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DAVIDE MASOERO

## Essays on the Painlevé first equation and the cubic oscillator

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PhD Thesis

# Essays on the Painlevé First Equation and the Cubic Oscillator

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A l'issue d'une immense réunion tenue au bal Bullier (six mille personnes s'y trouvaient entassées) un ordre du jour fut adopté encourageant les orateurs et les organisateurs de cette manifestation à se rendre auprès des citoyens Painlevé et Herriot, ministres de Poincaré, pour leur demander [...] si, dans un sursaut de vrai républicanisme, ils ne crieraient pas un "Non!" à l'Espagne, via l'Argentine. Painlevé est gêné. Il bredouille: "Oui..., assurément..." Nous pouvons compter sur lui comme sur une planche pourrie.

Henry Torrès (lawyer of Ascaso, Durruti and Jover), in "The short summer of anarchy" by H.M. Enzensberger

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# Chapter 1

## Introduction

### 1.1 Painlevé-I and the Cubic Oscillator

This Thesis is based on four papers [Mas10a], [Mas10b], [Mas10c], [Mas10d]. It deals mainly with the monodromy problem of the cubic oscillator

$$\psi'' = V(\lambda; a, b)\psi, \quad V(\lambda; a, b) = 4\lambda^3 - a\lambda - b, \quad a, b, \lambda \in \mathbb{C}, \quad (1.1)$$

and its relation with the distribution of poles of solutions of the Painlevé first equation (P-I)

$$y''(z) = 6y^2 - z, \quad z \in \mathbb{C}. \quad (1.2)$$

In particular we are interested in studying the poles of the tritronquée solution of P-I (also called *intégrale tritronquée*) and the cubic oscillators related to them.

**Painlevé-I** It is well-known that any local solution of P-I extends to a global meromorphic function  $y(z), z \in \mathbb{C}$ , with an essential singularity at infinity [GLS00]. Global solutions of P-I are called Painlevé-I transcendents, since they cannot be expressed via elementary functions or classical special functions [Inc56]. The *intégrale tritronquée* is a special P-I transcendent, which was discovered by Boutroux in his classical paper [Bou13] (see [JK88] and [Kit94] for a modern review). Boutroux characterized the *intégrale tritronquée* as the unique solution of P-I with the following asymptotic behaviour at infinity

$$y(z) \sim -\sqrt{\frac{z}{6}}, \quad \text{if } |\arg z| < \frac{4\pi}{5}.$$

Nowadays P-I is studied in many areas of mathematics and physics. Indeed, it is remarkable that special solutions of P-I describe scaling asymptotics of a wealth of different important problems.

For example, let us consider the  $n \times n$  Hermitean random matrix model with a polynomial potential. In 1989 three groups of researchers [DS90],

[BK90], [GM90] showed that these matrix models are extremely important in nonperturbative string-theory and 2d gravity. They also discovered that, if the polynomial is quartic, in the large  $n$ -limit the singular part of the susceptibility is a solution of P-I.

Indeed, it turns out [IKF90] that the partition function of the matrix model is the  $\tau$ -function of a difference analogue (i.e. a discretization) of P-I. The authors of [IKF90] proved that in the appropriate continuous limit (that here is the large  $n$ -limit) solutions of the difference analogue of P-I converge to solutions of P-I.

In the framework of the random matrix approach to string theory, it is also important to represent P-I as a "quantization" of finite-gap potentials of KdV. This view-point was developed in [Moo90] [Nov90] [GN94].

It has been shown recently [Dub08][CG09] that Painlevé equations play a big role also in the theory of nonlinear waves and dispersive equations. In particular, recently [DGK09] Dubrovin, Grava and Klein discovered that the intégrale tritronquée provides the universal correction to the semiclassical limit of solutions to the focusing nonlinear Schrödinger equation.

This elegant description of the semiclassical limit is effective for relatively big values of the semiclassical parameter  $\varepsilon$  if the intégrale tritronquée does not have any large pole in the sector  $|\arg \alpha| < \frac{4\pi}{5}$ . In this direction, theoretical and numerical evidences led the authors of [DGK09] to the following

**CONJECTURE.** [DGK09] *If  $\alpha \in \mathbb{C}$  is a pole of the intégrale tritronquée then  $|\arg \alpha| \geq \frac{4\pi}{5}$ .*

This conjecture has been a major source of inspiration for our work.

**The Cubic Oscillator** The cubic oscillator is a prototype for the general anharmonic oscillator (or Schrödinger equation with a polynomial potential).

In this thesis we deal only with the cubic anharmonic oscillator (1.1); in particular we are interested in the monodromy problem for the cubic oscillator. As most good mathematical problems, it is simple to state and hard to solve. We introduce it briefly here.

We let  $S_k = \left\{ \lambda : \left| \arg \lambda - \frac{2\pi k}{5} \right| < \frac{\pi}{5} \right\}$ ,  $k \in \mathbb{Z}_5$ . We call  $S_k$  the  $k$ -th Stokes sector. Here, and for the rest of the thesis,  $\mathbb{Z}_5$  is the group of the integers modulo five. We will often choose as representatives of  $\mathbb{Z}_5$  the numbers  $-2, -1, 0, 1, 2$ .

For any Stokes sector, there is a unique (up to a multiplicative constant) solution of the cubic oscillator that decays exponentially inside  $S_k$ . We call such solution the  $k$ -th *subdominant solution* and let  $\psi_k(\lambda; a, b)$  denote it.

The asymptotic behaviour of  $\psi_k$  is known explicitly in a bigger sector of the complex plane, namely  $S_{k-1} \cup \overline{S_k} \cup S_{k+1}$ :

$$\lim_{\substack{\lambda \rightarrow \infty \\ |\arg \lambda - \frac{2\pi k}{5}| < \frac{3\pi}{5} - \varepsilon}} \frac{\psi_k(\lambda; a, b)}{\lambda^{-\frac{3}{4}} \exp \left\{ -\frac{4}{5} \lambda^{\frac{5}{2}} + \frac{a}{2} \lambda^{\frac{1}{2}} \right\}} \rightarrow 1, \quad \forall \varepsilon > 0.$$

Here the branch of  $\lambda^{\frac{1}{2}}$  is chosen such that  $\psi_k$  is exponentially small in  $S_k$ .

Since  $\psi_{k-1}$  grows exponentially in  $S_k$ , then  $\psi_{k-1}$  and  $\psi_k$  are linearly independent. Then  $\{\psi_{k-1}, \psi_k\}$  is a basis of solutions, whose asymptotic behaviours is known in  $S_{k-1} \cup S_k$ .

Fixed  $k^* \in \mathbb{Z}_5$ , we know the asymptotic behaviour of  $\{\psi_{k^*-1}, \psi_{k^*}\}$  only in  $S_{k^*-1} \cup S_{k^*}$ . If we want to know the asymptotic behaviours of this basis in all the complex plane, it is sufficient to know the linear transformation from basis  $\{\psi_{k-1}, \psi_k\}$  to basis  $\{\psi_k, \psi_{k+1}\}$  for any  $k \in \mathbb{Z}_5$ .

From the asymptotic behaviours, it follows that these changes of basis are triangular matrices: for any  $k$ ,  $\psi_{k-1} = \psi_{k+1} + \sigma_k \psi_k$  for some complex number  $\sigma_k$ , called *Stokes multiplier*. The quintuplet of Stokes multipliers  $\sigma_k, k \in \mathbb{Z}_5$  is called the monodromy data of the cubic oscillator.

It is well-known (see Chapter 2) that the Stokes multipliers satisfy the following system of quadratic relations

$$-i\sigma_{k+3}(a, b) = 1 + \sigma_k(a, b)\sigma_{k+1}(a, b), \quad \forall k \in \mathbb{Z}_5, \quad \forall a, b \in \mathbb{C}. \quad (1.3)$$

Hence, it turns out that the monodromy data of any cubic oscillator is a point of a two-dimensional smooth algebraic subvariety of  $\mathbb{C}^5$ , called *space of monodromy data*, which we denote by  $V_5$ .

The monodromy problem is two-fold: on one side we have the *direct monodromy problem*, namely the problem of computing the Stokes multipliers of a given cubic oscillator; on the other side we have the *inverse monodromy problem*, viz, the problem of computing which cubic polynomials are such that the corresponding cubic oscillators have a given set of Stokes multipliers.

The monodromy problem can be easily generalized to anharmonic oscillators of any order. It has been deeply studied in mathematics and in quantum physics and a huge literature is devoted to it.

From the very beginning of quantum mechanics, physicists studied anharmonic oscillators as perturbations of the harmonic oscillator

$$\frac{d^2\psi(x)}{dx^2} = (x^2 - E) \psi(x), \quad x \in \mathbb{R}.$$

To this regard the reader may consult [BW68], [Sim70], [BB98].

In early thirties Nevanlinna [Nev32] showed that anharmonic oscillators classify coverings of the sphere with a finite number of logarithmic branch points. In particular the cubic oscillators classify coverings with five logarithmic branch points. Recently [EG09a] Eremenko and Gabrielov applied Nevanlinna's theory to studying the surfaces  $\Gamma_k = \{(a, b) \in \mathbb{C}^2 \mid \sigma_k(a, b) = 0\}$ . They succeeded in giving a complete combinatorial description of the (branched) covering map  $\pi : \Gamma_k \rightarrow \mathbb{C}$ ,  $\pi(a, b) = a$ .

In late nineties, Dorey and Tateo [DDT01] and Bazhanov, Lukyanov and Zamolodchikov [BLZ01] discovered a remarkable link between anharmonic



oscillator (with a potential  $\lambda^n - E$ ) and integrable models of Statistical Field Theory, that has been called 'ODE/IM Correspondence'. The 'ODE/IM' correspondence has been widely generalized (see for example [DDM<sup>+</sup>09]) and it is now a very active field of research.

**Poles of Solutions of P-I and the Cubic Oscillator** As it was mentioned before, the cubic oscillator (1.1) is strictly related to P-I. Such a correspondence will be thoroughly studied in Chapter 3. Here we explain it briefly.

It is well-known, and it will be important in the rest of the thesis, that P-I can be represented as the equation of isomonodromy deformation of an auxiliary linear equation; the choice of the linear equation is not unique, see for example [KT05], [Kap04], [FMZ92].

Here we follow [KT05] and choose the following auxiliary equation

$$\begin{aligned} \frac{d^2\psi(\lambda)}{d\lambda^2} &= Q(\lambda; y, y', z)\psi(\lambda), \lambda, y, y', z \in \mathbb{C} \\ Q(\lambda; y, y', z) &= 4\lambda^3 - 2\lambda z + 2zy - 4y^3 + y'^2 + \frac{y'}{\lambda - y} + \frac{3}{4(\lambda - y)^2}. \end{aligned} \quad (1.4)$$

We call such equation the *perturbed cubic oscillator*.

It turns out (see Chapter 2) that one can define subdominant solutions  $\psi_k$ , and Stokes multipliers  $\sigma_k, k \in \mathbb{Z}_5$  also for the perturbed cubic oscillator. Moreover, also the Stokes multipliers of the perturbed oscillator satisfy the system of quadratic relations (1.3); hence, the quintuplet of Stokes multipliers of any perturbed cubic oscillator is a point of the space of monodromy data  $V_5$ .

Since P-I is the equation of isomonodromy deformation of the perturbed cubic oscillator <sup>1</sup> we can define a map  $\mathcal{M}$  from the set of solutions of P-I to the space of monodromy data; fixed a solution  $y^*$ ,  $\mathcal{M}(y^*)$  is the monodromy data of the perturbed cubic oscillator with potential  $Q(y^*(z), y^{*'}(z), z)$ , for any  $z$  such that  $y^*$  is not singular. It is well-known that the map  $\mathcal{M}$  is a special case of Riemann-Hilbert correspondence [Kap04], [FMZ92].

In this thesis we are mainly interested in studying poles of solutions of P-I; we cannot use directly the perturbed oscillator in this study because the potential  $Q(\lambda; y, y', z)$  is not defined at poles, i.e. when  $y = y' = \infty$ .

However, inspired by a brilliant idea of Its et al. [IN86] about the Painlevé second equation, we study the auxiliary equation in the proximity of a pole of a solution  $y$  of P-I. We show that it has a well-defined limit and the limit is a cubic oscillator. More precisely, we will prove the following

---

<sup>1</sup>Let the parameters  $y = y(z), y' = \frac{dy(z)}{dz}$  of the potential  $Q(\lambda; y, y', z)$  be functions of  $z$ ; then  $y(z)$  solves P-I if and only if the Stokes multipliers of the perturbed oscillator do not depend on  $z$

**LEMMA (4.5).** *Let  $a$  be a pole of a fixed solution  $y^*(z)$  of P-I and let  $\psi_k(\lambda; z)$  denote the  $k$ -th subdominant solution of the perturbed cubic oscillator (1.4) with potential  $Q(\lambda; y^*(z), y'^*(z), z)$ . In the limit  $z \rightarrow a$ ,  $\psi_k(\lambda; z)$  converges (uniformly on compacts) to the  $k$ -th subdominant solution  $\psi_k(\lambda; 2a, 28b)$  of the cubic oscillator*

$$\psi'' = (4\lambda^3 - 2a\lambda - 28b) \psi. \quad (1.5)$$

*Here the parameter  $b$  is the coefficient of the  $(z - a)^4$  term in the Laurent expansion of  $y^*$ :  $y^* = \frac{1}{(z-a)^2} + \frac{a(z-a)^2}{10} + \frac{(z-a)^3}{6} + b(z-a)^4 + O((z-a)^5)$ .*

Lemma 4.5 is one of the most important technical parts of the thesis and Section 4.5 is entirely devoted to its proof.

Fixed arbitray Cauchy data  $\psi(\lambda_0), \psi'(\lambda_0)$ , it is rather easy to prove that the solution of the perturbed cubic oscillator (1.4) converges, as  $z \rightarrow a$ , to the solution of the cubic oscillator (1.5) with the same Cauchy data.

It is far more difficult, but it is necessary to show that the monodromy does not change in the limit, to prove the convergence of the subdominant solutions. This is due to the fact that subdominant solutions are not defined by a Cauchy problem but by an asymptotic behaviour, and (in some sense clarified in the proof of the Lemma) the limits  $\lambda \rightarrow \infty$  and  $z \rightarrow a$  do not commute .

In Chapter 3 we will be able to prove the following important consequences of Lemma 4.5, which define precisely the relation between P-I and the cubic oscillator; they are, therefore, the starting point of our research <sup>2</sup>.

**Theorem 3.2** Fix a solution  $y^*$  and call  $\sigma_k^*, k \in \mathbb{Z}_5$  its Stokes multipliers:  $\mathcal{M}(y^*) = \{\sigma_{-2}^*, \dots, \sigma_2^*\}$ .

The point  $a \in \mathbb{C}$  is a pole of  $y^*$  if and only if there exists  $b \in \mathbb{C}$  such that  $\sigma_k^*, k \in \mathbb{Z}_5$  are the monodromy data of the cubic oscillator

$$\psi'' = (4\lambda^3 - 2a\lambda - 28b) \psi.$$

The parameter  $b$  turns out to be the coefficient of the  $(z - a)^4$  term in the Laurent expansion of  $y^*$ .

**Theorem 3.3** Poles of intégrale tritronquée are in bijection with cubic oscillators such that  $\sigma_2 = \sigma_{-2} = 0$ . In physical terminology, these cubic oscillators are said to satisfy two "quantization conditions".

**Theorem 3.4** The Riemann-Hilbert correspondence  $\mathcal{M}$  is bijective. In other words,  $V_5$  is the moduli space of solutions of P-I.

---

<sup>2</sup>Even though the statement of Theorems 3.2 and 3.3 already appeared in [CC94] by D. Chudnovsky and G. Chudnovsky, in [Mas10a] we gave (perhaps the first) a rigorous proof. Theorem 3.4 can be proven also by other means (see for example [KK93], [Kit94]).

## 1.2 Aims

for the miserable and unhappy are those whose impulse to action  
is found in its reward.

in Bhagavadgita 2.49, translated by W. Q. Judge

Initially, our research was focused on the study of the distribution of poles of the intégrale tritronquée using the relation between P-I and the cubic oscillator.

Our first task was to obtain a qualitative picture of the distribution of poles. In this regard, we succeeded (see Chapter 5) in giving a very precise asymptotic description by means of the complex WKB methods, that we developed (see Chapter 4) following Fedoryuk [Fed93].

Eventually, it became clear that it was necessary to turn our attention to the general monodromy problem of the cubic oscillator in order to obtain more precise information on the poles of solutions of P-I close to the origin. Hence, the broader aim has been to give an effective solution to the monodromy problem of the cubic oscillator, a satisfactory solution both from the theoretical and from the computational view point.

Hence, after having developed the complex WKB method as a tool to compute approximately the monodromy problem (see Chapter 4), we decided to investigate the monodromy problem exactly generalizing the approach of Dorey and Tateo; they showed [DDT01] that if  $a = 0$ , the Stokes multiplier  $\sigma_k(0, b)$  satisfy a nonlinear integral equation, called Thermodynamic Bethe Ansatz. We were able to extend their construction in the case  $a \neq 0$ . We proved (see Chapter 6) that if  $a$  is fixed and small enough, then the Stokes multipliers  $\sigma_k(a, b)$  satisfy a deformation of the Thermodynamic Bethe Ansatz, that we called Deformed Thermodynamic Bethe Ansatz.

We have also focused our attention on the numerical solution of the monodromy problem, because this is one of the major tasks in view of possible applications.

On one side, in collaboration with A. Moro, we are studying the numerical solution the Deformed TBA (see Chapter 6 for preliminary results). On the other side, we invented a new numerical algorithm (see Chapter 7) to compute the Stokes multiplier without solving directly the cubic oscillator, but an associated nonlinear differential equation (2.11). This algorithm is based on an explicit relation (see Theorem 2.4), that we have discovered, between Stokes multipliers and the Nevanlinna's theory of the cubic oscillator.

## 1.3 Main Results

We now outline the main results, that we have achieved after having rigorously established the relation between poles of solution of P-I and cubic oscillators.

**WKB Analysis of the Cubic Oscillator** The complex WKB method is a rather powerful and well-known tool for the approximate solution of the monodromy problem of anharmonic oscillators. There are many possible approaches to it, see for example [Fed93], [BW68], [Sib75], [Vor83]. In our research we followed the Fedoryuk's approach.

The complex WKB method is essentially a method of steepest descent; indeed, a central role is played by the lines of steepest descent of the imaginary part of the action

$$S(\lambda; a, b) = \int_{\lambda^*}^{\lambda} \sqrt{V(\mu; a, b)} d\mu, \quad V(\lambda; a, b) = 4\lambda^3 - a\lambda - b. \quad (1.6)$$

These lines of steepest descent are called Stokes lines; the union of the Stokes lines is called the Stokes complex of the potential  $V(\lambda; a, b)$ . The Fedoryuk's approach clearly shows that the asymptotic behaviour of the Stokes multipliers depends on the topology of the Stokes complex.

In the huge literature devoted to the complex WKB analysis, the main application has been the study of the eigenvalues distribution for large value of the "energy"; in our notation, most authors studied the surfaces  $\Gamma_k = \{(a, b) \in \mathbb{C}^2 \mid \sigma_k(2a, 28b) = 0\}$  in the limit  $b \rightarrow \infty$  with  $a$  bounded.

According to the above-mentioned Theorem 3.3, the set of poles of the intégrale tritronquée is exactly the intersection  $\Gamma_2 \cap \Gamma_{-2}$ ; however (see Lemma 4.5), such intersection is eventually empty in the limit  $b \rightarrow \infty$  with  $a$  fixed, even though the intégrale tritronquée has an infinite number of poles: the scaling  $b \rightarrow \infty$  with  $a$  fixed **cannot be used** to study the poles of the intégrale tritronquée or of any given solution of P-I<sup>3</sup>. Therefore there were no results in the literature that could be readily used to study the poles of the intégrale tritronquée.

Indeed, it had been a challenging task to develop a fully rigorous complex WKB method up to the point that we could correctly describe the asymptotic distribution of the poles of the intégrale tritronquée. We achieved this goal through the following steps.

1. We obtained (see Theorem 4.2) a complete topological classification of the Stokes complexes of the general cubic potential. According to our classification, there are (modulo the action of  $\mathbb{Z}_5$ ) seven of such topologies.
2. We identified (see Section 4.2.3) one topology, that we called "Boutroux" graph, as the unique topology compatible with the system of "quanti-

---

<sup>3</sup>This should not come as a surprise; no meromorphic function can be approximated by a sequence of functions, that have a pole (the parameter  $a$  of the potential  $V(\lambda; 2a, 28b)$ ) inside some bounded set of the complex plane but such that a term (the parameter  $b$ ) of their Laurent expansions diverges. Indeed, as we explain below, solutions of the system  $\sigma_{\pm 2}(2a, 28b) = 0$  have asymptotically the following scaling behaviour  $\frac{a}{b} \rightarrow 0$ ,  $\frac{a^3}{b^2}$  bounded.

zation conditions"  $\sigma_{\pm 2}(2a, 28b) = 0$ , which describes the poles of the intégrale tritronquée.

3. We computed in the WKB approximation the Stokes multipliers  $\sigma_k(a, b)$  for all the potentials  $V(\lambda; a, b)$  whose Stokes complex is the Boutroux graph. In this way, we eventually derive the WKB analogue of the system  $\sigma_{\pm 2}(2a, 28b) = 0$ . It is the following pair of *Bohr-Sommerfeld* quantization conditions, that we have called Bohr-Sommerfeld-Boutroux (B-S-B) system<sup>4</sup>

$$\begin{aligned} \oint_{a_1} \sqrt{V(\lambda; 2a, 28b)} d\lambda &= i\pi(2n - 1) \\ \oint_{a_{-1}} \sqrt{V(\lambda; 2a, 28b)} d\lambda &= -i\pi(2m - 1) \end{aligned}$$

Here  $m, n$  are positive natural numbers and the paths of integration are shown in Figure 1.1.

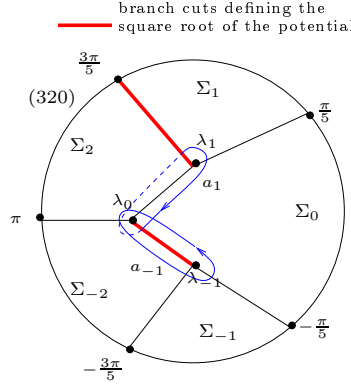


Figure 1.1: Riemann surface  $\mu^2 = V(\lambda; 2a, 28b)$

**B-S-B System and the Poles of Tritronquée** After having derived the B-S-B system, we showed (see Chapter 5) that the poles of intégrale tritronquée are well approximated by solutions of B-S-B system: the distance between a pole and the corresponding solution of the Bohr-Sommerfeld-Boutroux system vanishes asymptotically. Let us introduce precisely our result.

Solutions of the B-S-B system have a simple classification; they are in correspondence with ordered pairs  $(q, k)$ , where  $q = \frac{2n-1}{2m-1}$  is a positive rational number and  $k$  a positive integer. Here  $(a_k^q, b_k^q)$  denotes the general

---

<sup>4</sup>The B-S-B system reproduces the description of the poles of the intégrale tritronquée obtained by Boutroux [Bou13] through a completely different approach (for similar results, see also [JK88] [KK93] [Kit94]).

solution of B-S-B. Fixed  $q$ , the sequence of solutions  $\{(a_k^q, b_k^q)\}_{k \in \mathbb{N}^*}$  has a multiplicative structure:

$$(a_k^q, b_k^q) = ((2k+1)^{\frac{4}{5}} a_1^q, (2k+1)^{\frac{6}{5}} b_1^q) .$$

We were able to prove the following theorem.

**THEOREM (5.2).** *Let  $\varepsilon$  be an arbitrary positive number. If  $\frac{1}{5} < \mu < \frac{6}{5}$ , then it exists  $K \in \mathbb{N}^*$  such that for any  $k \geq K$  inside the disc  $|a - a_k^q| < k^{-\mu} \varepsilon$  there is one and only one pole of the intégrale tritronquée.*

**Deformed Thermodynamic Bethe Ansatz** In a seminal paper [DDT01] Dorey and Tateo proved that the Stokes multipliers  $\sigma_k(0, b)$  satisfy the Thermodynamic Bethe Ansatz equation, introduced by Zamolodchikov [Zam90] to describe the thermodynamics of the 3-state Potts model and of the Lee-Yang model. We succeeded in generalizing Dorey and Tateo approach to the general cubic potential.

Fix  $a \in \mathbb{C}$  and define  $\varepsilon_k(\vartheta) = \ln \left( i\sigma_0(e^{-k\frac{2\pi i}{5}} a, e^{\frac{6\vartheta}{5}}) \right)$ . Following the convention of Statistical Field Theory we call pseudo-energies the functions  $\varepsilon_k$ . We proved that the pseudo-energies satisfy the *nonlinear nonlocal Riemann-Hilbert problem* (6.11), which is equivalent (at least for small value of the parameter  $a$ ) to the following system of nonlinear integral equations that we called Deformed Thermodynamic Bethe Ansatz:

$$\chi_l(\sigma) = \int_{-\infty}^{+\infty} \varphi_l(\sigma - \sigma') \Lambda_l(\sigma') d\sigma', \quad \sigma, \sigma' \in \mathbb{R}, \quad l \in \mathbb{Z}_5 = \{-2, \dots, 2\} .$$

Here

$$\begin{aligned} \Lambda_l(\sigma) &= \sum_{k \in \mathbb{Z}_5} e^{i\frac{2lk\pi}{5}} L_k(\sigma) \quad , \quad L_k(\sigma) = \ln \left( 1 + e^{-\varepsilon_k(\sigma)} \right) , \\ \varepsilon_k(\sigma) &= \frac{1}{5} \sum_{l \in \mathbb{Z}_5} e^{-i\frac{2lk\pi}{5}} \chi_l(\sigma) + \frac{\sqrt{\frac{\pi}{3}} \Gamma(1/3)}{2^{\frac{5}{3}} \Gamma(11/6)} e^{\sigma} + a \frac{\sqrt{3\pi} \Gamma(2/3)}{4^{\frac{2}{3}} \Gamma(1/6)} e^{\frac{\sigma}{5} - i\frac{2k\pi}{5}} , \\ \varphi_0(\sigma) &= \frac{\sqrt{3}}{\pi} \frac{2 \cosh(2\sigma)}{1 + 2 \cosh(2\sigma)} \quad , \quad \varphi_1(\sigma) = -\frac{\sqrt{3}}{\pi} \frac{e^{-\frac{9}{5}\sigma}}{1 + 2 \cosh(2\sigma)} \\ \varphi_2(\sigma) &= -\frac{\sqrt{3}}{\pi} \frac{e^{-\frac{3}{5}\sigma}}{1 + 2 \cosh(2\sigma)} \quad , \quad \varphi_{-1}(\sigma) = \varphi_1(-\sigma), \varphi_{-2}(\sigma) = \varphi_2(-\sigma) . \end{aligned}$$

**A Numerical Algorithm** We have developed a new algorithm to compute the Stokes multipliers of the cubic oscillator (1.1) and of the perturbed cubic

oscillator (1.4). The algorithm is based on the formula <sup>5</sup> (1.8) below, that we discovered in [Mas10d].

