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On Transfer Operators for Anosov Flows

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On Transfer Operators for Anosov Flows

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Chapter 1

Foreword

"Order out of chaos is our natural striving,
and when we achieve it we treasure it."

The mathematical experience
By P. J. Davis, R. Hersh, E. Marchisotto

In the past years, dynamical systems have attracted a lot of attentions for their ability to melt in one framework both the properties of the spaces in which they live in and the properties of the transformations which define their development. In this sense, from time to time, solutions came naturally by focusing on one or another of these two elements. I believe that this in an unavoidable and lucky characteristic of our field; in fact most methods yields best results once a "correct couple" is found; moreover, at the end, by studying their interaction one knows more about them even singularly. To some extent in dynamical system we are particularly lucky since we can retain eye-naked visible properties (such as the undefined "chaotic" structures) and at the same time we can hide all the machinery behind the scene.

From this point of view transfer operators have many characteristics which make their study worthwhile. They bear connections with quantum chaos and statistical mechanics, they have a deep connection with a class of zeta functions, and their properties, obtained through spectral theory techniques, often reflect physical behaviors of the system, such as decay of correlations.

I structured this thesis to reflect this viewpoint.

The main purpose of the first chapter is to introduce the reader to the relationship between continuous time hyperbolic systems and zeta functions and guide it through the scattered bibliography available today. To do so I highlighted as much as possible recent developments as well as connection with other fields.

The next chapter contains the following original result.

Theorem. *For any $\mathcal{C}^r$ Anosov flow ϕ_t with $r \geq 2$ on a d-dimensional manifold of strictly negative curvature, the zeta function $\zeta_{Ruelle}(z)$ is meromorphic in a region*

$$\Re(z) > d_u \ln(\|D\phi_1\|_\infty) - \frac{\ln(\lambda)}{2}(r-1)$$

where λ is the coefficient of the Anosov splitting and $d_u \leq d$ is an integer.

As a corollary, one obtains that for a C^∞ Anosov flow the Ruelle zeta function is meromorphic in the entire complex plane. To prove our theorems one studies directly the transfer operator on suitable Banach spaces of anisotropic currents and resort to regularized traces of operators which are not of trace class.

In the last chapter I shortly introduced two research problems which I began to investigate and will be developed during the next years.

Chapter 2

Survey: Zeta Functions and Continuous Time Dynamics

2.1 A Continuous Time viewpoint

The idea of a survey on the relationship between continuous dynamical systems and zeta functions arose in two different ways. First it was a necessity in the preparation of my thesis, second, in several discussions, some colleagues complained about the lack of an overall references on the topic. In fact this paper addresses the questions: "What is it known about dynamical zeta function for a continuous dynamical system? And what it is not known?" In the last twenty years there were some advances in understanding the theory of zeta functions as invariants of dynamical systems; while the search for a unifying framework is still not concluded, we can now recognize few familiar patterns.

Rather then being exhaustive in every possible direction, I will try to guide the reader, assuming is either a mathematician or a physicist from the classical results to the most recent ones. In doing so I tried to outline, when possible, the connection between the dynamical viewpoint and other fields such as hyperbolic geometry, noncommutative geometry, number theory and quantum chaos. At this stage I want to remark that while this survey is not the first one to outline the relationship between dynamical systems and zeta functions, it is perhaps the first one to take firmly the continuous dynamic viewpoint. In fact if the reader is interested in the discrete case, he can read through the many excellent introductions available (for example [6], [7] or [8]) and the references contained therein. Going back to this survey, its viewpoint allows me to highlight precisely phenomena which cannot be seen in the other context, since typically the dynamics in the flow direction is neither expanding nor contracting.

The statements of the theorems includes the necessary, even if sometimes technical, conditions. For what concerns the proof, I omitted most of them, but sometimes I included a rough outline of the papers, so that a casual reader could appreciate the elegance of the ideas, while those who are already acquainted with the techniques can use this survey as a reference to original papers. On the other side I included, where possible, examples of how zeta functions were used to solve, with a reasonable approximation, problems arising from semiclassical analysis. In such context the use of dynamical zeta functions has been quite successful, since its zeroes are directly related to underlying physical phenomena. For example, in the study of how a wave scatters around an obstacle one wants to compute the resonances, and this turn out to be directly related to the zeroes of dynamical zeta functions. Some ideas, like looking at averages on phase space as sums over a reasonable set of periodic orbits, are exploited in many different situations, and are recurring "leitmotiv" in the following.

In many settings, I will show that the properties of the dynamical zeta functions in considerations become crucial to understand the evolution of densities in the chosen space, i.e. to understand the "chaotic evolution" of the system. The next sections are arranged in chronological order. Also within each section I tried to follow a chronological order, though at some point I did not respect it to show the links between ideas. The needed definitions are introduced along the way, following or preceding the discussion of results.

2.2 The framework and the early days

Before moving to the core matter I need to "state the obvious" and setup a bit of notation. As said before, this survey is about continuous dynamical systems that is, an action of $\mathbb{R}$ on X a compact metric space, we will denote our flows by ϕ_t. The attention here is restricted to metric spaces to simplify the presentation, but in some occasions one could relax such hypothesis. Throughout the following τ will be an orbit and τ_p be the prime orbit associated to it. Moreover $\lambda(\tau)$ indicates the length of a orbit, and $\mu(\tau)$ is its multiplicity with respect to its prime orbit so that $\lambda(\tau) = \mu(\tau)\lambda(\tau_p)$. We will use $\mathcal{T}$ to indicate the set of orbits and $\mathcal{T}_p$ to indicate the set of prime orbits, γ will be reserved to closed geodesics. Moreover $\widehat{f}$ is used for the Fourier transform, and $C > 0$ is a running constant, in particular it could change value even within the same sentence.

In 1956 Selberg [72] produced a cornerstone of the relationship between the objects we are interested in. He proved that for a surface of constant negative curvature one has the following striking theorem

Theorem 2.2.1 (Selberg [72]). *Let $h : \mathbb{C} \to \mathbb{C}$ be a suitable test function such that $h(s) = h(-s)$, $h(s)$ is holomorphic in a strip $\Im(s) \leq \frac{1}{2} + \epsilon$ for $\epsilon > 0$ and $|h(s)| \leq a(1 + |s|^2)^{-1-\epsilon}$ for some $a > 0$. Then*

$$\sum_{j=0}^{\infty} h(\rho_j) = \frac{Area(M)}{4\pi} \int_{-\infty}^{\infty} h(\rho) \tanh(\pi\rho)\rho d\rho + \sum_{\gamma\in\mathcal{T}_p} \sum_{n=1}^{\infty} \frac{\lambda(\gamma)\widehat{n\lambda(\gamma)}}{2\sinh(n\lambda(\gamma)/2)} \tag{2.2.1}$$

where ρ_j are the eigenvalues of the Laplacian (see Marklof [45] for a neat introduction to the trace formula). This result is achieved by first identifying $\gamma \in \mathcal{T}_P$ with certain conjugacy class in $\Gamma/\mathbb{H}^2$ where Γ is a Fuchsian group, thus the requirement on constant curvature and the inability (so far) to generalize the theorem. The formula above is the starting point to prove that ζ_{Selberg} is meromorphic in the entire complex plane.

$$\zeta_{\text{Selberg}} = \prod_{\gamma} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\lambda(\gamma)}\right) \tag{2.2.2}$$

In fact Selberg was able to show more: each element $\prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\lambda(\tau_p)}\right)$ has zeroes precisely at $s = \frac{1}{2} + i\nu\frac{2\pi}{\lambda(\gamma)}$ with $m \in \mathbb{N}, \nu \in \mathbb{Z}$ and their multiplicities can be computed exactly. Moreover he proved the following

Theorem 2.2.2 (Selberg [72])**.** *$\zeta_{Selberg}$ can be analytically extended to an entire function on the whole $\mathbb{C}$. Moreover it satisfies the following functional equation*

$$\zeta_{Selberg}(s) = \zeta_{Selberg}(s-1) \exp\left(\int_0^{s-\frac{1}{2}} u \tan(\pi u) du\right)$$

He was able to do so by following a pattern which has been fruitful later on in many different situations. First of all he tried to study the trace of $R(z) = (\Delta - zI)^{-1}$, where Δ is the Laplacian of the surface. However the test functions h, with the properties required by the statement of the theorem, do not behave well with respect to such trace: in fact he was forced to construct an approximated (but regularized) resolvent which is in trace class. Note that constructing an approximate trace or an approximate resolvent will always be a fundamental step in many of the theorems which we will encounter.

Ten years later, the properties of the dynamical zeta functions were discussed again in the seminal paper of Smale [74] where he asked if given an isolated, compact, hyperbolic set for a flow on a manifold M, one could find a meromorphic extension of ζ_{Selberg} once the closed geodesics are replaced by closed orbits. In such case he found that

Theorem 2.2.3 (Smale-Narasimhan [74])**.** *Let ϕ_t be a suspension of $f : M \to M$, where f is Anosov diffeomorphism with an associated rational Artin-Mazur zeta function. Then Selberg Zeta (for closed orbits) is meromorphic in an half plane.*

This theorem is cited here as an example of reduction from continuous time dynamics to discrete time dynamics, then the argument relies on counting fixed points and then using Lefschetz trace formula. This reduction is not always possible since one has to keep in mind the Anosov alternative [3], which says that either an Anosov flow has strong stable and strong unstable manifold everywhere dense or the flow is a suspension of an Anosov diffeomorphism by a constant roof function. The most studied example of Anosov flow is the geodesic flow on a surface.

In his 1970 thesis, Margulis finds the following asymptotic estimates

Theorem 2.2.4 (Margulis [44])**.** *Given a geodesic flow on a surface of negative curvature one has for $C > 0$ and $L > 0$*

$$\{\tau \in \mathcal{T} : \lambda(\tau) < L\} \sim \frac{e^{CL}}{CL}.$$

While Margulis did not openly state so, this implies that there is always an half-plane of convergence. Note that he obtained up to a constant, the correct asymptotic estimate, while Sinai [73], few years before, obtained distinct upper and lower bound for such estimate.

At this stage we recall that an Anosov flow is a flow such that there exists a $D\phi_t$-invariant continuous splitting $TM = E^0 \oplus E^s \oplus E^u$, constants $C > 0$ and $\lambda > 0$, such that for $t \geq 0$, E^0 is the one-dimensional subspace tangent to the flow and

$$\begin{array}{ll} \|D\phi_t(v)\| & \leq C\|v\|e^{-\lambda t} \text{ if } t \geq 0, v \in E^s \\ \|D\phi_{-t}(v)\| & \leq C\|v\|e^{-\lambda t} \text{ if } t \geq 0, v \in E^u. \end{array}$$

A major breakthrough in studying dynamical zeta functions was obtained by Ruelle [65] who introduced the zeta which bears his name after choosing orbits as characters. That is

$$\zeta_{\text{Ruelle}} = \prod_{\tau_p} \left(1 - e^{-s\lambda(\tau_p)}\right)^{-1} \tag{2.2.3}$$

Recall that by real analytic function in a neighborhood of a point we mean that in such neighborhood the function is infinitely many time differentiable and the Taylor expansion of $f(x)$ converges to $f(x)$. Thus, a real-analytic manifold is a manifold such that the charts are real-analytic and a real-analytic foliation is a foliation such that the map from each leaf to $\mathbb{R}^n$ is real-analytic. Ruelle obtained the following

Theorem 2.2.5 (Ruelle [65]). *Let ϕ_t be a real analytic Anosov flow on a real analytic manifold such that the stable and unstable manifolds form real-analytic foliations. Then ζ_{Ruelle} extends meromorphically to the whole complex plane. Moreover is the quotient between two entire functions d_1, d_2 such that $|d_i(z) - 1| \leq e^{-C\Re(z)}$.*

Note that whenever ζ_{Ruelle} and ζ_{Selberg} are both defined we have

$$\zeta_{\text{Ruelle}} = \frac{\zeta_{\text{Selberg}}(s)}{\zeta_{\text{Selberg}}(s+1)} \qquad ; \qquad \zeta_{\text{Selberg}} = \prod_{k=0}^{\infty} \zeta_{\text{Ruelle}}(s+k) \tag{2.2.4}$$

To prove his result Ruelle introduced suitable Markov partitions (on how to code an hyperbolic flow in symbols see [13]), then he constructed dynamical determinants which can be formally defined as

$$d_l(z) = \exp\left(-\sum_{\tau} \frac{1}{\mu(\tau)} \frac{\operatorname{tr}\left(\wedge^l(D_\tau\phi)\right) e^{-z\lambda(\tau)}}{\det\left(\mathbb{1} - D_\tau\phi\right)}\right) \tag{2.2.5}$$

Then he shows (here lies one big novelty) that the properties of (3.2.1) can be deduced from those of a "transfer operator".

Definition 2.2.6. Let $g : M \to R$ a continuous "weight" function. Let f be a function in a suitable Banach space. Then the Ruelle transfer operator is defined as

$$\mathcal{L}_{t,g} f(x) = \sum_{\phi_t(y)=x} e^{g(y)} f(y)$$

To understand one of the main ideas behind transfer operators, one should consider the definition above as a "formal one", and then struggle to find suitable spaces on which such operator is well-behaved. The following equation is the key of the paper, and of many of the works which followed this approach

$$\begin{aligned}\zeta_{\mathrm{Ruelle}}(z) &= \prod_{l=0}^{(\dim M)-1} d_l(z)^{(-1)^{l+1}} \\ &= \prod_{l=0}^{(\dim M)-1} \exp\left(-\sum_{\tau\in\mathcal{T}} \frac{1}{\mu(\tau)} \frac{\mathrm{tr}\left(\wedge^l(D_{\mathrm{hyp}}\phi_{\lambda(\tau)})\right)\lambda(\tau)^n e^{-z\lambda(\tau)}}{\det\left(\mathbb{1} - D_{\mathrm{hyp}}\phi_{\lambda(\tau)}\right)}\right) \\ &= \prod_{l=0}^{(\dim M)-1} \exp\left(-\mathrm{Trace}(R(z))\right)\end{aligned} \tag{2.2.6}$$

where $R(z)$ is the resolvent of the operator $\mathcal{L}_{t,g}$. Note here that we wrote Trace, since, a priori, one cannot use the standard trace of a finite rank linear operator. In fact one has some freedom at this stage, either one constructs a family of approximated resolvents $R_\epsilon(z)$ or one can construct an approximation of the trace. Moreover, note the presence of $\mathrm{tr}\left(\wedge^l(D_\tau\phi)\right)$ and of the weight $\det(I - D_\tau\phi)$. Thus, with respect to the definition of the transfer operator above, along the calculations one is forced to use spaces of forms and to choose a suitable weight. The spectral properties of $\mathcal{L}_t^{(l)}$ are obtained through Grothendieck theory of nuclear operators, and then are translated to the dynamical determinants $d_l(z)$.

On the other hand, while many people struggled to prove meromorphic extensions of such functions, Gallavotti [29] constructed a suspension flow such that ζ_{Ruelle} associated to it has an essential singularity in a negative neighborhood of the origin. He does so by first considering symbolic dynamics, specifically a full one sided shift. Then he constructs a suitable roof function inspired by a Fisher potential (for more details and the droplet model see [25]) of regularity $r \neq \infty$. For such system, one can explicitly perform calculations on the associated zeta functions and find that the constructed mixing Axiom A flow does not have a meromorphic extension to the entire complex plane.

2.3 Middle age

The years between 1983 and 1990 were very important and produced a serious of results about zeta functions. The techniques which became available at the time were related to a better understanding of Markov partitions, and the analysis of the spectrum of $\mathcal{L}_t$ on Hölder functions. I will go rather fast on a remarkable series of results obtained by Parry and Pollicott in this time frame (either independently or as joint work) since the interested reader can already find an optimal presentation of all such results in the monograph [53], written by the authors themselves.

Nevertheless a good starting point for this section is [52], where Parry and Pollicott show that if a subshift of finite type is weak mixing then one has a non-zero analytic extension to $\Re(s) \geq 1$ except for a simple pole at $s = 1$. Here they are able to show that if one begins with a weakly mixing flows then one obtains a prime orbit theorem, that is the number of prime orbits of length less then a given $L > 0$ are asymptotically e^{hL}/hL where h is the topological entropy. Recall that

Definition 2.3.1. Let X be a nonempty compact metric space and T a continuous map. A set $A \subseteq X$ is said to be (n,ϵ)-separated, if for all $x, y \in A$ with $x \neq y$ the $d(T^i x, T^i y) \geq \epsilon$ for some $i < n$. Let $\omega(n,\epsilon)$ be the maximal cardinality of a (n,ϵ)-separated set in X. The topological entropy is defined as

$$h_{\text{top}}(T) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log(\omega(n,\epsilon))}{n}.$$

For a discussion on equivalent definitions of topological entropy see Bowen [11]. Note that this result is both a refinement (here the constant is exactly the topological entropy) and a generalization of that of Margulis cited before. Moreover note that with the result above allows one to have another equivalent definition of topological entropy as exactly the constant which satisfies that inequality. This feature can, as matter of fact, be exploited in the context of zeta functions to actually compute the topological entropy.

One can also find a more precise result about spatial distribution. In fact one finds that given a test function f, and with respect to the measure of maximal entropy (for the definition and the relationship with topological entropy see [44] and references therein) then we have for $x \to \infty$

$$\sum_{\lambda(\tau) \leq x} \int_\tau f dm \sim \frac{e^{hx}}{h} \int f dm \tag{2.3.1}$$

One more example of ζ_{Ruelle} which has an essential singularity around $z = 0$ was given by Pollicott in [57]. He constructed a flow starting from a full two symbols shift by choosing an opportune roof function based on

$$f(x) = \begin{cases} -\log p & \text{if } x = 1 \\ -\log q & \text{if } x = 2 \end{cases}$$

For such system He computed explicitly that ζ_{Ruelle} has poles at $p^z + q^z = 1$. Following the study of the zeroes and of the poles of the zeta functions one would like to highlight the interconnection between such entities and the correlation spectrum.

In the same year Ruelle [66] constructed a flow which is mixing but not exponentially mixing. He did so by constructing a suspension flow starting from an Anosov diffeomorphism encoded by symbolic dynamic as usual. The key ingredient is to choose as a roof function for the suspension two different values for the two symbols. That, in turns, gives only a mixing property. The zeta function for such a system is computed and its zeroes and poles satisfy the equation $e^{\lambda_0 z} + e^{\lambda_1 z} = 1$ where λ_0, λ_1 are the values chosen for the roof function.

Settled the fact that not all hyperbolic flows mix exponentially fast, that is, in our language, that there can be a pole arbitrarily close to 0 one still aims to find instances of such behaviour. Recall that ϕ_t topologically weak mixing if restricted to a basic set Λ there is $a > 0$ and a non-trivial function f such that $f(\phi_t(x)) = e^{ia} f(x)$. In this sense we have the following theorems

Theorem 2.3.2 (Pollicott [58])**.** *Given a weak-mixing Anosov flow, if ζ_{Ruelle} has an analytic extension to a domain $\Re(z) > h - \varepsilon$ except for a pole at $z = h$ then $\rho(t) \to 0$ exponentially fast for every Hölder continuous function.*

Moreover we obtain that if ϕ_t is topologically weak mixing then there are no other poles on the line $\Re(s) = 1$. On the other hand if ϕ_t is not topologically weak mixing then it has poles at $s = 1 + iak$, $k \in \mathbb{Z}$ and ζ_{Ruelle} has a meromorphic extension to $\mathbb{C}$ with periodicity $s = (s + iak)$, for $k \in \mathbb{Z}$.

Here we recall that, roughly speaking, an Axiom A flow is a flow such that the nonwandering set $\Omega(\phi)$ is hyperbolic and the periodic orbits of ϕ are dense in $\Omega(\phi)$.

Theorem 2.3.3 (Pollicott [59])**.** *Let ϕ_t be an Axiom A flow of topological entropy h and contraction coefficient λ. Then ζ_{Ruelle} is meromorphic on the half plane $\Re(s) > h - C$ for an explicit constant $C = C(h, \lambda)$. Moreover ζ_{Ruelle} has a simple zero at $s = 1$ and has zeros and poles determined by a family of transfer operators.*

The proof of this result follows a scheme which was well established at the time. Pollicott showed that the spectrum of $\mathcal{L}_t$ is quasi-compact on the space of Hölder continuous functions, then constructed a suspended flow and concluded encoding an Axiom A through symbolic dynamics. Again in the monograph [53] all the details are nicely included. At the same time Ruelle kept studying the poles of the correlation function, and called the poles of the Fourier transform resonances. To make it clearer we have to give some definition. Given C^{+1} flow we can define the correlation function for two observables f, g in some reasonable class as

$$\rho_t(x) = \int_0^\infty f(\phi_t(x))g(x)d(\mu) \tag{2.3.2}$$

Then one can define its Fourier Transform to be

$$\widehat{\rho}(\omega) = \int_{-\infty}^{+\infty} e^{i\omega t}\rho(t)dt \tag{2.3.3}$$

He showed in [67] that $\hat{\rho}$ is meromorphic in a strip $|\Im\omega| < \delta$. The poles of $\hat{\rho}$ are called resonances and their residues can be understood as some special Gibbs distribution. From now on we will talk about mixing properties in the following sense. If $\rho_t(x) \to 0$ for $t \to \infty$ we say that the flow is mixing. on the other hand if for $\alpha \in (0, 1)$ exists $C_\alpha > 0$ and $\sigma_\alpha > 0$ such that $|\rho_t(x)| \leq \|f\|_\alpha \|g\|_\alpha e^{-\sigma_\alpha t}$ we say that the flow is exponentially mixing (again, coherently with the definition of the transfer operator one has to find some reasonable norm $\| \cdot \|_\alpha$).

In 1986 Fried published two papers. He relaxed a little the requirement on analyticity and obtained the following theorem.

Theorem 2.3.4 (Fried [27])**.** *Let ϕ_t be a real analytic Anosov flow on a real analytic manifold such that the unstable manifolds form a real-analytic foliation. Then ζ_{Ruelle} extends meromorphically to the whole complex plane.*

This result was used in the same year to connect geodesic flows and torsion. He shows ([26]) that the value of $\zeta_{\text{Ruelle}}(0)$ is equivalent to the value of torsion (either Reidemeister or Ray-Singer, since they are known to be equivalent) for a closed oriented hyperbolic manifold.

In the same year Tangerman [80] showed by using heat kernel that if one begins with $\sigma : M \to M$ a continuous expanding map and construct an opportune semiflow with cross-section M and return time $r(x)$, one finds that

Theorem 2.3.5. *If (M, σ, r) are of class C^k with a k large with respect to the dimension of M then ζ_{Ruelle} is meromorphic in*

$$D(\zeta_{Ruelle}) = \left\{ s | e^P(s) \leq \frac{\delta^{k/3d}}{\deg \sigma} \right\}$$

where λ is the expansion coefficient of σ and P is the pressure.

Note that the requirement on regularity steams from the fact that the foliations must be C^k. Tangerman also uses Ruelle's approach through forms by constructing suitable exteriors operators and shows that these operator, once normalized, can be recollected through a product formula.

In [60] Pollicott improves his previous results, here Theorem 2.3.3, and shows that the extension is analytical and not just meromorphic in the same region, i.e. $\Re(z) > h - \delta$. This is obtained by showing that $\mathcal{L}_t$ does not allow a sequence of poles which accumulates near the eigenvalue one but distinct of one.

In the case of expanding maps, which trivially extends to expanding semi-flows, in [68] Ruelle improved Tangerman estimates by introducing functions close in spirit to Fredholm determinant (but with finite radius of convergence). With this approach Ruelle shows that if one starts with $f, r \in C^{(k,\alpha)}$[1] where $f : M \to M$ is an expanding map and $r : M \to \mathbb{R}^+$ then one can construct a semiflow by obviously suspending f with respect to r. Ruelle's result says that zeta function associated to the semiflow admits a meromorphic extension to the half-plane $\Re(s) > \eta$ where η is smaller than the topological entropy and is the unique number such that $P(f, -\eta \cdot r) = \log \theta^{-(k+\alpha)}$ where P is the topological pressure and θ is the expansion coefficient. In the same year, it was shown by Baladi [5] that the results of Ruelle are optimal in the expanding case.

While some authors focused on the relationship between ζ_{Ruelle} and the physical properties of the system, some other tried to understand better its algebraic structure.

Sarnak [71] and Voros [83] in two independent papers, showed that how one can decompose the ζ_{Selberg} into a product formula over its zeroes, as it is usually done

[1] $C^{(k,\alpha)}$ being the class of function k-times differentiable such that the k-th derivative has Hölder exponent α.

in number theory with the Hadamard product for the ζ_{Riemann}, provided that we consider the spectrum of the Laplacian on a surface of constant negative curvature. In the development of such formulas the main ingredients are the realization of a functional determinant, which encodes the spectrum of the Laplacian, and the employment of Barnes double gamma function.

These results add to the expected properties of ζ_{Selberg}, and one might argue that the study of the regularity of the factors of ζ_{Selberg} might be easier. Moreover they might provide insight in how to generalize a Selberg trace in the case of non-constant curvature.

In the next year Pollicott, following Parry, slightly modified the definition of ζ_{Ruelle} and studied the properties of

$$\zeta(s) = \prod_{\tau \in \mathcal{T}_p} 1 - e^{-s\lambda^u(\tau)}$$

where one replaces the least period with the expansion along the unstable manifold around the flow. He obtains that

Theorem 2.3.6 (Pollicott [61])**.** *Let $\phi_t : \Lambda \to \Lambda$ be a smooth Axiom A flow restricted to an attractor Λ, for which the unstable bundle is one-dimensional; then the zeta function above has a meromorphic extension to the entire complex plane.*

One finds an interesting application of zeta functions in a joint paper by Katok, Knieper, Pollicott and Weiss ([40]). There ζ_{Ruelle} it is used to show that if one has a real analytic Anosov flow and a real analytic perturbation of it, then also the topological entropy varies in a real analytic sense. This is achieved by carefully studying the dependence of the poles of ζ_{Ruelle} under the action of an external parameter. Next we need to recall the following definitions. On a real analytic manifold for a real analytic Anosov flow let $E^u(x) = \lim_{t\to 0} \frac{1}{t} \log \operatorname{Jac}(D_x\phi_t|_{E^u_x})$ be the expansion coefficient. Let μ be the SRB measure i.e. the unique ϕ_t-invariant probability measure μ such that there is a set of positive Lebesgue measure such that for every continuous observable ψ one has

$$\mu(\psi) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \psi(\phi_t(x))dt$$

(see [88] for an introduction to SRB measures and reference therein). Then $h(\phi, \mu) = \int E^u(x)d\mu(x)$ is the metric entropy. In this setup, Pollicott showed in [62] following the same scheme of studying dependence from an external parameter, that the metric entropy varies in a real analytic manner.

