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Simple linear compactifications of spherical homogeneous spaces

Dottorato in Matematica, Roma «La Sapienza» (2010).

<http://www.bdim.eu/item?id=tesi_2010_GandiniJacopo_1>

SAPIENZA
UNIVERSITÀ DI ROMA

Simple linear compactifications of spherical homogeneous spaces

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A thesis submitted in fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics

December 15, 2010

Thesis not yet defended

Jacopo Gandini. *Simple linear compactifications of spherical homogeneous spaces.*
Ph.D. thesis. Sapienza – University of Rome

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Acknowledgments

I express my gratitude to A. Maffei, who proposed me the problems I deal with in this thesis, for his continuous advice and constant help. As well, I express my gratitude to P. Bravi, G. Pezzini and A. Ruzzi for their advice and for a lot of discussions. Finally, I thank all that people who made this thesis possible, and especially all my friends of Ph.D. in Rome.

Luna diagrams have been made with the package lunadiagrams*, made by P. Bravi and available on his personal website http://www.mat.uniroma1.it/∼bravi.*

Contents

Introduction

Homogeneous varieties of algebraic groups are an object which arises naturally in algebraic geometry and in representation theory. Among them, those understood better are the compact ones, like projective spaces and Grassmannians, which play a fundamental role in the representation theory of semisimple groups.

When it is given a homogeneous variety G/H for an algebraic group G, it is natural to study its compactifications, i.e. to construct complete varieties acted by G and possessing an open orbit isomorphic to G/H. By a theorem of Chevalley, any closed subgroup H is the stabilizer of a line in a finite dimensional rational G-module V: hence the homogeneous variety G/H can be realized as an orbit in the projective space $\mathbb{P}(V)$. Therefore a very natural class of compactifications to study is the class of those arising as an orbit closure in the projective space of a finite dimensional rational G-module V: similar compactifications are called *linear*.

A basic case is that of a compactification possessing a unique closed orbit: such compactifications are called *simple*: for instance, if G is connected, by a theorem of Sumihiro any simple normal compactification of G/H is linear.

If G is a connected and reductive group over an algebraically closed field $\Bbbk$ of characteristic zero, D. Luna and Th. Vust developed in [LV 83] a theory for classifying the normal equivariant compactifications (and more generically the normal equivariant embeddings) of a given homogeneous variety G/H. An important invariant attached to G/H which in a certain sense controls the complexity of its embedding theory is the minimal codimension of an orbit for a Borel subgroup $B \subset G$, which is called the *complexity* of G/H.

A homogeneous variety G/H for a connected reductive group G is called *spherical* if it has complexity zero. More generally, we will call *spherical variety* any embedding of a spherical homogeneous space. In the case of spherical varieties, the theory of normal equivariant embeddings developed in [LV 83] becomes particularly elegant and it can be formulated in purely combinatorial terms, generalizing the theory of normal embeddings of toric varieties which are spherical regarding $T = B = G$.

However many natural examples of equivariant compactifications (e.g. the linear ones) of a spherical homogeneous variety need not to be normal. The main object of the present work will be a special class of simple linear compactifications of a spherical homogeneous variety, namely those embedded in the projective space of a simple G-module. For these compactifications, we will study their orbit structure and that of their normalizations, giving as well necessary and sufficient conditions for their normalization morphisms to be bijective.

An important example which we will analyze in details in the last part of this work is that of the group G itself, regarded as a spherical $G \times G$-variety via the isomorphism

$$G \simeq G \times G \big/ \mathrm{diag}(G),$$

where $\mathrm{diag}(G) \subset G \times G$ is the diagonal: its sphericity follows from the Bruhat decomposition of G. In the particular case of a semisimple adjoint group, we will propose a strategy to classify all its simple linear compactifications and we will accomplish the aim in the case of an orthogonal group.

Spherical varieties and wonderful varieties.

Spherical homogeneous varieties can be defined by many equivalent properties. Their most important characterizations are the followings:

– Every Borel subgroup has an open orbit in G/H;

– Every equivariant completion of G/H possesses finitely many G-orbits;

– Given any G-linearized line bundle $\mathcal{L} \in \mathrm{Pic}(G/H)$, its space of global sections $\Gamma(G/H, \mathcal{L})$ is a multiplicity free G-module, i.e. every isotypic component is irreducible.

Because of the latter property, spherical varieties are sometimes called *multiplicity-free*: as shown by M. Brion in [Bri 87] they can be regarded as the algebraic counterpart of the *multiplicity-free manifolds* introduced in Hamiltonian geometry by V. Guillemin and S. Sternberg in [GS 84].

We will say that a subgroup $H \subset G$ is *spherical* if G/H is so. Spherical homogeneous varieties include many important examples, e.g.:

– *Flag varieties*, i.e. the compact homogeneous spaces;

– *Tori* (i.e. if G is a torus and H is trivial);

– *Symmetric varieties*, i.e. if H is the set of fixed points of an algebraic involution of G: we will say then that H is a *symmetric subgroup*;

– *Model varieties*, i.e. if G/H is quasi-affine and its coordinate ring contains every irreducible representation of G exactly once: we will say then that H is a *model subgroup*.

A very special class of compactifications of a spherical homogeneous variety which generalizes the class of flag varieties is that of *wonderful varieties*, which were first introduced by C. De Concini and C. Procesi in [DCP 83] in the context of symmetric varieties and then studied in generality by D. Luna in [Lu 96] and [Lu 01]. A compactification M of a homogeneous space G/H is called *wonderful* if it has the following properties:

– M is smooth and projective;

– the complement of the open orbit is a union of smooth prime divisors having non-empty transversal intersection;

– any orbit closure in M equals the intersection of the prime divisors containing it.

If G/H admits a wonderful completion M, then H is spherical and M is maximal among its simple compactifications: if X is any other simple compactification of G/H, then it is dominated by M. We will say that a spherical subgroup is *wonderful* if it occurs as the generic stabilizer of a wonderful variety.

Not every spherical homogeneous space admits a wonderful completion. However every self-normalizing spherical subgroup is wonderful and every wonderful subgroup has finite index in its normalizer.

Wonderful varieties play a fundamental role in the classification of spherical varieties, which can be reduced to that of wonderful varieties. In [Lu 01], D. Luna started a program for classifying them via a triple of combinatorial invariants and developed an appropriate combinatorial and diagrammatic language (*spherical systems* and *Luna diagrams*) for the study of wonderful varieties.

Spherical orbit closures in simple projective spaces.

Consider the simplest case of a linear compactification X of a spherical variety G/H which is embedded in the projective space of a simple G-module V: we will call then $\mathbb{P}(V)$ a *simple projective space*. Since the center of G acts trivially on any simple projective space, we may assume that G is semisimple.

P. Bravi and D. Luna showed in [BL 08] that any spherical subgroup which occurs as the stabilizer of a point in a simple projective space is wonderful. Since $\mathbb{P}(V)$ possesses a unique closed orbit, if M is the wonderful completion of G/H, then the morphism $G/H \to X$ extends to M and, if $\widetilde{X} \to X$ is the normalization, we get a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & \widetilde{X} \arrow[d] \\ & X \subset \mathbb{P}(V) \end{tikzcd}$$

Examining such morphisms, in Section 3.2 we will obtain a description of the set of orbits of X and of $\widetilde{X}$ in terms of their spherical systems and spherical diagrams. Moreover this will lead to a combinatorial criterion to establish whether or not two orbits in M map on the same orbit in X, which in particular implies that different orbits in X are never G-equivariantly isomorphic.

Under some assumptions on H (e.g. if it contains a symmetric subgroup or a model subgroup of G), if the highest weight of V is as regular as possible among those weights whose associated module possesses a line fixed by H, then X is the wonderful compactification of G/H. Wonderful varieties admitting an embedding in a simple projective space are called *strict* and they have been introduced by G. Pezzini in [Pe 07]: a wonderful variety M is strict if and only if the isotropy group of any point $x \in M$ is self-normalizing.

Our main theorem concerning the closure of a spherical orbit in a simple projective space is a combinatorial criterion for the normalization $\widetilde{X} \to X$ to be bijective (Theorem 3.3.9): this is done under the assumption that M is strict. The condition of bijectivity involves the double links of the Dynkin diagram of G and is easily read off by the Luna diagram of M. In particular it is trivially fulfilled whenever G is of type ADEG or if H contains a symmetric subgroup of G, while the most significant examples where bijectivity fails arise if H is the normalizer of a model subgroup: the general strict case is substantially deduced from this case.

A very paradigmatic case is that of the *wonderful model variety* M_G^{mod}, introduced by D. Luna in [Lu 07]: this is a wonderful variety whose orbits naturally parametrize up to isomorphism the model varieties for G. More precisely, every orbit of M_G^{mod} is of the shape $G/N_G(H)$, where G/H is a model variety, and conversely this correspondence gives a bijection up to isomorphism. In particular, this constructions highlights a special model subgroup $H_G^{\mathrm{mod}} \subset G$ (defined up to conjugation) which determines every model variety for G, namely that which fixes a point in the open orbit of M_G^{mod}.

In order to illustrate the above mentioned criterion of bijectivity in this case, let's set up some further notation. Fix a maximal torus $T \subset G$ and a Borel subgroup $B \supset T$, denote S the associated set of simple roots. If $G_i \subset G$ is a simple factor of type B or C, number the associated subset of simple roots $S_i = \{\alpha_1^i, \ldots, \alpha_{r_i}^i\}$ starting from the extreme of the Dynkin diagram of G_i which contains the double link; define moreover $S_i^{\mathrm{even}}, S_i^{\mathrm{odd}} \subset S_i$ as the subsets of those elements whose index is respectively even and odd. If λ is a dominant weight, define its *support* as the set of simple roots non-orthogonal to it and, if they are defined, set

$$e_i(\lambda) = \min\{k \leqslant r_i \,:\, \alpha_k^i \in \mathrm{Supp}(\lambda) \cap S_i^{\mathrm{even}}\}$$

$$o_i(\lambda) = \min\{k \leqslant r_i \,:\, \alpha_k^i \in \mathrm{Supp}(\lambda) \cap S_i^{\mathrm{odd}}\}$$

or set $e_i(\lambda) = +\infty$ (resp. $o_i(\lambda) = +\infty$) otherwise. Finally, if G_i is of type F_4, number the simple roots in $S_i = \{\alpha_1^i, \alpha_2^i, \alpha_3^i, \alpha_4^i\}$ starting from the extreme of the Dynkin diagram which contains a long root.

Theorem (see Theorem 3.3.9)**.** *Suppose that V is a simple G-module of highest weight λ and suppose that $v \in V$ generates a line whose stabilizer is the normalizer of H_G^{mod}; denote $X = \overline{G[v]} \subset \mathbb{P}(V)$. Then the normalization $\widetilde{X} \to X$ is bijective if and only if the following conditions are fulfilled, for every connected component $S_i \subset S$:*

b) If S_i is of type B, then either $o_i(\lambda) = 1$ or $e_i(\lambda) = +\infty$.

c) If S_i is of type C, then $o_i(\lambda) \geqslant e_i(\lambda) - 1$.

f) If S_i is of type F_4 and $\alpha_2^i \in \mathrm{Supp}(\lambda)$, then $\alpha_3^i \in \mathrm{Supp}(\lambda)$ as well.

Simple linear compactifications of semisimple adjoint groups.

Suppose that G is semisimple and simply connected and denote G_{ad} the corresponding adjoint group. We will now consider a very special case of the previous situation. Fix a maximal torus $T \subset G$ and a Borel subgroup $B \supset T$ and denote S the associated set of simple roots; denote $\mathcal{X}(B)^+$ the set of dominant weights. If $\lambda \in \mathcal{X}(B)^+$, denote $V(\lambda)$ the simple G-module of highest weight λ and consider the $G \times G$-variety

$$X_\lambda = \overline{(G \times G)[\mathrm{Id}]} \subset \mathbb{P}(\mathrm{End}(V(\lambda)))$$

which is a simple compactification of a quotient of G_{ad}: since $\mathrm{End}(V(\lambda))$ is a simple $G \times G$-module, this is a particular case of the situation considered above.

More generally, consider the following situation. Denote $\leqslant$ the *dominance order* on $\mathcal{X}(B)^+$, defined by $\mu \leqslant \lambda$ if and only if $\lambda - \mu \in \mathbb{N}S$, and if $\lambda \in \mathcal{X}(B)^+$ denote

$$\Pi^+(\lambda) = \{\mu \in \mathcal{X}(B)^+ \,:\, \mu \leqslant \lambda\}.$$

If $\Pi \subset \mathcal{X}(B)^+$, call it *simple* if there exists $\lambda \in \Pi$ such that $\Pi \subset \Pi^+(\lambda)$, i.e. if it possesses a unique maximal element w.r.t. $\leqslant$. Suppose that this is the case and denote $E(\Pi) = \bigoplus_{\mu \in \Pi} \mathrm{End}(V_\mu)$ and $\mathrm{Id}_\Pi = (\mathrm{Id}_\mu)_{\mu \in \Pi} \in E(\Pi)$ and consider the $G \times G$-variety

$$X_\Pi = \overline{(G \times G)[\mathrm{Id}_\Pi]} \subset \mathbb{P}(E(\Pi)).$$

In [Ka 02] S. S. Kannan proved that $X_{\Pi^+(\lambda)}$ is projectively normal, while in [DC 04] C. De Concini proved that $X_{\Pi^+(\lambda)} = \widetilde{X}_\lambda$ is the normalization of X_λ. In particular, if Π is simple with maximal element λ, we get equivariant morphisms

$$\widetilde{X}_\lambda \longrightarrow X_\Pi \longrightarrow X_\lambda$$

and X_Π is a simple variety with the same normalization of X_λ.

Since any simple linear compactification of a quotient of G is of the shape X_Π for some simple subset Π, this gives a strategy to classify these varieties, namely by classifying the simple subsets which give rise to isomorphic compactifications.

In case λ is a regular weight, then $X_\lambda = M$ is the wonderful compactification of G_{ad}. Together with P. Bravi, A. Maffei and A. Ruzzi, in [BGMR 10] we studied the degenerate cases and we gave a complete classification of the normality and of the smoothness of these varieties. In particular, we proved that X_λ depends only on the support $\mathrm{Supp}(\lambda)$ and that it is normal if and only if λ satisfies the following condition

($\star$) For every non-simply laced connected component $S' \subset S$, if $\mathrm{Supp}(\lambda) \cap S'$ contains a long root, then it contains also the short root which is adjacent to a long simple root.

In particular, if the Dynkin diagram of G is simply laced, it follows that X_λ is always normal and every simple linear compactification of G_{ad} is normal. Otherwise, if G possesses a non-simply laced simple factor, excepted some very special cases it possesses a lot of simple linear compactifications which are not normal.

If $\lambda \in \mathcal{X}(B)^+$, a weight $\mu \leqslant \lambda$ is called *trivial* if $X_{\{\lambda,\mu\}}$ is equivariantly isomorphic to X_λ. Denote $\Pi_{tr}(\lambda) \subset \Pi^+(\lambda)$ the subset of trivial weights, including there λ as well.

If G is a simple group of type B_r, by using Schur-Weyl duality for orthogonal groups, we will give a combinatorial description of $\Pi_{tr}(\lambda)$ (Theorem 4.2.2). In this case, we will deduce then the classification of the simple linear compactifications of $\mathrm{SO}(2r+1)$, classifying as well their linear embeddings.

Suppose that $\nu \leqslant \mu \leqslant \lambda$ are non-trivial weights. Then it is possible to define a partial order relation $\leqslant_\lambda$ on $\Pi^+(\lambda) \smallsetminus \Pi_{tr}(\lambda)$ with the following geometrical meaning:

$$\nu \leqslant_\lambda \mu \qquad \text{if and only if} \qquad \begin{array}{c}\text{there exists a } G \times G\text{-morphism}\\ X_{\{\lambda,\mu\}} \longrightarrow X_{\{\lambda,\nu\}}\end{array}$$

This is done as follows.