Consider the following Schwarzian equation

$$\{f(\lambda), \lambda\} = -2V(\lambda; a, b). \quad (1.7)$$

Here  $\{f(\lambda), \lambda\} = \frac{f'''(\lambda)}{f'(\lambda)} - \frac{3}{2} \left( \frac{f''(\lambda)}{f'(\lambda)} \right)^2$  is the Schwarzian derivative.

For every solution of the Schwarzian equation (1.7) the following limit exists

$$w_k(f) = \lim_{\lambda \rightarrow \infty, \lambda \in S_k} f(\lambda) \in \mathbb{C} \cup \infty,$$

provided the limit is taken along a curve non-tangential to the boundary of  $S_k$ .

In Chapter 2, we will prove that the following formula holds for any solution of the Schwarzian equation (1.7)

$$\sigma_k(a, b) = i(w_{1+k}(f), w_{-2+k}(f); w_{-1+k}(f), w_{2+k}(f)). \quad (1.8)$$

Here  $(a, b; c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$  is the cross ratio of four points on the sphere.

## 1.4 Structure of the Thesis

The Thesis consists of six chapters other than the Introduction.

**Chapter 2** Chapter 2 is introductory. In this Chapter we deal with the basic asymptotic theory of cubic oscillators and we set the notation that we will use throughout the thesis. We define precisely the Stokes multipliers, the space of monodromy data and the monodromy problem. We then introduce the geometric theory of the cubic oscillator and present some original contributions which are mainly drawn from [Mas10d][Mas10c].

**Chapter 3** In Chapter 3 we study the relation among poles of solutions of P-I and the cubic oscillator. The major source, but with some important modifications, is [Mas10a]. The main tool used is the isomonodromy deformation method.

As it was already explained, Painlevé-I is represented as the equation of isomonodromy deformation of the auxiliary Schrödinger equation

$$\begin{aligned} \frac{d^2\psi(\lambda)}{d\lambda^2} &= Q(\lambda; y(z), y'(z), z)\psi(\lambda), \\ Q(\lambda; y, y', z) &= 4\lambda^3 - 2\lambda z + 2zy - 4y^3 + y'^2 + \frac{y'}{\lambda - y} + \frac{3}{4(\lambda - y)^2}. \end{aligned}$$

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<sup>5</sup>For simplicity of notation we present here the theory for the cubic oscillator. Any statement remains valid if one substitutes  $V(\lambda; a, b)$  with  $Q(\lambda; y, y', z)$ .

We study the auxiliary equation in the proximity of a pole of a solution  $y$  of P-I and we prove the above-mentioned Theorems 3.2, 3.3, 3.4.

We warn the reader that the proof of the main technical Lemma of the Chapter, namely Lemma 3.6, is postponed at the end of Chapter 4 because the proof depends heavily on the WKB analysis.

**Chapter 4** The fourth Chapter is devoted to the WKB analysis of the cubic oscillator. Again, the major source is [Mas10a]. We develop the complex WKB method by Fedoryuk [Fed93] and give a complete topological classification of Stokes complexes for the cubic oscillator, an algorithmic construction of the Maximal Domains and we introduce the small parameter of the approximation. After that, we show by examples how to compute approximately asymptotic values (hence Stokes multipliers) of the cubic oscillator and eventually derive the Bohr-Sommerfeld-Boutroux system.

**Chapter 5** The fifth Chapter deals with the approximation of poles of the intégrale tritronquée by the solutions of the Bohr-Sommerfeld-Boutroux system. The material of this Chapter is taken from [Mas10a] and [Mas10b]. A section of the Chapter is devoted to the study of the poles of the tritronquée on the real axis.

**Chapter 6** In the sixth Chapter we derive the Deformed Thermodynamic Bethe Ansatz equation following [Mas10d]. We also show a numerical solution of the equation. This Chapter is, to a great extent, independent on all other chapters, but Chapter 2.

**Chapter 7** Chapter 7 is devoted to describe an algorithm for solving the direct monodromy problem for the perturbed and unperturbed cubic oscillator. The algorithm is based on the geometric theory of the cubic oscillator developed in Section 2.2. It also contains a numerical experiment. This shows that the WKB approximation is astonishingly precise. The algorithm originally appeared in [Mas10c].

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## Chapter 2

# The Cubic Oscillator

The present Chapter is introductory. In this Chapter we deal with the basic asymptotic theory of cubic oscillators and we set the notation that we will use throughout the thesis. We define precisely the Stokes multipliers, the space of monodromy data and the monodromy problem. We then introduce the geometric theory of the cubic oscillator and present some original contributions which are mainly drawn from [Mas10d][Mas10c].

The monodromy problem for the anharmonic oscillators (in particular the cubic one) is a fundamental and rather interesting problem in itself and a large literature is devoted to it. The interested reader may consult the following papers [BW68], [Sim70], [Vor83], [BB98], [DT99], [BLZ01], [EG09a] and the monograph [Sib75].

Here we do not review all the literature but introduce the elements of the theory that are needed in order to study the relation of the cubic oscillator with the Painlevé first equation; this relation will be explained thoroughly in Chapter 3.

The cubic oscillator is the following linear differential equation in the complex plane

$$\frac{d^2\psi(\lambda)}{d\lambda^2} = V(\lambda; a, b)\psi(\lambda), \quad V(\lambda; a, b) = 4\lambda^3 - a\lambda - b, \quad a, b, \lambda \in \mathbb{C}. \quad (2.1)$$

Since it will be useful in the study of Painlevé-I equation, together with the cubic oscillator we will study also its following perturbation

$$\begin{aligned} \frac{d^2\psi(\lambda)}{d\lambda^2} &= Q(\lambda; y, y', z)\psi(\lambda), \\ Q(\lambda; y, y', z) &= 4\lambda^3 - 2\lambda z + 2zy - 4y^3 + y'^2 + \frac{y'}{\lambda - y} + \frac{3}{4(\lambda - y)^2}. \end{aligned} \quad (2.2)$$

Here  $y, y', z$  are complex parameters.

Remarkably, in some limit relevant for studying the poles of the solutions of Painlevé-I equation (2.2) becomes the cubic oscillator (2.1) (see Lemma 4.10).

**Definition 2.1.** We call any cubic polynomial of the form  $V(\lambda; a, b) = 4\lambda^3 - a\lambda - b$  a cubic potential. The above formula identifies the space of cubic potentials with  $\mathbb{C}_2 \ni (a, b)$ . We call  $Q(\lambda; y, y', z)$  a deformed cubic potential.

The Chapter deals is divided in two Sections. The first one is devoted to the analytic theory in the spirit of Sibuya [Sib75]. In the second one we introduce the geometric or Nevanlinna's theory of the cubic oscillator.

## 2.1 Analytic Theory

Here we introduce the concepts of subdominant solutions, of Stokes multipliers and of eigenvalue problems.

### 2.1.1 Subdominant Solutions

In this subsection we introduce the subdominant solutions of the perturbed and unperturbed cubic oscillators (2.2, 2.1).

We define the Stokes Sector  $S_k$  as

$$S_k = \left\{ \lambda : \left| \arg \lambda - \frac{2\pi k}{5} \right| < \frac{\pi}{5} \right\}, k \in \mathbb{Z}_5. \quad (2.3)$$

We remark that in Chapter 4 we will name Stokes sector and denote it  $\Sigma_k$  a slightly different object.

**Lemma 2.1.** Fix  $k \in \mathbb{Z}_5 = \{-2, \dots, 2\}$ , define the branch of  $\lambda^{\frac{1}{2}}$  by requiring

$$\lim_{\substack{\lambda \rightarrow \infty \\ \arg \lambda = \frac{2\pi k}{5}}} \operatorname{Re} \lambda^{\frac{5}{2}} = +\infty$$

and choose one of the branch of  $\lambda^{\frac{1}{4}}$ . Then there exists a unique solution  $\psi_k(\lambda; a, b)$  of equation (2.1) such that

$$\lim_{\substack{\lambda \rightarrow \infty \\ \left| \arg \lambda - \frac{2\pi k}{5} \right| < \frac{3\pi}{5} - \varepsilon}} \frac{\psi_k(\lambda; a, b)}{\lambda^{-\frac{3}{4}} e^{-\frac{4}{5}\lambda^{\frac{5}{2}} + \frac{a}{2}\lambda^{\frac{1}{2}}}} \rightarrow 1, \forall \varepsilon > 0. \quad (2.4)$$

*Proof.* The proof can be found in Section 4.4 or in Sibuya's monograph [Sib75].  $\square$

A very similar Lemma is valid also for the perturbed oscillator.

**Lemma 2.2.** Fix  $k \in \mathbb{Z}_5 = \{-2, \dots, 2\}$  and define a cut in the  $\mathbb{C}$  plane connecting  $\lambda = y$  with  $\lambda = \infty$  such that its points eventually do not belong to  $S_{k-1} \cup \overline{S_k} \cup S_{k+1}$ . Choose the branch of  $\lambda^{\frac{1}{2}}$  by requiring

$$\lim_{\substack{\lambda \rightarrow \infty \\ \arg \lambda = \frac{2\pi k}{5}}} \operatorname{Re} \lambda^{\frac{5}{2}} = +\infty,$$

while choose arbitrarily one of the branch of  $\lambda^{\frac{1}{4}}$ . Then there exists a unique solution  $\psi_k(\lambda; y, y', z)$  of equation (2.2) such that

$$\lim_{\substack{\lambda \rightarrow \infty \\ |\arg \lambda - \frac{2\pi k}{5}| < \frac{3\pi}{5} - \varepsilon}} \frac{\psi_k(\lambda; y, y', z)}{\lambda^{-\frac{3}{4}} e^{-\frac{4}{5}\lambda^{\frac{5}{2}} + \frac{\varepsilon}{2}\lambda^{\frac{1}{2}}}} \rightarrow 1, \quad \forall \varepsilon > 0. \quad (2.5)$$

*Proof.* The proof can be found in Section 4.5.  $\square$

**REMARK.** Equation (2.2) has a fuchsian singularity at the pole  $\lambda = y$  of the potential  $Q(\lambda; y, y', z)$ . However this is an apparent singularity (see Lemma 3.1): the monodromy around the singularity of any solution is  $-1$ .

According to the previous Lemmas,  $\psi_k(\lambda; y, y', z)$  (or  $\psi_k(\lambda; a, b)$ ) is exponentially small inside the Stokes sector  $S_k$  and exponentially big inside  $S_{k\pm 1}$ . Due to their different asymptotics  $\psi_k$  and  $\psi_{k+1}$  are linearly independent for any  $k \in \mathbb{Z}_5$ . Hence,  $\psi_k$  is, modulo a multiplicative constant, the unique exponentially small solution in the  $k$ -th sector  $S_k$ .

**Definition 2.2.** We denote  $\psi_k(\lambda; a, b)$  the solution of equation (2.1) uniquely defined by (2.4). We denote  $\psi_k(\lambda; y, y', z)$  the solution of equation (2.2) uniquely defined by (2.5). We call them  $k$ -th subdominant solutions.

### 2.1.2 The monodromy problem

If one fixes the same branch of  $\lambda^{\frac{1}{4}}$  in the asymptotics (2.4) of  $\psi_{k-1}, \psi_k, \psi_{k+1}$  then the following equation hold true

$$\psi_{k-1}(\lambda; a, b) = \psi_{k+1}(\lambda; a, b) + \sigma_k(a, b)\psi_k(\lambda; a, b). \quad (2.6)$$

Moreover the *Stokes multipliers*  $\sigma_k$  satisfy the following system of quadratic equation

$$-i\sigma_{k+3} = 1 + \sigma_k\sigma_{k+1}, \quad \forall k \in \mathbb{Z}_5. \quad (2.7)$$

We can introduce Stokes multipliers also for the perturbed cubic oscillator (2.2). Define a cut in the  $\mathbb{C}$  plane connecting  $\lambda = y$  with  $\lambda = \infty$  such that its points eventually do not belong to  $S_{k-1} \cup \overline{S_k} \cup S_{k+1}$ . If one fixes the same branch of  $\lambda^{\frac{1}{4}}$  in the asymptotics (2.5) of  $\psi_{k-1}, \psi_k, \psi_{k+1}$  then the following equation hold true

$$\frac{\psi_{k-1}(\lambda; y, y', z)}{\psi_k(\lambda; y, y', z)} = \frac{\psi_{k+1}(\lambda; y, y', z)}{\psi_k(\lambda; y, y', z)} + \sigma_k(y, y', z). \quad (2.8)$$

The Stokes multipliers  $\sigma_k(y, y', z)$  satisfy the same system of quadratic equations (2.7). The reader should notice the ratio of two solutions of the perturbed oscillator is a single-valued meromorphic function.

**Definition 2.3.** The functions  $\sigma_k(a, b), \sigma_k(y, y', z)$  are called *Stokes multipliers*. The quintuplet of Stokes multipliers  $\sigma_k(a, b), k \in \mathbb{Z}_5$  (resp.  $\sigma_k(y, y', z)$ ) are called the *monodromy data of equation (2.1)* (resp. of equation (2.2)).

Observe that only 3 of the algebraic equations (2.7) are independent.

**Definition 2.4.** We denote  $V_5$  the smooth algebraic variety of quintuplets of complex numbers satisfying (2.7) and call *admissible monodromy data* the elements of  $V_5$ .

The Stokes multipliers of the cubic oscillator are entire functions of the two parameters  $(a, b)$  of the potential. Hence we define the following *monodromy map*

$$\begin{aligned} \mathcal{T} &: \mathbb{C}^2 \rightarrow V_5, \\ \mathcal{T}(a, b) &= (\sigma_{-2}(a, b), \dots, \sigma_2(a, b)). \end{aligned} \tag{2.9}$$

**Theorem 2.1.** The map  $\mathcal{T}$  is surjective. The preimage of any admissible monodromy data is a countable infinite subset of the space of cubic potentials.

*Proof.* See [Nev32]. □

We have collected all the elements to state the direct and inverse monodromy problem for the cubic oscillator

**PROBLEM.** We call *Direct Monodromy Problem* the problem of computing the monodromy map  $\mathcal{T}$ . We call *Inverse Monodromy Problem* the problem of computing the inverse of the monodromy map.

Until now, neither of the problems have been satisfactorily solved. However, we have made substantial progress towards the solution. For what concerns the inverse problem, we will show in Chapter 3 that to any admissible monodromy data  $v$  there corresponds one and only one solution  $y$  of the Painleve-I equation, such that  $\mathcal{T}^{-1}(v) = \{(2\alpha_i, 28\beta_i)\}_{i \in \mathbb{N}}$ , where  $\{(2\alpha_i, 28\beta_i)\}_{i \in \mathbb{N}}$  is the set of poles of  $y$ . Here  $\alpha_i$  is the location of a pole of  $y$ ,  $\beta_i$  is a coefficient of the Laurent expansion of  $y$  around  $\alpha_i$ .

We have also made many progress in the understanding of the direct problem: we have developed the asymptotic theory of the Stokes multipliers (see Chapter 4), and we have built an analytic tool called Deformed Thermodynamic Bethe Ansatz (see Chapter 6) and a numerical algorithm (see Chapter 7) to solve the monodromy problem.

## Eigenvalue Problems

The surfaces  $\{\sigma_k(a, b) = 0, k \in \mathbb{Z}_5\}$  are particularly important in the theory of the cubic oscillator. Indeed,  $\sigma_k(a, b) = 0$  if and only if there exists a

solution of the following boundary value problem

$$\frac{d^2\psi(\lambda)}{d\lambda^2} = V(\lambda; a, b)\psi(\lambda) , \quad \lim_{\lambda \rightarrow \infty, \lambda \in \gamma_{k-1} \cup \gamma_{k+1}} \psi(\lambda) = 0 .$$

Here  $\gamma_{k\pm 1}$  is any ray contained in the  $(k\pm 1)$ -th Stokes sector. The boundary value problem is also called *lateral connection problem*.

Fixed  $a$ , the boundary value problem is equivalent to the eigenvalue problem for the Schrödinger operator  $-\frac{d^2}{d\lambda^2} + 4\lambda^3 - a\lambda$  defined on  $L^2(\gamma_1 \cup \gamma_2)$ . Since the eigenvalues are a discrete set, the equation  $\sigma_k(a, b)$  is often called a *quantization condition*.

The eigenvalues problems are invariant under an anti-holomorphic involution: let  $\omega = e^{i\frac{2\pi}{5}}$ , then  $\sigma_k(a, b) = 0 \Leftrightarrow \sigma_k(\omega^{4k}\bar{a}, \omega^{6k}\bar{b}) = 0$ ; here  $\bar{\phantom{x}}$  stands for the complex conjugation.

If  $a$  is a fixed point of the involution  $a \rightarrow \omega^{4k}\bar{a}$ , i.e. if  $\omega^{2k}a$  is real, then the eigenvalue problem is said to be PT symmetric, because  $b$  is an eigenvalue if and only if  $\omega^{6k}\bar{b}$  is (the study of PT symmetric oscillators began in the seminal paper [BB98]).

It is natural to ask if all the eigenvalues  $b$  of a PT symmetric operator are invariant under the involution  $b \rightarrow \omega^{6k}\bar{b}$ . This is not the case in general (see for example [BB98]). However, in [BB98] the authors conjectured that if  $\omega^{2k}a$  is real and non-negative then all the eigenvalues  $b$  are such that  $\omega^{3k}b$  is real and negative.

Dorey, Dunning, Tateo [DDT01] proved the conjecture in the case  $a = 0$  and Shin [Shi02] extended the result to the general case.

This theorem will be fundamental in Chapter 6 for deriving the Deformed Thermodynamic Bethe Ansatz.

**Theorem 2.2.** *Fix  $k \in \mathbb{Z}_5$ . Suppose  $\sigma_k(a, b) = 0$  and  $\omega^{2k}a$  is real and non negative. Then  $\omega^{3k}b$  is real and negative.*

*Proof.* See [Shi02] (see also [DDT01] for the case  $a = 0$ ). □

## 2.2 Geometric Theory

In the analytic theory, the monodromy data of equation (2.1) are expressed in terms of Stokes multipliers, which are defined by means of a special set of solutions of the equation. In this section, following Nevanlinna [Nev32] and author's paper [Mas10d], we study the monodromy data from a geometric (hence invariant) viewpoint. Eventually, we realize the Stokes multipliers of the cubic oscillators as natural coordinates on the quotient  $W_5/PSL(2, \mathbb{C})$ , where  $W_5$  is a dense open subset of  $(\mathbb{P}^1)^5$  (the Cartesian product of five copies of  $\mathbb{P}^1$ ). The core of Nevanlinna theory is based on the interrelation among anharmonic oscillators and branched coverings of the sphere. We will not introduce the correspondence here. The interested reader may consult

the original works of Nevanlinna [Nev32] [Nev70] and Elfving [Elf34] or the remarkable recent papers of Gabrielov and Eremenko [EG09a],[EG09b].

In the present Section we consider the cubic oscillator (2.1) and the cubic potential  $V(\lambda; a, b)$  as particular cases of the perturbed oscillator (2.2) and of the potential  $Q(\lambda; y, y', z)$ . We use the convention that the cubic oscillator is the particular case of the perturbed cubic oscillator determined by  $y = \infty$ .

### 2.2.1 Asymptotic Values

The main geometric object of Nevanlinna's theory is the Schwarzian derivative of a (non constant) meromorphic function  $f(\lambda)$

$$\{f(\lambda), \lambda\} = \frac{f'''(\lambda)}{f'(\lambda)} - \frac{3}{2} \left( \frac{f''(\lambda)}{f'(\lambda)} \right)^2. \quad (2.10)$$

The Schwarzian derivative is strictly related to the Schrödinger equation (2.2). Indeed, the following Lemma is true.

**Lemma 2.3.** *The (non constant) meromorphic function  $f : \mathbb{C} \rightarrow \overline{\mathbb{C}}$  solves the Schwarzian differential equation*

$$\{f(\lambda), \lambda\} = -2Q(\lambda; y, y', z). \quad (2.11)$$

*iff  $f(\lambda) = \frac{\phi(\lambda)}{\chi(\lambda)}$  where  $\phi(\lambda)$  and  $\chi(\lambda)$  are two linearly independent solutions of the Schrödinger equation (2.2).*

Every solution of the Schwarzian equation (2.11) has limit for  $\lambda \rightarrow \infty$ ,  $\lambda \in S_k$ . More precisely we have the following

**Lemma 2.4** (Nevanlinna). *(i) Let  $f(\lambda) = \frac{\phi(\lambda)}{\chi(\lambda)}$  be a solution of (2.11) then for all  $k \in \mathbb{Z}_5$  the following limit exists*

$$w_k(f) = \lim_{\lambda \rightarrow \infty, \lambda \in S_k} f(\lambda) \in \mathbb{C} \cup \infty, \quad (2.12)$$

*provided the limit is taken along a curve non-tangential to the boundary of  $S_k$ .*

*(ii)  $w_{k+1}(f) \neq w_k(f)$ ,  $\forall k \in \mathbb{Z}_5$ .*

*(iii) Let  $g(\lambda) = \frac{af(\lambda)+b}{cf(\lambda)+d} = \frac{a\phi(\lambda)+b\chi(\lambda)}{c\phi(\lambda)+d\chi(\lambda)}$ ,  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(2, \mathbb{C})$ . Then*

$$w_k(g) = \frac{a w_k(f) + b}{c w_k(f) + d}. \quad (2.13)$$

*(iv) If the function  $f$  is evaluated along a ray contained in  $S_k$ , the convergence to  $w_k(f)$  is super-exponential.*

*Proof.* (i-iii) Let  $\psi_k$  be the solution of equation (2.1) subdominant in  $S_k$  and  $\psi_{k+1}$  be the one subdominant in  $S_{k+1}$ . We have that  $f(\lambda) = \frac{\alpha\psi_k(\lambda) + \beta\psi_{k+1}(\lambda)}{\gamma\psi_k(\lambda) + \delta\psi_{k+1}(\lambda)}$ , for some  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Gl(2, \mathbb{C})$ . Hence  $w_k(f) = \frac{\beta}{\delta}$  if  $\delta \neq 0$ ,  $w_k(f) = \infty$  if  $\delta = 0$ . Similarly  $w_{k+1}(f) = \frac{\alpha}{\gamma}$ . Since  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Gl(2, \mathbb{C})$  then  $w_k(f) \neq w_{k+1}(f)$

(iv) From equation (2.4) we know that inside  $S_k$ ,

$$\left| \frac{\psi_k(\lambda)}{\psi_{k+1}(\lambda)} \right| \sim e^{-Re\left(\frac{8}{5}\lambda^{\frac{5}{2}} - a\lambda^{\frac{1}{2}}\right)},$$

where the branch of  $\lambda^{\frac{1}{2}}$  is chosen such that the exponential is decaying.  $\square$

**Definition 2.5.** Let  $f(\lambda)$  be a solution of the Schwarzian equation (2.11) and  $w_k(f)$  be defined as in (2.12). We call  $w_k(f)$  the  $k$ -th asymptotic value of  $f$ .

## 2.2.2 Space of Monodromy Data

**Definition 2.6.** We define

$$W_5 = \{(z_{-2}, z_{-1}, z_0, z_1, z_2), z_k \in \mathbb{C} \cup \infty, z_k \neq z_{k+1}, z_2 \neq z_{-2}\}.$$

The group of automorphism of the Riemann sphere, called Möbius group or  $PSL(2, \mathbb{C})$ , has the following natural free action on  $W_5$ : let  $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$  then

$$T(z_{-2}, \dots, z_2) = \left( \frac{az_{-2} + b}{cz_{-2} + d}, \dots, \frac{az_2 + b}{cz_2 + d} \right).$$

After Definition 2.5 and Lemma 2.4(iii) every basis of solution of (2.2) determines a point in  $W_5$ . After the transformation law (2.13), the Schrödinger equation (2.2) determines an orbit of the  $PSL(2, \mathbb{C})$  action.

Below we prove that the quotient  $W_5/PSL(2, \mathbb{C})$  is isomorphic, as a complex manifold, to the space of monodromy data  $V_5$  defined by the system of quadratic equations (2.7) (see Definition 2.4). To this aim we introduce the following R functions

$$\begin{aligned} R_k : W_5 &\rightarrow \mathbb{C}, \quad k \in \mathbb{Z}_5, \\ R_k(z_{-2}, \dots, z_2) &= (z_{1+k}, z_{-2+k}; z_{-1+k}, z_{2+k}), \end{aligned} \quad (2.14)$$

where  $(a, b; c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$  is the cross ratio of four points on the sphere.

Functions  $R$  will be studied in details in Chapter 6. We collect here their main properties



**Lemma 2.5.** [Mas10d]

(i) The functions  $R_k$  are invariant under the  $PSL(2, \mathbb{C})$  action. Hence they are well defined on  $V_5$ : with a small abuse of notation we let  $R_k$  denote also the functions defined on  $V_5$ .

(ii) They satisfy the following set of quadratic relation

$$R_{k-2}R_{k+2} = 1 - R_k, \quad \forall k \in \mathbb{Z}_5. \quad (2.15)$$

(iii) The pair  $R_k, R_{k+1}$  is a coordinate system of  $W_5/PSL(2, \mathbb{C})$  on the open subset  $R_{k-2} \neq 0$ . The pair of coordinate systems  $(R_k, R_{k+1})$  and  $(R_{k+2}, R_{k-2})$  form an atlas of  $W_5/PSL(2, \mathbb{C})$ .

(iv)

$$\begin{aligned} R_k(z_{-2}, \dots, z_2) &\neq \infty, \quad \forall (z_{-2}, \dots, z_2) \in W_5, \\ R_k(z_{-2}, \dots, z_2) &= 0 \text{ iff } z_{k-1} = z_{k+1}, \\ R_k(z_{-2}, \dots, z_2) &= 1 \text{ iff } z_{k-1} = z_{k+2} \text{ or } z_{k+1} = z_{k-2}. \end{aligned} \quad (2.16)$$

We can now prove the following

**Theorem 2.3.** [Mas10d] The space of monodromy data  $V_5$  is isomorphic as a complex manifold to the quotient  $W_5/PSL(2, \mathbb{C})$ .