On a different direction, Mayer develops further the algebraic side of the problem, in particular we find that given the Gauss map $G(x) = x^{-1} \mod 1$ on the unit interval and the geodesic flow of a modular surface i.e. the surface of constant negative curvature constructed from a modular group one has the following striking theorem

Theorem 2.3.7 (Mayer [47]). *$\zeta_{Selberg}$ for the geodesic flow related to the modular group $PSL(2,\mathbb{Z})$ can be written as $\zeta_{Selberg}(z) = \det(1 - L_z)\det(1 + L_z)$ with L_z the transfer operator of the Gauss map. Moreover $\zeta_{Selberg}$ is meromorphic in the entire complex plane with (possibly removable) singularities at the points $z_k = (1 - k)/2$, for $k \in \mathbb{N}$.*

The importance of studying generalization of zeta functions, such as L-functions it's clear in the paper of Katsuda and Sunada [41]. In fact, one construct a dynamical L-function associated to a unitary character $\chi : H_1(X,\mathbb{Z}) \to U(1)$ as

$$L(s,\chi) = \prod_{\tau_p} \left(1 - \chi([\tau_p]e^{-s\lambda(\tau_p)}\right)^{-1} \tag{2.3.4}$$

Given this definition, which ties the orbits to be in a specific homology class, Katsuda and Sunada prove an equidistribution theorem on orbits, as one is used to see with primes, by carefully studying the location of poles. First they study the poles located near the real axis by the means of perturbation theory, then they study the location of zeroes or poles on and near the line of topological entropy. In this way they can estimate the number of orbits in an homology class.

Zeta functions, as said before have also been investigated in other context. One situations where they have proven to be quite successful is in the study of energy levels of billiards. The eigenenergies of such systems are often calculated through the boundary element method. In [31] Harayama and Shudo proved that one can define a zeta function of the type of ζ_{Selberg} and that the zeroes of such function encode the eigenenergies of the system. Their proof is constructive, in the sense that the zeta function is actually derived from the boundary element method. Thus, one is allowed to think of zeta functions as a good tool to numerically compute eigenenergies, since the actual value of the eigenenergies can be computed with some accuracy even by valuating a small number of orbits.

In a series of paper Ikawa ([34] ,[35] ,[36], [37] and references therein) studied the relationship between poles of the scattering matrix and zeroes of a dynamical zeta function for perturbed symbolic flows. Here the main concern is the study of periodic rays i.e. periodic trajectories which involve a number of bounces on several strictly convex bodies. Ikawa shows that one can define a suitable dynamical determinants and shows that gives rises to a "correct" zeta function which has zeroes at the poles of the scattering matrix.

Our interest in this paper also comes from the fact that, in this situation as well as in many other, the usual $\det(I - A)^{-1}$ in the expression of the dynamical determinant, thus also in any trace formula, is substituted by $\det(I - A)^{-1/2}$. This fact, which is often present in the "semiclassical" systems, is of its own interest and tell us that, in understanding dynamical zeta functions, one should not constrain oneself to study transfer operators with a unique weight, even when one restricts himself to the symbolic setting.

Next I want to mention the work of Cvitanović and Eckhardt [18]. There they look at the power spectra i.e. the resonances of the auto-correlation for generic smooth flows and show that also in such setting the zeta functions can be used to carry on such computation. In the semiclassical sense Cvitanović and Vattay ([19]) construct a family of multiplicative evolution operators for which the trace is meaningful and that can be extended meromorphically. They introduce a functional determinant which is expected to be entire for Anosov flows. In fact they computed Ruelle resonances for a family of scattering systems and found that such resonances were better located through the use of dynamical determinants then it would be with the Guitzwiller Trace formula.

They refrain from constructing an ad hoc zeta function, instead they show numerically that the zeroes of such determinant correspond to those of the Gutzwiller-Voros zeta function (as in [84]). There, roughly, one looks at a zeta function as a regularized functional determinant. For example, Voros shows that ζ_{Selberg} in the case of surfaces can be factored into the product of two functional determinants, one related to the Laplacian of the sphere and one to the Laplacian of the surface itself.

2.4 Modern age

In [69] Rugh studied a real analytic axiom A flow on a 3-dimensional real analytic manifold (along with a 2-dimensional diffeomorphism on a surface). By using Markov partitions he constructed a Fredholm determinant and showed that there is a cancellation effect so that ζ_{Selberg} is entire on the whole $\mathbb{C}$. That is if $h(s)$ is a complex valued function i.e. a weight, analytic on its hyperbolic set Λ, then one has

$$\zeta_{\text{Selberg}}{}^{-1}(s) = \exp\left(-\sum_{m\in\mathbb{N}} \frac{1}{m} \sum_{\tau\in\mathcal{T}_p} \frac{e^{mh}}{|\det\left(\mathbb{1} - D_\tau\phi_{-m\lambda(\tau)}\right)|} \right)$$

This result was later generalized to arbitrary dimension by [28] by using a mixture of techniques on negatively curve real analytic manifold and adapted Markov partition. Grouped together we find that

Theorem 2.4.1 (Rugh [69], Fried [28])**.** *Let ϕ_t be a real analytic Anosov flow on a real analytic three dimensional manifold. Then ζ_{Ruelle} extends meromorphically to the whole complex plane.*

Following the idea of exploiting analytic flows, and the results of Rugh and Fried, in [70] Morgado proves a stronger version of the theorem which relates the torsion with the zeroes of the zeta function already obtained by Fried. With this new approach he is able to get rid of the requirement of the extra regularity on the foliations.

Theorem 2.4.2 (Morgado [70] , Fried [26])**.** *Let ϕ_t be an analytic transitive Anosov flow on an orientable closed 3-manifold M. Let $\rho : \pi_1(M) \to U(n)$ be an acyclic representation. Suppose there is a periodic orbit τ such that 1 and the holonomy of τ are not eigenvalues of $\rho(\tau)$. Then the L-function is regular at the origin and $Torsion_\rho(M) = |L_{\phi,\rho}(0)|$.*

While the abstract framework of studying transfer operators for suitable symbolic dynamics through Markov partition had already been exploited at the time, zeta functions have been kept under investigation for their role in more physical framework.

Recall that given K_i disjoint compact subsets of $\mathbb{R}^3$ with smooth boundary and given $\overline{\Omega} \doteq \overline{\mathbb{R} \setminus (\cup_i K_i)}$ one can consider reflections as the usual geometrical optics. Let $M = \overline{\Omega} \times S^2$ for ϕ_t. This dispersing billiard flow is the typical model to which the result of the following papers apply.

With respect to such construction, Dahlqvist [20] approximates the zeta functions by determinants with different weights and shows that for such family the trace can be dominated by isolated zeroes or by the continuous spectra. He also notes that there is a phase transition between exponential decay and polynomial decay (reflected by the properties of the zeta functions) for different values of the largest eigenvalue of the weighted operator. Such approach seems numerically stable, he uses this approximation of the zeta functions to compute topological entropy (recall the remark after definition 2.3.1) and the method seems adaptable to compute other interesting features, such as Lyapunov exponents or rate of decay of correlations. In the same setting Petkov ([54]) studied analytic singularities of dynamical zeta functions. He showed that the properties of the semiclassical zeta function near the line of absolute convergence are similar to the properties of the ${\zeta_{\text{Riemann}}}^{-1}$ near $\Re(s) = 1$.

In this sense, Pollner and Vattay [64] begin by observing that is difficult to compute topological pressure of a dynamical system by first finding an explicit Markov partition. This problem can be avoided by recurring to transfer operators and showing that its largest eigenvalue is directly related to the topological pressure. Thus one is able to compute such pressure by summing over a reasonable number of orbits.

Recall that the topological pressure is the leading zero of

$$\zeta(z, s)^{-1} = \prod_{\tau \in \mathcal{T}_p} \left(1 - \frac{e^{z\lambda(\tau)}}{\Lambda^s}\right)$$

where Λ is the largest eigenvalue of $D\phi_t|_\tau$. Note that for $s = 0$ we recover the topological entropy, for $s = 1$ it is the escape rate for open systems and its related to the metric entropy. This approach allows them to see resonances which were hidden, if one had to use the Gutzwiller Trace formula. Thus again we see that the zeroes of zeta functions are intimately related to the properties of the system.

Wirzba and Henseler ([87]) analyze the flow of a scattering particle. and highlights relationship between zeta-functions based on semiclassical matrices and the quantum mechanical counterpart. The zeta functions can be decomposed into the product of distinct determinants, some of them collect the incoherent data of the scattering while the others take into consideration the "smooth" scattering problem. Here again one is led to approximate a determinant by an expansion through trace class operators.

In 1998, Chernov ([17]) showed in a remarkable paper, that the rate of mixing for Anosov flow is, at worst, stretched exponentially. Dolgopyat, as Chernov before him, did not translate his breakthrough papers (for example [22] [23] [24]) into the language of dynamical zeta functions. However even if Dolgopyat does not state so his results can be translated in the language of ζ_{Ruelle}. In fact his results prove that ζ_{Ruelle} is analytic in the half plane to the right of the topological pressure (except for the pole of at the topological pressure) for $C^{2+\varepsilon}$ weak-mixing Anosov flow with C^1 stable and unstable foliations. In proving decay of correlations for C^∞ weak-mixing flows, Dolgopyat obtains that the opportunely weighted zeta function is analytic (except as usual for the point of topological pressure) in a region $|\Re(z) - h_{\text{top}}| \leq |\Im(z)|^{-c}$ for some $c > 0$. That is, there is a small strip free of zeroes at the left of the half plane of convergence. Moreover, the work of Dolgopyat has been directly implemented by Pollicott and Sharp ([48], [49]) to improve the result of Margulis and Parry and Pollicott by estimating the error term, obtaining that

Theorem 2.4.3. *Let $\phi_t : M \to M$ be a weak-mixing transitive Anosov flow. Then there exists $\delta > 0$ such that*

$$\pi(L) = \frac{e^{hT}}{hT}\left(1 + \mathcal{O}\left(\frac{1}{T^\delta}\right)\right)$$

In fact the proofs of this result relies on precise estimates on the location of the poles of ζ_{Ruelle} through the means of Dolgopyat calculations.

In Naud [50], one finds that for an open billiard flow in $\mathbb{R}^3$ there is a generic Diophantine condition which grants an analytic extension of ζ on a strip to the left of entropy of width polynomially decreasing. He define two generic determinants

$$Z_D(s) = \sum_{\tau \in \mathcal{T}} (-1)^{m_\tau} \frac{\lambda(\tau_p) e^{-s\lambda(\tau)}}{|\det(I - P_\tau)|^{\frac{1}{2}}}$$

$$Z_0(s) = \sum_{m=1}^{\infty} \sum_{\tau_p \in \mathcal{T}} (-1)^{mr_\tau} \lambda(\tau_p) e^{-s\lambda(\tau_p) + \delta_t au}$$

where m_τ are the number of reflections, $r_\tau = 0$ if $\lambda(\tau_p)$ has an even number of reflections, 0 otherwise and $\delta_\tau = -\frac{1}{2}\log(e_1 e_2)$ where e_1 and e_2 are eigenvalues of P_τ. He defines a dynamical zeta close in spirit to ζ_{Ruelle} such that $Z_0(s) = -\zeta'(s)$ To show meromorphic continuation of such zeta one then introduces symbolic dynamics into the billiard flow along with an irrationality condition.

Definition 2.4.4. An irrational number $x \in \mathbb{R}$ is Diophantine if there exist $\nu > 0$ and $M > 0$ such that for all $(p, q) \in \mathbb{Z} \times \mathbb{N}*$ we have $\left|x - \frac{p}{q}\right| > \frac{M}{q^{2+\nu}}$.

Theorem 2.4.5. *Assume ϕ_t has two primitive orbits $\tau_{p,1}$ and $\tau_{p,2}$ such that $\frac{\lambda(\tau_{p,1})}{\lambda(\tau_{p,2})}$ is a Diophantine number. Then there exist $C, \rho > 0$ such that Z_D has an analytic continuation up to the domain $\{\sigma + it \in \mathbb{C} : |t| \geq 1, x_c - \frac{C}{|t|^\rho} \leq \sigma \leq x_c\}$.*

The condition on the orbits is met with ease since can be deduced if we have three obstacles such that $d_{1,2}/d_{1,3}$ is Diophantine. Here one estimate the resolvent of the transfer operator using the regularity of the Gibbs measure, following the same method of Dolgopyat ([22]). Since the resolvent acts naturally on S^1 then one uses the irrationality condition to prove that $(\xi I - \mathcal{L}_t)$ is invertible.

In the same time frame, and along the same line of reasoning, one finds Stoyanov [77] who studies billiard flows in $\mathbb{R}^3$ with a visibility condition. In such a case by using a Dolgopyat estimates they are able to prove exponential decay of correlations for Holder functions. They prove their results through the use of horocycle foliations, and requiring them to be smooth jointly non-integrable. In this framework they get a meromorphic continuation of the dynamical zeta function derived from the billiard upon $\Re(s) < h_{\text{top}} - \epsilon$.

In a series of a papers T. Harayama, A. Shudo and S. Tasaki ([79],[32]) studied zeta functions in semiclassical terms for strongly chaotic billiards. Continuing and improving the work presented in ([31]), it is possible to define a Fredholm determinant starting from the boundary element method. Here the authors are able to show that if one chooses the symbolic dynamic for the flow in a opportune manner, than it is possible to show that such Fredholm determinant agrees completely with the zeta function defined by Gutzwiller-Voros. Moreover their enquiry shows that, in some cases, there is a substantial difference between the classical and semiclassical weight in the resonances uncovered. Numerically the nice computability properties of such determinant are presented for the case of a concave completely asymmetric triangle billiard.

One more example of zeta functions which are not meromorphic come from Buzzi [16]. He studies ζ_{Ruelle} for a random map i.e. for a family of maps f_ω with $\omega \in \Omega$ where $(\Omega, \mathbb{P})$ it's a probability space. Thus one is left to study random orbits. In fact he proves even more, that ζ_{Ruelle} cannot be extended outside a disk of obvious convergence, thus it does not define poles. Moreover, Lyapunov exponents can't be determined almost surely for the series defining ζ_{Ruelle}.

2.5 State of the Art

Following the specialization along the years, at the present moment there are mainly two line of investigations, one of them regards billiards and their power spectra, the other one regards distributions and homology classes for orbits.

For what concerns a more physical point of view and billiards we start from the work of Baillif. There the role of zeta functions is highlighted in [4] where one can see that it is possible to consider, in some weak sense, the zeta function directly as a power series of orbits, as the inverse of the determinant of the kneading matrix (up to a polynomial function) by using results of Kitaev on kneading matrices.

Moreover the Casimir effect, that is the attractive or repulsive effect observed when two neutral metallic plates are pulled very close to each other, can be computed from quantum-mechanical billiard-type framework [86] hence its strength can be deduced from the appropriate zeta-functions built starting from suitable dynamical determinants as it is been done before. One in fact find a suitable trace which only highlights the physically interesting eigenvalues (around an infrared wavelength) and shows that the operator can be approximated for such a trace. They show that in principle one could apply such calculations to any length of given billiards, though good results are obtained only for medium to large separations of scatterers.

Next I want to mention that in Petkov and Stoyanov ([55]) it is possible to find an estimate on equidistribution of lengths of periodic orbits in no-eclipse billiards on the plane. No-eclipse billiards are billiards where the the convex hull of any two scatterers has empty intersection with any other scatterer. The billiard defined in this way has many nicer properties. The approach of Petkov and Stoyanov relies on the same arguments of Pollicott and Sharp, which we will discussed at a later stage.

Last, for what concerns physical billiards, Stoyanov [78] takes the ideas of Ikawa and he is able to show that on $\mathbb{R}^3$, under suitable hypothesis on the scatterers, the semiclassical dynamical zeta function shows an infinite number of poles on a strip near the real axis.

Anantharaman [1] uses a dynamical zeta to construct an asymptotic expansion of the functions which counts closed geodesics (under cohomological constraint on surfaces of negative curvature), they are able to do so by using the result of Dolgopyat and Chernov-Dolgopyat on standard pairs. The strategy there offers one of the few examples where the properties of zeta functions are used to obtain results on the orbits rather than the other way around.

Theorem 2.5.1. *Given a manifold M such that $\dim M = 3$ and an Anosov flow on it. Suppose moreover that the characteristic foliations are of class C^1 and uniformly jointly non integrable. Then there are analytic functions c_n for $n \neq 0$ in $D \times \mathbb{R}^d \times \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have*

$$\pi(\xi, \alpha, \delta, T) = \frac{e^{TH(\xi^T) - \langle u^T | \alpha \rangle}}{T^{d/2+1}} \left(c_0(\xi^T, \delta) + \sum_{k=i}^{N} \frac{c_k(\xi^T, \alpha, \delta}{T^k} + \mathcal{O}(T^{1-n}) \right) \tag{2.5.1}$$

Note that the requirements of foliations are coherent with the Anosov alternative in the sense that if the characteristic foliations are jointly integrable then the flow is a suspension of an Anosov diffeomorphism and is not topologically mixing. The strategy of the proof it involves mainly two ingredient. First of all the regularity of the foliations cannot be thrown away, and one has to ensure the regularity of the partitions as Dolgopyat does. Then a zeta function is introduced naturally as the inverse of the Laplace transform both for $\lambda(\tau) - T$ and for the homology class of τ. Then the transfer operator is defined and it is shown that suitably $|Z - \sum_k L_k| \leq e^{\theta}$. Hence by proving properties of the transfer operator one can then estimate the number of orbits and then obtain the required results.

The first paper on Anosov flows where one finds exponential decay of correlation, without requiring the extraregularity of foliations, restricts the attention to contact Anosov flow. This result can be found in Liverani ([42]). He builds on the work of

Dolgopyat, and uses techniques which exploit the existence of invariant dynamical cones, so he is able to study the transfer operator without passing to the usual Bowen symbolic coding. Thus again with respect to zeta functions we obtain a small strip free of zeroes at the left of the half plane of convergence. This approach has been used again by the same author ([15]) where for generic Anosov flows one can bound the essential spectrum of the transfer operator according to the regularity of the flow, shrinking it to zero for smooth flows.

One more application of the work of Dolgopyat is given by Pollicott and Sharp [63]. Here again the cancellation of oscillatory integrals plays a remarkable role in proving estimates concerning the distribution on the lengths of closed geodesic on a compact surface of negative curvature. In fact if one consider the fundamental group of the surface and its generators, one can write for each closed geodesic an element of the fundamental group conjugated to it. In their paper they prove that given two geodesic which are close in the fundamental group (that is they both have a conjugated element which is the product of the same number of generators) then there is an equidistribution theorem with respect to their difference in length.

One important result is that contained in [2] where the authors are able for a compact hyperbolic surface to show that given eingenfunctions of the Laplacian there is relation between Wigner distributions and Patterson-Sullivan distributions i.e. residues of weighted dynamical zeta functions. In fact they show that such distributions are asymptotically the same, in the region where they are both defined, on any line parallel to the imaginary axis. Note that from this viewpoint it seems that invariance under the reasonable wave group can be interpreted as invariance under the geodesic flow.

Going back to directly studying the transfer operator I would like to mention two works by Tsujii([81], [82]). In the first one he considers suspension semi-flows of angle-multiplying maps and in the latter contact Anosov Flows. In both framework he is able to construct suitable anisotropic Sobolev spaces which for which one can find optimal bounds for the essential spectral radius of the transfer operator.

Last from the theoretical point of view I want to mention the recent work of Naud ([51]). Here the fundamental theorem says that one can find infinitely many resonances for $(\hat{\rho})$ in a strip $\{-2h - \epsilon \leq \Re(z) \leq 0\}$ for real analytic suspension semiflows over uniformly expanding real-analytic map of the interval. Moreover one has that $(\hat{\rho})$ extends meromorphically to the whole complex plane.

I don't think that at this point I need to give any other motivation to persuade the reader of the general interest of such questions. Nevertheless I would like to conclude this survey by the following "extraordinary path". Alain Connes in "Non-commutative Geometry and the Riemann zeta function" suggests that one could use a suitable transfer operator to study the action of an opportune "Riemann flow". In fact, in his context he is able to define a suitable trace, similar to what we used so far which coherently relies on what we called dynamical determinant. Next he guesses that if we could find a replacement for the standard Selberg trace formula for such Riemann flow, we could probably be on the right track to reformulate the Riemann hypothesis in dynamical terms.

Chapter 3

Anosov flows and Dynamical Zeta Functions

3.1 Introduction

In the theory of dynamical zeta functions Selberg defined for a surface of constant curvature $\kappa = -1$ the following zeta function

$$\zeta_{\text{Selberg}}(z) = \prod_{\gamma} \prod_{n=0}^{\infty} \left(1 - e^{-(z+n)l(\gamma)}\right) \ , z \in \mathbb{C} \tag{3.1.1}$$

where we denote by $l(\gamma)$ the length of a closed geodesic γ. After noticing that it converges to a non-zero analytic function on the half-plane $\mathrm{Re}(z) > 1$, he showed that ζ_{Selberg} has an analytic extension to the entire complex plane, using the trace formula which bears his name [72]. He showed that the zeros of ζ_{Selberg} encode the spectral properties of the surface, in the sense that they correspond to eigenvalues of the Laplacian ($\lambda_n \in \mathrm{Sp}(-\Delta)$) and thus can be extracted from the information provided by the geodesics, their lengths and their distribution. While this formulation of a dynamical zeta function, which arose in the context of a surface of constant curvature, can be generalized, there are few results, since Selberg's methods are not directly exploitable, partly due to the lack of a suitable trace formula.

In 1976, Ruelle [65] proposed a dynamical version, in which the Selberg closed geodesics were replaced by the closed orbits of an Anosov flow $\phi_t : M \to M$, where M is a C^∞, d-dimensional compact manifold. We recall that an Anosov flow is a flow such that there exists a $D\phi_t$-invariant continuous splitting $TM = E^0 \oplus E^s \oplus E^u$ and a constant $\lambda > 0$, such that for $t \geq 0$, E^0 is the one-dimensional subspace tangent to the flow and[1]

$$\begin{aligned} \|D\phi_t(v)\| &\leq \|v\|e^{-\lambda t} && \text{if } t \geq 0, v \in E^s \\ \|D\phi_{-t}(v)\| &\leq \|v\|e^{-\lambda t} && \text{if } t \geq 0, v \in E^u. \\ \|D\phi_t(v)\| &= \|v\| && \text{if } t \in \mathbb{R}, v \in E^c \end{aligned} \tag{3.1.2}$$

[1]The usual definition allows for a constant C on the right hand side of the equations, yet one can find a Riemannian metric in which $C = 1$ by the Mather's construction [46, 65].

For such a flow the zeta function took the form

$$\zeta_{\text{Ruelle}}(z) = \prod_{\tau \in \mathcal{T}_p} \left(1 - e^{-z\lambda(\tau)}\right)^{-1}, z \in \mathbb{C} \tag{3.1.3}$$

where $\mathcal{T}_p$ denotes the set of prime[2] orbits and $\lambda(\tau)$ denotes the period (length) a closed orbit τ. Note that ζ_{Ruelle} converges to a well defined non-zero analytic function for $\Re(z) > h_{\text{top}}$, where h_{top} denotes the topological entropy as in [13]. It is easy to see that we can relate the Ruelle and Selberg zeta functions by

$$\zeta_{\text{Ruelle}}(z) = \zeta_{\text{Selberg}}(z+1)/\zeta_{\text{Selberg}}(z)$$

when they are both defined. The basic example being geodesic flows on manifold of constant curvature, which are special cases of mixing Anosov flows. Note that, whenever they are both defined, it is possible to reconstruct Selberg's zeta function from Ruelle's through the identity

$$\zeta_{\text{Selberg}}(z) = \prod_{i=0}^{\infty} \zeta_{\text{Ruelle}}(z+i)^{-1}.$$

Explicitly, if $\Re(z) > -(n - h_{\text{top}})$ then

$$\zeta_{\text{Selberg}}(z) = \zeta_{\text{Selberg}}(z+n) \prod_{i=0}^{n-1} \zeta_{\text{Ruelle}}(z+i)^{-1}$$

so that regularity results for one translate to the other. Moreover, we can write ζ_{Selberg} in terms of a sum

$$\zeta_{\text{Selberg}}(z) = \exp\left(-\sum_{\tau \in \mathcal{T}} \sum_{m=1}^{\infty} \frac{1}{m} \frac{e^{-zm\lambda(\tau)}}{1 - e^{-m\lambda(\tau)}}\right).$$

However, Ruelle's definition of a dynamical zeta function is more attractive to us, since it resembles closer the Riemann zeta function

$$\zeta_{\text{Riemann}}(z) = \prod_{p \text{ prime}} (1 - p^{-z})^{-1}$$

and allows us to expect properties shared by most zeta functions, such as extension to the whole $\mathbb{C}$, existence of functional equations and tight localization of zeros. To begin such project for ζ_{Ruelle} we have that

$$\begin{aligned} \zeta_{\text{Ruelle}}(z) &= \prod_{\tau \in \mathcal{T}_p} (1 - e^{-z\lambda(\tau)})^{-1} = \exp\left(\sum_{\tau \in \mathcal{T}_p} \sum_{m=1}^{\infty} \frac{1}{m} e^{-zm\lambda(\tau)}\right) \\ &= \exp\left(\sum_{\tau \in \mathcal{T}} \frac{1}{\mu(\tau)} e^{-z\lambda(\tau)}\right), \end{aligned} \tag{3.1.4}$$

[2] An orbit τ is a closed curve parametrized with respect to the arch length i.e. $\tau : [0, \infty) \to M$. A prime orbit τ_p is the restriction such that $\tau_p \doteq \tau|_{[0,\lambda(\tau)]}$ is one-to-one with its image. The support of τ is indicated by $\text{supp}(\tau)$ and its the image of τ.

where $\mu(\tau)$ is the multiplicity of the associated orbit τ (as in Ruelle [65] and Bowen [13]) and $\mathcal{T}$ is the whole set of periodic orbits on M.

In the very special case of Anosov flows with real analytic stable and unstable foliations, Ruelle already showed that his zeta function has a meromorphic extension to $\mathbb{C}$; this result was generalized by Fried ([27],[28]) still assuming strong regularity conditions on the foliations, particularly analyticity of unstable foliations.

