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Resonances and direct limits in Loewner equations

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Resonances and direct limits in Loewner equations

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Contents

0.1 Radial Loewner theory . . . 1
0.2 Recent developments . . . 3
0.3 Abstract viewpoint . . . 4
0.4 Analogies with Schröder equations . . . 6
0.5 Notations . . . 9
0.6 Ringraziamenti . . . 10

1 Background on complex hyperbolic and taut manifolds 11

2 An abstract approach to Loewner chains 15
2.1 Direct limits of univalent direct systems . . . 15
2.2 Evolution families and Loewner chains . . . 18
2.3 Kernels . . . 22
2.4 From Loewner chains to evolution families . . . 24
2.5 From evolution families to Loewner chains . . . 25
2.6 Herglotz vector fields and the Loewner-Kufarev PDE . . . 27
2.7 Kobayashi metric and complex structure of the Loewner range . . . 31

3 Resonances in Loewner equations 35
3.1 Dilation $\mathbb{N}$-evolution families . . . 35
3.2 Triangular $\mathbb{N}$-evolution families . . . 40
3.3 Parametric Poincaré-Dulac method . . . 43
3.4 Dilation $\mathbb{R}^+$-evolution families . . . 51
3.5 Counterexamples . . . 53

A 55

Bibliography 57

Introduction

0.1 Radial Loewner theory

Radial Loewner theory in the unit disc $\mathbb{D} \subset \mathbb{C}$ was introduced by C. Loewner in 1923 [19] and developed with contributions of P.P. Kufarev in 1943 [17] and C. Pommerenke in 1965 [21], and has been since then used to prove several deep results in geometric function theory [15].

The main motivation which led Loewner to develop his theory was the celebrated Bieberbach conjecture:

Conjecture 0.1.1 (Bieberbach)**.** *Let $f\colon \mathbb{D} \to \mathbb{C}$ be a univalent mapping given by*

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j.$$

Then

$$|a_n| \leq n.$$

The case $n = 2$ is a consequence of the Area Theorem and is due to Gronwall. In his paper Loewner proved the case $n = 3$ of the conjecture, and it is remarkable that the proof given by De Branges in 1985 [10] relies in part on Loewner theory.

Among the extensions of radial Loewner theory we recall the chordal Loewner theory [18], and the theory of Schramm-Loewner evolution [24] introduced in 1999 by Oded Schramm.

We recall the definitions of the three main objects in radial Loewner theory.

Definition 0.1.2. A *radial evolution family* is a family $(\varphi_{t,s})_{0\leq s\leq t}$ of holomorphic self-maps of the unit disc $\mathbb{D}$ satisfying

i) $\varphi_{s,s} = \mathsf{id}$ for all $s \geq 0$,

ii) $\varphi_{t,s} = \varphi_{t,u} \circ \varphi_{u,s}$ for all $0 \leq s \leq u \leq t$.

iii) $\varphi_{t,s}(z) = e^{s-t}z + O(|z|^2)$ for all $0 \leq s \leq t$.

A *radial Herglotz vector field* is a function $H\colon \mathbb{D}\times\mathbb{R}^+\to\mathbb{C}$ of the form

$$H(z,t) = -zp(z,t),$$

where the function $t\mapsto p(z,t)$ is measurable for all fixed $z\in\mathbb{D}$ and satisfies

i) for a.e. $t\geq 0$ the mapping $z\mapsto p(z,t)$ is holomorphic and $\mathsf{Re}\, p(z,t)\geq 0$ for all $z\in M$,

ii) for a.e. $t\geq 0$ we have $p(0,t)=1$.

A *radial Loewner chain* is a family $(f_s)_{0\leq s}$ of univalent mappings $f_s\colon \mathbb{D}\to\mathbb{C}$ satisfying

i) $f_s(\mathbb{B})\subset f_t(\mathbb{B})$ for all $0\leq s\leq t$,

ii) $f_s(z)=e^s z+O(|z|^2)$.

Among these concepts there are natural one-to-one correspondences. Namely if H is a radial Herglotz vector field, then the solutions of the *Loewner-Kufarev ODE*

$$\begin{cases}\frac{\partial\varphi_{t,s}}{\partial t}(z) = H(\varphi_{t,s}(z),t) & \text{a.e. } t\geq s,\\ \varphi_{s,s}(z)=z,\end{cases}$$

form a radial evolution family $(\varphi_{t,s})$. Conversely given any radial evolution family $(\varphi_{t,s})$ there exists an (essentially) unique radial Herglotz vector field H such that $(\varphi_{t,s})$ is the family of solutions of the associated Loewner-Kufarev ODE.

A radial Loewner chain and a radial evolution family are *associated* if

$$f_s = f_t\circ\varphi_{t,s},\quad 0\leq s\leq t.$$

This gives a one-to-one correspondence between radial evolution families and radial Loewner chain. Namely if (f_s) is a radial Loewner chain there is a unique associated radial evolution family:

$$\varphi_{t,s}\doteq f_t^{-1}\circ f_s,\quad 0\leq s\leq t.$$

Conversely given a radial evolution family $(\varphi_{t,s})$ there exists a unique associated radial Loewner chain:

$$f_s\doteq\lim_{t\to\infty} e^t\varphi_{t,s}.$$

Composing the two correspondences we obtain a one-to-one correspondence between Loewner chains and Herglotz vector fields, which is given by the *Loewner-Kufarev PDE*

$$\frac{\partial f_s}{\partial s}(z) = -\frac{\partial f_s}{\partial z}(z)H(z,s)\quad \text{a.e. } s\geq 0, \forall z\in M.$$

0.2 Recent developments

In [5] [6] Bracci, Contreras and Díaz-Madrigal propose a generalization of the radial (and chordal) Loewner theory. For the sake of clearness we treat the case L^∞ but most of the results apply in the case L^d, with $d \in [1,\infty]$.

From now on M will be a q-dimensional complete hyperbolic complex manifold endowed with a Hermitian metric and d_M will denote the associate distance. We further assume that the Kobayashi metric $\kappa_M \in C^1(M \times M \smallsetminus \mathrm{Diag})$.

Definition 0.2.1. A family $(\varphi_{t,s})_{0\le s\le t}$ of holomorphic self mappings of M is an L^∞*-evolution family* if

i) $\varphi_{s,s} = \mathsf{id}$ for all $s \geq 0$,

ii) $\varphi_{t,s} = \varphi_{t,u} \circ \varphi_{u,s}$ for all $0 \le s \le u \le t$,

iii) for any $T > 0$ and for any compact set $K \subset M$ there exists $c_{T,K} \geq 0$ such that

$$\sup_{z\in K} d_M(\varphi_{t,s}(z),\varphi_{u,s}(z)) \le c_{T,K}(t-u),$$

for all $0 \le s \le u \le t \le T$.

A L^∞*-weak holomorphic vector field* on M is a function $H\colon M \times \mathbb{R}^+ \to TM$ with the following properies:

i) for all $z \in M$ the function $t \mapsto H(z,t)$ is measurable,

ii) for all $t \geq 0$ the function $z \mapsto H(z,t)$ is holomorphic,

iii) for all compact set $K \subset\subset M$ and all $T > 0$ there exists $C_{K,T} \geq 0$ such that $\|H(z,t)\| \le C_{K,T}$ for all $z \in K$ and almost every $t \in [0,T]$.

A L^∞-weak holomorphic vector field $H(z,t)$ on M is a L^∞*-Herglotz vector field* if

$$d_{(z,w)}k_M(H(z,t),H(w,t)) \le 0, \quad z,w \in M,\ z \ne w \ \text{ and a.e. } t \geq 0.$$

They obtain a one-to-one correspondence between L^∞-evolution families and L^∞-Herglotz vector fields, given by the Loewner-Kufarev ODE.

In order to completely generalize the classical theory, one has to investigate Loewner chains in this framework. In [9] Contreras, Díaz-Madrigal and Gumenyuk define L^∞-Loewner chains in the case $M = \mathbb{D}$. A L^∞-Loewner chain in $\mathbb{D}$ is defined as a radial Loewner chain with property ii) replaced by

ii') for any compact set $K \subset \mathbb{D}$ and any $T > 0$ there exists $k_{K,T} \geq 0$ such that

$$|f_s(z) - f_t(z)| \le k_{K,T}(t-s)$$

for all $z \in K$ and for all $0 \le s \le t \le T$.

They prove that in the unit disc the functional equation $f_s = f_t \circ \varphi_{t,s}$ gives (up to biholomorphisms) a one-to-one correspondence between L^∞-Loewner chains and L^∞-evolution families. Moreover, the Loewner-Kufarev PDE provides the correspondence between L^∞-Loewner chains and L^∞-Herglotz vector fields.

0.3 Abstract viewpoint

To complete the picture one has to define Loewner chains on a complete hyperbolic complex manifold M. The first chapter of this thesis is devoted to some background on complex hyperbolic and taut manifolds. In the second chapter we study Loewner chains from an abstract point of view obtaining existence and uniqueness results, as well as a generalized Loewner-Kufarev PDE. The results of this chapter are, unless otherwise stated, a joint work with F. Bracci, H. Hamada and G. Kohr [3].

Since there exist complete hyperbolic complex manifolds (even non-compact ones) which do not embed as an open set in $\mathbb{C}^q$, requiring each f_s to be a univalent mapping from M to $\mathbb{C}^q$ would be unnecessarily restrictive. Hence we give the following definition.

Definition 0.3.1. Let N be a complex manifolds of dimension q endowed with a Hermitian metric. Let d_N denote the induced distance. A family $(f_t)_{t\geq 0}$ of univalent mappings $f_t \colon M \to N$ is a L^∞-*Loewner chain* if

i) $f_s(M) \subset f_t(M)$ for all $0 \leq s \leq t$,

ii) for any compact set $K \subset M$ and any $T > 0$ there exists $k_{K,T} \geq 0$ such that

$$d_N(f_s(z), f_t(z)) \leq k_{K,T}(t-s)$$

for all $z \in K$ and for all $0 \leq s \leq t \leq T$.

In order to study the functional equation

$$f_s = f_t \circ \varphi_{t,s}, \quad 0 \leq s \leq t, \tag{0.3.1}$$

we focus on the algebraic aspects rather than on L^d-estimates.

Definition 0.3.2. A $\mathbb{R}^+$-*Loewner chain* is a family (f_s) of univalent mappings from M to an arbitrary q-dimensional complex manifold N such that $f_s(M) \subset f_t(M)$ for all $0 \leq s \leq t$.

A $\mathbb{R}^+$-*evolution family* is a family $(\varphi_{t,s})$ of univalent self-mappings of M such that $\varphi_{s,s} = \mathsf{id}$ and $\varphi_{t,u} \circ \varphi_{u,s} = \varphi_{t,s}$ for all $0 \leq s \leq u \leq t$. We say that a $\mathbb{R}^+$-Loewner chain and a $\mathbb{R}^+$-evolution family are *associated* if (0.3.1) is satisfied.

Theorem 0.3.3 ([3]). *If a $\mathbb{R}^+$-evolution family $(\varphi_{t,s})$ and a $\mathbb{R}^+$-Loewner chain (f_t) on M are associated then $(\varphi_{t,s})$ is a L^d-evolution family if and only if (f_s) is a L^d-Loewner chain.*

This result allows us to reduce the study of (0.3.1) beween L^∞-evolution families and L^∞-Loewner chains to the study of (0.3.1) between $\mathbb{R}^+$-Loewner chains and $\mathbb{R}^+$-evolution families.

Given a $\mathbb{R}^+$-Loewner chain (f_s) there exists a unique associated $\mathbb{R}^+$-evolution family:

$$\varphi_{t,s} \doteq f_t^{-1} \circ f_s, \quad 0 \le s \le t.$$

Conversely we prove the following result.

Theorem 0.3.4. *If $(\varphi_{t,s})$ is a $\mathbb{R}^+$-evolution family on M, there exists an associated $\mathbb{R}^+$-Loewner chain (f_s).*

This chain is constructed as the direct limit in the following way. Define an equivalence relation on the product $M \times \mathbb{R}^+$:

$$(x,s) \sim (y,t) \quad \text{iff} \quad \varphi_{t,s}(x) = y.$$

Let $\pi_\sim \colon M \times \mathbb{R}^+ \to (M \times \mathbb{R}^+)/_\sim$ be the quotient projection, and let $i_s \colon M \to M \times \mathbb{R}^+$ be the injection $i_s(x) = (x,s)$. The chain is then defined as

$$f_s \doteq \pi_\sim \circ i_s, \quad s \ge 0.$$

Equation (0.3.1) holds since

$$\pi_\sim \circ i_s = \pi_\sim \circ i_t \circ \varphi_{t,s}, \quad 0 \le s \le t.$$

It is easy to endow the quotient $(M \times \mathbb{R}^+)/_\sim = \bigcup_{s \ge 0} f_s(M)$ with a complex structure which makes the mappings f_s holomorphic. If (g_s) is another $\mathbb{R}^+$-Loewner chain associated with $(\varphi_{t,s})$, then it defines a mapping on $M \times \mathbb{R}^+$ which is compatible with the equivalence relation $\sim$. This mapping passes thus to the quotient defining a biholomorphism

$$\Psi \colon \bigcup_{s \ge 0} f_s(M) \to \bigcup_{s \ge 0} g_s(M)$$

such that

$$g_s = \Psi \circ f_s, \quad s \ge 0.$$

Hence we can define the *Loewner range* $\mathsf{Lr}(\varphi_{t,s})$ of the $\mathbb{R}^+$-evolution family $(\varphi_{t,s})$ as the biholomorphism class of $\bigcup_{s \ge 0} f_s(M)$, where (f_s) is any associated $\mathbb{R}^+$-Loewner chain.

If $(\varphi_{t,s})$ is a $\mathbb{R}^+$-evolution family on the unit disc $\mathbb{D}$ the Loewner range has to be simply connected and cannot be compact, thus by the uniformization theorem it has to be either $\mathbb{D}$ or $\mathbb{C}$. As noticed in [9], whether we have $\mathbb{D}$ or $\mathbb{C}$ depends on the dynamics of $(\varphi_{t,s})$. Generalizing this result we prove that on a complete hyperbolic complex manifold M, if (f_s) and $(\varphi_{t,s})$ are associated, then

$$f_s^*\kappa_{\mathsf{Lr}(\varphi_{t,s})} = \lim_{t\to\infty} \varphi_{t,s}^*\kappa_M, \quad s \geq 0,$$

where κ_M and $\kappa_{\mathsf{Lr}(\varphi_{\mathrm{t,s}})}$ are the Kobayashi metrics of M and $\mathsf{Lr}(\varphi_{t,s})$ respectively.

If M is a domain in $\mathbb{C}^q$ it is natural to investigate whether it is possible to give conditions on a $\mathbb{R}^+$-evolution family $(\varphi_{t,s})$ ensuring the existence of an associated $\mathbb{R}^+$-Loewner chain (f_s) with image in $\mathbb{C}^q$. In other words: when is the Loewner range $\mathsf{Lr}(\varphi_{t,s})$ a domain of $\mathbb{C}^q$?

The question has not a trivial answer since we prove the following results.

Proposition 0.3.5. *There exists a $\mathbb{R}^+$-evolution family $(\varphi_{t,s})$ on $\mathbb{B}^q$ such that $\mathsf{Lr}(\varphi_{t,s})$ is not a domain in $\mathbb{C}^q$.*

Theorem 0.3.6. *Let $(\varphi_{t,s})$ be a $\mathbb{R}^+$-evolution family on $\mathbb{B}^q$. Assume that there exist $z \in \mathbb{B}^q$,$s \geq 0$ such that*

$$\dim\{v \in T_z\mathbb{B}^q : \lim_{t\to\infty} \kappa_M(\varphi_{t,s}(z), d_z\varphi_{t,s}(v)) = 0\} \leq 1.$$

Then $\mathsf{Lr}(\varphi_{t,s})$ is a domain in $\mathbb{C}^q$.

0.4 Analogies with Schröder equations

In the third chapter we focus on a special type of $\mathbb{R}^+$-evolution family on $\mathbb{B}^q$. The result of this chapter are, unless otherwise stated, in [2].

Definition 0.4.1. A $\mathbb{R}^+$-evolution family $(\varphi_{t,s})$ on $\mathbb{B}^q$ is a dilation evolution family if

$$\varphi_{t,s}(z) = e^{\Lambda(t-s)}(z) + O(|z|)^2, \quad 0 \leq s \leq t,$$

where the eigenvalues α_i of Λ satisfy $\mathsf{Re}\,(\alpha_i) < 0$.

A $\mathbb{R}^+$-Loewner chain (f_s) on $\mathbb{B}^q$ such that $f_s(z) = e^{-\Lambda s}(z) + O(|z|^2)$ is *normal* if the family given by the mappings

$$h_s \doteq e^{\Lambda s} \circ f_s, \quad s \geq 0$$

is normal.

For the sake of clearness we shall assume that Λ is diagonal, even if all results hold without this assumption.

We investigate the following

Problem 0.4.2. *Given a dilation $\mathbb{R}^+$-evolution family, does there exist an associated Loewner chain with values in $\mathbb{C}^q$?*

An affirmative answer to Problem 0.4.2 would yield as a consequence that any Loewner-Kufarev PDE

$$\frac{\partial f_t}{\partial t}(z) = -d_z f_t(H(z,t)), \quad \text{a. e. } t \geq 0,\ z \in \mathbb{B}, \tag{0.4.1}$$

where $H(z,t) = \Lambda z + O(|z|^2)$, admits global solutions.

Loewner theory in the unit ball $\mathbb{B}^q$ has been extensively studied [20] [11] [14]. A partial answer to Problem 0.4.2 is obtained by Graham, Hamada, M.Kohr and G.Kohr [14, Theorems 2.3, 2.6]:

Theorem 0.4.3. *Let $(\varphi_{t,s})$ be a dilation $\mathbb{R}^+$-evolution family such that the eigenvalues of Λ satisfy*

$$2\mathsf{Re}\,\alpha_1 < \mathsf{Re}\,\alpha_N. \tag{0.4.2}$$

Then there exists a normal $\mathbb{R}^+$-Loewner chain (f_s) associated with $(\varphi_{t,s})$, such that $\bigcup_{s\geq 0} f_s(\mathbb{B}) = \mathbb{C}^q$, hence $\mathsf{Lr}(\varphi_{t,s}) = \mathbb{C}^q$. This chain is given by

$$f_s = \lim_{t\to+\infty} e^{-\Lambda t} \circ \varphi_{t,s}, \quad s \geq 0, \tag{0.4.3}$$

and it is the unique normal Loewner chain associated with $(\varphi_{t,s})$.

The main result of this chapter gives an affirmative answer to Problem 0.4.2, without assuming condition (0.4.2): the Lowner range of a dilation $\mathbb{R}^+$-evolution family is $\mathbb{C}^q$.

Theorem 0.4.4. *Let $(\varphi_{t,s})$ be a dilation $\mathbb{R}^+$-evolution family. Then there exists a $\mathbb{R}^+$-Loewner chain (f_s) associated with $(\varphi_{t,s})$, such that $\bigcup_s f_s(\mathbb{B}) = \mathbb{C}^q$. If no real resonance occurs among the eigenvalues of Λ, then (f_s) is a normal chain, not necessarily unique.*

An analogous result is obtained indepentently with different methods by M. Voda in his paper "Solution of a Loewner chain equation in several variables" (arXiv:1006.3286v1 [math.CV], 2010).

Notice that (0.4.2) is a classical condition ensuring the existence of a solution of the Schröder functional equation. In fact we will see that normal Loewner chains correspond to solutions of a parametric Schröder equation. Let us first recall some facts about linearization of germs.

Let $\varphi(z) = e^{\Lambda}(z) + O(|z|^2)$ be a holomorphic germ at the origin of $\mathbb{C}^q$, where the eigenvalues α_i of Λ satisfy $\mathsf{Re}\,(\alpha_i) \leq 0$. If $h(z) = z + O(|z|^2)$ is a solution of the Schröder equation

$$h \circ \varphi = e^{\Lambda} \circ h, \tag{0.4.4}$$

we say that h linearizes φ. It is not always possible to solve this equation, indeed there can occur complex resonances among the eigenvalues of Λ, that is algebraic identities

$$\sum_{j=1}^{q} k_j \alpha_j = \alpha_l,$$

where $k_j \geq 0$ and $\sum_j k_j \geq 2$, which are obstructions to linearization. Indeed a celebrated theorem of Poincaré [22] states that if no complex resonance occurs, then there exists a solution h for (0.4.4). If moreover $2\mathsf{Re}\,\alpha_1 < \mathsf{Re}\,\alpha_q$ then h is given by $\lim_{n\to+\infty} e^{-\Lambda n} \circ \varphi^{\circ n}$.