*Proof.* Define the map  $\varphi : W_5/PSL(2, \mathbb{C}) \rightarrow V_5$ ,  $\varphi(\cdot) = i(R_{-2}(\cdot), \dots, R_2(\cdot))$ . Due to Lemma 2.5(i-iii)  $\varphi$  is bi-holomorphic.  $\square$

**REMARK.** From the construction of  $V_5$  as a quotient space it is evident that  $\mathcal{M}_{0,5} \subset V_5 \subset \overline{\mathcal{M}_{0,5}}$ . Here  $\mathcal{M}_{0,5}$  is the moduli space of genus 0 curves with five marked points and  $\overline{\mathcal{M}_{0,5}}$  is its compactification (see [Knu83] for the definition of  $\overline{\mathcal{M}_{0,5}}$ ).

With a slight abuse of notation we call  $R_k(a, b)$  the value of  $R_k$  when the asymptotic values are calculated via the Schwarzian equation with potential  $V(\lambda; a, b)$ . It is easily seen that  $R_k$  is an entire function of two variables. Moreover, it coincides essentially with the Stokes multiplier  $\sigma_k(a, b)$  defined previously.

**Theorem 2.4.** [Mas10d] For any  $a, b \in \mathbb{C}^2$ ,

$$\sigma_k(a, b) = iR_k(a, b). \quad (2.17)$$

*Proof.* Let  $\psi_{k+1}$  be the solution of (2.1) subdominant in  $S_{k+1}$  and  $\psi_{k+2}$  be the one subdominant in  $S_{k+2}$  (see the Appendix for the precise definition). By choosing  $f(\lambda) = \frac{\psi_{k+1}(\lambda)}{\psi_{k+2}(\lambda)}$ , one verifies easily that the identity (2.17) is satisfied.  $\square$

**REMARK.** According to previous Theorem and Lemma 2.5 (iv) the  $k$ -th lateral connection problem, i.e.  $\sigma_k(a, b) = 0$ , is solved if and only if for any solution  $f$  of the Schwarzian equation  $w_{k-1}(f) = w_{k+1}(f)$ .

**Singularities** We end the Chapter with an observation which will be used later on in Chapter 7. Since the Schwarzian differential equation is linearized (see Lemma 2.3) by the Schrödinger equation, any solution is a meromorphic function and has an infinite number of poles [Nev70]. The poles, however, are localized near the boundaries of the Stokes sectors  $S_k, k \in \mathbb{Z}_5$ . Indeed, using the complex WKB theory one can prove the following

**Lemma 2.6.** *Let  $f(\lambda)$  be any solution of the Schwarzian equation (2.11). Fix  $\varepsilon > 0$  and define  $\tilde{S}_k = \{\lambda : |\arg \lambda - \frac{2\pi k}{5}| \leq \frac{\pi}{5} - \varepsilon\}, k \in \mathbb{Z}_5$ . Then,  $\forall w \in \mathbb{C} \cup \infty$ ,  $f(\lambda) = w$  has a finite number of solutions inside  $\tilde{S}_k$ . In particular, there are a finite number of rays inside  $\tilde{S}_k$  on which  $f(\lambda)$  has a pole.*

## Chapter 3

# Painlevé First Equation

In this chapter we study the relation among poles of solutions  $y = y(z)$  of Painlevé first equation (P-I)

$$y'' = 6y^2 - z, \quad z \in \mathbb{C}, \quad (3.1)$$

and the cubic oscillator (2.1).

In particular we introduce the special solution called *intégrale tritronquée* and we show that its poles are described by cubic oscillators that admit the simultaneous solution of two quantization conditions.

As it is well-known, any local solution of P-I extends to a global meromorphic function  $y(z), z \in \mathbb{C}$ , with an essential singularity at infinity [GLS00]. Global solutions of P-I are called Painlevé-I transcendents, since they cannot be expressed via elementary functions or classical special functions [Inc56].

The *intégrale tritronquée* is a special P-I transcendent, which was discovered by Boutroux in his classical paper [Bou13] (see [JK88] and [Kit94] for a modern review). Boutroux characterized the *intégrale tritronquée* as the unique solution of P-I with the following asymptotic behaviour at infinity

$$y(z) \sim -\sqrt{\frac{z}{6}}, \quad \text{if } |\arg z| < \frac{4\pi}{5}.$$

We summarize hereafter the content of the Chapter.

Let us recall from Chapter 2 that the space of monodromy data (see Definition 2.4) is the variety of points in  $(z_{-2}, \dots, z_2) \in \mathbb{C}^5$  satisfying the system of quadratic equations  $-iz_{k+3} = 1 + z_k z_{k+1}, \forall k \in \mathbb{Z}_5$ .

The monodromy map  $\mathcal{T}$  (see equation (2.9)) is a holomorphic surjection of  $\mathbb{C}^2$  into  $\mathbb{C}^5$ .  $\mathcal{T}(a, b)$  are the Stokes multipliers  $(\sigma_{-2}(a, b), \dots, \sigma_2(a, b))$  of the cubic oscillator (2.1)

$$\frac{d^2 \psi(\lambda)}{d\lambda^2} = V(\lambda; a, b) \psi(\lambda), \quad V(\lambda; a, b) = 4\lambda^3 - a\lambda - b.$$

The main results of the present Chapter are enumerated here below <sup>1</sup>.

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<sup>1</sup>For what concerns the originality of these results see the Introduction

Theorem 3.2 Fix a solution  $y^*$  and call  $\sigma_k^*, k \in \mathbb{Z}_5$  its Stokes multipliers:  $\mathcal{M}(y^*) = \{\sigma_{-2}^*, \dots, \sigma_2^*\}$ .

The point  $a \in \mathbb{C}$  is a pole of  $y^*$  if and only if there exists  $b \in \mathbb{C}$  such that  $\sigma_k^*, k \in \mathbb{Z}_5$  are the monodromy data of the cubic oscillator

$$\psi'' = (4\lambda^3 - 2a\lambda - 28b)\psi.$$

The parameter  $b$  turns out to be the coefficient of the  $(z - a)^4$  term in the Laurent expansion of  $y^*$ .

Theorem 3.3 Poles of intégrale tritronquée are in bijection with cubic oscillators such that  $\sigma_2 = \sigma_{-2} = 0$ . In physical terminology, these cubic oscillators are said to satisfy two "quantization conditions".

Theorem 3.4 The Riemann-Hilbert correspondence  $\mathcal{M}$  is bijective. In other words,  $V_5$  is the moduli space of solutions of P-I.

The rest of the Chapter is devoted to the proof of these three results.

### 3.1 P-I as an Isomonodromic Deformation

In this section we show that any solution  $y$  of the Painlevé-I equation gives rise to an isomonodromic deformation of equation of the perturbed cubic oscillator (2.2)

$$\begin{aligned} \frac{d^2\psi(\lambda)}{d\lambda^2} &= Q(\lambda; y, y', z)\psi(\lambda), \\ Q(\lambda; y, y', z) &= 4\lambda^3 - 2\lambda z + 2zy - 4y^3 + y'^2 + \frac{y'}{\lambda - y} + \frac{3}{4(\lambda - y)^2}. \end{aligned}$$

Even though this fact is well-known in the literature about P-I (see for example [KT05] and [Mas10a]), we discuss it for convenience of the reader.

**Lemma 3.1.** *The perturbed cubic oscillator is Gauge-equivalent to the following ODE*

$$\vec{\Phi}_\lambda(\lambda; y, y', z) = \begin{pmatrix} y' & 2\lambda^2 + 2\lambda y - z + 2y^2 \\ 2(\lambda - y) & -y' \end{pmatrix} \vec{\Phi}(\lambda; y, y', z). \quad (3.2)$$

Moreover, the point  $\lambda = y$  of the perturbed oscillator (2.2) is an apparent singularity: the monodromy around  $\lambda = y$  of any solution is  $-1$ .

*Proof.* Define the following Gauge transform

$$G(\lambda; y, y', z) = \begin{pmatrix} \frac{y' + \frac{1}{2(\lambda - y)}}{\sqrt{2(\lambda - y)}} & \frac{1}{\sqrt{2(\lambda - y)}} \\ \sqrt{2(\lambda - y)} & 0 \end{pmatrix}. \quad (3.3)$$

Then  $\vec{\Phi}(\lambda; z) = G(\lambda; y, y', z) \vec{\Psi}(\lambda, z)$  satisfies (3.2) if and only if  $\vec{\Psi}(\lambda; z)$  satisfies the following equation

$$\Psi_\lambda(\lambda, z) = \begin{pmatrix} 0 & 1 \\ Q(\lambda; y, y', z) & 0 \end{pmatrix} \Psi(\lambda, z) .$$

Let  $\psi$  denote the first component of  $\vec{\Psi}$ . Then  $\psi$  satisfies the perturbed cubic oscillator equation.

The unique singular point of equation (3.2) is  $\lambda = \infty$ ; therefore any solution of (3.2) is an entire function. The Gauge transform itself has a square root singularity at  $\lambda = y$ , hence any solution of the perturbed oscillator is two-valued.  $\square$

**Lemma 3.2.** *For any Stokes sector  $S_k$ , there exists a unique normalized subdominant solution of equation 3.2. We call it  $\vec{\Phi}_k(\lambda; y, y', z)$ . The subdominant solutions satisfy the following monodromy relations*

$$\vec{\Phi}_{k-1}(\lambda; y, y', z) = \vec{\Phi}_{k+1}(\lambda; y, y', z) + \sigma_k(y, y', z) \vec{\Phi}_k(\lambda; y, y', z) ,$$

where  $\sigma_k(y, y', z)$  is the  $k$ -th Stokes multiplier of the perturbed cubic oscillator 2.2 (see Definition 2.3).

*Proof.* Choose  $\vec{\Phi}_k(\lambda; y, y', z)$  as the (inverse) gauge transform of  $\psi_k(\lambda; y, y', z)$ , the  $k$ -th subdominant solution of (2.2). From WKB analysis (see Sections 4.4 and 4.5) we know that also  $\psi'_k(\lambda; y, y', z)$  is subdominant; more precisely  $\frac{\psi'_k(\lambda)}{\lambda^{\frac{3}{2}} \psi_k(\lambda)} \rightarrow -1$  in  $S_{k-1} \cup \overline{S_k} \cup S_{k+1}$ . Since  $G(\lambda; y, y', z)$  is algebraic in  $\lambda$ , then  $\vec{\Phi}_k(\lambda; y, y', z)$  decays exponentially in  $S_k$  and grows exponentially in  $S_{k\pm 1}$ . Hence it is the unique (up to normalization)  $k$ -th subdominant solution of (3.2).

The equation for the Stokes multiplier is unchanged since the gauge transform is a linear operation.  $\square$

**Lemma 3.3.** *Let  $y = y(z)$  be a holomorphic function of  $z \in U \subset \mathbb{C}$ , let  $y'(z)$  be its derivative and let  $\sigma_k(z) = \sigma_k(y(z), y'(z), z)$ ,  $k \in \mathbb{Z}_5$  be the Stokes multipliers (2.6) of the perturbed cubic oscillator. If  $y(z)$  satisfies the Painlevé-I equation (3.1) then  $\frac{d\sigma_k(z)}{dz} = 0$ .*

*Proof.* We prove the statement using equation (3.2), and not directly equation (2.2). A straightforward computation shows  $y(z)$  satisfy P-I if and only if the following system admits a non trivial solution

$$\begin{aligned} \vec{\Phi}_\lambda(\lambda, z) &= \begin{pmatrix} y'(z) & 2\lambda^2 + 2\lambda y(z) - z + 2y^2(z) \\ 2(\lambda - y(z)) & -y'(z) \end{pmatrix} \vec{\Phi}(\lambda, z) , \\ \vec{\Phi}_z(\lambda, z) &= - \begin{pmatrix} 0 & 2y(z) + \lambda \\ 1 & 0 \end{pmatrix} \vec{\Phi}(\lambda, z) . \end{aligned}$$

Obviously the first equation of above system is equation (3.2), with  $y = y(z), y' = y'(z)$ .

Let  $z_0$  belong to  $U$ . Consider the solution  $\vec{\Phi}(\lambda; z)$  of the system of linear equation with the following Cauchy data  $\vec{\Phi}(\lambda; z_0) = \vec{\Phi}_k(\lambda; y(z_0), y'(z_0), z_0)$ . A simple calculation shows that locally <sup>2</sup>  $\vec{\Phi}(\lambda; z) = \vec{\Phi}_k(\lambda; y(z), y'(z), z)$ .

Therefore

$$\vec{\Phi}_{k-1}(\lambda; y(z), y'(z), z) = \vec{\Phi}_{k+1}(\lambda; y(z), y'(z), z) + \sigma_k(z) \vec{\Phi}_k(\lambda; y(z), y'(z), z) .$$

Differentiating by  $z$ , we obtain the thesis.  $\square$

**Definition 3.1.** According to Lemma 3.3, to any solution  $y$  of P-I we can associate a set of Stokes multipliers, i.e. a point of the space of monodromy data  $V_5$ . We denote this map  $\mathcal{M}$

$$\mathcal{M} : \{P\text{-I transcendentals}\} \rightarrow V_5 .$$

We say that  $\mathcal{M}(y)$  are the Stokes multipliers of  $y$ .

The map  $\mathcal{M}$  is a special case of a Riemann-Hilbert correspondence. In particular the following lemma is valid.

**Lemma 3.4.**  $\mathcal{M}$  is injective.

*Proof.* See [Kap04].  $\square$

The Stokes multipliers of the tritronquée solution are well-known. Indeed, the following Theorem holds true.

**Theorem 3.1.** (Kapaev) The image under  $\mathcal{M}$  of the intégrale tritronquée are the monodromy data uniquely characterized by the following equalities

$$\sigma_2 = \sigma_{-2} = 0 . \tag{3.4}$$

*Proof.* See [Kap04]. The Theorem was already stated, without proof, in [CC94].  $\square$

## 3.2 Poles and the Cubic Oscillator

Here we suppose that we have fixed a solution  $y$  of P-I. In the previous section we have shown that if we restrict  $y$  to a domain  $U$  where it is regular, then it gives rise to an isomonodromic deformation of the perturbed cubic oscillator (2.2).

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<sup>2</sup>hence globally; indeed  $\vec{\Phi}_k(\lambda; y(z), y'(z), z)$  is a single-valued function since  $y(z)$  is single-valued.

In the present section we study the behaviour of solutions of the perturbed cubic oscillator (2.2) in a neighborhood of a pole of a solution  $y$  of P-I. Let  $a$  denote a pole of a fixed solution  $y^*(z)$  of P-I. We prove that, in the limit  $z \rightarrow a$ , the perturbed cubic oscillator turns (*without changing the monodromy*) into the cubic oscillator (2.1) with potential  $V(\lambda; 2a, 28b)$  (here  $b$  is the coefficient of the  $(z - a)^4$  term in the Laurent expansion of  $y$  around  $a$ ). We then analyze some important consequences of this fact.

In order to be able to describe the behaviour of solution to the perturbed cubic oscillator near a pole  $a$  of  $y(z)$ , we have to know the local behavior of  $y(z)$  close to the same point  $a$ .

**Lemma 3.5** (Painlevé). *Let  $a \in \mathbb{C}$  be a pole of  $y$ . Then in a neighborhood of  $a$ ,  $y$  has the following convergent Laurent expansion*

$$y(z) = \frac{1}{(z-a)^2} + \frac{a(z-a)^2}{10} + \frac{(z-a)^3}{6} + b(z-a)^4 + \sum_{j \geq 5} c_j(a, b)(z-a)^j. \quad (3.5)$$

Here  $b$  is some complex number and  $c_j(\cdot, \cdot)$  are polynomials with real coefficients, which are independent on the particular solution  $y$ .

Conversely, fixed arbitrary  $a, b \in \mathbb{C}$ , the above expansion has a non zero radius of convergence and solves P-I.

*Proof.* See [GLS00]. □

**Definition 3.2.** We define the Laurent map

$$\mathcal{L} : \mathbb{C}^2 \rightarrow \{P\text{-I transcendentals}\}.$$

$\mathcal{L}(a, b)$  is the unique analytic continuation of the Laurent expansion (3.5).

We have already collected all elements necessary to formulate the important

**Lemma 3.6.** *Fix a solution  $y$  of P-I and let  $\mathcal{M}(y) = (\sigma_{-2}, \dots, \sigma_2)$  be its Stokes multipliers. Let  $a$  be a pole of  $y$  and  $b$  be such that the Laurent expansion (3.5) is valid. Then  $(\sigma_{-2}, \dots, \sigma_2)$  are the monodromy data of the cubic oscillator (2.1) with potential  $4\lambda^3 - 2a - 28b$ .*

*In other words,  $\mathcal{T}(2a, 28b) = \mathcal{M} \circ \mathcal{L}(a, b)$ . Here  $\mathcal{T}$  is the monodromy map of the cubic oscillator (see equation 2.9),  $\mathcal{M}$  is the Riemann-Hilbert correspondence for P-I (see Definition 3.1) and  $\mathcal{L}$  is the Laurent map (see Definition 3.2).*

*Proof.* Recall the definition of the  $k$ -th subdominant solutions  $\psi_k(\lambda; y, y', z)$  and  $\psi_k(\lambda; a, b)$  of the perturbed and unperturbed cubic oscillator (see Definition 2.2). Here  $y, y'$  are functions of  $z$ , hence we write  $\psi_k(\lambda; z) = \psi_k(\lambda; y, y', z)$ . To prove the Lemma it is sufficient to show that

$$\lim_{z \rightarrow a} \frac{\psi_{k+1}(\lambda; z)}{\psi_k(\lambda; z)} = \frac{\psi_{k+1}(\lambda; 2a, 28b)}{\psi_k(\lambda; 2a, 28b)}.$$

Since the proof of the desired limit requires some knowledge of WKB analysis, we postpone it in Section 4.5.  $\square$

The previous Lemma has many important consequences.

**Theorem 3.2.** *Let  $y$  be any solution of P-I. Then  $a \in \mathbb{C}$  is a pole of  $y$  iff there exists  $b \in \mathbb{C}$  such that  $\mathcal{M}(y) = \mathcal{T}(2a, 28b)$ .*

*Proof.* One implication is exactly the content of Lemma 3.6. Conversely suppose that  $\mathcal{M}(y) = \mathcal{T}(2a, 28b)$  for some  $a, b$ . Consider the solution  $\tilde{y} = \mathcal{L}(a, b)$  of P-I given by the Laurent expansion (3.5). As a consequence of Lemma 3.6,  $\mathcal{M}(y) = \mathcal{M}(\tilde{y})$ . Due to the fact that  $\mathcal{M}$  is injective (see Lemma 3.4) we have that  $y = \tilde{y}$ .  $\square$

As a corollary of Theorem 3.1 and Theorem 3.2, we can characterize the poles of the *intégrale tritronquée* as very particular cubic potentials.

**Theorem 3.3.** *The point  $a \in \mathbb{C}$  is a pole of the *intégrale tritronquée* if and only if there exists  $b \in \mathbb{C}$  such that the Schrödinger equation with the cubic potential  $V(\lambda; 2a, 28b)$  admits the simultaneous solution of two different quantization conditions, namely  $\sigma_{\pm 2}(2a, 28b) = 0$ . Equivalently, the asymptotic values associated to the *tritronquée intégrale* can be chosen to be*

$$w_0 = 0, w_1 = w_{-2} = 1, w_2 = w_{-1} = \infty. \quad (3.6)$$

As a consequence of Theorem 3.2, we can show that the Riemann-Hilbert correspondence  $\mathcal{M}$  is bijective.

**Theorem 3.4** (stated in [KK93]). *The map  $\mathcal{M}$  is bijective: solutions of P-I are in 1-to-1 correspondence with admissible monodromy data.*

*Proof.* We already know (see Lemma 3.4) that  $\mathcal{M}$  is injective. According to Lemma 3.6,  $\mathcal{T}(2a, 28b) = \mathcal{M} \circ \mathcal{L}(a, b)$ . Since  $\mathcal{T}$  is surjective (see Theorem 2.1) then  $\mathcal{M}$  is surjective too.  $\square$



## Chapter 4

# WKB Analysis of the Cubic Oscillator

The present Chapter is devoted to the complex WKB analysis of the cubic oscillator

$$\frac{d^2\psi(\lambda)}{d\lambda^2} = V(\lambda; a, b)\psi(\lambda), \quad V(\lambda; a, b) = 4\lambda^3 - a\lambda - b.$$

We develop the complex WKB analysis because it is an efficient method to solve approximately the direct monodromy problem for the cubic oscillator.

Indeed, our purpose is to *compute* the poles of the intégrale tritronquée after having characterized them as cubic oscillators that admit the simultaneous solutions of two quantization conditions (see Theorem 3.3). We succeed in our goal and we eventually show (see Section 4.2 and Chapter 5) that poles of intégrale tritronquée are described approximately by the solutions of a pair of *Bohr-Sommerfeld* quantization conditions, namely system (4.7,4.8) (more intelligibly rewritten as system (5.2)).

We remark that the theory developed here has a much wider range of applications than the study of poles of the tritronquée; for example, we will use the WKB analysis also in Chapter 6 in the derivation of the Deformed Thermodynamic Bethe Ansatz.

The Chapter is organized as follows. Section 4.1 is devoted to the topological classification of Stokes complexes. In Section 4.2 we calculate the monodromy data of the cubic oscillator in WKB approximation, and we derive the correct Bohr-Sommerfeld conditions for the poles of the tritronquée solution of P-I. In Section 4.3 we introduce the "small parameter" of the approximation. Section 4.4 and 4.5 deal with the proofs of Theorem 4.2 and Lemma 3.6.

**REMARK.** *Most of the present Chapter can be read independently of the other Chapters of the thesis. However, the reader must at least recall from*

Chapter 2 the definitions of Stokes multipliers (see Definition 2.3) and of asymptotic values (see Definition 2.5). We warn the reader that in the present Chapter we call Stokes sector and denote it  $\Sigma_k$  a rather different object than the Stokes Sector  $S_k$  defined in Chapter 2.

## 4.1 Stokes Complexes

In the complex WKB method a prominent role is played by the *Stokes and anti-Stokes lines*, and in particular by the topology of the *Stokes complex*, which is the union of the Stokes lines.

The main result of this section is the Classification Theorem, where we show that the topological classification of Stokes complexes divides the space of cubic potentials into seven disjoint subsets.

Even though Stokes and anti-Stokes lines are well-known objects, there is no standard convention about their definitions, so that some authors call Stokes lines what others call anti-Stokes lines. We follow here the notation of Fedoryuk [Fed93].

**REMARK.** *To simplify the notation and avoid repetitions, we study the Stokes lines only. Every single statement in the following section remains true if the word Stokes is replaced with the word anti-Stokes, provided in equation (4.1) the angles  $\varphi_k$  are replaced with the angles  $\varphi_k + \frac{\pi}{5}$ .*

**Definition 4.1.** *A simple (resp. double, resp. triple) zero  $\lambda_i$  of  $V(\lambda) = V(\lambda; a, b)$  is called a simple (resp. double, resp. triple) turning point. All other points are called generic.*

*Fix a generic point  $\lambda_0$  and a choice of the sign of  $\sqrt{V(\lambda_0)}$ . We call action the analytic function*

$$S(\lambda_0, \lambda) = \int_{\lambda_0}^{\lambda} \sqrt{V(u)} du$$

*defined on the universal covering of  $\lambda$ -plane minus the turning points.*

Let  $\tilde{i}_{\lambda_0}$  be the level curve of the real part of the action passing through a lift of  $\lambda_0$ . Call its projection to the punctured plane  $i_{\lambda_0}$ . Since  $i_{\lambda_0}$  is a one dimensional manifold, it is diffeomorphic to a circle or to a line. If  $i_{\lambda_0}$  is diffeomorphic to the real line, we choose one diffeomorphism  $i_{\lambda_0}(x), x \in \mathbb{R}$  in such a way that the continuation along the curve of the imaginary part of the action is a monotone increasing function of  $x \in \mathbb{R}$ .

**Lemma 4.1.** *Let  $\lambda_0$  be a generic point. Then  $i_{\lambda_0}$  is diffeomorphic to the real line, the limit  $\lim_{x \rightarrow +\infty} i_{\lambda_0}(x)$  exists (as a point in  $\mathbb{C} \cup \infty$ ) and it satisfies the following dichotomy:*

- (i) Either  $\lim_{x \rightarrow +\infty} i_{\lambda_0}(x) = \infty$  and the curve is asymptotic to one of the following rays of the complex plane

$$\lambda = \rho e^{i\varphi_k}, \varphi_k = \frac{(2k+1)\pi}{5}, \rho \in \mathbb{R}^+, k \in \mathbb{Z}_5, \quad (4.1)$$

- (ii) or  $\lim_{x \rightarrow +\infty} i_{\lambda_0}(x) = \lambda_i$ , where  $\lambda_i$  is a turning point.

Furthermore,

- (iii) if  $\lim_{x \rightarrow \pm\infty} i_{\lambda_0}(x) = \infty$  then the asymptotic ray in the positive direction is different from the asymptotic ray in the negative direction.
- (iv) Let  $\varphi_k, k \in \mathbb{Z}_5$  be defined as in equation (4.1). Then  $\forall \varepsilon > 0, \exists K \in \mathbb{R}^+$  such that if  $\varphi_{k-1} + \varepsilon < \arg \lambda_0 < \varphi_k - \varepsilon$  and  $|\lambda_0| > K$ , then  $\lim_{x \rightarrow \pm\infty} i_{\lambda_0}(x) = \infty$ . Moreover the asymptotic rays of  $i_{\lambda_0}$  are the ones with arguments  $\varphi_k$  and  $\varphi_{k-1}$ .

*Proof.* See [Str84]. □

**Definition 4.2.** We call Stokes line the trajectory of any curve  $i_{\lambda_0}$  such that there exists at least one turning point belonging to its boundary.

We call a Stokes line internal if  $\infty$  does not belong to its boundary.

We call Stokes complex the union of all the Stokes lines together with the turning points.

We state all important properties of the Stokes lines in the following

**Theorem 4.1.** The following statements hold true

- (i) The Stokes complex is simply connected. In particular, the boundary of any internal Stokes line is the union of two different turning points.
- (ii) Any simple (resp. double, resp. triple) turning point belongs to the boundary of 3 (resp. 4, resp 5) Stokes lines.
- (iii) If a turning point belongs to the boundary of two different non-internal Stokes lines then these lines have different asymptotic rays.
- (iv) For any ray with the argument  $\varphi_k$  as in equation (4.1), there exists a Stokes line asymptotic to it.

*Proof.* See [Str84]. □

### 4.1.1 Topology of Stokes complexes

In what follows, we give a complete classification of the Stokes complexes, with respect to the orientation preserving homeomorphisms of the plane.