The structure of the paper is as follow: in section 3.2 we will develop the proof of the main statement assuming few lemmas which will be proved later; in section 3.3, "Cones and Banach spaces", we construct the spaces on which our operators we will act; in section 3.4, "Transfer operators and Resolvents", we prove our estimates on the operator; in section 3.5 "Extending the determinants" we produce a relationship between our operator and a sum over the orbits of the considered flow; last, in section 3.6, "Linearity of extended determinants" we show that our operators behave, with respect to the standard trace, like linear operators.

3.2 Statement of Results

Our main result is as follow:

Theorem 3.2.1. *For any $\mathcal{C}^r$ Anosov flow ϕ_t with $r \geq 2$, the zeta function $\zeta_{Ruelle}(z)$ is meromorphic in a region*

$$\Re(z) > d_u \ln(\|D\phi_1\|_\infty) - \frac{\ln(\lambda)}{2}(r-1)$$

where λ is the coefficient of the Anosov splitting and $d_u \leq d$ is an integer.

Note that d_u is given precisely by equation (3.3.7) in the next section. Moreover it is well known that $\zeta_{\text{Ruelle}}(z)$ is analytic and non zero for $\Re(z) \geq h(\phi)$ apart for a single pole at $z = h(\phi)$, [53, Page 143].[3] Hence we have the following corollaries

Corollary 3.2.2. *For any C^∞ Anosov flow the zeta function $\zeta_{Ruelle}(z)$ is meromorphic in the entire complex plane, moreover it is analytic and non zero for $\Re(z) \geq h(\phi)$ apart for a single pole at $z = h(\phi)$.*

Corollary 3.2.3. *$\zeta_{Ruelle}(z)$ and $\zeta_{Selberg}(z)$ are meromorphic in the entire complex plane for any smooth flow on a compact manifold with variable strictly negative sectional curvatures.*

In our proof we will use *dynamical determinants*, as it has been done already starting from Ruelle [65], which arise naturally in the dynamical context and are formally of the general form

$$d_\ell(z) = \exp\left(-\sum_{\tau\in\mathcal{T}} \frac{1}{\mu(\tau)} \frac{\operatorname{tr}\left(\wedge^\ell(D_{\text{hyp}}\phi_{\lambda(\tau)})\right) e^{-z\lambda(\tau)}}{\det\left(\mathbb{1} - D_{\text{hyp}}\phi_{\lambda(\tau)}\right)}\right). \tag{3.2.1}$$

[3]By $h(\phi)$ we mean the topological entropy of the flow.

The above quantity is well defined if $\Re(z)$ is large enough where the symbol $D_{\text{hyp}}\phi_{\lambda(\tau)}$ indicates, in some orthonormal base, a $(d-1)$ dimensional matrix associated to the flow on a local transverse section to the orbit τ at the time $\lambda(\tau)$ (see equation (3.4.16) for a precise definition and properties). That is we are considering Poincaré maps and we will show that the quantities we are interested in depend only on the orbit. We denote by $\wedge^\ell A$ the matrix associated to the action of A on the standard ℓ-th exterior product.

As a direct consequence of the linear algebra identity, for $n \times n$ matrices

$$\det(\mathbb{1} - A) = \sum_{\ell=0}^{n} (-1)^\ell \operatorname{tr}(\wedge^\ell A) \tag{3.2.2}$$

(see [85] for more details) one obtains from (3.1.4), (3.2.1) and (3.2.2) a product formula à la Atiyah-Bott

$$\zeta_{\text{Ruelle}}(z) = \prod_{\ell=0}^{(\dim M)-1} d_\ell(z)^{(-1)^{\ell+1}}. \tag{3.2.3}$$

Thus Theorem 3.2.1 follows by the analogous statement on the dynamical determinants $d_\ell(z)$. To prove that $d_\ell(z)$ is meromorphic we will proceed in the following roundabout manner. First of all we define the following more general object.

Definition 3.2.4. For w sufficiently small, let

$$\widetilde{d}_\ell(w, z) \doteq \exp\left(-\sum_{n=1}^{\infty} \frac{w^n}{n!} \sum_{\tau \in \mathcal{T}} \frac{1}{\mu(\tau)} \frac{\operatorname{tr}\left(\wedge^\ell (D_{\text{hyp}}\phi_{\lambda(\tau)})\right) \lambda(\tau)^n e^{-z\lambda(\tau)}}{\det\left(\mathbb{1} - D_{\text{hyp}}\phi_{\lambda(\tau)}\right)} \right) \tag{3.2.4}$$

which converges trivially for $|w|$ sufficiently small and $\Re(z)$ sufficiently large. Then we establish a relation between $\widetilde{d}_\ell(w, z)$ and $d_\ell(z)$ by the following lemma

Lemma 3.2.5. *Let $0 \le \ell \le d-1, \xi, z \in \mathbb{C}$, $\Re(z)$ sufficiently large and $|\xi - z|$ sufficiently small. Then we can write*

$$\tilde{d}_\ell(\xi - z, \xi) = \frac{d_\ell(z)}{d_\ell(\xi)}. \tag{3.2.5}$$

Proof. By a direct calculation we can see that

$$\begin{aligned}
\tilde{d}_\ell(\xi - z, \xi) &= \exp\left(-\sum_{n=1}^{\infty} \frac{(\xi - z)^n}{n!} \sum_{\tau \in \mathcal{T}} \frac{1}{\mu(\tau)} \frac{\operatorname{tr}\left(\wedge^\ell (D_{\text{hyp}}\phi_{\lambda(\tau)})\right) \lambda(\tau)^n e^{-\xi\lambda(\tau)}}{\det\left(\mathbb{1} - D_{\text{hyp}}\phi_{\lambda(\tau)}\right)} \right) \\
&= \exp\left(-\sum_{\tau \in \mathcal{T}} \frac{1}{\mu(\tau)} \frac{\operatorname{tr}\left(\wedge^\ell (D_{\text{hyp}}\phi_{\lambda(\tau)})\right)}{\det\left(\mathbb{1} - D_{\text{hyp}}\phi_{\lambda(\tau)}\right)} \left(e^{-z\lambda(\tau)} - e^{-\xi\lambda(\tau)}\right) \right) = \frac{d_\ell(z)}{d_\ell(\xi)}
\end{aligned}$$

since the sum runs from $n = 1$. □

Hence if all the $\tilde{d}_\ell(\xi_0 - z, \xi_0)$ are meromorphic in z for some ξ_0, then the meromorphic extension of ζ_{Ruelle} is given by the product formula (3.2.3). More precisely, Theorem 3.2.1 follows from the following.

Proposition 3.2.6. *For any $\mathcal{C}^r$ Anosov flow, with $r \geq 2$, the function $\tilde{d}_\ell(w, z)$ is meromorphic, for $\Re(z)$ sufficiently large, in a region*

$$\Re(w) > d_u \ln(\|D\phi_1\|_\infty) - \frac{\ln(\lambda)}{2}(r-1)$$

with λ, d_u as in Theorem 3.2.1.

The rest of the paper is devoted to the proof of such a proposition.

Note that Ruelle's original definitions and proofs are based on a suitable determinant and its properties are studied through Markov partitions and Grothendieck theory, hence the requirement of analyticity.

Here instead, to deal with the finite smoothness case, we prove Proposition 3.2.6 using the methods of extending determinants through the choice of suitable Banach spaces, based on the approach of Liverani-Gouezel [30], Butterley-Liverani [15] and Liverani-Tsujii [43]. This approach allows us to apply transfer operator methods directly to the manifold M by resorting to currents, avoiding the obstructions of other methods arising from studying the regularity of foliations or the differentiability of the potential associated to the measure of maximal entropy.

We begin by constructing a family of spaces $\mathcal{B}^{p,q,\ell}$ as the closure of a proper subspace $\Omega^\ell_{0,s}(M) \subset \Omega^\ell_s(M)$ with respect to a suitable anisotropic norm so that the spaces $\mathcal{B}^{p,q,\ell}$ are an extension of the spaces in [30]. We start from an opportune cone structure considering an equivalence relation on TM with respect to a preferred direction, which will later on play the role of the flow direction. Here $\Omega^\ell_s(M)$ is the space of ℓ-forms on M i.e. the $\mathcal{C}^s$ sections of $\Lambda^l(T^*M)$ and $\Omega^\ell_{0,s}(M)$ is a subspace which only considers forms null with respect to our preferred direction.

In the next section we prove that in such spaces we can define a family of operators for $h \in \Omega^\ell 0, s(M)$ as

$$\mathcal{L}_t^{(\ell)}(h) = \phi^*_{-t} h$$

where $\phi^*_{-t}h$ is the pull-back of h with respect to ϕ_{-t}. This generalize the action of the transfer operator $\mathcal{L}_t$ on the anisotropic Banach spaces $\mathcal{B}^{p,q}$ of [30].

We prove that for each ℓ the operators $\mathcal{L}_t^{(\ell)}$ form a semigroup. Note that in this way we mimic the action of standard transfer operators on sections transverse to the flow, in fact we morally project our forms on a Poincaré section. In this setting we have that $R^{(\ell)}$ satisfies

$$R^{(\ell)}(z)^n = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-zt} \mathcal{L}_t^{(\ell)} dt. \tag{3.2.6}$$

Next we evaluate a trace for an operator $A : \Lambda^\ell(TM/\sim) \to \Lambda^\ell(TM/\sim)$. First we construct a local isomorphism $\boldsymbol{i}_\psi : \Pi_\psi \mathcal{B}^{p,q,\ell} \to (\Pi_\psi \mathcal{B}^{p,q,d})^{\binom{d-1}{\ell}}$ (see lemma 3.3.4 for the details) where ψ is a partition if unity which allows the localization. Given the identification $\boldsymbol{i}_\psi$ above we can write $A_{\bar{i},\bar{j}}$ for the elements in a matrix representation of A with respect to a basis, ordered by the index $\bar{i}$, of the product space $(\Pi_\psi \mathcal{B}^{p,q})^{\binom{d-1}{\ell}}$. Let ψ_α, ψ_β suitable partitions of unity for $\mathcal{B}^{p,q,\ell}$. We define an operator $\mathrm{tr}^{(\ell)}(A)(f) : \mathcal{B}^{p,q} \to \mathcal{B}^{p,q}$ by

$$\mathrm{tr}^{(\ell)}(A)(f) \doteq \sum_{\bar{k},\alpha,\beta} \left((\boldsymbol{i}_{\psi_\beta} \Pi_{\psi_\beta}) B (\Pi_{\psi_\alpha} \boldsymbol{i}_{\psi_\alpha}^{-1}) \right)_{\bar{k},\bar{k}} \tag{3.2.7}$$

where we note that $A_{\bar{i},\bar{i}} = e_{\bar{i}}^T (A e_{\bar{i}})$ for an element of a basis and its dual. Then we can define our "flat trace" as

$$\mathrm{Trace}^{(\ell)}(A) = \lim_{\epsilon \to 0} \int_{M \times M} j_{x,\epsilon}(y) \left(\mathrm{tr}^{(\ell)}(A) \right) (j_{x,\epsilon})(y) \omega(dx) \omega(dy) \tag{3.2.8}$$

by properly choosing the family of approximations $j_{x,\epsilon}$[4]. In section 3.5 we show that for the flat trace defined above on the quotient space just constructed we have the following remarkable lemma.

Lemma 3.2.7. *For $\Re(z) > d_u \ln(\|D\phi_1\|_\infty) - \frac{\ln(\lambda)}{2}(r-1)$ and $n \in \mathbb{N}$, $\mathrm{Trace}\left(R^{(\ell)}(z)^n\right) < \infty$. In addition,*

$$\mathrm{Trace}\left(R^{(\ell)}(z)^n\right) = \frac{1}{(n-1)!} \sum_{\tau \in \mathcal{T}} \frac{1}{\mu(\tau)} \frac{\mathrm{tr}\left(\wedge^\ell \left(D_{\mathrm{hyp}} \phi_{\lambda(\tau)}\right)\right) \lambda(\tau)^n e^{-z\lambda(\tau)}}{\det\left(\mathbb{1} - D_{\mathrm{hyp}} \phi_{\lambda(\tau)}\right)}. \tag{3.2.9}$$

Hence, we have that

$$\tilde{d}_\ell\,(\xi - z, \xi) = \exp\left(- \sum_{n=1}^{\infty} \frac{(\xi - z)^n}{n} \mathrm{Trace}(R^{(\ell)}(\xi)^n) \right) \tag{3.2.10}$$

In section 3.3 and 3.4 we define precisely what we mean by

$$\sigma_\ell = \lambda_+^{\min\{d_u, d-1-\ell\}} e^{-\lambda \max\{d_s - \ell, 0\}}, \quad \sigma_{p,q} = e^{-\lambda \min\{p,q\}}.$$

In section 3.4 we prove the following lemma

Lemma 3.2.8. *For $\xi \in \mathbb{C}$, $\Re(\xi) > 0$, the operator $R^{(\ell)}(\xi)$ is quasi compact on each $B^{p,q,\ell}$, $p, q \in \mathbb{N}$. Moreover*

$$\rho_{ess}(R^{(\ell)}) \leq c(\Re(\xi) - \ln(\sigma_{p,q}) - \ln(\sigma_\ell))^{-1}$$

The result above will be used together with the following lemma from section 3.6.

[4]From now on we will abuse of the notation and drop the index ℓ from $\mathrm{Trace}^{(\ell)}(A)$ since it will always be clear from the context to which space of ℓ-forms we are referring.

Lemma 3.2.9. *There exists a finite rank operator $P_\xi^{(\ell)}$ such that*

$$\textit{Trace}(R^{(\ell)}(\xi)^n) = tr\left((P^{(\ell)}(\xi))^n\right) + \mathcal{O}\left(\left[\sigma_{p,q}\sigma_\ell \Re(\xi)^{-1}\right]^n\right) \tag{3.2.11}$$

where "tr" is the standard trace.

Once we have established these results we can prove theorem 3.2.6.

Proof of Theorem 3.2.6. Let ξ such that $a = \Re(\xi) > d_u \ln(\|D\phi_1\|_\infty) - \frac{\ln(\lambda)}{2}(r-1)$ is sufficiently large. Let $|\xi - z| < (\max_i \lambda_{i,\ell})^{-1}$, $\lambda_{i,\ell}(\xi)$ be the eigenvalues of $P^{(\ell)}(\xi)$. Let $\rho_{\ell,p,q} = \ln(\sigma_\ell) - \ln(\sigma_{p,q})$. Then $\lambda_{i,\ell} \in B\left(\xi, \rho_{\ell,p,q}^{-1}\right)$ for all i we can conclude that as long as $|\xi - z| \leq \frac{c}{(d_u \ln(\|D\phi_1\|_\infty) - \frac{\ln(\lambda)}{2}(r-1)}$ we have

$$\begin{aligned}
\tilde{d}_\ell(\xi - z, \xi) &= \exp\left(\sum_{n=1}^{\infty} \frac{(\xi - z)^n}{n} \mathrm{Trace}(R^{(\ell)}(\xi)^n)\right) = \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{(\xi - z)^n}{n}\left(\sum_{\lambda_i \in B(\xi, \rho_{\ell,p,q}^{-1})} \frac{1}{(\xi - \lambda_i)^n} + \mathcal{O}\left(\left[\rho_{\ell,p,q}\Re(\xi)^{-1}\right]^n\right)\right)\right) \\
&= \exp\left(\sum_{\lambda_i \in B(\xi, \rho_{\ell,p,q}^{-1})} \log\left(1 - \frac{\xi - z}{\xi - \lambda_i}\right) + \sum_{n=1}^{\infty} \frac{(\xi - z)^n}{n}\left[\rho_{\ell,p,q}\Re(\xi)^{-1}\right]^n)\right) \\
&= \left(\prod_{\lambda_i \in B(\xi, \rho_{\ell,p,q}^{-1})} \frac{\xi - \lambda_i}{z - \lambda_i}\right)\psi(\xi, z)
\end{aligned} \tag{3.2.12}$$

where $\psi(w, z)$ is analytic in $B(\xi, \rho_{\ell,p,q}^{-1})$. Now we can choose ξ where the convergence of $d_\ell(\xi)$ is granted, as noted after equation (3.2.1), and we can freely move it along a line parallel to the imaginary axis. Hence we obtain that $\tilde{d}_\ell(\xi - z, \xi)$ is meromorphic in a strip beyond the original region of convergence by using the identity (3.2.5) which concludes the proof of proposition 3.2.6. It follows that for C^∞ flows we found that d_ℓ are analytic for all $z \in \mathbb{C}$ since $B(\xi, \rho_{\ell,p,q}^{-1})$ can be arbitrarily large. ☐

3.3 Cones and Banach spaces

Our first task is to introduce appropriate Banach spaces in which the various operators we are interested in have the expected spectral properties. It is very convenient to outline a general construction of such spaces based only on an abstract cone structure.

Consider an atlas given by $\{(U_\alpha, \Theta_\alpha)\}_{\alpha=1,\dots,N}$ where $\delta > 0$ is fixed (and to be chosen later) such that

(i) $\Theta_\alpha(U_\alpha) = B(0, 4\delta)$

(ii) $\cup_\alpha \Theta_\alpha^{-1}(B(0,\delta)) = M$.

Define $\{\psi_\alpha\}_{\alpha=1,\dots,N}$ to be a smooth partition of unity induced by our atlas such that $\mathrm{supp}(\psi_\alpha) \subseteq U_\alpha$ and $\psi_\alpha|_{\Theta_\alpha^{-1}(B(0,\delta))} = 1$. Moreover we choose the charts so that there is a "preferred coordinate", in the sense that for any two charts Θ_α, Θ_β, given $\tilde{x} \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$ we have

$$\Theta_\alpha \circ \Theta_\beta^{-1}(\tilde{x} + s x_d) = \Theta_\alpha \circ \Theta_\beta^{-1}(\tilde{x}) + s x_d. \tag{3.3.1}$$

Note that the above is equivalent to saying that there exists a vector field V, $\|V(x)\|_x = 1$, such that, for all α, $(\Theta_\alpha)_*(\partial_{x_d}) = V$. In other words the Θ_α are flow box charts for the flow generated by a vector field V.

Let $l \in \{0, \cdots, d\}$ and $\Lambda^l(T^*M)$ be the algebra of the exterior forms on M. Given two l-forms $v_1 \wedge \cdots \wedge v_l, w_1 \wedge \cdots \wedge w_l \in \Lambda^l(T_x^*M)$ we define the scalar product[5]

$$\langle v_1 \wedge \cdots \wedge v_l, w_1 \wedge \cdots \wedge w_l \rangle_x = \det \begin{pmatrix} \langle v_1, w_1 \rangle_x & \dots & \langle v_1, w_l \rangle_x \\ \vdots & \ddots & \vdots \\ \langle v_l, w_1 \rangle_x & \dots & \langle v_l, w_l \rangle_x \end{pmatrix}, \tag{3.3.2}$$

which, by linearity, defines a scalar product on each $\Lambda(T_x^*M)$, $x \in M$, (see [38, Section 2] for more details).

Let $\Omega_s^l(M)$ be the space of $\mathcal{C}^s$ sections of $\Lambda^l(T^*M)$. It is helpful to introduce, in each $\Omega_s^\ell(M)$, the scalar products

$$\langle h, g \rangle_{\Omega^\ell} = \int_M \langle h(x), g(x) \rangle_x \omega(x) \tag{3.3.3}$$

where ω is the Riemannian volume form.

Let $h = \sum_\alpha \psi_\alpha h \doteq \sum_\alpha h_\alpha$ so that $h_\alpha \in \Omega_s^\ell(U_\alpha)$. Let $\bar{i} = i_1 < i_2 < \ldots < i_l$ belong to the set $\mathcal{I}_l$ of l-multiindices ordered by the standard lexicographic order. Also, define the local bases $dx_{\alpha,\bar{i}} = (\Theta_\alpha)^*(dx_{\bar{i}})$, $e_{\alpha,\bar{i}} = (\Theta_\alpha^{-1})_*(e_{\bar{i}})$ where we have introduced the notation $dx_{\bar{i}} = dx_{i_1} \wedge \cdots \wedge dx_{i_l}$, $e_{\bar{i}} = \partial_{x_{i_1}} \wedge \cdots \wedge \partial_{x_{i_l}}$.[6] Given $h_\alpha \in \Omega_s^l(U_\alpha)$ and $(\Theta_\alpha^{-1})^* h_\alpha \in \Omega_s^l(B(0, 4\delta))$ we can write them uniquely as

$$h_\alpha = \sum_{\bar{i} \in \mathcal{I}_l} h_{\alpha,\bar{i}} dx_{\alpha,\bar{i}} \quad \text{and} \quad (\Theta_\alpha^{-1})^* h_\alpha = \sum_{\bar{i} \in \mathcal{I}_l} h_{\alpha,\bar{i}}(\Theta_\alpha^{-1}) dx_{\bar{i}}. \tag{3.3.4}$$

Let us now introduce the following equivalence relation: for $v, w \in T_xM$, $v \sim w$ if and only if there exists $\lambda \in \mathbb{R}$ such that $v = w + \lambda V(x)$. Let $\ell \in \{0, \cdots, d-1\}$, let $\Omega_s^{\ell,\sim}(M)$ be the space of $\mathcal{C}^s$ sections of $\Lambda^\ell((TM/\sim)^*)$. Let $\Psi : TM \to TM/\sim$ be defined as $\Psi(u) \doteq [u]$, hence $\Psi^* : \Omega_s^{\ell,\sim}(M) \to \Omega_s^\ell(M)$

$$\Psi^*(h)(v_1, \ldots, v_\ell) \doteq h([v_1], \ldots, [v_\ell]).$$

[5] In the following we will drop the subscript x in the scalar product as this does not create any confusion.

[6] It is understood that f^* denotes as usual the pullback and f_* the pushforward. Moreover $dx_{\alpha,\bar{i}}(e_{\alpha,\bar{j}}) = \delta_{\bar{i},\bar{j}}$.

Note that $\Psi^*(h)(\dots, V, \dots) = 0$ by construction, on the other hand any section with such a property acts naturally on equivalence classes. Hence Ψ^* is an isomorphism between $\Omega_s^{\ell,\sim}(M)$ and the subspace

$$\Omega_{0,s}^{\ell}(M) = \{h \in \Omega_s^{\ell}(M) \ : \ h(\dots, V, \dots) = 0\}. \tag{3.3.5}$$

Remark 3.3.1. From now on we will identify $\Omega_{0,s}^{\ell}(M)$ and $\Omega_s^{\ell,\sim}(M)$ without further comments.

Moreover, if we consider the usual map $i_V : \Omega_s^{\ell+1}(M) \to \Omega_s^{\ell}(M)$ defined by $i_V(h)(v_1, \dots, v_\ell) = h(V, v_1, \dots, v_\ell)$ we have $i_V : \Omega_s^{\ell+1}(M) \xrightarrow{\text{su}} \Omega_{0,s}^{\ell}(M)$. Indeed, it is clear that $i_V(\Omega_s^{\ell+1}(M)) \subset \Omega_{0,s}^{\ell}(M)$, while, if $h \in \Omega_{0,s}^{\ell}(M)$, $(-1)^{d-1} i_V(h \wedge dV) = h$ where dV is any one form such that $dV(V) = 1$. In particular, i_V is an isomorphism between $\Omega_s^d(M)$ and $\Omega_{0,s}^{d-1}(M)$. For further use we set

$$\tilde{\omega} = i_V \omega, \tag{3.3.6}$$

where ω is the Riemannian volume.

Next we assume that there exists a continuous family of cones $\mathcal{C}(x) \subset T_x M$ and sufficiently small $\rho > \rho_- > 0$ such that for each $x \in U_\alpha$ we have that

$$\{(s,u) \in \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \ : \ \|u\| \le \rho_- \|s\|\} \subset (\Theta_\alpha)_* \mathcal{C}(x) \subset \{(s,u) \ : \ \|u\| \le \rho \|s\|\}, \tag{3.3.7}$$

where $d_s + d_u = \dim(M)$.

Given such data we are going to construct several Banach spaces. To do so we first define norms on smooth forms and then we take the completion of such norms to define less regular elements in the new spaces. For sufficiently large $L > 0$, $\xi \in B(0, \delta) \subseteq \mathbb{R}^{d_s}$ and $d_{uc} = d_u + 1$ let us define

$$\mathcal{F}_r \doteq \{F : B(0, 3\delta) \to \mathbb{R}^{d_{uc}} \ : \ F(0) = 0\, ; \ |F|_{\mathcal{C}^1} \le \rho\, ; \ |F|_{\mathcal{C}^r} \le L\}. \tag{3.3.8}$$

Moreover for each $F \in \mathcal{F}_r$ let $G_{x,F}(\xi) \doteq x + (\xi, F(\xi))$. Let us also define $\widetilde{\Sigma} \doteq \{G_{x,F} \ : \ x \in B(0,\delta), F \in \mathcal{F}_r\}$. For each chart, indexed by α, and $G \in \widetilde{\Sigma}$ we associate the leaf $W_{\alpha,G} = \{\Theta_\alpha^{-1} G(\xi)\}_{\xi \in B(0,2\delta)}$, which form our set of stable leaves Σ (not to be confused with stable manifolds), its reduced and enlarged version $W_{\alpha,G}^{\pm} = \{\Theta_\alpha^{-1} G(\xi)\}_{\xi \in B(0,(2\pm 1)\delta)}$.

Also, we denote by $\Gamma_0^{\ell,s}(\alpha, G)$ the $\mathcal{C}^s$ sections of the fiber bundle on $W_{\alpha,G}$, with fibers $\Lambda^\ell(T^*M)$, which vanish in a neighborhood of $\partial W_{\alpha,G}$.

Let $\mathcal{V}(\alpha, G)$ be the set of $\mathcal{C}^r$ vector fields defined in a neighborhood of $W_{\alpha,G}^+$. Write L_v for the Lie derivative along a vector field v. Next, let $d(\text{vol})_{W_{\alpha,G}}$ be the volume form induced on $W_{\alpha,G}$ by the Riemannian structure of M. Finally, for all α, for all $W_{\alpha,G}$, and $G \in \widetilde{\Sigma}$, $g \in \Gamma_0^{l,0}(\alpha, G)$, $v_1, \dots, v_p \in \mathcal{V}(\alpha, G)$ and $h \in \Omega_s^{\ell,\sim}(M)$ we define

$$\jmath_{\alpha,G,g,v_1,\dots,v_p}(h) \doteq \int_{W_{\alpha,G}} \langle g, L_{v_1} \cdots L_{v_p} h \rangle \, d\,(\text{vol})_{W_{\alpha,G}} \in \mathbb{R}. \tag{3.3.9}$$

Next, for $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, $p+q < r-1$ let

$$\mathbb{U}_{p,q,\ell} \doteq \left\{ \jmath_{\alpha,G,g,v_1,\ldots,v_p} \mid 1 \leq \alpha \leq N,\, G \in \widetilde{\Sigma},\, |g|_{\Gamma_0^{l,p+q}(\alpha,G)} \leq 1,\, |v_j|_{\mathcal{C}^{q+p}} \leq 1 \right\} \tag{3.3.10}$$

where by $|v_j|_{C^{q+p}} \leq 1$ we mean that there exists an open set $U'_\alpha \supset W^+_{\alpha,G}$ such that v_j is defined on U'_α and $|v_j|_{\mathcal{C}^{q+p}(U'_\alpha)} \leq 1$.