The analogy between equation (0.3.1) and the Schröder equation (0.4.4) is illustrated by the following remark.

Remark 0.4.5. There exists a normal $\mathbb{R}^+$-Loewner chain (f_s) associated with the dilation $\mathbb{R}^+$-evolution family $(\varphi_{t,s})$ if and only if there exists a normal family (h_s) of univalent mappings $h_s(z) = z + O(|z|^2)$ such that

$$h_t \circ \varphi_{t,s} = e^{\Lambda(t-s)} \circ h_s, \quad 0 \leq s \leq t. \tag{0.4.5}$$

Indeed, (f_s) and (h_s) are related by

$$f_s = e^{-\Lambda s} \circ h_s, \quad s \geq 0.$$

In order to solve equation (0.4.5) we discretize times, obtaining the *parametric Schröder equation*:

$$h_m \circ \varphi_{m,n} = e^{\Lambda(m-n)} \circ h_n, \quad 0 \leq n \leq m \in \mathbb{N}. \tag{0.4.6}$$

There are surprising differences between the Schröder equation (0.4.4) and the parametric one (0.4.6). Namely, while in the first complex resonances are obstructions to the existence of formal solutions, in the latter there always exists the holomorphic solution

$$h_n \doteq e^{\Lambda n} \circ \varphi_{n,0}^{-1}, \quad n \geq 0,$$

but the domain of definition of the mapping h_n shrinks as n grows.

If, as we need, we look for a family h_n of solutions defined in the unit ball $\mathbb{B}$, then we find as obstructions the *real resonances* among the eigenvalues of Λ, that is algebraic identities

$$\mathsf{Re}\,(\sum_{j=1}^{q} k_j \alpha_j) = \mathsf{Re}\,\alpha_l,$$

where $k_j \geq 0$ and $\sum_j k_j \geq 2$. If no real resonance occurs we find a normal discrete Loewner chain associated with $(\varphi_{m,n})$.

If real resonances occur we solve a slightly different equation:

$$h_m \circ \varphi_{m,n} = T_{m,n} \circ h_n, \quad 0 \leq n \leq m$$

where $T_{m,n}$ is a suitable triangular discrete evolution family, finding in this way a non necessarily normal discrete Loewner chain associated with $(\varphi_{m,n})$.

Once we solved the problem for discrete times, it is easy to extend (f_n) to all real positive times obtaining a $\mathbb{R}^+$-Loewner chain (f_s) and Theorem 3.4.6 above.

We also prove that if the dilation $\mathbb{R}^+$-evolution family is periodic (for example if it is associated with a semigroup), then the only obstructions to the existence of an associated normal $\mathbb{R}^+$-Loewner chain are complex resonances.

We conclude giving examples of

1. a dilation $\mathbb{R}^+$-evolution family with no real resonances and several associated normal $\mathbb{R}^+$-Loewner chains,
2. a semigroup-type dilation $\mathbb{R}^+$-evolution family with complex resonances which does not admit any associated normal $\mathbb{R}^+$-Loewner chain,
3. a discrete dilation evolution family with pure real resonances (real non-complex resonances) which does not admit any associated discrete normal Loewner chain.

0.5 Notations

$\mathbb{R}^+$ denotes the set of the real numbers t such that $t \geq 0$.

$\mathcal{L}(V,W)$ denotes the set of the $\mathbb{C}$-linear mappings from V to W.

$\mathcal{A}(V)$ denotes the set of the $\mathbb{C}$-linear automorphisms of V.

$\mathsf{hol}(X,Y)$ denotes the set of the holomorphic mappings from X to Y.

$\mathsf{aut}(X)$ denotes the set of the holomorphic automorphisms of X.

$M_{q,p}(\mathbb{C})$ denotes the set of complex $(q \times p)$-matrices.

$GL_q(\mathbb{C})$ denotes the set of invertible complex $(q \times q)$-matrices.

$\mathbb{B}^q$ (or $\mathbb{B}$) denotes the unit ball in $\mathbb{C}^q$.

$\|\cdot\|$ (or $\|\cdot\|_2$) denotes, unless differently stated, the norm induced by the euclidean norm on $\mathbb{C}^q$ on $\mathcal{L}(\mathbb{C}^q,\mathbb{C}^q)$ and on $M_{q,q}(\mathbb{C})$.

I denotes a multi-index $(i_1, \ldots, i_q)$. If $z = (z_1, \ldots, z_q)$ then $z^I \doteq z_1^{i_1} \ldots z_q^{i_q}$.

$(\varphi_{\beta,\alpha})$ with $0 \leq \alpha \leq \beta$ denotes an evolution family. Notice that it is not the same notation as in [2] [3] [5] [6] [9]. If each $\varphi_{\beta,\alpha} \in \mathsf{aut}(\mathbb{C}^q)$ for all $0 \leq \alpha \leq \beta$ then we denote $\varphi_{\alpha,\beta} \doteq \varphi_{\beta,\alpha}^{-1}$.

0.6 Ringraziamenti

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Chapter 1

Background on complex hyperbolic and taut manifolds

A reference for hyperbolic complex manifolds and taut manifolds is [1]. We recall some basic definitions and results.

Definition 1.0.1. Let X and Y be complex manifolds. A family $\mathcal{F}$ is *normal* if for every sequence (f_n) in $\mathsf{hol}(X,Y)$, either

i) there exists a subsequence (f_{n_k}) converging uniformly on compacta, or

ii) for each compact set $K \subset X$ and each compact set $L \subset Y$ there exists an integer N such that
$$f_n(K) \cap L = \varnothing$$
for all $n \geq N$.

A complex manifold Y is *taut* if $\mathsf{hol}(\mathbb{D},Y)$ is normal.

Theorem 1.0.2 ([1, Theorem 2.1.2]). *Let X be a taut manifold. Then $\mathsf{hol}(Y,X)$ is a normal family for every complex manifold Y.*

Proposition 1.0.3 ([1, Corollary 2.1.17]). *Let X be a taut manifold. Then the topology of pointwise convergence on $\mathsf{hol}(X,Y)$ coincides with the compact-open topology for every complex manifold Y.*

Theorem 1.0.4 ([1, Theorem 2.4.1]). *Let X be a taut manifold, and let $f \in \mathsf{hol}(X,X)$ with fixed point z_0. Then the sequence of iterates (f^k) converges to z_0 if and only if*
$$\mathsf{sp}(d_{z_0} f) \subset \mathbb{D}.$$

Definition 1.0.5. A *pseudodistance* on a set X is a function $d\colon X \times X \to \mathbb{R}^+$ such that

i) $d(x,y) = d(y,x)$ for all $x,y \in X$,

ii) $d(x,y) \leq d(x,y) + d(y,x)$ for all $x,y,z \in X$,

iii) $d(x,x) = 0$ for all $x \in X$.

An *analytic chain* $\alpha = \{\zeta_0, \dots \zeta_m; \eta_0, \dots ,\eta_m; \varphi_0, \dots ,\varphi_m\}$ connecting two points z_0 and w_0 in a complex manifold X is a sequence of points $\zeta_0, \dots \zeta_m,\eta_0, \dots ,\eta_m$ in the unit disc $\mathbb{D}$ and holomorphic maps $\varphi_0, \dots ,\varphi_m \in \mathsf{hol}(\mathbb{D},X)$ such that $\varphi_0(\zeta_0) = z_0,\varphi_j(\eta_j) = \varphi_{j+1}(\zeta_{j+1})$ for $j = 0, \dots ,m-1$ and $\varphi_m(\eta_m) = w_0$. The *lenght* $\omega(\alpha)$ of the chain is

$$\omega(\alpha) = \sum_{j=0}^{m} \omega(\zeta_j,\eta_j).$$

If $z,w \in X$ the *Kobayashi pseudodistance* $k_X(z,w)$ is defined as

$$k_X(z,w) = \inf\{\omega(\alpha)\},$$

where the infimum is taken with respect to all analytic chains connecting z to w.

A complex manifold X is *hyperbolic* if k_X is a distance, and is *complete hyperbolic* if k_X is a complete distance.

Proposition 1.0.6 ([1, Propositions 2.3.10, 2.3.17])**.** *Let X be a complex manifold. The manifold X is hyperbolic if and only if k_X induces the topology of X. The manifold X is complete hyperbolic if and only if every cloded Kobayashi ball is compact.*

Definition 1.0.7. Let X be a complex manifold, and TX its tangent bundle. The *Kobayashi pseudometric* $\kappa_X \colon TX \to \mathbb{R}^+$ is defined as

$$\kappa_X(z,v) = \inf\{|\zeta| \text{ s.t. } \exists\varphi \in \mathsf{hol}(\mathbb{D},X) : \varphi(0) = z,d_0\varphi(\zeta) = v\}$$

for every $z \in X$ and $v \in T_zX$.

The following properties follow easily from the definitions.

Proposition 1.0.8. *Let X,Y be complex manifolds.*

i) $\kappa_X(z,\lambda v) = |\lambda|\kappa_X(z,v)$, for all $z \in X$, $v \in T_zX$ and $\lambda \in \mathbb{C}$,

ii) $\kappa_Y(f(z),d_zf(v)) \leq \kappa_X(z,v)$, for all $z \in X$, $v \in T_zX$ and $f \in \mathsf{hol}(X,Y)$.

As the next result shows, the Kobayashi pseudodistance is the integrated form of the Kobayashi pseudometric.

Theorem 1.0.9 ([1, Theorems 2.3.29, 2.3.32]). *Let X be a complex manifold. Then κ_X is upper semicontinuous, and if $z,w \in X$,*

$$k_X(z,w) = \inf_{\gamma} \int_a^b \kappa_X(\gamma(t),\gamma'(t))dt,$$

where γ is any piecewise C^1 curve in X such that $\gamma(a) = z$ and $\gamma(b) = w$.

As a consequence hyperbolicity affects the behaviour of the Kobayashi pseudo-metric.

Theorem 1.0.10 ([1, Propositions 2.3.33, 2.3.34]). *Let X be a hyperbolic manifold. Then $\kappa_X(z,v) > 0$ for every $z \in X$, $v \in T_zX \setminus \{0\}$. Let X be complete hyperbolic. Then $\kappa_X : TX \to \mathbb{R}^+$ is continuous.*

Remark 1.0.11. If X is a hyperbolic manifold, we call κ_X the *Kobayashi metric*. Notice however that the function $\kappa_X(z,\cdot) : T_zX \to \mathbb{R}^+$ is not a norm, since in general the triangular inequality $\kappa_X(z,v+w) \leq \kappa_X(z,v) + \kappa_X(z,w)$ does not hold.

We shall need two results relating hyperbolicity and tautness.

Theorem 1.0.12 ([1, Theorem 2.3.18]). *Every complete hyperbolic manifold X is taut.*

Lemma 1.0.13 ([13, Lemma 2.1]). *Let X be a hyperbolic manifold and assume that $X/\mathsf{aut}(X)$ is compact. Then X is complete hyperbolic and hence taut.*

Chapter 2

An abstract approach to Loewner chains

2.1 Direct limits of univalent direct systems

Definition 2.1.1. Let $(I, \leq)$ be a directed set. Let $(E_\alpha)_{\alpha \in I}$ be a family of complex manifolds of the same dimension q indexed by I and let $(f_{\beta,\alpha})_{\alpha \leq \beta}$ be a family of univalent mappings $f_{\beta,\alpha} \colon E_\alpha \to E_\beta$ satisfying

i) $f_{\alpha,\alpha} = \mathsf{id}$ for any $\alpha \in I$

ii) $f_{\gamma,\alpha} = f_{\gamma,\beta} \circ f_{\beta,\alpha}$ for any $\alpha \leq \beta \leq \gamma$ in I.

Then we call the pair $(E_\alpha, f_{\beta,\alpha})$ a *univalent direct system*.

Definition 2.1.2. Let F be a q-dimensional complex manifold and let (u_α) be a family of holomorphic mappings $u_\alpha \colon E_\alpha \to F$. The pair (F, u_α) is *compatible* with the univalent direct system $(E_\alpha, f_{\beta,\alpha})$ if for all $\alpha \leq \beta$,

$$u_\beta \circ f_{\beta,\alpha} = u_\alpha.$$

Definition 2.1.3. A *direct limit* of $(E_\alpha, f_{\beta,\alpha})$ is a compatible pair $\varinjlim E_\alpha = (E, f_\alpha)$ such that if (F, u_α) is any other compatible pair then there exists a unique holomorphic mapping $u \colon E \to F$ such that for all $\alpha \in I$,

$$u_\alpha = u \circ f_\alpha.$$

Proposition 2.1.4. *Assume that I admits a countable cofinal subset $\Gamma = \{\gamma_n\}_{n \in \mathbb{N}}$. Then any univalent direct system $(E_\alpha, f_{\beta,\alpha})_{\alpha \in I}$ admits a direct limit $\varinjlim E_\alpha = (E, f_\alpha)$.*

Proof. Let $\bigsqcup_\alpha E_\alpha$ denote the disjoint union of the family (E_α) and let

$$i_\alpha\colon E_\alpha \to \bigsqcup_\alpha E_\alpha$$

be the canonical injection. Consider the following equivalence relation on $\bigsqcup_\alpha E_\alpha$:

$$i_\alpha(x) \sim i_\beta(y) \quad \text{iff} \quad \exists\, \gamma \geq \alpha, \gamma \geq \beta \text{ s.t. } f_{\gamma,\alpha}(x) = f_{\gamma,\beta}(y).$$

Define $E \doteq \bigsqcup_\alpha E_\alpha / \sim$, let $\pi_\sim$ be the quotient projection, and define f_α as

$$\pi_\sim \circ i_\alpha\colon E_\alpha \to E.$$

If $\alpha \leq \beta$, then for each $x \in E_\alpha$ we have $f_{\beta,\beta}(f_{\beta,\alpha}(x)) = f_{\beta,\alpha}(x)$ and therefore the elements $i_\alpha(x)$ and $i_\beta(f_{\beta,\alpha}(x))$ are congruent modulo the relation $\sim$. Thus

$$f_\alpha = f_\beta \circ f_{\beta,\alpha}.$$

Endow E with the quotient topology. The mappings f_α are easily seen to be continuous and open, and are injective since the mappings $f_{\beta,\alpha}$ are injective. This shows that each f_α is an homeomorphism on its image. Let ζ,η be two points in E. Then there exist $x \in E_\alpha$ and $y \in E_\beta$ such that $\zeta = f_\alpha(x)$ and $\eta = f_\beta(y)$. Let $\gamma \geq \beta, \gamma \geq \alpha$. Then $\zeta = f_\gamma(f_{\gamma,\alpha}(x))$ and $\eta = f_\gamma(f_{\gamma,\beta}(y))$. Hence ζ and η are both contained in the open set $f_\gamma(E_\gamma)$. This shows that E is arc-connected and Hausdorff, since $f_\gamma(E_\gamma)$ is arc-connected and Hausdorff. Moreover if Γ is a countable cofinal subset of the index set I, we have

$$E = \bigcup_{\gamma \in \Gamma} f_\gamma(E_\gamma),$$

thus E is second countable.

We define a structure of q-dimensional complex manifold on E by means of the charts (f_α^{-1}). It is easy to see that we can check compatibility of charts only for couples of indices $\alpha \leq \beta$. But then $f_\beta^{-1} \circ f_\alpha = \varphi_{\beta,\alpha}$, which is holomorphic.

Now assume that (F,u_α) is any other compatible pair. The mappings u_α define a mapping $v\colon \bigsqcup_\alpha E_\alpha \to F$ which is compatible with $\sim$. Hence there exists a unique mapping $u\colon E \to F$ such that $v = u \circ \pi_\sim$. On each open set $f_\alpha(E_\alpha) \subset E$ we have

$$u = u_\alpha \circ f_\alpha^{-1}, \tag{2.1.1}$$

hence u is holomorphic. □

Remark 2.1.5. It is easy to see that the image of u is $\bigcup_\alpha u_\alpha(E_\alpha)$, and that u is univalent if and only if each u_α is univalent. If u is a biholomorphism then also (F,u_α) is a direct limit for $(E_\alpha,f_{\beta,\alpha})$. Hence an equivalent characterization of direct limits can be given as follows: a compatible pair (E,f_α) is a direct limit if and only if

i) $(f_\alpha)_{\alpha\in I}$ is univalent,

iii) $\cup_\alpha f_\alpha(E_\alpha) = E$.

From now on we assume that I admits a countable cofinal subset $\Gamma = \{\gamma_n\}_{n\in\mathbb{N}}$.

Definition 2.1.6. Let $(E_\alpha,f_{\beta,\alpha})$ and $(F_\alpha,g_{\beta,\alpha})$ be two univalent direct systems relative to the same index set I. Let $(E,f_\alpha) = \varinjlim E_\alpha$, and $(F,g_\alpha) = \varinjlim F_\alpha$. For each $\alpha \in I$ let $u_\alpha \colon E_\alpha \to F_\alpha$ be a holomorphic mapping such that if $\alpha \le \beta$ then

$$g_{\beta,\alpha} \circ u_\alpha = u_\beta \circ f_{\beta,\alpha}.$$

Then we call (u_α) a *direct system of holomorphic maps*.

Remark 2.1.7. A direct system of holomorphic maps allows us to pull-back compatible pairs. Namely if (u_α) is a direct system of holomorphic maps from $(E_\alpha,f_{\beta,\alpha})$ to $(F_\alpha,g_{\beta,\alpha})$ and (G,h_α) is a pair compatible with $(F_\alpha,g_{\beta,\alpha})$, the pair

$$(G,h_\alpha \circ u_\alpha)$$

is compatible with $(E_\alpha,f_{\beta,\alpha})$. Indeed define $v_\alpha \doteq h_\alpha \circ u_\alpha$. If $\alpha \le \beta$ we have

$$v_\beta \circ f_{\beta,\alpha} = h_\beta \circ u_\beta \circ f_{\beta,\alpha} = h_\beta \circ g_{\beta,\alpha} \circ u_\alpha = h_\alpha \circ u_\alpha = v_\alpha.$$

Proposition 2.1.8. *Given a direct system of holomorphic maps (u_α) there exists a unique holomorphic mapping $u\colon E \to F$ such that for all $\alpha \in I$*

$$g_\alpha \circ u_\alpha = u \circ f_\alpha.$$

Such mapping u is called the direct limit of (u_α) and is denoted $\varinjlim u_\alpha$. If each u_α is univalent then $\varinjlim u_\alpha$ is univalent, and if each u_α is surjective then $\varinjlim u_\alpha$ is surjective.

Proof. The pair $(F,g_\alpha \circ u_\alpha)$ is compatible with $(E_\alpha,f_{\beta,\alpha})$ by Remark 2.1.7. By Definition 2.1.3 there exists a unique holomorphic mapping $u\colon E \to F$ such that

$$u \circ f_\alpha = v_\alpha = g_\alpha \circ u_\alpha,$$

hence $u = \varinjlim u_\alpha$.