We define the map  $L$  from the  $\lambda$ -plane to the interior of the unit disc as

$$\begin{aligned} L &: \mathbb{C} \rightarrow D_1 \\ L(\rho e^{i\varphi}) &= \frac{2}{\pi} e^{i\varphi} \arctan \rho. \end{aligned} \quad (4.2)$$

The image under the map  $L$  of the Stokes complex is naturally a decorated graph embedded in the closed unit disc. The vertices are the images of the turning points and the five points on the boundary of the unit disc with arguments  $\varphi_k$ , with  $\varphi_k$  as in equation (4.1). The bonds are obviously the images of the Stokes lines. We call the first set of vertices *internal* and the second set of vertices *external*. External vertices are decorated with the numbers  $k \in \mathbb{Z}_5$ . We denote  $\mathcal{S}$  the decorated embedded graph just described. Notice that due to Theorem 4.1 (iii), there exists not more than one bond connecting two vertices.

The combinatorial properties of  $\mathcal{S}$  are described in the following

**Lemma 4.2.**  *$\mathcal{S}$  possesses the following properties*

- (i) *the sub-graph spanned by the internal vertices has no cycles.*
- (ii) *Any simple (resp. double, resp. triple) turning point has valency 3 (resp. 4, resp. 5).*
- (iii) *The valency of any external vertex is at least one.*

*Proof.* (i) Theorem 4.1 part (i)

(ii) Theorem 4.1 part (ii)

(iii) Theorem 4.1 part (iv)

□

**Definition 4.3.** *We call an admissible graph any decorated simple graph embedded in the closure of the unit disc, with three internal vertices and five decorated external vertices, such that (i) the cyclic-order inherited from the decoration coincides with the one inherited from the counter-clockwise orientation of the boundary, and (ii) it satisfies all the properties of Lemma 4.2. We call two admissible graphs equivalent if there exists an orientation-preserving homeomorphism of the disk mapping one graph into the other.*

**Theorem 4.2.** *Classification Theorem*

*All equivalence classes of admissible maps are, modulo a shift  $k \rightarrow k + m, m \in \mathbb{Z}_5$  of the decoration, the ones depicted in Figure 4.1.*

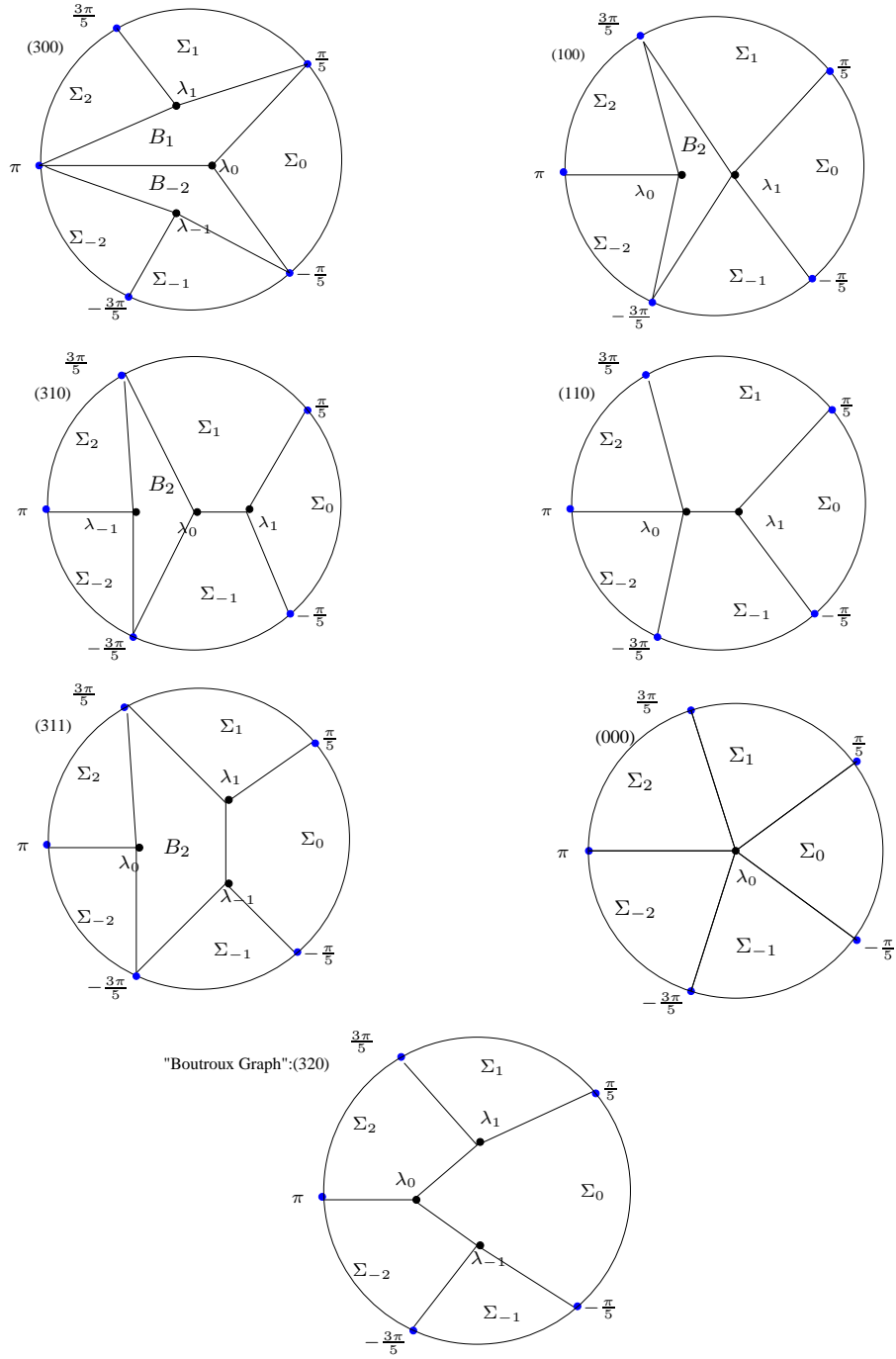


Figure 4.1: All the equivalence classes of admissible graphs.

*Proof.* Let us start analyzing the admissible graphs with three internal vertices and no internal edges.

Any internal vertex is adjacent to a triplet of external vertices. Due to the Jordan curve theorem, there exists an internal vertex, say  $\lambda_0$ , adjacent to a triplet of non consecutive external vertices. Performing a shift, they can be chosen to be the ones labelled by  $0, 2, -1$ . Call the respective edges  $e_0, e_{-1}, e_2$ .

The disk is cut in three disjoint domains by those three edges. No internal vertices can belong to the domain cut by  $e_0$  and  $e_4$ , since it could be adjacent only to two external vertices, namely the ones labelled with  $0$  and  $-1$ . By similar reasoning it is easy to show that one and only one vertex belong to each remaining domains.

Such embedded graph is equivalent to the graph (300).

Classifications for all other cases may proved by similar methods. □

The equivalence classes are encoded by a triplet of numbers  $(a \ b \ c)$ :  $a$  is the number of simple turning points,  $b$  is the number of internal Stokes lines, while  $c$  is a progressive number, distinguishing non-equivalent graphs with same  $a$  and  $b$ . Some additional information shown Figure 4.1 will be explained in the next section.

**REMARK.** *For any admissible graph there exists a real polynomial with an equivalent Stokes complex.*

**REMARK.** *Notice that the automorphism group of every graph in Figure 4.1 is trivial. Therefore the unlabelled vertices can be labelled. In the following we will label the turning points as in figure 4.1. We denote "Boutroux graph" the graph (320)*

#### 4.1.2 Stokes Sectors

**REMARK.** *We warn the reader that in the present Section and for the rest of the Chapter we call Stokes Sector and denote it by  $\Sigma_k$  a rather different object than the Stokes Sector  $S_k$  defined in Chapter 2.*

In the  $\lambda$ -plane the complement of the Stokes complex is the disjoint union of a finite number of connected and simply-connected domains, each of them called a *sector*.

Combining Theorem 4.1 and the Classification Theorem we obtain the following

**Lemma 4.3.** *All the curves  $i_{\lambda_0}$ , with  $\lambda_0$  belonging to a given sector, have the same two asymptotic rays. Moreover, two different sectors have different pairs of asymptotic rays.*

For any  $k \in \mathbb{Z}_5$  there is a sector, called the  $k$ -th Stokes sectors, whose asymptotic rays have arguments  $\varphi_{k-1}$  and  $\varphi_k$ . This sector will be denoted  $\Sigma_k$ . The boundary  $\partial\Sigma_k$  of each  $\Sigma_k$  is connected.

Any other sector has asymptotic rays with arguments  $\varphi_{k-1}$  and  $\varphi_{k+1}$ , for some  $k$ . We call such a sector the  $k$ -th sector of band type, and we denote it  $B_k$ . The boundary  $\partial B_k$  of each  $B_k$  has two connected components.

Choose a sector and a point  $\lambda_0$  belonging to it. The function  $S(\lambda_0, \lambda)$  is easily seen to be bi-holomorphic into the image of this sector. In particular, with one choice of the sign of  $\sqrt{V}$  it maps a Stokes sector into the half plane  $\operatorname{Re} S > c$ , for some  $-\infty < c < 0$  while it maps a  $B_k$  sector in the vertical strip  $c < \operatorname{Re} S < d$ , for some  $-\infty < c < 0 < d < +\infty$ .

**Definition 4.4.** We call a differentiable curve  $\gamma : [0, 1] \rightarrow \mathbb{C}$  an admissible path provided  $\gamma$  is injective on  $[0, 1[$ ,  $\lambda_i \notin \gamma([0, 1])$ , for all turning points  $\lambda_i$ , and  $\operatorname{Re} S(\gamma(0), \gamma(t))$  is a monotone function of  $t \in [0, 1]$ .

We say that  $\Sigma_j \rightleftharpoons \Sigma_k$  if there exist  $\mu_j \in \Sigma_j$ ,  $\mu_k \in \Sigma_k$  and an admissible path such that  $\gamma(0) = \mu_j$ ,  $\gamma(1) = \mu_k$ .

The relation  $\rightleftharpoons$  is obviously reflexive and symmetric but it is not in general transitive.

Notice that  $\Sigma_j \rightleftharpoons \Sigma_k$  if and only if for every point  $\mu_j \in \Sigma_j$  and every point  $\mu_k \in \Sigma_k$  an admissible path exists.

**Lemma 4.4.** The relation  $\rightleftharpoons$  depends only on the equivalence class of the Stokes complex  $\mathcal{S}$ .

*Proof.* Consider an admissible path from  $\Sigma_j$  to  $\Sigma_k$ ,  $j \neq k$ . The path is naturally associated to the sequence of Stokes lines that it crosses. We denote the sequence  $l_n, n = 0, \dots, N$ , for some  $N \in \mathbb{N}$ . We continue analytically  $S(\mu_j, \cdot)$  to a covering of the union of the Stokes sectors crossed by the path together with the Stokes lines belonging to the sequence. Since  $S(\mu_j, \cdot)$  is constant along each connected component of the boundary of every lift of a sector crossed by the path, then each of such connected components cannot be crossed twice by the path. Hence, due to the classification theorem no admissible path is a loop. Therefore, the union of the Stokes sectors crossed by the path together with the Stokes lines belonging to the sequence is simply connected.

Conversely, given any injective sequence of Stokes lines  $l_n, n = 0 \dots, N$  such that for any  $0 \leq n \leq N - 1$ ,  $l_n$  and  $l_{n+1}$  belong to two different connected components of the boundary of a same sector, there exists an admissible path with that associated sequence. This last observation implies that the relation  $\rightleftharpoons$  depends only on the topology of the graph  $\mathcal{S}$ . Moreover, if the sequence exists it is unique; indeed, if there existed two admissible paths, joining the same  $\mu_j$  and  $\mu_k$  but with different sequences, then there would be an admissible loop.  $\square$

Map	Pairs of non consecutive Sectors not satisfying the relation $\leftrightsquigarrow$
300	None
310	$(\Sigma_0, \Sigma_2), (\Sigma_0, \Sigma_{-2})$
311	$(\Sigma_1, \Sigma_{-1})$
320	$(\Sigma_1, \Sigma_{-1}), (\Sigma_1, \Sigma_{-2}), (\Sigma_{-1}, \Sigma_2)$
100	$(\Sigma_1, \Sigma_{-1}), (\Sigma_0, \Sigma_{-2}), (\Sigma_0, \Sigma_2)$
110	All but $(\Sigma_1, \Sigma_{-1})$
000	All

Table 4.1: Computation of the relation  $\leftrightsquigarrow$

With the help of Lemma 4.4 and of the Classification Theorem, relation  $\leftrightsquigarrow$  can be easily computed, as it is shown in Table 4.1. As it is evident from Figure 4.1, for any graph type we have that  $\Sigma_k \leftrightsquigarrow \Sigma_{k+1}, \forall k \in \mathbb{Z}_5$ .

## 4.2 Complex WKB Method and Asymptotic Values

In this section we introduce the *WKB functions*  $j_k, k \in \mathbb{Z}_5$  and use them to evaluate the asymptotic values of equation (2.1). The topology of the Stokes complex will show all its importance in these computations.

On any Stokes sector  $\Sigma_k$ , we define the functions

$$S_k(\lambda) = S(\lambda^*, \lambda), \quad (4.3)$$

$$L_k(\lambda) = -\frac{1}{4} \int_{\lambda^*}^{\lambda} \frac{V'(u)}{V(u)} du, \quad (4.4)$$

$$j_k(\lambda) = e^{-S_k(\lambda) + L_k(\lambda)}. \quad (4.5)$$

Here  $\lambda^*$  is an arbitrary point belonging to  $\Sigma_k$  and the branch of  $\sqrt{V}$  is such that  $\text{Re} S_k(\lambda)$  is bounded from below.

We call  $j_k$  the  $k$ -th *WKB function*.

### 4.2.1 Maximal Domains

In this subsection we construct the  $k$ -th *maximal domain*, that we denote  $D_k$ . This is the domain of the complex plane where the  $k$ -th WKB function approximates a solution of equation (2.1).

The construction is done for any  $k$  in a few steps (see Figure 4.2 for the example of the Stokes complex of type (300)):

- (i) for every  $\Sigma_l$  such that  $\Sigma_l \leftrightsquigarrow \Sigma_k$ , denote  $D_{k,l}$  the union of the sectors and of the Stokes lines crossed by any admissible path connecting  $\Sigma_l$  and  $\Sigma_k$ .



- (ii) Let  $\widehat{D}_k = \bigcup_l D_{l,k}$ . Hence  $\widehat{D}_k$  is a connected and simply connected subset of the complex plane whose boundary  $\partial\widehat{D}_k$  is the union of some Stokes lines.
- (iii) Remove a  $\delta$ -tubular neighborhood of the boundary  $\partial\widehat{D}_k$ , for an arbitrarily small  $\delta > 0$ , such that the resulting domain is still connected.
- (iii) For all  $l \neq k, l \neq k-1$ , remove from  $\widehat{D}_k$  an angle  $\lambda = \rho e^{i\varphi}, |\varphi - \varphi_l| < \epsilon, \rho > R$ , for  $\epsilon$  arbitrarily small and  $R$  arbitrarily big, in such a way that the resulting domain is still connected. The remaining domain is  $D_k$ .

#### 4.2.2 Main Theorem of WKB Approximation

We can now state the main theorem of the WKB approximation. Our Theorem is a slight improvement of a Theorem by F. Olver [Olv74], but whose origin goes back to G. D. Birkhoff [Bir33].

**Theorem 4.3.** *Continue the WKB function  $j_k$  to  $D_k$ . Then there exists a solution  $\psi_k(\lambda)$  of (2.1), such that for all  $\lambda \in D_k$*

$$\left| \frac{\psi_k(\lambda)}{j_k(\lambda)} - 1 \right| \leq g(\lambda) (e^{2\rho(\lambda)} - 1)$$

$$\left| \frac{\psi'_k(\lambda)}{j_k(\lambda)\sqrt{V(\lambda)}} + 1 \right| \leq \left| \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}} \right| + \left( 1 + \left| \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}} \right| \right) g(\lambda) (e^{2\rho(\lambda)} - 1)$$

Here  $\rho_k$  is a bounded positive continuous function, called the error function, satisfying

$$\lim_{\substack{\lambda \rightarrow \infty \\ \varphi_{k-1} < \arg \lambda < \varphi_{k+1}}} \rho_k(\lambda) = 0,$$

and  $g(\lambda)$  is a positive function such that  $g(\lambda) \leq 1$  and

$$\lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in D_k \cap \Sigma_{k \pm 2}}} g(\lambda) = \frac{1}{2}.$$

*Proof.* The proof is in the appendix 4.4. □

Notice that  $j_k$  is *sub-dominant* (i.e. it decays exponentially) in  $\Sigma_k$  and *dominant* (i.e. it grows exponentially) in  $\Sigma_l, \forall l \neq k$ .

For the properties of the error function,  $\psi_k$  is subdominant in  $\Sigma_k$  and dominant in  $\Sigma_{k \pm 1}$ . Therefore, in any Stokes sector  $\Sigma_k$  there exists a subdominant solution, which is defined uniquely up to multiplication by a non zero constant.

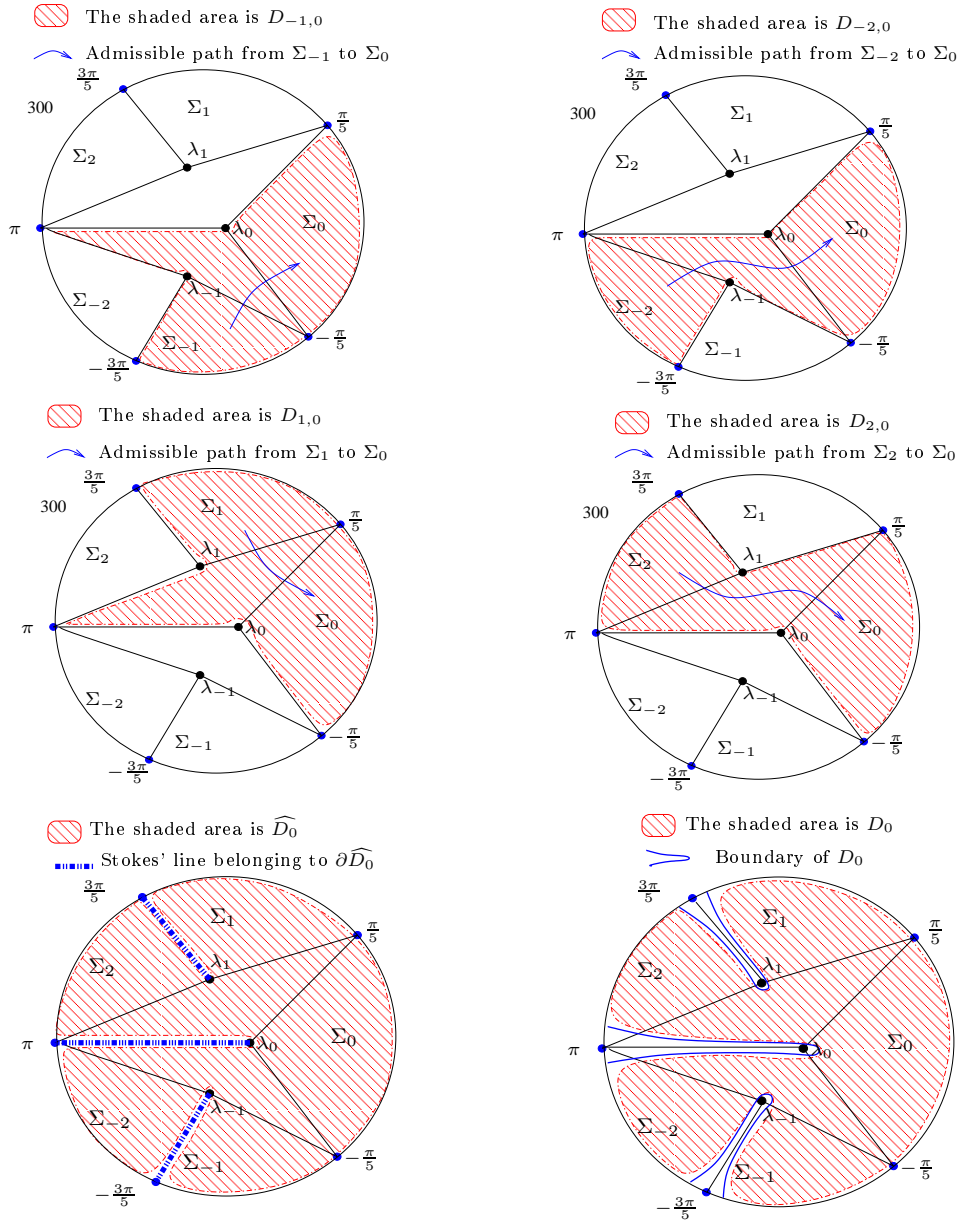


Figure 4.2: In the drawings, the construction of  $D_0$  for a graph of type (300) is depicted.

### 4.2.3 Computations of Asymptotic Values in WKB Approximation

The aim of this paragraph is to compute the asymptotic values for the Schrödinger equation (2.1) in WKB approximation. We explicitly work out the example of the Stokes complex of type (320), relevant to the study of poles of the intégrale tritronquée.

**Definition 4.5.** *Define the relative errors*

$$\rho_l^k = \begin{cases} \lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Sigma_k \cap D_l}} \rho_l(\lambda), & \text{if } \Sigma_l \rightleftharpoons \Sigma_k \\ \infty, & \text{otherwise} \end{cases}$$

and the asymptotic values (recall Definition 2.5)

$$w_k(l, m) \stackrel{\text{def}}{=} w_k\left(\frac{\psi_l}{\psi_m}\right). \quad (4.6)$$

We say that  $\Sigma_k \sim \Sigma_l$  provided  $\rho_l^k < \frac{\log 3}{2}$ . The relation  $\sim$  is a sub-relation of  $\rightleftharpoons$ .

Notice that  $\rho_l^{l+1} = 0$  and  $\rho_l^m = \rho_m^l$  (see Appendix 4.4).

In order to compute the asymptotic value  $w_k(l, m)$ , we have to know the asymptotic behavior of  $\psi_l$  and  $\psi_m$  in  $\Sigma_k$ . By Theorem 4.3,

$$\lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Sigma_k \cap D_l}} \frac{\psi_l(\lambda)}{j_l(\lambda)} \neq 0, \text{ if } \frac{1}{2}(e^{2\rho_k^l} - 1) < 1.$$

Hence the asymptotic behavior of  $\psi_l$  in  $\Sigma_k$  can be related to the asymptotic behavior of  $j_l$  in  $\Sigma_k$  if the relative error  $\rho_l^k$  is so small that the above inequality holds true, i.e. if  $\Sigma_k \sim \Sigma_l$ .

**REMARK.** *Depending on the type of the graph  $\mathcal{S}$ , there may not exist two indices  $k \neq l$  such that all the relative errors  $\rho_l^n, \rho_k^n, n \in \mathbb{Z}_5$  are small. However it is often possible to compute an approximation of all the asymptotic values  $w_n(l, k)$  using the strategy below.*

- (i) We select a pair of non consecutive Stokes sectors  $\Sigma_l, \Sigma_{l+2}$ , with the hypothesis that the functions  $\psi_l$  and  $\psi_{l+2}$  are linearly independent, so that  $w_l(l, l+2) = 0, w_{l+2}(l, l+2) = \infty$ . Since  $\rho_l^{l+1} = \rho_{l+2}^{l+1} = 0$  then

$$w_{l+1}(l, l+2) = \lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Sigma_{l+1} \cap D_l \cap D_{l+2}}} \frac{j_l(\lambda)}{j_{l+2}(\lambda)}.$$

Therefore, we find three exact and distinct asymptotic values.

- (ii) For any  $k \neq l, l+1, l+2$  such that  $\Sigma_l \sim \Sigma_k$  and  $\Sigma_{l+2} \sim \Sigma_k$ , we define the approximate asymptotic value

$$\widehat{w}_k(l, m) = \lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Sigma_k \cap D_l \cap D_{l+2}}} \frac{j_l(z)}{j_m(z)}.$$

The spherical distance between  $w_k(l, m)$  and  $\widehat{w}_k(l, m)$  may be easily estimated from above knowing the relative errors  $\rho_k^l$  and  $\rho_k^{l+2}$ .

If for any  $k \neq l, l+1, l+2$ ,  $\Sigma_l \sim \Sigma_k$  and  $\Sigma_{l+2} \sim \Sigma_k$ , then the we can compute, approximately, all Stokes multipliers using formula 2.14 and Theorem 2.4. In the sequel we let  $\widehat{\sigma}_k$  denote the approximate  $k$ -th Stokes multipliers.

- (iii) If for some pair  $(l, l+2)$  the assumption  $\Sigma_l \sim \Sigma_k$ ,  $\Sigma_{l+1} \sim \Sigma_k$  fails to be true for just one value of the index  $k = k^*$ , and, for another pair  $(l', l'+2)$  the assumption  $\Sigma_{l'} \sim \Sigma_{k'}$ ,  $\Sigma_{l'+2} \sim \Sigma_{k'}$  fails to be true for just one valued of the index  $k' = k'^*$ , with  $k'^* \neq k^*$ , then we can complete our calculations. Indeed we can compute both  $\widehat{\sigma}_{k^*}$  and  $\widehat{\sigma}_{k'^*}$  using formula 2.14 and Theorem 2.4. After that we calculate all other Stokes multipliers using the quadratic relations 2.7.

**REMARK.** As shown in Table 4.1, the relation  $\leftrightsquigarrow$  is uniquely characterized by the graph type. For the sake of computing the asymptotic values the important relation is  $\sim$  and not  $\leftrightsquigarrow$ . Indeed, the calculations for a given graph type, say  $(a \ b \ c)$ , are valid for (and only for) all the potentials whose relation  $\sim$  is equivalent to the relation  $\leftrightsquigarrow$  characterizing the graph type  $(a \ b \ c)$ .

Due to the above remark, in what follows we suppose that the relation  $\sim$  is equivalent to the relation  $\leftrightsquigarrow$ . We have the following

**Lemma 4.5.** *Let  $V(\lambda; a, b)$  such that the type of the Stokes complex is (300), (310), (311); moreover, suppose that the  $\sim$  relation coincides with  $\leftrightsquigarrow$ . Then all the asymptotic values of equation (2.1) are pairwise distinct, but for at most one pair.*

*Proof.* For a graph of type (300) or (311) the thesis is trivial. For a graph of type (320), it may be that  $w_0 = w_2$  or  $w_0 = w_{-2}$ . Since  $w_2 \neq w_{-2}$  the thesis follows.  $\square$

We completely work out the case of Stokes complex of type (320), while for the other cases we present the results only. Due to Lemma 4.5, we omit the results for potentials whose graph type is (300), (310) and (311).