Definition. Suppose that $G = \mathrm{Spin}(2r+1)$ and suppose that $\nu \leqslant \mu \leqslant \lambda$ are non-trivial weights and set $\mu - \nu = \sum_{i=1}^{r} a_i \alpha_i$. Then we say that μ and ν are λ-*comparable* and we write $\nu \leqslant_\lambda \mu$ if following conditions are fulfilled:

(λ-C1) If α_p is the first simple root in $\mathrm{Supp}(\lambda)$, then $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_p$

(λ-C2) If $\alpha_s, \alpha_t \in \mathrm{Supp}(\lambda)$ $(s < t)$ are such that $\alpha_i \notin \mathrm{Supp}(\lambda)$ for every $s < i < t$, then

$$\sum_{i=s}^{t-1} |a_i - a_{i+1}| \leqslant a_s + a_t$$

(λ-C3) If α_q is the last simple root in $\mathrm{Supp}(\lambda)$ and if $I_q = \{i \geqslant q \, : \, a_i < a_{i+1}\}$, then

$$2 \sum_{i \in I_q} (a_{i+1} - a_i) \leqslant a_r$$

Extend trivially $\leqslant_\lambda$ to a partial order relation on $\Pi^+(\lambda)$ by setting $\nu \leqslant_\lambda \mu$ if and only if $\mu = \nu$ or $\mu = \lambda$, for all trivial weights $\nu, \mu \in \Pi^+(\lambda)$. If $\Pi \subset \mathcal{X}(B)^+$ is simple with maximal element λ, denote

$$\Pi_{\mathrm{red}} = \{\mu \in \Pi \, : \, \mu \text{ is maximal w.r.t. } \leqslant_\lambda\}.$$

If $\Pi = \Pi_{\mathrm{red}}$ we will say that Π is *reduced*.

Suppose that Π, Π' are simple reduced subsets with maximal elements λ and λ'. Then Π and Π' are called *equivalent* if $\mathrm{Supp}(\lambda) = \mathrm{Supp}(\lambda')$ and if there exists a bijection $\mu \mapsto \mu'$ between $\Pi \smallsetminus \{\lambda\}$ and $\Pi' \smallsetminus \{\lambda'\}$ such that $\mu' - \mu = \lambda' - \lambda$ for every $\mu \in \Pi$.

Theorem (Theorem 4.3.8)**.** *Suppose that* $G = \mathrm{Spin}(2r+1)$.

i) If $\Pi \subset \mathcal{X}(B)^+$ *is simple, then* $X_\Pi \simeq X_{\Pi_{\mathrm{red}}}$.

ii) Let $\Pi, \Pi' \subset \mathcal{X}(B)^+$ *be simple subsets with the same maximal element* λ*. Then* X_Π *dominates* $X_{\Pi'}$ *if and only if for every* $\mu' \in \Pi'$ *there exists* $\mu \in \Pi$ *such that* $\mu' \leqslant_\lambda \mu$.

Corollary. *Simple linear compactifications of* $\mathrm{SO}(2r+1)$ *are classified by simple reduced subsets* $\Pi \subset \mathcal{X}(B)^+$ *up to equivalence.*

* * *

We now briefly explain how the work is organized and how the material is divided into chapters.

In Chapter 1, we overview the theory of spherical varieties and their embeddings. In order to simplify the exposition, we restrict ourselves to the simple case. Main references for this chapter are [LV 83], [Kn 91] and [Bri 97].

In the literature, the considered spherical embeddings are generally assumed to be normal: in this case indeed the combinatorial counterpart of the embedding theory is simpler and more elegant. However, as normal affine toric varieties can be classified by means of cones and non-normal affine toric varieties can be classified by means of semigroups, it seems to be possible to classify linear simple spherical varieties by means of *colored semigroups* as normal simple spherical varieties are classified by means of *colored cones*. Indeed, a big part of the theory of normal spherical embeddings can be deduced by a local structure theorem due to M. Brion, D. Luna and Th. Vust in [BLV 86], which allows to reduce the geometry of a general spherical variety to that of an affine spherical variety. Such theorem holds much more generally than in the normal case: rather than normality, the property which is really needed is the local linearity of the action.

A detailed investigation on a possible extension of the theory of spherical embeddings in the linear case and in the locally linear case goes beyond the aim of this work. However, in order to clarify the general spherical context of the analysis that we will develop in Chapter 4 in the particular case of the compactifications of semisimple adjoint groups, we partially reformulate the theory of spherical embeddings in the case of a simple linear projective spherical variety. The hypothesis of simpleness is not really needed and is mainly motivated by a matter of exposition. On the other hand, the real need to use the hypothesis of compactness of the closed orbit is not considered in this work and should be analyzed apart.

In Chapter 2, we overview the theory of wonderful varieties. Main references for this chapter are [Lu 01] and [BL 08].

In Chapter 3, we study the orbit structure of a linear compactification of a spherical homogeneous space embedded in a simple projective space. In particular, if the generic stabilizer is a strict subgroup, we prove the mentioned criterion of bijectivity of the normalization map, while in the non-strict case we give some necessary/sufficient conditions of bijectivity. However, following the description of the orbits, it is always possible to state if the normalization of a given orbit closure is bijective or not.

In Chapter 4, we analyze in details the case of the simple linear compactifications of semisimple adjoint groups. In the particular case of an odd orthogonal group, we classify all its simple linear compactifications.

Notations

Let G be a reductive group over an algebraically closed field $\Bbbk$ of characteristic zero. We will denote by $T \subset G$ a fixed maximal torus and by $B \subset G$ a fixed Borel subgroup containing T. Moreover we will denote U the unipotent radical of B and by B^- the opposite Borel subgroup of B w.r.t. T. Given any algebraic group denoted with a caption latin letter, we will denote its Lie algebra with the corresponding lower-case german letter. Denote $R \subset \mathfrak{t}^*$ the root system of G associated to T and $S \subset R$ the basis associated to B. Denote $\Lambda \subset \mathfrak{t}^*$ the weight lattice of R and $\Lambda^+ \subset \Lambda$ the semigroup of dominant weights associated to S. If $\alpha \in S$ is a simple root, let $\omega_\alpha \in \Lambda^+$ be the corresponding fundamental dominant weight and let $e_\alpha, \alpha^\vee, f_\alpha$ be an $\mathfrak{sl}(2)$-triple of T-weights $\alpha, 0, -\alpha$.

If K is any algebraic group, we will denote by $\mathcal{X}(K)$ its character group and by K^u its unipotent radical. If V is a rational K-module, we will denote by $V^{(K)}$ the set of K-eigenvectors in V (we will call such vectors *K-semiinvariants*) and by V^K the set of K-invariant vectors. If $\chi \in \mathcal{X}(K)$,

$$V_\chi^{(K)} = \{v \in V \,:\, gv = \chi(g)v \text{ for all } g \in K\}$$

will denote the set of K-eigenvectors in V of weight χ.

Denote $\leqslant$ the *dominance order* on Λ defined by

$$\mu \leqslant \lambda \qquad \text{if and only if} \qquad \lambda - \mu \in \mathbb{N}S.$$

If $\lambda \in \mathcal{X}(B)^+ = \mathcal{X}(B) \cap \Lambda^+$, we will denote by $V(\lambda)$ the simple G-module of highest weight λ and

$$\Pi^+(\lambda) = \{\mu \in \Lambda^+ \,:\, \mu \leqslant \lambda\}.$$

We will denote by $* : \Lambda \longrightarrow \Lambda$ the involution defined by $V(\lambda^*) \simeq V(\lambda)^*$ for $\lambda \in \mathcal{X}(B)^+$. Every G-module will be assumed to be rational and finite dimensional.

By a *variety* we will always mean an irreducible algebraic variety over $\Bbbk$. If X is any variety and $Z \subset X$ is a subvariety, $\overline{Z}$ will denote the closure of Z in X. If G acts rationally on a variety X, we will say that X is a *G-variety.* By an *embedding* of an homogeneous space G/H we mean a G-variety X together with an open equivariant embedding $G/H \hookrightarrow X$. If X is a G-variety, then G acts rationally in the algebra of regular functions $\Bbbk[X]$ and in the field of rational functions $\Bbbk(X)$ by

$$(gf)(x) = f(g^{-1}x).$$

If V is a vector space and $S \subset V$ is any subset, then $\langle S \rangle$ denotes the subspace generated by S. A subset $\mathcal{C} \subset V$ is called a *convex cone* if it is closed under addition and multiplication by $\mathbb{Q}^+$; its *dual cone* is the convex cone

$$\mathcal{C}^\vee = \{\phi \in V^* \,:\, \phi(v) \geqslant 0 \text{ for all } v \in \mathcal{C}\}.$$

The *linear part* of a convex cone $\mathcal{C}$ is the maximum linear subspace of V contained in it; $\mathcal{C}$ is called *strictly convex* if its linear part is reduced to zero. A cone $\mathcal{C}$ is called *finitely generated* if there are finitely many elements $v_1, \dots, v_n \in \mathcal{C}$ such that $\mathcal{C} = \mathbb{Q}^+ v_1 + \dots + \mathbb{Q}^+ v_n$. A *face* of $\mathcal{C}$ is a subset of the form

$$\mathcal{C} \cap \{v \in V \,:\, \phi(v) = 0\}$$

for some $\phi \in \mathcal{C}^\vee$; the *relative interior* of $\mathcal{C}$ is the subset $\mathcal{C}^\circ$ obtained by removing all its proper faces.

If Ω is a *lattice* (i. e. a finitely generated free abelian group), then $\Omega^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Omega, \mathbb{Z})$ denotes the dual lattice and $\Omega_{\mathbb{Q}} = \Omega \otimes \mathbb{Q}$ denotes the rational vector space generated by Ω. If $\Gamma \subset \Omega$ is a subsemigroup, denote $\mathcal{C}(\Gamma)$ the cone generated by Γ in $\Omega_{\mathbb{Q}}$. The *saturation* of Γ in Ω is the semigroup $\overline{\Gamma} = \mathcal{C}(\Gamma) \cap \Omega$; we will say that Γ is *saturated* in Ω if $\Gamma = \overline{\Gamma}$.

Chapter 1

Spherical varieties

Throughout this chapter, G will be a reductive connected algebraic group over an algebraically closed field $\Bbbk$ of characteristic zero. Let $B \subset G$ be a fixed Borel subgroup and $T \subset B$ a maximal torus. Denote $R \subset \mathfrak{t}^*$ the root system of G associated to T and $S \subset R$ the basis associated to B.

1.1 First definitions

A G-variety X is called *linear* if there exists a finite dimensional rational G-module V such that X is G-equivariantly isomorphic to a G-stable locally closed subvariety of the projective space $\mathbb{P}(V)$; X is called *locally linear* if it can be covered by G-stables linear open subsets. If it possesses a unique closed orbit, then X is called *simple*.

Theorem 1.1.1 ([Su 74] Thm. 1, [KKLV 89] Thm. 1.1 and Cor. 2.6)**.** *If a G-variety X is normal, then it is locally linear. If moreover X is simple or quasi-projective, then it is linear.*

Definition 1.1.2. A G-variety X is called *spherical* if it possesses an open B-orbit. A subgroup $H \subset G$ is called *spherical* if the homogeneous space G/H is spherical.

Usually in the literature spherical varieties are assumed to be normal. We will not require this property, however all the spherical varieties we will deal with will be simple and linear: by previous theorem this situation includes every simple normal spherical variety.

Every spherical variety can be regarded as an embedding of its open orbit: hence to any spherical variety is naturally attached a spherical subgroup (defined up to conjugation), namely the stabilizer of a point in the open orbit.

Given a G-variety X, consider the homomorphism $\Bbbk(X)^{(B)} \to \mathcal{X}(B)$ which associate to a rational B-eigenfunction f its character χ_f. This defines an exact sequence

$$0 \to \Bbbk(X)^B \smallsetminus \{0\} \to \Bbbk(X)^{(B)} \to \Lambda_X \to 0,$$

where $\Lambda_X \subset \mathcal{X}(B)$ is a sublattice.

Definition 1.1.3. If X is a G-variety, its *rank*, denoted by $\mathrm{rk}(X)$, is the rank of the lattice Λ_X.

Suppose that X is spherical: then $\Bbbk(X)^B \smallsetminus \{0\} = \Bbbk^*$ and $\Bbbk(X)^{(B)}$ depends only on the open orbit and. Moreover, if $Bx_0 \subset X$ is the open B-orbit and if $H = \mathrm{Stab}(x_0)$, we have a short exact sequence

$$0 \to \Lambda_X \to \mathcal{X}(B) \to \mathcal{X}(B \cap H) \to 0$$

which identifies $\Lambda_X = \Lambda_{G/H}$ with the kernel of the restriction $\mathcal{X}(B) \to \mathcal{X}(B \cap H)$.

Suppose that X is a flag variety: then X possesses a U-open orbit, where U denotes the unipotent radical of B. Let $f \in \Bbbk(X)^{(B)}$: since f is U-invariant, it follows that f is costant on the open orbit, hence it is costant on X. Therefore the rank of a flag varieties is zero. More precisely, rank zero G-varieties are described as follows:

Theorem 1.1.4 ([Bri 97] Cor.1.4.1)**.** *A G-variety X has rank zero if and only if every G-orbit is compact.*

Definition 1.1.5. If X is a G-variety, its *complexity*, denoted by $\mathrm{c}(X)$, is the minimal codimension of a B-orbit in X.

Therefore the class of spherical varieties coincides with the class of G-varieties of complexity 0.

Theorem 1.1.6 ([Vi 86])**.** *Let X be a G-variety and let $X' \subset X$ be a B-stable closed subvariety. Then $\mathrm{c}(X') \leqslant \mathrm{c}(X)$ and $\mathrm{rk}(X') \leqslant \mathrm{rk}(X)$.*

Corollary 1.1.7. *If a G-variety X is spherical, then it contains finitely many B-orbits. In particular every G-stable subvariety of X is spherical.*

Proof. Suppose that X contains infinitely many B-orbits and take $X' \subset X$ a B-stable closed subvariety containing infinitely many B-orbits which is minimal with these properties. By previous theorem X' contains an open B-orbit Bx; therefore an irreducible component of $X' \smallsetminus Bx$ must contain infinitely many B-orbits, contradicting the minimality of X'. □

It follows that a G-variety is spherical if and only if it contains finitely many B-orbits. More generally, spherical varieties can be characterized by any of the following conditions.

Theorem 1.1.8. *Let $H \subset G$ be a subgroup. The following conditions are equivalent:*

i) G/H is spherical.

ii) For any G-variety X and for any fixed point $x \in X^H$, the closure $\overline{Gx}$ contains finitely many G-orbits.

iii) For any G-variety X and for any fixed point $x \in X^H$, the closure $\overline{Gx}$ contains finitely many B-orbits.

While the equivalence between i) and iii) stems by previous corollary, the equivalence between i) and ii) has been shown in [Ak 85].

Another important characterization of spherical subgroups, due to Vinberg and Kimelfeld [VK 78], can be given in terms of representation theory. We will say that a G-module is *multiplicity-free* if every isotypic component is irreducible.

Theorem 1.1.9. *Let $H \subset G$ be a subgroup. The following conditions are equivalent:*

i) *G/H is spherical.*

ii) *For any $\lambda \in \mathcal{X}(B)^+$ the set $\mathbb{P}(V(\lambda))^H$ is finite.*

iii) *For any $\lambda \in \mathcal{X}(B)^+$ and for any $\chi \in \mathcal{X}(H)$ it holds $\dim V(\lambda)^{(H)}_\chi \leqslant 1$.*

iv) *For any G-linearized line bundle $\mathcal{L} \in \mathrm{Pic}(G/H)$, the G-module $\Gamma(G/H, \mathcal{L})$ is multiplicity-free.*

If moreover G/H is quasi-affine, then iii) and iv) can be weakened as follows:

iii') *For any $\lambda \in \mathcal{X}(B)^+$ it holds $\dim V(\lambda)^H \leqslant 1$.*

iv') *The G-module $\Bbbk[G/H]$ is multiplicity-free.*

Suppose that $G = T$ is an algebraic torus and let X be a normal toric variety for T: then by a theorem of Sumihiro [Su 74, Corollary 3.2] X can be covered by T-stables affine open subsets. This is not true in general for a connected reductive group G; however, if the considered variety is locally linear, it is always possible to cover it by translating affine open subsets which are stables for the action of a Borel subgroup.

Theorem 1.1.10 ([Kn 91] Thm. 2.3)**.** *Let $X \subset \mathbb{P}(V)$ be a locally linear G-variety and let $Y \subset X$ be an orbit. There exists a B-stable affine open subset $X^\circ \subset X$ which intersects Y and such that the restriction*

$$\Bbbk[X^\circ]^{(B)} \longrightarrow \Bbbk[X^\circ \cap Y]^{(B)}$$

is surjective.