If each u_α is univalent, then $v_\alpha = g_\alpha \circ u_\alpha$ is univalent, hence by Remark 2.1.5 u is also univalent. If each u_α is surjective

$$F = \bigcup_\alpha g_\alpha(F_\alpha) = \bigcup_\alpha g_\alpha(u_\alpha(E_\alpha)) = \bigcup_\alpha u(f_\alpha(E_\alpha)) = u(\bigcup_\alpha f_\alpha(E_\alpha)) = u(E).$$

□

Corollary 2.1.9. *Let $(E_\alpha,f_{\beta,\alpha})$,$(F_\alpha,g_{\beta,\alpha})$,$(G_\alpha,h_{\beta,\alpha})$ be three univalent direct systems indexed by I. Let (E,f_α),(F,g_α),(G,h_α) be respective direct limits. If (u_α),(v_α) are two direct systems of holomorphic maps $u_\alpha\colon E_\alpha \to F_\alpha$, $v_\alpha : F_\alpha \to G_\alpha$, then the maps $v_\alpha \circ u_\alpha\colon E_\alpha \to G_\alpha$ form a direct system of holomorphic maps, and*

$$\varinjlim\, (v_\alpha \circ u_\alpha) = (\varinjlim\, v_\alpha) \circ (\varinjlim\, u_\alpha).$$

Proof. Set $w_\alpha \doteq v_\alpha \circ u_\alpha$. If $\alpha \leq \beta$ then

$$h_{\beta,\alpha} \circ w_\alpha = (h_{\beta,\alpha} \circ v_\alpha) \circ u_\alpha = (v_\beta \circ g_{\beta,\alpha}) \circ u_\alpha = v_\beta \circ (u_\beta \circ f_{\beta,\alpha}) = w_\beta \circ f_{\beta,\alpha},$$

hence (w_α) is a direct system of holomorphic mappings. If $u = \varinjlim u_\alpha$ and $v = \varinjlim v_\alpha$ then

$$(v \circ u) \circ f_\alpha = v \circ (g_\alpha \circ u_\alpha) = h_\alpha \circ (v_\alpha \circ u_\alpha) = h_\alpha \circ w_\alpha,$$

thus $\varinjlim\, w_\alpha = v \circ u$. □

Proposition 2.1.10. *Let $(E_\alpha,f_{\beta,\alpha})$ be a univalent direct system indexed by I and let $\Gamma \subset I$ be a countable cofinal subset. Then if $(E,g_\gamma)_{\gamma\in\Gamma}$ is a direct limit for the univalent direct system restricted to Γ, there exists a unique direct limit $(E,f_\alpha)_{\alpha\in I}$ which extends (E,g_γ) in the sense that for all $\gamma \in \Gamma$ we have $f_\gamma = g_\gamma$.*

Proof. Set for all $\alpha \in I$,

$$f_\alpha \doteq f_\gamma \circ f_{\gamma,\alpha},$$

where γ is any element in Γ such that $\alpha \leq \gamma$. This is a good definition: indeed if $\alpha \leq \gamma \leq \gamma'$, where $\alpha \in I$ and $\gamma,\gamma' \in \Gamma$, then

$$f_{\gamma'} \circ f_{\gamma',\alpha} = f_{\gamma'} \circ f_{\gamma',\gamma} \circ f_{\gamma,\alpha} = f_\gamma \circ f_{\gamma,\alpha}.$$

The family (f_α) just defined satisfies $f_\alpha = f_\beta \circ f_{\beta,\alpha}$ for all $\alpha,\beta \in I$: indeed if $\alpha \leq \beta \leq \gamma$, where $\alpha,\beta \in I$ and $\gamma \in \Gamma$, then

$$f_\alpha = f_\gamma \circ f_{\gamma,\alpha} = f_\gamma \circ f_{\gamma,\beta} \circ f_{\beta,\alpha} = f_\beta \circ f_{\beta,\alpha}.$$

□

2.2 Evolution families and Loewner chains

Let I be $\mathbb{N}$ or $\mathbb{R}^+$.

Definition 2.2.1. Let M be a q-dimensional complex manifold. A *I-evolution family* is a family $(\varphi_{\beta,\alpha})_{\alpha\leq\beta\in I}$ of univalent self-mappings of M such that

i) $\varphi_{\alpha,\alpha} = \mathsf{id}$ for all $\alpha \geq 0$,

ii) $\varphi_{\beta,\alpha} = \varphi_{\beta,\gamma} \circ \varphi_{\gamma,\alpha}$ for all $0 \leq \alpha \leq \gamma \leq \beta$.

Remark 2.2.2. If we define $M_\alpha = M$ for $\alpha \in I$, the pair $(M_\alpha,\varphi_{\beta,\alpha})$ is a univalent direct system.

Definition 2.2.3. Let M, N be two complex manifolds of the same dimension. Given two holomorphic mappings $f,g\colon M \to N$ we say that f is *subordinate* to g if there exists a holomorphic mapping $v\colon M \to M$ such that $f = g \circ v$. Note that if g is univalent this is equivalent to $f(M) \subset g(M)$.

A *I-subordination chain* (f_α) is a family of holomorphic mappings $f_\alpha\colon M \to N$ such that if $0 \leq \alpha \leq \beta$ then f_α is subordinate to f_β. We denote it also (f_α,M,N).

A I-subordination chain is *associated* with a I-evolution family $(\varphi_{\beta,\alpha})$ if for all $0 \leq \alpha \leq \beta$ we have $f_\alpha = f_\beta \circ \varphi_{\beta,\alpha}$.

Definition 2.2.4. A *I-Loewner chain* is a univalent I-subordination chain. Equivalently a family (f_α) of univalent mappings $f_\alpha\colon M \to N$ is a *I-Loewner chain* if for all $0 \leq \alpha \leq \beta$,

$$f_\alpha(M) \subset f_\beta(M).$$

A I-Loewner chain is sometimes denoted (f_α,M,N).

The *range* $\mathsf{rg}\,(f_\alpha)$ of a *I-Loewner chain* (f_α) is

$$\bigcup_{\alpha\geq 0} f_\alpha(M).$$

Proposition 2.2.5. *Let (f_α,M,N) be a I-Loewner chain. Then there exists a unique I-evolution family $(\varphi_{\beta,\alpha})$ such that (f_α) is associated with $(\varphi_{\beta,\alpha})$.*

Proof. Define

$$\varphi_{\beta,\alpha} = f_\beta^{-1} \circ f_\alpha.$$

It is easy to see that it is a I-evolution family. □

Remark 2.2.6. Let $(\varphi_{t,s},M)$ be a $\mathbb{R}^+$-evolution family. Then the pair $(M,\varphi_{t,s})$ is a univalent direct system with index set $I = \mathbb{R}^+$. If (f_t,M,N) is an associated $\mathbb{R}^+$-Loewner chain then by Remark 2.1.5 the pair $(\mathsf{rg}\,(f_t),f_t)$ is a direct limit.

Theorem 2.2.7 ([3])**.** *Let $(\varphi_{\beta,\alpha})$ be a I-evolution family on M. Then there exists an associated I-Loewner chain (f_α,M,N).*

Proof. Set $M_\alpha = M$ for $\alpha \geq 0$. Then $(M_\alpha,\varphi_{\beta,\alpha})$ is a univalent direct system. Let (E,f_α) be a direct limit. The mappings $f_\alpha\colon M \to E$ are univalent by Remark 2.1.5 and satisfy for all $0 \leq \alpha \leq \beta$

$$f_\alpha = f_\beta \circ \varphi_{\beta,\alpha}.$$

□

Theorem 2.2.8 ([3]). *Let $(\varphi_{\beta,\alpha})$ be a I-evolution family on M and let (f_α,M,N) be an associated I-Loewner chain. If (g_α,M,T) is a subordination chain associated with $(\varphi_{\beta,\alpha})$ then there exists a holomorphic mapping $\Psi\colon \mathsf{rg}\,(f_\alpha) \to T$ such that*

$$g_\alpha = \Psi \circ f_\alpha.$$

The mapping Ψ is univalent if and only if each g_α is univalent and its image is $\bigcup_{\alpha\geq 0} g_\alpha(M)$.

Proof. Set $M_\alpha = M$ for $\alpha \geq 0$. Then $(M_\alpha,\varphi_{\beta,\alpha})$ is a univalent direct system. Since $(\mathsf{rg}\,(f_\alpha),f_\alpha)$ is a direct limit, Definition 2.1.3 gives the existence of Ψ. The last statement follows from Remark 2.1.5. □

Definition 2.2.9. Let $(\varphi_{\beta,\alpha})$ be a I-evolution family on M and (f_α,M,N) an associated I-Loewner chain. Then $\mathsf{rg}\,(f_\alpha)$ is the *Loewner range* of $(\varphi_{\beta,\alpha})$ and is denoted $\mathsf{Lr}(\varphi_{\beta,\alpha})$. By Theorem 2.2.8 the Loewner range is well-defined up to biholomorphisms.

Corollary 2.2.10. *Let $(\varphi_{\beta,\alpha})$ be a I-evolution family on the unit disc $\mathbb{D}$. Then $\mathsf{Lr}(\varphi_{\beta,\alpha})$ is equal to $\mathbb{D}$ or to $\mathbb{C}$.*

Proof. Any Loewner range for $(\varphi_{\beta,\alpha})$ is a simply connected non-compact Riemann surface. The result follows from the uniformization theorem. □

Definition 2.2.11. Let $d \in [1,\infty]$. Let M be a complex manifold endowed with a Hermitian metric and let d_M be the associate distance. A family $(\varphi_{t,s})_{0\leq s\leq t}$ of holomorphic self mappings of M is an L^d*-evolution family* if

i) $\varphi_{s,s} = \mathsf{id}$ for all $s \geq 0$,

ii) $\varphi_{t,s} = \varphi_{t,u} \circ \varphi_{u,s}$ for all $0 \leq s \leq u \leq t$,

iii) for any $T > 0$ and for any compact set $K \subset M$ there exists a function $c_{T,K} \in L^d([0,T],\mathbb{R}^+)$ such that

$$\sup_{z\in K} d_M(\varphi_{t,s}(z),\varphi_{u,s}(z)) \leq \int_u^t c_{T,K}(\xi)d\xi,$$

for all $0 \leq s \leq u \leq t \leq T$.

Proposition 2.2.12 ([6, Lemma 2.8]). *Let $d \in [1,+\infty]$. Let $(\varphi_{t,s})$ be a L^d-evolution family on a taut manifold M. Let $\Delta \doteq \{(s,t) : 0 \leq s \leq t\}$. Then the mapping*

$$(s,t) \mapsto \varphi_{t,s}$$

from Δ to $\mathsf{hol}(M,M)$ endowed with the topology of uniform convergence on compacta is jointly continuous. Hence the mapping $\Phi(z,s,t) \doteq \varphi_{t,s}(z)$ from $M \times \Delta$ to M is jointly continuous.

Remark 2.2.13. The proof of Proposition 2.2.12 relies on Proposition 1.0.3.

Proposition 2.2.14. *Let $d \in [1,+\infty]$. Let $(\varphi_{t,s},M)$ be a L^d-evolution family. Then for all $0 \leq s \leq t$ the mapping $(\varphi_{t,s})$ is univalent.*

Proof. Suppose that there exists $0 < s_0 < t_0$ and $z_0 \neq w_0$ in M such that

$$\varphi_{t_0,s_0}(z_0) = \varphi_{t_0,s_0}(w_0).$$

Set

$$t_1 \doteq \inf\{t \in [s_0,t_0] : \varphi_{t,s_0}(z_0) = \varphi_{t,s_0}(w_0)\}.$$

By Proposition 2.2.12 we have

$$\lim_{t \to s_0} \varphi_{t,s_0} = \varphi_{s_0,s_0} = \mathsf{id}$$

uniformly on compacta, thus $t_1 > s_0$. If $t \in (s_0,t_1)$,

$$\varphi_{t_1,t}(\varphi_{t,s_0}(z_0)) = \varphi_{t_1,t}(\varphi_{t,s_0}(w_0)),$$

and since $\varphi_{t,s_0}(z_0) \neq \varphi_{t,s_0}(w_0)$, the mapping $\varphi_{t_1,t}$ is not univalent. By Proposition 2.2.12,

$$\lim_{t \to t_1} \varphi_{t_1,t} = \varphi_{t_1,t_1} = \mathsf{id}$$

uniformly on compacta which is a contradiction since the identity mapping is univalent. □

Corollary 2.2.15. *Let $d \in [1,+\infty]$. A L^d-evolution family is a $\mathbb{R}^+$-evolution family.*

Definition 2.2.16. Let $d \in [1,+\infty]$. Let M be a complex manifold and let N be a complex manifolds of the same dimension endowed with a Hermitian metric. Let d_N denote the induced distance. A $\mathbb{R}^+$-Loewner chain (f_t,M,N) is a *L^d-Loewner chain* if for any compact set $K \subset M$ and any $T > 0$ there exists a $k_{K,T} \in L^d([0,T],\mathbb{R}^+)$ such that

$$d_N(f_s(z),f_t(z)) \leq \int_s^t k_{K,T}(\xi)d\xi, \quad \text{for all } z \in K \text{ and for all } 0 \leq s \leq t \leq T. \quad (2.2.1)$$

Remark 2.2.17. Condition (2.2.1) implies that if (f_t) is a L^d-Loewner chain then the mapping $t \mapsto f_t(z)$ is locally absolutely continuous, uniformly on compacta with respect to $z \in M$. It is easy to see that the mapping $t \mapsto f_t$ is continuous with respect to the topology of uniform convergence on compacta on $\mathsf{hol}(M,N)$ and that the mapping $\Psi\colon M \times \mathbb{R}^+ \to N$ defined as $\Psi(z,t) = f_t(z)$ is jointly continuous.

2.3 Kernels

Proposition 2.3.1. *Let $U \subset \mathbb{C}^q$ be an open set. Let $f_n : U \to N$ be a sequence of univalent mappings. Assume that $f_n \to f$ uniformly on compacta and that f is univalent. Then for all $z \in U$ and $0 < s < r$ such that $B(z_0,s) \subset\subset B(z_0,r) \subset\subset U$ there exists $m = m(z_0,s,r)$ such that if $n > m$ then*

$$f(B(z_0,s)) \subset f_n(B(z_0,r)).$$

Proof. Let $K = f\left(\overline{B(z_0,s)}\right)$, $\gamma = \partial B(z_0,r)$ and $\Gamma = f(\gamma)$. Then $K \cap \Gamma = \varnothing$ since f is univalent on U.

Let η be the distance between Γ and K. Then $\eta > 0$ and

$$\eta = \min_{w \in K, |z-z_0|=r} |f(z) - w|.$$

If $u_0 \in K$ then $|f(z) - u_0| \geq \eta$ for $z \in \gamma$, and since $f_n \to f$ uniformly on γ there exists $m > 0$ such that if $n \geq m$ and $z \in \gamma$ then

$$|f(z) - f_n(z)| < |f(z) - u_0|.$$

Rouché theorem in several complex variables yields then that $f_n(z) - u_0$ and $f(z) - u_0$ have the same number of zeros on $B(z_0,r)$ counting multiplicities. But $f(z) - u_0$ has a zero in $B(z_0,r)$ since $u_0 \in K$, and thus $u_0 \in f_n(B(z_0,r))$ for $n \geq m$. The constant $m > 0$ does not depend on $u_0 \in K$, hence we have the result. □

Corollary 2.3.2. *Let (f_n) be a sequence of holomorphic mappings $f_n : U \to \mathbb{C}^q$ converging uniformly on compacta to an univalent mapping f. Then any compact set $K \subset f(U)$ is eventually contained in $f_n(U)$.*

Proof. All the balls $B(z,s) \subset\subset U$ give an open cover of U. Since K is compact there is a finite number of balls $B(z_i,s_i) \subset\subset U$ such that $K = \bigcup_i f(B(z_i,s_i))$, hence Proposition 2.3.1 yields the result. □

Definition 2.3.3. Let (Ω_n) be a sequence of open subset of a manifold M. The *kernel* Ω of (Ω_n) is the biggest open set such that for all compact sets $K \subset \Omega$ there exists $m = m(K)$ such that if $n \geq m$ then $K \subset \Omega_n$.

The kernel might be empty as the following example shows.

Example 2.3.4. Let $M = \mathbb{C}$ and $f_n : \mathbb{D} \to \mathbb{C}$ defined by $f_n(z) = \frac{1}{n}z$. Then (f_n) is a sequence of univalent mappings converging uniformly on compacta to 0. The kernel of the sequence $(f_n(\mathbb{D}))$ is then empty.

Theorem 2.3.5 ([3]). *Let (f_n) be a sequence of univalent mappings from a complete hyperbolic complex manifold M to a complex manifold N of the same dimension. Suppose that (f_n) converges uniformly on compacta to an univalent mapping f. Then $f(M)$ is a connected component of the kernel of the sequence $(f_n(M))$, and $(f_n^{-1}|_{f(M)})$ converges uniformly on compacta to $(f^{-1}|_{f(M)})$.*

Proof. Let $K \subset f(M)$ be a compact set. We want to prove that eventually $K \subset f_n(M)$. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of M such that any U_α is biholomorphic to $\mathbb{B}$, and let $\mathcal{H}$ be the open cover of M given by all open subsets H satisfying the following property: there exists $U_\alpha \in \mathcal{U}$ such that $H \subset\subset U_\alpha$. Note that f is an open mapping since M and N have the same dimension, thus $f_*\mathcal{U} = \{f(U_\alpha)\}_{U_\alpha \in \mathcal{U}}$ is an open cover of $f(M)$.

Since K is compact there exist a finite number of open subsets $H_i \in \mathcal{H}$ such that $K \subset \bigcup_i f(H_i)$. Note that on H_i the sequence f_n takes eventually values in some $f(U_{\alpha_i})$ thanks to uniform convergence on compacta. By using a partition of unity it is easy to see that there exist a finite number of compact set K_i such that $K_i \subset f(H_i)$ and $K = \bigcup_i K_i$. Thus we can assume $M \subset \mathbb{C}^q$ and $N = \mathbb{C}^q$, and the claim follws from corollary 2.3.2.

Thus $f(M)$ is a subset of the kernel of the sequence $(f_n(M))$. This implies that on any compact set $K \subset f(M)$ the sequence $f_n^{-1} \colon K \to M$ is eventually defined. Let Ω be the connected component of the kernel which contains $f(M)$. We want to prove that $(f_n^{-1}|_\Omega)$ admits a subsequence converging uniformly on compacta. Assume that $(f_n^{-1}|_\Omega)$ is compactly divergent. Since M is complete hyperbolic, this is equivalent to assume that for all fixed $z_0 \in M$ and compact sets $K \subset \Omega$ we have

$$\liminf_{n\to\infty} \left(\min_{w \in K} k_M(f_n^{-1}(w), z_0) \right) = +\infty. \tag{2.3.1}$$

Let $j \geq 0$ and let

$$K(j) \doteq \{f(z_0)\} \cup \bigcup_{n \geq j} \{f_n(z_0)\}.$$

Since $f_n(z_0) \to f(z_0)$, the set $K(j)$ is compact. Since $f(M)$ is open there exists $m > 0$ such that $K(m) \subset f(M) \subset \Omega$. But

$$k_M\left(f_n^{-1}(f_n(z_0)), z_0\right) = 0,$$

in contradition with (2.3.1).

Let $(f_{n_k}^{-1}|_\Omega)$ be a converging subsequence and let $w_0 \in \Omega$. The sequence $(f_{n_k}^{-1}(w_0))$ is eventually defined and converging to some $z = g(w_0)$. Thus $w_0 = f_{n_k}(f_{n_k}^{-1}(w_0)) \to f(z)$, which implies that $\Omega = f(M)$ and that $g(w_0) = f^{-1}(w_0)$, hence $(f_n^{-1}|_\Omega)$ converges to $f^{-1}|_\Omega$ uniformly on compacta. $\square$

2.4 From Loewner chains to evolution families

Theorem 2.4.1 ([3]). *Let $d \in [1,+\infty]$ and let (f_t,M,N) be an L^d-Loewner chain. Assume that M is complete hyperbolic. Define for $0 \leq s \leq t$ the mapping $\varphi_{t,s} \doteq f_t^{-1} \circ f_s$. Then $(\varphi_{t,s})$ is an L^d-evolution family on M associated with (f_t).*

Proof. We need to prove iii) of Definition 2.2.11. Fix $T > 0$. Let $0 \leq t \leq T$ and let H be a compact subset of $f_t(M)$. Set

$$L(H,t) \doteq \sup_{\eta,\zeta \in H, \eta \neq \zeta} \frac{d_M(f_t^{-1}(\zeta), f_t^{-1}(\eta))}{d_N(\zeta,\eta)} < +\infty$$

since $w \mapsto f_t^{-1}(w)$ is locally Lipschitz.