Finally, given $h \in \Omega^{\ell,\sim}_{p+q}(M)$, we define the following norms

$$\begin{aligned} \|h\|^-_{p,q,\ell} &:= \sup_{\jmath \in \mathbb{U}_{p,q,\ell}} \jmath(h) \qquad && \forall\, p \in \mathbb{N},\ q \in \mathbb{R}^+ \\ \|h\|_{p,q,\ell} &:= \sup_{n \leq p} \|h\|^-_{n,q,\ell} && \forall\, p \in \mathbb{N},\ q \in \mathbb{R}^+. \end{aligned} \tag{3.3.11}$$

For each $q \in \mathbb{R}^+, p \in \mathbb{N},\ \ell \in \{0,\ldots,d-1\}$ we define the spaces $\mathcal{B}^{p,q,\ell}$ to be the closure of $\Omega^{\ell,\sim}_\infty(M)$ with respect to the norm $\|\cdot\|_{p,q,\ell}$.

The above spaces are the natural extensions of the spaces defined in [30] to the case of ℓ-forms, the case that we need in the following. There the Banach space $\mathcal{B}^{p,q}$ was defined for $h \in C^r(M,\mathbb{R})$ as[7]

$$\|h\|_{\mathcal{B}^{p,q}} = \sup_{0 \leq k \leq p} \sup_{W \in \tilde{\Sigma}} \sup_{\substack{v_1,\ldots,v_k \in \mathcal{V}(W) \\ |v_i|_{C^r} \leq 1}} \sup_{\substack{\varphi \in C^q_0(W,\mathbb{R}) \\ |\varphi|_{C^{k+q}} \leq 1}} \int_W L_{v_1} \cdots L_{v_k}(h) \cdot \varphi d(\mathrm{vol})_W.$$

Yet, if we define the isomorphism $\boldsymbol{i} : \mathcal{C}^r(M) \to \Omega^{d-1}_{0,r}(M)$ by $\boldsymbol{i}(f) = f \cdot \tilde{\omega}$ one can verify that $\boldsymbol{i}$ extends to an isomorphism between the Banach space $\mathcal{B}^{p,q}$ in [30] and the present Banach space $\mathcal{B}^{p,q,d-1}$.

Remark 3.3.2. From now on, throughout this section and in the following ones as well, c will be used in multiple situations and will represent a generic nonspecific constant which could change from time to time, even within the same equation.

Note that any function $\psi \in \mathcal{C}^\infty(M)$ is a bounded multiplier on $\mathcal{B}^{p,q,\ell}$ (hence also on $\mathcal{B}^{p,q}$). More precisely we have the following lemma.

Lemma 3.3.3. *Let* $\Pi_\psi(h) \doteq \psi \cdot h$. *Then*

$$\|\Pi_\psi h\|_{\mathcal{B}^{p,q,\ell}} = \|\psi \cdot h\|_{\mathcal{B}^{p,q,\ell}} \leq c\|h\|_{\mathcal{B}^{p,q,\ell}}.$$

[7] In [30] the coordinate charts are chosen with slightly different properties. However for our purposes they are equivalent.

Proof. We have the following inequality for $\|\cdot\|_{\mathcal{B}^{1,q,\ell}}$.

$$
\begin{aligned}
&\sup_{|g|_{\Gamma_0^{\ell,q+1}(\alpha,G)}\leq 1}\ \sup_{W_{\alpha,G}} \int_{W_{\alpha,G}} \langle g, L_v(\psi\cdot h)\rangle\, d\,(\mathrm{vol})_{W_{\alpha,G}} \\
&= \sup_{|g|_{\Gamma_0^{\ell,q+1}(\alpha,G)}\leq 1}\ \sup_{W_{\alpha,G}} \Bigg(|gL_v(\psi)|_{q+1} \int_{W_{\alpha,G}} \langle \frac{g(L_v\psi)}{|g(L_v\psi)|_{q+1}}, h\rangle\, d\,(\mathrm{vol})_{W_{\alpha,G}} \\
&\qquad\qquad + |g\psi|_{q+1} \int_{W_{\alpha,G}} \langle \frac{g\psi}{|g\psi|_{q+1}}, L_v h\rangle\, d\,(\mathrm{vol})_{W_{\alpha,G}} \Bigg) \\
&\leq c\,\|h\|^-_{0,q,l} + c\,\|h\|^-_{1,q,l} \leq c\,\|h\|^-_{1,q,l}\,.
\end{aligned}
\tag{3.3.12}
$$

Then we can repeat the argument by induction with respect to the several Lie derivatives $L_{v_1},\ldots,L_{v_p}$ present in $\langle g, L_{v_1}\cdots L_{v_p}\psi_h\rangle$ thus obtaining our estimates for $\|\cdot\|_{p,q,\ell}$. □

The following structure Lemma clarifies the relations between $\mathcal{B}^{p,q,l}$ and $\mathcal{B}^{p,q}$.

Lemma 3.3.4. *There exists an isomorphism* $\boldsymbol{i}_{\psi_\alpha} : \Pi_{\psi_\alpha}(\mathcal{B}^{p,q,\ell}) \to [\Pi_{\psi_\alpha}(\mathcal{B}^{p,q})]^{\binom{d-1}{\ell}}$.

Proof. Let us consider the map $\boldsymbol{j}_\alpha : \Omega_s^{\ell,\sim}(U_\alpha) \to \mathcal{C}^\ell(U_\alpha, \mathbb{R}^{\binom{d-1}{\ell}})$. First we recall that we have an obvious bijective map s from $\{1,\ldots,\binom{d-1}{\ell}\}$ to the set $\mathcal{I}$ of lexicographic orderings of $i_1,\ldots,i_l$. Thus, given (3.3.4), we can define

$$
\boldsymbol{j}_\alpha(h_\alpha) \doteq (h_{\alpha,s(1)},\ldots,h_{\alpha,s(\binom{d-1}{\ell})})
$$

that is, colloquially, we mapped a form to the vector made of its local coefficients. Then we set $\boldsymbol{i}_{\psi_\alpha}(\Pi_{\psi_\alpha}h) \doteq \boldsymbol{j}_\alpha(\psi_\alpha h) = \boldsymbol{j}_\alpha(h_\alpha) \in [\Pi_{\psi_\alpha}(\mathcal{C}^l)]^{\binom{d-1}{\ell}}$. We define a norm in $[\Pi_{\psi_\alpha}(\mathcal{B}^{p,q})]^{\binom{d-1}{\ell}}$ as

$$
\|h_\alpha\|_{(\mathcal{B}^{p,q})^{\binom{d-1}{\ell}}} \doteq \sum_{\bar{i}\in\mathcal{I}} \|h_{\alpha,\bar{i}}\|_{\mathcal{B}^{p,q}}. \tag{3.3.13}
$$

Then for each $|g|_{\Gamma_0^{\ell,q+1}(\alpha,G)} \leq 1$ and $W_{\alpha,G}$ we have

$$
\begin{aligned}
&\int_{W_{\alpha,G}} \langle g, L_{v_1}\cdots L_{v_p} h_\alpha\rangle d\,(\mathrm{vol})_{W_{\alpha,G}} \\
&= \sum_{\bar{i}} \int_{W_{\alpha,G}} \langle g, L_{v_1}\cdots L_{v_p}(h_{\alpha,\bar{i}}dx_{\alpha,\bar{i}})\rangle d\,(\mathrm{vol})_{W_{\alpha,G}} \\
&\leq \sum_{\bar{i}} \int_{W_{\alpha,G}} \langle g, dx_{\alpha,\bar{i}}\rangle L_{v_1}\cdots L_{v_p}(h_{\alpha,\bar{i}}) + c\sum_{k=0}^{p-1} \|h_{\alpha,\bar{i}}\|_{k,q+p-k} \\
&\leq c\sum_{\bar{i}} \|h_{\alpha,\bar{i}}\|_{p,q}
\end{aligned}
$$

To prove the opposite inequality chose $g = \varphi dx_{\alpha,\bar{j}}$, $|\varphi|_{\mathcal{C}^{p+q}} \leq 1$,

$$\int_{W_{\alpha,G}} \langle g, L_{v_1} \cdots L_{v_p} h_\alpha \rangle d\,(\mathrm{vol})_{W_{\alpha,G}}$$
$$\geq \sum_{\bar{i}} \int_{W_{\alpha,G}} \varphi L_{v_1} \cdots L_{v_p}(h_{\alpha,\bar{i}}) - c \sum_{k=0}^{p-1} \|h_{\alpha,\bar{i}}\|_{k,q+p-k}.$$

Taking the sup on φ and $W_{\alpha,G}$, summing on $\bar{i}$ and using the previous upper bound yields

$$c\|h_\alpha\|^-_{p,q,\ell} \geq \sum_{i\in\mathcal{I}} \|h_{\alpha,\bar{i}}\|_{p,q} - c \sum_{k=0}^{p-1} \|h_\alpha\|^-_{k,q+p-k}.$$

Hence it follows immediately that there exists $C > 0$ such that

$$C^{-1}\|h\|_{(\mathcal{B}^{p,q})^{\binom{d-1}{\ell}}} \leq \|h\|_{\mathcal{B}^{p,q,\ell}} \leq C\|h\|_{(\mathcal{B}^{p,q})^{\binom{d-1}{\ell}}},$$

which proves the Lemma. □

The product structure of our space $\mathcal{B}^{p,q,\ell}$ will be useful later on in several situations. Hence most of the theorems obtained in [30] extend to the present situation by using this identification. For example, the compactness lemma used there can now be stated in our context.

Lemma 3.3.5. *For each $\ell \in \{0, \ldots, d-1\}$, the unit ball of $\mathcal{B}^{p,q,\ell}$ is relatively compact in $\mathcal{B}^{p-1,q+1,\ell}$.*

Proof. Given our atlas $(U_\alpha, \Theta_\alpha)$ with an induced partition of unity we have that $\boldsymbol{i}_{\psi_\alpha}\left(\mathcal{B}^{p,q,\ell}\right)$ is relatively compact in $(\Pi_{\psi_\alpha}(\mathcal{B}^{p-1,q+1}))^{\binom{d-1}{\ell}}$ by the results of [30]. Then Lemma 3.3.4 implies the result. □

Given a form h of degree ℓ, we can define a functional

$$\jmath_h(g) \doteq \langle g, h\rangle_{\Omega^\ell} \qquad \text{where} \qquad g \in \Omega^\ell_s(M) \tag{3.3.14}$$

The space of such functionals, equipped with the $*$-weak topology of $\Omega^\ell_s(M)'$, gives rise to the space of *currents* of regularity s. We indicate the space of currents of regularity s and degree ℓ by $\mathcal{E}^\ell_s$. In analogy with [30, Proposition 4.1] and [30, Lemma 2.1] we have the following

Lemma 3.3.6. *For each $\ell \in \{0, \ldots, d-1\}$, there is a canonical injection from the space $\mathcal{B}^{p,q,\ell}$ to a subspace of $\mathcal{E}^\ell_{p+q}$.*

Proof. We begin by recalling that $\jmath : \Omega^\ell_s(M) \to \mathcal{E}^\ell_{p+q}$ is such that if we foliate locally M with d-dimensional manifolds with tangent space in the constructed cones it readily follows that $|\jmath_h(g)| \leq C\|h\|_{\mathcal{B}^{0,q,\ell}}\|g\|_{\Omega^\ell_s(M)}$. Thus $\jmath$ can be extended to a continuous immersion of $\mathcal{B}^{p,q,\ell}$ in $\mathcal{E}^\ell_{p+q}$. Consider a sequence $\{h_n\} \subset \Omega^\ell_{0,s}$ that converges to h in $\mathcal{B}^{p,q,\ell}$ such that $\jmath(h) = 0$. Then, for each manifold $W_{\alpha,G} \in \widetilde{\Sigma}$ and test form $g = \bar{g}_\alpha dx_{\alpha,\bar{i}}$ we have

$$\int_{W_{\alpha,G}} \langle g, h_n\rangle = \int_{B(0,\delta)} \bar{g}_\alpha(\xi, F(\xi))h_{n,\alpha,\bar{i}}(\xi, F(\xi))JF(\xi)d\xi$$

where JF takes into account the change of variables. Now consider a smooth approximation of the δ at zero, and let $G_{\eta,F}(\xi) = (\xi, F(\xi) + \eta) = G(\xi) + (0, \eta)$. Then

$$\int_{W_{\alpha,G}} \langle g, h_n\rangle = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{d_u+1}} \int_{W_{\alpha,G_\eta}} \langle \kappa_\varepsilon(\eta)g_\eta, h_n\rangle = \lim_{\varepsilon\to 0} \jmath_{h_n}(\hat{g}_\varepsilon)$$

where $g_\eta(z) \doteq g(z - (0, \eta))$ and $\hat{g}_\varepsilon(\xi, \eta) \doteq \kappa_\varepsilon(\eta)g_\eta(G_{\eta,F}(\xi))$. By the above representation it follows

$$\begin{aligned}|\jmath_{h_n}(\hat{g}_\varepsilon) - \jmath_{h_m}(\hat{g}_\varepsilon)| &\le \int_{\mathbb{R}^{d_u+1}} d\eta\kappa_\varepsilon(\eta) \left|\int_{W_{\alpha,G_\eta}} \langle h_n - h_m, g_\eta\rangle\right| \\ &\le \|g_\eta\|_{\Gamma_0^{\ell,q}(\alpha,G_\eta)} \|h_n - h_m\|_{0,q,\ell} \\ &\le C\|g\|_{\Gamma_0^{\ell,q}(\alpha,G)} \|h_n - h_m\|_{0,q,\ell} \le C\|h_n - h_m\|_{0,q,\ell}\end{aligned}$$

Now, given the previous inequality, we can consider the limit for $n \to \infty$ i.e. $h_n \to h$ and obtain

$$\int_{W_{\alpha,G}} \langle g, h\rangle \doteq \lim_{n\to\infty} \int_{W_{\alpha,G}} \langle g, h_n\rangle = \lim_{\varepsilon\to 0} \jmath_h(\hat{g}_\varepsilon) = 0.$$

Thus by taking the sup on α, G, g we obtain $\|h\|_{0,q,\ell} = 0$. By similar computations (involving the necessary derivatives) it follows $\|h\|_{p,q,\ell} = 0$. Thus $\jmath$ is injective and we obtain the statement of the theorem. □

3.4 Transfer operators and Resolvents

Let $\phi_t : M \to M$ be a C^r flow on a C^r Riemannian d-dimensional compact manifold with $r \ge 3$. Without loss of generality we can take $|V(x)| = 1$ where $V(x)$ is the vector field generating the Anosov flow.

Remark 3.4.1. The definition of Anosov flows already implies a suitable "cone" field structure, in the sense that the Anosov splitting can be used to construct the required cone fields of the previous section. In fact without loss of generality we can assume that atlas introduced has the following extra property

$$\begin{cases} D_0\Theta_\alpha^{-1}\{(0, u, 0) \;:\; u \in \mathbb{R}^{d_u}\} = E^u(\Theta_\alpha^{-1}(0)) \\ D_0\Theta_\alpha^{-1}\{(s, 0, 0) \;:\; s \in \mathbb{R}^{d_s}\} = E^s(\Theta_\alpha^{-1}(0)) \\ \Theta_\alpha^{-1}((s, u, t)) = \phi_t\Theta_\alpha^{-1}((s, u, 0)) \end{cases} \tag{3.4.1}$$

where $d_s = \dim E^s$, $d_u = \dim E^u$ and $t \in \mathbb{R}$. Moreover, $\mathcal{C}^s(x) = \{(s, u, t) \in TM_x \;:\; \|u\| + \|t\| \le \frac{1}{2}\|s\|\}$ is an invariant cone field associated to the stable direction with the property that any vector in the cone is strictly expanded by $(\phi_{-t})_*$. By an harmless linear change of coordinates we can finally assume that $\mathcal{C}^s(x)$ satisfies the condition (3.3.7) with $\rho_- = 1$. Thus we can use section 3.3 to define a spaces $\mathcal{B}^{p,q,\ell}$ in the present context.

Let $h \in \Omega^\ell_{0,s}(M)$ as in the previous section. Let $\tilde{h} \in \Omega^{\ell+1}_s(M)$ such that $i_V(\tilde{h}) = h$, then

$$\begin{aligned} i_V(\phi_t^*\tilde{h})(v_1, \dots v_\ell) &= \phi_t^*\tilde{h}(V, v_1, \dots v_\ell) = \tilde{h}(V, (\phi_t)_*v_1, \dots, (\phi_t)_*v_\ell) \\ &= \phi_t^* i_V \tilde{h}(v_1, \dots v_\ell) = \phi_t^* h(v_1, \dots v_\ell). \end{aligned}$$

Thus $\phi_t^*\Omega^\ell_{0,s}(M) \subset \Omega^\ell_{0,s}(M)$ for all $t \in \mathbb{R}$. We can then define the operators $\mathcal{L}_t^{(\ell)} : \Omega^\ell_{0,s}(M) \to \Omega^\ell_{0,s}(M)$, $t \in \mathbb{R}_+$, by

$$\mathcal{L}_t^{(\ell)} h \doteq \phi_{-t}^* h. \tag{3.4.2}$$

Hence locally we can write[8]

$$\begin{aligned} (\mathcal{L}_t^{(\ell)}(h))_{\alpha,\bar{j}}(x) &= \psi_\alpha(x) \sum_{\beta,\bar{i}} h_{\beta,\bar{i}}(\phi_{-t}(x)) \cdot (\phi_{-t})^*(dx_{\beta,\bar{i}})(e_{\alpha,\bar{j}}(x)) \\ &= \psi_\alpha(x) \sum_{\beta,\bar{i}} h_{\beta,\bar{i}}(\phi_{-t}(x)) \; \det{(D_x\phi_{-t})^{\alpha,\beta}_{\bar{i},\bar{j}}} \end{aligned} \tag{3.4.3}$$

where $\bar{i} = (i_1, \dots i_\ell)$ with $i_1 < i_2 < \cdots < i_\ell < d$.

In the special case of a $d-1$ form, for $h \in \Omega^{d-1}_{0,s}(U_\alpha)$ we have $h = \bar{h}\tilde{\omega}$ where $\bar{h} \in C^s(M, \mathbb{R})$ and $\tilde{\omega}$ is defined in (3.3.6). Then [9]

$$\mathcal{L}_t^{(d-1)} h = \bar{h} \circ \phi_{-t} \det(D\phi_{-t})\tilde{\omega} = (\mathcal{L}_t\bar{h})\tilde{\omega}.$$

Thus we recover the standard Ruelle-Perron-Frobenius.

Analogously, we can argue in coordinates. Setting $\phi_t^{\alpha,\beta} = \Theta_\beta \circ \phi_t \circ \Theta_\alpha^{-1}$ and $x = (\tilde{x}, x_d) \in \mathbb{R}^d$ we have $\phi_t^{\alpha,\beta}(x) = (\tilde{\phi}_t^{\alpha,\beta}(\tilde{x}), r^{\alpha,\beta}(\tilde{x}) + t)$, thus

$$D\phi_t^{\alpha,\beta} = \begin{pmatrix} D_{\tilde{x}}\tilde{\phi}_t^{\alpha,\beta} & 0 \\ \nabla_{\tilde{x}} r^{\alpha,\beta} & 1 \end{pmatrix} \doteq \begin{pmatrix} (\tilde{D}_x\phi)_t^{\alpha,\beta} & 0 \\ \nabla_{\tilde{x}} r^{\alpha,\beta} & 1 \end{pmatrix}. \tag{3.4.4}$$

[8] Here $\det{(D\phi_{-t})^{\alpha,\beta}_{\bar{i},\bar{j}}}$ is the determinant of the "minor" matrix obtained by choosing the $\bar{i}$-columns and the $\bar{j}$-rows from the matrix $D\phi_{-t}$ with respect to the atlas and the partition of unity indexed by α, β.

[9] Note that $\det(D\phi_t)\omega = \phi_t^*\omega$. Hence,

$$\begin{aligned} \phi_t^*\tilde{\omega}(v_1, \dots, v_{d-1}) &= \tilde{\omega}((\phi_t)_*v_1, \dots, (\phi_t)_*v_{d-1}) = \omega((\phi_t)_*V, (\phi_t)_*v_1, \dots, (\phi_t)_*v_{d-1}) \\ &= (\phi_t^*\omega)(V, v_1, \dots v_{d-1}) = \det(D\phi_t)\omega(V, v_1, \dots v_{d-1}) \\ &= \det(D\phi_t)\tilde{\omega}(v_1, \dots, v_{d-1}). \end{aligned}$$

That is $\phi_t^*\tilde{\omega} = \det(D\phi_t)\tilde{\omega}$.

Note that in equation (3.4.3) if $j_\ell = d$, then since $i_\ell < d$, the determinant would have a zero column and hence be zero. It follows that we can restrict to multiindexes with values in the set $\{1, \dots, d-1\}$ and

$$\begin{aligned}(\mathcal{L}_t^{(\ell)}(h))_{\alpha,\bar{j}}(x) &= \psi_\alpha(x) \sum_{\beta,\bar{i}} h_{\beta,\bar{i}}(\phi_{-t}(x)) \cdot (\phi_{-t})^*(dx_{\beta,\bar{i}})(e_{\alpha,\bar{j}}(x)) \\ &= \psi_\alpha(x) \sum_{\beta,\bar{i}} h_{\beta,\bar{i}}(\phi_{-t}(x)) \, \det\left(\tilde{D}_x\phi_{-t}\right)^{\alpha,\beta}_{\bar{i},\bar{j}} .\end{aligned} \tag{3.4.5}$$

For further use let us define

$$\begin{aligned}&\lambda_+ = \|D\phi_1\|_\infty, \\ &\sigma_\ell = \lambda_+^{\min\{d_u, d-1-\ell\}} e^{-\lambda \max\{d_s - \ell, 0\}}, \\ &\sigma_{p,q} = e^{-\lambda \min\{p,q\}},\end{aligned} \tag{3.4.6}$$

where λ is the expansion rate in (3.1.2).

We begin with the following Lasota-Yorke type of result for $\mathcal{L}_t^{(\ell)}$.

Lemma 3.4.2. *For each $p+q < r-1$, $\ell \in \{0, \dots, d-1\}$ the linear operators $\mathcal{L}_t^{(\ell)}$ are bounded in the $\|\cdot\|_{p,q,\ell}$ norm. Accordingly, they can be uniquely extended to bounded operators*[10] $\mathcal{L}_t^{(\ell)} : \mathcal{B}^{p,q,\ell} \to \mathcal{B}^{p,q,\ell}$. *They satisfy*

$$\|\mathcal{L}_t^{(\ell)} h\|_{\mathcal{B}^{0,q,\ell}} \le c\sigma_\ell^t \|h\|_{\mathcal{B}^{0,q,\ell}} \tag{3.4.7}$$

$$\|\mathcal{L}_t^{(\ell)} h\|_{\mathcal{B}^{p,q,\ell}} \le c\sigma_\ell^t \|h\|_{\mathcal{B}^{p-1,q+1,\ell}} + c\sigma_\ell^t \sigma_{p,q}^t \|h\|_{\mathcal{B}^{p,q,\ell}} + c\sigma_\ell^t \|X^p h\|_{\mathcal{B}^{0,p+q,\ell}}. \tag{3.4.8}$$

In particular, $\|\mathcal{L}_t^{(\ell)}\| \le c\sigma_\ell^t$ for all $t \in \mathbb{R}_+$. Moreover, $\mathcal{L}_t^{(\ell)}$ is strongly continuous semigroup.

Proof. From the definition (3.4.2) and equation (3.4.5)[11]

$$\begin{aligned}\left\|\mathcal{L}_t^{(\ell)} h\right\|_{\mathcal{B}^{0,q,l}} \le &\sup_{W_{\alpha,G} \in \tilde{\Sigma}} \sup_{|g|_{\Gamma_0^{l,q}(\alpha,G)} \le 1} \int_{W_{\alpha,G}} \sum_{\beta,\bar{i},\bar{j}} \frac{\det\left(\tilde{D}\phi_{-t}\right)^{\alpha,\beta}_{\bar{i},\bar{j}} g_{\bar{j}}}{\det(\tilde{D}\phi_{-t})} \cdot h_{\beta,\bar{i}} \circ \phi_{-t} \\ &\times \det(\tilde{D}\phi_{-t}) \, d(\mathrm{vol})_{W_{\alpha,G}} \\ \le \sup_{\alpha,G \in \tilde{\Sigma}} &\sup_{|g|_{\Gamma_0^{l,q}(\alpha,G)} \le 1} \sum_{\beta,\bar{i}} \left\| \frac{\det\left(\tilde{D}\phi_{-t}\right)^{\alpha,\beta}_{\bar{i},\bar{j}} g_{\bar{j}}}{\det(\tilde{D}\phi_{-t})} \right\|_{\Gamma_0^{\ell,q}(\alpha,G)} \|\mathcal{L}_t h_{\beta,\bar{i}}\|_{\mathcal{B}^{0,q}}.\end{aligned} \tag{3.4.9}$$

Note that [12]

[10] That, by an harmless abuse of notation, we still designate by $\mathcal{L}_t^{(\ell)}$.

[11] In the last line we identify $h_{\beta,\bar{i}}$ with the element $h_{\beta,\bar{i}}\omega$ of $\mathcal{B}^{0,p,d}$, where ω is the Riemannian volume form.