Let X be a spherical G-variety. If $Y \subset X$ is an orbit, then it contains finitely many B-orbits; we denote $Y_B \subset Y$ the open B-orbit. Denote

$$\Delta(X) = \{B\text{-stable prime divisors of } X \text{ which are not } G\text{-stable}\}$$

its elements are called the *colors* of X. If $Y \subset X$ is any orbit, denote $\Delta_Y(X)$ the set of colors which contain Y. Suppose that $Bx_0 \subset X$ is the open B-orbit and set $H = \mathrm{Stab}(x_0)$: since the irreducible components of $X \smallsetminus Gx_0$ are G-stables, the sets of colors $\Delta(X)$ and $\Delta(G/H)$ are naturally identified. Moreover, since $BH/H \simeq B/B \cap H$ is an affine open subset of G/H, the colors of G/H are identified with the irreducible components of $G/H \smallsetminus BH/H$.

Finally denote

$$\mathcal{B}(X) = \{G\text{-stable prime divisors of } X\}.$$

If $Y \subset X$ is any orbit, $\mathcal{B}_Y(X)$ denotes the set of G-stable prime divisors which contain Y.

Example 1.1.11 (The group G as a spherical $G \times G$-variety). The group $G \times G$ acts transitively on G by $(g_1, g_2)g = g_1 g g_2^{-1}$. Since the stabilizer of the identity is the diagonal $\mathrm{diag}(G)$, we get an isomorphism

$$G \simeq {G \times G}/_{\mathrm{diag}(G)}.$$

Denote B^- the opposite Borel subgroup to B w.r.t. T. Then $B \times B^-$ is a Borel subgroup in $G \times G$ and $T \times T$ is a maximal torus contained in it. Since $BB^- \subset G$ is open, it follows that G is a $G \times G$-spherical variety.

Denote $\hat{G}$ the set of irreducible representations of G. As $G \times G$-module, there is a canonical isomorphism

$$\Psi : \bigoplus_{V \in \hat{G}} V^* \otimes V \xrightarrow{\sim} \Bbbk[G]$$

defined by $\Psi(\phi \otimes v) = \langle \phi, gv \rangle$ where $v \in V, \phi \in V^*$ and $g \in G$. It follows that

$$\Bbbk[G]^{(B \times B^-)}/_{\Bbbk^*} = \{(\lambda, -\lambda) \, : \, \lambda \in \mathcal{X}(B)^+\}.$$

Since G is an affine variety, by [VP 94, Theorem 3.3] it follows that any rational $B \times B^-$-semiinvariant function on G is the quotient of two regular $B \times B^-$-semiinvariant functions: hence we get

$$\Lambda_G = \{(\lambda, -\lambda) \, : \, \lambda \in \mathcal{X}(B)\} \simeq \mathcal{X}(B)$$

and the rank of G as a $G \times G$-variety equals the rank of G as an algebraic group.

Denote W the Weyl group associated to $T \subset G$. If $\alpha \in S$ is a simple root, denote $s_\alpha \in W$ the associated simple reflection, by the Bruhat decomposition (see [Sp 98, Theorem 8.3.8]) it follows that

$$G \smallsetminus BB^- = \bigcup_{\alpha \in S} \overline{B s_\alpha B^-}$$

and every $D_\alpha = \overline{B s_\alpha B^-}$ is a prime divisor: hence we get

$$\Delta(G) = \{D_\alpha \, : \, \alpha \in S\}.$$

1.2 The local structure of a spherical variety

Let V be a finite dimensional rational G-module and let $Y \subset \mathbb{P}(V)$ be a closed orbit; let $y_0 = [v^-] \in Y^{B^-}$ be the unique fixed point by B^- (where $v^- \in V^{(B^-)}$ is a lowest weight vector) and let $\eta \in (V^*)^{(B)}$ be a highest weight vector such that $\langle \eta, v^- \rangle = 1$.

If P is the stabilizer of $[\eta]$, then P and $\mathrm{Stab}(y)$ are opposite parabolic subgroups; denote $L = P \cap \mathrm{Stab}(y)$ the associated Levi subgroup. Denote $\mathbb{P}(V)_\eta \subset \mathbb{P}(V)$ the open affine subset defined by the non-vanishing of η: then $\mathbb{P}(V)_\eta \cap Y = By_0$ is the open B-orbit of Y.

Consider the L-stable affine subvariety W_η defined by

$$W_\eta = \mathbb{P}(\Bbbk v^- \oplus (\mathfrak{g}\eta)^\perp) \cap \mathbb{P}(V)_\eta.$$

Proposition 1.2.1 ([BLV 86] Prop. 1.2)**.** *The action of P induces a P-equivariant isomorphism as follows*

$$\begin{array}{ccc} P^u \times W_\eta & \longrightarrow & \mathbb{P}(V)_\eta \\ (g,s) & \longmapsto & gs \end{array}$$

Suppose that $X \subset \mathbb{P}(V)$ is a projective spherical variety containing Y and set

$$X_Y^\circ = X \cap \mathbb{P}(V)_\eta :$$

then X_Y° is a P-stable affine open subset and by previous proposition we get a P-equivariant isomorphism

$$\begin{array}{ccc} P^u \times W_Y & \longrightarrow & X_Y^\circ \\ (g,s) & \longmapsto & gs, \end{array}$$

where $W_Y = W_\eta \cap X_Y^\circ$ is a L-stable affine subvariety of X_Y°.

Since X possesses an open B-orbit, it follows that W_Y possesses an open $(B \cap L)$-orbit, hence an open L-orbit; since W_Y is affine, we get then

$$\Bbbk[W_Y /\!/ L] = \Bbbk[W_Y]^L = \Bbbk.$$

Therefore W_Y is a L-spherical variety possessing a unique closed L-orbit, namely the fixed point y.

Theorem 1.2.2 ([BLV 86] Thm. 1.4)**.** *Let X be a linear projective spherical variety and let $Y \subset X$ be a closed orbit. In the previous notations, the complement $X \smallsetminus X_Y^\circ$ is the union of the B-stable (possibly G-stable) prime divisors of X which do not contain Y. Moreover, it holds the following description:*

$$X_Y^\circ = \{x \in X \,:\, \overline{Bx} \supset Y\}.$$

Proof. Notice that By_0 is the unique closed B-orbit in X_Y°. Indeed if $Z \subset X_Y^\circ$ is a closed B-orbit, then $Z \cap W_Y$ is a closed $(B \cap L)$-orbit in W_Y, thus $L(Z \cap W_Y)$ is a closed L-orbit in W_Y. Since $\{y_0\} \subset W_Y$ is the unique closed L-orbit, it follows then $Z = By_0$.

Let's show that $X \smallsetminus X_Y^\circ$ is the union of the B-stable prime divisors of X which do not contain Y. Since X_Y° is affine, B-stable and intersects the closed orbit Y, the complement $X \smallsetminus X_Y^\circ$ is a union of B-stable prime divisors which do not contain Y. Suppose that D is a B-stable prime divisor such that $D \cap X_Y^\circ \neq \varnothing$: since it is closed and B-stable, it follows that $D \cap X_Y^\circ$ contains a closed B-orbit Z. By the remark at the beginning of the proof, it follows that $Z = By_0$, thus $D \supset Y$.

Set $X_Y^B = \{x \in X \,:\, \overline{Bx} \supset Y\}$. Then

$$X \smallsetminus X_Y^B = \bigcup_{x \notin X_Y^B} \overline{Bx} :$$

since X possesses finitely many B-orbits it follows that $X_Y^B \subset X$ is an open subset. Let $x \in X_Y^B$: then it must be $x \in X_Y^\circ$, since otherwise it would be $Y \subset \mathbb{P}(\ker \eta)$. Suppose that the inclusion $X_Y^B \subset X_Y^\circ$ is proper: since X_Y^B is an open subset, it follows that there exists a closed B-orbit $Z \subset X_Y^\circ \smallsetminus X_Y^B$. But this is absurd since by the remark at the beginning of the proof it follows $Z = By_0$. $\square$

Consider

$$GX_Y^\circ = \{x \in X \,:\, \overline{Gx} \supset Y\}:$$

since X contains finitely many G-orbits, it is a G-stable open subset of X which is sperical and which possesses a unique closed orbit, namely Y. Therefore we can cover X with a finite number of simple spherical varieties, one for each closed orbit $Y \subset X$.

Suppose now that X is a normal spherical variety (non-necessarily linear). Then a result of local structure analogous to Theorem 1.2.2 can be stated for any orbit $Z \subset X$. Indeed denote

$$X_Z^\circ = \{x \in X \,:\, \overline{Bx} \supset Z\}$$

and denote

$$P = \{g \in G \,:\, gX_Z^\circ = X_Z^\circ\}$$

the stabilizer of X_Z°, which is a parabolic subgroup of G.

Theorem 1.2.3 ([Bri 89] Thm. 1)**.** *Let X be a normal spherical variety and let $Z \subset X$ be an orbit. In the previous notations:*

i) X_Z° is a P-stable affine open subset of X which intersects Z in a B-orbit and it is minimal in X with these properties;

ii) The complement $X \smallsetminus X_Z^\circ$ is the union of the B-stable (possibly G-stable) prime divisors of X which do not contain Z;

iii) There exists a Levi subgroup $L \subset P$ and a closed L-stable subvariety $W_Z \subset X_Z^\circ$ such that the P-action induces an isomorphism

$$\begin{array}{ccc} P^u \times W_Z & \longrightarrow & X_Z^\circ \\ (g, s) & \longmapsto & gs \end{array}$$

Proposition 1.2.4 ([BP 87] Prop. 3.5)**.** *Let X be a normal spherical variety and let $Z \subset X$ be an orbit. Then the closure $\overline{Z}$ is normal.*

Proof. By previous theorem, we may assume that X is simple and affine. Consider the algebra of U-invariants $\Bbbk[X]^U$: by [Gr 97, Theorem 9.6] it is a finitely generated $\Bbbk$-algebra, while by [Vu 76, Theorem 1.1] it is integrally closed; hence $X/\!/U = \mathrm{Spec}(\Bbbk[X]^U)$ is a normal B/U-toric variety. Let $I \subset \Bbbk[X]$ be the ideal of regular functions which vanish on $\overline{Z}$: then $\Bbbk[\overline{Z}]^U \simeq \Bbbk[X]^U/I^U$; hence $\overline{Z}/\!/U = \mathrm{Spec}(\Bbbk[\overline{Z}]^U)$ is a B/U-orbit closure in $X/\!/U$ and it is normal by [Oda 88, Corollary 1.7]. Therefore $\Bbbk[\overline{Z}]^U$ is integrally closed and by [Vu 76, Theorem 1.1] it follows the normality of $\overline{Z}$. □

Definition 1.2.5. A spherical variety X is called *toroidal* if $\Delta_Z(X) = \varnothing$ for every orbit $Z \subset X$.

Suppose that X is normal and toroidal: for such a variety the local structure reduces to that of a toric variety.

Theorem 1.2.6 ([BP 87] Prop. 3.4, [Bri 97] Prop. 2.4.1)**.** *Let X be a toroidal variety; denote*

$$X^\circ = X \smallsetminus \bigcup_{D \in \Delta(X)} D$$

and set P the stabilizer of X°.

i) There exists a Levi subgroup $L \subset P$ and a L-stable closed subvariety $W_X \subset X^\circ$ such that the P-action induces an isomorphism as follows

$$P^u \times W_X \longrightarrow X^\circ.$$

ii) The derived subgroup $[L, L]$ acts trivially on W_X, which is a toric variety for a quotient of $L/[L, L]$.

iii) Every G-orbit of X intersects W_X in a single L-orbit.

1.3 The G-invariant valuation cone

Consider a spherical homogeneous space G/H. A *rational discrete valuation* of G/H is a map $\nu : \Bbbk(G/H)^* \to \mathbb{Q}$ with the following properties:

- $\nu(f_1 + f_2) \geqslant \min\{\nu(f_1), \nu(f_2)\}$ for all $f_1, f_2 \in \Bbbk(G/H)^*$ s.t. $f_1 + f_2 \in \Bbbk(G/H)^*$;
- $\nu(f_1 f_2) = \nu(f_1) + \nu(f_2)$ for all $f_1, f_2 \in \Bbbk(G/H)^*$;
- $\nu(f) = 0$ for all $f \in \Bbbk^*$.

Suppose that $G/H \hookrightarrow X$ is an equivariant embedding of G/H and let $D \subset X$ be a prime divisor; denote $\mathcal{O}_{D,X} \subset \Bbbk(X)$ the associated one-dimensional local ring. If $f \in \Bbbk(G/H)^*$, write $f = f_1/f_2$ with $f_1, f_2 \in \mathcal{O}_{D,X}$. Then D defines a discrete valuation ν_D of G/H by

$$\nu_D(f) = l\left(\mathcal{O}_{D,X}/_{(f_1)}\right) - l\left(\mathcal{O}_{D,X}/_{(f_2)}\right),$$

where l denotes the length of the $\mathcal{O}_{D,X}$-module in parentheses (see [Fu 98, §1.2]).

In case X is a normal variety, then $\mathcal{O}_{D,X}$ is a discrete valuation ring and ν_D coincides with the usual valuation associated to D. In case X is not normal, for $f \in \Bbbk(G/H)^*$, the valuation ν_D is described as follows, where $p : \widetilde{X} \to X$ is the normalization map:

$$\nu_D(f) = \sum_{\widetilde{D} \in \mathrm{Irr}(p^{-1}(D))} [\Bbbk(\widetilde{D}) : \Bbbk(D)]\ \nu_{\widetilde{D}}(f),$$

where $\mathrm{Irr}(p^{-1}(D))$ denotes the set of irreducible components of $p^{-1}(D)$ and where $[\Bbbk(\widetilde{D}) : \Bbbk(D)]$ denotes the degree of the field extension.

Any rational discrete valuation ν of G/H defines an element $\rho(\nu) \in (\Lambda^\vee_{G/H})_{\mathbb{Q}}$ by

$$\langle \rho(\nu), \chi \rangle = \nu(f_\chi)$$

where $f_\chi \in \Bbbk(G/H)^{(B)}$ is any B-semiinvariant function of weight χ: since G/H possesses an open B-orbit, the definition does not depend on the function, but only on the weight.

If $G/H \hookrightarrow X$ is an equivariant embedding and if $D \subset X$ is any B-stable prime divisor, then by an abuse of notation we will denote $\rho(D) = \rho(\nu_D) \in \Lambda^\vee_{G/H}$ the image of the associated valuation ν_D.

A rational discrete valuation ν of G/H is said *G-invariant* if $\nu(g \cdot f) = \nu(f)$, for every $f \in \Bbbk(G/H)$ and for every $g \in G$. If $G/H \hookrightarrow X$ is an equivariant embedding and if $D \subset X$ is any G-stable prime divisor, then the associated discrete valuation ν_D is G-invariant. Denote $\mathcal{V}_{G/H}$ the set of G-invariant rational valuations of G/H.

Recall that two roots $\alpha, \beta \in R$ are called *strongly orthogonal* if $\alpha \pm \beta \notin R \cup \{0\}$. Strongly orthogonal roots are always orthogonal: indeed if $\langle \alpha, \beta^\vee \rangle < 0$ then $\alpha + \beta \in R$, while if $\langle \alpha, \beta^\vee \rangle > 0$ then $\alpha - \beta \in R$.

Theorem 1.3.1 ([LV 83] Prop. 7.4, [BP 87] Cor. 3.2, Cor. 4.1 and Prop. 4.2)**.**

i) The restriction

$$\rho : \mathcal{V}_{G/H} \longrightarrow (\Lambda^\vee_{G/H})_{\mathbb{Q}}$$

is injective and identifies $\mathcal{V}_{G/H}$ with a finitely generated convex cone which generates $(\Lambda^\vee_{G/H})_{\mathbb{Q}}$ as a vector space.

ii) The dual cone $\mathcal{V}^\vee_{G/H} \subset (\Lambda_{G/H})_{\mathbb{Q}}$ is generated by negative roots and by sums of two strongly orthogonal negative roots.

Together with such embedding, $\mathcal{V}_{G/H}$ is called the *G-invariant valuation cone of G/H*.

More precisely, the dual cone $\mathcal{V}^\vee_{G/H}$ can be described as follows. Since $BH \subset G$ is an open subset, up to a scalar factor every eigenfunction $f \in \Bbbk[G]^{(B\times H)}$ is uniquely determined by its weight $(\lambda, \chi) \in \mathcal{X}(B) \times \mathcal{X}(H)$: we will denote then by $f_{\lambda,\chi}$ the unique eigenfunction in $\Bbbk[G]^{(B\times H)}$ of weight (λ, χ) such that $f_{\lambda,\chi}(1) = 1$.

