 6.3.1 and rearranging. ■

Corollary 6.3.3. *Let $N \subset\subset \bar{N}$ be a compact subset with non empty interior, $int(N) \neq \emptyset$, of the n-dimensional Riemannian manifold (N,g) isometrically embedded in some $\mathbb{R}^S$ (by Nash Embedding Theorem) and fix $m \leq n-1$.*

Assume that the space (N,g) does not contain any non zero m-varifold with null generalized second fundamental form and consider a function $F : G_m(N) \times \mathbb{R}^{S^3} \to \mathbb{R}^+$ satisfying (6.6.2), (6.27) and a corresponding minimizing sequence of curvature m-varifolds $\{V_k\}_{k\in\mathbb{N}} \subset CV_m(N)$ with generalized second fundamental forms $\{A_k\}_{k\in\mathbb{N}}$ such that

$$\int_{G_m(N)} F(x,P,A_k)dV_k \downarrow \alpha^m_{N,F}$$

(for the definition of $\alpha^m_{N,F}$ see (6.22)). Then, called μ_k the spatial measures associated to V_k, there exists a constant $a_{N,F,m} > 0$ depending only on N,F and m such that

$$i) \qquad diam_{\bar{N}}(\operatorname{spt}\mu_k) \geq a_{N,F,m} \tag{6.31}$$

$$ii) \qquad |V_k| \geq a_{N,F,m}. \tag{6.32}$$

Proof. From Theorem 6.2.1 and the finiteness of $\alpha^m_{N,F}$, since (N,g) does not contain any non zero m-varifold with null generalized second fundamental form,

$$|V_k| \leq C_{N,F,m} \int_{G_m(N)} F(x,P,A_k)dV_k \leq C_{N,F,m}$$

for some $C_{N,F,m} > 0$ depending only on N, F and m.
Moreover, since (by assumption (6.27)) $F(x, P, A) \geq C|A|^p$ for some $p > m$ and $C > 0$, the boundness of $\int_{G_m(N)} F(x, P, A_k)dV_k$ implies that

$$\int_{G_m(N)} |A_k|^p dV_k \leq C_{N,F,m}$$

for some $C_{N,F,m}>0$ depending only on N, F and m.

The conclusion follows putting the last two inequalities into Theorem 6.3.2. ■

6.3.2 Existence and regularity of the minimizer

Collecting Corollary 6.2.4 and Corollary 6.3.3 we can finally state and prove the first main Theorem 1.0.17.

Proof of Theorem 1.0.17

If $a)$ is true we are done, so we can assume that $a)$ is not satisfied.

Let $\{V_k\}_{k\in\mathbb{N}} \subset CV_m(N)$ with generalized second fundamental forms $\{A_k\}_{k\in\mathbb{N}}$ be a minimizing sequence of $\alpha^m_{N,F}$:

$$\int_{G_m(N)} F(x, P, A_k)dV_k \downarrow \alpha^m_{N,F}.$$

Called μ_k the spatial measures associated to V_k notice that, since the integrand F is non negative, we can assume that the spatial supports $\operatorname{spt}\mu_k$ are connected (indeed, from Definition 6.6.5, using cut off functions it is clear that every connected component of $\operatorname{spt}\mu_k$ is the spatial support of a curvature varifold). From Corollary 6.3.3 we have the lower bounds:

$$\begin{aligned} &i) \qquad diam_{\bar{N}}(\operatorname{spt}\mu_k) \geq a_{N,F,m} \\ &ii) \qquad |V_k| \geq a_{N,F,m}, \end{aligned}$$

for a constant $a_{N,F,m} > 0$ depending only on N,F and m. Corollary 6.2.4 implies the existence of a curvature m-varifold $V = V(M, \theta) \in CV_m(N)$ with generalized second fundamental form A such that, up to subsequences,

i) $(V_k, A_k) \rightharpoonup (V, A)$ in the weak sense of measure-function pairs of N,

ii) $\int_{G_m(N)} F(x, P, A)dV \leq \alpha^m_{N,F}$.

The measure-function pair convergence implies the varifold convergence of $V_k \to V$ and the convergence of the associated spatial measures

$$\pi_\sharp V_k =: \mu_k \to \mu := \pi_\sharp V \quad \text{weak convergence of Radon measures on } N.$$

It follows that

$$0 < a_{N,F,m} \leq |V_k| = \mu_k(N) \to \mu(N) = |V|,$$

thus $V \neq 0$ is a minimizer for $\alpha^m_{N,F}$.

Notice that, since $N \hookrightarrow \mathbb{R}^S$ is properly embedded, the weak convergence $\mu_k \to \mu$ on N implies the weak convergence of $\mu_k \to \mu$ as Radon measures on $\mathbb{R}^S$. From mass bound on the V_k and the bound on $\int_{G_m(N)} |A_k|^p dV_k$ given by the assumption (6.27) on F, Lemma 6.3.1 allows us to apply Proposition 6.1.7 and we can say that the spatial supports

$$\operatorname{spt}\mu_k \to \operatorname{spt}\mu \quad \text{Hausdorff convergence as subsets of } \mathbb{R}^S.$$

Notice that, since $\bar{N} \hookrightarrow \mathbb{R}^S$ is embedded, the Hausdorff convergence of $M_k \to M$ as subsets of $\mathbb{R}^S$ implies

$$\operatorname{spt}\mu_k \to \operatorname{spt}\mu \quad \text{Hausdorff convergence as subsets of } \bar{N},$$

and this implies that

$$0 < a_{N,F,m} \leq \lim_k \operatorname{diam}_{\bar{N}}(\operatorname{spt}\mu_k) = diam_{\bar{N}}(\operatorname{spt}\mu),$$

hence $b2$). Moreover the Hausdorff limit of connected subsets is connected thus also $b1$) is proved.