[12] We used the fact that, for $A \in GL(d, \mathbb{R})$,

$$\det(A)_{\bar{i},\bar{j}} = (-1)^{\ell(d-\ell)} \varepsilon(\bar{i})\varepsilon(\bar{j}) \det(A) \det\left(A^{-1}\right)_{\bar{i}^c,\bar{j}^c}, \tag{3.4.10}$$

$$\det\left(\tilde{D}\phi_{-t}\right)^{\alpha,\beta}_{\bar{i},\bar{j}} \cdot \det(\tilde{D}\phi_{-t})^{-1} = (-1)^{\ell(d-\ell-1)}\varepsilon(\bar{i})\varepsilon(\bar{j})\det\left(\tilde{D}\phi_t\right)^{\alpha,\beta}_{\bar{j}^c,\bar{i}^c}.$$

Since $\left(\tilde{D}\phi_t\right)^{\alpha,\beta}_{\bar{j}^c,\bar{i}^c}$ corresponds to the action of the dynamics on the $d-1-\ell$ forms it follows that its norm is bounded by $c\sigma_\ell^t$, since σ_ℓ clearly bounds the rate of expansion of ℓ volumes. Note that $\mathcal{L}_t$ is the usual Ruelle transfer operator, the one which action on the spaces $\mathcal{B}^{p,q}$ (the same as the current $\mathcal{B}^{p,q,d-1}$) is studied in [15]. Hence, using [15, Lemma 4.1] and Lemma 3.3.4, we obtain

$$\|\mathcal{L}_t^{(\ell)}h\|_{\mathcal{B}^{0,q,\ell}} \leq c\sigma_\ell^t \sum_{\beta,\bar{i}} \|\mathcal{L}_t^{(d)}h_{\beta,\bar{i}}\|_{\mathcal{B}^{0,q,d}} \leq c\sigma_\ell^t\|h\|_{\mathcal{B}^{0,q,\ell}}.$$

For the second equation, involving p-derivatives, we consider here the case $p=1$ and the other ones can be computed by induction on p. We have that, for all $W_{\alpha,G}$ and $|g|_{\Gamma_0^{\ell,1+q}} \leq 1$,

$$\begin{aligned}
&\int_{W_{\alpha,G}} \sum_{\beta,\bar{i},\bar{j}} \left\langle g_{\alpha,\bar{j}}dx_{\alpha,\bar{j}}, L_v\left[(h_{\beta,\bar{i}}\circ\phi_{-t})\det\left(\tilde{D}\phi_{-t}\right)^{\alpha,\beta}_{\bar{i},\bar{j}} dx_{\alpha,\bar{i}}\right]\right\rangle \\
&= \pm\sum_{\beta,\bar{i},\bar{j}} \left|\left\langle g_{\alpha,\bar{j}}dx_{\alpha,\bar{j}}, L_v\left[\det\left(\tilde{D}\phi_{-t}\right)^{\alpha,\beta}_{\bar{i}^c,\bar{j}^c} dx_{\alpha,\bar{i}}\right]\right\rangle\right|_{q+1} \\
&\times \int_{W_{\alpha,G}} \frac{\left\langle g_{\alpha,\bar{j}}dx_{\alpha,\bar{j}}, L_v\left[\det\left(\tilde{D}\phi_{-t}\right)^{\alpha,\beta}_{\bar{i}^c,\bar{j}^c} dx_{\alpha,\bar{i}}\right]\right\rangle}{\left|\left\langle g_{\alpha,\bar{j}}dx_{\alpha,\bar{j}}, L_v\left[\det\left(\tilde{D}\phi_{-t}\right)^{\alpha,\beta}_{\bar{i}^c,\bar{j}^c} dx_{\alpha,\bar{i}}\right]\right\rangle\right|_{q+1}} \mathcal{L}_t h_{\beta,\bar{i}} \\
&\pm\sum_{\beta,\bar{i}} \left|\det\left(\tilde{D}\phi_{-t}\right)^{\alpha,\beta}_{\bar{i}^c,\bar{i}^c} g_{\alpha,\bar{i}}\right|_{q+1} \int_{W_{\alpha,G}} \frac{\det\left(\tilde{D}\phi_{-t}\right)^{\alpha,\beta}_{\bar{i}^c,\bar{i}^c} g_{\alpha,\bar{i}}}{\left|\det\left(\tilde{D}\phi_{-t}\right)^{\alpha,\beta}_{\bar{i}^c,\bar{i}^c} g_{\alpha,\bar{i}}\right|_{q+1}} \\
&\qquad\qquad \times L_v(\mathcal{L}_t h_{\beta,\bar{i}})\, d(\mathrm{vol})_{W_{\alpha,G}} \\
&\leq c\sigma_\ell^t\|h\|_{\mathcal{B}^{0,q+1,\ell}} + c\sigma_\ell^t\left(\sum_{\beta,\bar{i}}\left\|L_v(\mathcal{L}_t h_{\beta,\bar{i}})\right\|_{\mathcal{B}^{1,q}}\right)
\end{aligned} \tag{3.4.11}$$

---

where $\varepsilon(\bar{i})$ is the sign of the permutation that sends $\{1,\dots,d\}$ into $\{i_1,\dots,i_\ell,i_1^c,\dots,i_{d-\ell}^c\}$. Indeed, for each $\omega\in\Lambda^d(\mathbb{R}^d)$, $A^*\omega = \det(A)\omega$ and for each $u\in\Lambda^\ell(\mathbb{R}^d)$, $v\in\Lambda^{d-\ell}(\mathbb{R}^d)$

$$A^*u\wedge v = A^*\left(u\wedge(A^*)^{-1}v\right) = \det(A)\left(u\wedge(A^{-1})^*v\right).$$

Thus for $u = dx_{i_1}\wedge\ldots\wedge dx_{i_l}$ and $v = dx_{j_1}\wedge\ldots\wedge dx_{j_{d-l}}$, where $\bar{i} = (i_1,\dots,i_\ell)$ and $\bar{j} = (j_1,\dots,j_{d-\ell})$ are ordered multiindexes, we have

$$\varepsilon(\bar{j})\det(A)_{\bar{j}^c,\bar{i}} = \det(A)\det(A^{-1})_{\bar{i}^c,\bar{j}}\varepsilon(\bar{i}),$$

the results follows since $\varepsilon(\bar{j}^c) = (-1)^{\ell(d-\ell)}\varepsilon(\bar{j})$.

where in the last inequality we have used standard distortion estimates. Next, [15, Lemma 4.1] implies

$$\|L_v(\mathcal{L}_t h_{\beta,\bar{i}})\|_{\mathcal{B}^{1,q}} \leq c\sigma_{1,q}^t \sigma_\ell^t \|h\|_{\mathcal{B}^{1,q}} + c\sigma_\ell^t \|X^p h_{\beta,\bar{i}}\|_{\mathcal{B}^{0,1+q}}$$

Thus, from the equation above, Lemma 3.3.4, equation (3.3.13) , equation (3.4.11) and the induction on p, we obtain

$$\|\mathcal{L}_t^{(\ell)} h\|_{\mathcal{B}^{p,q,\ell}} \leq c\sigma_\ell^t \|h\|_{\mathcal{B}^{p-1,q+1,\ell}} + c\sigma_\ell^t \sigma_{p,q}^t \|h\|_{\mathcal{B}^{p,q,\ell}} + c\sigma_\ell^t \|X^p h\|_{\mathcal{B}^{0,p+q,\ell}}.$$

The general case follows similarly remembering Lemma 3.3.4. In particular, from the above estimates, follows $\|\mathcal{L}_t^{(\ell)}\|_{p,q,\ell} \leq c\sigma_\ell^t$, for all $t \geq 0$.

We now need to prove that operators $\mathcal{L}_t^{(\ell)}$ form a strongly continuous semigroup. It is easy to verify that $\mathcal{L}_t^{(\ell)}$ is strongly continuous on $\Omega_s^\ell(M)$. The result follows then by a standard density argument: let h_ε be an approximation of h

$$\lim_{t\to 0} \mathcal{L}_t^{(\ell)} h = \lim_{t\to 0} \mathcal{L}_t^{(\ell)} h_\varepsilon + \mathcal{O}(\|h - h_\varepsilon\|_{p,q,\ell}) = h_\varepsilon + \mathcal{O}(\|h - h_\varepsilon\|_{p,q,\ell})$$

and then considering $\varepsilon \to 0$. □

Given the two lemmas above one cannot use directly Hennion result [33] for the spectral radius of $\mathcal{L}_t^{(\ell)}$, instead we have to look at their generators $X^{(\ell)}$ and at the related resolvent. By standard results, see for example Davies [21], $X^{(\ell)}$ is a closed operator on $\mathcal{B}^{p,q,\ell}$ such that $X^{(\ell)}\mathcal{L}_t^{(\ell)} = \frac{d}{dt}\mathcal{L}_t^{(\ell)}$. Hence we can define the resolvent by $R^{(\ell)}(z) \doteq \left(z - X^{(\ell)}\right)^{-1}$, where z lies outside the spectrum of $X^{(\ell)}$. This is again a linear operator on $\mathcal{B}^{p,q,\ell}$. If we compute $X^{(\ell)}R^{(\ell)}(z)$ and $R^{(\ell)}(x)X^{(\ell)}$ we obtain the following identity

$$R^{(\ell)}(z) = \int_0^\infty e^{-zt}\mathcal{L}_t^{(\ell)} dt$$

for each $z \in \mathbb{C}$ and $\Re(z)$ sufficiently large. This expression is generalized to

Lemma 3.4.3. *For $n \in \mathbb{N}$ we can write*

$$R^{(\ell)}(z)^n = \frac{1}{(n-1)!}\int_0^\infty t^{n-1} e^{-zt}\mathcal{L}_t^{(\ell)} dt$$

In particular, for $\Re(z) > \ln \sigma_\ell$ we have that $R^{(\ell)}(z)$ is a bounded linear operator. These two facts, along with the next lemma, are easily obtained arguing as in Lemma 3.4.2 and using [15, Lemma 4.3].

Lemma 3.4.4. *Let $p, q \in \mathbb{R}$, $p + q \leq r$, $z \in \mathbb{C}$, for $a = \Re(z) > \hat{\sigma}_\ell$ we have*

$$\|R^\ell(z)^n\|_{\mathcal{B}^{0,q,\ell}} \leq c(a - \hat{\sigma}_\ell)^{-n}$$
$$\|R^\ell(z)^n h\|_{\mathcal{B}^{p,q,\ell}} \leq c(a - \hat{\sigma}_\ell - \hat{\sigma}_{p,q})^{-n}\|h\|_{\mathcal{B}^{p,q,\ell}} + c(a - \hat{\sigma}_\ell)^{-n}|z|\|h\|_{\mathcal{B}^{p-1,q+1,\ell}},$$

where $\hat{\sigma}_\ell = \ln \sigma_\ell$ and $\hat{\sigma}_{p,q} = \ln \sigma_{p,q}$.

Thus, arguing as in [42, Proposition 2.10, Corollary 2.11].

Lemma 3.4.5. *For $p+q \leq r$ the spectrum of the generator $X^{(\ell)}$ of the semigroup $\mathcal{L}_t^{(\ell)}$ acting on $\mathcal{B}^{p,q,\ell}$ lies on the left of the line $\{\hat{\sigma}_\ell + ib\}_{b\in\mathbb{R}}$ and in the strip $\hat{\sigma}_\ell \geq Re(z) > \hat{\sigma}_\ell - \lambda \min\{p,q\}$ consists of isolated eigenvalues of finite multiplicity.*

To conclude this section we define and compute $\mathrm{tr}^{(\ell)}\mathcal{L}_t^{(\ell)}$. Given a vector space V^d over $\mathbb{R}$ and a matrix representation of a linear operator $A : V^d \to V^d$, we can naturally construct, by the standard external product, a matrix representation of $\Lambda^l A : \Lambda^l(V^d) \to \Lambda^l(V^d)$ of elements $a_{\bar{i},\bar{j}}$. In this framework, we can define the following operator $\mathrm{tr}^{(\ell)} : L(\Lambda^l(V^d), \Lambda^l(V^d)) \to L(V^d, V^d)$ as

$$\mathrm{tr}^{(\ell)}(\Lambda^l A) \doteq \sum_{\bar{i}} \det(A_{\bar{i},\bar{i}}) \tag{3.4.12}$$

where, again, $A_{\bar{i},\bar{i}}$ is the minor matrix corresponding to the choice of $\bar{i}$-rows and $\bar{i}$-columns from A. This is due to the fact that $a_{\bar{i},\bar{j}} = \det(A_{\bar{i},\bar{j}})$ (for more details see [9]). Now given $B : \mathcal{B}^{p,q,\ell} \to \mathcal{B}^{p,q,\ell}$ we can extend $\mathrm{tr}^{(\ell)}$, with a slight abuse of notation, to an operator $\mathrm{tr}^{(\ell)} : L(\mathcal{B}^{p,q,\ell}, \mathcal{B}^{p,q,\ell}) \to L(\mathcal{B}^{p,q}, \mathcal{B}^{p,q})$. Let α, β be indexes of two atlas coherently with what has been done so far. Recalling Lemma 3.3.4 we have

$$\mathrm{tr}^{(\ell)}(B) = \sum_{\bar{k},\alpha,\beta} \left((\boldsymbol{i}_{\psi_\beta} \Pi_{\psi_\beta}) B (\Pi_{\psi_\alpha} \boldsymbol{i}_{\psi_\alpha}^{-1}) \right)_{\bar{k},\bar{k}} \tag{3.4.13}$$

Let $e_{\bar{k}}$ be a basis for $(\mathcal{B}^{p,q})^{\binom{d-1}{\ell}}$ over $\mathcal{B}^{p,q}$. Then by using (3.4.2) and (3.4.5) for $f \in \mathcal{B}^{p,q}$ we have that

$$\begin{aligned}
\mathrm{tr}^{(\ell)}(\mathcal{L}_t^{(\ell)}) f &= \sum_{\bar{k},\alpha,\beta} \left((\boldsymbol{i}_{\psi_\beta} \Pi_{\psi_\beta}) \mathcal{L}_t^{(\ell)} (\Pi_{\psi_\alpha} \boldsymbol{i}_{\psi_\alpha}^{-1}) \right)_{\bar{k},\bar{k}} f \\
&= \sum_{\bar{k},\alpha,\beta} \left(e_{\bar{k}}^T \left((\boldsymbol{i}_{\psi_\beta} \Pi_{\psi_\beta}) \phi_{-t}^* (\Pi_{\psi_\alpha} \boldsymbol{i}_{\psi_\alpha}^{-1}) \right) e_{\bar{k}} \right) f \\
&= \sum_{\bar{k},\alpha,\beta} (\tilde{e}_{\bar{k}})^T \left((\boldsymbol{i}_{\psi_\beta} \Pi_{\psi_\beta}) \sum_{\bar{n}} (f_\alpha \circ \phi_{-t}) d(\phi_{-t})_{\alpha,\bar{k}} (\tilde{e}_{\alpha,\bar{n}}) \right) \\
&= \sum_{\bar{k},\alpha,\beta} (f_{\alpha,\beta} \circ \phi_{-t}) \det(D\phi_{-t})_{\bar{k},\bar{k}}^{\alpha,\beta} = (f \circ \phi_{-t}) \,\mathrm{tr}(\Lambda^l(D\phi_{-t}))
\end{aligned} \tag{3.4.14}$$

Note that $\mathrm{tr}^{(\ell)}$ is invariant with respect to change of coordinates A, since one can write $\hat{e}_{\bar{k}} = A\tilde{e}_{\bar{k}}$ and linearly apply both A and A^{-1} in the previous computation. Last we compute $\mathrm{tr}(\Lambda^l(D\phi_{-t}))$ in terms of $D_{\mathrm{hyp}}\phi_{-t}$ where, locally, we have

$$D_{\mathrm{hyp}}\phi_{-t} = \begin{pmatrix} \frac{\partial_1 \phi_{-t}}{\partial x_{\alpha,1}} & \cdots & \frac{\partial_1 \phi_{-t}}{\partial x_{\alpha,d-1}} \\ \vdots & & \vdots \\ \frac{\partial_{d-1} \phi_{-t}}{\partial x_{\alpha,1}} & \cdots & \frac{\partial_{d-1} \phi_{-t}}{\partial x_{\alpha,d-1}} \end{pmatrix} \tag{3.4.15}$$

Note that locally we have chosen the one forms $dx_1, \dots, dx_d$ such that $dx_d(V(x)) = 1$, $dx_d(K(x)) = 0$ for $K(x) \neq \lambda V(x)$ and $\{dx_i(V(x)) = 0\}_{i=1,\dots,d-1}$. Thus we have

$$\begin{aligned}\operatorname{tr}(\Lambda^l(D\phi_{-t})) = \sum_{\bar{k}} \det(D\phi_{-t})_{\bar{k},\bar{k}} &= \sum_{\bar{k}} \det(D_{\text{hyp}}\phi_{-t})_{\bar{k},\bar{k}} \\ &= \operatorname{tr}(\Lambda^l(D_{\text{hyp}}\phi_{-t}))\end{aligned} \tag{3.4.16}$$

3.5 Extending the Determinants

We would like to define something akin to the trace and the determinant for operators which are not of trace class. For example note that the linear operators $A \doteq R^{(d-1)}(\xi)^{-1}R^{(d-1)}(z)$ are bounded, provided $z, \xi \in \mathbb{C}$ are not in the spectrum of $X^{(d-1)}$, but not trace class. Recall that in our case $A : \mathcal{B}^{p,q,\ell} \to \mathcal{B}^{p,q,\ell}$.

We start by choosing a suitable approximation of δ_x. For $x \in M$, $\epsilon > 0$, let $B_{M,\epsilon}(x) \doteq \{y \in M : d(x,y) < \epsilon\}$; let $j_{x,\epsilon} \in C^\infty(M, \mathbb{R}^+)$ be a family of approximations to the usual δ_x supported on $B_{M,\epsilon}(x)$ such that $\lim_{\epsilon\to0} \int_M j_{\epsilon,x}(y) f(y)\omega(dy) = f(x)$ for all $f \in C^0$, where $\omega(dx)$ stands for the Riemannian volume. We recall that our trace has been defined in Section 3.1 as

$$\operatorname{Trace}^{(\ell)}(A) = \lim_{\epsilon\to0} \int_{M^2} j_{x,\epsilon}(y) \left(\operatorname{tr}^{(\ell)}(A)\right)(j_{x,\epsilon})(y)\omega(dx)\omega(dy) \tag{3.5.1}$$

where $\operatorname{tr}^{(\ell)}(A)$ is defined by (3.4.12).

In order to make precise our definition of trace we specify an explicit family of approximate identities $j_{\epsilon,x}(y)$. Let $k_s \in C^\infty(\mathbb{R}^s, \mathbb{R}^+)$ such that $\int_{\mathbb{R}^s} k_s(\xi)d\xi = 1$ where $\operatorname{supp}(k_s) \subset \{x \in \mathbb{R}^s : \|x\| \leq r\}$ for some appropriate $r \in (0,1)$. Moreover, we define $k \in C^\infty(\mathbb{R}^d, \mathbb{R}^+)$ as $k(x_1, \dots x_d) = k_{d-1}(x_1, \dots, x_{d-1})k_1(x_d)$. Note that given $\epsilon \geq 0$, $f, h \in C^0$, one has for k_s

$$\begin{aligned}\lim_{\epsilon\to0} \epsilon^{-s} \int_{\mathbb{R}^s} k_s(\epsilon^{-1}(x + \epsilon h(x))f(x)dx &= \lim_{\epsilon\to0} \int_{\mathbb{R}^s} k_s(z + h(\epsilon z))f(\epsilon z)dz \\ &= f(0) \int_{\mathbb{R}^s} k_s(z + h(0))dz = f(0).\end{aligned} \tag{3.5.2}$$

Let $\{\Theta_\alpha, U_\alpha, \psi_\alpha\}$ be the atlas with the related partition of unity already used in Section 3.3. Recall that $\exists\, \epsilon_0 > 0$ such that if $x \in \operatorname{supp} \psi_\alpha$, then $B_{M,\epsilon_0}(x) \subset U_\alpha$ for all α. Let $\epsilon < \epsilon_0$ and define

$$j_{\epsilon,x}(y) \doteq g_\epsilon(x)^{-1} \sum_\alpha \epsilon^{-d} k\left(\epsilon^{-1}\left(\Theta_\alpha(x) - \Theta_\alpha(y)\right)\right)\psi_\alpha(x) \tag{3.5.3}$$

where g_ϵ is the normalization factor

$$g_\epsilon(x) = \sum_\alpha \int_{V_\alpha} \epsilon^{-d}\psi_\alpha(x) k\left(\epsilon^{-1}(\Theta_\alpha(x) - z)\right)\omega_\alpha(z)dz. \tag{3.5.4}$$

such that $\omega_\alpha(z)dz = (\Theta_\alpha^{-1})^*(\omega)$ so that

$$\lim_{\epsilon\to0} g_\epsilon(x) = \sum_\alpha \psi_\alpha(x)\omega_\alpha \circ \Theta_\alpha(x).$$

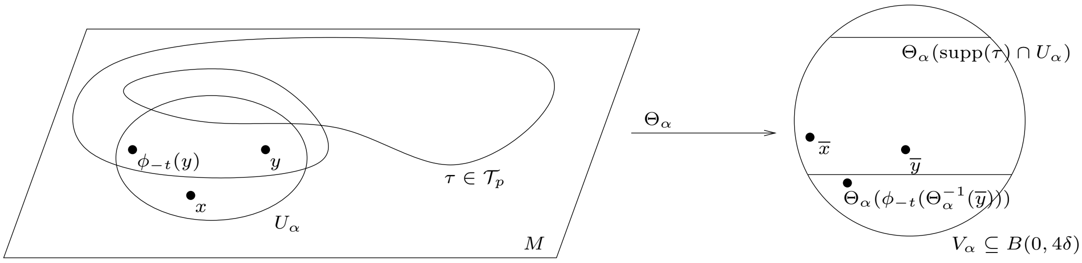

Figure 3.1. The framework of our lemma

After recalling that we used $\mathcal{T}$ for the set of orbits for the flow and $\mathcal{T}_p$ for the set of prime orbits, we now establish the following

Lemma 3.5.1. *For $\Re(z)$ sufficiently large and $n \in \mathbb{N}$, $0 \le \ell \le d-1$ we have* $\mathrm{Trace}^{(\ell)}(R^{(\ell)}(z)^n) < \infty$. *In addition,*

$$\mathrm{Trace}^{(\ell)}(R^{(\ell)}(z)^n) = \frac{1}{(n-1)!} \sum_{\tau \in \mathcal{T}} \frac{1}{\mu(\tau)} \frac{\mathrm{tr}\left(\wedge^\ell(D_{\mathrm{hyp}}\phi_{\lambda(\tau)})\right) \lambda(\tau)^n e^{-z\lambda(\tau)}}{\det\left(\mathbb{1} - D_{\mathrm{hyp}}\phi_{\lambda(\tau)}\right)}$$

Proof. Recall that by equation (3.3.1), $(\Theta_\alpha, U_\alpha)$ have been chosen such that we can write $\overline{x} \doteq \Theta_\alpha(x) = (\tilde{x}, x_d) \in V_\alpha \subset \mathbb{R}^{d-1} \times \mathbb{R}$ and $\Theta_\alpha^{-1}(\tilde{x}, x_d + s) = \phi_s(\Theta_\alpha^{-1}(\tilde{x}, x_d))$ for a neighborhood of x. Let y be in a $B_{M,\epsilon}(x)$, from equation (3.2.8) and Lemma 3.4.3 we can write

$$\begin{aligned} \mathrm{Trace}^{(\ell)}(R^{(\ell)}(z)^n) = \lim_{\epsilon \to 0} \int_{M^2} \omega(dx)\omega(dy) \frac{j_{x,\epsilon}(y)}{(n-1)!} \\ \times \mathrm{tr}^{(\ell)}\left(\int_0^\infty dt\, t^{n-1} e^{-zt} \mathcal{L}_t^{(\ell)}\right)(j_{x,\epsilon})(y) \end{aligned} \tag{3.5.5}$$

Notice that the integrand is zero when $d(x,y) > \epsilon$ or $d(x, \phi_{-t}(y)) > \epsilon$. Hence, for any t, integrating on M^2 is the same as integrating on (see figure 3.1)

$$D_{\epsilon,t} \doteq \{(x,y) \in M^2 : d(x,y) \le \epsilon, d(x, \phi_{-t}(y)) \le \epsilon\}.$$

Now we recall the shadowing theorem for flows as in Bowen [12], with the formulation explicitly given by Plugging in [56] as theorem 1.5.1, adapted to our case. First of all we define the (ϵ_0, L)-pseudo-orbits $\mathfrak{t}_t$. Let $\mathfrak{t}(t) : \mathbb{R} \to M$ be a map such that given $\epsilon_0 > 0, L > 0$ for any $t' \in \mathbb{R}$ we have $d(\phi_t(\mathfrak{t}_{t'}), \mathfrak{t}(t+t')) \le \epsilon_0$ if $|t| < L$. Note that we did not require $\mathfrak{t}(t)$ to be continuous. Then we have the following

Theorem 3.5.2 ([56])**.** *Let M be a smooth manifold and ϕ_t a C^2 Anosov flow. There exists positive numbers ϵ_0, c such that given a (ϵ, c)-pseudo orbit $\mathfrak{t}_t$ with $\epsilon < \epsilon_0$ there is a unique orbit τ, a point $p \in \tau$ and a reparametrization $\sigma(t)$ for all $t \in \mathbb{R}$, we have*

$$d(\mathfrak{t}_t, \phi_{\sigma(t)}(p)) \le c\epsilon \text{ where } |\sigma(t) - t| \le c\epsilon|t|.$$

Now for each $\tau \in \mathcal{T}$ we define

$$\Delta_{\epsilon,\tau} \doteq \left\{(x,y) \in M^2 : d(x,y) < \epsilon,\ d(\phi_t(y),\tau) < c\epsilon \text{ for } t \in [0, \lambda(\tau)(1+c\epsilon)]\right\}$$

then, by the theorem above for all $t, \epsilon > 0$ we have that

$$D_{\epsilon,t} \subset \bigcup_{\tau\in\mathcal{T}:\lambda(\tau)\in[t(1-c\epsilon),t(1+c\epsilon)]} \Delta_{\epsilon,\tau}. \tag{3.5.6}$$

Note that we can choose ϵ so that there are no orbits τ such that $\lambda(\tau) \leq c\epsilon$, since there exists a minimum period for the orbits. Now we can establish the following lemma.