Proposition 1.3.2 ([BP 87] Prop. 4.1)**.** *The dual cone $\mathcal{V}^\vee_{G/H}$ is the convex hull of the differences $\nu - \mu - \mu'$ such that there exist $\chi, \chi' \in \mathcal{X}(H)$ with*

$$f_{\nu,\chi+\chi'} \ \in \ \langle G f_{\mu,\chi} \rangle \langle G f_{\mu',\chi'} \rangle.$$

Example 1.3.3 (**The group G as a spherical $G \times G$-variety, II**)**.** Following Example 1.1.11, regard G as a spherical $G \times G$-variety and identify the lattice Λ_G with $\mathcal{X}(B)$. Up to a finite covering, we may assume that G is the direct product of a torus by a semisimple simply connected group, i.e. that the algebra $\Bbbk[G]$ is factorial (see [Me 98, Proposition 1.10]). Hence, for $\alpha \in S$, the $B \times B^-$-stable divisor $D_\alpha = \overline{B s_\alpha B^-}$ has an equation $f_{\omega_\alpha} \in \Bbbk[G]$ which is a $B \times B^-$-eigenvector of weight $(\omega_\alpha, -\omega_\alpha)$. If $f_\lambda \in \Bbbk[G]$ is a $B \times B^-$-eigenvector of weight $(\lambda, -\lambda)$, it follows then that $\nu_{D_\alpha}(f_\lambda)$ is the multiplicity of f_{ω_α} in f_λ, i.e. the coefficient of ω_α in λ: hence $\rho(D_\alpha)$ is identified with the simple coroot $\alpha^\vee$.

If V is a G-module, define its matrix coefficient $c_V : V^* \otimes V \to \Bbbk[G]$ by $c_V(\psi \otimes v)(g) = \langle \psi, gv \rangle$. If we multiply functions in $\Bbbk[G]$ of this type then we get

$$c_V(\psi \otimes v) \cdot c_W(\chi \otimes w) = c_{V \otimes W}\big((\psi \otimes \chi) \otimes (v \otimes w)\big).$$

Notice that $c_{V\otimes W}$ is a linear combination with positive coefficients of the matrix coefficients C_M, where M is a simple module occurring in the decomposition of

$V \otimes W$. If we identify $\operatorname{End}(V)$ with $V^* \otimes V$, then we get that the multiplication $\operatorname{End}(V(\lambda)) \operatorname{End}(V(\mu)) \subset \Bbbk[G]$ is the sum of all $\operatorname{End}(V(\nu))$ with $V(\nu) \subset V(\lambda) \otimes V(\mu)$.

If $\alpha \in S$, take λ and μ such that $\langle \lambda, \alpha^\vee \rangle \neq 0$ and $\langle \mu, \alpha^\vee \rangle \neq 0$: then by [Bo 75, §VIII.7, Exercise 17] it follows that $V(\lambda) \otimes V(\mu)$ contains a simple submodule of highest weight $\lambda + \mu - \alpha$: hence by Proposition 1.3.2 we get $-\alpha \in \mathcal{V}_G^\vee$. Therefore $\mathcal{V}_G$ is identified with the negative Weyl chamber of $\mathcal{X}(B)_\mathbb{Q}$.

Example 1.3.4 (Symmetric spaces). Suppose that G is semisimple and let $\sigma : G \to G$ be an algebraic involution; let $H \subset G$ be a subgroup such that $H^\sigma \subset H \subset N_G(H^\sigma)$. Choose a maximal torus T_1 such that $\sigma(t) = t^{-1}$ for all $t \in T_1$ and fix a maximal torus T of G which contains T_1. Fix a Borel subgroup $B \supset T$ in such a way that the dimension of $\sigma(B) \cap B$ is the minimal possible. Then σ fixes the set of roots R, where we still denote by σ the induced involution on $\mathfrak{t}^*$.

Denote $R_1 = \{\alpha \in R : \sigma(\alpha) \neq \alpha\}$ and consider the intersection $S_1 = S \cap R_1$. If $\alpha \in R$, denote $\overline{\alpha} = \alpha - \sigma(\alpha)$ and set

$$\overline{R} = \{\overline{\alpha} : \alpha \in R_1\} :$$

this is a (possibly non-reduced) root system in $\mathcal{X}(T_1)_\mathbb{R}$, called the *restricted root system*, and $\overline{S} = \{\overline{\alpha} : \alpha \in S_1\}$ is a basis for $\overline{R}$. Moreover, the weight lattice and the root lattice of $\overline{R}$ are respectively identified with

$$\mathcal{X}\left(T_1 \big/ T_1 \cap G^\sigma\right) \qquad \text{and} \qquad \mathcal{X}\left(T_1 \big/ T_1 \cap N_G(G^\sigma)\right),$$

which are subgroups of $\mathcal{X}(T_1)$ of finite index (see [Vu 90, Lemmas 2.2, 2.3 and 3.1]).

If $\alpha \in R$, denote $\mathfrak{g}_\alpha \subset \mathfrak{g}$ the eigenspace of weight α and denote $R_1^+ = R_1 \cap \mathbb{N}S$. Then it holds the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}_1 \oplus \bigoplus_{\alpha \in R_1^+} \mathfrak{g}_\alpha :$$

in particular it follows that $\mathfrak{g} = \mathfrak{h} + \mathfrak{b}$, i.e. $BH \subset G$ is an open subset and H is a spherical subgroup (see [DCP 83, Proposition 1.3]). Moreover, restriction of characters gives an isomorphism

$$\Lambda_{G/H} \simeq \mathcal{X}\left(T_1 \big/ T_1 \cap H\right).$$

Via this identification, the following descriptions hold (see [Vu 90, Propositions 2.1 and 2.2]):

i) The image of $\rho : \Delta(G/H) \to \mathcal{X}\left(T_1 / T_1 \cap H\right)^\vee$ is the set of simple restricted coroots $\overline{S}^\vee$ and the fibers of ρ contain at most two colors.

ii) The valuation cone $\mathcal{V}_{G/H}$ is the negative Weyl chamber in $\mathcal{X}\left(T_1 / T_1 \cap H\right)^\vee_\mathbb{Q}$.

This generalizes the case of a semisimple group G regarded as $G \times G$ variety treated in Example 1.3.3, which is a symmetric variety for the involution $\sigma : G \times G \to G \times G$ defined by $\sigma(g_1, g_2) = (g_2, g_1)$.

There exists a very tight connection between $\mathcal{V}_{G/H}$ and the normalizer of H in G, which is explained by following theorem. Consider the projection $G/H \to G/N_G(H)$; correspondingly we get an embedding $\Lambda_{G/N_G(H)} \hookrightarrow \Lambda_{G/H}$. Consider the natural right action of $N_G(H)$ on G/H defined by

$$g \cdot (g'H) = g'g^{-1}H :$$

such action identifies the G-equivariant automorphism group $\mathrm{Aut}_G(G/H)$ with the quotient group $N_G(H)/H$. If $g \in N_G(H)$ and if $\lambda \in \Lambda_{G/H}$, let $f \in \Bbbk(G/H)^{(B)}$ be a B-eigenfunction of weight λ and consider its translated $g \cdot f$: since it is still a B-eigenfunction of weight λ, there exists $\Theta_g(\lambda) \in \Bbbk^*$ such that

$$g \cdot f = \Theta_g(\lambda) f.$$

This defines an homomorphism

$$\Theta_g : \Lambda_{G/H} \longrightarrow \Bbbk^*$$

which is trivial restricted to $\Lambda_{G/N_G(H)}$.

Theorem 1.3.5 ([BP 87] §5, [Bri 97] Thm. 4.3)**.** *In the previous notations:*

i) The map

$$\Theta : {}^{N_G(H)}\!\big/_{H} \longrightarrow \mathrm{Hom}\left({}^{\Lambda_{G/H}}\!\big/_{\Lambda_{G/N_G(H)}}, \Bbbk^*\right)$$

defined by $\Theta(gH) = \Theta_g$ is an isomorphism of algebraic group. In particular, $N_G(H)/H$ is a diagonalizable group.

ii) The annihilator $\Lambda^{\perp}_{G/N_G(H)} \subset (\Lambda^{\vee}_{G/H})_{\mathbb{Q}}$ equals the linear part of $\mathcal{V}_{G/H}$. In particular, the dimension of $N_G(H)/H$ equals the dimension of the linear part of $\mathcal{V}_{G/H}$.

iii) If H° is the identity component of H, then $N_G(H) = N_G(H^\circ)$.

iv) If B is any Borel subgroup such that BH is open in G, then $N_G(H)$ equals the right stabilizer of BH.

1.4 Simple normal embeddings and colored cones

We here overview the theory of normal spherical embeddings. Since we are interested in the simple linear compactifications of a spherical homogeneous space, we will restrict the exposition of the theory to the simple case. However, if it is not needed, we will not require that the considered embedding is simple. For a complete exposition of the theory, see [Kn 91] or [Bri 97].

Let $G/H \hookrightarrow X$ be a normal embedding of a spherical homogeneous space G/H. If $Z \subset X$ is an orbit, denote

$$X_Z^\circ = X \smallsetminus \bigcup_{\Delta(X) \smallsetminus \Delta_Z(X)} D :$$

following Theorem 1.2.3, if P is the parabolic subgroup of G defined by

$$P = \{g \in G \,:\, gX_Z^\circ = X_Z^\circ\},$$

there exist a Levi subgroup $L \subset P$ and a L-stable affine subvariety $W_Z \subset X_Z^\circ$ such that the P-action induces a P-equivariant isomorphism

$$X_Z^\circ \simeq P^u \times W_Z.$$

Denote $\Omega_Z(X) \subset \Lambda_{G/H}$ the semigroup defined by

$$\Omega_Z(X) = \Bbbk[X_Z^\circ]^{(B)} \big/ \Bbbk^* = \Bbbk[W_Z]^{(B\cap L)} \big/ \Bbbk^*.$$

Denote $\mathcal{C}_Z(X) \subset (\Lambda_{G/H}^\vee)_\mathbb{Q}$ the cone generated by the images of B-stable (possibly G-stable) prime divisors of X containing Z, i.e. by $\rho(\Delta_Z(X))$ together with $\rho(\mathcal{B}_Z(X)) \subset \mathcal{V}_{G/H}$.

Theorem 1.4.1 ([Kn 91] Thm. 3.5)**.** *Suppose that X is a normal embedding of G/H and let $Z \subset X$ an orbit. Then*

i) $\Omega_Z(X) = \mathcal{C}_Z(X)^\vee \cap \Lambda_{G/H}$.

ii) $\mathcal{C}_Z(X)$ *is a strictly convex cone such that* $\mathcal{C}_Z(X)^\circ \cap \mathcal{V}_{G/H} \neq \varnothing$.

Combining previous theorem with [Oda 88, Proposition 1.1] we get the following corollary. Later we will see a more general proof which makes use of the local structure theorem (see Proposition 1.5.4).

Corollary 1.4.2. *$\Omega_Z(X)$ is a finitely generated semigroup which is saturated in $\Lambda_{G/H}$ and which generates $\Lambda_{G/H}$.*

The codimension of the cone $\mathcal{C}_Z(X)$ has the following geometrical interpretation: it expresses the rank of the orbit Z. This is the content of following theorem.

Theorem 1.4.3 ([Kn 91] Thm. 7.3)**.** *Suppose that X is a normal spherical variety and let $Z \subset X$ be an orbit. Then*

$$\Lambda_Z = \Lambda_X \cap \mathcal{C}_Z(X)^\perp.$$

In particular, $\mathrm{rk}(Z) = \mathrm{rk}(X) - \dim \mathcal{C}_Z(X)$ *and Λ_Z is saturated in Λ_X.*

Proof. Let $f \in \Bbbk(X)^{(B)}$. Since X is normal, both the zero locus and the non-definiton locus of f are B-stables subvarieties of pure codimension 1. Therefore, if f is zero or undefined on Z, then there exists a B-stable prime divisor $D \in \Delta_Z(X) \cup \mathcal{B}_Z(X)$ such that $v_D(f) \neq 0$. This shows the inclusion $\Lambda_X \cap \mathcal{C}_Z(X)^\perp \subset \Lambda_Z$, while the surjectivity follows by Theorem 1.1.10. □

If $Z \subset X$ is an orbit, consider the couple $(\mathcal{C}_Z(X), \Delta_Z(X))$: it is a colored cone in the sense of following definition.

Definition 1.4.4. A *colored cone* for G/H is a pair $(\mathcal{C}, \Delta_\square)$ with $\mathcal{C} \subset (\Lambda_{G/H}^\vee)_\mathbb{Q}$ and $\Delta_\square \subset \Delta(G/H)$ having the following properties:

(CC1) $\mathcal{C}$ is a convex cone generated by $\rho(\Delta_\square)$ together with finitely many elements of $\mathcal{V}_{G/H}$.

(CC2) $\mathcal{C}^\circ \cap \mathcal{V}_{G/H} \neq \varnothing$.

A colored cone is called *strictly convex* if $\mathcal{C}$ is strictly convex and $0 \notin \rho(\Delta_\square)$; it is called *complete* if $\mathcal{C}$ contains the G-invariant valuation cone $\mathcal{V}_{G/H}$.

If X is simple with closed orbit Y, denote $\mathcal{C}^c(X) = (\mathcal{C}_Y(X), \Delta_Y(X))$: we will call $\mathcal{C}^c(X)$ the *colored cone* of X.

Theorem 1.4.5 ([LV 83] Prop. 4.10, [Kn 91] Thm. 4.1)**.** *The map $X \mapsto \mathcal{C}^c(X)$ defines a bijection between isomorphism classes of simple normal embeddings of G/H and strictly convex colored cones.*

Following the natural identification $\Lambda_Y = \Lambda_X \cap \mathcal{C}_Y(X)^\perp$ of Theorem 1.4.3, consider the homomorphism $\Lambda_Y \longrightarrow \mathbb{Z}[\Delta(X) \smallsetminus \Delta_Y(X)]$ defined by

$$\chi \longmapsto \sum_{\Delta(X)\smallsetminus\Delta_Y(X)} \langle \rho(D), \chi\rangle D.$$

Since any divisor in the image of previous map is principal, we get an exact sequence

$$\Lambda_Y \longrightarrow \mathbb{Z}[\Delta(X) \smallsetminus \Delta_Y(X)] \longrightarrow \mathrm{Pic}(X).$$

Theorem 1.4.6 ([Bri 89] §2)**.** *Suppose that X is a simple normal spherical variety with closed orbit Y.*

i) A divisor δ is a Cartier divisor if and only if it is linearly equivalent to a B-stable divisor $\delta' \in \mathbb{Z}[\Delta(X) \smallsetminus \Delta_Y(X)]$. Moreover, we have an exact sequence

$$\Lambda_Y \longrightarrow \mathbb{Z}[\Delta(X) \smallsetminus \Delta_Y(X)] \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

ii) A Cartier divisor δ is generated by global sections (resp. ample) if and only if it is linearly equivalent to a B-stable divisor $\delta' \in \mathbb{Z}[\Delta(X) \smallsetminus \Delta_Y(X)]$ with non-negative (resp. positive) coefficients.

Combining previous theorem with Theorem 1.1.4 and Theorem 1.4.3, we get the following corollary.

Corollary 1.4.7. *If X is a simple normal spherical variety with compact closed orbit Y, then*

$$\mathrm{Pic}(X) \cong \mathbb{Z}[\Delta(X) \smallsetminus \Delta_Y(X)].$$

Colored cones allow as well to express combinatorially the existence of a morphism between two given spherical embeddings. Indeed, let $H' \supset H$ be another spherical subgroup and denote $\phi : G/H \to G/H'$ the projection. We get then two natural maps

$$\phi^* : \Lambda_{G/H'} \hookrightarrow \Lambda_{G/H} \qquad \text{and} \qquad \phi_* : (\Lambda^\vee_{G/H})_\mathbb{Q} \twoheadrightarrow (\Lambda^\vee_{G/H'})_\mathbb{Q}$$

If $\Delta_\phi \subset \Delta(G/H)$ is the subset of colors which map dominantly on G/H', then ϕ induces as well a map

$$\phi_* : \Delta(G/H) \smallsetminus \Delta_\phi \longrightarrow \Delta(G/H')$$

Definition 1.4.8. Suppose that $(\mathcal{C}, \Delta_\square)$ and $(\mathcal{C}', \Delta'_\square)$ are colored cones respectively for G/H and for G/H'. Then we say that $(\mathcal{C}, \Delta_\square)$ *maps to* $(\mathcal{C}', \Delta'_\square)$ if the following conditions hold:

(CM1) $\phi_*(\mathcal{C}) \subset \mathcal{C}'$.