Now the minimizer $V \in CV(N)$ is a non null curvature varifold on N with generalized second fundamental form A (relative to $\bar{N}$) in $L^p(V)$ for some $p > m$. Since $N \hookrightarrow \mathbb{R}^S$, V can also be seen as a varifold in $\mathbb{R}^S$ and Remark 6.6.6 tell that V is actually a varifold with generalized curvature function B given by

$$B_{ijk} = A^k_{ij} + A^j_{ik} + P_{jl}P_{iq}\frac{\partial Q_{lk}}{\partial x_q}(x) + P_{kl}P_{iq}\frac{\partial Q_{lj}}{\partial x_q}(x)$$

where the terms of the type $P_{jl}P_{iq}\frac{\partial Q_{lk}}{\partial x_q}(x)$ represent the extrinsic curvature of $\bar{N}$ as a submanifold of $\mathbb{R}^S$ and, of course, are bounded on N from the compactness:

$$\sup_{x\in N}\left|P_{jl}P_{iq}\frac{\partial Q_{lk}}{\partial x_q}(x) + P_{kl}P_{iq}\frac{\partial Q_{lj}}{\partial x_q}(x)\right| \leq C_N.$$

Hence, from triangle inequality,

$$|B| \leq 2|A| + C_N$$

and

$$|B|^p \leq C_{N,p}\left(|A|^p + 1\right).$$

Using the mass bound $|V| = \lim_k |V_k| \leq C < \infty$, with an integration we get

$$\int_{G_m(\mathbb{R}^S)} |B|^p dV < \infty.$$

Under this conditions Hutchinson shows in [Hu2] that V is a locally a graph of multivalued $C^{1,\alpha}$ functions and that $b3$) holds. ■

6.4 Existence of an integral m-varifold with weak mean curvature minimizing $\int |H|^p$ for $p > m$

As before, throughout this Section $(\bar{N}, g)$ stands for a compact n-dimensional Riemannian manifold isometrically embedded in some $\mathbb{R}^S$ (by Nash Embedding Theorem) and $N \subset\subset \bar{N}$ is a compact subset with non empty interior (as subset of N). Fix $m \leq n-1$; analogously to Section 6.3 we will focus our attention to the case

$$\begin{aligned} F \quad &: \quad G_m(N) \times \mathbb{R}^S \to \mathbb{R}^+ \text{ is a function satisfying (6.6.2)} \\ F(x,P,H) \quad &\geq \quad C|H|^p \text{ for some } p > m \text{ and } C > 0. \end{aligned} \tag{6.33}$$

Recall that we are considering the minimization problem

$$\beta^m_{N,F} := \inf\left\{\int_{G_m(N)} F(x,P,H^N)dV : V \in HV_m(N), V \neq 0 \text{ with weak mean curvature } H^N \text{ relative to } \bar{N}\right\}.$$

Our goal is to prove the existence of a minimizer for $\beta^m_{N,F}$, F as in (6.33).

As in Section 6.3 we consider a minimizing sequence $\{V_k\}_{k\in\mathbb{N}} \subset HV_m(N)$ of integral m-varifolds with weak mean curvatures $\{H^N_k\}_{k\in\mathbb{N}}$ relative to $\bar{N}$ such that

$$\int_{G_m(N)} F(x,P,H^N_k)dV_k \downarrow \beta^m_{N,F};$$

from Corollary 6.2.8 we already know that there exists $V \in HV_m(N)$ with with weak mean curvature H^N relative to $\bar{N}$ such that, up to subsequences,

i) $(V_k, H^N_k) \rightharpoonup (V, H^N)$ in the weak sense of measure-function pairs,

ii) $\int_{G_m(N)} F(x,P,H^N)dV \leq \beta^m_{N,F}$.

In order to have the existence of a minimizer we only have to check that V is not the zero varifold; this will be done analogously to Subsection 6.3.1 using the estimates of Section 6.1.

As before, since $N \subset \mathbb{R}^S$, an integral m-varifold V of N with weak mean curvature H^N relative to $\bar{N}$ can be seen as integral m-varifold of $\mathbb{R}^S$ with weak mean curvature $H^{\mathbb{R}^S}$. We write $V = V(M,\theta)$ where M is a rectifiable set and θ is the integer multiplicity function; finally, as in Section 6.1, let us denote by $\mu = \mu_V = \mathcal{H}^m \llcorner \theta = \pi_\sharp V$ the spatial measure associated to V and with $\operatorname{spt}\mu$ the spatial support of V.

Lemma 6.4.1. *Let $N \subset\subset \bar{N}$ be a compact subset of the n-dimensional Riemannian manifold (N, g) isometrically embedded in some $\mathbb{R}^S$ (by Nash Embedding Theorem) and fix $p > 1$. Then there exists a constant $C_{N,p} > 0$ depending only on p and N such that for every $V = V(M, \theta) \in HV_m(N)$ integral m-varifold of N with weak mean curvature H^N relative to $\bar{N}$*

$$\int_M \left|H^{\mathbb{R}^S}\right|^p d\mu \leq C_{N,p}\left(|V| + \int_{G_m(N)} |H^N|^p dV\right).$$

Proof. By Definition 6.6.11 we can express

$$(H^{\mathbb{R}^S})_i = (H^N)_i + P_{jk}\frac{\partial Q_{ij}}{\partial x^k}$$

and from the triangle inequality

$$\left|H^{\mathbb{R}^S}\right| \leq \left|H^N\right| + \left|P_{jk}\frac{\partial Q_{ij}}{\partial x^k}\right|; \tag{6.34}$$

as vectors in $\mathbb{R}^S$. The second summand of the right hand side is a smooth function on the compact set $G_m(N)$ hence bounded by a constant C_N depending on N:

$$\left|P_{jk}\frac{\partial Q_{ij}}{\partial x^k}\right| \leq C_N.$$

Using the standard inequality $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ for $a, b \geq 0$ and $p > 1$ we get

$$\left|H^{\mathbb{R}^S}\right|^p \leq C_{N,p}(1 + \left|H^N\right|^p)$$

which gives the thesis with an integration. ■

Remark 6.4.2. *An analogous result to Theorem 6.3.2 holds, just replace $V = V(M, \theta) \in CV_m(N)$ with $V = V(M, \theta) \in HV_m(N)$ and $\int_{G_m(N)} |A|^p dV$ with $\int_{G_m(N)} |H^N|^p dV$.*

Now we can show the non degeneracy of the minimizing sequence for $\beta^m_{N,F}$, F as in 6.6.2, (6.33).