Lemma 3.5.3. *For all sufficiently small ϵ and sufficiently large $\Re(z)$,*

$$\lim_{L\to\infty} \sup_{\epsilon>0} \left| \int_{M^2} \omega(dx)\omega(dy) \int_L^\infty dt \frac{j_{x,\epsilon}(y)}{(n-1)!} t^{n-1} e^{-zt} \mathrm{tr}^{(\ell)} \left(\mathcal{L}_t^{(\ell)}\right)(j_{x,\epsilon})(y) \right| = 0.$$

Proof. By (3.5.6) we have

$$\left\{(x,y,t) \in M^2 \times \mathbb{R}^+ \ :\ (x,y) \in D_{\epsilon,t}, t \geq L\right\} \subset \bigcup_{\{\tau\in\mathcal{T}:\lambda(\tau)>L(1-c\epsilon)\}} \Delta_{\epsilon,\tau} \times [\lambda(\tau)(1-c\epsilon), \lambda(\tau)(1+c\epsilon)]\,.$$

By the definition of the transfer operator and the linearity of $\mathrm{tr}^{(\ell)}$, given $\sigma_{p,q,\ell}$ derived from Lemma 3.4.2, we can apply Fubini-Tonelli to obtain[13]

$$\begin{aligned}
&\left| \int_{M^2} \omega(dx)\omega(dy) \frac{j_{x,\epsilon}(y)}{(n-1)!} \int_L^\infty dt\ t^{n-1} e^{-zt} \left(\mathrm{tr}^{(\ell)} \mathcal{L}_t^\ell\right)(j_{x,\epsilon})(y) \right| \\
&\leq c \sum_{\{\tau\in\mathcal{T}:\lambda(\tau)>L(1+c\epsilon)\}} \int_{\lambda(\tilde\tau)(1-c\epsilon)}^{\lambda(\tilde\tau)(1+c\epsilon)} dt \int_{\Delta_{\epsilon,\tau}} \omega(dx)\omega(dy)\ \epsilon^{-2d} t^{n-1} e^{\left(-\Re(z)+\log(|\sigma_{p,q,\ell}|)\right)t} \\
&\leq c \sum_{\{\tau\in\mathcal{T}:\lambda(\tau)>L(1+c\epsilon)\}} \left| \lambda(\tau)^n e^{(-\Re(z)+\log(|\sigma_{p,q,\ell}|))\lambda(\tau)} \right|.
\end{aligned}$$

Therefore, since the growth of the number of orbits of periodicity $\lambda(\tau) \leq L$ for Anosov flows is at most exponential (see [39]) we can establish absolute convergence, given $\Re(z)$ sufficiently large. $\square$

If we define

$$K_{\epsilon,\ell,n,z}(x,y,t) \doteq \frac{j_{x,\epsilon}(y)}{(n-1)!} t^{n-1} e^{-zt} \left(\mathrm{tr}^{(\ell)} \mathcal{L}_t^\ell\right)(j_{x,\epsilon})(y) \tag{3.5.7}$$

the equation (3.5.5) can be rewritten by Lemma 3.5.3 as

$$\mathrm{Trace}(R^\ell(z)^n) = \lim_{L\to\infty} \lim_{\epsilon\to 0} \int_0^L dt \int_{D_{\epsilon,t}} \omega(dx)\omega(dy)\ K_{\epsilon,\ell,n,z}(x,y,t).$$

[13] Note that $m(\Delta_{\epsilon,\tau})$ is bounded by the measure of a $c\epsilon$ neighborhood of τ times the volume of a ϵ ball i.e. $m(\Delta_{\epsilon,\tau}) \leq \lambda(\tau)\epsilon^{d-1}\epsilon^d$

Now let us define

$$\widetilde{\Delta}_{\epsilon,\tau} \doteq \{(x,y,t) \in M^2 \times \mathbb{R}^+ : (x,y) \in \Delta_{\epsilon,\tau}, |t-\lambda(\tau)| \le \lambda(\tau)c\epsilon|\}$$

Lemma 3.5.4. *Let $\tau_1, \tau_2 \in \mathcal{T}$ with $\lambda(\tau_i) < L$, $i = 1,2$. There exists $\epsilon_L > 0$ such that for all $\epsilon < \epsilon_L$ if we have $\widetilde{\Delta}_{\epsilon,\tau_1} \cap \widetilde{\Delta}_{\epsilon,\tau_2} \neq \varnothing$ then $\tau_1 = \tau_2$. Moreover $\bigcup_{t\in[0,L]} D_{\epsilon,t} \times \{t\} \subset \bigcup_{\{\tau\in\mathcal{T}:\tau<L\}} \widetilde{\Delta}_{\epsilon,\tau}$.*

Proof. Let $(x,y,t) \in \widetilde{\Delta}_{\epsilon,\tau_1} \cap \widetilde{\Delta}_{\epsilon,\tau_2}$. Then $t \in [\lambda(\tau_1)(1-c\epsilon), \lambda(\tau_1)(1+c\epsilon)] \cap [\lambda(\tau_2)(1-c\epsilon), \lambda(\tau_2)(1+c\epsilon)]$, thus $|\lambda(\tau_1)-\lambda(\tau_2)| \le 2c\epsilon L$. Now let $(x,y) \in \Delta_{\epsilon,\tau_1} \cap \Delta_{\epsilon,\tau_2}$, then by the previous theorem there exists $\tau \in \mathcal{T}_p$ such that $\operatorname{supp}(\tau) \subseteq (\operatorname{supp}(\tau_1) \cap \operatorname{supp}(\tau_2))$. Thus since both the support and times must coincide we have $\tau_1 = \tau_2$ for $\epsilon < \epsilon_L$. Note that this is granted as soon as $\epsilon_L \le \nu L^{-1} c$ for ν the minimum period of an orbit on M. Last, if $(x,y,t) \in \bigcup_{t\in[0,L]} D_{\epsilon,t} \times \{t\}$ by (3.5.6) we have that there exists $\tau \in \mathcal{T}$ such that $(x,y) \in \Delta_{\epsilon,\tau}$ and $|t-\lambda(\tau)| \le \lambda(\tau)(1-c\epsilon)$. Thus $(x,y,t) \in \widetilde{\Delta}_{\epsilon,\tau}$. □

Hence we obtain

$$\operatorname{Trace}(R^{(\ell)}(z)^n) = \lim_{L\to\infty} \lim_{\epsilon\to 0} \sum_{\substack{\tau\in\mathcal{T} \\ \lambda(\tau)<L}} \int_{\widetilde{\Delta}_{\epsilon,\tau}} K_{\epsilon,\ell,n,z}(x,y,t)\omega(dx)\omega(dy)dt$$

Given the definition of $\widetilde{\mathcal{L}}_t^{(\ell)}$ and passing to coordinate charts we set

$$\Omega_{\alpha,\epsilon,\tau} = (\Theta_\alpha \times \Theta_\alpha \times \mathbb{1}) \left((U_\alpha \times U_\alpha \times \mathbb{R}) \cap \widetilde{\Delta}_{\epsilon,\tau} \right)$$

so we can rewrite our integral as

$$\begin{aligned} \operatorname{Trace}(R^{(\ell)}(z)^n) = \sum_\alpha \lim_{L\to\infty} \lim_{\epsilon\to 0} &\sum_{\{\tau\in\mathcal{T}:\lambda(\tau)<L\}} \int_{\Omega_{\alpha,\epsilon,\tau}} (g_\epsilon \circ \Theta_\alpha^{-1}(\overline{x}))^{-1} \\ &\times (\psi_\alpha \circ \Theta_\alpha^{-1}(\overline{x}))\epsilon^{-d} k \left(\epsilon^{-1}(\overline{x}-\overline{y}) \right) \frac{t^{n-1}e^{-zt}}{(n-1)!} \\ &\times \left(\operatorname{tr}^{(\ell)} \mathcal{L}_t^{(\ell)} \right) (j_{\Theta_\alpha^{-1}(\overline{x}),\epsilon})(\Theta_\alpha^{-1}(\overline{y}))\omega_\alpha(\overline{x})\omega_\alpha(\overline{y})d\overline{x}d\overline{y}dt. \end{aligned} \tag{3.5.8}$$

Next we compute $(\operatorname{tr}^{(\ell)} \mathcal{L}_t^{(\ell)})(j_{\Theta_\alpha^{-1}(\overline{x}),\epsilon})(\Theta_\alpha^{-1}(\overline{y}))$ for $(\overline{x},\overline{y},t) \in \Omega_{\alpha,\epsilon,\tau}$. From the definition of $j_{\Theta_\alpha^{-1}(\overline{x}),\epsilon}$, equations (3.4.14), (3.4.16) and (3.5.3), we obtain

$$\begin{aligned} (\operatorname{tr}^{(\ell)} \mathcal{L}_t^{(\ell)})(j_{\Theta_\alpha^{-1}(\overline{x}),\epsilon})(\Theta_\alpha^{-1}(\overline{y})) &= (j_{\Theta_\alpha^{-1}(\overline{x}),\epsilon}) \circ \phi_{-t}(\Theta_\alpha^{-1}(\overline{y})) \operatorname{tr}(\wedge^l(D_{\text{hyp}}\phi_{-t})) \\ = \operatorname{tr}\left(\wedge^\ell (D_{\text{hyp}}\phi_{-t}) \right) &\left(g_\epsilon \circ \Theta_\alpha^{-1}(\overline{x}) \right)^{-1} \sum_\beta \left(\psi_\beta \circ \Theta_\alpha^{-1}(\overline{x}) \right) \\ &\epsilon^{-d} k \left(\epsilon^{-1} \left(\Theta_\beta \circ \Theta_\alpha^{-1}(\overline{x}) - \Theta_\beta \circ \phi_{-t} \circ \Theta_\alpha^{-1}(\overline{y}) \right) \right). \end{aligned}$$

For each V_α let $\Sigma_\alpha = \{\overline{v} = (\tilde{v}, v_d) \in V_\alpha : v_d = 0\}$. Let $\bar{\tau}_{\alpha,i}$ be the connected components of $\Theta_\alpha(\operatorname{supp}(\tau) \cap U_\alpha)$ indexed by i. Let $\Omega_{\alpha,\varepsilon,\tau,i} \supset \bar{\tau}_{\alpha,i}$ be the related neighborhoods such that $\Omega_{\alpha,\varepsilon,\tau} = \bigcup_i \Omega_{\alpha,\varepsilon,\tau,i}$. Furthermore let $\overline{\eta}_{\alpha,i} = \overline{\tau}_{\alpha,i} \cap \Sigma_\alpha$ and $|\overline{\eta}_{\alpha,i} - \tilde{x}| < c\epsilon$. Let $\hat{\phi}_{\alpha,-t}(\overline{v}) \doteq \Theta_\alpha \circ \phi_{-t} \circ \Theta_\alpha^{-1}(\overline{v})$ where it is well defined. Define $r_{\alpha,\bar{\tau},i} : \Sigma_\alpha \to \mathbb{R}^{+}$[14] as $r_{\alpha,\bar{\tau},i}(\tilde{v}) \doteq \inf\{t \in [\lambda(\tau)(1-c\epsilon), \lambda(\tau)(1+c\epsilon)] : \hat{\phi}_{\alpha,-t} \circ \Theta_\alpha^{-1}(\tilde{v}, 0) \in \Sigma_\alpha\}$ and note that $r_{\alpha,\bar{\tau},i}(\overline{\eta}_{\alpha,i}) = \lambda(\tau)$. Let $P_\alpha : V_\alpha \to \Sigma_\alpha$ as $P_\alpha(\overline{v}) = \hat{\phi}_{\alpha,-v_d}(\overline{v})$ so that

$$\hat{\phi}_{\alpha,-t}(\overline{y}) = \hat{\phi}_{\alpha,-t+y_d}(P_\alpha(\overline{y})) = \hat{\phi}_{\alpha,-t+y_d+r_{\alpha,\bar{\tau},i}(P_\alpha(\overline{y}))} \circ \hat{\phi}_{\alpha,-r_{\alpha,\bar{\tau},i}(P_\alpha(\overline{y}))}(P_\alpha(\overline{y}))$$

In particular we have that $[\hat{\phi}_{\alpha,-t}(\overline{y})]_d = -t + y_d + r_{\alpha,\bar{\tau},i}(P_\alpha(\overline{y}))$ provided $|t - \lambda(\tau)| \leq c\epsilon$.

Let $\Pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ such that $\Pi(v_1, \ldots, v_d) = (v_1, \ldots, v_{d-1})$ and recall that we have invertible maps $G_{\alpha,\beta} : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ and maps $F_{\alpha,\beta} : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\Theta_\beta \circ \Theta_\alpha^{-1}(z) = (G_{\alpha,\beta}(z_1, \ldots, z_{d-1}), F_{\alpha,\beta}(z_1, \ldots, z_{d-1}) + z_d)\,.$$

Hence we have the decomposition

$$\begin{aligned}&k\left(\epsilon^{-1}\left(\Theta_\beta \circ \Theta_\alpha^{-1}(\overline{x}) - \Theta_\beta \circ \phi_{-t} \circ \Theta_\alpha^{-1}(\overline{y})\right)\right)\\ &= k_{d-1}(\epsilon^{-1}(G_{\alpha,\beta}(\tilde{x}) - G_{\alpha,\beta} \circ \Pi \circ \hat{\phi}_{\alpha,-t}(\overline{y})))k_1\left(\epsilon^{-1} h_{\alpha,\beta,\bar{\tau},i}(\overline{x}, \overline{y}, t)\right)\end{aligned} \tag{3.5.9}$$

where we have set

$$h_{\alpha,\beta,\bar{\tau},i}(\overline{x}, \overline{y}, t) = \left(F_{\alpha,\beta}(\tilde{x}) + x_d - F_{\alpha,\beta}(\Pi \circ \hat{\phi}_{\alpha,-t}(\overline{y})) - y_d - r_{\alpha,\bar{\tau},i}(P_\alpha(\overline{y})) + t\right)$$

For $(\overline{x}, \overline{y}, t) \in \Omega_{\alpha,\epsilon,\tau}$ we can define the following transformation $\Xi : \Omega_{\alpha,\epsilon,\tau,i} \to \mathbb{R}^{2d+1}$ as

$$\begin{aligned}\overline{\Xi}(\overline{x}, \overline{y}, t) &= \left(\tilde{x} - \tilde{y}, x_d - y_d, G_{\alpha,\beta}(\tilde{x}) - G_{\alpha,\beta} \circ \Pi \circ \hat{\phi}_{\alpha,-t}(\overline{y}), x_d, t\right)\\ &= (\xi, \xi_d, \rho, \rho_d, t').\end{aligned} \tag{3.5.10}$$

Lemma 3.5.5. $\overline{\Xi}$ *is a diffeomorphism between* $\Omega_{\alpha,\varepsilon,\tau,i}$ *and its image.*

Proof. To see that it is a local diffeomorphism we begin by writing explicitly

$$D\overline{\Xi} = \left(\begin{array}{c:c:c:c:c} \mathbb{1} & 0 & -\mathbb{1} & 0 & 0 \\ \hdashline 0 & 1 & 0 & -1 & 0 \\ \hdashline D_{\tilde{x}} G_{\alpha,\beta} & 0 & -D_{\Pi \circ \hat{\phi}_{\alpha,-t}(\overline{y})} G_{\alpha,\beta} \cdot D_{\hat{\phi}_{\alpha,-t}(\overline{y})} \Pi \cdot D_{d-1,\overline{y}} \hat{\phi}_{\alpha,-t'} & 0 & * \\ \hdashline 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

We use $D_y f$ for the standard derivative matrix of f at y. Note that by construction $G_{\alpha,\beta} \circ \Pi \circ \hat{\phi}_{\alpha,-t}(\overline{y})$ does not depend on y_d. Moreover

$$D_{\hat{\phi}_{\alpha,-t}(\overline{y})} \Pi \cdot D_{d-1,\overline{y}} \hat{\phi}_{\alpha,-t} =$$

$$\left(\begin{array}{ccc:c} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{array}\right) \begin{pmatrix} \frac{\partial_1 \hat{\phi}_{\alpha,-t}(\overline{y}))}{\partial y_1} & \cdots & \frac{\partial_1 \hat{\phi}_{\alpha,-t}(\overline{y}))}{\partial y_{d-1}} \\ \vdots & & \vdots \\ \frac{\partial_d \hat{\phi}_{\alpha,-t}(\overline{y}))}{\partial y_1} & \cdots & \frac{\partial_d \hat{\phi}_{\alpha,-t}(\overline{y}))}{\partial y_{d-1}} \end{pmatrix} = D_{\mathrm{hyp}} \hat{\phi}_{\alpha,-t}$$

[14] With a little abuse we write $r_{\alpha,\bar{\tau},i}$ for $r_{\alpha,\bar{\tau}_{\alpha,i}}$, thus avoiding repeated and double subscripts.

where $D_{\text{hyp}}\hat{\phi}_{\alpha,-t}$ is an hyperbolic matrix. Hence for this transformation, using the properties of determinants, we have that

$$\begin{aligned}
|\det(D\overline{\Xi})| &= \left|\begin{array}{c:c:c} \mathbb{1} & -\mathbb{1} & 0 \\ \hdashline 0 & 0 & -\mathbb{1} \\ \hdashline D_{\tilde{x}}G_{\alpha,\beta} & -D_{\Pi\circ\hat{\phi}_{\alpha,-t}(\overline{y})}G_{\alpha,\beta}\cdot D_{\text{hyp}}\hat{\phi}_{\alpha,-t} & 0 \end{array}\right| \\
&= \left|\begin{array}{c:c} \mathbb{1} & -\mathbb{1} \\ \hdashline D_{\tilde{x}}G_{\alpha,\beta} & -D_{\Pi\circ\hat{\phi}_{\alpha,-t}(\overline{y})}G_{\alpha,\beta}\cdot D_{\text{hyp}}\hat{\phi}_{\alpha,-t} \end{array}\right| \\
&= \left|\begin{array}{c:c} \mathbb{1} & 0 \\ \hdashline D_{\tilde{x}}G_{\alpha,\beta} & D_{\tilde{x}}G_{\alpha,\beta} - D_{\Pi\circ\hat{\phi}_{\alpha,-t}(\overline{y})}G_{\alpha,\beta}\cdot D_{\text{hyp}}\hat{\phi}_{\alpha,-t} \end{array}\right| \\
&= |D_{\tilde{x}}G_{\alpha,\beta}|\cdot\left|\mathbb{1} - (D_{\tilde{x}}G_{\alpha,\beta})^{-1}D_{\Pi\circ\hat{\phi}_{\alpha,-t}(\overline{y})}G_{\alpha,\beta}\cdot D_{\text{hyp}}\hat{\phi}_{\alpha,-t}\right|
\end{aligned}$$

Now observe that $(D_{\tilde{x}}G_{\alpha,\beta})^{-1}D_{\Pi\circ\hat{\phi}_{\alpha,-t}(\overline{y})}G_{\alpha,\beta} = \mathbb{1} + o(\epsilon)$ so that

$$\mathbb{1} - (D_{\tilde{x}}G_{\alpha,\beta})^{-1}D_{\Pi\circ\hat{\phi}_{\alpha,-t}(\overline{y})}G_{\alpha,\beta}\cdot D_{\text{hyp}}\hat{\phi}_{\alpha,-t} = \left(\mathbb{1} - D_{\text{hyp}}\hat{\phi}_{\alpha,-t}\right)(\mathbb{1} - \epsilon A)$$

for some uniformly bounded matrix A. Therefore

$$|\det D\overline{\Xi}| = \frac{\omega_\alpha}{\omega_\beta}\left|\mathbb{1} - D_{\text{hyp}}\hat{\phi}_{\alpha,-t}\right||\mathbb{1} - \epsilon A| \neq 0 \tag{3.5.11}$$

Now let $\overline{\Xi}(x,y,t) = \overline{\Xi}(w,z,s)$ for opportune variables x,y,w,z,t,s. By applying the transformation we must have trivially $t = s$, $x_d = w_d$ and $y_d = z_d$. Since our transformation is defined on $\Omega_{\alpha,\epsilon,\tau,i}$ we must have $t \sim \lambda(\tau)$. Moreover x, y arbitrarily close i.e. $|x-y| < \varepsilon$. Thus $|x-w| < \varepsilon, |y-z| < \varepsilon$. Hence $x = w, y = z$ since Ξ is a local diffeomorphism. □

To simplify our expression let

$$\begin{aligned}
H_{\alpha,\beta,\ell,\epsilon,\tau}(\overline{x},\overline{y},t) &= \operatorname{tr}\left(\wedge^\ell(D_{\text{hyp}}\phi_{-t})\right)(g_\epsilon\circ\Theta_\alpha^{-1}(\overline{x}))^{-2} \\
&\quad\times(\psi_\alpha\circ\Theta_\alpha^{-1}(\overline{x}))(\psi_\beta\circ\Theta_\alpha^{-1}(\overline{x}))\frac{t^{n-1}e^{-zt}}{(n-1)!}\omega_\alpha(\overline{x})\omega_\alpha(\overline{y})
\end{aligned} \tag{3.5.12}$$

Given $\overline{\eta}_{\alpha,i}$, recall that $|\overline{\eta}_{\alpha,i} - \tilde{x}| < c\epsilon$ then, for all $\eta_{\alpha,i}$ we have

$$\begin{cases}
F_{\alpha,\beta}(\tilde{x}) = F_{\alpha,\beta}(\overline{\eta}_{\alpha,i}) + D_{\overline{\eta}_{\alpha,i}}F_{\alpha,\beta}(\tilde{x} - \overline{\eta}_{\alpha,i}) + o(\epsilon) \\
F_{\alpha,\beta}(\Pi\circ\hat{\phi}_{\alpha,-t}(\overline{y})) = F_{\alpha,\beta}(\overline{\eta}_{\alpha,i}) + D_{\overline{\eta}_{\alpha,i}}F_{\alpha,\beta}(\Pi\circ\hat{\phi}_{\alpha,-t}(\overline{y}) - \overline{\eta}_{\alpha,i}) + o(\epsilon) \\
r_{\alpha,\bar{\tau},i}(P_\alpha(\overline{y})) = \lambda(\tau) + (D_{\overline{\eta}_{\alpha,i}}r_{\alpha,\bar{\tau},i})(\tilde{y} - \overline{\eta}_{\alpha,i}) + o(\epsilon).
\end{cases} \tag{3.5.13}$$

Moreover, after recalling that $G_{\alpha,\beta}$ are invertible maps of at least class C^1 and recalling the definition of $\overline{\Xi}$ we also have that $\rho = G_{\alpha,\beta}(\tilde{x}) - G_{\alpha,\beta}\circ\Pi\circ\hat{\phi}_{\alpha,-t}(\overline{y}) = D_{\overline{\eta}_{\alpha,i}}G_{\alpha,\beta}(\tilde{x} - \Pi\circ\hat{\phi}_{\alpha,-t}(\overline{y})) + o(\epsilon)$ so that $\tilde{x} - \Pi\circ\hat{\phi}_{\alpha,-t}(\overline{y}) = D_{\overline{\eta}_{\alpha,i}}G_{\alpha,\beta}^{-1}\rho + o(\epsilon)$. From this equality and by (3.5.13) and (3.5.10) we have

$$\tilde{y} - \overline{\eta}_{\alpha,i} = \left(\mathbb{1} - D_{\overline{\eta}_{\alpha,i}}(\Pi\circ\hat{\phi}_{\alpha,-t})\right)^{-1}\left(D_{\overline{\eta}_{\alpha,i}}G_{\alpha,\beta}^{-1}\rho - \xi\right) + o(\epsilon)$$

Hence we can write

$$
\begin{aligned}
k_1(\epsilon^{-1}h_{\alpha,\beta,\tau,i}(\overline{x},\overline{y},t)) &= k_1\Big(\epsilon^{-1}\Big(\xi_d + t - \lambda(\tau) \\
&+ D_{\overline{\eta}_{\alpha,i}} r_{\alpha,\bar{\tau},i}(\mathbb{1} - D_{\overline{\eta}_{\alpha,i}}(\Pi \circ \hat{\phi}_{\alpha,-t}))^{-1}(D_{\overline{\eta}_{\alpha,i}} G^{-1}_{\alpha,\beta}\rho - \xi) \\
&+(D_{\overline{\eta}_{\alpha,i}} F_{\alpha,\beta})(D_{\overline{\eta}_{\alpha,i}} G^{-1}_{\alpha,\beta})\rho + o(\epsilon)\Big)\Big)
\end{aligned}
$$

Hence after setting $\overline{\Omega}_{\alpha,\epsilon,\tau,i} \doteq \Xi(\Omega_{\alpha,\epsilon,\tau,i})$ by rewriting equation (3.5.8) our integral reads

$$
\begin{aligned}
\operatorname{Trace}(R^{(\ell)}(z)^n) &= \sum_{\alpha,\beta} \lim_{L\to\infty} \lim_{\epsilon\to 0} \sum_{\{\tau\in\mathcal{T}:\lambda(\tau)<L\}} \int_{\overline{\Omega}_{\alpha,\epsilon,\tau,i}} \epsilon^{-2d} \\
&\times H_{\alpha,\beta,\ell,\epsilon,\tau} \circ \overline{\Xi}^{-1}(\xi,\xi_d,\rho,\rho_d,t)\, k_{d-1}\left(\epsilon^{-1}\xi\right) k_1\left(\epsilon^{-1}\xi_d\right) k_{d-1}\left(\epsilon^{-1}\rho\right) \\
&\times k_1\left(\epsilon^{-1}\left(\xi_d + t - \lambda(\tau) + D_{\overline{\eta}_{\alpha,i}} r_{\alpha,\tau}(\mathbb{1} - D_{\overline{\eta}_{\alpha,i}}(\Pi\circ\hat{\phi}_{\alpha,-t}))^{-1}\right.\right. \\
&\qquad \left.\left.\times\, ((D_{\overline{\eta}_{\alpha,i}} G^{-1}_{\alpha,\beta})\rho - \xi)\ + (D_{\overline{\eta}_{\alpha,i}} F_{\alpha,\beta})(D_{\overline{\eta}_{\alpha,i}} G^{-1}_{\alpha,\beta})\rho + o(\epsilon)\right)\right) \\
&\times \left|\det(D\Xi)^{-1}\right| d\xi d\xi_d d\rho d\rho_d dt
\end{aligned}
\tag{3.5.14}
$$

Now we rescale the variables by setting $\breve{\xi} = \epsilon^{-1}\xi$, $\breve{\xi}_d = \epsilon^{-1}\xi_d$, $\breve{\rho} = \epsilon^{-1}\rho$, $s = \epsilon^{-1}(t - \lambda(\tau))$. We set

$$
\begin{aligned}
\breve{h}_{\alpha,\beta,\epsilon,\tau,\epsilon s+\lambda(\tau),i}(\breve{\xi},\breve{\rho}) \doteq D_{\overline{\eta}_{\alpha,i}} r_{\alpha,\tau}(\mathbb{1} - D_{\overline{\eta}_{\alpha,i}}(\Pi\circ\hat{\phi}_{\alpha,-\epsilon s-\lambda(\tau)}))^{-1}((D_{\overline{\eta}_{\alpha,i}} G^{-1}_{\alpha,\beta})\breve{\rho} - \breve{\xi}) \\
+ (D_{\overline{\eta}_{\alpha,i}} F_{\alpha,\beta})(D_{\overline{\eta}_{\alpha,i}} G^{-1}_{\alpha,\beta})\breve{\rho} + \mathcal{O}(\epsilon).
\end{aligned}
$$