**Boutroux Graph = 320** We suppose that  $\Sigma_0 \sim \Sigma_{\pm 2}$ .

Let us consider first the pair  $\Sigma_0$  and  $\Sigma_{-2}$ . In Figure 4.3 the maximal domains  $D_0$  and  $D_{-2}$  are depicted by colouring the Stokes lines not belonging to them blue and red respectively. In particular  $S_0, L_0, j_0$  (resp.  $S_{-2}, L_{-2}, j_{-2}$ ) can be extended to all  $D_0$  (resp.  $D_{-2}$ ) along any curve that does not intersect any blue (resp. red) Stokes line.

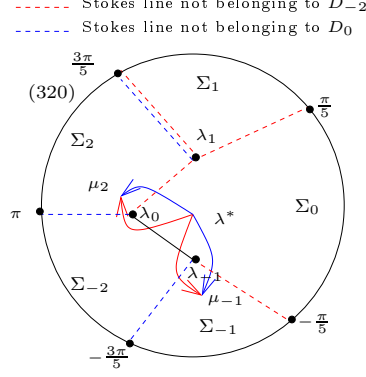


Figure 4.3: Calculation of  $w_{-1}(0, -2)$  and of  $\widehat{w}_2(0, -2)$

We fix a point  $\lambda^* \in \Sigma_0$  such that  $S_0(\lambda^*) = S_{-2}(\lambda^*) = L_0(\lambda^*) = L_{-2}(\lambda^*) = 0$ .

By definition

$$\begin{aligned} \widehat{w}_k(0, -2) &= \lim_{\lambda \rightarrow \infty_k} \frac{j_0(\lambda)}{j_{-2}(\lambda)} \\ &= \lim_{\lambda \rightarrow \infty_k} e^{-S_0(\lambda) + S_{-2}(\lambda)} e^{L_0(\lambda) - L_{-2}(\lambda)} \quad , \end{aligned}$$

Here  $\lambda \rightarrow \infty_k$  is a short-hand notation for  $\lambda \rightarrow \infty, \lambda \in \Sigma_k \cap D_0 \cap D_{-2}$ . We calculate  $\widehat{w}_k(0, -2)$  for  $k = -1, 2$ .

We first calculate  $\lim_{\lambda \rightarrow \infty_k} e^{-S_0(\lambda) + S_{-2}(\lambda)}$ .

Notice that  $\frac{\partial S_0}{\partial \lambda} = \frac{\partial S_{-2}}{\partial \lambda}$  in  $\Sigma_k$ . Hence

$$\lim_{\lambda \rightarrow \infty_k} -S_0(\lambda) + S_{-2}(\lambda) = -S_0(\mu_k) + S_{-2}(\mu_k), k = -1, 2,$$

where  $\mu_k$  is any point belonging to  $\Sigma_k$  (in Figure 4.3, the paths of integration defining  $S_0(\mu_k)$  and  $S_{-2}(\mu_k)$  are coloured blue and red respectively).

On the other hand, since  $\frac{\partial S_0}{\partial \lambda} = -\frac{\partial S_{-2}}{\partial \lambda}$  in  $\Sigma_0 \cup \Sigma_{-2}$ , we have that

$$-S_0(\mu_k) + S_{-2}(\mu_k) = -2S_0(\lambda_s), s = -1 \text{ if } k = -1 \text{ and } s = 0 \text{ if } k = 2.$$

We now compute  $\lim_{\lambda \rightarrow \infty_k} e^{L_0(\lambda) - L_{-2}(\lambda)}$ . Since  $\frac{\partial L_0}{\partial \lambda} = \frac{\partial L_{-2}}{\partial \lambda}$  in  $D_0 \cap D_{-2}$ ,

we have that

$$\begin{aligned}\lim_{\lambda \rightarrow \infty_k} L_0(\lambda) - L_{-2}(\lambda) &= L_0(\mu_k) - L_{-2}(\mu_k) , \\ L_0(\mu_k) - L_{-2}(\mu_k) &= -\frac{1}{4} \oint_{c_k} \frac{V'(\mu)}{V(\mu)} d\mu, k = -1, 2 .\end{aligned}$$

Here  $c_k$  is the blue path connecting  $\lambda^*$  with  $\mu_k$  composed with the inverse of the red path connecting  $\lambda^*$  with  $\mu_k$  (see Figure 4.3).

Therefore, we have

$$\lim_{\lambda \rightarrow \infty_k} L_0(\lambda) - L_{-2}(\lambda) = -\sigma \frac{2\pi i}{4}, \sigma = -1 \text{ if } k = -1 \text{ and } \sigma = +1 \text{ if } k = 2 .$$

Combining the above computations and formulas (2.14, 2.17) , we get

$$\begin{aligned}w_{-1}(0, -2) &= i e^{-2S_0(\lambda_{-1})} , & \widehat{w}_2(0, -2) &= -i e^{-2S_0(\lambda_0)} , \\ \widehat{\sigma}_1 &= -i e^{-2(S_0(\lambda_0) - S_0(\lambda_{-1}))} .\end{aligned}$$

We stress that  $w_{-1}(0, -2)$  is exact while  $\widehat{w}_2(0, -2)$  is an approximation.

Performing the same computations for the pair  $\Sigma_0$  and  $\Sigma_2$ , we obtain

$$\begin{aligned}w_1(0, 2) &= -i e^{-2S_0(\lambda_1)} , & \widehat{w}_{-2}(0, 2) &= i e^{-2S_0(\lambda_0)} , \\ \widehat{\sigma}_{-1} &= -i e^{-2(S_0(\lambda_0) - S_0(\lambda_1))} .\end{aligned}$$

Using the quadratic relation (2.7) among Stokes multipliers, we eventually compute all other Stokes multipliers

$$\widehat{\sigma}_{\pm 2} = -i \frac{1 + e^{-2(S_0(\lambda_0) - S_0(\lambda_{\pm 1}))}}{e^{-2(S_0(\lambda_0) - S_0(\lambda_{\mp 1}))}} , \widehat{\sigma}_0 = i (1 + \widehat{\sigma}_2 \widehat{\sigma}_{-2}) .$$

**Quantization Conditions** The computations above provides us with the following quantization conditions:

$$\widehat{\sigma}_2 = 0 \Leftrightarrow e^{-2(S_0(\lambda_1) - S_0(\lambda_0))} = -1 \quad (4.7)$$

$$\widehat{\sigma}_{-2} = 0 \Leftrightarrow e^{-2(S_0(\lambda_{-1}) - S_0(\lambda_0))} = -1 \quad (4.8)$$

$$\begin{aligned}\widehat{\sigma}_0 &= 0 \Leftrightarrow e^{-2(2S_0(\lambda_0) - S_0(\lambda_{-1}) - S_0(\lambda_1))} = \\ &\quad -(1 + e^{-2(S_0(\lambda_0) - S_0(\lambda_1))})(1 + e^{-2(S_0(\lambda_0) - S_0(\lambda_{-1}))})\end{aligned} \quad (4.9)$$

We notice that equation (4.9) is incompatible both with (4.7) and (4.8). Equations (4.7) and (4.8) are Bohr-Sommerfeld quantizations.

As was shown in equation (3.6), the poles of the intégrale tritonquée are related to the polynomials such that  $w_1 = w_{-2}$  and  $w_{-1} = w_2$ . Since equations (4.7) and (4.8) can be simultaneously solved, solutions of system (4.7, 4.8) describe, in WKB approximation, polynomials related to the

intégrale tritronquée. System (4.7,4.8) was found by Boutroux in [Bou13] (through a completely different analysis), to characterize the asymptotic distribution of the poles of the intégrale tritronquée. Therefore we call (4.7,4.8) the *Bohr-Sommerfeld-Boutroux system*.

Equation (4.9) will not be studied in this thesis, even though is quite remarkable. Indeed, it describes the breaking of the PT symmetry (see [DT00] and [BBM<sup>+</sup>01]).

**Case (100)**

$$\begin{aligned} w_0(1, -1) &= -1 \\ \widehat{w}_{-2}(1, -1) &= \widehat{w}_2(1, -1) = 1 \end{aligned}$$

Since  $w_0 \neq \widehat{w}_{\pm 2}$  and  $w_2 \neq w_{-2}$ , if the error  $\rho_1^{-2}$  or  $\rho_{-1}^2$  is small enough, then all the asymptotic values are pairwise distinct.

**Case (110)**

$$\begin{aligned} \widehat{w}_{-1}(1, -2) &= 1 \\ w_2(1, -2) &= -1 \end{aligned}$$

In this case, it is impossible to calculate  $w_0$  with the WKB method that has been here developed. Hence it may be that either  $w_0 = w_2$  or  $w_0 = w_{-2}$ .

Notice, however, that (110) is the graph only of a very restricted class of potentials namely  $V(\lambda) = (\lambda + \lambda_0)^2(\lambda - 2\lambda_0)$ , where  $\lambda_0$  is real and positive. Since the potential is real then  $w_0 \neq w_{\pm 2}$ .

**Case (000)** In this case, no asymptotic values can be calculated. Notice, however, that  $V(\lambda) = \lambda^3$  is the only potential with graph (000). For this potential the asymptotic values can be computed exactly, simply using symmetry considerations. Indeed one can choose  $w_k = e^{\frac{2k\pi}{5}i}$ ,  $k \in \mathbb{Z}_5$ .

## 4.3 The Small Parameter

The WKB method normally applies to problem with an *external small parameter*, usually denoted  $\hbar$  or  $\varepsilon$ . In the study of the distributions of poles of a given solution  $y$  of P-I there is no external small parameter and we have to explore the whole space of cubic potentials. The aim of this section is to introduce an *internal small parameter* in the space of cubic potentials, that greatly simplifies our study. The results of the present Section will be extremely important when studying the poles of the intégrale tritronquée (see Chapter 5).

On the linear space of cubic potentials in canonical form

$$V(\lambda; a, b) = 4\lambda^3 - a\lambda - b,$$

we define the following action of the group  $\mathbb{R}^+ \times \mathbb{Z}_5$  (similar to what is called Symanzik rescaling in [Sim70])

$$(x, m)[V(\lambda; a, b)] = V(\lambda; \Omega^{2m} x^2 a, \Omega^{3m} x^3 b), \quad x \in \mathbb{R}^+, \quad m \in \mathbb{Z}_5, \quad \Omega = e^{\frac{2\pi}{5}i} \quad (4.10)$$

The induced action on the graph  $\mathcal{S}$ , on the relative error  $\rho_l^m$ , and on the difference  $S_i(\lambda_j) - S_i(\lambda_k)$  is described in the following

**Lemma 4.6.** *Let the action of the group  $\mathbb{R}^+ \times \mathbb{Z}_5$  be defined as above. Then*

(i)  *$(x, m)$  leaves the graph  $\mathcal{S}$  invariant, but for a shift of the labels  $k \rightarrow k + m$  of the external vertices.*

(ii)  *$(x, m)[S_i(\lambda_j) - S_i(\lambda_k)] = x^{\frac{5}{2}} (S_i(\lambda_j) - S_i(\lambda_k))$ .*

(iii)  *$(x, m)[\rho_l^k] = x^{-\frac{5}{2}} \rho_l^k$ .*

*Proof.* The proof of (i) and (ii) follows from the following equality

$$\sqrt{V(\lambda; \Omega^{2k} x^2 a, \Omega^{3k} x^3 b)} d\lambda = x^{\frac{5}{2}} \sqrt{V(\lambda'; a, b)} d\lambda', \quad \lambda = x\lambda'.$$

The proof of point (iii) follows from a similar scaling law of the 1-form  $\alpha(\lambda)d\lambda$  (see equation (4.16) in Appendix 4.4).  $\square$

Due to Lemma 4.6(iii),  $\varepsilon = \left| \frac{a}{b} \right|$  plays the role of the small parameter. Indeed, along any orbit of the action of the group  $\mathbb{R} \times \mathbb{Z}_5$ , all the (finite) relative errors go to zero uniformly as  $\left| \frac{a}{b} \right| \rightarrow 0$ .

Since all the relevant information is encoded in the quotient of the space of cubic potentials with respect to the group action, we define the following change of variable

$$\nu(a, b) = \frac{b}{a}, \quad \mu(a, b) = \frac{b^2}{a^3}. \quad (4.11)$$

The induced action on these coordinates is simple, namely

$$(x, m)[\mu(a, b)] = \mu(a, b) \text{ and } (x, m)[\nu(a, b)] = \Omega^m x \nu(a, b).$$

Moreover, the orbit of the set  $\{(\nu, \mu) \in \mathbb{C}^2 \text{ s.t. } |\nu| = 1, |\arg \nu| < \frac{\pi}{5}, \mu \neq 0\}$  is a dense open subset of the space of cubic potentials.



## 4.4 Proof of the Main Theorem of WKB Analysis

The aim of this appendix is to prove Theorem 4.3. Our approach is similar to the approach of Fedoryuk [Fed93].

Notations are as in sections 4.1 and 4.2, except for  $\infty_k$ . In what follows, we suppose to have fixed a certain cubic potential  $V(\lambda; a, b)$  and a maximal domain  $D_k$ . To simplify the notation we write  $V(\lambda)$  instead of  $V(\lambda; a, b)$ .

### 4.4.1 Gauge Transform to an L-Diagonal System

The strategy is to find a suitable gauge transform of equation (2.1) such that for large  $\lambda$  it simplifies. We rewrite the Schrödinger equation

$$-\psi''(\lambda) + V(\lambda)\psi(\lambda) = 0 , \quad (4.12)$$

in first order form:

$$\begin{aligned} \Psi'(\lambda) &= E(\lambda)\Psi(\lambda) , \\ E(\lambda) &= \begin{pmatrix} 0 & 1 \\ V(\lambda) & 0 \end{pmatrix} . \end{aligned} \quad (4.13)$$

**Lemma 4.7** (Fedoryuk). *In  $D_k$*

(i) *the gauge transform*

$$\begin{aligned} Y(\lambda) &= A(\lambda)U(\lambda) , \\ A(\lambda) &= j_k(\lambda) \begin{pmatrix} 1 & 1 \\ \sqrt{V(\lambda)} - \frac{V'(\lambda)}{4V(\lambda)} & -\sqrt{V(\lambda)} - \frac{V'(\lambda)}{4V(\lambda)} \end{pmatrix} , \end{aligned} \quad (4.14)$$

*is non singular and*

(ii) *the system (4.13) is transformed into the following one*

$$\begin{aligned} U'(\lambda) &= F(\lambda)U(\lambda) = (A^{-1}EA - A^{-1}A')U , \\ F(\lambda) &= 2\sqrt{V(\lambda)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha(z) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} , \end{aligned} \quad (4.15)$$

$$\alpha(\lambda) = \frac{1}{32\sqrt{V(\lambda)}^5} (4V(\lambda)V''(\lambda) - 5V'^2(\lambda)) . \quad (4.16)$$

*Proof.* (i) Indeed  $\det A(\lambda) = 2j_k^2(\lambda)\sqrt{V(\lambda)} \neq 0$ ,  $\forall \lambda \in D_k$ , by construction of  $j_k$  and  $D_k$ .

(ii) It is proven by a simple calculation. □

#### 4.4.2 Some Technical Lemmas

Before we can begin the proof of Theorem 4.3, we have to introduce the compactification of  $D_k$  and the preparatory Lemmas 4.8 and 4.9.

**Compactification of  $D_K$**  Since  $D_k$  is simply connected, it is conformally equivalent to the interior of the unit disk  $D$ . We denote  $U$  the uniformisation map,  $U : D \rightarrow D_k$ .

By construction, the boundary of  $D_k$  is the union of  $n$  free Jordan curves, all intersecting at  $\infty$ . Here  $n$  is equal to the number of sectors  $\Sigma_l$  such that  $\Sigma_l \rightleftharpoons \Sigma_k$  minus 2.

Due to an extension of Carathéodory's Theorem ([Car], §134-138), the map  $U$  extends to a continuous map from the closure of the unit circle to the closure of  $D_k$ . The map is injective on the closure of  $D$  minus the  $n$  counterimages of  $\infty$ . Hence, the uniformisation map realizes a  $n$  point compactification of  $D_k$ , that we call  $\overline{D_k}$ . In  $\overline{D_k}$  there are  $n$  points at  $\infty$ :  $\infty_k$  denotes the point at  $\infty$  belonging to the closure of  $U(\Sigma_{k-1} \cup \Sigma_k \cup \Sigma_{k+1})$ ;  $\infty_l$  denotes the eventual point at  $\infty$  belonging to the closure of  $U(\Sigma_l)$  (here  $\lambda = k \pm 2$ , and  $\Sigma_l \rightleftharpoons \Sigma_k$ ).

**Definition 4.6.** Let  $H$  be the space of function holomorphic in  $D_k$  and continuous in  $\overline{D_k}$ .  $H$  endowed with the sup norm is a Banach space  $(H, \|\cdot\|_H)$ .

Let  $\Gamma(\lambda), \lambda \in \overline{D_k} - \infty_k$  be the set of injective piecewise differentiable curves  $\gamma : [0, 1] \rightarrow \overline{D_k}$ , such that

1.  $\gamma(0) = \lambda, \gamma(1) = \infty_k$ ,
2.  $\text{Re}S_k(\gamma(0), \gamma(t))$  is eventually non decreasing,
3. there is an  $\varepsilon > 0$  such that eventually  $|\arg \gamma(t) - \frac{2\pi k}{5}| < \frac{\pi}{5} - \varepsilon$ ,
4. the length of the curve restricted to  $[0, T]$  is  $O(|\gamma(T)|)$ , as  $t \rightarrow 1$ .

Let  $\tilde{\Gamma}(\lambda)$  be the subset of  $\Gamma(\lambda)$  of the paths along which  $\text{Re}S_k(\gamma(0), \gamma(t))$  is non decreasing.

Let  $K_1 : H \rightarrow H$  and  $K_2 : H \rightarrow H$  be defined (for the moment formally)

$$K_1[h](\lambda) = - \int_{\gamma \in \Gamma(\lambda)} e^{2S_k(\mu, \lambda)} \alpha(\mu) h(\mu) d\mu, \quad (4.17)$$

$$K_2[h](\lambda) = \int_{\gamma \in \Gamma(\lambda)} \alpha(\mu) h(\mu) d\mu. \quad (4.18)$$

Let  $\rho : \overline{D_k} \rightarrow \overline{D_k}$ :

$$\rho(\lambda) = \begin{cases} \inf_{\gamma \in \tilde{\Gamma}(\lambda)} \int_0^1 \left| \alpha(\gamma(t)) \frac{d\gamma(t)}{dt} \right| dt, & \text{if } \lambda \neq \infty_k \\ 0, & \text{if } \lambda = \infty_k. \end{cases}$$

**REMARK.** Since along rays of fixed argument  $\varphi$ , with  $|\varphi - \frac{2\pi k}{5}| < \frac{\pi}{5} - \varepsilon$ ,  $\text{Re}S_k$  is eventually increasing, there are paths satisfying point (1) through (4) of the above definition. Moreover, by construction of  $D_k$ ,  $\tilde{\Gamma}(\lambda)$  is non empty for any  $\lambda$ .

Before beginning the proof of the theorem, we need two preparatory lemmas.

**Lemma 4.8.** Fix  $\varepsilon > 0$ , an angle  $|\arg \varphi - \frac{2\pi l}{5}| < \frac{\pi}{5} - \varepsilon$ , and let  $\Omega = \Sigma_l \cap \{\lambda \in \mathbb{C}, |\lambda - \frac{2\pi l}{5}| < \frac{\pi}{5} - \varepsilon\}$ . Denote  $i(R) = i_{\text{Re}^{i\varphi}} \cap \Omega$ ,  $R \in \mathbb{R}^+$ , and let  $L(R)$  be the length with respect to the euclidean metric of  $i(R)$ . Then  $L(R) = O(R)$  and  $\inf_{\lambda' \in i(R)} |\lambda'| = O(R)$ .

Let  $r$  be any level curve of  $S_l(\lambda^*, \cdot)$  asymptotic to the ray of argument  $\frac{2\pi l}{5}$ ,  $\Omega(R) = \{\lambda \in \Omega, \text{Re}S_l(\lambda, \text{Re}^{i\varphi}) \geq 0\}$ , and  $M(R)$  be the length of  $r \cap \Omega(R)$ . Then  $M(R) = O(R)$ .

*Proof.* [Str84], chapter 3. □

**Lemma 4.9.** (i)  $\rho$  is a continuous function.

(ii)  $K_1$  and  $K_2$  are well-defined bounded operator. In particular

$$|K_i[h](\lambda)| \leq \rho(\lambda) \|h\|_H, \quad i = 1, 2 \quad (4.19)$$

(iii)  $K_2[h](\infty_k) = K_1[h](\infty_k) = K_1(\infty_{k\pm 2}) = 0, \forall h \in H$

*Proof.* (i) Since  $\alpha(\lambda)d\lambda = O(|\lambda|^{-\frac{7}{2}})$ , then  $\alpha(\lambda)d\lambda$  is integrable along any curve  $\gamma \in \tilde{\Gamma}(\lambda)$ . Therefore  $\rho$  is a continuous function on  $\overline{D_k}$ .

(ii) We first prove that (a)  $K_i[h](\lambda)$  does not depend on the integration path for any  $\lambda \in \overline{D_k}$  minus the points at infinity. A result that easily implies that  $K_i[h](\cdot)$  is an analytic function on  $D_k$ , continuous on  $\lambda \in \overline{D_k}$  minus the points at  $\infty$ . We then prove (b) the estimates (4.19) and (c) the existence of the limits  $K_i[h](\infty_l)$ ,  $l = \infty_k, \infty_{k\pm 2}$ .

To simplify the notation, we prove the theorem for the operator  $K_1$ . The proof for  $K_2$  is almost identical.

(a) Let  $\gamma_a, \gamma_b \in \Gamma(\lambda)$ . The curve  $i_{\gamma_a(T)}$ , where  $T = 1 - \varepsilon$  for some small  $\varepsilon > 0$  intersect  $\gamma_b$  at some  $\gamma_b(T')$ . Therefore we can decompose  $-\gamma_b \circ \gamma_a$  into two different paths with the help of a segment of  $i_{\gamma_a(T)}$ ,  $\int_{-\gamma_b \circ \gamma_a} e^{2S_k(\mu, \lambda)} \alpha(\mu) h(\mu) d\mu = \int_{\gamma_1} + \int_{\gamma_2} e^{2S_k(\mu, \lambda)} \alpha(\mu) h(\mu) d\mu$ . One path  $\gamma_1$  is the loop based at  $\lambda$  and the other  $\gamma_2$  is the loop based at  $\infty_k$ . Since  $\gamma_1 \subset D_k$ , then  $\int_{\gamma_1} e^{2S_k(\mu, \lambda)} \alpha(\mu) h(\mu) d\mu = 0$ . Along  $\gamma_2$ ,  $e^{2S_k(\gamma_2(t), \lambda)} \leq 1$  therefore the integrand can be estimated just by  $|\alpha(\gamma_2(t))|$ . Due to lemma 4.8,  $\int_{\gamma_2} |\alpha(\mu) h(\mu) d\mu| = O(|\gamma_a(T)|^{-\frac{5}{2}})$ . Since  $\varepsilon$  is arbitrary, then  $K_1[h](\lambda)$  does not depend on the integration path.

(b) Clearly for any path  $\gamma \in \tilde{\Gamma}(\lambda)$ ,  $|K_1[h](\lambda)| \leq \int_0^1 |\alpha(\lambda) h(\lambda) d\lambda| dt$ . Since  $K_1[h](\lambda)$  does not depend on  $\gamma$ , then estimate (4.19) follows.

(c) Let  $\lambda_n$  be a sequence converging to  $\infty_l, l = k + 2$  or  $l = k - 2$ ; without losing any generality we suppose that the sequence is ordered such that  $ReS_k(\lambda_n) \leq ReS_k(\lambda_{n+1})$ . Fix a curve  $r$ , as defined in Lemma 4.8. By construction of  $D_k$ , it is always possible to connect two points  $\lambda_n$  and  $\lambda_{n+m}$  with a union of segments of the curves  $i_{\lambda_n}, i_{\lambda_{n+m}}$  and of  $r$ . We denote by  $\gamma$  the union of this three segment. By construction of  $D_k$  (see Subsection 4.2.1 (iii)), there exists  $\varepsilon > 0$  such that  $|\arg \lambda_n - \frac{2\pi l}{5}| < \frac{\pi}{5} - \varepsilon, \forall n$ . Therefore, due to Lemma 4.8,  $\gamma$  has length of order  $|\lambda_n| + |\lambda_{n+m}|$ . Hence  $|K_1[h](\lambda_n) - K_1[h](\lambda_{n+m})| \leq \int_\gamma |h(\lambda)\alpha(\lambda)d\lambda| = O(|\lambda_n|^{-\frac{5}{2}})$ . Then  $K_1[h](\lambda_n)$  is a Cauchy sequence and the limit is well defined.

We now prove that this limit is zero by calculating it along a fixed ray  $\lambda = xe^{i\varphi}$  inside  $\Sigma_{k\pm 2}$ . Let us fix a point  $x^*$  on this ray in such a way that the function  $ReS_k(x^*, x)$  is monotone decreasing in the interval  $[x^*, +\infty[$ . Along the ray we have

$$K_1[h](x) = -\frac{\int_{x^*}^x e^{2S_k(y, x^*)}\alpha(y)h(y)dy + g(x^*)}{e^{2S_k(x^*, x)}},$$

where  $g(x^*)$  is a constant, namely  $\int_{\gamma \in \Gamma(x^*)} e^{2S_k(\mu, x^*)}\alpha(\mu)h(\mu)d\mu$ . Hence  $\lim_{x \rightarrow \infty} K_1[h](x) = \lim_{x \rightarrow \infty} \frac{\alpha(x)h(x)}{\sqrt{V(x)}} = 0$ .

With similar methods the reader can prove that the limit  $K_1[h](\infty_k)$  exists and is zero. □

We are now ready to prove Theorem 4.3.

**Theorem 4.4.** *Extend the WKB function  $j_k$  to  $D_k$ . There exists a unique solution  $\psi_k$  of (2.1) such that for all  $\lambda \in D_k$*

$$\begin{aligned} \left| \frac{\psi_k(\lambda)}{j_k(\lambda)} - 1 \right| &\leq g(\lambda)(e^{2\rho(\lambda)} - 1), \\ \left| \frac{\psi'_k(\lambda)}{j_k(\lambda)\sqrt{V(\lambda)}} + 1 \right| &\leq \left| \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}} \right| + (1 + \left| \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}} \right|)g(\lambda)(e^{2\rho(\lambda)} - 1), \end{aligned}$$

where  $g(\lambda)$  is a positive function,  $g(\lambda) \leq 1$  and  $g(\infty_{k\pm 2}) = \frac{1}{2}$ .