Lemma 6.4.3. *Let $N \subset\subset \bar{N}$ be a compact subset with non empty interior, $int(N) \neq \emptyset$, of the n-dimensional Riemannian manifold (N, g) isometrically embedded in some $\mathbb{R}^S$ (by Nash Embedding Theorem) and fix $m \leq n - 1$.*

Assume that the space (N, g) does not contain any non zero m-varifold with null weak mean curvature H^N relative to $\bar{N}$ and consider a function $F : G_m(N) \times \mathbb{R}^S \to \mathbb{R}^+$ satisfying (6.6.2), (6.33) and a corresponding minimizing sequence of integral m-varifolds $\{V_k\}_{k\in\mathbb{N}} \subset HV_m(N)$ with weak mean curvatures $\{H^N_k\}_{k\in\mathbb{N}}$ relative to $\bar{N}$ such that

$$\int_{G_m(N)} F(x, P, H^N_k)dV_k \downarrow \beta^m_{N,F}$$

(for the definition of $\beta^m_{N,F}$ see (6.26)). Then, called μ_k the spatial measures of V_k, there exists a constant $b_{N,F,m} > 0$ depending only on N,F and m such that

$$i) \quad \operatorname{diam}_{\bar{N}}(\operatorname{spt}\mu_k) \geq b_{N,F,m} \tag{6.35}$$

$$ii) \quad |V_k| \geq b_{N,F,m}. \tag{6.36}$$

Proof. From Theorem 6.2.6 and the finiteness of $\beta^m_{N,F}$, since (N, g) does not contain any non zero m-varifold with null weak mean curvature H^N relative to $\bar{N}$,

$$|V_k| \leq C_{N,F,m} \int_{G_m(N)} F(x, P, H^N_k)dV_k \leq C_{N,F,m}$$

for some $C_{N,F,m} > 0$ depending only on N, F and m.
Moreover, since (by assumption (6.33)) $F(x, P, H^N) \geq C|H^N|^p$ for some $p > m$ and $C > 0$, the boundness of $\int_{G_m(N)} F(x, P, H_k^N) dV_k$ implies that

$$\int_{G_m(N)} |H_k^N|^p dV_k \leq C_{N,F,m}$$

for some $C_{N,F,m}{>}0$ depending only on N, F and m.

The conclusion follows from the last two inequalities and Remark 6.4.2. ■

Now, collecting Corollary 6.2.8 and Lemma 6.4.3 we can finally state and prove Theorem 1.0.15, namely the existence of a non trivial minimizer for $\beta_{N,F}^m$, F as in 6.6.2, (6.33).

Proof of Theorem 1.0.15

If a) is true we are done, so we can assume that a) is not satisfied.

Let $\{V_k\}_{k \in \mathbb{N}} \subset HV_m(N)$ with weak mean curvatures $\{H_k^N\}_{k \in \mathbb{N}}$ be a minimizing sequence of $\beta_{N,F}^m$:

$$\int_{G_m(N)} F(x, P, H_k^N) dV_k \downarrow \beta_{N,F}^m.$$

Called μ_k the spatial measures of V_k notice that, since the integrand F is non negative, we can assume that the spatial supports $\operatorname{spt} \mu_k$ are connected (indeed, as for the curvature varifolds, every connected component of $\operatorname{spt} \mu_k$ is the spatial support of a mean curvature varifold). From Lemma 6.4.3 we have the lower bounds:

$$\begin{array}{ll} i) & diam_{\bar{N}}(\operatorname{spt} \mu_k) \geq b_{N,F,m} > 0 \\ ii) & |V_k| \geq b_{N,F,m} > 0, \end{array}$$

for a constant $b_{N,F,m} > 0$ depending only on N,F and m. Corollary 6.2.8 implies the existence of an integral m-varifold $V \in HV_m(N)$ with weak mean curvature H^N relative to $\bar{N}$ such that, up to subsequences,

i) $(V_k, H_k^N) \rightharpoonup (V, H^N)$ in the weak sense of measure-function pairs of N,

ii) $\int_{G_m(N)} F(x, P, H^N) dV \leq \beta_{N,F}^m$.

Analogously to the proof of Theorem 1.0.17, one shows that

$$0 < b_{N,F,m} \leq |V_k| = \mu_k(N) \to \mu(N) = |V|,$$

thus $V \neq 0$ is a minimizer for $\beta_{N,F}^m$. The proof of $b1$) and $b2$) are again analogous to the proof of the corresponding sentences in Theorem 1.0.17: from the mass bound on the V_k and the bound on $\int_{G_m(N)} |H_k^N|^p dV_k$ given by the assumption (6.33) on F, Lemma 6.4.1 allows us to apply Proposition 6.1.7 and, using the same tricks of Theorem 1.0.17 we can say that the spatial supports

$$\operatorname{spt} \mu_k \to \operatorname{spt} \mu \quad \text{Hausdorff convergence as subsets of } \bar{N},$$

and this implies that

$$0 < b_{N,F,m} \leq \lim_k \operatorname{diam}_{\bar{N}}(\operatorname{spt} \mu_k) = \operatorname{diam}_{\bar{N}}(\operatorname{spt} \mu),$$

hence $b2$). Moreover the Hausdorff limit of connected subsets is connected thus also $b1$) is proved. ■

6.5 Examples and Remarks

First of all let us point out that our setting includes, speaking about ambient manifolds, a large class of Riemannian manifolds with boundary.

Remark 6.5.1. *Notice that if N is a compact n-dimensional manifold with boundary then there exists an n-dimensional (a priori non compact) manifold $\bar{N}$ without boundary such that N is a compact subset of $\bar{N}$ (to define $\bar{N}$ just extend N a little beyond ∂N in the local boundary charts). Hence, given a compact n-dimensional Riemannian manifold (N, g) with boundary such that the metric g can be extended in a smooth and non degenerate way (i.e. g positive definite) up to the boundary ∂N, then N is isometric to a compact subset of an n-dimensional Riemannian manifold $(\bar{N}, \bar{g})$ without boundary.*

Thus all the Lemmas, Propositions and Theorems apply to the case in which the ambient space is a Riemannian manifold with boundary with the described non degeneracy property at ∂N.

Now let us show that the main results Theorem 1.0.17 and Theorem 1.0.15 are non empty, i.e we have examples of compact subsets of Riemannian manifolds where do not exist non zero varifolds with null weak mean curvature relative to $\bar{N}$ and a fortiori there exists no non zero varifold with null generalized second fundamental form. Let us start with an easy Lemma:

Lemma 6.5.2. *Let $N \subset\subset \bar{N}$ be a compact subset of the n-dimensional Riemannian manifold (N, g) isometrically embedded in some $\mathbb{R}^S$ (by Nash Embedding Theorem), fix $m \leq n-1$ and assume that N contains no non zero m-varifold with null weak mean curvature relative to $\bar{N}$. Then N does not contain any non zero m-varifold with null generalized second fundamental form.*