We can take the limit for $\epsilon \to 0$ and obtain

$$
\begin{aligned}
\operatorname{Trace}(R^{(\ell)}(z)^n) &= \sum_{\alpha,\beta} \lim_{L\to\infty} \sum_{\{\tau\in\mathcal{T}:\lambda(\tau)<L\}} \int_{\mathbb{R}^{2d}} d\breve{\xi} d\breve{\xi}_d d\breve{\rho} d\rho_d \int_{-c\lambda(\tau)}^{c\lambda(\tau)} ds \\
&H_{\alpha,\beta,\ell,0,\tau} \circ \overline{\Xi}^{-1}\left(0,0,0,\rho_d,\lambda(\tau)\right) k_{d-1}\left(\breve{\xi}\right) k_1\left(\breve{\xi}_d\right) k_{d-1}\left(\breve{\rho}\right) \\
&k_1\left(s + \breve{\xi}_d + \breve{h}_{\alpha,\beta,0,\tau,\lambda(\tau),i}(\breve{\xi},\breve{\rho})\right)\left|\det(D\overline{\Xi})^{-1}\right|.
\end{aligned}
\tag{3.5.15}
$$

Note that $\breve{h}$ is uniformly bounded for all $\epsilon s+\lambda(\tau)$. Thus in the above equation we can choose the constant c large enough so that all the s for which $k_1\left(s + \breve{\xi}_d + \breve{h}_{\alpha,\beta,0,\tau,\lambda(\tau)}(\breve{\xi},\breve{\rho})\right) \neq 0$ belongs to $[-c\lambda(\tau), c\lambda(\tau)]$. Last, we integrate with respect to $ds, d\breve{\xi}, d\breve{\xi}_d, d\breve{\rho}$. Note that if $\overline{\Xi}(\overline{x},\overline{y},\lambda(\tau)) = (0,0,0,\rho_d,\lambda(\tau))$ then $\overline{x} = \overline{y} = (\overline{\eta}_{\alpha,i},\rho_d)$. Thus, setting $p_{\alpha,i} = \Theta_\alpha^{-1}(\bar{\eta}_{\alpha,i},0)$, we obtain

$$
\begin{aligned}
\operatorname{Trace}(R^{(\ell)}(z)^n) &= \sum_{\alpha,\beta,i} \lim_{L\to\infty} \sum_{\{\tau\in\mathcal{T}:\lambda(\tau)<L\}} \int_{-4\delta}^{4\delta} d\rho_d \operatorname{tr}\left(\wedge^\ell(D_{\mathrm{hyp}}\phi_{-\lambda(\tau)})\right) \\
&\times \frac{\lambda(\tau)^{n-1}e^{-z\lambda(\tau)}}{(n-1)!} \frac{\psi_\alpha\circ\phi_{\rho_d}(p_{\alpha,i})\,\psi_\beta\circ\phi_{\rho_d}(p_{\alpha,i})}{\left(\sum_\gamma \psi_\gamma\circ\phi_{\rho_d}(p_{\alpha,i})\omega_\gamma\circ\Theta_\gamma\circ\phi_{\rho_d}(p_{\alpha,i})\right)^2} \\
&\times \omega_\alpha(\Theta_\alpha\circ\phi_{\rho_d}(p_{\alpha,i}))\,\omega_\beta(\Theta_\beta\circ\phi_{\rho_d}(p_{\alpha,i}))\det\left(\mathbb{1} - D_{\mathrm{hyp}}\phi_{-\lambda(\tau)}\right)^{-1}.
\end{aligned}
$$

We can then choose $p_\tau \in \operatorname{supp}\tau$. Then $\{\phi_t(p_\tau)\}_{t\in[0,\lambda(\tau)]}$ cross each connected component of $\operatorname{supp}\psi_\alpha \cap \operatorname{supp}\tau$ (and hence each $p_{\alpha,i}$) exactly $\mu(\tau)$ times. Accordingly,

$$\begin{aligned}
\operatorname{Trace}(R^{(\ell)}(z)^n) &= \sum_{\alpha,\beta} \lim_{L\to\infty} \sum_{\{\tau\in\mathcal{T}:\lambda(\tau)<L\}} \int_0^{\lambda(\tau)} d\rho_d \operatorname{tr}\left(\wedge^\ell(D_{\mathrm{hyp}}\phi_{-\lambda(\tau)})\right) \\
&\quad \times \frac{\lambda(\tau)^{n-1}e^{-z\lambda(\tau)}}{\mu(\tau)(n-1)!} \frac{\psi_\alpha\circ\phi_{\rho_d}(p_\tau)\,\psi_\beta\circ\phi_{\rho_d}(p_\tau)}{\left(\sum_\gamma \psi_\gamma\circ\phi_{\rho_d}(p_\tau)\omega_\gamma\circ\Theta_\gamma\circ\phi_{\rho_d}(p_\tau)\right)^2} \\
&\quad \times \omega_\alpha(\Theta_\alpha\circ\phi_{\rho_d}(p_\tau))\,\omega_\beta(\Theta_\beta\circ\phi_{\rho_d}(p_\tau)) \det\left(\mathbb{1} - D_{\mathrm{hyp}}\phi_{-\lambda(\tau)}\right)^{-1} \\
&= \lim_{L\to\infty} \sum_{\substack{\tau\in\mathcal{T}\\ \lambda(\tau)<L}} \int_0^{\lambda(\tau)} \frac{\operatorname{tr}(\wedge^\ell(D_{\mathrm{hyp}}\phi_{-\lambda(\tau)}))\lambda(\tau)^{n-1}e^{-z\lambda(\tau)}}{(n-1)!\mu(\tau)\det\left(\mathbb{1} - D_{\mathrm{hyp}}\phi_{-\lambda(\tau)}\right)}.
\end{aligned}$$

By taking the limit for $L\to\infty$ we finally obtain

$$\operatorname{Trace}(R^{(\ell)}(z)^n) = \frac{1}{(n-1)!}\sum_{\tau\in\mathcal{T}} \frac{1}{\mu(\tau)} \frac{\operatorname{tr}(\wedge^\ell(D_{\mathrm{hyp}}\phi_{-\lambda(\tau)}))\lambda(\tau)^n e^{-z\lambda(\tau)}}{\det\left(\mathbb{1} - D_{\mathrm{hyp}}\phi_{-\lambda(\tau)}\right)}.$$

□

3.6 Linearity of extended determinants

We will extend the methods of Liverani-Tsujii [43] to the dynamical determinants just studied. To do so we first introduce the adjoint operator of $\mathcal{L}_t^{(\ell)}$, we will compute its expression and then we will construct a suitable "product" operator. We start by computing

$$\begin{aligned}
\langle \mathcal{L}_t^{(\ell)} f, g\rangle_{\Omega_s^\ell} &= \int_M \langle \mathcal{L}_t^{(\ell)} f, g\rangle_x \omega(x) = \int_M (\phi_{-t}^* f)(x)\wedge *g(x) \\
&= \int_M \phi_{-t}^*(f\wedge\phi_t^*(*g))(x) = \int_M f(x)\wedge\phi_t^*(*g)(x) \\
&= \int_M (-1)^{\ell(d-\ell)}\langle f, *\phi_t^*(*g)\rangle_x\omega(x) = \langle f, \overline{\mathcal{L}}_t^{(\ell)} g\rangle_{\Omega_s^\ell}
\end{aligned} \tag{3.6.1}$$

where d is the dimension of M, $*$ indicates the Hodge dual and $**g = (-1)^{\ell(d-\ell)}g$ for each ℓ-from g, see [38, Lemma 2.1.1], and we have defined

$$\overline{\mathcal{L}}_t^{(\ell)} g \doteq (-1)^{\ell(d-\ell)} *(\phi_t^*(*g)). \tag{3.6.2}$$

Thus given $g\in\Omega_{0,s}^\ell(M)$ we have the local expression[15]

$$(\overline{\mathcal{L}}_t^{(\ell)} g)_{\alpha,\bar{j}} \doteq \psi_\alpha \sum_{\beta,\bar{i}} \left(g_{\beta,\bar{i}}\circ\phi_t\right)(-1)^{\ell(d-\ell)}\varepsilon(\bar{i})\varepsilon(\bar{j})\det(D\phi_t)^{\alpha,\beta}_{\bar{j}^c,\bar{i}^c}, \tag{3.6.3}$$

[15]It follows from (3.4.5) applied to $*g$, with ϕ_t instead of ϕ_{-t}, and noticing that $*dx_{\bar{i}} = (-1)^{\varepsilon(\bar{i})}dx_{\bar{i}^c}$, see [38, Section 2] for more details.

where $\bar{i}^c$ is an ordered $(d-\ell)$-multiindex such that $(dx_{\bar{i}}, d_{x_{\bar{i}^c}})$ form a complete base and $\varepsilon(\bar{i})$ is the sign of the permutation that maps $\{1, \dots, d\}$ to $\{i_1, \dots, i_\ell, i_1^c, \dots, i_{d-1-\ell}^c\}$. Note that, if $g \in \Omega_{0,s}^\ell(M)$, then $i_k < d$ thus, recalling the representation (3.4.4), it follows that in (3.6.3) $\bar{i}^c$ has always the last component equal to d. Thus, only the $\bar{j}$ such that $j_\ell < d$ contribute to the sum. In other words $\overline{\mathcal{L}}_t^{(\ell)}(\Omega_{0,s}^\ell) \subset \Omega_{0,s}^\ell$ and on such a subspace we can write

$$(\overline{\mathcal{L}}_t^{(\ell)} g)_{\alpha,\bar{j}} \doteq \psi_\alpha \sum_{\beta,\bar{i}} \left(g_{\beta,\bar{i}} \circ \phi_t\right) (-1)^{\ell(d-\ell)} \varepsilon(\bar{i})\varepsilon(\bar{j}) \det(\tilde{D}\phi_t)_{\bar{j}^c,\bar{i}^c}^{\alpha,\beta}, \tag{3.6.4}$$

Remark 3.6.1. Note that $\bar{i}^c$ in (3.6.3) is a $d-\ell$ ordered multi-index while in (3.6.4) is a $d-1-\ell$ multi-index since its components can take value only in the set $\{1, \dots, d-1\}$. The latter is the situation in the present section, and we will not warn the reader any further.

Our goal is to use the operators $\mathcal{L}_t^{(\ell)}, \overline{\mathcal{L}}_t^{(\ell)}$ to construct an operator that acts naturally on an appropriate subspace of $\Omega_s^{2\ell}(M^2)$. As a first step we define the subspace of interest.

Consider the projections $\pi_i : M^2 \to M$, $i \in \{1, 2\}$, such that $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. For each pair of ℓ-forms f, g in $\Omega_s^l(M)$ we have that $\pi_1^* f \wedge \pi_2^* g \in \Omega_s^{2l}(M^2)$. We define then $\Omega_{2,s}^\ell(M) = \text{span}\{\pi_1^* f \wedge \pi_2^* g \;:\; f, g \in \Omega_{0,s}^\ell(M)\}$.[16]

Next, we want to use section 3.3 to define a norm that reflects the relevant dynamical properties. On M we consider two systems of charts: the one used in section 3.4, that we will call $(U_\alpha, \Theta_\alpha^-)$ and the other constructed exactly in the same way but with respect to the flow ϕ_{-t} rather than ϕ_t, that we will call $(U_\alpha, \Theta_\alpha^+)$.[17] Also we consider the associated adapted partition of unity $\{\psi_\alpha\}$. We consider then the atlas, of M^2, $(U_\alpha \times U_\beta, \Theta_\alpha^- \times \Theta_\beta^+)$, subordinates partitions $\{\psi_{\alpha,\beta}\}$, $\psi_{\alpha,\beta}(x, y) = \psi_\alpha(x)\psi_\beta(y)$, and the cones $\mathcal{C}$ such that

$$(\Theta_\alpha^- \times \Theta_\beta^+)_* \mathcal{C}(x, y) \doteq \left\{(\xi, \eta) \in \mathbb{R}^{2d} : |\xi_c| + |\eta_c| + |\xi_u| + |\eta_s| \le |\xi_s| + |\eta_u|\right\}.$$

Note that we can assume without loss of generality that $\mathcal{C}$ are strictly invariant cones with respect to the $\mathbb{R}_+^2$ action $\phi_{-t} \times \phi_s$.

We use the above structure on M^2 to define norms $\|\cdot\|_{p,q,\ell,2}$ using the construction in Section 3.3. We define then the Banach space $\mathcal{B}_2^{p,q,\ell}$ as the closure of $\Omega_{2,s}^\ell(M)$ with respect to the norm $\|\cdot\|_{p,q,\ell,2}$.

We can finally construct operators $\mathcal{L}_{t,s}^{(\ell)} = \mathcal{L}_t^{(\ell)} \otimes \overline{\mathcal{L}}_s^{(\ell)}$ in analogy with what has been done in [43] for the case of diffeomorphisms.[18]

[16] Locally, the $\mathcal{C}^s$ closure of $\Omega_{2,s}^\ell(M)$ contains all the forms of the type $\sum_{\bar{i},\bar{j}} \omega(x, y) dx_{i_1} \wedge \dots dx_{i_\ell} \wedge dy_{i_1} \wedge \dots dy_{i_\ell}$ with $\omega \in \mathcal{C}^{s'}(M^2)$, for each $s' \ge s$.

[17] In particular, the relevant cone is $\mathcal{C}^u$, not $\mathcal{C}^s$, and its representation in chart will contain $\{\|s\| + |t| \le \|u\|\}$. Note that one can always reduce to this case by eventually refining the covering.

[18] Note that in the present setting we are forced to introduce a $\mathbb{R}_+^2$ action since the product of two flows has a two dimensional central bundle.

Given $f, g \in \Omega^{\ell}_{0,s}(M)$ we define

$$\mathcal{L}^{(\ell)}_{t,s}(\pi_1^* f \wedge \pi_2^* g) \doteq (\pi_1^*(\mathcal{L}^{(\ell)}_t f)) \wedge (\pi_2^*(\overline{\mathcal{L}}^{(\ell)}_t g)) \tag{3.6.5}$$

which extends by linearity to an operator $\mathcal{L}^{(\ell)}_{t,s} : \Omega^{\ell}_{2,s}(M) \to \Omega^{\ell}_{2,s}(M)$.

To express $h \in \Omega^{\ell}_{2,s}(M)$ locally, we adopt the natural extension of the previous notation, i.e. $h_{\alpha,\beta} = \psi_{\alpha,\beta} h$ and

$$h_{\alpha,\beta}(x,y) = \sum_{\bar{i},\bar{j} \in \mathcal{I}_\ell} h_{\alpha,\beta,\bar{i},\bar{j}}(x,y) dx_{\alpha,\bar{i}} \wedge dy_{\beta,\bar{j}}. \tag{3.6.6}$$

Accordingly, we can write $\mathcal{L}^{(\ell)}_{t,s} h$ as

$$\begin{aligned}(\mathcal{L}^{(\ell)}_{t,s} h)_{\alpha,\beta,\bar{i},\bar{j}} = &\sum_{\gamma,\delta,\bar{n},\bar{m}} \psi_{\alpha,\beta} \cdot h_{\gamma,\delta,\bar{m},\bar{n}} \circ \phi_{-t} \times \phi_s \\ &\times (-1)^{\ell(d-\ell)} \varepsilon(\bar{j}) \varepsilon(\bar{n}) \det(\tilde{D}\phi_{-t})^{\alpha,\gamma}_{\bar{n},\bar{i}} \det(\tilde{D}\phi_s)^{\beta,\delta}_{\bar{j}^c,\bar{n}^c}.\end{aligned} \tag{3.6.7}$$

In the case of $d-1$-forms, from equation (3.6.1) and setting $\omega_2 = \pi_1^*\tilde{\omega} \wedge \pi_2^*\tilde{\omega}$, we have

$$\mathcal{L}^{(d)}_{t,t}(f\omega_2) = f \circ (\phi_{-t} \times \phi_t) \det(D\phi_{-t}) \omega_2. \tag{3.6.8}$$

That is, we recover the same type of operator studied in [43].

We can then proceed exactly as in section 3.4 to prove the following Lemma.

Lemma 3.6.2. *For each $\ell \in \{0, \dots, d-1\}$, $\mathcal{L}^{(\ell)}_{t,s} : \mathcal{B}^{p,q,\ell}_2 \to \mathcal{B}^{p,q,\ell}_2$ are bounded operators. Furthermore, for each $\varrho > 2$ there exists $C > 0$ such that, for $\Re(z) \geq \varrho\hat{\sigma}_\ell$, the operator*

$$R^{(\ell)}_2(z)^n = \frac{1}{(n-1)!^2} \int_0^\infty \int_0^\infty (ts)^{n-1} e^{-z(t+s)} \mathcal{L}^{(\ell)}_{t,s} dt ds \tag{3.6.9}$$

is a linear quasi-compact operator on $\mathcal{B}^{p,q,\ell}_2$ with spectral radius bounded by $C(\sigma_\ell a^{-1})^{2n}$ and essential spectral radius bounded by $C(\sigma_\ell \sigma_{p,q} a^{-1})^{2n}$.

To follow the scheme of [43] we need to define a suitable delta function in $\mathcal{B}^{p,q,\ell}_2$. For each $f, g \in \Omega^{\ell}_s(M)$, let

$$\delta^{\ell}_2(\pi_1^* f \wedge \pi_2^* g) \doteq \int_M \langle f, g \rangle_x \omega(dx).$$

Such a definition extends by linearity to all sections in $\Omega^{\ell}_{2,s}(M)$, thus $\delta^{\ell}_2 \in \Omega^{\ell}_{2,s}(M)'$. Since for $h = \Omega^{\ell}_{2,s}(M)$ we have the coordinate expression

$$h = \sum_{\alpha,\beta} \sum_{\bar{i},\bar{j}} \psi_\alpha(x) \psi_\beta(y) h^{\alpha,\beta}_{\bar{i},\bar{j}}(x,y) dx_{\alpha,\bar{i}} \wedge dy_{\beta,\bar{j}}, \tag{3.6.10}$$

we obtain

$$\begin{aligned}\delta_2^\ell(h) &= \sum_{\alpha,\beta}\sum_{\bar{i},\bar{j}}\int_M \psi_\alpha(x)\psi_\beta(x)h^{\alpha,\beta}_{\bar{i},\bar{j}}(x,x)\langle dx_{\alpha,\bar{i}}, dx_{\beta,\bar{j}}\rangle\omega(dx)\\ &= \sum_{\alpha,\beta}\sum_{\bar{i}}\int_M \psi_\alpha(x)\psi_\beta(x)h^{\alpha,\alpha}_{\bar{i},\bar{i}}(x,x)\omega(dx)\\ &= \sum_{\alpha}\sum_{\bar{i}}\int_M \psi_\alpha(x)h^{\alpha,\alpha}_{\bar{i},\bar{i}}(x,x)\omega(dx).\end{aligned} \tag{3.6.11}$$

Next we need the equivalent of Lemma 3.3.6, the proof is omitted since it is exactly the same as before starting from the space $\Omega^\ell_{2,s}(M)$.

Lemma 3.6.3. *There exists an injective immersion* $\iota : \mathcal{B}_2^{p,q,\ell} \to \left(\Omega^\ell_{2,s}(M)\right)'$

Next we have the following

Lemma 3.6.4. *The current* δ_2^ℓ *extends uniquely to an element of* $\left(\mathcal{B}_2^{p,q,\ell}\right)'$.

Proof. Since $\mathcal{B}_2^{p,q,\ell}$ is defined by the closure of the sections on which δ_2^ℓ is defined, it suffices to prove that there exists $c > 0$ such that $|\delta_2^\ell(h)| \leq c\|h\|_{p,q,2}$. This follows immediately from (3.6.11) since δ_2^ℓ corresponds to integrating on the manifold $W_D := \{(x,y) \in M^2 \ : \ x = y\}$. If $x \in U_\alpha$ we can foliate W_D, in the local chart V_α, with the manifolds $W_{\alpha,G_s} \in \tilde{\Sigma}$ given by the graph of the functions $G_s(x^s, y^u) = (x^s, y^u, s, x^s, y^u, s)$. Accordingly,

$$|\delta_2^\ell(h)| \leq \sum_\alpha |\delta_2^\ell(\psi_\alpha h)| \leq \sum_{\alpha,\bar{i}} \int ds \left|\int_{W_{\alpha,G_s}} \langle \psi_\alpha dx_{\alpha,\bar{i}} \wedge dy_{\alpha,\bar{i}}, h\rangle\right| \leq c\|h\|_{\mathcal{B}_2^{0,q,\ell}}.$$

□

Next we want to see that, if $\Re(z)$ large enough, $R_2(z)\delta_2^\ell$ can be naturally identified with an element of $\mathcal{B}_2^{p,q,\ell}$. Let us begin with a definition

$$J_\varepsilon(x,y) = \sum_\alpha\sum_{\bar{i}} \psi_\alpha(x) j_{\varepsilon,x}(y) dx_{\alpha,\bar{i}} \wedge dy_{\alpha,\bar{i}}. \tag{3.6.12}$$

Then, $J_\varepsilon \in \Omega^\ell_{2,s}(M)$ and given $h \in \Omega^\ell_{2,s}(M)$, remembering the representation (3.6.10),

$$\begin{aligned}\langle J_\varepsilon, h\rangle_{\Omega^{(\ell)}_{2,s}} &= \sum_{\alpha,\bar{i}}\int_M \psi_\alpha(x) j_{\varepsilon,x}(y)\langle dx_{\alpha,\bar{i}} \wedge dy_{\alpha,\bar{i}}, h\rangle_{(x,y)}\omega(x)\omega(y)\\ &= \sum_{\alpha,\bar{i}}\int_M \psi_\alpha(x) j_{\varepsilon,x}(y) h^{\alpha,\alpha}_{\bar{i},\bar{i}}(x,y)\omega(x)\omega(y).\end{aligned}$$

Thus, for each $h \in \Omega^\ell_{2,s}(M)$, $\lim_{\varepsilon\to 0} \iota(J_\varepsilon)(h) = \delta_2^\ell(h)$.

Lemma 3.6.5. *There exists $n_0 \in \mathbb{N}$ such that each $z \in \mathbb{C}$, $\Re(z) \geq \varrho\hat{\sigma}_\ell$, we have that $R_2^{(\ell)}(z)^{n_0} J_\varepsilon$ form a Cauchy sequence in $\mathcal{B}_2^{p,q,\ell}$. We call $\bar{\delta}_2^\ell(z)$ the limit of such a sequence. Moreover, $\iota(J_\varepsilon)$ converges to δ_2^ℓ in $(\mathcal{B}_2^{p,q,\ell})'$.*

Proof. First of all a direct computation shows

$$(\mathcal{L}_{t,s}^{(\ell)} J_\varepsilon)_{\alpha,\beta,\bar{i},\bar{j}} = \sum_{\gamma,\bar{n}} \psi_{\alpha,\beta} \cdot \psi_\gamma(\phi_{-t}(x)) j_{\varepsilon,\phi_{-t}(x)}(\phi_s(y))(-1)^{\ell(d-\ell)}\varepsilon(\bar{j})\varepsilon(\bar{n}) \\ \times \det(\tilde{D}_x\phi_{-t})_{\bar{n},\bar{i}}^{\alpha,\gamma} \det(\tilde{D}_y\phi_s)_{\bar{j}^c,\bar{n}^c}^{\beta,\gamma}.$$

Given a d dimensional admissible manifold $W_{\alpha,\beta,G}$[19] test form $g \in \Gamma_0^{2\ell}(\alpha,\beta,G)$ we can write

$$\begin{aligned}\langle R_2^{(\ell)}(z)^{n_0} J_\varepsilon, g\rangle =& \frac{1}{(n_0-1)!^2}\int_{\mathbb{R}_+^2} ds\, dt\, (ts)^{n_0} e^{-z(t+s)} \sum_{\alpha,\beta,\gamma,\bar{i},\bar{j},\bar{n}} \int_{W_{\alpha,\beta,G}} \psi_\gamma(\phi_{-t}(x)) g_{\alpha,\beta,\bar{i},\bar{j}}(x,y) \\ & \times j_{\varepsilon,\phi_{-t}(x)}(\phi_s(y))(-1)^{\ell(d-\ell)}\varepsilon(\bar{j})\varepsilon(\bar{n}) \det(\tilde{D}_x\phi_{-t})_{\bar{n},\bar{i}}^{\alpha,\gamma} \det(\tilde{D}_y\phi_s)_{\bar{j}^c,\bar{n}^c}^{\beta,\gamma} \\ =& \int_{\mathbb{R}_+^2} ds\, dt\, \frac{e^{-z(t+s)}(ts)^{n_0}}{(n-1)!^2} \sum_{\alpha,\beta,\gamma,\bar{i},\bar{j},\bar{n}} \int_{\phi_{-t}\times\phi_s(W_{\alpha,\beta,G})} \psi_\gamma(x) g_{\alpha,\beta,\bar{i},\bar{j}}(\phi_t(x),\phi_{-s}(y)) \\ & \times j_{\varepsilon,x}(y)(-1)^{\ell(d-\ell)}\varepsilon(\bar{j})\varepsilon(\bar{n}) \det(\tilde{D}_{\phi_t(x)}\phi_{-t})_{\bar{n},\bar{i}}^{\alpha,\gamma} \det(\tilde{D}_{\phi_{-s}(y)}\phi_s)_{\bar{j}^c,\bar{n}^c}^{\beta,\gamma} \\ & \times K_{t,s}(x,y),\end{aligned} \tag{3.6.13}$$

where $K_{t,s}(x,y)$ is the Jacobian of the change of coordinates. Next, note that $\phi_{-t}\times\phi_s(W_{\alpha,\beta,G})$, in coordinates, is a graph of the type

$$G(x^s,y^u) = (x^s, G_u(x^s,y^u), G_{0,1}(x^s,y^u), G_s(x^s,y^u), y^u, G_{0,2}(x^s,y^u)),$$

where $\|DG\| < 1$. Then we perform the change of variables

$$(\xi,\eta) = (x^s - G_s(x^s,y^u), G_u(x^s,y^u) - y^u).$$

Such a change of variables is invertible with determinant bounded by $(ts)^{-n_0}$ for some $n_0 \in \mathbb{N}$. The $\mathcal{B}_2^{0,q,\ell}$ norm is uniformly bounded in ε, via a direct computation in the style of (3.5.10) Section 3.5 . The extension to $\mathcal{B}_2^{p,q,\ell}$ is treated similarly after integrating by part p times. Now given ε,ε' we directly apply our estimates to $\|R_2^{(\ell)}(z)^{n_0}J_\varepsilon - R_2^{(\ell)}(z)^{n_0}J_{\varepsilon'}\|_{p,q,\ell,2}$ to prove the Lemma. □

Corollary 3.6.6. *For each $z \in \mathbb{C}$, $\Re(z) \geq \varrho\hat{\sigma}_\ell$, $\iota(\bar{\delta}_2^\ell(z)) = R_2^\ell(z)'\delta_2^\ell$.*

Lemma 3.6.7. *For each $z \in \mathbb{C}$, such that $\Re(z) \geq \varrho\hat{\sigma}_\ell$, holds true*

$$\delta_2^{(\ell)}\left(\left(R_2^{(\ell)}\right)^{n-n_0}\delta_2^{(\ell)}(z)\right) = \mathrm{Trace}^{(\ell)}(R^{(\ell)}(z)^{2n}), \tag{3.6.14}$$

[19] To make the following calculations more readable, from now on every time we drop the integration variables we mean that we are integrating with respect to the volume of the integration bound.