(CM2) $\phi_*(\Delta_\square \smallsetminus \Delta_\phi) \subset \Delta'_\square$.

If $\mathcal{C}^c = (\mathcal{C}, \Delta_\square)$ is a colored cone for a spherical homogeneous space G/K, define its *support* as

$$\operatorname{Supp} \mathcal{C}^c = \mathcal{V}_{G/K} \cap \mathcal{C}.$$

Theorem 1.4.9 ([Kn 91] Thm. 5.1 and Thm. 5.2)**.** *Suppose $H' \supset H$ are spherical subgroups and let X and X' be simple normal embeddings of G/H and G/H' respectively.*

i) The projection $\phi : G/H \to G/H'$ extends to a morphism $\phi : X \to X'$ if and only if $\mathcal{C}^c(X)$ maps to $\mathcal{C}^c(X')$.

ii) If such an extension exists, then it is proper if and only if

$$\operatorname{Supp} \mathcal{C}^c(X) = \phi_*^{-1}(\operatorname{Supp} \mathcal{C}^c(X')).$$

In particular, X is complete if and only if $\mathcal{C}^c(X)$ is complete.

1.5 Simple linear compactifications and colored semigroups

We begin this section with some general remarks about (possibly non-linear) quasi-projective spherical varieties. Suppose that X is a quasi-projective embedding of a spherical homogeneous space G/H and denote $p : \widetilde{X} \to X$ the normalization.

Proposition 1.5.1 ([Ti 03] Prop. 1)**.** *If X is a quasi-projective embedding of G/H, then the normalization $\widetilde{X} \to X$ is bijective on the set of G-orbits.*

If $Z \subset X$ is an orbit, denote $Z' = p^{-1}(Z)$ its inverse image in $\widetilde{X}$. If $D \in \Delta(G/H)$, denote $\overline{D}$ (resp. $\widetilde{D}$) its closure in X (resp. in $\widetilde{X}$).

Proposition 1.5.2. *Let $D \in \Delta(G/H)$, let $Z \subset X$ be an orbit. Then $\overline{D} \supset Z$ if and only if $\widetilde{D} \supset Z'$.*

Proof. By Corollary 1.1.7, every orbit in a spherical variety is spherical. Thus we may fix base points $z_0 \in Z$ and $z'_0 \in p^{-1}(z_0)$ such that $Bz_0 \subset Z$ and $Bz'_0 \subset Z'$ are open. Denote $K = \operatorname{Stab}(z_0)$ and $K' = \operatorname{Stab}(z'_0)$: since p is a finite morphims, it follows that $[K : K'] < \infty$. Hence $K^\circ = (K')^\circ$ and by Theorem 1.3.5 it follows $K' \subset N_G(K)$ and $BK' = BK$. Therefore $p^{-1}(Bz_0) = Bz'_0$ is open in Z'.

Suppose that $\overline{D} \supset Z$: since $\widetilde{D}$ is B-stable, by previous discussion we get $\widetilde{D} \supset Bz'_0$, thus $\widetilde{D} \supset Z$. Suppose conversely that $\widetilde{D} \supset Z'$: then $\overline{D} = p(\widetilde{D}) \supset p(Z') = Z$. □

As in the normal case, if $Z \subset X$ is an orbit, denote $\mathcal{C}_Z(X) \subset (\Lambda^\vee_{G/H})_\mathbb{Q}$ the cone generated by the images of B-stable (possibly G-stable) prime divisors of X containing Z, i.e. by $\rho(\Delta_Z(X))$ together with $\rho(\mathcal{B}_Z(X)) \subset \mathcal{V}_{G/H}$.

Corollary 1.5.3. *Let X be a simple quasi-projective embedding of an homogeneous space G/H and let $p : \widetilde{X} \to X$ be the normalization. If $Z \subset X$ is an orbit and $Z' \subset \widetilde{X}$ is the corresponding orbit, then*

$$\Big(\mathcal{C}_Z(X), \Delta_Z(X)\Big) = \Big(\mathcal{C}_{Z'}(\widetilde{X}), \Delta_{Z'}(\widetilde{X})\Big).$$

Proof. If $D \subset X$ is a G-stable prime divisor, by Proposition 1.5.1 it follows that $p^{-1}(D) = \widetilde{D}$ is a G-stable prime divisor of $\widetilde{X}$; in particular it follows that the valuations of G/H defined by D and $\widetilde{D}$ are proportional. On the other hand, by Proposition 1.5.2 it follows that $\Delta_Y(X) = \Delta_{Y'}(\widetilde{X})$ are identified with the same subset of $\Delta(G/H)$. Therefore $\mathcal{C}(X)$ and $\mathcal{C}(\widetilde{X})$ up to proportionality are generated by the same subset of $(\Lambda^\vee_{G/H})_\mathbb{Q}$ and, after the identifications $\Delta(X) = \Delta(\widetilde{X}) = \Delta(G/H)$, we get the equality. $\square$

Consider now the case of a simple linear compactification of a spherical homogeneous space G/H and denote $Y \subset X$ the compact orbit. Denote $p : \widetilde{X} \to X$ the normalization, if $Z \subset X$ is an orbit denote $Z' = p^{-1}(Z)$ the corresponding orbit in $\widetilde{X}$. As in the normal case, denote

$$X^\circ_Y = X \smallsetminus \bigcup_{\Delta(X)\smallsetminus\Delta_Y(X)} D :$$

following Theorem 1.2.2 if P is the parabolic subgroup of G defined by

$$P = \{g \in G \,:\, gX^\circ_Y = X^\circ_Y\},$$

there exist a Levi subgroup $L \subset P$ and a L-stable affine subvariety $W_Y \subset X^\circ_Y$ such that the P-action induces a P-equivariant isomorphism

$$X^\circ_Y \simeq P^u \times W_Y.$$

Consider the semigroup $\Omega_Y(X) \subset \Lambda_{G/H}$ defined by

$$\Omega_Y(X) = \mathbb{k}[X^\circ_Y]^{(B)}\big/\mathbb{k}^* = \mathbb{k}[W_Y]^{(B\cap L)}\big/\mathbb{k}^*.$$

Proposition 1.5.4. *$\Omega_Y(X)$ is a finitely generated semigroup which generates $\Lambda_{G/H}$.*

Proof. Since W_Y is an affine variety, by [Gr 97, Theorem 9.4] it follows that

$$\mathbb{k}[W_Y]^{U\cap L} = \mathbb{k}[\Omega_Y(X)]$$

is a finitely generated $\mathbb{k}$-algebra, where the latter denotes the semigroup algebra of $\Omega_Y(X)$: therefore $\Omega_Y(X) \subset \Lambda_{G/H}$ is a finitely generated semigroup. Since every B-semiinvariant rational function on X can be written as a quotient of two B-semiinvariant regular functions on X°_Y, it follows that $\Omega_Y(X)$ generates $\Lambda_{G/H}$. $\square$

Denote

$$\Omega^c(X) = \big(\Omega_Y(X), \Delta_Y(X)\big) :$$

we will call $\Omega^c(X)$ the *colored semigroup* of X. Following previous results together with the results in previous section, we get that if $G/H \hookrightarrow X$ is a simple normal embedding or a simple linear compactification, then $\Omega^c(X)$ is a colored semigroup in the sense of following definition.

Definition 1.5.5. A *colored semigroup* for G/H is a pair $\Omega^c = (\Omega, \Delta_\square)$ where

(CS1) $\Omega \subset \Lambda_{G/H}$ is a finitely generated semigroup which generates $\Lambda_{G/H}$.

(CS2) $(\mathcal{C}(\Omega)^\vee, \Delta_\square)$ is a strictly convex colored cone for G/H.

We will say that a colored semigroup is *complete* if $\Omega \subset \mathcal{V}_{G/H}^{\vee}$.

Proposition 1.5.6. *Let X be a simple linear compactification of G/H. Then $\Omega^c(X)$ is complete and it uniquely determines X among the simple linear compactifications of G/H.*

Proof. The completeness of $\Omega^c(X)$ follows by Theorem 1.4.9. To show the second claim, recall the P-equivariant isomorphism $X_Y^\circ \simeq P^u \times W_Y$ of Theorem 1.2.2, where W_Y is an affine L-spherical variety with open L-orbit $L/L\cap H$. Since $H \cap L \subset L$ is spherical, we get

$$\Bbbk[W_Y] = \bigoplus_{\lambda \in \Omega_Y(X)} \Bbbk[L/L\cap H]_{(\lambda)} \cong \bigoplus_{\lambda \in \Omega_Y(X)} V_L(\lambda),$$

where $\Bbbk[L/L\cap H]_{(\lambda)}$ denotes the isotypic component of weight λ and where $V_L(\lambda)$ is the simple L-module of highest weight λ. In particular, $\Omega_Y(X)$ uniquely determines W_Y. On the other hand, since P is the stabilizer of X_Y°, it is uniquely determined by $\Delta_Y(X)$, so $\Omega^c(X)$ uniquely determines X_Y°. Since $X = GX_Y^\circ$ is a simple spherical variety, it follows that X as well is uniquely determined by $\Omega_Y^c(X)$. □

Following proposition explains the link between $\Omega_Y(X)$ and $\Omega_{Y'}(\widetilde{X})$.

Proposition 1.5.7. *If X is a simple linear compactification of G/H and if $\widetilde{X}$ is its normalization, then $\Omega_{Y'}(\widetilde{X})$ is the saturation of $\Omega_Y(X)$ in $\Lambda_{G/H}$, where $Y \subset X$ and $Y' \subset \widetilde{X}$ are the compact orbits. In particular X is normal if and only if $\Omega_Y(X)$ is saturated.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{X}_{Y'}^\circ & \xrightarrow{\sim} & P^u \times \widetilde{W}_{Y'} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{Id} \times p} \\ X_Y^\circ & \xrightarrow{\sim} & P^u \times W_Y \end{array}$$

Then $\widetilde{W}_{Y'}$ and W_Y are affine L-spherical varieties and the restriction $p : \widetilde{W}_{Y'} \to W_Y$ is the normalization map. Set $U_0 = U \cap L$ and denote $A = \Bbbk[W_Y]$ and $\widetilde{A} = \Bbbk[\widetilde{W}_{Y'}]$: let's show that $\widetilde{A}^{U_0} \supset A^{U_0}$ is an integral extension.

Following [Maf 09, Lemma 3], consider the ideal

$$I = \{a \in A \,:\, a\widetilde{A} \subset A\},$$

which is non-zero since $\widetilde{A}$ is a finite extension of A, and take $a \in I^{U_0}$. Then $\widetilde{A}^{U_0} \simeq a\widetilde{A}^{U_0}$ as an A^{U_0}-module, and the latter is a finitely generated A^{U_0}-module since it is an ideal in A^{U_0}. Therefore $\widetilde{A}^{U_0} \supset A^{U_0}$ is an integral extension.

Let $\lambda \in \Omega_{Y'}(\widetilde{X}) \smallsetminus \{0\}$ and let $f_\lambda \in \widetilde{A}^{U_0}$ be a B-eigenfunction of weight λ; let

$$p(t) = t^n + f_{\mu_1} t^{n-1} + \ldots + f_{\mu_{n-1}} t + f_{\mu_n} \in A^{U_0}[t]$$

be a monic polinomial annihilated by f_λ, where f_{μ_i} is a B-eigenfunction of weight $\mu_i \in \Omega_Y(X)$. We may assume that $p(f_\lambda)$ is T-homogeneous; since $\widetilde{A}^{U_0}$ is a domain, take i such that $\mu_i \neq 0$. Therefore $n\lambda = \mu_i + (n-i)\lambda$, i.e. $i\lambda = \mu_i$.

The second claim follows by the first one together with Proposition 1.5.6. □

Following lemma will allow us to establish the link between the lattice of an orbit $Z \subset X$ and that of the corresponding orbit $Z' \subset \widetilde{X}$.

Lemma 1.5.8. *Let $K' \subset K$ be two spherical subgroups of G with K' normal in K; fix a Borel subgroup B such that BK' is open in G and consider the projection $\pi : G/K' \to G/K$. Then $\pi^{-1}(BK/K) = BK'/K'$ and $\pi^* : \Lambda_{G/K} \to \Lambda_{G/K'}$ identifies $\Lambda_{G/K}$ with a sublattice of $\Lambda_{G/K'}$ such that*

$$\Lambda_{G/K'} \big/ \Lambda_{G/K} \simeq \mathcal{X}\left(K/K'\right).$$

Proof. First claim follows by the equality $BK' = BK$, which stems immediately from Theorem 1.3.5 iv).

If $B' \subset B$, denote $\mathcal{X}(B)^{B'}$ the kernel of the restriction $\mathcal{X}(B) \to \mathcal{X}(B')$, which is surjective by the following argument: if $U \subset B$ is the unipotent radical, then $\mathcal{X}(B) = \mathcal{X}(B/U)$ and $\mathcal{X}(B') = \mathcal{X}(B'/B' \cap U)$ and the restriction $\mathcal{X}(B/U) \to \mathcal{X}(B'/B' \cap U)$ is surjective since $B'/B' \cap U \subset B/U$ is a diagonalizable subgroup of the torus B/U.

By definition, we have isomorphisms $\Lambda_{G/K} \simeq \mathcal{X}(B)^{B\cap K}$ and $\Lambda_{G/K'} \simeq \mathcal{X}(B)^{B\cap K'}$; thus the restriction gives a surjective homomorphism

$$\Lambda_{G/K'} \to \mathcal{X}(B \cap K)^{B\cap K'} = \mathcal{X}\left(B \cap K / B \cap K'\right)$$

whose kernel is $\Lambda_{G/K}$. On the other hand $BK/K \simeq B/(B \cap K)$ and $BK'/K' \simeq B/(B \cap K')$, hence the equality $BK' = BK$ implies

$$B \cap K \big/ B \cap K' \simeq K/K'.$$

Therefore we get

$$\Lambda_{G/K'} \big/ \Lambda_{G/K} \simeq \mathcal{X}\left(B \cap K \big/ B \cap K'\right) \simeq \mathcal{X}\left(K/K'\right). \qquad \square$$

Proposition 1.5.9. *Let X be a linear spherical variety and let $p : \widetilde{X} \to X$ be the normalization. Let $Z \subset X$ be an orbit and let $Z' = p^{-1}(Z)$ be the corresponding orbit in $\widetilde{X}$.*

i) Z and Z' are isomorphic if and only if $\Lambda_Z = \Lambda_{Z'}$.

ii) $\Lambda_{Z'}$ is the saturation of Λ_Z in $\Lambda_{G/H}$.

iii) If $Z \simeq Z'$, then $\overline{Z'} \subset \widetilde{X}$ is the normalization of $\overline{Z} \subset X$.

Proof. First claim follows straightforward by previous lemma. By Theorem 1.4.3, we may identify $\Lambda_{Z'}$ with a saturated sublattice of $\Lambda_{G/H}$. Thus the second claim follows from previous lemma together with the fact that p is a finite map. Finally, the third claim stems from Proposition 1.2.4 together with the fact that the restriction $p : \overline{Z'} \to \overline{Z}$ is finite and birational. $\square$

1.5.1 Multiplicatively saturated colored semigroups

In this subsection we explain a necessary and sufficient condition so that a colored semigroup for a spherical homogeneous space G/H is associated to a simple linear compactification. Actually the condition is not satisfactory since it is not very expendable from a combinatorial point of view, however it sets the general spherical context of what will be done in Chapter 4 in the case of the adjoint group G_{ad}. For simplicity, we will assume that G is semisimple and simply connected.

Regard the coordinate ring $\Bbbk[G]$ as a $G \times G$-module, where the two factors act respectively on the left and on the right. If $\pi : G \to G/H$ is the projection and if $D \in \Delta(G/H)$, then every divisor $\pi^{-1}(D)$ has an equation $f_D \in \Bbbk[G]^{(B\times H)}$, which is uniquely defined up to a scalar factor since $\Bbbk[G]^* = \Bbbk^*$. Since $BH \subset G$ is open it must be $f_D(1) \neq 0$, so we may assume $f_D(1) = 1$. Denote $\omega : \Bbbk(G)^{(B\times H)} \to \mathcal{X}(B)$ and $\psi : \Bbbk(G)^{(B\times H)} \to \mathcal{X}(H)$ the maps which associate to every eigenfunction the respective characters.