Proof. We show that if the varifold V has null generalized second fundamental form relative to $\bar{N}$ then V also has null weak mean curvature relative to $\bar{N}$. Indeed let V be a varifold on N with generalized curvature function B and second fundamental form A relative to $\bar{N}$, then, from Remark 6.6.6,

$$B_{ijk} = A_{ij}^k + A_{ik}^j + P_{jl}P_{ip}\frac{\partial Q_{lk}}{\partial x_p}(x) + P_{kl}P_{ip}\frac{\partial Q_{lj}}{\partial x_p}(x)$$

where P and $Q(x)$ are the projection matrices on $P \in G_m(N)$ and $T_x\bar{N}$. Moreover, from Remark 6.6.10, V has weak mean curvature as a varifold in $\mathbb{R}^S$

$$(H^{\mathbb{R}^S})_i = B_{jij};$$

hence, if the generalized second fundamental form A is null, then

$$(H^{\mathbb{R}^S})_i = P_{il}P_{jk}\frac{\partial Q_{lj}}{\partial x_k}(x) + P_{jl}P_{jk}\frac{\partial Q_{li}}{\partial x_k}(x).$$

It is not hard to check that the first summand of the right hand side is null (fix a point x of $\bar{N}$ and choose a base of $T_x\bar{N}$ in which the Christoffel symbols of $\bar{N}$ vanish at x; write down the orthogonal projection matrix Q with respect to this base and check the condition in this frame). Thus $H_i^{\mathbb{R}^S} = P_{jk}\frac{\partial Q_{ij}}{\partial x^k}$ and Definition 6.6.11 gives

$$(H^N)_i = (H^{\mathbb{R}^S})_i - P_{jk}\frac{\partial Q_{ij}}{\partial x_k}(x) = 0.$$

■

Collecting Lemma 6.5.2 and Remark 6.6.12 we can say that if a compact subset $N \subset\subset \bar{N}$ has a non zero m-varifold with null generalized second fundamental form, then a fortiori N contains a non zero m-varifold with null weak mean curvature relative to $\bar{N}$, then a fortiori N contains a non zero m-varifold with null first variation relative to $\bar{N}$ (recall, see Remark 6.6.13, that a varifold with null first variation is also called stationary varifold). Hence it is enough to give examples of compact subsets of Riemannian manifolds which do not contain any non zero m-varifold with null first variation relative to $\bar{N}$.

First, we mention two examples given by White in [Whi] (for the proofs we refer to the original paper) next we will propose a couple of new examples which can be seen as a sort of generalization of White's ones. Recall that if N is a compact Riemannian manifold with smooth boundary, N is said to be *mean convex* provided that the mean curvature vector at each point of ∂N is an nonnegative multiple of the inward-pointing unit normal.

Example 6.5.1. *Suppose that N is a compact, connected, mean convex Riemannian manifold with smooth, nonempty boundary, and that no component of ∂N is a minimal surface. Suppose also that the dimension n of N is at most 7 and that the Ricci curvature of N is everywhere positive. Then N contains no non zero $(n-1)$-varifold with null first variation relative to N (i.e. stationary $n-1$-varifold).*

More generally, if N has nonnegative Ricci curvature, then the same conclusion holds unless N contains a closed, embedded, totally geodesic hypersurface M such that $Ric(\nu, \nu) = 0$ for every unit normal ν to M (where Ric is the Ricci tensor of N).

Minimal surfaces in ambient manifolds of the form $M \times \mathbb{R}$ have been deeply studied in recent years (see for example [MeRo04], [MeRo05] and [NeRo02]); notice that $M \times \mathbb{R}$ is foliated by the minimal surfaces $M \times \{z\}$. In the second example we can see that very general compact subsets of ambient spaces admitting such foliations do not contain non zero codimension 1 varifolds with null first variation.

Example 6.5.2. *Let $\bar{N}$ be an n-dimensional Riemannian manifold. Let $f : \bar{N} \to \mathbb{R}$ be a smooth function with nowhere vanishing gradient such that the level sets of f are minimal hypersurfaces or, more generally, such that the sublevel sets $\{x : f(x) \leq z\}$ are mean convex. Let N be a compact subset of $\bar{N}$ such that for each $z \in \mathbb{R}$, no connected component of $f^{-1}(z)$ is a minimal hypersurface lying entirely in N. Then N contains no non zero $n-1$-varifold with null first variation relative to $\bar{N}$.*

Observe that both examples concern the non-existence of *codimension 1* stationary varifolds: next we propose a couple of new examples in higher codimension. We need the following maximum principle for stationary (i.e. with null first variation) varifolds given by White, for the proof see [Whi2], Theorem 1. Before stating it recall that if N is an n-dimensional Riemannian manifold with boundary ∂N, N is said *strongly m-convex* at a point $p \in \partial N$ provided

$$k_1 + k_2 + \ldots + k_m > 0$$

where $k_1 \leq k_2 \leq \ldots \leq k_{n-1}$ are the principal curvatures of ∂N at p with respect to the unit normal ν_N that points into N.

Theorem 6.5.3. *Let $\bar{N}$ be a smooth Riemannian manifold of dimension n, let $N \subset \bar{N}$ be a smooth Riemannian n-dimensional manifold with boundary, and assume p to be a point in ∂N at which N is strongly m-convex. Then p is not contained in the support of any m-varifold in N with null first variation relative to $\bar{N}$.*

Actually the Theorem of White is more general and precise, but for our purposes this weaker version is sufficient.

Now are ready to state and prove the two examples.

Theorem 6.5.4. *Let $\bar{N}$ be an n-dimensional Riemannian manifold and consider as ambient manifold $\bar{N} \times \mathbb{R}^S, s > 1$ with the product metric. Then any compact subset $N \subset\subset \bar{N} \times \mathbb{R}^S$ does not contain any non null stationary $n+k$-varifold, $k = 1, \ldots, s-1$ (i.e. $n+k$-varifold with null first variation relative to $\bar{N} \times \mathbb{R}^S$).*