*Proof.* We seek a particular solution to the linear system (4.15) via successive approximation.

If  $U(\lambda) = U^{(1)} \oplus U^{(2)} \in H \oplus H$  satisfies the following integral equation of Volterra type

$$\begin{aligned} U(\lambda) &= U_0 + K[U](\lambda), \quad U_0 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ K[U](\lambda) &= \begin{pmatrix} K_1[U^{(1)} + U^{(2)}](\lambda) \\ K_2[U^{(1)} + U^{(2)}](\lambda) \end{pmatrix}, \end{aligned} \tag{4.20}$$

then  $U(\lambda)$  restricted to  $D_k$  satisfies (4.15).

We define the the *Neumann series* as follows

$$U_{n+1} = U^0 + K[U_n], U_{n+1} = \sum_{i=0}^{n+1} K^i[U^0]. \quad (4.21)$$

More explicitly,

$$K^n[U_0](\lambda) = \left( \int_{\lambda}^{\infty_k} d\mu_1 \int_{\mu_1}^{\infty_k} d\mu_2 \dots \int_{\mu_{n-1}}^{\infty_k} d\mu_n \begin{array}{cc} -e^{2S(\mu_1, z)} \alpha(\mu_1) & \times \\ \alpha(\mu_1) & \times \end{array} \right. \\ \left. \begin{array}{c} \alpha(\mu_2)(1 - e^{2S(\mu_2, \mu_1)}) \dots \alpha(\mu_n)(1 - e^{2S(\mu_n, \mu_{n-1})}) \\ \alpha(\mu_2)(1 - e^{2S(\mu_2, \mu_1)}) \dots \alpha(\mu_n)(1 - e^{2S(\mu_n, \mu_{n-1})}) \end{array} \right).$$

Here the integration path  $\gamma$  belong to  $\Gamma(\lambda)$ . For any  $\gamma \in \tilde{\Gamma}(\lambda)$  and any  $n \geq 1$

$$|K^n[U_0]^{(i)}(\lambda)| \leq \frac{1}{2} \int_{\lambda}^{\infty_k} \int_{\mu_1}^{\infty_k} \dots \int_{\mu_{n-1}}^{\infty_k} \prod_{i=1}^n |2\alpha(\mu_i) d\mu_i| = \frac{2^{n-1}}{n!} \left( \int_{\gamma} d\mu_1 |\alpha(\mu_1)| \right)^n,$$

where  $K^n[U_0]^{(i)}$  is the i-th component of  $K^n[U_0]$ . Hence

$$|K^n[U]_i(\lambda)| \leq \frac{1}{2} \frac{1}{n!} (2\rho(\lambda))^n \quad (4.22)$$

Thus the sequence  $U^n$  converges in  $H$  and is a solution to (4.20); call  $U$  its limit. Due to Lemma 4.9,  $U^{(1)}(\infty_{k\pm 2}) = 0$ .

Let  $\Psi_k$  be the solution to (4.13) whose gauge transform is  $U$  restricted to  $D_k$ ; The first component  $\psi_k$  of  $\Psi_k$  satisfies equation (4.12).

From the gauge transform (4.14), we obtain

$$\begin{aligned} \frac{\psi_k(\lambda)}{j_k(\lambda)} - 1 &= U_1(\lambda) + U_2(\lambda) - 1, \\ \frac{\psi'_k(\lambda)}{j_k(\lambda)\sqrt{V(\lambda)}} + 1 &= U_1(\lambda)(1 - \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}) - (U_2(\lambda) - 1)(1 + \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}) + \\ &\quad - \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}, \end{aligned}$$

The thesis follows from these formulas, inequality (4.22) and from the fact that  $U_1(\infty_{k\pm 2}) = 0$ . □

**REMARK.** The solution  $\psi_k(\lambda)$  of equation (2.1) described in Theorem 4.3 may be extended from  $D_k$  to the whole complex plane, since the equation is linear with entire coefficients. The continuation is constructed in the following Corollary.

**Corollary 4.1.** For any  $\lambda \in \mathbb{C}$ ,  $\lambda$  not a turning point, we define  $\Gamma(\lambda)$  as in Definition 4.6. Fixed any  $\gamma \in \Gamma(\lambda)$  and  $h$  a continuous function on  $\gamma$ , we define the functionals  $K_i[h](\lambda)$  as in equations (4.17) and (4.18). We define the Neumann series as in equations (4.20) and (4.21), and we continue  $j_k$  along  $\gamma$ .

Then the Neumann series converges and we call  $U^{(1)}(\lambda)$  and  $U^{(2)}(\lambda)$  the first and second component of its limit.

Moreover,  $\psi_k(\lambda) = (U^{(1)}(\lambda) + U^{(2)}(\lambda)) j_k(\lambda)$  solves equation (2.1) and for any  $\varepsilon > 0$

$$\lim_{|\lambda| \rightarrow \infty, |\arg \lambda - \frac{2\pi k}{5}| < \frac{3\pi}{5} - \varepsilon} (U^{(1)}(\lambda) + U^{(2)}(\lambda)) = 1$$

The reader should notice that if  $\lambda \notin D_k$ , then  $\tilde{\Gamma}(\lambda)$  is empty and we cannot estimate  $\frac{\psi_k(\lambda)}{j_k(\lambda)}$ .

## 4.5 Proof of Lemma 3.6

In this Section we suppose to have fixed a solution  $y = y(z)$  of P-I and a pole  $a$  of  $y$ . The Laurent expansion of  $y$  around  $a$  is as follows (see Lemma 3.5)

$$y(z) = \frac{1}{(z-a)^2} + \frac{a(z-a)^2}{10} + \frac{(z-a)^3}{6} + b(z-a)^4 + \text{higher order} ,$$

for some  $b \in \mathbb{C}$ .

Here we denote  $\psi_k(\lambda; z)$  (see Definition 2.2) the subdominant solution of the perturbed oscillator

$$\frac{d^2 \psi(\lambda)}{d\lambda^2} = Q(\lambda; y, y', z) \psi(\lambda) , \quad (4.23)$$

$$Q(\lambda; z) = 4\lambda^3 - 2\lambda z + 2zy(z) - 4y^3(z) + y'^2(z) + \frac{y'(z)}{\lambda - y(z)} + \frac{3}{4(\lambda - y(z))^2} \quad (4.24)$$

Similarly,  $\psi_k(\lambda; 2a, 28b)$  is the subdominant solution of the cubic oscillator

$$\frac{d^2 \psi(\lambda)}{d\lambda^2} = (4\lambda^3 - 2a\lambda - 28b) \psi(\lambda) .$$

In the present Section we complete the proof of Lemma 3.6. As was shown in Chapter 3, it is sufficient to show

$$\lim_{z \rightarrow a} \frac{\psi_{k+1}(\lambda; z)}{\psi_k(\lambda; z)} = \frac{\psi_{k+1}(\lambda; 2a, 28b)}{\psi_k(\lambda; 2a, 28b)} .$$

This is the content of Lemma 4.5 below.

To achieve our goal we use the explicit construction of the subdominant solutions by means of Neumann series of the some functionals: we show that, as  $z \rightarrow a$ , the functionals defining  $\psi_k(\lambda; z)$  converge in norm to the functionals defining  $\psi_k(\lambda; 2a, 28b)$ .

**Preliminary Lemmas** We summarize some property of the perturbed potential, which can be easily verified using the Laurent expansion of  $y$ .

**Lemma 4.10.** *Let  $\varepsilon^2 = \frac{1}{y(z)} = (z - a)^2 + O((z - a)^6)$  then*

- (i)  *$Q(\lambda; z)$  has a double pole at  $\lambda = \frac{1}{\varepsilon^2}$ . It is an apparent fuchsian singularity for equation (4.23): the local monodromy around it is  $-1$ .*
- (ii)  *$Q(\lambda; z)$  has two zeros at  $\lambda = \frac{1}{\varepsilon^2} + O(\varepsilon^2)$*
- (iii)  *$Q(\lambda; z) = 4\lambda^3 - 2(a + \varepsilon)\lambda - 28b + c(\varepsilon) - \frac{2\lambda\varepsilon^{-1}}{\lambda - \varepsilon^{-2}} + \frac{3}{4(\lambda - \varepsilon^{-2})^2}$ , where  $c(\varepsilon)$  is a  $O(\varepsilon)$  constant.*

Equation (4.23) is a perturbation of the cubic Schrödinger equation (2.1) and the asymptotic behaviours of solutions to the two equations are very similar. Indeed the terms  $4\lambda^3$  and  $-2z\lambda^3$  are the only relevant in the asymptotics of the subdominant solutions.

More precisely, the equivalent of Corollary 4.1 in Lemma 4.4 is valid also for the perturbed Schrödinger equation.

**Definition 4.7.** *For any  $z$ , define a cut from  $\lambda = y(z)$  to  $\infty$  such that it eventually does not belong to the angular sector  $|\arg \lambda - \frac{2\pi k}{5}| \leq \frac{3\pi}{5}$ .*

*Fix  $\lambda^*$  in the cut plane.  $S_k(\lambda; z) = \int_{\lambda^*}^{\lambda} \sqrt{Q(\mu; z)} d\mu$  is well-defined for  $|\arg \lambda - \frac{2\pi k}{5}| < \frac{3\pi}{5}$  and  $\lambda \gg 0$ . Here the branch of  $\sqrt{Q}$  is chosen so that  $\operatorname{Re} S_k(\lambda) \rightarrow +\infty$  as  $|\lambda| \rightarrow \infty$ ,  $\arg \lambda = \frac{2\pi k}{5}$ . We define the "WKB functions"  $j_k(\lambda; z)$  as in equation (4.5).*

*For any  $\lambda$  in the cut plane, let  $\Gamma(\lambda)$  be the set of piecewise differentiable curves  $\gamma : [0, 1]$  to the cut plane,  $\gamma(0) = \lambda$ ,  $\gamma(1) = \infty$ , satisfying properties (2)(3) and (4) of Definition 4.6.*

*We define  $\alpha(\lambda; z), z \neq a$  as in equation (4.16), but replacing  $V(\lambda)$  with  $Q(\lambda; z)$ . For any  $\gamma \in \Gamma(\lambda)$ , let  $H$  be the Banach space of continuous functions on  $\gamma$  that have a finite limit as  $t \rightarrow 1$ . Formulae (4.17) and (4.18) define two bounded functionals on  $H$ . We call such functionals  $K_1(\lambda; z)$  and  $K_2(\lambda; z)$ .  $K_1(\lambda; a)$  and  $K_2(\lambda; a)$  are defined similarly substituting  $Q(\lambda; z)$  with  $V(\lambda; 2a, 28b)$ .*

Following the proof of Theorem 4.3, the reader can prove the following

**Lemma 4.11.** *Let  $\lambda$  belong to the cut plane,  $\lambda$  not a zero of  $Q(\cdot; z)$ . Fixed any  $\gamma \in \Gamma(\lambda)$ , we define the Neumann series as in equations (4.20) and (4.21), and we continue  $j_k$  along  $\gamma$ .*

*Then the Neumann series converges and  $\tilde{\psi}_k(\lambda) = (U_1(\lambda) + U_2(\lambda)) j_k(\lambda)$  solves equation (4.23). Moreover, for any  $\varepsilon > 0$*

$$\lim_{|\lambda| \rightarrow \infty, \left| \arg \lambda - \frac{2k\pi}{5} \right| < \frac{3\pi}{5} - \varepsilon} \left( U^{(1)}(\lambda) + U^{(2)}(\lambda) \right) = 1$$

**REMARK.** *We notice that if the cuts are continuous in  $z$ , then  $\tilde{\psi}_k(\lambda, z) = c(z)\psi_k(\lambda; z)$ , where  $\psi_k(\lambda)$  is the solution constructed in Lemma 4.11 and  $c(z)$  is a bounded holomorphic function.*

**Theorem 4.5.**

$$\lim_{z \rightarrow a} \frac{\psi_{k+1}(\lambda; z)}{\psi_k(\lambda; z)} = \frac{\psi_{k+1}(\lambda; 2a, 28b)}{\psi_k(\lambda; 2a, 28b)}, \text{ uniformly on compact subsets.}$$

*Proof.* To keep notation simple, we prove the pointwise convergence; uniform convergence is a straightforward corollary of our proof, that we leave to the reader.

Let  $\lambda$  be any point in the complex plane which is not a zero of  $V(\lambda; a, b)$ . For any sequence  $\varepsilon_n$  converging to zero, we choose two fixed rays  $r_1$  and  $r_2$  of different argument  $\varphi_1$  and  $\varphi_2$ ,  $|\varphi_i - \frac{2k\pi}{5}| < \frac{\pi}{5}$ . We denote  $D_{R,\varepsilon}$  a disk of radius  $R$  with center  $\lambda = y(a + \varepsilon) \approx \frac{1}{\varepsilon^2}$  and we split the sequence  $\varepsilon_n$  into two subsequences  $\varepsilon_n^i$  such that  $r_i \cap D_{R,\varepsilon_n^i} = \emptyset$  for any  $n$  big enough.

For  $i = 1, 2$ , we choose the cuts defined in Definition 4.7 in such a way that there exists a differentiable curve  $\gamma_i : [0, 1] \rightarrow \overline{\mathbb{C}}$ ,  $\gamma_i(0) = \lambda$ ,  $\gamma_i(1) = \infty$  with the following properties: (i)  $\gamma_i$  avoids the zeroes of  $Q(\lambda, \varepsilon_n^i)$  and a fixed, arbitrarily small, neighborhood of the zeroes of  $V(\lambda; 2a, 28b)$ , (ii)  $\gamma_i$  does not intersect any cut, and (iii)  $\gamma_i$  eventually lies on  $r_i$ .

The proof of the thesis relies on the following estimates:

$$\begin{aligned} \sup_{\lambda \in \mathbb{C} - D_{R,\varepsilon}} \left| \lambda^{-\delta} \right| |Q(\lambda; a + \varepsilon) - V(\lambda; 2a, 28b)| &= O(\varepsilon^{2\delta-3}), \quad (4.25) \\ \sup_{\lambda \in \mathbb{C} - D_{R,\varepsilon}} \left| \lambda^{-\delta} \right| |Q_\lambda(\lambda; a + \varepsilon) - V_\lambda(\lambda; 2a, 28b)| &= O(\varepsilon^{2\delta-3}), \\ \sup_{\lambda \in \mathbb{C} - D_{R,\varepsilon}} \left| \lambda^{-\delta} \right| |Q_{\lambda\lambda}(\lambda; a + \varepsilon) - V_{\lambda\lambda}(\lambda; 2a, 28b)| &= O(\varepsilon^{2\delta-3}). \end{aligned}$$

Due the above estimates it is easily seen that  $\gamma_i \in \Gamma(\lambda)$ ,  $\forall \varepsilon_n^i$ . Due to Lemma 4.11 and Corollary 4.1, to prove the thesis it is sufficient to show that the functionals  $K_1(\lambda; a + \varepsilon_n^i)$  and  $K_2(\lambda; a + \varepsilon_n^i)$  converge in norm to  $K_1(\lambda; a)$  and  $K_2(\lambda; a)$ . We notice that the norm of the functionals are just the  $L^1(\gamma_i)$  norm of their integral kernels.

We first consider the functionals  $K_2(\lambda; a + \varepsilon_n^i)$ . Due to the above estimates

$$\lambda^{\frac{7}{2}} \alpha(\mu; a + \varepsilon_n^i) \rightarrow \lambda^{\frac{7}{2}} \alpha(\mu; 2a, 28b), \text{ uniformly on } \gamma_i([0, 1]) \text{ as } n \rightarrow \infty.$$



Hence the sequence  $\alpha(\mu; a + \varepsilon_n^i)$  converges in norm  $L^1(\gamma_i)$  to  $\alpha(\mu; 2a, 28b)$  and the sequence  $K_2(\lambda; a + \varepsilon_n^i)$  converges in operator norm to  $K_2(\lambda; a)$ .

We consider now the sequence  $K_1(\lambda; a + \varepsilon_n^i)$ .

To prove the convergence of the above sequence of operators, it is sufficient to prove that

$$e^{S_k(\lambda; a + \varepsilon_n^i) - S_k(\mu; a + \varepsilon_n^i)} \rightarrow e^{S_k(\lambda; 2a, 28b) - S_k(\mu; 2a, 28b)},$$

uniformly on  $\gamma_i([0, 1]) \ni \mu$  as  $n \rightarrow \infty$ .

We first note that

$$\begin{aligned} e^{S_k(\lambda; 2a, 28b) - S_k(\mu; 2a, 28b)} - e^{S_k(\lambda; a + \varepsilon_n^i) - S_k(\mu; a + \varepsilon_n^i)} &= \\ e^{S_k(\lambda; 2a, 28b) - S_k(\mu; 2a, 28b)} \left( 1 - e^{g(\mu; \varepsilon)} \right), \\ g(\mu, \varepsilon) &= \int_{\lambda, \gamma_i}^{\mu} \frac{Q(\nu, \varepsilon) - V(\nu; a, 28b)}{\sqrt{Q(\nu, \varepsilon)} + \sqrt{V(\nu; 2a, 28b)}} d\nu. \end{aligned}$$

Using estimate (4.25), it is easy to show that  $g(\mu; \varepsilon) = f(\varepsilon)O(\mu^\delta)$ , where  $f(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and  $0 < \delta \ll 1$ . Therefore the difference of the exponential functions converges uniformly to 0.  $\square$

## Chapter 5

# Poles of Intégrale Tritronquée

The aim of the present Chapter is to study the distribution of poles of the intégrale tritronquée using the WKB analysis of the cubic oscillator developed in Chapter 4.

The reader should recall from Chapter 3 that the intégrale tritronquée is the unique solution of P-I with the following asymptotic behaviour at infinity

$$y(z) \sim -\sqrt{\frac{z}{6}}, \quad \text{if } |\arg z| < \frac{4\pi}{5}.$$

In Chapter 3, it was shown that the point  $a \in \mathbb{C}$  is a pole of the tritronquée solution if and only if there exists  $b \in \mathbb{C}$  such that the following Schrödinger equation

$$\frac{d^2\psi(\lambda)}{d\lambda^2} = V(\lambda; 2a, 28b)\psi(\lambda), \quad V(\lambda; a, b) = 4\lambda^3 - a\lambda - b, \quad (5.1)$$

admits the simultaneous solutions of two different quantization conditions, namely  $\sigma_{\pm 2}(2a, 28b) = 0$ .

In Section 4.2.3, we studied this system of quantization conditions using the WKB approximation. We showed that the WKB analogue of the system  $\sigma_{\pm 2}(2a, 28b) = 0$  is a pair of *Bohr-Sommerfeld* quantization conditions (4.7, 4.8)<sup>1</sup>, that we have called Bohr-Sommerfeld-Boutroux (B-S-B) system.

We rewrite it in the following equivalent form :

$$\begin{aligned} \oint_{a_1} \sqrt{V(\lambda; 2a, 28b)} d\lambda &= i\pi(2n - 1), \\ \oint_{a_{-1}} \sqrt{V(\lambda; 2a, 28b)} d\lambda &= -i\pi(2m - 1). \end{aligned} \quad (5.2)$$

Here  $m, n$  are positive natural numbers and the paths of integration are shown in figure 5.1. It is natural to suppose that poles of intégrale tritronquée

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<sup>1</sup>if we make the change of variable  $a \rightarrow 2a, b \rightarrow 28b$

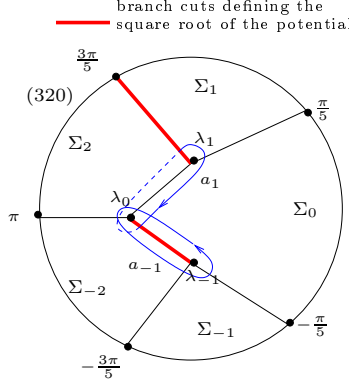


Figure 5.1: Riemann surface  $\mu^2 = V(\lambda; 2a, 28b)$

are in bijection with solution of B-S-B system. We are not yet able to prove this but we will prove (see Theorem 5.1 and 5.2 below) that *poles are asymptotically close to solution of the B-S-B system*.

Let us introduce precisely our result.

For any pair of quantum numbers there is one and only one solution to the Bohr-Sommerfeld-Boutroux system; this is proven for example in [Kap03].

Solutions of B-S-B system have naturally a multiplicative structure.

**Definition 5.1.** Let  $(a^*, b^*)$  be a solution of the B-S-B system with quantum numbers  $n, m$  such that  $2n - 1$  and  $2m - 1$  are coprime. We call  $(a^*, b^*)$  a primitive solution of the system and denote it  $(a^q, b^q)$ , where  $q = \frac{2n-1}{2m-1} \in \mathbb{Q}$ . Due to Lemma 4.6, we have that

$$(a_k^q, b_k^q) = ((2k+1)^{\frac{4}{5}} a^q, (2k+1)^{\frac{6}{5}} b^q), k \in \mathbb{N},$$

is another solution of the B-S-B system. We call it a descendant solution. We call  $\{(a_k^q, b_k^q)\}_{k \in \mathbb{N}}$  the  $q$ -sequence of solutions.

**Definition 5.2.** Let  $D_{\epsilon, \delta}^{(a', b')} = \{|a - a'| < \epsilon, |b - b'| < \delta, \epsilon, \delta \neq 0\}$  denote the polydisc centered at  $(a', b')$ .

Our main result concerning the poles of the intégrale tritronquée is the following

**Theorem 5.1.** Let  $\epsilon, \delta$  be arbitrary positive numbers. Let  $\frac{1}{5} < \mu < \frac{6}{5}$ ,  $-\frac{1}{5} < \nu < \frac{4}{5}$ , then it exists a  $K \in \mathbb{N}^*$  such that for any  $k \geq K$ , inside the polydisc  $D_{k^{-\mu\epsilon, k^{-\nu\delta}}}^{(a_k^q, b_k^q)}$  there is one and only one solution of the system  $\sigma_{\pm 2}(2a, 28b) = 0$ .

*Proof.* The proof is in section 5.2 below. □

As a corollary of above Theorem, we have the following asymptotic characterization of the location of poles of the intégrale tritronquée:

**Theorem 5.2.** *Let  $\varepsilon$  be an arbitrary positive number. If  $\frac{1}{5} < \mu < \frac{6}{5}$ , then it exists  $K \in \mathbb{N}^*$  such that for any  $k \geq K$  inside the disc  $|a - a_k^q| < k^{-\mu}\varepsilon$  there is one and only one pole of the intégrale tritronquée.*

## 5.1 Real Poles

In this section, we compute all the real solutions of system (5.2) and compare them with some numerical results from [JK01]. We note that the accuracy of the WKB method is astonishing also for small  $a$  and  $b$  (see Table 5.1 below).

In the paper [JK01], the authors showed that the intégrale tritronquée has no poles on the real positive axis. The real poles are a decreasing sequence of negative numbers  $\alpha_n$  and some of them are evaluated numerically in the same paper.

For the subset of real potentials, we have

$$\oint_{a_1} \sqrt{V(\lambda; a, b)} d\lambda = \overline{\oint_{a_{-1}} \sqrt{V(\lambda; a, b)} d\lambda} \quad ,$$

where  $\overline{\phantom{x}}$  stands for complex conjugation.

Therefore system (5.2) reduces to one equation and the real poles of tritronquée are approximated by the 1-sequence of solution of the B-S-B system (see Definition 5.1). The real primitive solution is computed numerically as  $a^1 \cong -2,34, b^1 \cong -0,064$ . Hence real solutions of the B-S-B system are the following sequence

$$(a_k^1, b_k^1) = (-2,34(2k+1)^{\frac{4}{5}}, -0,064(2k+1)^{-\frac{6}{5}}), k \in \mathbb{N}.$$

After Theorem 5.1, above sequence approximates the sequence of real poles of the intégrale tritronquée, if  $k$  is big enough. However, it turns out that the approximation is very good already for the first real poles  $(a^1, b^1)$ .

In Table 5.1 below, we compare the first two real solutions to system (5.2) with the numerical evaluation of the first two poles of the intégrale tritronquée in [JK01].

## 5.2 Proof of Theorem 5.1

**Multidimensional Rouché Theorem** The main technical tool of the proof is the following generalization of the classical Rouché theorem.

**Theorem 5.3** ([AY83]). *Let  $D, E$  be bounded domains in  $\mathbb{C}^n$ ,  $\overline{D} \subset E$ , and let  $f(z), g(z)$  be holomorphic maps  $E \rightarrow \mathbb{C}^n$  such that*

- $f(z) \neq 0, \forall z \in \partial D$ ,

	WKB estimate	Numeric [JK01]	Error %
$\alpha_1$	-2, 34	-2, 38	1, 5
$\beta_1$	-0, 064	-0, 062	2
$\alpha_2$	-5, 65	-5, 66	0, 2
$\beta_2$	-0, 23	unknown	unknown

Table 5.1: Comparison between numerical and WKB evaluation of the first two real poles of the intégrale tritronquée.

- $|g(z)| < |f(z)|, \forall z \in \partial D,$

then  $w(z) = f(z) + g(z)$  and  $f(z)$  have the same number (counted with multiplicities) of zeroes inside  $D$ . Here  $|f(z)|$  is any norm on  $\mathbb{C}^n$ .

**A function whose zeroes are the pole of tritronquée solution** The reader should recall the definition of  $k$ -th subdominant function  $\psi_k(\lambda; a, b)$  (see Definition 2.2), of  $k$ -th asymptotic values  $w_k(l, m)$  and of the relative errors  $\rho_l^k$  (see Definition 4.5).

The reader should also remember that, if  $\psi_l$  and  $\psi_{l+2}$  are linearly independent then  $w_{k-1}(l, l+2) = w_{k+1}(l, l+2)$  if and only if  $\sigma_k = 0$ .

**Definition 5.3.** Let  $E$  be the (open) subset of the  $(a, b)$  plane such that  $\psi_0(\lambda; a, b)$  and  $\psi_{\pm 2}(\lambda; a, b)$  are linearly independent (its complement in the  $(a, b)$  plane is the union of two smooth surfaces [EG09a]). On  $E$  we define the following functions

$$u_2(a, b) = \frac{w_2(0, -2)}{w_{-1}(0, -2)} \quad (5.3)$$

$$u_2(a, b) = \frac{w_{-2}(0, 2)}{w_1(0, 2)} \quad (5.4)$$

$$U(a, b) = \left( \frac{u_2(a, b) - 1}{u_{-2}(a, b) - 1} \right). \quad (5.5)$$

All the functions are well defined and holomorphic. Indeed, due to WKB theory we have that  $w_{l+1}(l, l+2)$  is always different from 0 and  $\infty$ .