Given a compact subset $K \subset M$ define for $0 \leq t \leq T$

$$K_t \doteq \bigcup_{s \in [0,t]} f_s(K).$$

The set K_t is a compact subset of $f_t(M)$ by Remark 2.2.17 since $K_t = \Psi(K \times [0,t])$. We claim that the function $L(K_t,t)$ on $0 \leq t \leq T$ is bounded by a constant $C(K,T) < +\infty$. We argue by contradiction. Assume that $L(K_t,t)$ is unbounded. Then there exists a sequence $(t_n) \subset [0,T]$ such that

$$L(K_{t_n},t_n) \geq n+1, \quad \forall n \geq 0.$$

Hence for any $n \geq 0$ there exist $\zeta_n,\eta_n \in K_{t_n}$ such that $\zeta_n \neq \eta_n$ and

$$\frac{d_M(f_{t_n}^{-1}(\zeta_n), f_{t_n}^{-1}(\eta_n))}{d_N(\zeta_n,\eta_n)} \geq n. \tag{2.4.1}$$

By passing to a subsequence we may assume that the sequence $(t_n, f_{t_n}^{-1}(\zeta_n))$ in the compact set $[0,T] \times K$ converges to $(t,z) \in [0,T] \times K$. Hence the sequence $(\Psi(t_n, f_{t_n}^{-1}(\zeta_n))) = (\zeta_n)$ converges to $\zeta \doteq \Psi(t,z) = f_t(z) \in K_t$. In the same way we see that $\eta_n \to \eta \doteq \Psi(t,w) = f_t(w) \in K_t$. By (2.4.1) we have $\eta = \zeta$, since otherwise

$$\frac{d_M(f_{t_n}^{-1}(\zeta_n), f_{t_n}^{-1}(\eta_n))}{d_N(\zeta_n,\eta_n)} \to \frac{d_M(z,w)}{d_N(\zeta,\eta)}.$$

Let U,V be two open subsets of $f_t(M)$, both biholomorphic to $\mathbb{B}$ such that

$$\zeta \in U \subset\subset V \subset\subset f_t(M),$$

and let $(\psi, f_t^{-1}(V)),(\varphi,V)$ be charts around z and ζ respectively. Since by Theorem 2.3.5 the sequence $(f_{t_n}^{-1})$ converges to f_t^{-1} uniformly on V we have that eventually

$f_{t_n}^{-1}(U) \subset f_t^{-1}(V)$. Therefore the sequence $g_{t_n} = \psi \circ f_{t_n}^{-1} \circ \varphi^{-1}$ is eventually defined on $\varphi(U)$ and converges uniformly to $g = \psi \circ f_t^{-1} \circ \varphi^{-1}$. Then the sequence (g_{t_n}) is equibounded and by Cauchy estimates it is equi-Lipschitz in a neighborhood of $\varphi(\zeta)$. Since two Hermitian metrics are equivalent on compact sets, this contradicts (2.4.1) and thus proves the claim.

Let $\Delta_T \doteq \{(s,t) : 0 \leq s \leq t \leq T\}$. Then the mapping $(s,t) \mapsto f_t^{-1} \circ f_s$ from Δ_T to $\mathsf{hol}(M,M)$ endowed with the topology of uniform convergence on compacta is continuous. Indeed, let $(s_n,t_n) \to (s,t)$. Let $K \subset M$ be a compact set. By Remark 2.2.17 the set

$$K(j) \doteq f_s(K) \cup \bigcup_{n\geq j} f_{s_n}(K) = \Psi(K,\{s\} \cup \bigcup_{n\geq j}\{s_n\})$$

is compact. There exists $m > 0$ such that $K(m) \subset f_t(M)$. By Theorem 2.3.5 the sequence $(f_{t_n}^{-1})$ converges to f_t^{-1} uniformly on $K(m)$. This proves that $(s,t) \mapsto f_t^{-1} \circ f_s$ is continuous.

This implies that the mapping $\Phi \colon M \times \Delta_T \to M$ defined as $\Phi(z,s,t) \doteq \varphi_{t,s}(z)$ is jointly continuous. Hence if $K \subset M$ is a compact set,

$$\hat{K} \doteq \bigcup_{0\leq a\leq b\leq T} \varphi_{b,a}(K) = \bigcup_{0\leq a\leq b\leq T} f_b^{-1}(f_a(K))$$

is compact in M. Therefore, since

$$d_M(\varphi_{u,s}(z),\varphi_{t,s}(z)) = d_M(\varphi_{u,s}(z),\varphi_{t,u}(\varphi_{u,s}(z))),$$

in order to estimate $d_M(\varphi_{u,s}(z),\varphi_{t,s}(z))$ for $z \in K$ and $0 \leq s \leq u \leq t \leq T$ it is enough to estimate $d_M(\zeta,\varphi_{t,u}(\zeta))$ for $\zeta \in \hat{K}$.

Since

$$\begin{aligned} d_M(\zeta,\varphi_{t,u}(\zeta)) = d_M(f_t^{-1}(f_t(\zeta)),f_t^{-1}(f_u(\zeta))) &\leq L(\hat{K}_t,t) d_N(f_t(\zeta),f_u(\zeta)) \\ &\leq C(\hat{K},T) d_N(f_t(\zeta),f_u(\zeta)) \\ &\leq C(\hat{K},T) \int_u^t k_{\hat{K},T}(\xi) d\xi. \end{aligned}$$

□

2.5 From evolution families to Loewner chains

Theorem 2.5.1 ([3]). *Let $d \in [1,+\infty]$. Let $(\varphi_{t,s})$ be a L^d-evolution family on a taut manifold M. Then if (E,f_t) is a direct limit for $(M,\varphi_{t,s})$ then (f_t) is a L^d-Loewner chain associated with $(\varphi_{t,s})$ with respect to any hermitian metric on E.*

Proof. Endow E with an hermitian metric and denote d_E the induced distance.

Since M is taut, for any compact subset $K \subset M$ and any $T > 0$ the set

$$\hat{K} \doteq \bigcup_{0 \le s \le t \le T} \varphi_{t,s}(K)$$

is compact. Indeed by Proposition 2.2.12 the mapping $\Phi(z,s,t) \doteq \varphi_{t,s}(z)$ is continuous on $M \times \Delta$, hence $\hat{K} = \Phi(K,\Delta_T)$ is compact.

Let $K \subset M$ be a compact set. Let $T > 0$ be fixed. Since f_T is locally Lipshitz there exists $C = C(\hat{K}) > 0$ such that

$$d_E(f_T(z),f_T(w)) \le C d_M(z,w) \quad \forall z,w \in \hat{K}.$$

Moreover there exists $C' = C(\hat{K}) > 0$ such that $\forall z,w \in \hat{K}, \forall t \in [0,T]$,

$$d_M(\varphi_{T,t}(z),\varphi_{T,t}(w)) \le C' d_M(z,w).$$

Indeed assume by contradiction that there exist sequences $(z_n),(w_n)$ in $\hat{K}$, and (t_n) in $[0,T]$ such that

$$\frac{d_M(\varphi_{T,t_n}(z_n),\varphi_{T,t_n}(w_n))}{d_M(z_n,w_n)} \ge n. \tag{2.5.1}$$

By passing to subsequences we can assume $t_n \to t$, $z_n \to z$ and $w_n \to w$, and by (2.5.1)to see that $z = w$.

Let U,V be two open subsets of M, both biholomorphic to $\mathbb{B}$ such that

$$z \in V \subset\subset U \subset\subset M,$$

and let $(\psi,U),(\nu,\varphi_{T,t}(U))$ be charts around z and $\varphi_{T,t}(z)$ respectively. Since the sequence (φ_{T,t_n}) converges to $\varphi_{T,t}$ uniformly on U we have that eventually $\varphi_{T,t_n}(V) \subset \varphi_{T,t}(U)$. Therefore the sequence $g_{t_n} = \nu \circ \varphi_{T,t_n} \circ \psi^{-1}$ is eventually defined on $\psi(V)$ and converges uniformly to $g = \nu \circ \varphi_{T,t} \circ \psi^{-1}$. Then the sequence (g_{t_n}) is equibounded and by Cauchy estimates it is equi-Lipschitz in a neighborhood of $\psi(z)$. Since two Hermitian metrics are equivalent on compact sets, this contradicts (2.5.1).

Hence for all $z \in K$ and $0 \le s \le t \le T$ we have

$$\begin{aligned} d_E(f_s(z),f_t(z)) = d_E(f_T(\varphi_{T,s}(z)),f_T(\varphi_{T,t}(z))) &\le C d_M(\varphi_{T,s}(z),\varphi_{T,t}(z)) \\ &= C d_M(\varphi_{T,t}(\varphi_{t,s}(z)),\varphi_{T,t}(z)) \\ &\le CC' d_M(\varphi_{t,s}(z),z) \\ &= CC' d_M(\varphi_{t,s}(z),\varphi_{s,s}(z)) \\ &\le CC' \int_s^t c_{K,T}(\xi) d\xi. \end{aligned}$$

□

Corollary 2.5.2. *Let (f_t,M,N) be a L^d-Loewner chain. Then $(f_t,M,\mathsf{rg}\,(f_t))$ is a L^d-Loewner chain for any hermitian metric on $\mathsf{rg}\,(f_t)$.*

Proof. Theorem 2.4.1 yields a L^d-evolution family $(\varphi_{t,s})$ associated with (f_t,M,N). Since $(\mathsf{rg}\,(f_t),f_t)$ is a direct limit for $(M,\varphi_{t,s})$, Theorem 2.5.1 proves that $(f_t,M,\mathsf{rg}\,(f_t))$ is a L^d-Loewner chain for any hermitian metric on $\mathsf{rg}\,(f_t)$. □

2.6 Herglotz vector fields and the Loewner-Kufarev PDE

Let M be a complex manifold of complex dimension q. Let $\|\cdot\|$ be a Hermitian metric along the fibers of TM and let d_M be the induced distance on M.

Definition 2.6.1. Let $d \in [1,\infty]$. A L^d-*weak holomorphic vector field* on M is a function

$$G\colon M \times \mathbb{R}^+ \to TM$$

with the following properies:

i) for all $z \in M$ the function $\mathbb{R}^+ \ni t \mapsto G(z,t)$ is measurable,

ii) for all $t \geq 0$ the function $M \ni z \mapsto G(z,t)$ is holomorphic,

iii) for all compact set $K \subset\subset M$ and all $T > 0$ there exists a function $C_{K,T} \in L^d([0,T],\mathbb{R}^+)$ such that

$$\|G(z,t)\| \leq C_{K,T}(t)$$

for all $z \in K$ and almost every $t \in [0,T]$.

Definition 2.6.2. Let $d \in [1,\infty]$. Let M be a complex manifold. Assume that $\kappa_M \in C^1(M \times M \setminus \mathrm{Diag})$. A L^d-weak holomorphic vector field $H(z,t)$ is a L^d-*Herglotz vector field* if

$$d_{(z,w)}k_M(H(z,t),H(w,t)) \leq 0, \quad z,w \in M,\ z \neq w \ \text{ and a.e. } t \geq 0.$$

Remark 2.6.3. According to [7], if $M = D$ is a strongly convex domain in $\mathbb{C}^q$ with smooth boundary then H is a L^d-Herglotz vector field if and only if it is a L^d-weak holomorphic vector field such that for almost every $t \geq 0$ the function $D \ni z \mapsto H(z,t)$ is an infinitesimal generator of a semigroup of holomorphic self-maps of D. See [1] for definitions and properties of semigroups.

The following results are proved in [6].

Theorem 2.6.4 ([6])**.** *Let $d \in [1,\infty]$. Let M be a complete hyperbolic manifold and assume that $\kappa_M \in C^1(M \times M \smallsetminus \mathrm{Diag})$. Then for any L^d-Herglotz vector field H there exists a unique L^d-evolution family $(\varphi_{t,s},M)$ such that*

$$\frac{\partial \varphi_{t,s}}{\partial t}(z) = H(\varphi_{t,s}(z),t), \quad z \in M, \text{ a.e } t \geq s. \tag{2.6.1}$$

Definition 2.6.5. Equation (2.6.1) is called the *Loewner-Kufarev ODE* associated with H.

Theorem 2.6.6 ([6])**.** *Let M be a taut manifold. Assume that $\kappa_M \in C^1(M \times M \smallsetminus$ Diag). Then for any evolution family $(\varphi_{t,s},M)$ there exists a L^∞-Herglotz vector field H such that $(\varphi_{t,s})$ satisfies the Loewner-Kufarev ODE associated with H. Moreover if G is a weak holomorphic vector field such that $(\varphi_{t,s})$ satisfies the Loewner-Kufarev ODE associated with G then $H(z,t) = G(z,t)$ for all $z \in M$ and almost every $t \geq 0$.*

We can now prove the following results.

Lemma 2.6.7. *Let $d \in [1,+\infty]$, let (f_t,M,N) be a L^d-Loewner chain. There exists a set $E \subset \mathbb{R}^+$ (independent of z) of zero measure such that for every $s \in \mathbb{R}^+ \setminus E$, the mapping*

$$M \ni z \mapsto \frac{\partial f_s}{\partial s}(z) \in T_{f_s(z)}N$$

is well-defined and holomorphic on M.

Proof. Let $\mathbb{R}^>$ denote the set of strictly positive real numbers. The manifold $M \times \mathbb{R}^>$ has a countable basis $\mathfrak{B}$ given by products of the type $B \times I$, where $B \subset M$ is an open subset biholomorphic to a ball, and $I \subset \mathbb{R}^>$ is an open interval. Let $\mathcal{V}$ be a countable covering of N by chart domains. The mapping $\Psi\colon M \times \mathbb{R}^> \to N$ is continuous, hence there exists a covering $\mathcal{U}$ of $M \times \mathbb{R}^>$ such that $\mathcal{U} \subset \mathfrak{B}$ and such that $\mathcal{U} < f^{-1}(\mathcal{V})$.

We will prove that for each $U = B \times I \in \mathcal{U}$ there exists a set of full measure $I' \subseteq I$ such that $B \ni z \mapsto \frac{\partial f_s}{\partial s}(z)$ is well defined and holomorphic for all $t \in I'$. Hence $M \ni z \mapsto \frac{\partial f_s}{\partial s}(z)$ will be well defined and holomorphic on $\bigcap_{\mathcal{U}} I'$ which is a set of full measure in $\mathbb{R}^+$.

We can assume that $M = \mathbb{B}^q$, $N = \mathbb{C}^q$, and that the distances d_M,d_N on M,N are given by the standard Hermitian metric on $\mathbb{C}^n$. Since $t \mapsto f_t(z)$ is locally absolutely continuous on $\mathbb{R}^+$ locally uniformly with respect to $z \in \mathbb{B}^n$, we deduce that for each $z \in \mathbb{B}^q$, there is a null set $E_1(z) \subset I$ such that for each $t \in I \setminus E_1(z)$, there exists the limit

$$\frac{\partial f_t}{\partial t}(z) = \lim_{h \to 0} \frac{f_{t+h}(z) - f_t(z)}{h}.$$

By Definition 2.2.16, there exists a function $p_k \in L^d_{loc}(I,\mathbb{R}^+)$ such that

$$\|f_s(z) - f_t(z)\| \leq \int_s^t p_k(\xi)d\xi, \quad \|z\| \leq 1 - \frac{1}{k}, \quad s \leq t \in I. \tag{2.6.2}$$

Also, since $p_k \in L^d_{loc}(I,\mathbb{R}^+)$, we may find a null set $E_2(k) \subset I$ such that for each $t \in I \setminus E_2(k)$, there exists the limit

$$p_k(t) = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} p_k(\xi)d\xi, \quad k \in \mathbb{N}. \tag{2.6.3}$$

Next, let Q be a countable set of uniqueness for the holomorphic functions on $\mathbb{B}^q$ and let

$$E = \left(\bigcup_{q\in Q} E_1(q)\right) \bigcup \left(\bigcup_{k=1}^{\infty} E_2(k)\right).$$

Then E is a null subset of $\mathbb{R}^+$, which does not depend on $z \in \mathbb{B}^n$. Arguing as in the proof of [9, Theorem 4.1(1)(a)], it is not difficult to see that (2.6.2) and (2.6.3) imply that for each $s \in I \setminus E$, the family

$$\{(f_{s+h}(z) - f_s(z))/h, \quad |h| < \mathsf{dist}(s,\partial I)\}$$

is relatively compact and has a unique accumulation point for $|h| \to 0$ by Vitali Theorem in several complex variables, proving the result.

□

Theorem 2.6.8 ([3]). *Let M be a complete hyperbolic manifold.*

i) *Let (f_t,M,N) be a L^∞-Loewner chain on M. Then there exist a L^∞-Herglotz vector field $G(z,t)$ and a set $E \subset \mathbb{R}^+$ (independent of z) of zero measure such that for every $s \in (0,\infty) \setminus E$ and every $z \in M$,*

$$\frac{\partial f_s}{\partial s}(z) = -d_z f_s(G(z,s)).$$

ii) *Let $d \in [1,\infty]$. Let $G(z,t)$ be a L^d-Herglotz vector field associated with the L^d-evolution family $(\varphi_{t,s})$. Assume that (f_t) is a family of univalent mappings from M into a complex manifold N of the same dimension, such that the mapping $t \mapsto f_t(z)$ is locally absolutely continuous on $\mathbb{R}^+$ locally uniformly with respect to $z \in M$. In addition, assume that*

$$\frac{\partial f_s}{\partial s}(z) = -d_z f_s(G(z,s)) \qquad a.e.\ s \geq 0, \forall z \in M.$$

Then (f_t,M,N) is a L^d-Loewner chain associated with the L^d-evolution family $(\varphi_{t,s})$.

Proof. i) By Theorem 2.4.1, there exists a L^∞-evolution family $(\varphi_{t,s})$ associated with (f_t). By Theorem 2.6.6, there exists a L^∞-Herglotz vector field $G : M \times \mathbb{R}^+ \to TM$ such that

$$\frac{\partial \varphi_{t,s}}{\partial t}(z) = G(\varphi_{t,s}(z),t) \quad \text{a.e. } t \geq s, \quad \forall z \in M.$$

Let $E_1 \subset \mathbb{R}^+$ be a null set such that the above equality holds for all $t \in [s,\infty) \setminus E_1$ and for all $z \in M$. By Lemma 2.6.7, there is a null set $E_2 \subset \mathbb{R}^+$ such that $z \mapsto \frac{\partial f_s}{\partial s}(z)$ is well defined and holomorphic for all $s \in \mathbb{R}^+ \setminus E_2$. The set $E = E_1 \cup E_2$ has also zero measure. It is clear that the mapping $L_t(z) = f_t(\varphi_{t,0}(z))$ is locally absolutely continuous on $\mathbb{R}^+$ locally uniformly with respect to $z \in M$. Also $L_t(z) = f_0(z)$ for $z \in M$. Differentiating the last equality with respect to $t \in (0,\infty) \setminus E$ we obtain

$$\begin{aligned} 0 &= d_{\varphi_{t,0}(z)} f_t \left(\frac{\partial \varphi_{t,0}}{\partial t}(z) \right) + \frac{\partial f_t}{\partial t}(\varphi_{t,0}(z)) \\ &= d_{\varphi_{t,0}(z)} f_t(G(\varphi_{t,0}(z),t)) + \frac{\partial f_t}{\partial t}(\varphi_{t,0}(z)), \end{aligned}$$

for all $t \in (0,\infty) \setminus E$ and for all $z \in M$. Hence

$$\frac{\partial f_t}{\partial t}(w) = -d_w f_t(G(w,t)),$$

for all w in the open set $\varphi_{t,0}(M)$ and for all $t \in (0,\infty) \setminus E$. The identity theorem for holomorphic mappings provides the result.

ii) Fix $s \geq 0$ and let $L_t(z) = f_t(\varphi_{t,s}(z))$ for $t \in [s,\infty)$ and $z \in M$. In view of the hypothesis, it is not difficult to deduce that

$$\frac{\partial L_t}{\partial t}(z) = 0, \quad \text{a.e.} \quad t \in \mathbb{R}^+, \quad \forall z \in M.$$

Hence $L_t(z) \equiv L_s(z)$, i.e.

$$f_t(\varphi_{t,s}(z)) = f_s(z),$$

for all $z \in M$ and $t \geq s$. Hence (f_t) is a L^d-Loewner chain by Theorem 2.5.1. □

Definition 2.6.9. The partial differential equation

$$\frac{\partial f_s}{\partial s}(z) = -d_z f_s(G(z,s)) \quad \text{a.e. } s \geq 0, \forall z \in M$$

is called the *Loewner-Kufarev PDE.*

2.7 Kobayashi metric and complex structure of the Loewner range

Let $(\varphi_{\beta,\alpha})$ be a I-evolution family on a hyperbolic manifold M. Let $\kappa_M : TM \to \mathbb{R}^+$ be the Kobayashi metric of M. For $v \in T_zM$ we define

$$\beta_v^\alpha(z) \doteq \lim_{\beta\to\infty} \kappa_M(\varphi_{\beta,\alpha}(z), d_z\varphi_{\beta,\alpha}(v)). \tag{2.7.1}$$

Remark 2.7.1. The Kobayashi metric is contracted by holomorpic mappings, hence if $0 \le \alpha \le \eta \le \beta$,

$$\begin{aligned}\kappa_M(\varphi_{\beta,\alpha}(z), d_z\varphi_{\beta,\alpha}(v)) &= \kappa_M(\varphi_{\beta,\eta}(\varphi_{\eta,\alpha}(z)), d_{\varphi_{\eta,\alpha}(z)}\varphi_{\beta,\eta} \circ d_z\varphi_{\eta,\alpha}(v)) \\ &\le \kappa_M(\varphi_{\eta,\alpha}(z), d_z\varphi_{\eta,\alpha}(v)),\end{aligned}$$

thus $\beta_v^\alpha(z)$ is well defined.