Proof. Assume by contradiction that V is a non null $n+k$-varifold in N with null first variation in $\bar{N} \times \mathbb{R}^S$ for some $1 \leq k \leq s-1$. Consider the function $\rho : \bar{N} \times \mathbb{R}^S \to \mathbb{R}^+$ defined as

$$\bar{N} \times \mathbb{R}^S \ni (x, y) \mapsto \rho(x, y) := |y|_{\mathbb{R}^S}$$

where of course $|y|_{\mathbb{R}^S}$ is the norm of y as vector of $\mathbb{R}^S$. With abuse of notation, call $M \subset N$ the spatial support of V (now M may not be rectifiable, it is just compact); observe that, since M is compact, then the function ρ restricted to M has a maximum $r > 0$ at the point $(x_0, y_0) \in M \subset \bar{N} \times \mathbb{R}^S$ (observe that the maximum r has to be strictly positive otherwise we would have a non null $n+k$-varifold in an n-dimensional space, which clearly is not possible by the very definition of varifold). It follows that, called $\bar{N}_r$ the tube of center $\bar{N}$ and radius r

$$\bar{N}_r := \{(x, y) : x \in \bar{N}, |y|_{\mathbb{R}^S} \leq r\},$$

the spatial support of V is contained in $\bar{N}_r$:

$$M \subset \bar{N}_r. \tag{6.37}$$

Moreover M is tangent to the hypersurface $C_r := \partial \bar{N}_r = \{(x, y) : x \in \bar{N}, |y|_{\mathbb{R}^S} = r\}$ at the point (x_0, y_0). Observe that C_r is diffeomorphic to $\bar{N} \times rS^{s-1}_{\mathbb{R}^S}$, where of course $rS^{s-1}_{\mathbb{R}^S}$ is the $s-1$-dimensional sphere of $\mathbb{R}^S$ of radius r centered in the origin.

Using normal coordinates in $\bar{N} \times \mathbb{R}^S$ it is a simple exercise to observe that the principal curvatures of C_r with respect to the inward pointing unit normal are constantly

$$k_1 = k_2 = \ldots = k_n = 0, \ k_{n+1} = k_{n+2} = \ldots = k_{s-1} = \frac{1}{r}$$

(just observe that the inward unit normal is $-\Theta$, where Θ is the radial vector which parametrizes $S^{s-1}_{\mathbb{R}^S}$; of course $-\Theta$ is constant respect to the x coordinates; using normal coordinates one checks that the second fundamental form is made of two blocks: the one corresponding to $\bar{N}$ is null and the other one coincides with the second fundamental form of $S^{s-1}_{\mathbb{R}^S}$ as hypersurface in $\mathbb{R}^S$).

It follows that $C_r = \partial \bar{N}_r$ is strongly $n+k$-convex in all of its points, for all $1 \leq k \leq n-1$; but V is a non null $n+k$-varifold in $\bar{N}_r$ with null first variation relative to $\bar{N}$ and tangent to C_r at the point $(x_0, y_0) \in C_r \cap M$. Fact which contradicts the maximum principle, Theorem 6.5.3. ■

As a corollary we have an example in all the codimensions in $\mathbb{R}^S$:

Theorem 6.5.5. *Let $N \subset\subset \mathbb{R}^S$ be a compact subset of $\mathbb{R}^S$, $s > 1$. Then, for all $1 \leq m \leq s-1$, N contains no non zero m-varifold with null first variation relative to $\mathbb{R}^S$.*

Proof. Just take $\bar{N} := \{x\}$ in the previous example, Theorem 6.5.4, and observe that $\{x\} \times \mathbb{R}^S$ is isometric to $\mathbb{R}^S$.

Otherwise argue by contradiction as in the proof of Theorem 6.5.4 and observe that the support of the non zero m-varifold with null first variation is contained in a ball of $\mathbb{R}^S$ and tangent to its boundary, namely a sphere. Of course the sphere is strongly m-convex; it follows a contradiction with the maximum principle, Theorem 6.5.3.

■

Remark 6.5.6. *Recall that if the ambient n-dimensional Riemannian manifold N is compact without boundary, then Almgren proved in [Alm] that for every $1 \leq m < n$ there exists an integral m-varifold with null first variation relative to N. Moreover, in the same setting of compact N and $\partial N = \emptyset$, Schoen and Simon [ShSim81], using the work of Pitts [Pit81], proved that N must contain a closed, embedded hypersurface with singular set of dimension at most $n-7$. Hence, the isoperimetric inequality Theorem 6.2.6 fails for such N and the Theorem 1.0.15 is trivially true. However, as written above, there are many interesting examples of ambient manifolds with boundary where the Theorem is non trivial.*

Remark 6.5.7. *It is known that the ambient Riemannian n-manifolds, $n \geq 3$ (with or without boundary) which contain a smooth m-dimensional submanifold, $m \geq 2$, with null second fundamental form (i.e a totally geodesic submanifold) are quite rare. It could be interesting to show the same in the context of varifolds, that is to prove that the ambient compact Riemannian n-manifolds, $n \geq 3$ (with or without boundary) which contain a non zero (a priori non rectifiable) m-varifold, $m \geq 2$, with null second fundamental form relative to N (see Definition 6.6.5) are quite rare. This fact would imply the existence of a larger class of ambient Riemannian manifolds where the isoperimetric inequality Theorem 6.2.1 holds and the main Theorem 1.0.17 is non trivial.*

6.6 Appendix: some basic facts about varifold theory

Since throughout the thesis we use the theory of varifolds, in order to make the exposition as much as possible self-contained, we recall here some basic useful facts. In particular we review the concept of curvature varifold introduced by Hutchinson in [Hu1] giving a slightly more general definition; namely Hutchinson defines the curvature varifolds as "special" integral varifolds in a Riemannian manifold but, as a matter of fact, the same definition makes sense for an even non rectifiable varifold in a subset of a Riemannian manifold. So we will define (a priori non rectifiable) varifolds with curvature, which are endowed with a generalized second fundamental form.

We start by recalling the basic definitions. For more material about the general theory, the interested reader may look at the standard references [Fed], [Mor], [SiGMT] or, for faster introductions, at [Mant] or the appendix of [Whi].

Consider a (maybe non compact) n-dimensional Riemannian manifold $(\bar{N}, g)$. Without loss of generality, by the Nash Theorem, we can assume that

$$(\bar{N}, g) \hookrightarrow \mathbb{R}^S \text{ isometrically embedded for some } S > 0.$$

We will be concerned with a subset $N \subset \bar{N}$ which, a fortiori, is also embedded in $\mathbb{R}^S$: $N \hookrightarrow \mathbb{R}^S$. Since throughout the thesis we reduce ourself to the case when $N \subset\subset \bar{N}$ is a compact subset (to avoid pathological behavior we will also assume that it has non empty interior $int(N) \neq \emptyset$) also in this appendix it is assumed to be so, even if most of the following definitions and properties are valid for more general subsets.