Lemma 1.5.10 ([Lu 01] Lemma 6.2.2, [Bri 07] Lemma 2.1.1)**.** *The multiplicative group $\Bbbk(G)^{(B\times H)}/\Bbbk^*$ is freely generated by the functions f_D with $D \in \Delta(G/H)$. Moreover, the commutative diagram*

$$\begin{tikzcd} \Bbbk(G)^{(B\times H)}/\Bbbk^* \arrow[r, "\psi"] \arrow[d, "\omega"'] & \mathcal{X}(H) \arrow[d] \\ \mathcal{X}(B) \arrow[r] & \mathcal{X}(B\cap H) \end{tikzcd}$$

identifies $\Bbbk(G)^{(B\times H)}/\Bbbk^*$ with the fiber product

$$\mathcal{X}(B) \times_{\mathcal{X}(B\cap H)} \mathcal{X}(H) = \Big\{(\lambda,\chi) \in \mathcal{X}(B) \times \mathcal{X}(H) \,:\, \lambda|_{B\cap H} = \chi|_{B\cap H}\Big\}.$$

If $(\lambda,\chi) \in \mathcal{X}(B) \times_{\mathcal{X}(B\cap H)} \mathcal{X}(H)$, we will denote by $f_{\lambda,\chi} \in \Bbbk[G]^{(B\times H)}$ the eigenfunction of weights λ and χ defined by $f_{\lambda,\chi}(1) = 1$.

Recall the canonical decompostion as a $G \times G$-module

$$\Bbbk[G] \simeq \bigoplus_{\lambda\in\mathcal{X}(B)^+} V(\lambda)^* \otimes V(\lambda)$$

A dominant weight $\lambda \in \mathcal{X}(B)^+$ is called *spherical* if $V(\lambda)^H \neq 0$, it is called *quasi-spherical* if $V(\lambda)^{(H)} \neq 0$. Denote $\Lambda^+_{G/H}$ the semigroup of spherical weights and $\Xi^+_{G/H}$ the semigroup of quasi-spherical weights: then we get the decompositions

$$\Bbbk[G/H] \simeq \bigoplus_{\lambda\in\Lambda^+_{G/H}} V(\lambda)^* \otimes V(\lambda)^H \qquad \text{and} \qquad \Bbbk[G]^{(H)} \simeq \bigoplus_{\lambda\in\Xi^+_{G/H}} V(\lambda)^* \otimes V(\lambda)^{(H)}.$$

Definition 1.5.11. Let $\Omega^c = (\Omega, \Delta_\Box)$ be a colored semigroup for G/H and let $\Gamma \subset \Omega$ be a finite subset. We say that Γ is a *set of multiplicative generators* for ω^c if there exists

$$f_{\lambda,\chi} = \Big(\prod_{\Delta(G/H)\smallsetminus\Delta_\Box} f_D\Big)^N \in \Bbbk[G]^{(B\times H)}$$

such that $f_{\gamma+\lambda,\chi} \in \Bbbk[G]$ for all $\gamma \in \Gamma$ and such that the following condition holds:

(MS) If $\mu \in \mathcal{X}(B)^+$ and $n = \sum_{\Gamma\cup\{0\}} n_\gamma$ (with $n_\gamma \in \mathbb{N}$ for all $\gamma \in \Gamma \cup \{0\}$) are such that
$$f_{\mu,n\chi} \in \prod_{\gamma\in\Gamma\cup\{0\}} \langle G f_{\gamma+\lambda,\chi}\rangle^{n_\gamma}$$
then $\mu - n\lambda \in \Omega$ and every $\nu \in \Omega$ arises in this way.

If Ω^c admits a set of multiplicative generators, we will say that it is *multiplicatively saturated.*

Proposition 1.5.12. *If $G/H \hookrightarrow X$ is a simple normal embedding, then the colored semigroup $\Omega^c(X)$ is multiplicatively saturated.*

Proof. Denote $Y \subset X$ the closed orbit and let $\Gamma \subset \Omega_Y(X)$ be a subset which generates $\Omega_Y(X)$ as a semigroup. Identify $\Delta_Y(X)$ with a subset of $\Delta(G/H)$ and denote
$$f = \prod_{D\in\Delta(G/H)\smallsetminus\Delta_Y(X)} f_D.$$
If $\gamma \in \Gamma$, let $f_\gamma \in \Bbbk(G/H)^{(B)} \subset \Bbbk(G)^{(B\times H)}$ be an eigenfunction of weight γ: by Theorem 1.4.1 it follows $\nu_{\pi^{-1}(D)}(f_\gamma) \geqslant 0$ for all $D \in \Delta_Y(X)$, thus there exists $N > 0$ such that $f_\gamma f^N \in \Bbbk[G]$ for all γ. If $f^N = f_{\lambda,\chi}$, we get then $f_{\gamma+\lambda,\chi} \in \Bbbk[G]$ for all γ.

To show **(MS)**, suppose that μ and $n = \sum_{\Gamma\cup\{0\}} n_\gamma$ are such that
$$f_{\mu,n\chi} \in \prod_{\gamma\in\Gamma\cup\{0\}} \langle G f_{\gamma+\lambda,\chi}\rangle^{n_\gamma}$$
Write $\mu - n\lambda = \left(\sum_{\gamma\in\Gamma} n_\gamma\gamma\right) - \sigma$, where by Proposition 1.3.2
$$\sigma = \sum_{\gamma\in\Gamma} n_\gamma\gamma + n\lambda - \mu \in -\mathcal{V}^\vee_{G/H}.$$
Suppose that $\nu \in \mathcal{C}_Y(X) \cap \mathcal{V}_{G/H}$: then by Theorem 1.4.1 it follows
$$\langle \nu, \mu - n\lambda\rangle = \sum_{\gamma\in\Gamma} n_\gamma \langle\nu,\gamma\rangle - \langle \nu,\sigma\rangle \geqslant \sum_{\gamma\in\Gamma} n_\gamma\langle\nu,\gamma\rangle \geqslant 0.$$
Suppose now that $D \in \Delta(G/H)$ is such that $\nu_D(f_{\mu,n\chi} f^{-n}_{\lambda,\chi}) \leqslant 0$: then
$$\nu_{\pi^{-1}(D)}(f_{\mu,n\chi} f^{-n}_{\lambda,\chi}) \leqslant 0$$
as well. Since $f_{\mu,n\chi}$ is regular, by the definition of $f_{\lambda,\chi}$ it follows $D \in \Delta(G/H) \smallsetminus \Delta_Y(X)$. Therefore $\langle \rho(v_D), \mu - n\lambda\rangle \geqslant 0$ for all $D \in \Delta_Y(X) \cup \mathcal{B}_Y(X)$ and by the definition of $\mathcal{C}_Y(X)$ we get $\mu - n\lambda \in \mathcal{C}_Y(X)^\vee \cap \Lambda_{G/H}$. Theorem 1.4.1 shows then $\mu - n\lambda \in \Omega_Y(X)$. □

Since G is semisimple and simply connected, every line bundle $\mathcal{L} \in \mathrm{Pic}(G/H)$ admitsa unique linearization and we have an isomorphism $\mathrm{Pic}(G/H) \simeq \mathcal{X}(H)$ (see [KKV 89, Proposition 3.2]). If $\mathcal{L} \in \mathrm{Pic}(G/H)$ and $\chi \in \mathcal{X}(H)$ is the character of H acting on the fiber of $\mathcal{L}$ over eH, we will write then $\mathcal{L} = \mathcal{L}_\chi$: notice that there is a natural isomorphism $\Gamma(G/H, \mathcal{L}_\chi) \simeq \Bbbk[G]^{(H)}_\chi$.

Following the lines of the proof of [Kn 91, Theorem 4.1], we prove the following theorem.

Theorem 1.5.13. *The map $X \to \Omega^c(X)$ is a bijection between isomorphism classes of simple linear compactifications of G/H and multiplicatively saturated complete colored semigroups.*

Proof. Suppose that X is a simple linear compactification of G/H with closed orbit Y and consider the colored semigroup $\Omega^c(X) = (\Omega_Y(X), \Delta_Y(X))$. Following Proposition 1.5.6, $\Omega^c(X)$ is complete and the correspondence $X \mapsto \Omega^c(X)$ is injective. Let's show that $\Omega^c(X)$ is multiplicatively saturated.

Suppose that $X \subset \mathbb{P}(V)$ is an equivariant embedding, by Theorem 1.1.9, we may assume that $V = \bigoplus_{i=0}^m V(\mu_i)$ is multiplicity free. Fix $v \in V^{(H)}$ such that $G[v] \subset X$ is the open orbit. If $\chi \in \mathcal{X}(H)$ is such that $v \in V_\chi^{(H)}$, then the restriction of the hyperplane bundle $\mathcal{O}(1)$ to $G[v]$ is identified with the linearized line bundle $\mathcal{L}_\chi \in \mathrm{Pic}(G/H) \simeq \mathcal{X}(H)$ and its space of global sections is isomorphic to $\Bbbk[G]_\chi^{(H)}$.

Denote μ_i^* the highest weight of $V(\mu_i)^*$ and let $\eta_i \in V(\mu_i)^*$ be a highest weight vector. Regarded as a section of $\mathcal{L}_\chi$, η_i is identified with the function

$$\langle \eta_i, gv \rangle = f_{\mu_i^*,\chi}(g) \in \Bbbk[G]^{(B\times H)}.$$

Suppose $Y \subset \mathbb{P}(V(\mu_0))$ and consider the associated B-stable affine open subset $X_Y^\circ = X \cap \mathbb{P}(V)_{\eta_0}$. Its coordinate ring is generated as an algebra by the elements of the shape $f'/f_{\mu_0^*,\chi}$, with $f' \in V^*$. Thus every function $f \in \Bbbk[X_Y^\circ]$ is the restriction of a quotient of the shape $s/f^n_{\mu_0^*,\chi}$, where $s \in S^n(V)$ is in the n-symmetric power of V. The inclusion $V \subset \Gamma(G/H, \mathcal{L}) \simeq \Bbbk[G]_\chi^{(H)}$ identifies the multiplication in $S(V)$ with the multiplication in the subalgebra of $\Bbbk[G]$ generated by $\Bbbk[G]_\chi^{(H)}$. Hence it follows that Ω^c is multiplicatively generated by $\{\mu_1^* - \mu_0^*, \ldots, \mu_m^* - \mu_0^*\}$.

Suppose conversely that $\Omega^c = (\Omega, \Delta_\Box)$ is a multiplicatively saturated complete colored semigroup. Let $\Gamma \subset \Omega$ be a set of multiplicative generators and let $N \in \mathbb{N}$ be as in Definition 1.5.11. Consider the function

$$f_{\lambda,\chi} = \Big(\prod_{D \in \Delta(G/H) \smallsetminus \Delta_\Box} f_D \Big)^N \in \Bbbk[G]^{(B\times H)}$$

and, for $\gamma \in \Gamma$, let $f_\gamma \in \Bbbk(G/H)^{(B)} \subset \Bbbk(G)^{(B\times H)}$ be a B-eigenfunction of weight γ: by the definition of N we get then $f_\gamma f_{\lambda,\chi} = f_{\gamma+\lambda,\chi} \in \Bbbk[G]^{(B\times H)}$.

Denote $W \subset \Bbbk[G]$ the G-module generated by $\{f_{\gamma+\lambda,\chi}\}_{\gamma \in \Gamma\cup\{0\}}$ (where G acts on the left) and denote $V = W^*$. Since $W \subset \Bbbk[G]_\chi^{(H)}$, we get a morphism

$$\phi : G/H \longrightarrow \mathbb{P}(V).$$

Denote

$$X^\circ = \overline{\phi(G/H)} \cap \mathbb{P}(V)_{f_{\lambda,\chi}} \qquad \text{and} \qquad X = GX^\circ.$$

By construction, X is a simple spherical variety and

$$\Bbbk[X^\circ]^{(B)} \big/ \Bbbk^* = \Omega :$$

indeed as an algebra $\Bbbk[X^\circ]$ is generated by the elements of the shape $f'/f_{\lambda,\chi}$, with $f' \in W$. Since Ω generates $\Lambda_{G/H}$ by **(CS1)**, we get that $\Lambda_X = \Lambda_{G/H}$ and by Lemma 1.5.8 it follows that $\phi : G/H \to X$ is an embedding. Denote $p : \widetilde{X} \to X$

the normalization. If $Y \subset X$ the closed orbit, by Proposition 1.5.1 we get that $Y' = p^{-1}(Y)$ is the closed orbit of $\widetilde{X}$. Since Ω^c is complete, it follows that $\mathcal{C}_{Y'}(\widetilde{X}) = \mathcal{C}_Y(X) = \mathcal{C}(\Omega)^\vee \supset \mathcal{V}_{G/H}$: therefore Theorem 1.4.9 shows that $\widetilde{X}$ and X are complete. Finally, it follows by the definition of $f_{\lambda,\chi}$ that $X \smallsetminus X^\circ$ is the union of the colors in $\Delta(X) \smallsetminus \Delta_\square$: by Theorem 1.2.2 this implies $\Delta_Y(X) = \Delta_\square$. □

Previous theorem is not satisfactory since, given $\chi \in \mathcal{X}(H)$ and quasi-spherical weights $\lambda, \mu \in \Xi^+_{G/H}$ such that $V(\lambda)^{(H)}_\chi \neq 0$ and $V(\mu)^{(H)}_\chi \neq 0$, it is not at all clear how to describe combinatorially the multiplication

$$m_\chi : V(\lambda) \otimes V(\mu) \longrightarrow \bigoplus_{\nu \in \Xi^+_{G/H} \,:\, \nu \leqslant \lambda + \mu} V(\nu)$$

induced by the multiplication in $\Bbbk[G]$ by the identifications $V(\lambda) \simeq \langle G f_{\lambda,\chi} \rangle$ and $V(\mu) \simeq \langle G f_{\mu,\chi} \rangle$. In the case of the adjoint group G_{ad} regarded as a spherical $G \times G$ variety, such a description is known (see Example 1.3.3). By using this description, in Chapter 4 we will see how the correspondence of previous theorem can be made much more concrete, allowing in principle to give an explicit classification of the simple linear compactifications of a semisimple adjoint group. In particular, we will examine in details the case of an odd orthogonal group and we will derive a such classification in this case.

1.6 Colored subspaces and coconnected inclusions

An inclusion $H \subset H'$ of spherical subgroups of G is called *coconnected* if H'/H is connected. As simple normal embeddings $G/H \hookrightarrow X$ are classified by strictly convex colored cones, the set of coconnected inclusions $H \subset H'$ is classified by a particular class of colored cones for G/H.

Suppose that $H \subset H'$ are spherical subgroups. Denote $\phi : G/H \to G/H'$ the projection and consider the induced maps

$$\phi^* : \Lambda_{G/H'} \hookrightarrow \Lambda_{G/H} \qquad \text{and} \qquad \phi_* : (\Lambda^\vee_{G/H})_\mathbb{Q} \twoheadrightarrow (\Lambda^\vee_{G/H'})_\mathbb{Q}$$

Denote $\Delta_\phi \subset \Delta(G/H)$ the subset of colors which map dominantly on G/H' and denote

$$\mathcal{C}_\phi = \{ v \in (\Lambda^\vee_{G/H})_\mathbb{Q} \,:\, v(\chi) = 0 \text{ for all } \chi \in \Lambda_{G/H'} \}.$$

the annihilator of $\Lambda_{G/H'}$.

Theorem 1.6.1 ([Kn 91] Lemma 5.3)**.** *Suppose that H'/H is connected.*

i) The projection $\phi : G/H \to G/H'$ identifies $\Delta(G/H')$ with $\Delta(G/H) \smallsetminus \Delta_\phi$. Moreover $\Lambda_{G/H'} = \Lambda_{G/H} \cap \mathcal{C}_\phi^\perp$ is saturated in $\Lambda_{G/H}$ and $\mathcal{V}_{G/H'}$ is the quotient of $\mathcal{V}_{G/H}$ by $\mathcal{C}_\phi$.

ii) $(\mathcal{C}_\phi, \Delta_\phi)$ is a colored cone for G/H.

Definition 1.6.2. A *colored subspace* for G/H is a colored cone $(\mathcal{C}, \Delta_\square)$ such that $\mathcal{C} \subset (\Lambda^\vee_{G/H})_\mathbb{Q}$ is a linear subspace.