Proposition 2.7.2. *Let $(\varphi_{\beta,\alpha})$ be a I-evolution family on a hyperbolic complex manifold M. Let (f_α, M, N) be an associated I-Loewner chain with range N. For all $v \in T_zM$*

$$f_\alpha^* \kappa_N(z,v) = \beta_v^\alpha(z).$$

Proof. Since N is the union of the growing sequence of manifolds $(f_m(M))_{m\in\mathbb{N}}$ it is easy to see that the Kobayashi metric satisfies

$$\kappa_N(f_\alpha(z), d_zf_\alpha(v)) = \lim_{m\to\infty} \kappa_{f_m(M)}(f_\alpha(z), d_zf_\alpha(v)).$$

The result follows from

$$\begin{aligned}\lim_{m\to\infty} \kappa_{f_m(M)}(f_\alpha(z), d_zf_\alpha(v)) &= \lim_{m\to\infty} \kappa_M(f_m^{-1}(f_\alpha(z)), d_{f_\alpha(z)}f_m^{-1} \circ d_zf_\alpha(v)) \\ &= \lim_{m\to\infty} \kappa_M(\varphi_{m,\alpha}(z), d_z\varphi_{m,\alpha}(v)).\end{aligned}$$

□

As a corollary we find (cf. [9, Theorem 1.6])

Corollary 2.7.3. *Let $(\varphi_{\beta,\alpha})$ be a I-evolution family in the unit disc $\mathbb{D}$. If there exists $z \in \mathbb{D}, v \in T_z\mathbb{D} = \mathbb{C}, \alpha \ge 0$ such that $\beta_v^\alpha(z) = 0$ then*

i) $\mathsf{Lr}(\varphi_{\beta,\alpha}) = \mathbb{C}$,

ii) $\beta_v^\alpha(z) = 0$ *for all* $z \in \mathbb{D}, v \in \mathbb{C}, \alpha \ge 0$.

If there exists $z \in \mathbb{D}, v \in T_z\mathbb{D} = \mathbb{C}, \alpha \ge 0$ such that $\beta_v^\alpha(z) \ne 0$ then *i)* $\mathsf{Lr}(\varphi_{\beta,\alpha}) = \mathbb{D}$,

ii) $\beta_v^\alpha(z) \neq 0$ *for all* $z \in \mathbb{D}, v \in \mathbb{C}, \alpha \geq 0$.

Proof. Since $\mathsf{Lr}(\varphi_{t,s})$ is non-compact and simply connected, by the uniformization Theorem it has to be biholomorphic either to $\mathbb{D}$ or to $\mathbb{C}$. Since

$$k_{\mathbb{C}}(z,v) = 0, \quad z \in \mathbb{C}, v \in \mathbb{C},$$

$$k_{\mathbb{D}}(z,v) \neq 0, \quad z \in \mathbb{D}, v \in \mathbb{C},$$

the result follows from Proposition 2.7.2. □

In order to further study the complex structure and the Kobayashi metric of the Loewner range we focus on the "union problem": let N be a complex manifold, exhausted by a sequence of open submanifolds

$$M_0 \subset M_1 \subset M_2 \ldots,$$

all biholomorphic to a fixed manifold M. What can be said about the complex structure of N?

A first answer is as follows. Recall that a complex manifold N is *Stein* if and only if

i) the algebra $\mathcal{O}(N)$ of holomorphic functions separates points of N,

ii) N is *holomorphically convex*, that is for every compact subset $K \subset N$ the $\mathcal{O}(N)$-envelope of K

$$\hat{K} \doteq \left\{ x \in N : |f(x)| \leq \max_K |f|, \quad \forall f \in \mathcal{O}(N) \right\}$$

is compact.

Theorem 2.7.4 ([4]). *If the manifold N is an open subset of $\mathbb{C}^q$ and $M = \mathbb{B}^q$ then N is Stein.*

For a general complex manifold N this is no longer true, indeed there is the following result.

Theorem 2.7.5 ([12]). *There exists a 3-dimensional complex manifold N enjoying the following properties:*

1. *N can be exhausted by open submanifolds $B_1 \subset B_2 \subset B_3 \ldots$, where each B_i is biholomorphic to $\mathbb{B} \subset \mathbb{C}^3$.*

2. *N is not holomorphically convex, hence it is not Stein.*

Definition 2.7.6. Let N be a complex manifold and κ_N its Kobayashi metric. The *null set* $\mathcal{N}_N(z)$ of κ_N in $z \in M$ is the complex subspace $\{v \in T_z M : \kappa_N(v) = 0\}$. The *corank* at z is $\dim \mathcal{N}_N(z)$.

Let M be a hyperbolic complex manifold such that $M/\mathsf{aut}(M)$ is compact. Choose a compact set K such that $M = \mathsf{aut}(M) \cdot K$. Then there exist a point $w_0 \in M_0$ and biholomorphic maps $g_n \colon M \to M_n$ such that $g_n^{-1}(w_0) \in K$. By Lemma 1.0.13 M is taut, hence by Proposition 1.0.2 the sequence (g_n^{-1}) converges up to a subsequence uniformly on compact sets on N to a mapping $\psi \colon N \to M$.

The maps $\alpha_n \colon M \to M$ defined by $\alpha_n = \psi \circ g_n$ converge up to a subsequence to a holomorphic map $\alpha \colon M \to M$ since M is taut and $\alpha_n(K)$ contains $\psi(w_0)$.

The following results are proved in [13].

Lemma 2.7.7. *The following facts hold:*

i) If $w \in N$ and $v \in T_w N$ then $k_N(w,v) = \psi^* k_M(w,v)$, hence $\mathcal{N}_N(w) = \ker d_w \psi$.

ii) The map ψ has constant rank k and k_N has constant corank $q - k$.

iii) We have $\psi = \alpha \circ \psi$. The set $Z \doteq \{z \in M : \alpha(z) = z\}$ is a connected closed k-dimensional complex submanifold of M and is equal to $\psi(M)$. The mapping α is a retraction of M to Z.

iv) The fibers of ψ are $(q-k)$-dimensional complex manifolds and topologically cells. Moreover for any $z \in Z$, $k_{\psi^{-1}(z)} \equiv 0$.

Theorem 2.7.8 ([13])**.** *Assume that M is hyperbolic and that $M/\mathsf{aut}(M)$ is compact. If there exists $z \in M$ such that $k_N(z,v) \neq 0$ whenever $v \neq 0$ in $T_z M$, then N is biholomorphic to M.*

Theorem 2.7.9 ([13])**.** *Assume that M is hyperbolic and that $M/\mathsf{aut}(M)$ is compact. If the corank of κ_N is one, then (N,Z,ψ) is a locally trivial holomorphic fiber bundle with fiber $\mathbb{C}$.*

Corollary 2.7.10. *Assume $M = \mathbb{B}^q$ and that the corank of κ_N is one. Then N is biolomorphic to $\mathbb{B}^{q-1} \times \mathbb{C}$.*

Remark 2.7.11. The manifold N in Proposition 2.7.5 has constant corank 2.

Remark 2.7.12. Let $(\varphi_{\beta,\alpha})$ be a I-evolution family on a hyperbolic manifold M such that $M/\mathsf{aut}(M)$ is compact, and let (f_α, M, N) be an associated I-Loewner chain such that $\cup_{\alpha \geq 0} f_\alpha(M) = N$. Assume that $(\varphi_{\beta,\alpha})$ admits a relatively compact trajectory. Then we can assume that $f_n = g_n, \psi = \lim_{n_k \to \infty} f_{n_k}^{-1}$, and $\alpha_n = \psi \circ f_n = \lim_{n_k \to \infty} \varphi_{n_k,n}$.

We want now to investigate the following problem.

Problem 2.7.13. *Give conditions on a I-evolution family $(\varphi_{\beta,\alpha})$ on $\mathbb{B}^q$ ensuring the existence of an associated Loewner chain $(f_\alpha,M,\mathbb{C}^q)$.*

Remark 2.7.14. The previous problem can be formulated in the following terms: when is $\mathsf{Lr}(\varphi_{\beta,\alpha})$ an open set in $\mathbb{C}^q$?

Using the previous results we give a (partial) answer to Problem 2.7.13.

Theorem 2.7.15 ([3])**.** *Let $(\varphi_{\beta,\alpha})$ be a I-evolution family on $\mathbb{B}^q$. Assume that there exist $z \in \mathbb{B}^q,\alpha \geq 0$ such that*

$$\dim\{v \in T_z\mathbb{B}^q : \beta_v^\alpha(z) = 0\} \leq 1.$$

Then $\mathsf{Lr}(\varphi_{\beta,\alpha})$ is an open set in $\mathbb{C}^q$.

Proof. The result follows from Proposition 2.7.2, Theorem 2.7.8 and Corollary 2.7.10. □

We have also a negative result.

Proposition 2.7.16. *There exists a $\mathbb{N}$-evolution family $(\varphi_{m,n})$ on $\mathbb{B}^3$ which does not admit any associated $\mathbb{N}$-Loewner chain $(f_n,\mathbb{B}^3,\mathbb{C}^3)$.*

Proof. Let $N = \cup_{n\geq 0}B_n$ given by Proposition 2.7.5. Let $f_i \colon \mathbb{B}^3 \to B_i$ be a biholomorphism, and define $\varphi_{n+1,n} = f_{n+1}^{-1} \circ f_n$. Assume that there exists a $\mathbb{N}$-Loewner chain $(g_n,\mathbb{B}^3,\mathbb{C}^3)$ associated with $(\varphi_{m,n})$. Then since both (N,f_n) and $(\bigcup_n g_n(\mathbb{B}),g_n)$ are direct limits for $(\mathbb{B}^3,\varphi_{t,s})$ we have that N is biholomorphic to $\bigcup_n g_n(\mathbb{B})$ which by Proposition 2.7.4 is Stein, contradiction. □

Remark 2.7.17. Setting $f_s \doteq f_{[s]}$ and $\varphi_{t,s} \doteq f_t^{-1} \circ f_s$ we obtain a $\mathbb{R}^+$-evolution family $(\varphi_{t,s})$ on $\mathbb{B}^3$ which does not admit any associated $\mathbb{R}^+$-Loewner chain $(f_s,\mathbb{B}^3,\mathbb{C}^3)$. It is not known whether there exists a L^d-evolution family with the same property.

In the next chapter we are going to give an answer to Problem 2.7.13 for a particular class of I-evolution families proving that they have Loewner range $\mathbb{C}^q$.

Chapter 3

Resonances in Loewner equations

3.1 Dilation $\mathbb{N}$-evolution families

We begin by studying Problem 2.7.13 in the discrete case.

Definition 3.1.1. Let $A \in \mathcal{L}(\mathbb{C}^q,\mathbb{C}^q)$. The *spectrum* $\sigma(A)$ of A is the set of its eigenvalues. The *spectral radius* $\rho(A)$ is $\max_{\lambda\in\sigma(A)} |\lambda|$.

Definition 3.1.2. Let D be a domain in $\mathbb{C}^q$ containing 0. A $\mathbb{N}$-evolution family $(\varphi_{m,n})$ on D is a *dilation* $\mathbb{N}$*-evolution family* if for all $0 \leq n$,

$$\varphi_{n+1,n}(z) = A(z) + O(|z|^2), \tag{3.1.1}$$

with $A \in \mathcal{A}(\mathbb{C}^q)$ such that $\rho(A) < 1$. A dilation $\mathbb{N}$-evolution family $(\varphi_{m,n},D)$ is of *semigroup type* if for all $n \geq 0$,

$$\varphi_{n+1,n} = \varphi_{1,0}.$$

A $\mathbb{N}$-Loewner chain $(f_n,D,\mathbb{C}^q)$ is a *dilation* $\mathbb{N}$*-Loewner chain* if for all $n \geq 0$,

$$f_n(z) = A^{-n}(z) + O(|z|^2),$$

with $A \in \mathcal{A}(\mathbb{C}^q)$ such that $\rho(A) < 1$. A dilation Loewner chain is *normal* if $(A^n \circ f_n)$ is a normal family.

Remark 3.1.3. Let $(\varphi_{m,n},D)$ be a dilation semigroup type $\mathbb{N}$-evolution family on a taut manifold D. Then by Poincaré-Dulac Theorem (see for example [23, Appendix]) there exists a univalent mapping $h(z) = z+O(|z|^2)$ defined in a neighborhood of the origin which conjugates $\varphi_{1,0}$ to a polynomial automorphism Q of $\mathbb{C}^q$. This mapping h is given by

$$\lim_{n\to\infty} Q^{-n} \circ k \circ \varphi_{n,0},$$

where k is a well-chosen polynomial mapping $k\colon \mathbb{C}^q \to \mathbb{C}^q$.

Since D is a taut manifold, by Theorem 1.0.4 it follows that $\lim_{n\to\infty} Q^{-n} \circ k \circ \varphi_{n,0}$ defines a univalent mapping h on the whole D such that

$$h \circ \varphi_{1,0} = Q \circ h.$$

Note that

$$h \circ \varphi_{m,n} = Q^{m-n} \circ h$$

and that

$$Q^{-m} \circ h \circ \varphi_{m,n} = Q^{-n} \circ h.$$

Thus $(Q^{-n} \circ h, D, \mathbb{C}^q)$ is a N-Loewner chain associated with $(\varphi_{m,n})$.

Motivated by this remark we are going to introduce a notion of conjugacy for general dilation type $\mathbb{N}$-evolution families.

Definition 3.1.4. Let $(\varphi_{m,n}, D)$ be a dilation $\mathbb{N}$-evolution family. An open subset $\Omega \subset D$ is *invariant* if for all $0 \leq n \leq m$,

$$\varphi_{m,n}(\Omega) \subset \Omega.$$

Note that $(\varphi_{m,n}, \Omega)$ defines a dilation $\mathbb{N}$-evolution family, called the *restriction* of $(\varphi_{m,n}, D)$ to Ω.

Definition 3.1.5. A dilation $\mathbb{N}$-evolution family $(\varphi_{m,n})$ is *shrinking* if

i) the family $(\varphi_{n+1,n})$ is uniformly bounded on compacta,

ii) $\lim_{p\to\infty} \varphi_{n+p,n} = 0$ uniformly on compacta, uniformly in $n \geq 0$.

Remark 3.1.6. A dilation semigroup type $\mathbb{N}$-evolution family $(\varphi_{m,n}, D)$ on a taut manifold D is shrinking by Theorem 1.0.4.

Remark 3.1.7. The property of being shrinking is invariant by biholomorphisms. Namely, if $(\varphi_{m,n}, D)$ is shrinking and $\Psi\colon D \to Q \subset \mathbb{C}^q$ is a biholomorphism fixing 0, then

$$(\Psi \circ \varphi_{m,n} \circ \Psi^{-1}, Q)$$

is shrinking.

We recall a classic result from matrix analysis.

Lemma 3.1.8 ([16, Lemma 5.6.10]). *Let $A \in M_{q,q}(\mathbb{C})$ and let $\delta > 0$. Then there exists $L \in GL_q(\mathbb{C})$ such that*

$$\|LAL^{-1}\|_2 \leq \rho(A) + \delta.$$

If $\mathcal{I}$ is the set of all matrix norms induced by a norm on $\mathbb{C}^q$, then

$$\rho(A) = \inf_{\|\cdot\|\in\mathcal{I}}\{\|A\|\}.$$

Furthermore if $\rho(A)$ denotes the spectral radius,

$$\lim_{k\to\infty} A^k = 0 \quad \textit{if and only if} \quad \rho(A) < 1.$$

Proposition 3.1.9. *Let $(\varphi_{m,n},\mathbb{B})$ be a dilation $\mathbb{N}$-evolution family. Then $(\varphi_{m,n})$ is shrinking.*

Proof. Since $\rho(A) < 1$ we have by Lemma 3.1.8,

$$\lim_{k\to\infty} A^k = 0.$$

Hence there exists an integer $p > 0$ such that $\|A^p\| < 1$. Let $n \in \mathbb{N}$, let $K \subset \mathbb{B}$ be a compact subset and let $0 < t < 1$. Consider the family $(\varphi_{(m+1)p+n,mp+n})$, $m \geq 0$. Lemma A.0.7 gives $m > 0$ (independent of n) such that $\varphi_{pm+n,n}(K) \subset t\mathbb{B}$. Thus for all $j \geq pm + n$,

$$\varphi_{j,n}(K) = \varphi_{j,pm+n} \circ \varphi_{pm+n,n}(K) \subset t\mathbb{B}.$$

□

Lemma 3.1.10. *Let $(\varphi_{m,n},D)$ be a shrinking $\mathbb{N}$-evolution family. Then for any neighborhood U of 0 there exists an open invariant set $\Omega \subset U$ biholomorphic to $\mathbb{B}$.*

Proof. Since $\rho(A) < 1$, by Lemma 3.1.8 there exists $L \in \mathcal{A}(\mathbb{C})$ such that

$$\|L \circ A \circ L^{-1}\| < 1.$$

The dilation $\mathbb{N}$-evolution family $(\psi_{m,n},Q) \doteq (L \circ \varphi_{m,n} \circ L^{-1},L(D))$ is also shrinking by Remark 3.1.7, thus in particular the family $(\psi_{n+1,n})$ is uniformly bounded on compacta. We can hence apply Lemma A.0.6 to $(\psi_{m,n},Q)$ obtaining an invariant ball $s\mathbb{B}$. Note that any ball $r\mathbb{B}$ with $0 < r \leq s$ is also invariant by Schwarz Lemma A.0.4. Thus for all $0 < r \leq s$,

$$\Omega_r \doteq L^{-1}(r\mathbb{B})$$

is invariant for $(\varphi_{m,n})$.

□

Definition 3.1.11. Two shrinking $\mathbb{N}$-evolution families $(\varphi_{m,n},D)$ and $(\psi_{m,n},Q)$ are *locally conjugate* if there exists, on an invariant open set $\Omega \subset D$ biholomorphic to $\mathbb{B}$, a family (h_n) of holomorphic mappings $h_n \colon \Omega \to Q$ such that

i) (h_n) is a uniformly bounded family,

ii) $h_n(z) = z + O(|z|^2), \quad n \geq 0,$

iii)

$$h_m \circ \varphi_{m,n} = \psi_{m,n} \circ h_n, \quad 0 \leq n \leq m. \tag{3.1.2}$$

The mappings of the family (h_n) are called *intertwining mappings*.

Proposition 3.1.12. *Each intertwining mapping h_n is univalent.*

Proof. Notations are as in previous definition. Assume that there exist $z \neq w$ in Ω and $n \geq 0$ such that $h_n(z) = h_n(w)$. Then by (3.1.2),

$$h_m(\varphi_{m,n}(z)) = h_m(\varphi_{m,n}(w)), \quad 0 \leq n \leq m. \tag{3.1.3}$$

By Lemma A.0.8 there exists a ball $s\mathbb{B} \subset \Omega$ such that for all $m \geq 0$ the mapping $h_m|_{s\mathbb{B}}$ is univalent. Since $(\varphi_{m,n})$ is shrinking, there exists $m \geq n$ such that $\varphi_{m,n}(z) \cup \varphi_{m,n}(w) \subset s\mathbb{B}$. But $\varphi_{m,n}(z) \neq \varphi_{m,n}(w)$ since $\varphi_{m,n}$ is a univalent mapping, hence (3.1.3) contradicts the univalence of $h_m|_{s\mathbb{B}}$.

□

Proposition 3.1.13. *Local conjugacy is an equivalence relation.*

Proof. Notations are as in previous definition.

i) Reflexivity: take $h_n = \mathsf{id}$ for all $n \geq 0$.

ii) Simmetry: by Lemma A.0.8 there exists a ball $t\mathbb{B}$ such that $t\mathbb{B} \subset h_n(s\mathbb{B})$ for all $n \geq 0$. Let $\Omega' \subset t\mathbb{B}$ be the invariant open set for $(\psi_{m,n})$ given by Lemma 3.1.10. On Ω' the family (h_n^{-1}) is uniformly bounded by s and satisfies

$$h_m^{-1} \circ \psi_{m,n} = \varphi_{m,n} \circ h_n^{-1}, \quad 0 \leq n \leq m.$$

iii) Transitivity: follows from Corollary 2.1.9 and the equicontinuity in 0 of the family (h_n).