Let us denote by $G(S, m)$ the Grassmannian of unoriented m-dimensional linear subspaces of $\mathbb{R}^S$, with

$$G_m(\bar{N}) := (\mathbb{R}^S \times G(S, m)) \cap \{(x, P) : x \in \bar{N}, P \subset T_x\bar{N} \text{ } m\text{-dimensional linear subspace}\}$$

and with

$$G_m(N) := G_m(\bar{N}) \cap \{(x, P) : x \in N\}.$$

We recall that a *m-varifold* V on N is a Radon measure on $G_m(N)$ and that the sequence of varifolds $\{V_k\}_{k\in\mathbb{N}}$ converges to the varifold V in varifold sense if $V_k \to V$ weak as Radon measures on $G_m(N)$; i.e.

$$\int_{G_m(N)} \phi \, dV_k \to \int_{G_m(N)} \phi \, dV$$

as $k \to \infty$, for all $\phi \in C_c^0(G_m(N))$. A special class of varifolds are the *rectifiable varifolds*: given a countably m-rectifiable, $\mathcal{H}^m$- measurable subset M of $N \subset \mathbb{R}^S$ and θ a non negative locally $\mathcal{H}^m$ integrable function on M, the *rectifiable varifold* V associated to M and θ is defined as

$$V(\phi) := \int_M \theta(x)\phi(x, T_x M) d\mathcal{H}^m \quad \forall \phi \in C_c^0(G_m(N))$$

and sometimes it is denoted with $V(M, \theta)$. Recall that if M, θ are as above then the approximate tangent space $T_x M$ exists for $\mathcal{H}^m$-almost every $x \in M$ (Theorem 11.6 in [SiGMT], for the definitions see 11.4 of the same book). If moreover θ is integer valued, then we say that V is an *integral varifold*; the set of the integral m-varifolds in N is denoted by $IV_m(N)$.

If V is a k-varifold, let $|V|$ denote its mass:

$$|V| := V(G_m(N)).$$

Observe that we have a natural projection

$$\pi : G_m(N) \to N \quad (x, P) \mapsto x, \tag{6.38}$$

and pushing forward the measure V via the projection π, we have a positive Radon measure μ_V on N

$$\mu_V(B) := V(\pi^{-1}(B)) = V(G_m(B)) \quad \forall B \subset N \text{ Borel set.}$$

Since V is a measure on $G_m(N)$, its support is a closed subset of $G_m(N)$. If we project that closed set on N by the projection π then we get the *spatial support* of V, which coincides with $\operatorname{spt} \mu_V$.

Now let us define the notion of measure-function pair.

Definition 6.6.1. *Let V be a Radon measure on $G_m(N)$ (i.e. a varifold) and $f : G_m(N) \to \mathbb{R}^\alpha$ be a well defined V almost everywhere $L^1_{loc}(V)$ function. Then we say that (V, f) is a* measure-function pair *over $G_m(N)$ with values in $\mathbb{R}^\alpha$.*

Given $\{(V_k, f_k)\}_{k\in\mathbb{N}}$ and (V, f) measure-function pairs over $G_m(N)$ with values in $\mathbb{R}^\alpha$, suppose $V_k \to V$ weak as Radon measures in $G_m(N)$ (or equivalently as varifolds in N). Then we say (V_k, f_k) converges to (V, f) in the weak sense *and write*

$$(V_k, f_k) \rightharpoonup (V, f)$$

if $V_k \lfloor f_k \to V \lfloor f$ weak convergence of Radon vector valued measures. In other words, if

$$\int_{G_m(N)} \langle f_k, \phi\rangle \, dV_k \to \int_{G_m(N)} \langle f, \phi\rangle \, dV$$

as $k \to \infty$, for all $\phi \in C_c^0(G_m(N), \mathbb{R}^\alpha)$, where $\langle ., . \rangle$ is the scalar product in $\mathbb{R}^\alpha$.

Definition 6.6.2. *Suppose $F : G_m(N) \times \mathbb{R}^\alpha \to \mathbb{R}$. We will denote the variables in $G_m(N) \times \mathbb{R}^\alpha$ by (x,P,q). We say that F satisfies the condition* (6.6.2) *if the following statements are verified:*
i) F is continuous,
ii) F is non negative (i.e. $F(x, P, q) \geq 0$ for all $(x, P, q) \in G_m(N) \times \mathbb{R}^\alpha$) and $F(x, P, q) = 0$ if and only if $q = 0$,
iii) F is convex in the q variables, i.e.

$$F(x, P, \lambda q_1 + (1 - \lambda)q_2) \leq \lambda F(x, P, q_1) + (1 - \lambda)F(x, P, q_2)$$

for all $\lambda \in (0, 1), (x, P) \in G_m(N), q_1 \in \mathbb{R}^\alpha, q_2 \in \mathbb{R}^\alpha$,
iv) F has non linear growth in the q variables, i.e. there exists a continuous function ϕ, where ϕ :

$G_m(N) \times [0,\infty) \to [0,\infty)$, $0 \le \phi(x,P,s) \le \phi(x,P,t)$ *for* $0 \le s \le t$ *and* $(x,P) \in G_m(N)$, $\phi(x,P,t) \to \infty$ *locally uniformly in* (x,P) *as* $t \to \infty$, *such that*

$$\phi(x,P,|q|)|q| \le F(x,P,q)$$

for all $(x,P,q) \in G_m(N) \times \mathbb{R}^\alpha$.

An example (trivial but fundamental for this thesis) of such an F is $F(x,P,q) := |q|^p$ for any $p > 1$.

Remark 6.6.3. *For simplicity, in Definition 6.6.2, we assumed the same conditions of Hutchinson ([Hu1] Definition 4.1.2) on* F *but some hypotheses can be relaxed. For example, about the results in this thesis, if* $F = F(q)$ *depends only on the* q *variables it is enough to assume (in place of i)) that* F *is lower semicontinuous (see Theorem 6.1 in [MantCVB]).*

In the aforementioned paper, Hutchinson proves the following useful compactness and lower semicontinuity Theorem (see Theorem 4.4.2 in [Hu1]):

Theorem 6.6.4. *Suppose* $\{(V_k, f_k)\}_{k\in\mathbb{N}}$ *are measure-function pairs over* $G_m(N)$ *with values in* $\mathbb{R}^\alpha$. *Suppose* V *is a Radon measure on* $G_m(N)$ *(i.e a varifold in* N*) and* $V_k \to V$ *weak converges as Radon measures (equivalently varifold converges in* N*). Suppose* $F : G_m(N) \times \mathbb{R}^\alpha \to \mathbb{R}$ *satisfies the condition* (6.6.2). *Then the following are true:*
i) If there exists $C > 0$ *such that*

$$\int_{G_m(N)} F(x,P,f_k(x,P))dV_k \le C \quad \forall k \in \mathbb{N} \tag{6.39}$$

then there exists a function $f \in L^1_{loc}(V)$ *such that, up to subsequences,* $(V_k, f_k) \rightharpoonup (V, f)$.
ii) if there exists $C > 0$ *such that* (6.39) *is satisfied and* $(V_k, f_k) \rightharpoonup (V, f)$ *then*

$$\int_{G_m(N)} F(x,P,f(x,P))dV \le \liminf_k \int_{G_m(N)} F(x,P,f_k(x,P))dV_k.$$