The fundamental result of Chapter 3 is the following characterization of the poles of the intégrale tritronquée, which is indeed equivalent to Theorem 3.3

**Lemma 5.1.** The point  $\alpha \in \mathbb{C}$  is a pole of the intégrale tritronquée if and only if there exists  $\beta \in \mathbb{C}$  such that  $(\alpha, \beta)$  belongs to the domain of  $U$  and  $U(2\alpha, 28\beta) = 0$ . In other words  $\psi_{-1}(\lambda; 2\alpha, 28\beta)$  and  $\psi_2(\lambda; 2\alpha, 28\beta)$  are linearly dependent and  $\psi_1(\lambda; 2\alpha, 28\beta)$  and  $\psi_{-2}(\lambda; 2\alpha, 28\beta)$  are linearly dependent.

We recall from Chapter 3 that the complex number  $\beta$  in previous lemma is the coefficient of the quartic term in the Laurent expansion of the tritronquée solution around  $\alpha$ .

**The WKB Approximation of U** We want to define a function  $\tilde{U}$  on the space of cubic potentials that approximates  $U$ . Then we compare the zeroes of  $U$  and  $\tilde{U}$  using Rouché Theorem. However, due to the nature of WKB approximation, we cannot build such a function globally but only in neighborhoods of potential whose Stokes graph is of type "320" (see Figure 5.2).

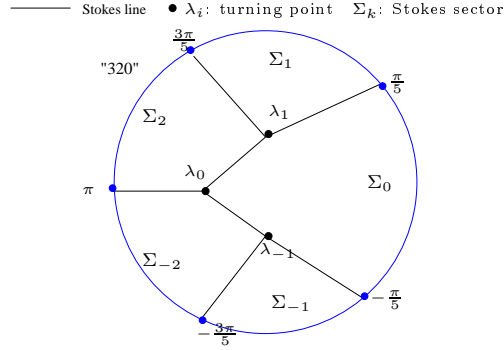


Figure 5.2: Graph "320": dots on the circle represents asymptotic directions in the complex plane

**Definition 5.4.** Let  $(a^*, b^*)$  be a point such that the Stokes graph of  $V(\cdot; 2a^*, 28b^*)$  is of type "320". On a sufficiently small neighborhood of  $(a^*, b^*)$  we define the following analytic functions

$$\chi_{\pm 2}(a, b) = \oint_{c_{\mp 1}} \sqrt{V(\lambda; 2a, 28b)} d\lambda, \quad (5.6)$$

$$\tilde{u}_{\pm 2}(a, b) = -e^{\chi_{\pm 2}(a, b)}, \quad (5.7)$$

$$\tilde{U}(a, b) = \begin{pmatrix} \tilde{u}_2(a, b) - 1 \\ \tilde{u}_{-2}(a, b) - 1 \end{pmatrix}. \quad (5.8)$$

The cycles  $c_{\pm 1}$  are depicted in Figure 1 and the branch of  $\sqrt{V}$  is chosen such that  $\operatorname{Re} \sqrt{V(\lambda)} \rightarrow +\infty$  as  $\lambda \rightarrow \infty$  along the positive semi-axis in the cut plane.

From above Definition it is clear that the B-S-B system is equivalent to the vanishing condition of function  $\tilde{U}$ .

**Lemma 5.2.** The B-S-B system (5.2) system is equivalent to the equation  $\tilde{U}(a, b) = 0$ .

In [Kap03] the following lemma was proven.

**Lemma 5.3.** *For any pair of quantum numbers  $n, m \in \mathbb{N} - 0$  there exists one and only one solution of the B-S-B system.*

We can compare the functions  $U$  and  $\tilde{U}$  defined above using Theorem 4.3 and the computations of Section 4.2.3. Indeed, they imply the following

**Lemma 5.4.** *Let  $(a, b)$  be such that the Stokes graph is of type "320". There exists a neighborhood of  $(a, b)$  and two continuous positive functions  $\rho_{\pm 2}$  such that  $\chi_{\pm 2}$  are holomorphic and*

$$|\tilde{u}_{\pm 2} - u_{\pm 2}| \leq \frac{1}{2}(e^{2\rho_{\pm 2}} - 1). \quad (5.9)$$

Moreover if  $\rho_{\pm 2} < \frac{\ln 3}{2}$  then  $\psi_0$  and  $\psi_{\pm 2}$  are linearly independent.

To simplify the notation we have denoted  $\rho_{\pm 2}$  what was denoted  $\rho_{\pm 2}^0$  in the previous Chapter.

Using classical relations of the theory of elliptic functions we have the following

**Lemma 5.5.** *The map  $\tilde{U}$  defined in (5.8) is always locally invertible (hence its zeroes are always simple) and*

$$\frac{\partial \chi_2}{\partial a}(a, b) \frac{\partial \chi_{-2}}{\partial b}(a, b) - \frac{\partial \chi_{-2}}{\partial a}(a, b) \frac{\partial \chi_2}{\partial b}(a, b) = -28\pi i.$$

*Proof.* On the compactified elliptic curve  $\mu^2 = V(\lambda; a, b)$ , consider the differentials  $\omega_a = -\frac{\lambda d\lambda}{\mu}$  and  $\omega_b = -\frac{d\lambda}{\mu}$ .

It is easily seen that

$$\frac{\partial \chi_{\pm 2}}{\partial a}(a, b) = \oint_{c_{\mp 1}} \omega_a, \quad \frac{\partial \chi_{\pm 2}}{\partial b}(a, b) = 14 \oint_{c_{\mp 1}} \omega_b.$$

Moreover we have that

$$J\tilde{U} = \left( \frac{\partial \chi_2}{\partial a}(a, b) \frac{\partial \chi_{-2}}{\partial b}(a, b) - \frac{\partial \chi_{-2}}{\partial a}(a, b) \frac{\partial \chi_2}{\partial b}(a, b) \right) \tilde{u}_2 \tilde{u}_{-2},$$

where  $J\tilde{U}$  is the Jacobian of the map  $\tilde{U}$ .

The statement of the lemma follows from the classical Legendre relation between complete elliptic periods of the first and second kind [EMOT53].  $\square$

Our aim is to locate the zeroes of  $U$  (the poles of the intégrale tritronquée after Theorem 5.1) knowing the location of zeroes of  $\tilde{U}$  (the solutions of the B-S-B system). We want to find a neighborhood of a given solution of the B-S-B system inside which there is one and only one zero of  $U$ . Due to estimate (5.9) and Rouché theorem, it is sufficient to find a domain on whose boundary the following inequality holds

$$\frac{1}{2}(e^{2\rho_2} - 1)|u_2| + \frac{1}{2}(e^{2\rho_{-2}} - 1)|u_{-2}| < |1 - \tilde{u}_2| + |1 - \tilde{u}_{-2}|. \quad (5.10)$$

**Scaling Law** In order to analyze the important inequality (5.10), we take advantage of the scaling laws introduced in Section 4.3 (the "small parameter").

**Lemma 5.6.** *Let  $(a^*, b^*)$  be such that the Stokes graph is of type "320" and  $E$  be a neighborhood of  $(a^*, b^*)$  such that the estimates (5.9) are satisfied. Then, for any real positive  $x$  the point  $(x^2 a^*, x^3 b^*)$  is such that the Stokes graph is of type "320" and in the neighborhood  $E_X = \{(x^2 a, x^3 b) : (a, b) \in E\}$  the estimates (5.9) are satisfied. Moreover for any  $(a, b) \in E$  the following scaling laws are valid*

- $\chi_{\pm 2}(x^2 a, x^3 b) = x^{\frac{5}{2}} \chi_{\pm 2}(a, b).$
- $\frac{\partial^{(n+m)} \chi_{\pm 2}}{\partial a^n \partial b^m}(x^2 a, x^3 b) = x^{\frac{5-4n-6m}{5}} \frac{\partial^{(n+m)} \chi_{\pm 2}}{\partial a^n \partial b^m}(a, b).$
- $\rho_{\pm 2}(x^2 a, x^3 b) = x^{-\frac{5}{2}} \rho_{\pm 2}(a, b).$

*Proof.* It is a corollary of Lemma 4.6. □

**Proof** From Lemma 5.6 we can extract the leading behaviour of  $\tilde{U}$  around solutions of the B-S-B system.

**Lemma 5.7.** *Let  $z = (2k+1)^\mu (a - a_k^q)$ ,  $c_{\pm 2} = \frac{\partial \chi_{\pm 2}}{\partial a}(a^q, b^q)$ ,  $w = (2k+1)^\nu (b - b_k^q)$ , and  $d_{\pm 2} = \frac{\partial \chi_{\pm 2}}{\partial b}(a^q, b^q)$ . If  $\mu > \frac{1}{5}$  and  $\nu > -\frac{1}{5}$ , then*

$$\begin{aligned} \tilde{u}_2(z, w) &= 1 + c_2(2k+1)^{\frac{1}{5}-\mu} z + d_2(2k+1)^{-\frac{1}{5}-\nu} w + O((2k+1)^{-\gamma'}) , \\ \tilde{u}_{-2}(z, w) &= 1 + c_{-2}(2k+1)^{\frac{1}{5}-\mu} z + d_{-2}(2k+1)^{-\frac{1}{5}-\nu} w + O((2k+1)^{-\gamma'}) , \\ \gamma' &> -\frac{1}{5} + \mu, \gamma' > \frac{1}{5} + \nu . \end{aligned} \tag{5.11}$$

*Proof.* It follows from Lemma 5.6. □

For convenience of the reader, we recall here the definition of polydisc

**Definition.** We denote  $D_{\epsilon, \delta}^{(a', b')} = \{|a - a'| < \epsilon, |b - b'| < \delta, \epsilon, \delta \neq 0\}$  the polydisc centered at  $(a', b')$ .

Theorem 5.1 is a corollary of the following

**Lemma 5.8.** *Let  $\epsilon, \delta$  be arbitrary positive numbers. If  $\frac{1}{5} < \mu < \frac{6}{5}$ ,  $-\frac{1}{5} < \nu < \frac{4}{5}$ , then there exists a  $K \in \mathbb{N}^*$  such that for any  $k \geq K$ ,  $U$  and  $\tilde{U}$  are well-defined and holomorphic on  $D_{k^{-\mu}\epsilon, k^{-\nu}\delta}^{(a_k^q, b_k^q)}$  and the following inequality holds true*

$$\left| U(a, b) - \tilde{U}(a, b) \right| < \left| \tilde{U}(a, b) \right|, \forall (a, b) \in \partial D_{k^{-\mu}\epsilon, k^{-\nu}\delta}^{(a_k, b_k)} . \tag{5.12}$$



*Proof.* The polydisc  $D_{k^{-\mu}\varepsilon, k^{-\nu}\delta}^{(a_k^q, b_k^q)}$  is the image under rescaling  $a \rightarrow (2k+1)^{\frac{4}{5}}a$ ,  $b \rightarrow (2k+1)^{\frac{6}{5}}b$  of a shrinking polydisc centered at  $(a^q, b^q)$ ; call it  $\tilde{D}_k$ . Hence due to Lemma 5.4, for  $k \geq K'$   $\tilde{D}_k$  is such that  $\rho_{\pm 2}$  are bounded,  $\chi_{\pm 2}$  are holomorphic and the estimates (5.9) hold. Call  $\rho^*$  the supremum of  $\rho_{\pm 2}$  on  $D_{K'}$ . Due to scaling property, for all  $k \geq K'$   $\rho_{\pm 2}$  is bounded from above by  $(2k+1)^{-1}\rho^*$  on  $D_{k^{-\mu}\varepsilon, k^{-\nu}\delta}^{(a_k, b_k)}$ ; such a bound is eventually smaller than  $\frac{\ln 3}{2}$ .

Then for a sufficiently large  $k$ ,  $D_{k^{-\mu}\varepsilon, k^{-\nu}\delta}^{(a_k, b_k)}$  is a subset of the domain of  $U$  and inside it  $U$  and  $\tilde{U}$  satisfy (5.9) and (5.11).

We divide the boundary in two subsets:  $\partial D_{k^{-\mu}\varepsilon, k^{-\nu}\delta}^{(a_k, b_k)} = D_0 \cup D_1$ ,

$$D_0 = \{|a - a_k^q| = k^{-\mu}\varepsilon; |b - b_k^q| \leq k^{-\nu}\delta\},$$

$$D_1 = \{|a - a_k| \leq k^{-\mu}\varepsilon; |b - b_k| = k^{-\nu}\delta\}.$$

Inequality (5.12) will be analyzed separately on  $D_0$  and  $D_1$ .

If  $|d_2| \leq |d_{-2}|$ , denote  $d_2 = d, d_{-2} = D, c = c_2, C = c_{-2}$ ; in the opposite case  $|d_2| > |d_{-2}|$ , denote  $d_{-2} = d, d_2 = D, c = c_{-2}, C = c_2$ . By the triangle inequality and expansion (5.11), we have that

$$\left| \tilde{U}(a, b) \right| \geq (2k+1)^{\frac{1}{5}-\mu}\varepsilon \left| c - \frac{Cd}{D} \right| + \text{higher order terms}, \quad (a, b) \in D_0.$$

Similarly, if  $|c_2| \leq |c_{-2}|$  denote  $d_2 = d, d_{-2} = D, c = c_2, C = c_{-2}$ ; in the opposite case  $|c_2| > |c_{-2}|$ , denote  $d_{-2} = d, d_2 = D, c = c_{-2}, C = c_2$ . By the triangle inequality and expansion (5.11), we have that

$$\left| \tilde{U}(a, b) \right| \geq (2k+1)^{-\frac{1}{5}-\nu}\delta \left| d - \frac{Dc}{C} \right| + \text{higher order terms}, \quad (a, b) \in D_1.$$

We observe that  $(c - \frac{Cd}{D}) \neq 0$  and  $(d - \frac{Dc}{C}) \neq 0$ , since (see Lemma 5.5)  $c_2 d_{-2} - c_{-2} d_2 = -28\pi i$ . By hypothesis  $-1 < \frac{1}{5} - \mu < 0$  and  $-1 < -\frac{1}{5} - \nu < 0$ .

Conversely,  $\left| U(a, b) - \tilde{U}(a, b) \right| \leq \frac{\rho^*}{2k+1} + \text{higher order terms}$ , for all  $(a, b) \in D_0 \cup D_1$ .

The Lemma is proven. □

**REMARK.** Using the proof of Lemma 5.8, we can alternatively prove Theorem 5.1 invoking the Banach-Caccioppoli contraction mapping principle instead of the multidimensional Rouché theorem.

## Chapter 6

# Deformed Thermodynamic Bethe Ansatz

In Chapters 4 and 5 we have developed the complex WKB method, in order to solve *approximately* the monodromy problem of the cubic oscillator

$$\frac{d^2\psi(\lambda)}{d\lambda^2} = V(\lambda; a, b)\psi(\lambda), \quad V(\lambda; a, b) = 4\lambda^3 - a\lambda - b.$$

The main purpose of the present Chapter is to introduce a novel instrument of analysis, that we call Deformed Thermodynamic Bethe Ansatz (Deformed TBA), to solve *exactly* the direct monodromy problem.

The first breakthrough towards an exact evaluation of the monodromy problem is the work of Dorey and Tateo [DT99]: they analyze anharmonic oscillators with a monomial potential  $\lambda^n - E$  ( $n$  not necessarily 3) via the Thermodynamic Bethe Ansatz and other nonlinear integral equations (called sometimes Destri - de Vega equations). Subsequently Bazhanov, Lukyanov and Zamolodchikov generalized the Dorey-Tateo analysis to monomial potentials with a centrifugal term [BLZ01]. In the present Chapter we generalize Dorey and Tateo approach to the general cubic potential.

Here we summarize the main result of the present Chapter. Let us recall from Chapter 2 the definition of the  $R$  functions.

$$\begin{aligned} R_k &: \mathbb{C}^2 \rightarrow \mathbb{C}, \\ R_k(a, b) &= (w_{1+k}(f), w_{-2+k}(f); w_{-1+k}(f), w_{2+k}(f)). \end{aligned} \quad (6.1)$$

Here  $(a, b; c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$  is the cross ratios of four points on the sphere,  $f = \frac{\varphi}{\chi}$  with  $\{\varphi, \chi\}$  an arbitrary basis of solutions of the cubic oscillator and  $w_k(f)$  is the  $k$ -th asymptotic value of  $f$  (see Definition 2.5).

They satisfy the following system of quadratic relations

$$R_{k-2}(a, b)R_{k+2}(a, b) = 1 - R_k(a, b), \quad \forall k \in \mathbb{Z}_5, \quad (6.2)$$

and, according to Theorem 2.4, we have that  $\sigma_k(a, b) = iR_k(a, b)$ .

Fix  $a \in \mathbb{C}$  and denote  $\varepsilon_k(\vartheta) = \ln \left( -R_0(e^{-k\frac{2\pi i}{5}}a, e^{\frac{6\vartheta}{5}}) \right)$ . Following the convention of Statistical Field Theory we call pseudo-energies the functions  $\varepsilon_k$ . In Theorem 6.2 below, we show that the pseudo-energies satisfy a *non-linear nonlocal Riemann-Hilbert problem*, which is equivalent (at least for small value of the parameter  $a$ ) to the following system of nonlinear integral equations that we call Deformed Thermodynamic Bethe Ansatz:

$$\chi_l(\sigma) = \int_{-\infty}^{+\infty} \varphi_l(\sigma - \sigma') \Lambda_l(\sigma') d\sigma', \quad \sigma, \sigma' \in \mathbb{R}, \quad l \in \mathbb{Z}_5 = \{-2, \dots, 2\}. \quad (6.3)$$

Here

$$\begin{aligned} \Lambda_l(\sigma) &= \sum_{k \in \mathbb{Z}_5} e^{i\frac{2lk\pi}{5}} L_k(\sigma) \quad , \quad L_k(\sigma) = \ln \left( 1 + e^{-\varepsilon_k(\sigma)} \right) , \\ \varepsilon_k(\sigma) &= \frac{1}{5} \sum_{l \in \mathbb{Z}_5} e^{-i\frac{2lk\pi}{5}} \chi_l(\sigma) + \frac{\sqrt{\frac{\pi}{3}} \Gamma(1/3)}{2^{\frac{5}{3}} \Gamma(11/6)} e^{\sigma} + a \frac{\sqrt{3\pi} \Gamma(2/3)}{4^{\frac{2}{3}} \Gamma(1/6)} e^{\frac{\sigma}{5} - i\frac{2k\pi}{5}} , \\ \varphi_0(\sigma) &= \frac{\sqrt{3}}{\pi} \frac{2 \cosh(2\sigma)}{1 + 2 \cosh(2\sigma)} \quad , \quad \varphi_1(\sigma) = -\frac{\sqrt{3}}{\pi} \frac{e^{-\frac{9}{5}\sigma}}{1 + 2 \cosh(2\sigma)} \\ \varphi_2(\sigma) &= -\frac{\sqrt{3}}{\pi} \frac{e^{-\frac{3}{5}\sigma}}{1 + 2 \cosh(2\sigma)} \quad , \quad \varphi_{-1}(\sigma) = \varphi_1(-\sigma), \varphi_{-2}(\sigma) = \varphi_2(-\sigma). \end{aligned}$$

For  $a = 0$  equations (6.3) reduce to the Thermodynamic Bethe Ansatz, introduced by Zamolodchikov [Zam90] to describe the thermodynamics of the 3-state Potts model and of the Lee-Yang model. We will discuss such reduction in Subsection 6.2.1.

The Chapter is divided in two Sections. The first is devoted to the introduction of the  $Y$  functions and of the Deformed  $Y$ -system. In the second we derive the Deformed TBA equations.

## 6.1 Y-system

Here we introduce the  $Y$ -system (6.7), which is a fundamental step in the derivation of the Deformed TBA.

We begin with an observation due to Sibuya ([Sib75]):

**Lemma 6.1.** *Let  $\omega = e^{i\frac{2\pi}{5}}$ . Then*

$$R_k(\omega^{-1}a, \omega b) = R_{k-2}(a, b) \quad (6.4)$$

*Proof.* Denote  $\varphi(\lambda; a, b)$  a solution of (2.1) whose Cauchy data do not depend on  $a, b$ . It is an entire function of three complex variables with some

remarkable properties. It is a simple calculation to verify that for any  $k \in \mathbb{Z}_5$   $\varphi(\omega^k \lambda; \omega^{2k} a, \omega^{3k} b)$  satisfies the same Schrödinger equation (2.1). Fix  $\varphi(\lambda; a, b), \chi(\lambda; a, b)$  linearly independent solutions and define the asymptotic values

$$\begin{aligned} w_k(a, b) &= w_k\left(\frac{\varphi(\lambda; a, b)}{\chi(\lambda; a, b)}\right), \\ \bar{w}_k(a, b) &= w_k\left(\frac{\varphi(\omega^l \lambda; \omega^{2l} a, \omega^{3l} b)}{\chi(\omega^l \lambda; \omega^{2l} a, \omega^{3l} b)}\right). \end{aligned}$$

Obviously  $w_k(\omega^{2l} a, \omega^{3l} b) = \bar{w}_{k-l}(a, b)$ . Choose  $l = 2$  and use the definition of the functions  $R$  (see equation (6.1)) to obtain the thesis.  $\square$

Due to equations (6.2) and relations (6.4), the holomorphic functions  $R_k(a, b)$  satisfy the following system of functional equations, first studied by Sibuya [Sib75]

$$R_k(\omega^{-1} a, \omega b) R_k(\omega a, \omega^{-1} b) = 1 - R_k(a, b), \quad \forall k \in \mathbb{Z}_5. \quad (6.5)$$

We have collected all the elements to introduce the important *Y-functions* and *Y-system*.

We fix  $a \in \mathbb{C}$  and define

$$Y_k(\vartheta) = -R_0(\omega^{-k} a, e^{\frac{6}{5}\vartheta}), \quad k \in \mathbb{Z}_5. \quad (6.6)$$

Sibuya's equation (6.5) is equivalent to the following system of functional equations, that we call *Deformed Y-system*:

$$Y_{k-1}(\vartheta - i\frac{\pi}{3}) Y_{k+1}(\vartheta + i\frac{\pi}{3}) = 1 + Y_k(\vartheta). \quad (6.7)$$

**REMARK.** If  $a = 0$ ,  $Y_k = Y_0, \forall k$  and the system (6.7) reduces to just one equation, called *Y-system*, which was introduced by Zamolodchikov [Zam91] in relation with the Lee-Yang and 3-state Potts models. Dorey and Tateo [DT99] studied the Zamolodchikov *Y-system* in relation with the Schrödinger equation with potential  $V(\lambda; 0, b) = 4\lambda^3 - b$ .

### 6.1.1 Analytic Properties of $Y_k$

In the following theorem we summarize the analytic properties of the *Y-functions*. For all  $a$  and  $k$ ,  $Y_k(\vartheta)$  is periodic with period  $i\frac{5\pi}{3}$ . Hence, from now on we restrict it to the strip  $\{|Im\vartheta| \leq \frac{5\pi}{6}\}$ .

**Theorem 6.1.** (i) For any  $a \in \mathbb{C}$  and  $k \in \mathbb{Z}_5$ ,  $Y_k$  is analytic and  $i\frac{5\pi}{3}$  periodic. If  $a$  is real then  $\overline{Y_k(\vartheta)} = Y_{-k}(\vartheta)$ , where  $-$  stands for complex conjugation.

(ii) For any  $a \in \mathbb{C}$  and  $k \in \mathbb{Z}_5$ , on the strip  $|Im\vartheta| \leq \frac{\pi}{2} - \varepsilon$

$$\begin{aligned} \left| \frac{Y_k(\vartheta)}{\tilde{Y}_k(\vartheta)} - 1 \right| &= O(e^{-Re\vartheta}), \text{ as } Re\vartheta \rightarrow +\infty, \\ \tilde{Y}_k(\vartheta) &= \exp \left( Ae^{\vartheta} + Bae^{\frac{\vartheta}{5} - i\frac{2k\pi}{5}} \right). \end{aligned} \quad (6.8)$$

Here  $A = \frac{\sqrt{\frac{\pi}{3}}\Gamma(1/3)}{2^{\frac{2}{3}}\Gamma(11/6)}$  and  $B = \frac{\sqrt{3\pi}\Gamma(2/3)}{4^{\frac{2}{3}}\Gamma(1/6)}$ .

(iii) For any  $a \in \mathbb{C}$  and any  $K \in \mathbb{R}$ ,  $Y_k(\vartheta)$  is bounded on  $Re\vartheta \leq K$ . If  $a = 0$ ,  $\lim_{\vartheta \rightarrow -\infty} Y_k(\vartheta) = \frac{1+\sqrt{5}}{2}$ .

(iv) If  $e^{i\frac{2k\pi}{5}}a$  is real non negative then  $Y_k(\vartheta) = 0$  implies  $Im\vartheta = \pm\frac{5\pi}{6}$ . If  $a = 0$  then  $Y_k(\vartheta) = -1$  implies  $\vartheta = \pm i\frac{\pi}{2}$ .

(v) Fix  $\varepsilon > 0$ . If  $a$  is small enough, then for any  $k \in \mathbb{Z}_5$ ,  $Y_k(\vartheta) \neq 0, -1$  for any  $\vartheta \in \{|Im\vartheta| \leq \frac{\pi}{2} - \varepsilon\}$ .

*Proof.* (i) Trivial.

(ii) These "WKB-like" estimates can be derived from Chapter 4 or can be found in [Sib75]. Due to Theorem 2.4, these "WKB-like" estimates are equivalent to those derived in [Shi05] Section 4 for the Stokes multipliers.

However, for convenience of the reader we briefly sketch an alternative proof here.

Using the tools developed in Sections 4.2.1 and 4.3, it is easily shown that the following asymptotic is valid

$$R_0(a, x^{\frac{6}{5}}e^{i\phi}) = -\exp \left\{ x \oint \sqrt{V(\lambda; ax^{-\frac{4}{5}}, e^{i\phi})} d\lambda \right\} (1 + O(x^{-1})),$$

if  $x \gg 0$  and  $|\phi| \leq \frac{3\pi}{5} - \varepsilon$ . Here the integration is taken along a path encircling the cut joining the roots  $\lambda_{\pm} = e^{\pm i\frac{2\pi}{3} + \frac{\phi}{3}} + O(x^{-\frac{2}{5}})$  of the cubic potential and the sign of the square root is chosen in such a way that  $Re \left\{ \oint \sqrt{V(\lambda; ax^{-\frac{4}{5}}, e^{i\phi})} d\lambda \right\} > 0$  for  $x$  big enough.