Proof. From Lemma 3.6.5 we obtain

$$\begin{aligned}\delta_2^\ell\left(\left(R_2^{(\ell)}\right)^{n-n_0}\delta_2^{(\ell)}(z)\right) &= \lim_{\varepsilon_1\to 0}\lim_{\epsilon_2\to 0}\int_{M^2}\langle J_\varepsilon, \left(R_2^{(\ell)}\right)^n J_\varepsilon\rangle \\ &= \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\frac{1}{(n-1)!^2}\int_{M^2}\int_{\mathbb{R}_+^2} ds\,dt(ts)^{n-1}e^{-z(t+s)}\langle J_\varepsilon, \mathcal{L}_{t,s}^{(\ell)}J_\epsilon\rangle.\end{aligned}$$

Next, by (3.6.12), (3.6.7) and remembering footnote 12, we have

$$\begin{aligned}\langle J_{\epsilon_1}, \mathcal{L}_{t,s}^{(\ell)}J_{\epsilon_2}\rangle_{(x,y)} &= \sum_{\alpha,\beta,\bar{i},\bar{j}}\psi_\alpha(x)j_{\epsilon_1,x}(y)\psi_\beta(\phi_{-t}(x))j_{\epsilon_2,\phi_{-t}(x)}(\phi_s(y)) \\ &\quad\times\det(\tilde{D}_x\phi_{-t})_{\bar{j},\bar{i}}^{\alpha,\beta}\det(\tilde{D}_y\phi_s)_{\bar{i}^c,\bar{j}^c}^{\alpha,\beta}\varepsilon(\bar{i})\varepsilon(\bar{j}) \\ &= \sum_{\alpha,\beta,\bar{i},\bar{j}}\psi_\alpha(x)j_{\epsilon_1,x}(y)\psi_\beta(\phi_{-t}(x))j_{\epsilon_2,\phi_{-t}(x)}(\phi_s(y)) \\ &\quad\times\det(\tilde{D}_x\phi_{-t})_{\bar{j},\bar{i}}^{\alpha,\beta}\det(\tilde{D}_{\phi_s(y)}\phi_{-s})_{\bar{j},\bar{i}}^{\alpha,\beta}\det(\tilde{D}_y\phi_s)\end{aligned}$$

Thus, remembering that the integrals are uniformly convergent with respect to time (see Lemma 3.5.3),

$$\begin{aligned}\delta_2^\ell\left(\left(R_2^{(\ell)}\right)^{n-n_0}\delta_2^{(\ell)}(z)\right) &= \frac{1}{(n-1)!^2}\int_{\mathbb{R}_+^2} ds\,dt(ts)^{n-1}e^{-z(t+s)} \\ &\times\lim_{\epsilon_1\to 0}\lim_{\epsilon_2\to 0}\int_{M^2}\sum_{\alpha,\beta,\bar{i},\bar{j}}\psi_\alpha(x)j_{\epsilon_1,x}(\phi_{-s}(y))\psi_\beta(\phi_{-t}(x))j_{\epsilon_2,\phi_{-t}(x)}(y) \\ &\quad\times\det(\tilde{D}_x\phi_{-t})_{\bar{j},\bar{i}}^{\alpha,\beta}\det(\tilde{D}_y\phi_{-s})_{\bar{j},\bar{i}}^{\alpha,\beta} \\ =&\lim_{\epsilon\to 0}\frac{1}{(n-1)!^2}\int_M\int_{\mathbb{R}_+^2} ds\,dt(ts)^{n-1}e^{-z(t+s)} \\ &\times\sum_{\alpha,\bar{i}}\psi_\alpha(x)j_{\epsilon,x}(\phi_{-s-t}(x))\det(\tilde{D}_x\phi_{-t-s})_{\bar{i},\bar{i}}^{\alpha,\alpha}.\end{aligned}$$

By changing variables to $(t', s') \doteq (t+s, t)$ and integrating by parts $n-1$ times w.r.t. μ we obtain

$$\begin{aligned}\delta_2^{(l)}\left((R_2^{(l)})^{n-n_0}\delta_2^{(l)}(z)\right) &= \lim_{\epsilon\to 0}\frac{1}{(n-1)!^2}\int_M\int_0^\infty dt\int_0^t ds(t-s)^{n-1}s^{n-1}e^{-zt} \\ &\times\sum_{\alpha,\bar{i}}\psi_\alpha(x)j_{\epsilon,x}(\phi_{-t}(x))\det(\tilde{D}_x\phi_{-t})_{\bar{i},\bar{i}}^{\alpha,\alpha} \\ =&\lim_{\epsilon\to 0}\int_M\int_0^\infty dt\frac{t^{2n-1}}{(2n-1)!}e^{-zt}j_{\epsilon,x}(\phi_{-t}(x))\left[\sum_{\alpha,\bar{i}}\psi_\alpha(x)\det(\tilde{D}_x\phi_{-t})_{\bar{i},\bar{i}}^{\alpha,\alpha}\right].\end{aligned}$$

Note that the integrand is different from zero only for $d(x, \phi_{-t}(x)) \leq \varepsilon$, we can then use the piece of orbit $\{\phi_{-s}(x)\}_{s\in[0,t]}$ to create an arbitrarily close ε-pseudoorbit. Then, for ε small enough, Theorem 3.5.2 implies that there exists a unique close orbit τ such that $d(\phi_{-s}(x),\tau)\leq c\varepsilon$ for all $s\in[0,t]$ and $|\lambda(\tau)-t|\leq c\varepsilon\lambda(\tau)$. Hence $(x,t)\in\Omega_\tau=\{(x,t)\in M\times\mathbb{R}_+ \;:\; |t-\lambda(\tau)|\leq\varepsilon c\lambda(\tau),\, d(\phi_{-s}(x),\tau)\leq c\varepsilon\,\forall s\in(0,(1+c\varepsilon)\lambda(\tau))\}$.[20] That is, setting $F_z(x,t)=\frac{t^{2n-1}}{(2n-1)!}e^{-zt}\sum_{\alpha,\bar{i}}\psi_\alpha(x)\det(D_x\phi_{-t})^{\alpha,\alpha}_{\bar{i},\bar{i}}$ and $M_+=M\times\mathbb{R}_+$ we have

$$\int_{M_+} j_{\epsilon,x}(\phi_{-t}(x))F_z(x,t)=\sum_{\tau\in\mathcal{T}}\int_{\Omega_\tau} j_{\epsilon,x}(\phi_{-t}(x))F_z(x,t).$$

Next, by (3.5.3) we have

$$\int_{M_+} j_{\epsilon,x}(\phi_{-t}(x))F_z(x,t)=\sum_{\tau\in\mathcal{T},\alpha}\int_{\Omega_\tau} F_z(x,t)g_\varepsilon(x)\psi_\alpha(x)k(\varepsilon^{-1}(\Theta_\alpha(x)-\Theta_\alpha(\phi_{-t}(x))).$$

Note that, for $\Re(z)$ large enough, the sum is uniformly convergent in ε since the number of closed orbits grows at most exponentially. It is then convenient to define

$$F_{z,\alpha,\varepsilon}(\xi,t)=F_z(\Theta_\alpha^{-1}(\xi),t)g_\varepsilon(\Theta_\alpha^{-1}(\xi))\psi_\alpha(\Theta_\alpha^{-1}(\xi))\bar{\omega}_\alpha(\xi)$$

where $\bar{\omega}_\alpha(\xi)d\xi$ is the induced volume form in V_α. We also define $\phi^\alpha_{-t}=\Theta_\alpha\circ\phi_{-t}\circ\Theta_\alpha^{-1}$ and recall that for our adapted charts we have $\Theta_\alpha(x,t)=(\Theta_\alpha(x),t)$. We can then write

$$\int_{M_+} j_{\epsilon,x}(\phi_{-t}(x))F_z(x,t)=\sum_{\tau\in\mathcal{T},\alpha}\int_{\bar{\Theta}_\alpha(\Omega_\tau\cap U_\alpha\times\mathbb{R})} F_{z,\alpha,\varepsilon}(\xi,t)k(\varepsilon^{-1}(\xi-\phi^\alpha_{-t}(\xi)))\,d\xi\,dt.$$

Note that $\bar{\Theta}_\alpha(\Omega_\tau\cap U_\alpha\times\mathbb{R}_+)$ consists of ε-neighborhood of a finite number of lines (the connected pieces of $\Theta_\alpha(\tau\cup U_\alpha)$), let us call $\{\bar{\Omega}_{\alpha,i}\}$ the collection of such connected components. Let us now consider the changes of variables $\Xi_{\alpha,i}:\bar{\Omega}_{\alpha,i}\to\mathbb{R}^{d+1}$ defined by

$$\begin{aligned}\Xi_{\alpha,i}(\xi,t)&=(\xi_1-\phi^\alpha_{-t}(\xi)_1,\ldots,\xi_{d-1}-\phi^\alpha_{-t}(\xi)_{d-1},\xi_d,t)\\&=(\zeta_1,\ldots,\zeta_{d-1},\rho,s).\end{aligned}$$

Let us introduce the matrices $\Lambda_{i,j}=\delta_{i,j}-\frac{\partial(\phi^\alpha_{-t})_i}{\partial\xi_j}$, $i,j\in\{1,\ldots,d-1\}$. Note that $\det(D\Xi_{\alpha,i})=\det(\Lambda)\neq 0$ since ϕ^α is hyperbolic. Moreover if $\Xi_{\alpha,i}(\xi,t)=\Xi_{\alpha,i}(\xi',t')$, then $t=t'$, $\xi_d=\xi'_d$, hence $\|\xi-\xi'\|\leq c\varepsilon$ and thus $\xi=\xi'$ since $\Xi_{\alpha,i}$ is a local diffeomorphism. Also note that for $(\xi,t)\in\bar{\Omega}_{\alpha,i}$, setting $\xi=(\tilde{\xi},\xi_d)$ we can define a smooth function $r_{\alpha,i}(\tilde{\xi})$ such that $\phi_{-r_{\alpha,i}(\tilde{\xi})}(\tilde{\xi},0)_d=0$. By construction, for each α,i, there exists a unique $p_{\alpha,i}\in\mathbb{R}^{d-1}$ such that $(p_{\alpha,i},0)\in\tau$ and $((p_{\alpha,i},0),\lambda(\tau))\in\Omega_{\alpha,i}$, then $r_{\alpha,i}(p_{\alpha,i})=\lambda(\tau)$ and $\phi^\alpha_{-t}(\xi)=\phi^\alpha_{-r_{\alpha,i}(\tilde{\xi})}(\tilde{\xi},0)+(0,-t+r_{\alpha,i}+\xi_d)$. Then $\Xi(p,0,t)=(0,0,t)$, hence

$$\begin{aligned}\tilde{\xi}&=p+\Lambda^{-1}(p)\zeta+\mathcal{O}(\zeta^2)\\ r_{\alpha,i}(\tilde{\xi})&=\lambda(\tau)+\partial_{\tilde{\xi}}r_{\alpha,i}(p)\Lambda^{-1}(p)\zeta+\mathcal{O}(\zeta^2).\end{aligned}$$

[20] Note that, by Theorem 3.5.2 again, $\Omega_\tau\cap\Omega_{\tau'}\neq\emptyset$ implies $\tau=\tau'$. Hence such sets are all disjoint.

It follows,

$$\int_{M_+} j_{\epsilon,x}(\phi_{-t}(x))F_z(x,t) = \sum_{\tau\in\mathcal{T},\alpha}\int_{\Xi_{\alpha,i}(\bar\Omega_{\alpha,i})} F_{z,\alpha,\varepsilon}\circ\Xi_{\alpha,i}^{-1}(\zeta,\rho,s) \\ \times k_{d-1}(\varepsilon^{-1}\zeta)k_1(\varepsilon^{-1}(s-r_{\alpha,i}(\tilde\xi))\,d\zeta\,ds\,d\rho.$$

At this point we make the change of variables $\eta=\varepsilon^{-1}\zeta$, $v=\varepsilon^{-1}(s-\lambda(\tau))$, yielding

$$\int_{M_+} j_{\epsilon,x}(\phi_{-t}(x))F_z(x,t) = \sum_{\tau\in\mathcal{T},\alpha}\int_{\Xi_{\alpha,i}(\bar\Omega_{\alpha,i})} F_{z,\alpha,\varepsilon}\circ\Xi_{\alpha,i}^{-1}(\varepsilon\eta,\rho,\lambda(\tau)+\varepsilon v) \\ \times k_{d-1}(\eta)k_1(v-\partial_{\tilde\xi}r_{\alpha,i}(p)\Lambda^{-1}(p)v+\mathcal{O}(\varepsilon v^2))\,d\eta\,dv\,d\rho.$$

We can now take the limit for $\varepsilon\to 0$ and obtain, after integrating first in v and then in η,

$$\lim_{\varepsilon\to 0}\int_{M_+} j_{\epsilon,x}(\phi_{-t}(x))F_z(x,t) = \sum_{\tau\in\mathcal{T}}\sum_{\alpha,i}\int_{-\delta}^{\delta} F_{z,\alpha,0}(\phi_\rho^\alpha(p_{\alpha,i},0),\lambda(\tau))d\rho.$$

Thus,

$$\delta_2^{(\ell)}\left((R_2^{(\ell)})^{n-n_0}\delta_2^{(\ell)}(z)\right) = \frac{1}{(2n-1)!}\sum_{\tau\in\mathcal{T}}\frac{1}{\mu(\tau)}\frac{\operatorname{tr}(\wedge^\ell(D_{\text{hyp}}\phi_{-\lambda(\tau)}))\lambda(\tau)^{2n}e^{-z\lambda(\tau)}}{\det\left(\mathbb{1}-D_{\text{hyp}}\phi_{-\lambda(\tau)}\right)}.$$

□

We obtained an expression only for the traces of even exponents of the resolvent. Completely analogous computations yields the following Lemma.

Lemma 3.6.8. *Let $\mathbb{1}$ be the identity operator on the space $\mathcal{B}_2^{p,q,\ell}$*

$$\delta_2^{(\ell)}\left(\left(R_2^{(\ell)}\right)^{n-n_0}\left(R_2^{(\ell)}(z)\times\mathbb{1}\right)\left(\delta_2^{(\ell)}(z)\right)\right) = \text{Trace}\left(R^{(\ell)}(z)^{2n+1}\right) \tag{3.6.15}$$

Finally we can harvest all the previous results. By Lemma 3.6.2 we have $R_2^{(\ell)}(z)=P_2^{(\ell)}(z)+U_2^{(\ell)}(z)$ where $P_2^{(\ell)}(z)$ is a finite rank operator, the spectral radius of $U_2^{(\ell)}(z)$ is bounded by $(\sigma_\ell\sigma_{p,q}a^{-1})^2$ and $P_2^{(\ell)}(z)U_2^{(\ell)}(z)=U_2^{(\ell)}(z)P_2^{(\ell)}(z)=0$.

Lemma 3.6.9. *Let $R^{(\ell)}(z)=P^{(\ell)}(z)+U^{(\ell)}(z)$ be the spectral decompositions of $R^{(\ell)}(z)$. Then*

$$\delta_2^{(\ell)}\left(\left(P_2^{(\ell)}\right)^{n-n_0}\delta_2^{(\ell)}(z)\right) = \operatorname{tr}\left(P^{(\ell)}(z)^{2n}\right). \tag{3.6.16}$$

Proof. By the previous lemma we already have that, for $h=\pi_1^*f\wedge\pi_2^*g$,

$$\delta_2^{(\ell)}\left(R_2^{(\ell)}(z)^n h\right) = \int_M\langle R^{(\ell)}(z)^2(f),g\rangle\omega(dx) \tag{3.6.17}$$

Now consider the Von Neumann expansion for the operator $(\xi\mathbb{1} - R_2^{(\ell)}(z))^{-1}$, a path Γ outside the essential spectrum of $R^2(z)$ and $R_2^{(\ell)}(z)$. Thus for h as before

$$\begin{aligned}\delta_2^{(\ell)}\left(U_2^{(\ell)}(z)h\right) =& \frac{1}{2\pi i}\int_\Gamma \delta_2^{(\ell)}\left(\left(\xi\mathbb{1} - R_2^{(\ell)}(z)\right)^{-1} h\right)\xi d\xi \\ &= \frac{1}{2\pi i}\int_M\int_\Gamma \langle\left(\xi\mathbb{1} - R^{(\ell)}(z)^{2n}\right)^{-1} f, g\rangle\omega(dx)d\xi \\ &= \int_M \langle U^{(\ell)}(z)^2 f, g\rangle\omega(dx)\end{aligned} \tag{3.6.18}$$

Hence, by the spectral decomposition $P^{(\ell)}(z) = R^{(\ell)}(z) - U^{(\ell)}(z)$, we obtain

$$\delta_2^{(\ell)}\left(P_2^{(\ell)}(z)^n h\right) = \int_M \langle P^{(\ell)}(z)^{2n} f, g\rangle\omega(dx) \tag{3.6.19}$$

Next, since $P^{(\ell)}$ is a finite rank operator, we have $P^{(\ell)}(z)(f) = \sum_k \nu_k u^k v^k(f)$. Thus for a general h we have

$$\delta_2^{(\ell)}\left(P_2^{(\ell)}(z)^n h\right) = \sum_k \nu_n^{2n} \sum_{\alpha,\beta,\bar{i},\bar{j}} \int_{M^2} u^k_{\alpha,\bar{i}}(x) v^k_{\beta,\bar{j}}(y) h^{\alpha,\beta}_{\bar{i},\bar{j}}(x,y)\omega(dx)\omega(dy)$$

Finally we have

$$\delta_2^{(\ell)}\left(\left(P_2^{(\ell)}\right)^{n-n_0}\delta_2^{(\ell)}(z)\right) = \lim_{\varepsilon\to 0}\delta_2^{(\ell)}\left(P^{(\ell)}(z)^{2n} J_\varepsilon\right) = \sum_k \nu_n^{2n} v^k(u_k).$$

$\square$

The same ideas can be applied to the operator $\left(R_2^{(\ell)}(z)^n(R^{(\ell)}(z) \times \mathcal{I})\right)$. Hence we can write, regardless if $R^{(\ell)}(z)^n$ is raised to an odd or to an even exponent.

$$\begin{aligned}\mathrm{Trace}(R^{(\ell)}(z)^n) &= \mathrm{tr}(P^{(\ell)}(z)^n) + \mathcal{O}\left(\left[\sigma_\ell\sigma_{p,q}\Re(z)^{-1}\right]^n\right) \\ &= \frac{1}{(n-1)!}\sum_\tau \frac{\mathrm{tr}\left(\wedge^l(D_{\mathrm{hyp}}\phi_{\lambda(\tau)})\right)\lambda(\tau)^n e^{-s\lambda(\tau)}}{\det(I - D_{\mathrm{hyp}}\phi_{\lambda(\tau)})}\end{aligned} \tag{3.6.20}$$

Chapter 4

Ongoing and Future Research Projects

For fifteen days I strove to prove that there could not be any functions like those I have since called Fuchsian functions. I was then very ignorant; every day I seated myself at my work table, stayed an hour or two, tried a great number of combinations and reached no results. One evening, contrary to my custom, I drank black coffee and could not sleep. Ideas rose in crowds; I felt them collide until pairs interlocked, so to speak, making a stable combination.
H. Poincaré

In this section I include some of the work that has been done in these years which lie the foundations for projects which are not yet completed.

4.1 L-functions

Selberg's and Ruelle zeta functions have been actively studied for their ability to collect information on the spacial distribution of orbits in the sense of uniformity and, as shown in the preceding chapters, many results have been obtained from this point of view. Here we are interested in studying the topological distribution of orbits. To capture such a behavior is necessary to go beyond the zeta functions and strive for results on more general L-functions

Let h be an l-form on some anisotropic space, ϕ is an Anosov flow, ω a closed 1-form, X be the vector field generating the Anosov flow then one could define

$$\mathcal{L}_t^{(l,\omega)}h(x) \doteq (\phi_t^* h) \cdot \exp\left(2\pi i \int_{-t}^{0} \omega(X(\phi_{-t}(x)))dt\right) \tag{4.1.1}$$

which, in the case of 0-forms, resemble the expected transfer operator

$$\mathcal{L}_t^{(\omega)}h(x) \doteq h \circ \phi_{-t} |\det(D\phi_t)|^{-1} \cdot \exp\left(2\pi i \int_{-t}^{0} \omega(X(\phi_{-t}))dt\right) \tag{4.1.2}$$

Let $\mathrm{tr}^{(l)}$ be a suitable trace, in the sense of the previous chapter. Then, by linearity,

$$\mathrm{tr}^{(l)}\left((\phi_t^* h)\cdot\exp\left(2\pi i\int_{-t}^{0}\omega(X(\phi_{-t}(x)))dt\right)\right)=\mathrm{tr}(\phi_t^* h)\cdot\exp\left(2\pi i\int_{-t}^{0}\omega(X(\phi_{-t}(x)))dt\right)$$

Following what we have previously done we can now estimate an approximated $\mathrm{Trace}(\mathcal{R}_t^{(l,\omega)})$ as following

$$\mathrm{Trace}(\mathcal{R}_t^{(l,\omega)}(z))=\sum_{\tau}\frac{\mathrm{tr}\wedge^l(D_\tau\phi)}{\det(I-D_\tau\phi)}\exp\left(-z\lambda(\tau)+2\pi i\int_{-t}^{0}\omega(X(\phi_{-t}(x)))dt\right) \tag{4.1.3}$$

Hence the results obtained in the previous work seems likely to extend to L-functions in the sense that

$$L(z,\chi)=\prod_{\tau}\det\left(\mathbb{1}-\chi([\tau])e^{-z\lambda(\tau)}\right)^{-1} \tag{4.1.4}$$

is meromorphic in the whole complex plane and that by studying this class of functions one is likely to recollect that $L(0,\chi)$ is in some sense the “torsion” of our space. Thus extending typical torsion result to the case of variable curvature.

4.2 Decay of Correlations in Economical systems

The aim of this research project is to study ergodic properties of economical dynamical systems with the tools of hyperbolic theory. The original proposed project was partly financed by AMaMeF, and some of the preliminary work has been carried out. The main focus was, and still is, to outline a strategy to approach the correlation function $\rho(t)$ and analyze how its properties can lead to averaging results relevant to economic analysis. Here I would like to thank AMaMeF for giving me the opportunity to start this project. I also would like to thank Prof. Mark Pollicott for hosting my visit.

In 1998 Stephen Smale prepared a list of mathematical problem as a response to the International Mathematical Union which included the following:

> “Extend the mathematical model of general equilibrium theory to include price adjustments.”

He believed that the solution, which he looked for in the framework of differentiable dynamical system ([75],[76]) as well as any major advance towards it, would lead to new insights both to mathematical knowledge and to economical understanding.

The spirit of this research project is along those lines: define and study a dynamical system model, whose states are economics variables, where the macroscopic overall behavior, compatible with the existing equilibrium theory, is obtained as the result of a deterministic dynamics. In well-established models the study is concentrated on the analysis of excess demand and supply, where convergence it is proven mainly through fixed point arguments; we identify these variables with standard macroscopic observables (as one would do for energy or temperature in physical deterministic systems) to related microscopic deterministic behavior. That is we are trying to set that the rational behavior of single agents produces stable phenomena at macroscopic level as well as large fluctuations or chaotic motions.

As it happened in nineties for general dynamical systems, shifting to a functional analysis approach might produce new and interesting results. In this sense equilibrium models provide an interesting class of models to which ergodic theories could be applied, as it is clearly pointed out by Brock and Dechert[14]. This viewpoint is also supported in literature where the emergence of chaotic behavior has already been studied in economic growth by Boldrin [10].

The main idea is to model the time evolution of aggregate variables as hyperbolic flows, and after choosing suitable Banach spaces ([30],[15]), we will study the properties of weighted transfer operators on such spaces. Note that with this approach, by studying the evolution of measures we are able to overcome difficulties, typical of economical analysis, related to choosing a “part” of the population in study. To be more precise let M be an m-dimensional manifold and ϕ_t an hyperbolic flow which represents the demands for m goods , where $D\phi_t$ can be modeled to reflect real demand. Here we are interested in studying the ergodic properties of $\mathcal{L}_t$. One strives for the standard chain of results, after obtaining the quasi-compactness of the operator, ergodic properties such as mixing, existence of central limit theorem, large deviations. Special attention will be posed to the perturbations of such model, since they have strong impact on applications.

After a suitable framework will be established, particular care will be spent in translating the features of the systems in understanding the financial implications of it. First, as it is the most obvious result but still necessary, it will be important to show that convergence of densities under the action of the operator in the chosen space is equivalent to equilibrium in real life situation. This first step will be achieved by carefully choosing the hypothesis of the model. Second, one aim will be to gain better insight in the problem of noisy data series, which is common in economics. In fact if the dynamics is strong enough, in the hyperbolic sense, noise will have a short term impact on the system, while if not it will show the impossibility of making forecast: through an explicit estimates of the decay of correlation of initial states one will be able to find the time span of the validity of the forecast. Third, it will be noticeable to show that how such flow may highlight resonance effects between the microscopical agents, a feature which is hard to see in stochastic models, due to the properties of Brownian motions, but commonly showed by markets. In particular it will be interesting to investigate the difference in estimates which come from looking at equilibrium theory using stochastic analysis or by using differentiable dynamical systems.

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