Now we want to define the varifolds of N with curvature. Observe that given $(x,P) \in G_m(N)$, the m-dimensional linear subspace $P \subset T_x\bar{N} \subset \mathbb{R}^S$ can be identified with the orthogonal projection matrix on $Hom(\mathbb{R}^S, \mathbb{R}^S) \cong \mathbb{R}^{S^2}$

$$P \equiv [P_{ij}] \in \mathbb{R}^{S^2}.$$

Similarly, the tangent space of $\bar{N}$ at x can be identified with its orthogonal projection matrix

$$T_x\bar{N} \equiv Q(x) := [Q_{ij}(x)] \in \mathbb{R}^{S^2}.$$

Before defining the varifolds with curvature let us introduce a bit of notation: given $\phi = \phi(x,P) \in C^1(\mathbb{R}^S \times \mathbb{R}^{S^2})$ we call the partial derivatives of ϕ with respect to the variables x_i and P_{jk} (freezing all other variables) by

$$D_i\phi \quad \text{and} \quad D^*_{jk}\phi \quad \text{for} \quad i,j,k = 1,\dots,S$$

respectively. In the following definition we will consider the quantity

$$P_{ij}\frac{\partial\psi}{\partial x_j}(x) \quad \text{for } \psi \in C^1(\bar{N});$$

we mean that ψ is extended to a C^1 function to some neighborhood of $x \in \mathbb{R}^S$ and, since P is the projection on a m-subspace of $T_x\bar{N}$, the definition does not depend on the extension. Observe moreover that the quantity depends on (x,P) so it is a function on $G_m(\bar{N})$.

Definition 6.6.5. *Let* V *be an* m*-varifold on* $N \subset \bar{N} \hookrightarrow \mathbb{R}^S$, $m \le n-1$. *We say that* V *is a* varifold with (generalized) curvature or with (generalized) second fundamental form *if there exist real-valued functions* B_{ijk}, *for* $1 \le i,j,k \le S$, *defined* V *almost everywhere in* $G_m(N)$ *such that on setting* $B = [B_{ijk}]$ *the following are true:*

i) (V, B) is a measure-function pair over $G_m(N)$ with values in $\mathbb{R}^{S^3}$
ii) For all functions $\phi = \phi(x, P) \in C^1_c(\mathbb{R}^S \times \mathbb{R}^{S^2})$ one has

$$0 = \int_{G_m(N)} [P_{ij} D_j \phi(x,P) + B_{ijk}(x,P) D^*_{jk}\phi(x,P) + B_{jij}(x,P)\phi(x,P)] dV(x,P) \quad \textit{for } i = 1, \dots, S. \tag{6.40}$$

In this case B is called (generalized) curvature *and we can also define the* (generalized) second fundamental form *of V (with respect to $\bar{N}$) as the $L^1_{loc}(V)$ function with values in $\mathbb{R}^{S^3}$*

$$\begin{aligned} A \quad &: \quad G_m(N) \to \mathbb{R}^{S^3}, \\ A^k_{ij}(x,P) \quad &:= \quad P_{lj} B_{ikl}(x,P) - P_{lj} P_{iq} \frac{\partial Q_{kl}}{\partial x_q}(x). \end{aligned} \tag{6.41}$$

We will denote the set of integral *m-varifolds of N with generalized curvature as $CV_m(N)$ and we will call them* curvature m-varifolds.

Observe that we use different notation from [Hu1]: we call B what Hutchinson calls A and vice versa; this is because we want to denote by A the second fundamental form with respect to $\bar{N}$. Moreover, as it is shown in Section 5 of [Hu1], if V is the integral varifold associated to a smooth immersed m-submanifold of N then A coincides with the classical second fundamental form with respect to N.

Remark 6.6.6. *By definition, the generalized second fundamental form A is expressed in terms of B but, as Hutchinson proved in [Hu1] Propositions 5.2.4 and 5.2.6, it is possible to express B in terms of A. Indeed, choosing appropriate test functions, with some easy computations one can prove that*

$$B_{ijk} = A^k_{ij} + A^j_{ik} + P_{jl} P_{iq} \frac{\partial Q_{lk}}{\partial x_q}(x) + P_{kl} P_{iq} \frac{\partial Q_{lj}}{\partial x_q}(x). \tag{6.42}$$

Now let us recall the fundamental compactness and lower semi continuity Theorem of Hutchinson (Theorem 5.3.2 in [Hu1])

Theorem 6.6.7. *Consider $\{V_k\}_{k\in\mathbb{N}} \subset CV_m(N)$ with generalized second fundamental forms $\{A_k\}_{k\in\mathbb{N}}$, V an integral m-varifold of N and suppose $V_k \to V$ in varifold sense. Let $F : G_m(N) \times \mathbb{R}^{S^3} \to \mathbb{R}$ be a function satisfying the condition* (6.6.2) *and assume that*

$$\int_{G_m(N)} F(x, P, A_k) dV_k \le C$$

for some $C > 0$ independent of k. Then
i) $V \in CV_m(N)$ with generalized second fundamental form A,
ii) $(V_k, A_k) \rightharpoonup (V, A)$ in the weak sense of measure-function pairs,
iii) $\int_{G_m(N)} F(x, P, A) dV \le \liminf_k \int_{G_m(N)} F(x, P, A_k) dV_k$.

Now we briefly recall the definition of first variation of an m-varifold V in $\mathbb{R}^S$; the original definitions are much more general, here we recall only the facts we need for this thesis.

Definition 6.6.8. *Let V be an m-varifold in $\mathbb{R}^S$ and let X be a $C^1_c(\mathbb{R}^S)$ vector field. We define* first variation *δV the linear functional on $C^1_c(\mathbb{R}^S)$ vector fields*

$$\delta V(X) := \int_{G_m(\mathbb{R}^S)} div_P X(x) dV(x,P);$$

where for every $P \in G(S, m)$,

$$div_P X := \sum_{i=1}^S \nabla^P_i X^i = \sum_{i,j=1}^S P_{ij} D_j X^i,$$

where $\nabla^P f = P(\nabla f)$ is the projection on P of the gradient in $\mathbb{R}^S$ of f and $\nabla^P_i := e_i \cdot \nabla^P$ (where $\{e_i\}_{i=1,\dots,S}$ is an orthonormal basis of $\mathbb{R}^S$).