Theorem 1.6.3 ([Kn 91] Thm. 5.4)**.** *Let H be a spherical subgroup. The map $H' \mapsto (\mathcal{C}_\phi, \Delta_\phi)$ induces a bijection between the set of subgroups $H' \supset H$ such that H'/H is connected and the set of colored subspaces for G/H.*

Chapter 2

Wonderful varieties

Throughout this chapter, G will denote a simply connected semisimple algebraic group over an algebraically closed field of characteristic zero. The action of the center of G on any G-variety will be assumed to be trivial: all the considered G-varieties will be G_{ad}-varieties, where G_{ad} denotes the adjoint group of G.

2.1 Wonderful varieties and spherical systems

Let G/H be a spherical homogeneous space and suppose that $\mathcal{V}_{G/H}$ is a strictly convex cone: then the couple $(\mathcal{V}_{G/H}, \varnothing)$ is a colored cone and by Theorem 1.4.9 it follows that the associated embedding $M(G/H)$ is simple, complete and toroidal. If it exists, $M(G/H)$ is called the *canonical embedding* of G/H. The following corollary to Theorem 1.3.5 explains when does a canonical embedding for G/H exist.

Corollary 2.1.1. *A spherical homogeneous space G/H admits a canonical embedding if and only if $[N_G(H) : H] < \infty$.*

A spherical subgroup H is called *sober* if it has finite index in its normalizer. Suppose that H is sober; then the canonical embedding $M(G/H)$ satisfies the following universal property: given any toroidal embedding X' and any simple completion X'' of G/H, there exist unique proper birational equivariant morphisms

$$X' \longrightarrow M(G/H) \longrightarrow X''$$

which extend the identity map on G/H.

If a canonical embedding is smooth, then it is called a *wonderful embedding.* A sober subgroup H will be called *wonderful* if the canonical embedding of G/H is wonderful.

Suppose that M is the canonical embedding of a spherical homogeneous space G/H; denote $Y \subset M$ the closed orbit and $y_0 \in Y^{B^-}$ the unique B^- fixed-point. Following Theorems 1.2.3 and 1.2.6, consider the decomposition

$$M^\circ = P^u \times W_M,$$

where $M^\circ = M \smallsetminus \bigcup_{\Delta(M)} D$ and where $W_M \subset M^\circ$ is an affine toric variety for a quotient of T with fixed point y_0 such that

$$\mathbb{k}[W_M]^{(T)} \big/ \mathbb{k}^* = \mathcal{V}^\vee_{G/H} \cap \Lambda_{G/H} = \Omega_Y(M).$$

Therefore, by [Oda 88, Theorem 1.10], it follows that M is smooth if and only if the semigroup $\Omega_Y(M)$ is freely generated by a (uniquely defined) basis of $\Lambda_{G/H}$, which is contained in the root lattice since the center of G acts trivially. Moreover, if Σ_M is the opposite of this basis, then Theorem 1.3.1 shows that $\Sigma_M \subset \mathbb{N}S$ and every element in Σ_M is either a positive root or a sum of two strongly orthogonal positive roots.

Definition 2.1.2. Suppose that M is the wonderful embedding of a spherical variety. An element $\sigma \in \Sigma_M$ is called a *spherical root* of M (or equivalently of G/H).

Suppose that M is the wonderful embedding of a spherical variety G/H. Then G-stable divisors of M correspond with T-stable divisors on W_M, which are the coordinate hyperplanes relatively to the basis Σ_M. It follows that $M \smallsetminus G/H$ is the union of $r = \mathrm{rk}(M)$ smooth prime divisors having a non-empty transversal intersection and the following description holds:

$$\Sigma_M = \left\{ T\text{-weights of the } T\text{-module } {T_{y_0}M}\big/{T_{y_0}Y} \right\}.$$

If $\sigma \in \Sigma_M$, denote M^σ the associated G-stable prime divisor of M, defined by

$${T_{y_0}M}\big/{T_{y_0}M^\sigma} \cong V_T(\sigma),$$

where $V_T(\sigma)$ denotes the one dimensional T-module of weight σ. Equivalently, $M^\sigma \cap M^\circ$ is the principal divisor defined by a B-eigenfunction $f_{-\sigma} \in \Bbbk[M^\circ]^{(B)}$ of weight $-\sigma$.

Definition 2.1.3. A *wonderful variety* (of *rank* r) is a smooth projective G-variety having an open orbit which satisfies following properties:

i) the complement of the open orbit is the union of r smooth prime divisors having a non-empty transversal intersection;

ii) any orbit closure equals the intersection of the prime divisors containing it.

Theorem 2.1.4 ([Lu 96])**.** *A G-variety is wonderful if and only if it is the wonderful embedding of a spherical homogeneous space.*

Example 2.1.5 (**The wonderful completion of an adjoint symmetric space**)**.** Following Example 1.3.4, consider the case of an adjoint symmetric space G/H, where $H = N_G(G^\sigma)$ is the normalizer of the set of the points fixed by an algebraic involution $\sigma : G \to G$. By the isomorphism

$$\Lambda_{G/H} \simeq \mathcal{X}\left({T_1}\big/{T_1 \cap H}\right) = \mathbb{Z}\overline{S},$$

since $\mathcal{V}_{G/H} \subset \mathcal{X}(T_1/T_1 \cap H)^\vee_\mathbb{Q}$ is identified with the negative Weyl chamber, it follows that $\mathcal{V}_{G/H} \cap \Lambda^\vee_{G/H} = -\mathbb{N}\overline{S}$ is a free semigroup. Therefore G/H admits a wonderful embedding M.

The variety M was first considered by C. De Concini and C. Procesi in [DCP 83]. If λ is any dominant weight such that $\sigma(\lambda) = -\lambda$ and such that $\langle \lambda, \alpha^\vee \rangle \neq 0$ for every $\alpha \in S_1$, then H fixes (pointwise) a unique line in $V(\lambda)$: if $v_0 \in V(\lambda)^H$ is a non-zero representative of this line, then in [DCP 83] it is shown that $\mathrm{Stab}[v_0] = H$ and that $\overline{G[v_0]} \subset \mathbb{P}(V(\lambda))$ is a smooth G-variety which satisfies conditions i) and ii) of Definition 2.1.3.

Suppose that M is a wonderful variety. If $\Sigma' \subset \Sigma_M$, then the G-stable subvariety

$$M_{\Sigma'} = \bigcap_{\sigma \in \Sigma_M \smallsetminus \Sigma'} M^\sigma$$

is a wonderful G-variety whose spherical root set is Σ': this defines a bijection between subsets of spherical roots and G-stable irreducible subvarieties of M. We will call $M_{\Sigma'}$ the *localization* of M at Σ'.

Wonderful varieties of rank one are well known and classified [Ak 83]; in particular, for any fixed G, they are finitely many. Denote $\Sigma(G)$ the finite set of all possible spherical roots of a rank one wonderful G-variety and denote $\mathrm{Supp}(\sigma)$ the set of simple roots $\alpha \in S$ where σ is supported (see Table 2.1).

If M is a wonderful variety, then Σ_M is the set of spherical root of all possible rank one wonderful G-subvarieties of M: therefore $\Sigma_M \subset \Sigma(G)$.

Table 2.1. The set of spherical roots of G.

Type of $\mathrm{Supp}(\sigma)$	**Shape of** σ	**Type of** σ
A_1	α_1 $2\alpha_1$	$\mathsf{A}_1^{\mathrm{I}}$ $\mathsf{A}_1^{\mathrm{II}}$
$\mathsf{A}_1 \times \mathsf{A}_1$	$\alpha_1 + \alpha_1'$	$\mathsf{A}_1 \times \mathsf{A}_1$
A_r, $r \geqslant 2$	$\alpha_1 + \ldots + \alpha_r$	A_r
B_r, $r \geqslant 2$	$\alpha_1 + \ldots + \alpha_r$ $2\alpha_1 + \ldots + 2\alpha_r$	$\mathsf{B}_r^{\mathrm{I}}$ $\mathsf{B}_r^{\mathrm{II}}$
B_3	$\alpha_1 + 2\alpha_2 + 3\alpha_3$	$\mathsf{B}_3^{\mathrm{III}}$
C_r, $r \geqslant 3$	$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{r-1} + \alpha_r$	C_r
D_r, $r \geqslant 3$	$2\alpha_1 + \ldots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$	D_r
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	F_4
G_2	$2\alpha_1 + \alpha_2$ $4\alpha_1 + 2\alpha_2$ $\alpha_1 + \alpha_2$	$\mathsf{G}_2^{\mathrm{I}}$ $\mathsf{G}_2^{\mathrm{II}}$ $\mathsf{G}_2^{\mathrm{III}}$

If α is a simple root, denote P_α the associated minimal parabolic subgroup containing B and denote

$$\Delta(M)(\alpha) = \{D \in \Delta(M) \,:\, P_\alpha D \neq D\}.$$

If $D \in \Delta(M)(\alpha)$, we will say that D is *moved by* α.

Proposition 2.1.6 ([Lu 97] §3.2 and §3.4)**.** *For every $\alpha \in S$, the cardinality of $\Delta(M)(\alpha)$ is at most 2. Moreover:*

a) if $\Delta(M)(\alpha) = \{D_\alpha^+, D_\alpha^-\}$ has cardinality 2, then $\alpha \in \Sigma_M$ and

$$\rho(D_\alpha^+) + \rho(D_\alpha^-) = \alpha^\vee|_{\Lambda_M}.$$

2a) If $\Delta(M)(\alpha) = \{D_\alpha\}$ has cardinality 1 and if $2\alpha \in \Sigma_M$, then

$$\rho(D_\alpha) = \frac{1}{2}\alpha^\vee|_{\Lambda_M}.$$

b) If $\Delta(M)(\alpha) = \{D_\alpha\}$ has cardinality 1 and if $2\alpha \notin \Sigma_M$, then

$$\rho(D_\alpha) = \alpha^\vee|_{\Lambda_M}.$$

We will say that a simple root α is of *type* a, $2a$ or b if it holds respectively the condition a), $2a$) or b) of previous lemma, finally we will say that α is of type p if $\Delta(M)(\alpha) = \varnothing$. Denote $S^a_M, S^{2a}_M, S^b_M, S^p_M$ the set of simple roots of type respectively $a, 2a, b$ and p. Equivalently, S^p_M coincides with the set of simple roots associated with the stabilizer of the B^--fixed point y_0 in the closed orbit Y and $\alpha \in S^p_M$ if and only if $P_\alpha M^\circ = M^\circ$.

Denote $\Delta^a(M)$ (resp. $\Delta^{2a}(M)$, $\Delta^b(M)$) the union of the $\Delta(\alpha)$'s where α runs in S^a_M (resp. in S^{2a}_M, S^b_M).

Proposition 2.1.7 ([Lu 01] Prop. 3.2)**.** *Let $\alpha, \beta \in S$; then $\Delta(M)(\alpha) \cap \Delta(M)(\beta) \neq \varnothing$ if and only if it holds one of the followings:*

i) $\alpha, \beta \in S^a_M$ and $\Delta(M)(\alpha) \cup \Delta(M)(\beta)$ has cardinality 3.

ii) $\alpha, \beta \in S^b_M$ are orthogonal and $\alpha + \beta \in \Sigma_M$.

It follows that the union

$$\Delta(M) = \Delta(M)^a \cup \Delta(M)^{2a} \cup \Delta(M)^b$$

is disjoint. We will say that a color $D \in \Delta(M)$ is of *type* a, $2a$ or b according as $D \in \Delta(M)^a$, $D \in \Delta(M)^{2a}$ or $D \in \Delta(M)^b$.

Definition 2.1.8. The natural pairing $c_M : \Delta(M) \times \Sigma_M \to \mathbb{Z}$ between colors and spherical roots defined by

$$c_M(D, \sigma) = \langle \rho(D), \sigma \rangle$$

is called the *Cartan pairing* of M.

Regarding $\Delta(M)$ as a set of functionals (possibly containing some repeated elements) of the lattice $\Lambda_M = \mathbb{Z}\Sigma_M$, it turns out from Lemma 2.1.6 that the subsets of colors $\Delta(M)^{2a}$ and $\Delta(M)^b$ can be recovered from the set of spherical roots Σ_M together with the set of simple roots S^p_M. This remark leads to the following definition.

Definition 2.1.9. The *spherical system* of M is the triplet

$$\mathscr{S}_M = (\Sigma_M, S^p_M, \mathbf{A}_M),$$

where $\mathbf{A}_M = \Delta(M)^a$ is regarded as a multi-subset of $\Lambda^\vee_M$ via the map $c_M : \Delta(M)^a \times \Sigma_M \to \mathbb{Z}$. If H is any wonderful subgroup, by the *spherical system* of G/H we will mean that of its wonderful completion.

Following theorem was conjectured by D. Luna in [Lu 01].

Theorem 2.1.10 ([Lo 09] Thm. 1)**.** *Two G-wonderful varieties are equivariantly isomorphic if and only if they have the same spherical system.*

Consider now the case of a rank one wonderful variety: in this case the third datum of the spherical system is uniquely determined by the other data.

Definition 2.1.11. A spherical root $\sigma \in \Sigma(G)$ and a subset $S^p \subset S$ are called *compatible* if there exists a rank one wonderful variety X such that $\Sigma_X = \{\sigma\}$ and $S^p_X = S^p$.

Following the classification of rank one wonderful varieties given in [Ak 83], it follows that σ and S^p are compatible if and only if

$$S^{pp}(\sigma) \subset S^p \subset S^p(\sigma)$$

where $S^p(\sigma)$ denotes the set of simple roots orthogonal to σ and where

$$S^{pp}(\sigma) = \begin{cases} S^p(\sigma) \cap \mathrm{Supp}(\sigma) \smallsetminus \{\alpha_{i+r}\} & \text{if } \sigma \text{ is of type } \mathsf{B}^{\mathrm{I}}_r \\ S^p(\sigma) \cap \mathrm{Supp}(\sigma) \smallsetminus \{\alpha_{i+1}\} & \text{if } \sigma \text{ is of type } \mathsf{C}_r \\ S^p(\sigma) \cap \mathrm{Supp}(\sigma) & \text{otherwise} \end{cases}$$

where $\mathrm{Supp}(\sigma) = \{\alpha_{i+1}, \ldots, \alpha_{i+r}\}$ and simple roots are labelled following Bourbaki [Bo 75].

2.2 The Picard group of a wonderful variety

From now on, M will denote a wondeful variety with open B-orbit Bx_0 and generic stabilizer $H = \mathrm{Stab}(x_0)$. If this is not confusing, we will drop the indices relating to M form all those sets we associated to M in previous section: we will denote the spherical system of M by $\mathscr{S} = (\Sigma, S^p, \mathbf{A})$ and the set of colors of M by Δ, we will denote S^a, S^{2a}, S^b the respective sets of simple roots associated to M.

If $\sigma \in \Sigma$, consider $M^\sigma \cap M^\circ$: by its definition, it is the principal divisor of M° associated to f_σ^{-1}, where $f_\sigma \in \Bbbk(G/H)^{(B)}$ is a B-eigenfunction of weight σ.

Regard $\Bbbk(G/H)^{(B)}$ as a subgroup of $\Bbbk(G)^{(B\times H)}$ and recall from Lemma 1.5.10 that the latter is the generated by the functions f_D with $D \in \Delta$. Since G is semisimple it follows $\Bbbk[G]^* = \Bbbk$, thus by the definition of the Cartan pairing up to a scalar factor it holds the equality

$$f_\sigma = \prod_{D\in\Delta} f_D^{c(D,\sigma)}.$$

By considering the associated divisor, we get then $\mathrm{div}(f_\sigma) = [M^\sigma] + \sum_{D\in\Delta} c(D,\sigma)[D]$, hence

$$[M^\sigma] = \sum_{D\in\Delta} c(D,\sigma)[D]:$$

thus the Cartan pairing expresses the coefficients of the G-stable divisors with respect to the basis Δ.

Denote $Y \subset M$ the closed orbit and let $y_0 \in Y^{B^-}$ be the B^--fixed point. Since G is semisimple and simply connected, $\mathrm{Pic}(Y)$ is identified with a sublattice of $\mathcal{X}(B)$, while $\mathrm{Pic}(G/H)$ is identified with $\mathcal{X}(H)$ (see [KKV 89, Proposition 3.2]): if $\mathcal{L} \in \mathrm{Pic}(Y)$, then $\mathcal{L}$ will be identified with the character of B^- acting on the fiber of $\mathcal{L}$ over y_0, while if $\mathcal{L} \in \mathrm{Pic}(G/H)$, then $\mathcal{L}$ will be identified with the character of H acting on the fiber over eH.