□

Proposition 3.1.14. *Let $(\varphi_{m,n},D)$ be a shrinking $\mathbb{N}$-evolution family. Let Ω be an invariant set. Assume that (f_n,Ω,F) is a Loewner chain associated with $(\varphi_{m,n},\Omega)$. Then*

$$f_n^e \doteq \lim_{m\to\infty} f_m \circ \varphi_{m,n}$$

defines a Loewner chain (f_n^e,D,F) associated with $(\varphi_{m,n},D)$, such that

$$f_n^e|_\Omega = f_n.$$

Hence we can denote f_n^e by f_n.

Proof. Fix $n \geq 0$ and let $K \subset D$ be a compact subset. Since $(\varphi_{m,n}, D)$ is shrinking, there exists $k \geq n$ such that if $m \geq k$ then $\varphi_{m,n}(K) \subset \Omega$. The sequence $f_m \circ \varphi_{m,n}|_K$ is well defined for $m \geq k$ and satisfies

$$f_m \circ \varphi_{m,n}|_K = f_m \circ \varphi_{m,k} \circ \varphi_{k,n}|_K = f_k \circ \varphi_{k,n}|_K,$$

hence the uniform convergence on compacta is trivially verified.

The mapping f_n^e is trivially univalent for all $n \geq 0$. The sequence (f_n^e, D, F) is a Loewner chain associated with $(\varphi_{m,n}, D)$. Indeed if $0 \leq n \leq m$ then

$$f_n^e = \lim_{j \to \infty} f_j \circ \varphi_{j,n} = \lim_{j \to \infty} f_j \circ \varphi_{j,m} \circ \varphi_{m,n} = f_m^e \circ \varphi_{m,n}.$$

For all $n \geq 0$ we have $f_n^e|_\Omega = f_n$. Indeed $f_n^e|_\Omega = \lim_{m \to \infty} f_m \circ \varphi_{m,n}|_\Omega = f_n$ since (f_n, Ω, F) is a Loewner chain associated with $(\varphi_{m,n}, \Omega)$. □

Proposition 3.1.15. *Let $(\varphi_{m,n}, D)$ and $(\psi_{m,n}, Q)$ be two locally conjugate shrinking $\mathbb{N}$-evolution families. Then*

$$\mathsf{Lr}(\varphi_{m,n}) = \mathsf{Lr}(\psi_{m,n}).$$

Proof. Let (f_n, Q, F) be a Loewner chain associated with $(\psi_{m,n}, Q)$ with range F. Then by Remark 2.1.7 $(f_n \circ h_n, \Omega, F)$ is a Loewner chain associated with $(\varphi_{m,n}, \Omega)$.

Since $(\psi_{m,n})$ is shrinking we claim that

$$\mathsf{rg}\,(f_n \circ h_n, \Omega, F) = F.$$

Indeed, by Lemma A.0.8 there exists a ball $s\mathbb{B}$ such that

$$s\mathbb{B} \subset h_n(\Omega), \quad n \geq 0.$$

Given a point $x \in Q$, since $(\psi_{m,n})$ is shrinking we have $\psi_{m,n}(x) \in s\mathbb{B}$ when m is large enough. Hence for a fixed $n \geq 0$, we have, when m is large enough,

$$f_n(x) = f_m(\psi_{m,n}(x)) \in f_m(s\mathbb{B}).$$

Thus

$$\bigcup_{n \geq 0} f_n(h_n(\Omega)) \supset \bigcup_{n \geq 0} f_n(s\mathbb{B}) \supset \bigcup_{n \geq 0} f_n(x).$$

Since

$$\bigcup_{n \geq 0} f_n(Q) = \bigcup_{x \in Q} \bigcup_{n \geq 0} f_n(x)$$

we have $\mathsf{rg}\,(f_n \circ h_n, \Omega, F) = F$.

By previous Proposition we can extend this chain to the whole of D, thus obtaining the result. □

In order to prove that any dilation $\mathbb{N}$-evolution family $(\varphi_{m,n})$ has Loewner range $\mathbb{C}^q$ we are going to locally conjugate it with a well-behaved evolution family and then we use Proposition 3.1.15. In the next section we introduce this "special" evolution families.

3.2 Triangular $\mathbb{N}$-evolution families

Definition 3.2.1. Let Q be a polynomial mapping from $\mathbb{C}^q$ to $\mathbb{C}^q$. The *degree* $\deg Q$ is the maximum of the degree of its components.

A *polynomial* $\mathbb{N}$*-evolution family* is a dilation $\mathbb{N}$-evolution family $(Q_{m,n},\mathbb{C}^q)$ such that each $Q_{n+1,n}$, and hence every $Q_{m,n}$, is a polynomial automorphism of $\mathbb{C}^q$.

A polynomial $\mathbb{N}$-evolution family $(Q_{m,n})$ has *uniformly bounded coefficients* if the family $(Q_{n+1,n})$ has uniformly bounded coefficients. A polynomial $\mathbb{N}$-evolution family $(Q_{m,n})$ has *uniformly bounded degree* if the family $(Q_{n+1,n})$ has uniformly bounded degree.

Definition 3.2.2. A *triangular mapping* is a mapping $T\colon \mathbb{C}^q \to \mathbb{C}^q$ of the form

$$\begin{aligned}
T^{(1)}(z) &= \lambda_1 z_1,\\
T^{(2)}(z) &= \lambda_2 z_2 + t^{(2)}(z_1),\\
T^{(3)}(z) &= \lambda_3 z_3 + t^{(3)}(z_1,z_2),\\
&\vdots\\
T^{(q)}(z) &= \lambda_q z_q + t^{(q)}(z_1,z_2,\dots,z_{q-1}),
\end{aligned}$$

where $\lambda_i \in \mathbb{C}$ and $t^{(i)}$ is a polynomial in $i-1$ variables fixing the origin.

If $\lambda_i \neq 0$ for all $1 \le i \le q$, the mapping T is called a *triangular automorphism*. This is indeed an automorphism, since we can iteratively write its inverse, which is still a triangular mapping:

$$\begin{aligned}
z_1 &= w_1/\lambda_1,\\
z_2 &= w_2/\lambda_2 - (1/\lambda_2)t^{(2)}(z_1),\\
&\vdots\\
z_q &= w_q/\lambda_q - (1/\lambda_q)t^{(q)}(z_1,z_2,\dots,z_{q-1}).
\end{aligned} \tag{3.2.1}$$

Triangular automorphisms form a subgroup of $\mathsf{aut}(\mathbb{C}^q)$.

A *triangular* $\mathbb{N}$*-evolution family* is a dilation $\mathbb{N}$-evolution family $(T_{m,n},\mathbb{C}^q)$ such that each $T_{n+1,n}$, and hence every $T_{m,n}$, is a triangular automorphism of $\mathbb{C}^q$.

The following Lemmas are just adaptations of [23, Lemma 1, p.80].

Lemma 3.2.3. *If $(T_{m,n})$ is a triangular $\mathbb{N}$-evolution family of uniformly bounded degree, then* $\sup_n \deg T_{n,0} < \infty$.

Proof. Set

$$\mu^{(j)} = \max_n \deg T^{(j)}_{n+1,n}.$$

We denote by $S(m,k)$ the property

$$\deg T^{(j)}_{k,0} \le \mu^{(1)} \cdots \mu^{(j)}, \;\; 1 \le j \le m.$$

Since $T_{k+1,0} = T_{k+1,k} \circ T_{k,0}$, we have

$$T^{(j)}_{k+1,0} = \lambda_j T^{(j)}_{k,0} + t^{(j)}_{k+1,k}(T^{(1)}_{k,0}, \ldots, T^{(j-1)}_{k,0}),\ \ 2 \le j \le q,$$

thus $S(m,k+1)$ follows from $S(m,k)$ and $S(m-1,k)$. Since $S(1,k)$ and $S(m,1)$ are obviously true for all k and m (note that $\mu^{(1)} = 1$, e $\mu^{(j)} \ge 1$, for every j), $S(q,k)$ follows by induction. Hence

$$\deg T_{k,0} \le \mu^{(1)} \cdots \mu^{(q)}.$$

□

Proposition 3.2.4. *Let $(T_{m,n})$ be a triangular ℕ-evolution family of uniformly bounded degree and uniformly bounded coefficients. Let Δ be the unit polydisc. Then there exists a constant $\gamma \ge 0$ such that*

$$T_{0,k}(\Delta) \subset \gamma^k \Delta, \quad k \ge 0.$$

Proof. The family $(T_{n,n+1})$ of inverses of $(T_{n+1,n})$ has uniformly bounded coefficients. Indeed the family $(T_{n+1,n})$ has uniformly bounded coefficients, and the assertion follows by looking at (3.2.1). Likewise, $\sup_n \deg T_{0,n} < \infty$, since $\sup_n \deg T_{n,0} < \infty$.

Hence there exists $C \ge 1$ such that $|T^{(j)}_{n,n+1}(z)| \le C$ for $z \in \Delta$, $1 \le j \le q$, and there exists $d = \max_n \deg T_{0,n}$. Let M be the number of multi-indices $I = (i_1, \ldots, i_q)$ with $|I| \le d$, and set $\gamma = MC^d$, we claim that

$$|T^{(j)}_{0,k}(z)| \le \gamma^k. \tag{3.2.2}$$

We proceed by induction on k. Since $C \le \gamma$, (3.2.2) holds for $k = 1$. Assume (3.2.2) for some $k \ge 1$, then by Cauchy estimates the coefficients in $T^{(j)}_{0,k}(z) = \sum_{|I| \le d} a_I z^I$ satisfy

$$|a_I| \le \gamma^k.$$

Since $T_{0,k+1} = T_{0,k} \circ T_{k,k+1}$, we have

$$T^{(j)}_{0,k+1} = T^{(j)}_{0,k}(T^{(1)}_{k,k+1}, \ldots, T^{(q)}_{k,k+1}) = \sum_{|I| \le d} a_I (T^{(1)}_{k,k+1})^{i_1} \cdots (T^{(q)}_{k,k+1})^{i_q}.$$

Then

$$|T^{(j)}_{0,k+1}| \le MC^d \gamma^k = \gamma^{k+1}.$$

□

Corollary 3.2.5. *Let $(T_{m,n})$ be a triangular ℕ-evolution family of uniformly bounded degree and uniformly bounded coefficients. Then there exists $\beta \ge 0$ such that for all $k \ge 0$,*

$$|T_{0,k}(z) - T_{0,k}(z')| \le \beta^k |z - z'|, \quad z,z' \in (1/2)\Delta.$$

Proof. Recall that $\Delta \subset \sqrt{q}\mathbb{B}$ and that if $B = (b_{ij}) \in M_{q,q}(\mathbb{C})$, then

$$\|B\| \leq q \max_{i,j} |b_{ij}|.$$

If $z \in (1/2)\Delta$, then by Cauchy estimates

$$\|d_z T_{0,k}\| \leq 2q\sqrt{q}\gamma^k.$$

The result follows setting $\beta \doteq 2q\sqrt{q}\gamma$. □

Proposition 3.2.6. *Let $(T_{m,n})$ be a triangular $\mathbb{N}$-evolution family, with uniformly bounded degree and uniformly bounded coefficients, then $T_{n,0}(z) \to 0$ uniformly on compacta. Hence for each neighborhood V of 0 we have*

$$\bigcup_{n=1}^{\infty} T_{0,n}(V) = \mathbb{C}^q.$$

Proof. Let K be a compact set in $\mathbb{C}^q$. Notice that $T^{(1)}_{k,0}(z) = \lambda_1^k z_1$, hence if $\|\cdot\|$ denotes the sup-norm on K, we have $\|T^{(1)}_{k,0}\| \to 0$. Let $1 < i \leq q$ and assume that $\lim_{k\to\infty} \|T^{(j)}_{k,0}\| = 0$, for $1 \leq j < i$. On K,

$$\lim_{k\to\infty} \|t^{(i)}_{k+1,k}(T^{(1)}_{k,0}, \dots, T^{(i-1)}_{k,0})\| = 0,$$

since $(t^{(i)}_{k+1,k})$ has uniformly bounded coefficients and uniformly bounded degree.

Notice that

$$T^{(i)}_{k+1,0} = \lambda_i T^{(i)}_{k,0} + t^{(i)}_{k+1,k}(T^{(1)}_{k,0}, \dots, T^{(i-1)}_{k,0}), \ \ 2 \leq i \leq q. \tag{3.2.3}$$

For each $\varepsilon > 0$, $|T^{(i)}_{k+1,0}| \leq |\lambda_i||T^{(i)}_{k,0}| + \varepsilon$, on K for a large enough k. Therefore

$$\limsup_{k\to\infty} \|T^{(i)}_{k,0}\| \leq \frac{\varepsilon}{1 - |\lambda_i|}.$$

Induction on i yields the Lemma. □

Corollary 3.2.7. *A triangular $\mathbb{N}$-evolution family $(T_{m,n})$ with uniformly bounded degree and uniformly bounded coefficients is shrinking.*

Remark 3.2.8. A triangular $\mathbb{N}$-evolution family $(T_{m,n})$ admits as associated $\mathbb{N}$-Loewner chain $(T_{0,n})$ and thus $\mathsf{Lr}(T_{m,n}) = \mathbb{C}^q$. In the next sections we will prove that any dilation $\mathbb{N}$-evolution family is locally conjugate to a triangular $\mathbb{N}$-evolution family with uniformly bounded degree and uniformly bounded coefficients.

3.3 Parametric Poincaré-Dulac method

For a detailed exposition of the classical Poincaré-Dulac method, see [23, Appendix]. In the following we identify an automorphism $A \in \mathcal{A}(\mathbb{C}^q)$ with its associated matrix with respect to the canonical basis. Recall that the ε-Jordan normal form is the classical lower triangular Jordan normal form with the underdiagonal multiplied by ε.

Definition 3.3.1. A *real multiplicative resonance* for $A \in \mathcal{A}(\mathbb{C}^q)$ with eigenvalues λ_i is an identity

$$|\lambda_j| = |\lambda_1^{i_1} \dots \lambda_q^{i_q}|,$$

where $i_j \geq 0$, and $\sum_j i_j \geq 2$. If for every $1 \leq j \leq q$ we have $|\lambda_j| < 1$, real multiplicative resonances can occur only in a finite number. Moreover, if $0 < |\lambda_q| \leq \dots \leq |\lambda_1| < 1$ then

$$|\lambda_j| = |\lambda_1^{i_1} \dots \lambda_q^{i_q}| \Rightarrow i_j = i_{j+1} = \dots = i_q = 0. \tag{3.3.1}$$

Definition 3.3.2. An automorphism $A \in \mathcal{A}(\mathbb{C}^q)$ is in *optimal form* if

i) A is in lower-triangular ε-Jordan normal form for some $\varepsilon > 0$,

ii) if the diagonal of A is $(\lambda_1, \dots, \lambda_q)$ then $1 > |\lambda_1| \geq \dots \geq |\lambda_q| > 0$,

iii) $\|A\|_2 < 1$.

Note that any linear automorphism can be put in optimal form by a linear change of coordinates.

Let $A \in \mathcal{A}(\mathbb{C}^q)$ be in optimal form. For $1 \leq j \leq q$ let $\pi_j \colon \mathbb{C}^q \to \mathbb{C}$ be the projection to the j-th coordinate. Let $i \geq 2$ and let $\mathcal{H}_i$ be the vector space of all holomorphic maps $H : \mathbb{C}^q \to \mathbb{C}^q$ whose components $\pi_j \circ H$ are homogeneus polynomials of degree i. A basis for this vector space is easily described: let $1 \leq j \leq q$, let $I \in \mathbb{N}^q$ be a multi-index of absolute value $|I| = i$, and define

$$\pi_l \circ X_I^j \doteq \delta_{l,j} z^I, \quad 1 \leq l \leq q.$$

The set $\mathfrak{B} \doteq \{X_I^j : \ 1 \leq j \leq q, |I| = q\}$ is a basis of $\mathcal{H}_i$. Next we define a splitting of $\mathcal{H}_i$ by specifying a partition of the basis $\mathfrak{B}$.

We set $X_I^j \in \mathfrak{B}_r$ if $|\lambda_j \lambda^{-I}| = 1$. The *real resonant space* $\mathcal{R}_i$ is the vector subspace spanned by the vectors in $\mathfrak{B}_r$.

We set $X_I^j \in \mathfrak{B}_s$ if $|\lambda_j \lambda^{-I}| < 1$. The *stable space* $\mathcal{S}_i$ is the vector subspace spanned by the vectors in $\mathfrak{B}_s$.

We set $X_I^j \in \mathfrak{B}_u$ if $|\lambda_j \lambda^{-I}| > 1$. The *unstable space* $\mathcal{U}_i$ is the vector subspace spanned by the vectors in $\mathfrak{B}_u$.

This defines the splitting

$$\mathcal{H}_i = \mathcal{R}_i \oplus \mathcal{S}_i \oplus \mathcal{U}_i,$$

with projections π_r, π_s, and π_u. The *real resonant part* of $H \in \mathcal{H}_i$ is $\pi_r(H)$. By 3.3.1 the mapping $\pi_r(H)$ is triangular.

If $F \in \mathcal{L}(\mathbb{C}^q)$, then $H \mapsto H \circ F$ and $H \mapsto F \circ H$ are endomorphisms of $\mathcal{H}_i$. We define the linear operator $\Gamma\colon \mathcal{H}_i \to \mathcal{H}_i$ as $H \mapsto A \circ H \circ A^{-1}$.

The next lemma justifies the terms "stable" and "unstable".

Lemma 3.3.3. *The stable space $\mathcal{S}_i$ is Γ-totally invariant and $\rho(\Gamma|_{\mathcal{S}_i}) < 1$. Indeed*

$$\mathsf{sp}(\Gamma|_{\mathcal{S}_i}) = \{\lambda_j \lambda^{-I} : X_I^j \in \mathfrak{B}_s\}.$$

The unstable space $\mathcal{U}_i$ is Γ-totally invariant and $\rho(\Gamma^{-1}|_{\mathcal{U}_i}) < 1$. Indeed

$$\mathsf{sp}(\Gamma|_{\mathcal{U}_i}) = \{\lambda_j \lambda^{-I} : X_I^j \in \mathfrak{B}_u\}.$$

Proof. Let $\lambda(1), \ldots, \lambda(p)$ be the distinct eigenvalues of A, with algebraic multiplicity $\mathfrak{m}_{\lambda(1)}, \ldots, \mathfrak{m}_{\lambda(p)}$. Consider the splitting $\mathbb{C}^q = E_{\lambda(1)} \oplus \cdots \oplus E_{\lambda(p)}$, where $E_{\lambda(l)}$ is the generalized eigenspace relative to $\lambda(l)$. Since A is in ε-Jordan normal form, for all $1 \le l \le p$ there exist $0 \le h_l \le k_l$ such that $E_{\lambda(l)} = \operatorname{Span}\{e_j : h_l \le j \le k_l\}$, $k_l - h_l = \mathfrak{m}_{\lambda(l)}$, and

$$\lambda_j = \lambda(l), \quad h_l \le j \le k_l. \tag{3.3.2}$$

Hence we identify $E_{\lambda(l)} \simeq \mathbb{C}^{\mathfrak{m}_{\lambda(l)}}$. Since the generalized eigenspaces are A-invariant, A induces automorphisms $A_{\lambda(l)} \in \mathcal{A}(\mathbb{C}^{\mathfrak{m}_{\lambda(l)}})$.

Let $\pi_{\lambda(l)}\colon \mathbb{C}^q \to \mathbb{C}^{\mathfrak{m}_{\lambda(l)}}$ be the spectral projection, that is

$$\pi_{\lambda(l)}\colon (z_1, \ldots z_q) \mapsto (z_{h_l}, \ldots, z_{k_l}).$$

We denote by $\pi_{\lambda(l)}\colon \mathbb{N}^q \to \mathbb{N}^{\mathfrak{m}_{\lambda(l)}}$ also the projection

$$\pi_{\lambda(l)}\colon (i_1, \ldots i_q) \mapsto (i_{h_l}, \ldots, i_{k_l})$$

acting on multi-indices. We have

$$\begin{aligned}(A^{-1}(z))^I &= [A^{-1}_{\lambda(1)} \circ \pi_{\lambda(1)}(z)]^{\pi_{\lambda(1)}(I)} \cdots [A^{-1}_{\lambda(p)} \circ \pi_{\lambda(p)}(z)]^{\pi_{\lambda(p)}(I)} \\ &= [\sum_{\alpha_1} c_{1,\alpha_1} \pi_{\lambda(1)}(z)^{J_{1,\alpha_1}}] \cdots [\sum_{\alpha_p} c_{p,\alpha_p} \pi_{\lambda(p)}(z)^{J_{p,\alpha_p}}],\end{aligned}$$

with $J_{l,\alpha_l} \in \mathbb{N}^{\mathfrak{m}_{\lambda(l)}}$, $|J_{l,\alpha_l}| = |\pi_{\lambda(l)}(I)|$ for all α_l, and with $c_{l,\alpha_l} \in \mathbb{C}$. Thus $(A^{-1}(z))^I$ is a linear combination of monomials of the type $\pi_{\lambda(1)}(z)^{J_{1,\alpha_1}} \cdots \pi_{\lambda(p)}(z)^{J_{p,\alpha_p}}$.