V is said to be of locally bounded first variation *in $\mathbb{R}^S$ if for every relatively compact open $W \subset\subset \mathbb{R}^S$ there exists a constant $C_W < \infty$ such that*

$$|\delta V(X)| \leq C_W \sup_W |X|$$

for all $X \in C^1_c(\mathbb{R}^S)$ vector fields with support in W.

An interesting subclass of varifolds with locally bounded first variation are the varifolds with weak mean curvature.

Definition 6.6.9. *Let V be an m-varifold in $\mathbb{R}^S$ and $H : G_m(\mathbb{R}^S) \to \mathbb{R}^S$ an $L^1_{loc}(V)$ function (in the previous notation we would say that (V, H) is a measure-function pair on $G_m(\mathbb{R}^S)$ with values in $\mathbb{R}^S$); then we say that V has* weak mean curvature *H if for any vector field $X \in C^1_c(\mathbb{R}^S)$ one has*

$$\delta V(X) := \int_{G_m(\mathbb{R}^S)} div_P X(x) dV(x,P) = -\int_{G_m(\mathbb{R}^S)} H \cdot X dV(x,P). \tag{6.43}$$

Observe that if $V = V(M, \theta)$ is a rectifiable varifold with weak mean curvature H then with abuse of notation we can write $H(x) = H(x, T_x M)$ and we get the following identities:

$$\int_M div_M X d\mu_V = \int_{G_m(\mathbb{R}^S)} div_{T_xM} X(x) dV = -\int_{G_m(\mathbb{R}^S)} H(x, T_xM) \cdot X dV = -\int_M H(x) \cdot X d\mu_V, \tag{6.44}$$

where $div_M X$ is the tangential divergence of the vector field X and is defined to be $div_M X(x) := div_{T_xM} X(x)$ where T_xM is the approximate tangent space to M at x (which exists for μ_V-a.e. x).

Remark 6.6.10. *As Hutchinson observed in [Hu1], if V is an m-varifold on $N \hookrightarrow \mathbb{R}^S$ with generalized curvature $B = [B_{ijk}]_{i,j,k=1,\dots,S}$ then, as a varifold in $\mathbb{R}^S$, V has weak mean curvature $H_i = \sum_{j=1}^S B_{jij}$ for $i = 1, \dots, S$. Indeed, for any relatively compact open subset $W \subset\subset \mathbb{R}^S$ and any vector field $X \in C^1_c(\mathbb{R}^S)$ with compact support in W, taking $\phi = X^i$, $i = 1, \dots, S$ in equation* (6.40) *and summing over i we get*

$$0 = \int_{G_m(\mathbb{R}^S)} [P_{ij} D_j X^i(x) + B_{jij}(x,P) X^i(x)] dV(x,P)$$

which implies

$$\delta V(X) := \int_{G_m(\mathbb{R}^S)} div_P X(x) dV(x,P) = -\int_{G_m(\mathbb{R}^S)} B_{jij}(x,P) X^i(x) \, dV(x,P);$$

the conclusion follows from the fact that $B \in L^1_{loc}(V)$.

Now let us define the varifolds with weak mean curvature in a compact subset $N \subset\subset \bar{N}$ of a Riemannian manifold (N, g) isometrically embedded in $\mathbb{R}^S$.

Definition 6.6.11. *Let V be an m-varifold on $N \subset \bar{N} \hookrightarrow \mathbb{R}^S$, $m \leq n-1$. We say that V is a* varifold with weak mean curvature H^N relative to $\bar{N}$ *if it has weak mean curvature $H^{\mathbb{R}^S}$ as varifold in $\mathbb{R}^S$. In this case the value of $(H^N)_i$, $i = 1, \dots, S$ is given by*

$$(H^N)_i = (H^{\mathbb{R}^S})_i - P_{jk} \frac{\partial Q_{ij}}{\partial x^k}. \tag{6.45}$$

Consistently with the notation introduced for the curvature varifolds, we denote by $HV_m(N)$ the set of integral *m-varifolds on N with weak mean curvature relative to $\bar{N}$; the elements of $HV_m(N)$ are called* mean curvature varifolds.

Observe that in case V is the varifold associated to a smooth submanifold of $\bar{N}$ then H^N coincides with the classical mean curvature relative to $\bar{N}$ (it is enough to trace the identity (i) of Proposition 5.1.1 in [Hu1] recalling that we denote by A,Q what Hutchinson calls B,S). Moreover, as an exercise, the reader may check that also in the general case the vector $\left(P_{jk} \frac{\partial Q_{ij}}{\partial x^k}\right)_{i=1,\dots,S}$ of $\mathbb{R}^S$ is orthogonal to $\bar{N}$ (fix a point x of $\bar{N}$ and choose a base of $T_x\bar{N}$ in which the Christoffel symbols of $\bar{N}$ vanish at x; write down the orthogonal projection matrix Q with respect to this base and check the orthogonality condition).

Remark 6.6.12. *If V is an m-varifold on $N \subset \bar{N} \hookrightarrow \mathbb{R}^S$, $m \leq n-1$ with weak mean curvature H^N relative to $\bar{N}$ then, for each compactly supported vector field $X \in C_c^1(\bar{N})$ tangent to $\bar{N}$,*

$$\delta V(X) = \int_{G_m(N)} div_P X(x) dV(x,P) = -\int_{G_m(N)} H^N \cdot X dV(x,P).$$

This fact gives consistency to Definition 6.6.11 and follows from Definition 6.6.11, from formula (6.43) and the orthogonality of $\left(P_{jk}\frac{\partial Q_{ij}}{\partial x^k}\right)_{i=1,\dots,S}$ to $\bar{N}$.

Remark 6.6.13. *If V is an m-varifold on $N \subset \bar{N} \hookrightarrow \mathbb{R}^S$, $m \leq n-1$ with null weak mean curvature $H^N = 0$ relative to $\bar{N}$ then, for each compactly supported vector field $X \in C_c^1(\bar{N})$ tangent to $\bar{N}$,*

$$\delta V(X) = \int_{G_m(N)} div_P X(x) dV(x,P) = 0.$$

In this case we say that V is an m-varifold in N with null weak mean curvature relative to $\bar{N}$ or, using more classical language, that V is a stationary *m-varifold in N (where stationary as to be intended in $\bar{N}$).*

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