A straightforward computation shows that

$$\begin{aligned} x \oint \sqrt{V(\lambda; ax^{-\frac{4}{5}}, e^{i\phi})} &= xe^{i\frac{5\phi}{6}} \oint \sqrt{4\lambda^3 - 1} d\lambda + \\ &- \frac{x^{\frac{1}{5}}e^{i\frac{\phi}{6}}a}{2} \oint \frac{\lambda}{\sqrt{4\lambda^3 - 1}} d\lambda + O(x^{-\frac{3}{5}}). \end{aligned}$$

If we let  $x^{\frac{6}{5}}e^{i\phi} = e^{\frac{6\vartheta}{5}}$  we obtain the thesis.

- (iii) The boundedness follows directly from the fact that  $R_k(a, b)$  is entire in  $(a, b)$  and, in particular, analytic at  $b = 0$ . If  $a = b = 0$ , then for symmetry reasons one can choose  $\varphi, \chi$  such that  $w_k(\frac{\varphi}{\chi}) = e^{i\frac{2k\pi}{5}}$ . This implies the thesis.
- (iv) The statement is equivalent to Theorem 2.2.
- (v) Since  $Y_k$  depends analytically on the parameter  $a$ , it follows from (iv).  $\square$

## 6.2 Deformed TBA

This section is devoted to the derivation of the Deformed Thermodynamic Bethe Ansatz equations (6.3) <sup>1</sup>.

In what follows we always make the following

**ASSUMPTIONS 6.1.** *We assume that there exists an  $\varepsilon > 0$  such that*

- (i) *every branch of  $\ln Y_k$  is holomorphic on  $|Im\vartheta| \leq \frac{\pi}{3} + \varepsilon$ , and bounded for  $\vartheta \rightarrow -\infty$ . And*
- (ii) *every branch of  $\ln(1 + \frac{1}{Y_k})$  is holomorphic on  $|Im\vartheta| \leq +\varepsilon$ , and bounded for  $\vartheta \rightarrow -\infty$ .*

From Theorem 6.1(iii, v) we know that the assumptions are valid if  $a$  is small enough. We briefly discuss what happens if the assumptions fail in Subsection 6.2.2 below.

We define the following bounded analytic functions on *the physical strip*  $|Im\vartheta| \leq \frac{\pi}{3}$

$$\begin{aligned}
\varepsilon_k(\vartheta) &= \ln Y_k(\vartheta), \\
\delta_k(\vartheta) &= \varepsilon_k(\vartheta) - \frac{\sqrt{\frac{\pi}{3}}\Gamma(1/3)}{2^{\frac{2}{3}}\Gamma(11/6)}e^{\vartheta} - a\frac{\sqrt{3\pi}\Gamma(2/3)}{4^{\frac{2}{3}}\Gamma(1/6)}e^{\frac{\vartheta}{5}-i\frac{2k\pi}{5}}, \\
L_k(\vartheta) &= \ln(1 + e^{-\varepsilon_k(\vartheta)}).
\end{aligned} \tag{6.9}$$

Here the branches of logarithms are fixed by requiring

$$\lim_{\sigma \rightarrow +\infty} \delta_k(\sigma + i\tau) = \lim_{\sigma \rightarrow +\infty} L_k(\sigma + i\tau) = 0, \quad \forall |\tau| \leq \frac{\pi}{3}. \tag{6.10}$$

We remark that by Theorem 6.1(ii), this choice is always possible.

Due to the Y-system (6.7), the functions  $\delta_k$  satisfy the following *nonlinear nonlocal Riemann-Hilbert (R-H) problem*

$$\delta_{k-1}(\vartheta - i\frac{\pi}{3}) + \delta_{k+1}(\vartheta + i\frac{\pi}{3}) - \delta_k(\vartheta) = L_k(\vartheta), \quad |Im\vartheta| \leq \varepsilon. \tag{6.11}$$

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<sup>1</sup>The reader who wants to repeat all the computations below, should remember that  $\sum_{l=0}^{n-1} e^{i\frac{2\pi l}{n}} = 0$ .

Here the boundary conditions are given by asymptotics (6.10).

The system (6.11) is  $\mathbb{Z}_5$  invariant. Hence we diagonalize its linear part (the left hand side) by taking its *discrete Fourier transform* (also called *Wannier transform*):

$$\begin{aligned}\chi_l(\vartheta) &= \sum_{k \in \mathbb{Z}_5} e^{i \frac{2kl\pi}{5}} \delta_k(\vartheta) \quad , \quad \delta_k(\vartheta) = \frac{1}{5} \sum_{l \in \mathbb{Z}_5} e^{-i \frac{2kl\pi}{5}} \chi_l(\vartheta) , \\ \Lambda_l(\vartheta) &= \sum_{k \in \mathbb{Z}_5} e^{i \frac{2kl\pi}{5}} L_k(\vartheta) \quad , \quad L_k(\vartheta) = \frac{1}{5} \sum_{l \in \mathbb{Z}_5} e^{-i \frac{2kl\pi}{5}} \Lambda_l(\vartheta) .\end{aligned}\tag{6.12}$$

The above defined functions satisfy the following Riemann-Hilbert problem

$$e^{-i \frac{2l\pi}{5}} \chi_l(\vartheta + i \frac{\pi}{3}) + e^{i \frac{2l\pi}{5}} \chi_l(\vartheta - i \frac{\pi}{3}) - \chi_l(\vartheta) = \Lambda_l(\vartheta) , \tag{6.13}$$

$$\lim_{\sigma \rightarrow +\infty} \chi_l(\sigma + i\tau) = 0 , \quad \forall |\tau| \leq \frac{\pi}{3} . \tag{6.14}$$

The system of functional equation (6.13), may be rewritten in the convenient form of a system of coupled integral equations (6.3).

**Theorem 6.2.** *If  $a$  is small enough, the functions  $\chi_l$  satisfy the Deformed Thermodynamic Bethe Ansatz*

$$\chi_l(\sigma) = \int_{-\infty}^{+\infty} \varphi_l(\sigma - \sigma') \Lambda_l(\sigma') d\sigma' , \quad \sigma, \sigma' \in \mathbb{R} . \tag{6.3}$$

Here  $\Lambda_l$  are defined as in (6.9, 6.12) and

$$\begin{aligned}\varphi_0(\sigma) &= \frac{\sqrt{3}}{\pi} \frac{2 \cosh(\sigma)}{1 + 2 \cosh(2\sigma)} \\ \varphi_1(\sigma) &= -\frac{\sqrt{3}}{\pi} \frac{e^{-\frac{9}{5}\sigma}}{1 + 2 \cosh(2\sigma)} \\ \varphi_2(\sigma) &= -\frac{\sqrt{3}}{\pi} \frac{e^{-\frac{3}{5}\sigma}}{1 + 2 \cosh(2\sigma)} \\ \varphi_{-1}(\sigma) &= -\frac{\sqrt{3}}{\pi} \frac{e^{\frac{9}{5}\sigma}}{1 + 2 \cosh(2\sigma)} \\ \varphi_{-2}(\sigma) &= -\frac{\sqrt{3}}{\pi} \frac{e^{\frac{3}{5}\sigma}}{1 + 2 \cosh(2\sigma)} .\end{aligned}\tag{6.15}$$

*Proof.* If  $a$  is small enough then the Assumptions 6.1 are valid. Hence the thesis follows from system (6.13) and the technical Lemma 6.2 below.  $\square$

**Lemma 6.2.** *Let  $f : \{|Im\vartheta| \leq \varepsilon\} \rightarrow \mathbb{C}$  be a bounded analytic function. Then for any  $l \in \mathbb{Z}_5$  there exists a unique function  $F$  analytic and bounded on  $|Im\vartheta| \leq \frac{\pi}{3} + \varepsilon$ , such that*

$$e^{-i\frac{2\pi l}{5}}F(\vartheta + i\frac{\pi}{3}) + e^{i\frac{2\pi l}{5}}F(\vartheta - i\frac{\pi}{3}) - F(\vartheta) = f(\vartheta), \forall |Im\vartheta| \leq \varepsilon.$$

Moreover,  $F$  is expressed through the following integral transform

$$F(\vartheta + i\tau) = \int_{-\infty}^{+\infty} \varphi_l(\vartheta + i\tau - \vartheta')f(\vartheta')d\sigma', \quad \forall |Im\vartheta| \leq \varepsilon, |\tau| \leq \frac{\pi}{3}, \quad (6.16)$$

provided  $|Im(\vartheta + i\tau - \vartheta')| < \frac{\pi}{3}$  and the integration path belongs to the strip  $|Im\vartheta'| \leq \varepsilon$ . Here  $\varphi_l$  is defined by formula (6.15).

*Proof.* Uniqueness: let  $F_1, F_2$  be bounded solution of the functional equation

$$e^{-i\frac{2\pi l}{5}}F_j(\vartheta + i\frac{\pi}{3}) + e^{i\frac{2\pi l}{5}}F_j(\vartheta - i\frac{\pi}{3}) - F_j(\vartheta) = f(\sigma), \quad j = 1, 2, |Im\vartheta| \leq \varepsilon.$$

Their difference  $G = F_1 - F_2$  satisfies

$$e^{-i\frac{2\pi l}{5}}G(\vartheta + i\frac{\pi}{3}) + e^{i\frac{2\pi l}{5}}G(\vartheta - i\frac{\pi}{3}) - G(\vartheta) = 0.$$

Then  $G$  extends to an entire function such that  $G(\vartheta + 2i\pi) = e^{i\frac{2\pi l}{5}}G(\vartheta)$ . Therefore  $G$  is bounded, hence a constant. The only constant satisfying the functional relation is zero.

Existence: one notices that if  $\theta \neq \pm in\frac{\pi}{3}$ ,  $n \in \mathbb{Z}$  then

$$e^{-i\frac{2\pi l}{5}}\varphi_l(\theta + i\frac{\pi}{3}) + e^{i\frac{2\pi l}{5}}\varphi_l(\theta - i\frac{\pi}{3}) - \varphi_l(\theta) = 0, \quad \forall l \in \mathbb{Z}_5.$$

Then a rather standard computation shows that the function  $F$  defined through formula (6.16) satisfies all the desired properties.  $\square$

**REMARK.** *Once the system of integral equations (6.3) is solved for  $\sigma \in \mathbb{R}$ , one can use the same set of integral equations as explicit formulas to extend the functions  $\chi_l(\theta)$  on  $|Im\theta| \leq \frac{\pi}{3}$ . Then one can use the  $Y$ -system (6.7) to extend the  $Y$  functions on the entire fundamental strip  $|Im\vartheta| \leq \frac{5\pi}{6}$ .*

**REMARK.** *While the  $Y$ -system equations do not depend on the parameter  $a$  (the coefficient of the linear term of the potential  $4\lambda^3 - a\lambda - b$ ), on the contrary the Deformed TBA equations depend on it since it enters explicitly into the definition of functions  $\Lambda_l$ .*



### 6.2.1 The case $a = 0$

If  $a = 0$  then  $\delta_k = \delta_0$ ,  $L_k = L_0$  for any  $k$ . Therefore,  $\delta_0$  satisfy the single functional equation

$$\delta_0(\vartheta - i\frac{\pi}{3}) + \delta_0(\vartheta + i\frac{\pi}{3}) - \delta_0(\vartheta) = L_0(\vartheta), \quad |Im\vartheta| \leq \varepsilon. \quad (6.17)$$

Similar reasoning as in Theorem 6.2 shows that  $\delta_0$  satisfies the following nonlinear integral equation (as it was firstly discovered by Dorey and Tateo [DT99])

$$\delta_0(\sigma) = \int_{-\infty}^{+\infty} \varphi_0(\sigma - \sigma') \ln(1 + \exp(-(\delta_0(\sigma') + Ae^{\sigma'}))) d\sigma', \quad \sigma, \sigma' \in \mathbb{R}. \quad (6.18)$$

Here  $\varphi_0$  is defined as in formula (6.15) and  $A = \frac{\sqrt{\frac{\pi}{3}}\Gamma(1/3)}{2^{\frac{2}{3}}\Gamma(11/6)}$ .

Equation (6.18) is called Thermodynamic Bethe Ansatz and was introduced by Zamolodchikov [Zam90] to describe the Thermodynamics of the 3-state Potts and Lee-Yang models.

### 6.2.2 Zeros of $Y$ in the physical strip

In case the Assumptions 6.1 are not satisfied, the functions  $\varepsilon_k$  and  $L_k$  (see equation (6.9)) are not well-defined on the physical strip  $|Im\vartheta| \leq \frac{\pi}{3}$ . Therefore we cannot transform the Y-system into a Riemann-Hilbert problem (6.11) using the same procedure shown above.

However, one can get a well-posed Riemann-Hilbert problem for the functions  $Y_k$  simply factorizing out their zeroes (this approach was developed for the Thermodynamic Bethe Ansatz in [BLZ97]). In this way, one eventually obtains a system of nonlinear integral equation similar to the Deformed TBA equation for the functions  $Y_k$  and an (essentially) algebraic system of equations for the location of their zeroes.

We postpone to a subsequent publication the discussion of these more general Deformed TBA equations, because we have not yet reached a satisfactorily knowledge about the region in the  $a$ -plane where the Assumptions 6.1 fail.

## 6.3 The First Numerical Experiment

In collaboration with A. Moro we are studying the numerical solution of the Deformed TBA equations (6.3). The work is still in progress but we can present some preliminary results. In Figure 6.3 below, we show the Stokes multiplier  $\sigma_0(0, b)$  for  $|b| \leq 15$  as computed by means of the numerical solution of the Thermodynamic Bethe Ansatz equation (6.18). More precisely

we plot the function  $F(z) = \frac{\left| \frac{1}{1+i\sigma_0(z)} \right|}{1 + \left| \frac{1}{1+i\sigma_0(z)} \right|}$ , with  $z = |b|^{\frac{5}{6}} e^{i \arg b}$ .

The green plateau corresponds to the sector of the  $z$  plane where the Stokes multiplier is exponentially small, the blue plateau to the sector where the Stokes multiplier is exponentially big (see Section 7.2 for the explicit formulas). The peaks correspond to the zeroes of  $i\sigma_0 + 1$ , which are, due to equation (2.7), the zeroes of  $\sigma_2 \times \sigma_{-2}$ . According to Theorem 2.2 there are two infinite series of such peaks along the rays with angles  $\pm \frac{3\pi}{5}$ .  $\sigma_0$  has an oscillatory behaviour along these two rays.

In the next Chapter we present an alternative algorithm for computing Stokes multipliers and we compare the results furnished by the two methods.

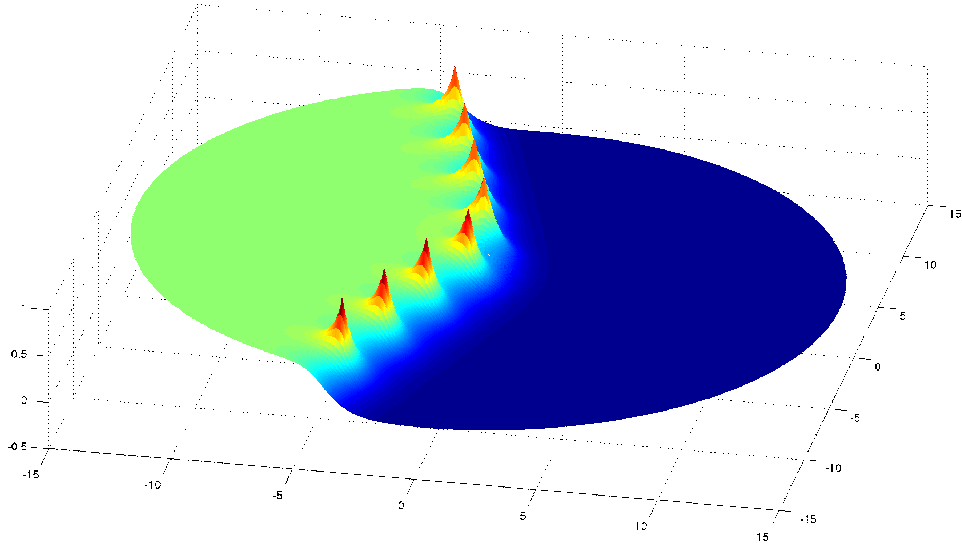


Figure 6.1:  $F(z) = \frac{\left| \frac{1}{1+i\sigma_0(z)} \right|}{1 + \left| \frac{1}{1+i\sigma_0(z)} \right|}$ , with  $z = |b|^{\frac{5}{6}} e^{i \arg b}$

## Chapter 7

# A Numerical Algorithm

The aim of the present Chapter is to introduce a new algorithm for computing the Stokes multipliers of the perturbed cubic oscillator

$$\begin{aligned}\frac{d^2\psi(\lambda)}{d\lambda^2} &= Q(\lambda; y, y', z)\psi(\lambda) , \\ Q(\lambda; y, y', z) &= 4\lambda^3 - 2\lambda z + 2zy - 4y^3 + y'^2 + \frac{y'}{\lambda - y} + \frac{3}{4(\lambda - y)^2} .\end{aligned}$$

As it was shown in Lemma 4.10, in some limit relevant for studying the poles of the solutions of P-I the perturbed cubic oscillator becomes the cubic oscillator

$$\frac{d^2\psi(\lambda)}{d\lambda^2} = V(\lambda; a, b)\psi(\lambda) , \quad V(\lambda; a, b) = 4\lambda^3 - a\lambda - b .$$

In the present Chapter we consider the cubic oscillator as a particular case of the perturbed cubic oscillator. We set the convention that the cubic oscillator is the particular case of the perturbed cubic oscillator determined by  $y = \infty$ .

As it was explained in Chapter 3, solutions of Painlevé-I give rise to isomonodromic deformations of the perturbed cubic oscillator. Hence, our algorithm gives a numerical solution of the *direct monodromy problem* for the Painlevé first equation: given the Cauchy data  $y(z), y'(z), z$  of a particular solution of P-I we are able to compute the corresponding Stokes multipliers, even when  $z$  is a pole of that solution. In the latter case we have to consider the cubic oscillator with potential  $V(\lambda; 2\alpha, 28\beta)$ , where  $z = \alpha$  is the pole and  $\beta$  is the coefficient of the  $(z - \alpha)^4$  term in the Laurent expansion of  $y(z)$  (see Lemma 3.6).

The Chapter is divided in two Sections. The first is devoted to the description of the Algorithm. In the second we test our algorithm against the WKB prediction and the Deformed TBA equations.

**REMARK.** *The algorithm we present here depends heavily on the theory developed in Chapter 2 and especially in Section 2.2. We refer to that Chapter for any definitions and theorems.*

## 7.1 The Algorithm

In Chapter 2 we have proved the following remarkable facts

- Along any ray contained in the Stokes Sector  $S_k$ , any solution  $f$  to the Schwarzian differential equation (2.11) converges super-exponentially to the asymptotic value  $w_k(f)$ . See Lemma 2.4 (iv).
- The Stokes multipliers of the perturbed cubic oscillator (2.2) are cross ratios of the asymptotic values  $w_k(f)$ . See Theorem (2.4).
- Inside any closed subsector of  $S_k$ ,  $f$  has a finite number of poles. See Lemma 2.6.

Hence the Simple Algorithm for Computing Stokes Multipliers goes as follows:

1. Set  $k = -2$ .
2. Fix arbitrary Cauchy data of  $f$ :  $f(\lambda^*)$ ,  $f'(\lambda^*)$ ,  $f''(\lambda^*)$ , with the conditions  $\lambda^* \neq y$ ,  $f'(\lambda^*) \neq 0$ <sup>1</sup>.
3. Choose an angle  $\alpha$  inside  $S_k$ , such that the singular point  $\lambda = y$  does not belong to the corresponding ray, i.e.  $\alpha \neq \arg y$ . Define  $t : \mathbb{R}^+ \cup 0 \rightarrow \mathbb{C}$ ,  $t(x) = f(e^{i\alpha}x + \lambda^*)$ . The function  $t$  satisfies the following Cauchy problem
$$\begin{cases} \{t(x), x\} = e^{2i\alpha} Q(e^{i\alpha}x + \lambda^*; y, y', z), \\ t(0) = f(\lambda^*), t'(0) = e^{i\alpha} f'(\lambda^*), t''(0) = e^{2i\alpha} f''(\lambda^*) . \end{cases} \quad (7.1)$$
4. Integrate equation (7.1) either directly<sup>2</sup> or by linearization (see Remark below), and compute  $w_k(f)$  with the desired accuracy and precision.
5. If  $k < 2, k++$ , return to point 3.
6. Compute  $\sigma_l$  using formula (2.17) for all  $l \in \mathbb{Z}_5$ .

**REMARK.** *As was shown in Lemma 2.3, any solution  $f$  of the Schwarzian equation is the ratio of two solutions of the Schrödinger equation. Hence, one can solve the nonlinear Cauchy problem (7.1) by solving two linear Cauchy problems.*

*Whether the linearization is more efficient than the direct integration of (7.1) will not be investigated here.*

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<sup>1</sup>Indeed, the derivative  $f'(\lambda)$  of any solution of the Schwarzian equation never vanishes if  $\lambda \neq \lambda^*$

<sup>2</sup>Integrating equation (7.1) directly, one can hit a singularity  $x^*$  of  $y$ . To continue the solution past the pole, starting from  $x^* - \varepsilon$  one can integrate the function  $\tilde{y} = \frac{1}{y}$ , which satisfies the same Schwarzian differential equation.

## 7.2 The Second Numerical Experiment

We have implemented our algorithm using MATHEMATICA's ODE solver NDSOLVE. We have chosen to integrate equation (7.1) with steps of length 0.1. We decided the integrator to stop at step  $n$  if

$$\begin{aligned} |t(0.1n) - t(0.1(n-1))| &< 10^{-13} \text{ and} \\ \left| \frac{t(0.1n) - t(0.1(n-1))}{t(0.1n)} \right| &< 10^{-13} . \end{aligned}$$

To test our algorithm we computed the Stokes multiplier  $\sigma_0(b)$  of the equation

$$\frac{d^2\psi(\lambda)}{d\lambda^2} = (4\lambda^3 - b)\psi(\lambda) . \quad (7.2)$$

Using the WKB theory developed in Chapter 4, one can easily show that the Stokes multiplier  $\sigma_0(b)$  has the following asymptotics

$$\sigma_0(b) \sim \begin{cases} -ie^{\frac{\sqrt{\pi}}{3}\Gamma(1/3)} b^{\frac{5}{6}} & , \text{ if } b > 0 \\ -2ie^{-\frac{\sqrt{\pi}\Gamma(1/3)}{2^{5/3}\Gamma(11/6)}(-b)^{\frac{5}{6}}} \cos\left(\frac{\sqrt{\pi}\Gamma(1/3)}{2^{5/3}\Gamma(11/6)}(-b)^{\frac{5}{6}}\right) & , \text{ if } b < 0 . \end{cases} \quad (7.3)$$

Our computations (see Figure 1 and 2 below) shows clearly that the WKB approximation is very efficient also for small value of the parameter  $b$ .

We also tested our results against the numerical solution (due to A. Moro) of the Deformed Thermodynamic Bethe Ansatz equations (Deformed TBA), which has been introduced in Chapter 6. The numerical solution of the Deformed TBA equations enable to a-priori set the absolute error in the evaluation of the Stokes multiplier  $\sigma_0(b)$  rescaled with respect to the WKB exponentials (7.3). Hence, in the range of  $-20 \leq b \leq 20$  we could verify that we had computed the rescaled  $\sigma_0(b)$  with an absolute error less than  $10^{-8}$ .

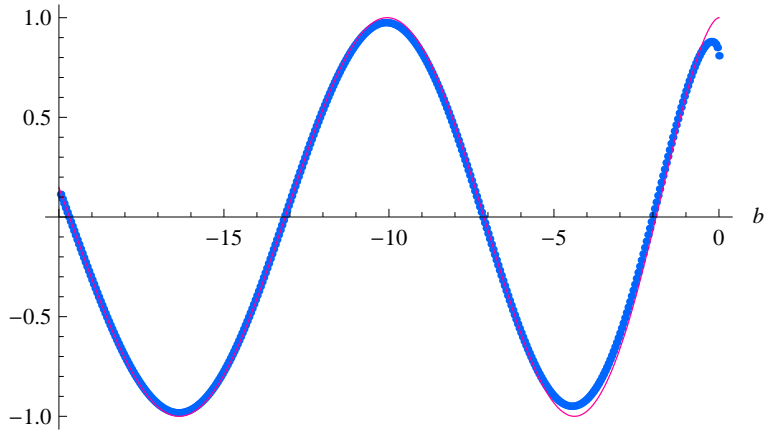


Figure 7.1: Thick dotted line: the rescaled Stokes multiplier  $\frac{i}{2} e^{\frac{\sqrt{\pi}\Gamma(1/3)}{2^{2/3}\Gamma(11/6)}(-b)^{5/6}} \sigma_0(b)$  evaluated with our algorithm; thin continuous line:  $\cos\left(\frac{\sqrt{\pi}\Gamma(1/3)}{2^{5/3}\Gamma(11/6)}(-b)^{5/6}\right)$ , i.e. the WKB prediction for the rescaled Stokes multiplier.

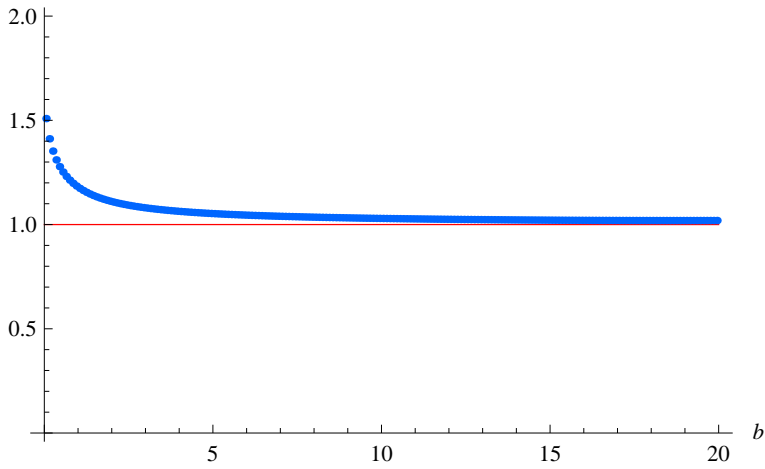


Figure 7.2: Thick dotted line: the rescaled Stokes multiplier  $e^{-\frac{\sqrt{\pi}\Gamma(1/3)}{2^{2/3}\Gamma(11/6)}b^{5/6}} \sigma_0(b)$  evaluated with our algorithm; thin continuous line: 1, the WKB prediction for the rescaled Stokes multiplier.

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