Assume $X_I^j \in \mathfrak{B}_s$. Then by definition $|\lambda_j| < |\lambda^I|$. Any monomial of the form $\pi_{\lambda(1)}(z)^{J_{1,\alpha_1}} \cdots \pi_{\lambda(p)}(z)^{J_{p,\alpha_p}}$ is in $\mathcal{S}_i$. Indeed by (3.3.2),

$$\lambda(1)^{|\pi_{\lambda(1)}(I)|} \cdots \lambda(p)^{|\pi_{\lambda(p)}(I)|} = \lambda^I,$$

and

$$|\lambda_j| < |\lambda^I| = \left|\lambda(1)^{|J_{1,\alpha_1}|} \cdots \lambda(p)^{|J_{p,\alpha_p}|}\right|.$$

The same argument works for $X_I^j \in \mathfrak{B}_u$. Thus $\mathcal{S}_i$ and $\mathcal{U}_i$ are invariant by the linear operator $H \mapsto H \circ A^{-1}$.

Now let $X_I^j \in \mathfrak{B}$. There exists $1 \leq l \leq p$ such that $h_l \leq j \leq k_l$. Then

$$A \circ X_I^j = (c_1 z^I, \ldots, c_q z^I) = \sum_\alpha c_\alpha X_I^\alpha$$

with $c_\alpha \in \mathbb{C}$ and $c_\alpha = 0$ if $\alpha < h_l$ or $\alpha > k_l$.

Assume $X_I^j \in \mathfrak{B}_s$. Then by definition $|\lambda_j| < |\lambda^I|$. Any X_I^α such that $h_l \leq \alpha \leq k_l$ satisfies $X_I^\alpha \in \mathfrak{B}_s$. Indeed by (3.3.2), we have

$$|\lambda_\alpha| = |\lambda_j| < |\lambda^I|, \quad h_l \leq \alpha \leq k_l$$

The same argument works for $X_I^j \in \mathfrak{B}_u$. Thus $\mathcal{S}_i$ and $\mathcal{U}_i$ are invariant by the linear operator $H \mapsto A \circ H$.

Now we want to find the spectrum of $\Gamma|_{\mathcal{S}_i}$. The automorphism A is conjugate to any automorphism obtained multiplying the underdiagonal by a positive constant. Thus there exists a continuous path $\gamma\colon [0,1] \to \mathcal{A}(\mathbb{C}^q)$ such that $\gamma(0) = A$ and $\gamma(1) = (\lambda_1 z_1, \ldots, \lambda_q z_q)$, with γ_0 conjugated to γ_t for all $t \in [0,1)$.

Let $M \in \mathcal{A}(\mathbb{C}^q)$. Define $\Xi(M) \in \mathcal{A}(\mathcal{S}_i)$ as $H \mapsto M \circ H \circ M^{-1}$. If $B = M \circ A \circ M^{-1}$, the linear operator $\Gamma|_{\mathcal{S}_i} = \Xi(A)$ is conjugate to the linear operator $\Xi(B)$. Indeed

$$B \circ H \circ B^{-1} = M \circ A \circ M^{-1} \circ H \circ M \circ A^{-1} \circ M^{-1},$$

thus $\Xi(B) = \Xi(M) \circ \Xi(A) \circ \Xi(M)^{-1}$.

We have $\lim_{t \to 1} \Xi(\gamma(t)) = \Xi(\lambda_1 z_1, \ldots, \lambda_q z_q)$ and $\Gamma|_{\mathcal{S}_i} = \Xi(A) = \Xi(\gamma(0))$ is conjugate to $\Xi(\gamma(t))$ for all $t \in [0,1)$. Thus

$$\mathsf{sp}(\Gamma|_{\mathcal{S}_i}) = \mathsf{sp}(\Xi(\lambda_1 z_1, \ldots, \lambda_q z_q)).$$

It is easy to see that the linear operator $\Xi(\lambda_1 z_1, \ldots, \lambda_q z_q)$ is diagonalizable and that the basis $\mathfrak{B}_s$ is a basis of eigenvectors such that

$$[\Xi(\lambda_1 z_1, \ldots, \lambda_q z_q)](X_I^j) = \lambda_j \lambda^{-I} X_I^j.$$

Thus

$$\mathsf{sp}(\Gamma|_{\mathcal{S}_i}) = \mathsf{sp}(\Xi(\lambda_1 z_1, \ldots, \lambda_q z_q)) = \{\lambda_j \lambda^{-I} : X_I^j \in \mathfrak{B}_s\}.$$

The same argument works for the spectrum of $\Gamma|_{\mathcal{U}_i}$. $\square$

Proposition 3.3.4. *Let $(\varphi_{m,n},D)$ be a shrinking $\mathbb{N}$-evolution family such that $\varphi_{n+1,n}(z) = A(z) + O(|z|^2)$ with A in optimal form. Then for each $i \geq 2$ there exist*

i) a family (k_n^i) of polynomial maps $k_n(z) = z + O(|z|^2)$ with uniformly bounded degree and uniformly bounded coefficients, and

ii) a triangular evolution family $(T_{m,n}^i)$ with $T_{n+1,n}^i(z) = A(z) + O(|z|^2)$,

$$\deg T_{n+1,n}^i \leq i - 1,$$

and uniformly bounded coefficients such that for all $n \geq 0$,

$$k_{n+1}^i \circ \varphi_{n+1,n} - T_{n+1,n}^i \circ k_n^i = O(|z|^i). \tag{3.3.3}$$

Proof. For $i = 2$ set $T_{n,n+1}^2 = A$, $k_{n+1,n}^2 = \mathsf{id}$, and we are done since A is a triangular mapping. Now assume that (3.3.3) holds for $i \geq 2$. We can rewrite (3.3.3) as

$$k_{n+1}^i \circ \varphi_{n+1,n} - T_{n+1,n}^i \circ k_n^i = P_{n+1,n} + O(|z|^{i+1}), \tag{3.3.4}$$

where $(P_{n+1,n})$ is a bounded sequence in $\mathcal{H}_i$. Define $R_{n+1,n} \doteq \pi_r(P_{n+1,n})$, and $N_{n+1,n} \doteq P_{n+1,n} - R_{n+1,n} \in \mathcal{S}_i \oplus \mathcal{U}_i$. Set

$$T_{n+1,n}^{i+1} \doteq T_{n+1,n}^i + R_{n+1,n},$$

which is still a triangular $\mathbb{N}$-evolution family with uniformly bounded degree and uniformly bounded coefficients since $R_{n+1,n}$ is a triangular mapping thanks to (3.3.1), and set

$$k_n^{i+1} \doteq k_n^i + H_n \circ k_n^i,$$

where (H_n) is an unknown bounded sequence in $\mathcal{H}_i$.

$$\begin{aligned}
k_{n+1}^{i+1} \circ \varphi_{n+1,n} - T_{n+1,n}^{i+1} \circ k_n^{i+1} = \\
&= (k_{n+1}^i + H_{n+1} \circ k_{n+1}^i) \circ \varphi_{n+1,n} - (T_{n+1,n}^i + R_{n+1,n}) \circ (k_n^i + H_n \circ k_n^i) \\
&= P_{n+1,n} - R_{n+1,n} + H_{n+1} \circ A - A \circ H_n + O(|z|^{i+1}) \\
&= N_{n+1,n} + H_{n+1} \circ A - A \circ H_n + O(|z|^{i+1}).
\end{aligned}$$

Thus to end the proof we need to prove the existence of a bounded sequence (H_n) of elements of $\mathcal{H}_i$ which satisfies

$$N_{n+1,n} = A \circ H_n - H_{n+1} \circ A, \tag{3.3.5}$$

that is a bounded solution (H_n) of the *homological difference equation*

$$H_{n+1} = A \circ H_n \circ A^{-1} - N_{n+1,n} \circ A^{-1}.$$

Define $B_n \doteq -N_{n+1,n} \circ A^{-1}$. In the proof of Lemma 3.3.3 we proved that $\mathcal{S}_i$ and $\mathcal{U}_i$ are invariant by the linear operator $H \mapsto H \circ A^{-1}$, thus $B_n \in \mathcal{S}_i \oplus \mathcal{U}_i$. Define $B_n^s \doteq \pi_s(B_n)$, $B_n^u \doteq \pi_u(B_n)$. If $n \geq 1$ it is easy to prove by induction that

$$H_n = \Gamma^n(H_0) + \sum_{j=0}^{n-1} \Gamma^j(B_{n-1-j}) = \Gamma^n(H_0) + \sum_{j=0}^{n-1} \Gamma^{n-1-j}(B_j). \tag{3.3.6}$$

We have

$$\begin{aligned} H_n &= \Gamma^n(H_0) + \sum_{j=0}^{n-1} \Gamma^j(B^s_{n-1-j}) + \sum_{j=0}^{n-1} \Gamma^j(B^u_{n-1-j}) \\ &= \sum_{j=0}^{n-1} \Gamma^j(B^s_{n-1-j}) + \Gamma^{n-1}(\Gamma(H_0) + \sum_{j=0}^{n-1} \Gamma^{-j}(B^u_j)). \end{aligned}$$

By Lemma 3.1.8 and Lemma 3.3.3 there exist a norm $\|\cdot\|_s$ on $\mathcal{S}_i$ and a norm $\|\cdot\|_u$ on $\mathcal{U}_i$ such that $\|\Gamma|_{\mathcal{S}_i}\|_s < 1$,$\|\Gamma^{-1}|_{\mathcal{U}_i}\|_u < 1$. Define a norm on $\mathcal{S}_i \oplus \mathcal{U}_i$ by

$$\|H\| \doteq \|\pi_s(H)\|_s + \ \|\pi_u(H)\|_u.$$

Since (B_n^s) is bounded there exists $C > 0$ such that

$$\|\sum_{j=0}^{n-1} \Gamma^j(B^s_{n-1-j})\| \leq \sum_{j=0}^{\infty} \|\Gamma^j(B^s_{n-1-j})\|_s \leq C, \quad n \geq 0.$$

Since (B_n^u) is bounded, $\sum_{j=0}^{\infty} \|\Gamma^{-j}(B^u_j)\|_u < +\infty$, thus we can define

$$H_0 \doteq -\Gamma^{-1}(\sum_{j=0}^{\infty} \Gamma^{-j}(B^u_j)) \in \mathcal{U}_i.$$

With this definition,

$$\|H_n\| \leq C + \|\Gamma^{n-1}(\sum_{j=n}^{\infty} \Gamma^{-j}(B^u_j))\|_u = C + \|\sum_{j=1}^{\infty} \Gamma^{-j}(B^u_{n-1+j})\|_u,$$

and since

$$\|\sum_{j=1}^{\infty} \Gamma^{-j}(B^u_{n-1+j})\|_u \leq \sum_{j=1}^{\infty} \|\Gamma^{-j}(B^u_{n-1+j})\|_u \leq C',$$

we have $\|H_n\| \leq C + C'$. □

Remark 3.3.5. Let $p \geq 0$ be the smallest integer such that $|\lambda_1^p| < |\lambda_q|$. Then if $i \geq p$ we have $\pi_r(P_{n+1,n}) = 0$ in $\mathcal{H}_i$. Hence $T^i_{n+1,n} = T^p_{n+1,n}$ for any $i \geq p$.

Theorem 3.3.6 ([2]). *Let $(\varphi_{m,n}, D)$ be a shrinking $\mathbb{N}$-evolution family such that $\varphi_{n+1,n}(z) = A(z) + O(|z|^2)$ with A in optimal form. Then there exists a triangular $\mathbb{N}$-evolution family $(T_{m,n})$ with uniformly bounded degree and uniformly bounded coefficients locally conjugate to $(\varphi_{m,n})$.*

Proof. Let $\|A\| < \alpha < 1$. Let $(T^i_{m,n})$ and (k^i_n) be the families given by Proposition 3.3.4. Let $p \geq 0$ be as in previous remark. Define $(T_{m,n}) \doteq (T^p_{m,n})$. Let $\beta > 0$ be the constant given by Corollary 3.2.5 for $(T_{m,n})$. Let $l \geq 0$ be an integer such that

$$\alpha^l < 1/\beta,$$

and define $(k_n) \doteq (k^l_n)$. By Proposition 3.3.4,

$$k_{n+1} \circ \varphi_{n+1,n} - T_{n+1,n} \circ k_n = O(|z|^l),$$

thus

$$T_{n,n+1} \circ k_{n+1} \circ \varphi_{n+1,n} - k_n = O(|z|^l).$$

Lemma A.0.6 gives us $r > 0$ (we can assume $0 < r < 1/2$) such that on $r\mathbb{B}$ we have $|\varphi_{n+1,n}(z)| \leq \alpha|z|$ and $|T_{n+1,n}(z)| \leq \alpha|z|$ for all $n \geq 0$.Thus for $\zeta \in r\mathbb{B}$ we have

$$|\varphi_{m,0}(\zeta)| < r\alpha^m.$$

Thanks to Lemma A.0.5 we have on $r\mathbb{B}$,

$$|T_{m,m+1} \circ k_{m+1} \circ \varphi_{m+1,m}(\zeta) - k_m(\zeta)| \leq C|\zeta|^l.$$

Hence

$$|T_{m,m+1} \circ k_{m+1} \circ \varphi_{m+1,0}(\zeta) - k_m \circ \varphi_{m,0}(\zeta)| \leq C|\varphi_{m,0}(\zeta)|^l \leq Cr^l\alpha^{lm}.$$

There exists $s\mathbb{B} \subset r\mathbb{B}$ such that

$$T_{m,m+1} \circ k_{m+1} \circ \varphi_{m+1,0}(s\mathbb{B}) \subset \frac{1}{2}\Delta$$

and

$$k_m \circ \varphi_{m,0}(s\mathbb{B}) \subset \frac{1}{2}\Delta.$$

Indeed the families (k_m) and $(T_{m,m+1})$ are uniformly bounded on $r\mathbb{B}$ and thus equicontinuous in 0.

Hence Corollary 3.2.5 can be applied to get on $s\mathbb{B}$,

$$|T_{0,m+1} \circ k_{m+1} \circ \varphi_{m+1,0}(\zeta) - T_{0,m} \circ k_m \circ \varphi_{m,0}(\zeta)| \leq Cr^l(\beta\alpha^l)^m.$$

Likewise it is easy to see that for all $m \geq n \geq 0$,

$$|T_{n,m+1} \circ k_{m+1} \circ \varphi_{m+1,n}(\zeta) - T_{n,m} \circ k_m \circ \varphi_{m,n}(\zeta)| \leq Cr^l(\beta\alpha^l)^{m-n}.$$

Since $\alpha^l < 1/\beta$ for all $n \geq 0$ there exists a holomorphic mapping h_n on $s\mathbb{B}$ such that

$$h_n = \lim_{m\to\infty} T_{n,m} \circ k_m \circ \varphi_{m,n}$$

uniformly on compacta. Each h_n is bounded by

$$|k_n| + \sum_{j=0}^{\infty} Cr^l(\beta\alpha^l)^j,$$

hence they are uniformly bounded. Moreover

$$h_m \circ \varphi_{n,m} = \lim_{j\to\infty} T_{m,j} \circ k_j \circ \varphi_{j,m} \circ \varphi_{m,n} = \lim_{j\to\infty} T_{m,n} \circ T_{n,j} \circ k_j \circ \varphi_{j,n} = T_{m,n} \circ h_n.$$

□

Remark 3.3.7. Theorem 3.3.6 actually generalizes the result in [2] which holds under the assumption that A is diagonal.

Remark 3.3.8. The sequence $T_{n,m} \circ k_m \circ \varphi_{m,n}$ converges uniformly on compacta on the whole D for $m \to \infty$, thus defining the intertwining mappings on the whole D.

Indeed since $(\varphi_{m,n})$ is shrinking, for each $n \geq 0$ and each compact set $K \subset D$ there exists $u \geq 0$ such that

$$\varphi_{u,n}(K) \subset s\mathbb{B}.$$

Then for $m \geq u$,

$$T_{n,m} \circ k_m \circ \varphi_{m,n}|_K = T_{n,u} \circ (T_{u,m} \circ k_m \circ \varphi_{m,u}) \circ \varphi_{u,n}|_K,$$

thus

$$\lim_{m\to\infty} T_{n,m} \circ k_m \circ \varphi_{m,n}|_K = T_{n,u} \circ (\lim_{m\to\infty} T_{u,m} \circ k_m \circ \varphi_{m,u}) \circ \varphi_{u,n}|_K = T_{n,u} \circ h_u \circ \varphi_{u,n}|_K,$$

which gives the uniform convergence on compacta. Define $h_n \doteq \lim_{m\to\infty} T_{n,m} \circ k_m \circ \varphi_{m,n}$. It is easy to see that each h_n is univalent and that (h_n) is a family of intertwining mappings.

The family (h_n) is normal. Indeed let $K \subset D$ be a compact set. On $s\mathbb{B}$ the family is uniformly bounded and since $(\varphi_{m,n})$ is shrinking there exists $u \geq 0$ such that if $m - n \geq u$, $\varphi_{m,n}(K) \subset s\mathbb{B}$. Hence

$$h_n|_K = T_{n,m} \circ h_m \circ \varphi_{m,n}|_K$$

which is uniformly bounded.

Corollary 3.3.9. *Let* $(\varphi_{m,n},D)$ *be a shrinking* $\mathbb{N}$*-evolution family. Then there exists a polynomial* $\mathbb{N}$*-evolution family* $(Q_{m,n})$ *with uniformly bounded degree and uniformly bounded coefficients locally conjugate to* $(\varphi_{m,n})$*. If no real resonance occurs, then* $(\varphi_{m,n})$ *is locally conjugate to its linear part* (A^{m-n})*.*

Proof. Let $L \in \mathcal{A}(\mathbb{C}^q)$ such that $L \circ A \circ L^{-1}$ is in optimal form. By applying Theorem 3.3.6 to the shrinking $\mathbb{N}$-evolution family $(L \circ \varphi_{m,n} \circ L^{-1}, L(D))$ we get the existence of a triangular $\mathbb{N}$-evolution family $(T_{m,n})$ with bounded degree and bounded coefficients locally conjugate to $(L \circ \varphi_{m,n} \circ L^{-1})$, by means of a normal family (h_n) of intertwining mappings defined on $L(D)$.

We have

$$h_m \circ L \circ \varphi_{n,m} = T_{m,n} \circ h_n \circ L,$$

hence

$$(L^{-1} \circ h_m \circ L) \circ \varphi_{n,m} = L^{-1} \circ T_{m,n} \circ L \circ (L^{-1} \circ h_n \circ L).$$

Define $(Q_{m,n}) \doteq (L^{-1} \circ T_{m,n} \circ L)$. Then $g_n \doteq L^{-1} \circ h_n \circ L$ defines a normal family on D of intertwining mappings between $(\varphi_{m,n})$ and $(Q_{m,n})$.

If no real resonance occurs, then $T_{m,n} = L \circ A^{m-n} \circ L^{-1}$, hence $Q_{m,n} = A^{m-n}$. □

As a consequence we get the following result.

Corollary 3.3.10. *Let* $(\varphi_{m,n},D)$ *be a shrinking* $\mathbb{N}$*-evolution family. Then there exists an associated* $\mathbb{N}$*-Loewner chain* $(f_n,D,\mathbb{C}^q)$ *with range* $\mathbb{C}^q$*. Furthermore there exist a polynomial evolution family* $(Q_{m,n})$ *of bounded degree and bounded coefficients and a family* (g_n) *of polynomial maps* $g_n(z) = z + O(|z|^2)$ *of uniformly bounded degree and uniformly bounded coefficients such that*

$$f_n = \lim_{m\to\infty} Q_{0,m} \circ g_m \circ \varphi_{m,n}.$$

If no real resonances occur, then $(Q_{m,n}) = (A^{m-n})$ *and the chain, given by*

$$f_n = \lim_{m\to\infty} A^{-m} \circ g_m \circ \varphi_{m,n},$$

is normal.

Proof. Simply notice that $f_n \doteq Q_{0,n} \circ g_n$ is an associated $\mathbb{N}$-Loewner chain, and that

$$g_n = \lim_{m\to\infty} Q_{n,m} \circ g_m \circ \varphi_{m,n}.$$

□

3.4 Dilation $\mathbb{R}^+$-evolution families

Let $\mathbb{H}_l$ denote the left half-plane in $\mathbb{C}$.

Definition 3.4.1. Let D be a domain in $\mathbb{C}^q$ containing 0. A $\mathbb{R}^+$-evolution family $(\varphi_{t,s})$ on D is a *dilation* $\mathbb{R}^+$*-evolution family* if for all $0 \leq s \leq t$,

$$\varphi_{t,s}(z) = e^{\Lambda(t-s)}(z) + O(|z|^2), \tag{3.4.1}$$

with $\Lambda \in \mathcal{L}(\mathbb{C}^q,\mathbb{C}^q)$ satisfying $\mathsf{sp}(\Lambda) \subset \mathbb{H}_l$.

A $\mathbb{R}^+$-Loewner chain $(f_t,D,\mathbb{C}^q)$ is a *dilation* $\mathbb{R}^+$*-Loewner chain* if for all $t \geq 0$,

$$f_t(z) = e^{-\Lambda t}(z) + O(|z|^2),$$

with $\Lambda \in \mathcal{L}(\mathbb{C}^q,\mathbb{C}^q)$ satisfying $\mathsf{sp}(\Lambda) \subset \mathbb{H}_l$. A dilation Loewner chain is *normal* if $(e^{\Lambda t} \circ f_t)$ is a normal family.

Definition 3.4.2. If we restrict time to integer values in a dilation $\mathbb{R}^+$-evolution family $(\varphi_{t,s})$ we obtain its *discretized* dilation $\mathbb{N}$-evolution family $(\varphi_{m,n})$. We have

$$\varphi_{n+1,n}(z) = e^{\Lambda}(z) + O(|z|^2).$$

An *additive real resonance* is an identity

$$\mathsf{Re}\,(\sum_{j=1}^{N} k_j\alpha_j) = \mathsf{Re}\,\alpha_l,$$

where $k_j \geq 0$ and $\sum_j k_j \geq 2$. Recall that α is an eigenvalue of Λ with algebraic molteplicity m if and only if e^α is an eigenvalue of e^Λ with algebraic molteplicity m. Hence additive real resonances of Λ correspond to multiplicative real resonances of e^Λ.

Lemma 3.4.3. *Let D a complete hyperbolic domain containing* 0*. Let $(\varphi_{t,s},D)$ be a dilation $\mathbb{R}^+$-evolution family, and let $(\varphi_{m,n},D)$ be its discretized evolution family. Assume there exists a $\mathbb{N}$-Loewner chain (f_n) associated with $(\varphi_{m,n})$. Then we can extend it in a unique way to a $\mathbb{R}^+$-Loewner chain associated with $(\varphi_{t,s})$. If (f_n) is a normal $\mathbb{N}$-Loewner chain, then also (f_s) is normal.*

Proof. By Proposition 2.1.10 there exists a unique $\mathbb{R}^+$-Loewner chain (f_s) associated with $(\varphi_{t,s})$ which extends (f_n). This chain is defined as

$$f_s = f_j \circ \varphi_{j,s},$$

where j is an integer such that $s \leq j$. Assume $(e^{\Lambda n} \circ f_n)$ is a normal family and let $\Omega(0,r)$ be the ball with respect to k_D of radius $r > 0$ centered in 0. Since D is complete hyperbolic, $\overline{\Omega(0,r)}$ is compact by Proposition 1.0.6. The family $(e^{\Lambda n} \circ f_n)$ is thus uniformly bounded on $\overline{\Omega(0,r)}$. For each $s \geq 0$ define m_s as the smallest integer greater than s. We have

$$e^{\Lambda s} \circ f_s = e^{\Lambda s} \circ f_{m_s} \circ \varphi_{m_s,s} = e^{\Lambda(s-m_s)} \circ e^{\Lambda m_s} \circ f_{m_s} \circ \varphi_{m_s,s},$$

which is uniformly bounded on $\overline{\Omega(0,r)}$ since $\varphi_{m_s,s}(\overline{\Omega(0,r)}) \subset \overline{\Omega(0,r)}$ and $m_s - s$ is smaller than 1. Hence $(e^{\Lambda s} \circ f_s)$ is a normal family. □

Definition 3.4.4. A *shrinking domain* is a complete hyperbolic domain D containing 0 such that any dilation $\mathbb{N}$-evolution family on D is shrinking. Notice that since we assume D to be complete hyperbolic, property i) of Definition 3.1.5 is satisfied for any $\mathbb{N}$-evolution family on D.

Remark 3.4.5. By Proposition 3.1.9 and Remark 3.1.7 any domain D biholomorphic to $\mathbb{B}$ and containing 0 is a shrinking domain.

The following result generalizes [14, Theorem 2.3] and [11, Theorem 3.1].

Theorem 3.4.6 ([2])**.** *Let D be a shrinking domain. Let $(\varphi_{t,s},D)$ be a dilation $\mathbb{R}^+$-evolution family. Then there exists a dilation $\mathbb{R}^+$-Loewner chain $(f_s,D,\mathbb{C}^q)$ associated with $(\varphi_{t,s})$, with range $\mathbb{C}^q$. Furthermore there exist a polynomial $\mathbb{N}$-evolution family $(Q_{m,n})$ of uniformly bounded degree and uniformly bounded coefficients and a family (g_n) of polynomial maps $g_n(z) = z + O(|z|^2)$ of uniformly bounded degree and uniformly bounded coefficients such that*

$$f_s = \lim_{m\to\infty} Q_{0,m} \circ g_m \circ \varphi_{m,s}.$$

If no real resonances occur, then $(Q_{m,n}) = (A^{m-n})$ and the chain, given by

$$f_s = \lim_{m\to\infty} A^{-m} \circ g_m \circ \varphi_{m,s},$$

is normal.

Proof. Let $(\varphi_{m,n},D)$ be the discretized of $(\varphi_{t,s},D)$. Since D is a shrinking domain, $(\varphi_{m,n})$ is a shrinking $\mathbb{N}$-evolution family. Since no additive real resonance occurs in Λ, no multiplicative real resonance occurs in $A = e^{\Lambda}$.

The result follows from Corollary 3.3.10 and Lemma 3.4.3. □

Remark 3.4.7. Theorem 3.4.6 actually generalizes the result in [2] which holds under the assumption that Λ is diagonal.

Definition 3.4.8. A $\mathbb{R}^+$-evolution family $(\varphi_{t,s},D)$ is called *1-periodic* if for all $t \geq s \geq 0$,

$$\varphi_{t,s} = \varphi_{t+1,s+1}.$$

Proposition 3.4.9. *Let $(\varphi_{t,s},D)$ be a 1-periodic $\mathbb{R}^+$-evolution family fixing 0 on a taut domain $D \subset \mathbb{C}^q$, and assume that*

$$\rho(d_0\varphi_{1,0}) < 1.$$

Then there exist an associated $\mathbb{R}^+$-Loewner chain (f_s), which is normal if no complex resonance occurs among the eigenvalues of $d_0\varphi_{1,0}$.

Proof. The discretized evolution family $(\varphi_{m,n})$ is of dilation semigroup type. Hence as in Remark 3.1.3 we have an univalent mapping $h(z) = z + O(|z|^2)$ defined on D and a polynomial automorphism Q such that $(f_n) \doteq (Q^{-n} \circ h)$ is a $\mathbb{N}$-Loewner chain associated with $(\varphi_{m,n})$. If no complex resonance occurs among the eigenvalues of $d_0\varphi_{1,0}$, then $Q = d_0\varphi_{1,0}$ by Poincaré-Dulac Theorem (see for example [23, Appendix]) and thus the family $(f_n) = ((d_0\varphi_{1,0})^{-n} \circ h)$ is a normal $\mathbb{N}$-Loewner chain.

Lemma 3.4.3 yields a $\mathbb{R}^+$-Loewner chain (f_s), which is normal if no complex resonance occurs. □

3.5 Counterexamples

Example 3.5.1. Let $\Lambda \in \mathcal{L}(\mathbb{C}^2,\mathbb{C}^2)$ given by $\mathrm{Diag}(\alpha_1,\alpha_2)$, with $\mathsf{Re}\,\alpha_1 < 0$ and $\mathsf{Re}\,\alpha_2 < 0$. If $(\varphi_{t,s},\mathbb{B})$ is a dilation $\mathbb{R}^+$-evolution family such that $\varphi_{t,s} = e^{\Lambda(t-s)}(z) + O(|z|^2)$ and

$$2\mathsf{Re}\,\alpha_1 < \mathsf{Re}\,\alpha_2,$$

then by Lemma 2.12 in [11] there exists a unique normal $\mathbb{R}^+$-Loewner chain associated with $(\varphi_{t,s})$. This is no longer true when $2\mathsf{Re}\,\alpha_1 \geq \mathsf{Re}\,\alpha_2$. Indeed, consider on $\mathbb{B} \subset \mathbb{C}^2$ the linear dilation $\mathbb{R}^+$-evolution family defined by

$$\varphi_{t,s}(z) = e^{\Lambda(t-s)}(z) = (e^{\alpha_1(t-s)}z_1, e^{\alpha_2(t-s)}z_2).$$

The family $(e^{-\Lambda s})$ is trivially a normal $\mathbb{R}^+$-Loewner chain associated with $(e^{\Lambda(t-s)})$. The univalent family

$$k_s(z) = (z_1, z_2 + e^{(\alpha_2 - 2\alpha_1)s}z_1^2),$$

satisfies $k_t \circ e^{\Lambda(t-s)} = e^{\Lambda(t-s)} \circ k_s$. Since $\mathsf{Re}\,\alpha_2 \leq 2\mathsf{Re}\,\alpha_1$, it is a uniformly bounded family, thus $(e^{-\Lambda s} \circ k_s)$ is another normal $\mathbb{R}^+$-Loewner chain associated with $(e^{\Lambda(t-s)})$.

Example 3.5.2. Let $\Lambda \in \mathcal{L}(\mathbb{C}^2,\mathbb{C}^2)$ given by $\mathrm{Diag}(\alpha_1,\alpha_2)$, with $\mathsf{Re}\,\alpha_1 < 0$ and $\mathsf{Re}\,\alpha_2 < 0$, and $\alpha_2 = 2\alpha_1$. There exists a dilation $\mathbb{R}^+$-evolution family $(\varphi_{t,s})$ such that $\varphi_{t,s} = e^{\Lambda(t-s)}(z) + O(|z|^2)$, which does not admit any associated normal $\mathbb{R}^+$-Loewner chain. Indeed, recall that for $c \in \mathbb{C}^*$ small enough, the family (ψ_t) defined by

$$\psi_t(z) = (e^{\alpha_1 t}z_1, e^{\alpha_2 t}(z_2 + ctz_1^2))$$

is a semigroup on $\mathbb{B} \subset \mathbb{C}^2$. Thus

$$\varphi_{t,s}(z) = \psi_{t-s}(z)$$

defines a dilation $\mathbb{R}^+$-evolution family. Let (f_s) be a normal $\mathbb{R}^+$-Loewner chain associated with $(\varphi_{t,s})$. The family $(h_s) = (e^{\Lambda s} \circ f_s)$ satisfies $h_t \circ \varphi_{t,s} = e^{\Lambda(t-s)} \circ h_s$, thus in particular

$$h_t \circ \varphi_{t,0} = e^{\Lambda t} \circ h_0. \tag{3.5.1}$$

Let a_s be the coefficient of the term z_1^2 in the second component of h_s. Then imposing equality of terms in z_1^2 in equation (3.5.1) we find $e^{\alpha_2 t} ct + a_t e^{2\alpha_1 t} = a_0 e^{\alpha_2 t}$, hence

$$a_t = e^{(\alpha_2 - 2\alpha_1)t}(a_0 - ct),$$

which gives $a_t = a_0 - ct$, so that (h_s) cannot be a normal family.

Example 3.5.3. Let $A \in \mathcal{A}(\mathbb{C}^2)$ given by $\mathrm{Diag}(\lambda_1,\lambda_2)$, with $\lambda_1,\lambda_2 \in \mathbb{D} \setminus \{0\}$ and $|\lambda_1|^2 = |\lambda_2|$, $\lambda_1^2 \neq \lambda_2$. There exists a $\mathbb{N}$-dilation evolution family $(\varphi_{m,n})$ such that $\varphi_{n+1,n}(z) = A(z) + O(|z|^2)$, which does not admit any associated $\mathbb{N}$-normal Loewner chain. Indeed, if $r > 0$ is sufficiently small, given any sequence $(a_{n+1,n})$ in $r\mathbb{B}$ there exist a $\mathbb{N}$-dilation evolution family defined by

$$\varphi_{n+1,n}(z) = (\lambda_1 z_1, \lambda_2 z_2 + a_{n+1,n} z_1^2).$$

If (f_n) is a normal $\mathbb{N}$-Loewner chain associated with $(\varphi_{m,n})$, then the family $(h_n) = (A^n \circ f_n)$ satisfies

$$h_{n+1} \circ \varphi_{n+1,n} = A \circ h_n. \tag{3.5.2}$$

Let α_n be the coefficient of the term z_1^2 in the second component of h_s, and set $\zeta = \lambda_1^2/\lambda_2$. Then imposing equality of terms in z_1^2 in equation (3.5.2) we obtain the recursive formula

$$\alpha_n \zeta^n \lambda_1^2 = \alpha_0 \lambda_1^2 - a_{1,0}\zeta - a_{2,1}\zeta^2 - \cdots - a_{n,n-1}\zeta^n.$$

For $1 \leq j \leq 8$ define $C_j = \{\zeta \in S^1 : 2\pi(j-1)/8 \leq \arg z \leq 2\pi j/8\}$. There exist a C_j which contains the images of a subsequence (ζ^{k_n}). Set

$$a_{m,m-1} = \begin{cases} r/2, \text{ if there exist } n \text{ such that } m = k_n, \\ 0, \text{ else,} \end{cases} \tag{3.5.3}$$

then the sequence $(\sum_{j=0}^n a_{j,j-1}\zeta^j)$ is not bounded, hence the sequence (α_n) is also not bounded. Thus for $(\varphi_{m,n})$ no normal family (h_n) can solve (3.5.2).

Appendix A

The following is a several variables version of the Schwarz Lemma [15, Lemma 6.1.28].

Lemma A.0.4. *Let $M > 0$ and $f\colon \mathbb{B} \to \mathbb{C}^q$ a holomorphic mapping fixing the origin and bounded by M. Then for z in the ball, $|f(z)| \leq M|z|$. If there is a point $z_0 \in \mathbb{B} \setminus \{0\}$ such that $|f(z_0)| = M|z_0|$, then $|f(\zeta z_0)| = M|\zeta z_0|$ for all $|\zeta| < 1/|z_0|$. Moreover, if $f(z) = O(|z|^k)$, $k \geq 2$, then for z in the ball, $|f(z)| \leq M|z|^k$.*

Lemma A.0.5. *Let $A \in \mathcal{L}(\mathbb{C}^q,\mathbb{C}^q)$. Let $\mathcal{F}$ be a family of holomorphic mappings $f\colon r\mathbb{B} \to \mathbb{C}^q$, bounded by $M > 0$, and let $k \geq 2$ such that*

$$f(z) - A(z) = O(|z|^k), \quad \forall f \in \mathcal{F}.$$

Then there exists $C_k > 0$ such that

$$|f(z) - A(z)| \leq C_k|z|^k, \quad \forall z \in r\mathbb{B}.$$

Proof. Set $g(z) = f(z) - A(z)$. Then $|g(rw)| \leq M + r\|A\|$, hence

$$|g(rw)| \leq (M + r\|A\|)|w|^k$$

thanks to previous lemma. Thus if $z \in r\mathbb{B}$,

$$|g(z)| \leq (M + r\|A\|)|(1/r)z|^k \leq (M + r\|A\|)(1/r)^k|z|^k.$$

$\square$

Lemma A.0.6. *Let $A \in \mathcal{L}(\mathbb{C}^q,\mathbb{C}^q)$, and let D be a domain containing 0. Let $\mathcal{F}$ be a family of holomorphic mappings $f\colon D \to \mathbb{C}^q$, bounded by $M > 0$, and satisfying $f(z) = A(z) + O(|z|^2)$. Let $\alpha > 0$ be such that $\|A\| < \alpha$. Then there exists $s > 0$ such that*

$$|f(z)| \leq \alpha|z|, \quad \forall f \in \mathcal{F}, |z| \leq s.$$

Proof. We proceed by contradiction: assume there exist a sequence $f_n \in \mathcal{F}$ and a sequence (z_n) converging to the origin verifying $|f_n(z_n)| > \alpha|z_n|$. The sequence (z_n) is eventually contained in some $r\mathbb{B} \subset D$. By Lemma A.0.5 we have that eventually

$$|f_n(z_n)| = |A(x_n) + f_n(z_n) - A(z_n)| \leq |A(z_n)| + C|z_n|^2,$$

thus

$$\alpha < \frac{|f_n(z_n)|}{|z_n|} \leq \frac{|A(z_n)|}{|z_n|} + C|z_n|,$$

but the right hand term has lim sup less or equal than $\|A\|$, which is the desired contradiction. □

Lemma A.0.7. *Let $A \in \mathcal{L}(\mathbb{C}^q,\mathbb{C}^q)$, and let $\mathcal{F}$ be a family of holomorphic self-mappings of $\mathbb{B}$ satisfying $f(z) = A(z) + O(|z|^2)$, with $\|A\| < 1$. Let $0 < s < 1$. Then there exists $k < 1$ such that*

$$|f(z)| \leq k|z|, \quad \forall f \in \mathcal{F}, |z| \leq s.$$

Proof. Assume the contrary: suppose there exist a sequence $f_n \in \mathcal{F}$ and a sequence of points z_n in $\overline{s\mathbb{B}}$ verifying $|f_n(z_n)| > (1 - 1/n)|z_n|$. Up to subsequences we have $z_n \to z'$ for some z' such that $|z'| \leq s$, and $f_n \to f \in \mathcal{F}$ uniformly on compacta since $\mathcal{F}$ is a compact family. If $z' \neq 0$ we have

$$1 - \frac{1}{n} < \frac{|f_n(z_n)|}{|z_n|} \to \frac{|f(z')|}{|z'|} < 1,$$

which is a contradiction. If $z' = 0$, using again Lemma A.0.5 we get

$$1 - \frac{1}{n} < \frac{|f_n(z_n)|}{|z_n|} \leq \frac{|A(z_n)|}{|z_n|} + C|z_n|,$$

and the right hand term has lim sup less or equal $\|A\|$, contradiction. □

Lemma A.0.8. *Let $A \in \mathcal{A}(\mathbb{C}^q)$, and let D be a domain containing 0. Let $\mathcal{F}$ be a family of holomorphic mappings $f\colon D \to \mathbb{C}^q$, bounded by $M > 0$, and satisfying $f(z) = A(z) + O(|z|^2)$. There exist $r > 0$ and $s > 0$ such that if $f \in \mathcal{F}$ then f is univalent on $r\mathbb{B}$, and such that $s\mathbb{B} \subset f(r\mathbb{B})$.*

Proof. Suppose there does not exist a ball $r\mathbb{B} \subset B$ such that every f is univalent on $r\mathbb{B}$. Since $\mathcal{F}$ is a compact family there exists a sequence $f_n \to f \in \mathcal{F}$ uniformly on compacta, and such that there does not exist a ball $r\mathbb{B} \subset B$ with the property that every f_n is univalent on $r\mathbb{B}$. By the inverse mapping theorem there exists a ball where f is univalent. We can now apply [15, Theorem 6.1.18], getting a contradiction. Assume now there does not exist a ball contained in each $f(r\mathbb{B})$. Again there is a sequence $f_n \to f \in \mathcal{F}$ uniformly on compacta, such that there does not exist a ball $s\mathbb{B} \subset \bigcap f_n(r\mathbb{B})$. The contradiction is then given by Proposition 2.3.1. □

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