Proposition 2.2.1 ([Bri 07] Prop. 2.2.1)**.** *i) There is an exact sequence*

$$0 \longrightarrow \mathbb{Z}\mathcal{B} \longrightarrow \mathrm{Pic}(M) \longrightarrow \mathrm{Pic}(G/H) \to 0,$$

where $\mathcal{B} = \mathcal{B}(M)$ is the set of G-stable prime divisors of M.

ii) Denote $\omega : \mathrm{Pic}(M) \to \mathcal{X}(B)$ *and* $\psi : \mathrm{Pic}(M) \to \mathcal{X}(H)$ *the restrictions to the closed and to the open orbit. Then the commutative diagram*

$$\begin{tikzcd} \mathrm{Pic}(M) \arrow[r, "\psi"] \arrow[d, "\omega"'] & \mathcal{X}(H) \arrow[d] \\ \mathcal{X}(B) \arrow[r] & \mathcal{X}(B \cap H) \end{tikzcd}$$

induces isomorphisms

$$\mathrm{Pic}(M) \simeq \mathcal{X}(B) \times_{\mathcal{X}(B\cap H)} \mathcal{X}(H) \simeq \mathbb{k}(G)^{(B\times H)} \big/ \mathbb{k}^*.$$

This identifies the exact sequence in i) with

$$0 \longrightarrow \mathcal{X}(B)^{B\cap H} \longrightarrow \mathcal{X}(B) \times_{\mathcal{X}(B\cap H)} \mathcal{X}(H) \longrightarrow \mathcal{X}(H) \to 0,$$

where $\mathcal{X}(B)^{B\cap H} = \Lambda_{G/H}$ *denotes the group of characters of* B *which are invariant under* $B \cap H$*.*

The isomorphisms of previous proposition can be explicitly described as follows. Let $\delta \in \mathbb{N}\Delta$ be a divisor generated by global sections and denote $\mathcal{O}(\delta) \in \mathrm{Pic}(M)$ be the associated line bundle. Let $s \in \Gamma(M, \mathcal{O}(\delta))^{(B)}$ be the canonical section: then the simple G-module $\langle Gs \rangle \subset \Gamma(M, \mathcal{O}(\delta))$ generated by s is identified with the simple module $V(\omega(\delta))$ and we get a morphism

$$\phi_\delta : M \longrightarrow \mathbb{P}(V(\omega(\delta))^*).$$

Take $v_0 \in (V(\omega(\delta))^*)^{(H)}$ such that $[v_0] = \phi_\delta(x_0)$: then (up to a scalar factor) the $B \times H$-eigenfunction $f_\delta \in \mathbb{k}[G]^{(B\times H)}$ associated to δ is

$$f_\delta(g) = \langle s, g v_0 \rangle.$$

In particular, the character $\psi(\delta) \in \mathcal{X}(H)$ coincides with the H-weight of v_0.

Denote $\Xi_M \subset \mathcal{X}(B)$ the image of the restriction $\omega : \mathrm{Pic}(M) \to \mathcal{X}(B)$. If ω is injective, then $\Xi_M \cap \mathcal{X}(B)^+ = \Xi^+_{G/H}$ is the semigroup of quasi-spherical weights of G/H and $\Lambda_M \cap \mathcal{X}(B)^+ = \Lambda^+_{G/H}$ is the semigroup of spherical weights (see §1.5.1).

Consider the partial order on Ξ_M defined as follows:

$$\mu \leqslant_\Sigma \lambda \qquad \text{if and only if} \qquad \lambda - \mu \in \mathbb{N}\Sigma.$$

Theorem 2.2.2 ([DCP 83] Thm. 8.3)**.** *Let* $\delta \in \mathbb{N}\Delta$ *be a divisor generated by its global sections.*

i) If $\mathrm{Hom}\,(V(\mu), \Gamma(M, \mathcal{O}(\delta))) \neq 0$*, then* $\mu \leqslant_\Sigma \omega(\delta)$*.*

ii) If the restriction ω *is injective, then the converse also is true:*

$$\Gamma(M, \mathcal{O}(\delta)) \simeq \bigoplus_{\mu \in \mathcal{X}(B)^+ \,:\, \mu \leqslant_\Sigma \omega(\delta)} V(\mu).$$

Proof. i). Consider the canonical section $s_\delta \in \Gamma\big(M, \mathcal{O}(\delta)\big)^{(B)}$: it is a highest weight vector of weight $\omega(\delta)$ and it determines a trivialization of the restiction $\mathcal{O}(\delta)\big|_{M^\circ}$.

If $s \in \Gamma\big(M, \mathcal{O}(\delta)\big)^{(B)}$ is a B-eigenvector, then $s = f s_\delta$ with $f \in k[M^\circ]^{(B)}$; if μ is the weight of s it holds then

$$\mu = \omega(\delta) - \sum_{\sigma \in \Sigma} a_\sigma \sigma,$$

where a_σ is the vanishing order of $f = \prod_\Sigma f_\sigma^{-a_\sigma}$ along the divisor M^σ. Therefore $\mu \leqslant_\Sigma \omega(\delta)$.

ii). Suppose that ω is injective and that $\mu \in \Xi_M^+$ is of the shape $\mu = \omega(\delta) - \sum a_\sigma \sigma$. Consider the B-stable divisor

$$\delta' = \delta - \sum_{\sigma \in \Sigma} a_\sigma M^\sigma.$$

Since ω is injective, Ξ_M^+ is a semigroup which is freely generated by the $\omega(D)$'s, $D \in \Delta$: thus by Theorem 1.4.6 a weight $\lambda \in \Xi_M$ is dominant if and only if $\omega^{-1}(\lambda) \in \mathbb{N}\Delta$ is generated by global sections. Therefore δ' is generated by global sections.

Let $s_\sigma \in \Gamma\big(M, \mathcal{O}(M^\sigma)\big)$ and $s_{\delta'} \in \Gamma\big(M, \mathcal{O}(\delta')\big)$ be the canonical sections: s_σ is a B-semiinvariant eigenvector of weight σ, while $s_{\delta'}$ is a B-semiinvariant eigenvector of weight μ. Then the claim follows since

$$s = s_{\delta'} \prod_\sigma s_\sigma^{a_\sigma} \in \Gamma\big(M, \mathcal{O}(\delta)\big) :$$

is a non-zero B-eigenvector of weight μ. □

A combinatorial description of the restriction ω is given by following theorem.

Theorem 2.2.3 ([Fo 98] Thm. 2.2)**.** *The map $\omega : \mathrm{Pic}(M) \to \mathcal{X}(B)$ is combinatorially described on colors as follows:*

$$\omega(D) = \begin{cases} \sum_{D \in \Delta(\alpha)} \omega_\alpha & \text{if } D \in \Delta \smallsetminus \Delta^{2a} \\ 2\omega_\alpha & \text{if } D = D_\alpha \in \Delta^{2a} \end{cases}$$

Corollary 2.2.4. *If M is a wonderful variety which does not possess simple spherical roots, then the restriction $\omega : \mathrm{Pic}(M) \longrightarrow \mathcal{X}(B)$ is injective.*

Remark 2.2.5. If $\Sigma' \subset \Sigma$ is a subset of spherical roots, consider the associated localization $M' = \bigcap_{\sigma \in \Sigma \smallsetminus \Sigma'} M^\sigma$: it is a wonderful variety whose spherical system is $\mathscr{S}' = (\Sigma', S^p, \mathbf{A}')$, where $\mathbf{A}' = \bigcup_{\alpha \in S \cap \Sigma'} \mathbf{A}(\alpha)$ (see [Lu 01, §2.2.3])

Denote Δ' the set of colors of M'; if $\alpha \in S \cap \Sigma'$ and $\beta \in S \smallsetminus (\Sigma' \cup S^p)$ set $\Delta'(\alpha) = \{\, 'D_\alpha^+, 'D_\alpha^- \}$ and $\Delta'(\beta) = \{\, D'_\beta \}$. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Pic}(M) \arrow[r, "q"] \arrow[dr, "\omega"'] & \mathrm{Pic}(M') \arrow[d, "\omega'"] \\ & \mathcal{X}(B) \end{tikzcd}$$

Then Theorem 2.2.3 shows that:

- if $\alpha \in S \cap \Sigma'$ then $q(D_\alpha^+)$ (resp. $q(D_\alpha^-)$) is supported on $'D_\alpha^+$ (resp. on $'D_\alpha^-$) with multiplicity one, while it is not supported on $'D_\alpha^-$ (resp. on $'D_\alpha^+$);
- if $\alpha \in S \cap (\Sigma \smallsetminus \Sigma')$ then $q(D_\alpha^+)$ and $q(D_\alpha^-)$ are supported on D'_α with multiplicity one;
- if $\alpha \in S \cap \frac{1}{2}\Sigma'$, then $q(D_\alpha) = D'_\alpha$;
- if $\alpha \in S \cap \frac{1}{2}(\Sigma \smallsetminus \Sigma')$, then $q(D_\alpha) = 2D'_\alpha$;
- if $\alpha \in S^b$, then $q(D_\alpha)$ is supported on D'_α with multiplicity one and on at most one more color.

2.3 The G-equivariant automorphism group

Let H be a spherical subgroup. Then the natural right action of $N_G(H)$ on G/H identifies the G-equivariant automorphism group $\mathrm{Aut}_G(G/H)$ with $N_G(H)/H$ and induces as well an action of $N_G(H)$ on $\Delta(G/H)$.

Definition 2.3.1. The kernel of the action of $N_G(H)$ on Δ is called the *spherical closure* of H. If H coincides with its spherical closure, then it is called *spherically closed.*

Denote $\overline{H}$ the spherical closure of a spherical subgroup H: then $\overline{H}$ is still a spherical subgroup of G and the projection $G/H \to G/\overline{H}$ identifies $\Delta(G/H)$ with $\Delta(G/\overline{H})$. Moreover, since the identity component $(N_G(H)/H)^\circ$ acts trivially on $\Delta(G/H)$, spherically closed subgroups are sober.

Theorem 2.3.2 ([Kn 96] Cor. 7.6)**.** *A spherically closed subgroup is wonderful.*

A wonderful variety (or equivalently a spherical homogeneous space) will be called *spherically closed* if its generic stabilizer is so. In particular, every self-normalizing spherical subgroup is spherically closed, thus wonderful. Following proposition gives another characterization of spherically closed subgroups. By a *simple projective space* we will mean the projective space of a simple G-module.

Proposition 2.3.3 ([BL 08] Cor. 2.4.2)**.** *A spherical subgroup is spherically closed if and only if it occurs as the stabilizer of a point in a simple projective space.*

A very special class of spherically closed subgroups arises by requiring that M can be embedded in a simple projective space. As shown by Example 2.1.5, for instance this is the case if M is the wonderful completion of an adjoint symmetric space.

Definition 2.3.4. A wonderful variety is called *strict* if the stabilizer of any point $x \in M$ is self-normalizing. A spherical subgroup is called *strict* if it occurs as the generic stabilizer of a strict wonderful variety.

Theorem 2.3.5 ([Pe 07] Thm. 2)**.** *Let M be a wonderful variety. Then M is strict if and only if there exists a simple module V together with a closed embedding $M \hookrightarrow \mathbb{P}(V)$.*

Theorem 2.3.6 ([Pe 07] Thm. 5 and Lemma 14)**.** *Let M be a strict wonderful variety.*

i) If $\delta \in \mathbb{N}\Delta$ is any ample divisor, then the associated morphism

$$\phi_\delta : M \longrightarrow \mathbb{P}(V(\omega(\delta))^*)$$

is a closed embedding.

ii) The restriction to the closed orbit $\omega : \mathrm{Pic}(M) \to \mathcal{X}(B)$ is injective.

Let M be a wonderful variety with spherical system $\mathscr{S} = (\Sigma, S^p, \mathbf{A})$. Then the G-equivariant automorphism group $\mathrm{Aut}_G(M)$ is naturally identified with $N_G(H)/H$.

Definition 2.3.7. A spherical root $\sigma \in \Sigma$ is called *loose* if one of the following conditions holds:

i) $\sigma \in \Sigma \smallsetminus S$, $2\sigma \in \Sigma(G)$ and the couple $(2\sigma, S^p)$ is compatible.

ii) $\sigma \in S \cap \Sigma$ and $c(D_\sigma^+, \sigma') = c(D_\sigma^-, \sigma')$ for every $\sigma' \in \Sigma$.

The set of loose spherical roots will be denoted by Σ_ℓ.

Following the classification of rank one wonderful varieties [Ak 83], non-simple loose spherical roots are easily described. They are those of the following types (where $S = \{\alpha_1, \ldots, \alpha_n\}$ and simple roots are labelled as in Bourbaki [Bo 75]):

– spherical roots $\sigma = \alpha_{i+1} + \ldots + \alpha_{i+r}$ of type $\mathsf{B}_\mathsf{r}^\mathrm{I}$ with $\alpha_{i+r} \in S^p$;

– spherical roots $\sigma = 2\alpha_{i+1} + \alpha_{i+2}$ of type G_2^I.

Fix a base point $x_0 \in M$ and set $H = \mathrm{Stab}(x_0)$. Denote $\overline{M} = M(G/N_G(H))$ the wonderful completion of $G/N_G(H)$ and denote $\overline{\Sigma}$ its set of spherical roots; then we get a morphism $M \to \overline{M}$ which determines an inclusion $\overline{\Sigma} \subset \mathbb{N}\Sigma$.

Recall the is isomorphism

$$\Theta : \mathrm{Aut}_G(M) \xrightarrow{\sim} \mathrm{Hom}\left({}^{\mathbb{Z}\Sigma}\!/_{\mathbb{Z}\overline{\Sigma}}, \Bbbk^*\right)$$

defined in Theorem 1.3.5: hence the description of $\mathrm{Aut}_G(M)$ follows from the description of $\overline{\Sigma}$.

Theorem 2.3.8 ([Lo 09] Thm. 2)**.** *$\overline{\Sigma}$ is obtained from Σ by doubling loose spherical roots:*

$$\overline{\Sigma} = (\Sigma \smallsetminus \Sigma_\ell) \cup 2\Sigma_\ell.$$

As a consequence we get the following description:

$$\mathrm{Aut}_G(M) \simeq \left({}^{\mathbb{Z}}\!/_{2\mathbb{Z}}\right)^{\mathrm{card}(\Sigma_\ell)}.$$

Remark 2.3.9. Let $\gamma \in \mathrm{Aut}_G(M)$ and denote $\Sigma(\gamma) = \{\sigma \in \Sigma_\ell \,:\, \Theta_\gamma(\sigma) = -1\}$. Consider the quotient variety M/γ: it is a simple, toroidal complete spherical variety whose weight lattice is described as follows:

$$\Lambda_{M/\gamma} = \{\lambda \in \Lambda_M \,:\, \Theta_\gamma(\lambda) = 1\} = \mathbb{Z}[\Sigma \smallsetminus \Sigma(\gamma)] \oplus 2\,\mathbb{Z}\Sigma(\gamma).$$

If $Y' \subset M/\gamma$ is the closed orbit, it follows that

$$\Omega_{Y'}(M/\gamma) = -\Big(\mathbb{N}[\Sigma \smallsetminus \Sigma(\gamma)] \oplus 2\,\mathbb{N}\Sigma(\gamma)\Big) :$$

hence M/γ is a wonderful variety with spherical system $\mathscr{S}/\gamma = (\Sigma/\gamma, S^p, \mathbf{A}/\gamma)$, where

$$\Sigma/\gamma = (\Sigma \smallsetminus \Sigma(\gamma)) \cup 2\Sigma(\gamma) \qquad \text{and} \qquad \mathbf{A}/\gamma = \bigcup_{\alpha \in S \cap \Sigma/\gamma} \mathbf{A}(\alpha).$$

Denote Δ' the set of colors of M/γ and consider the induced map $\Delta \to \Delta'$. Then Proposition 2.1.6 shows that γ acts transitively on $\Delta(\alpha)$ for every $\alpha \in S \cap \Sigma(\gamma)$, while it fixes every color $D \in \Delta(\alpha)$ with $\alpha \in S \smallsetminus \Sigma(\gamma)$.

If $\sigma \in \Sigma_\ell$, denote $\gamma(\sigma) \in \mathrm{Aut}_G(M)$ the unique automorphism such that

$$\Theta_{\gamma(\sigma)}(\sigma') = \left\{ \begin{array}{ll} 1 & \text{if } \sigma' \neq \sigma \\ -1 & \text{if } \sigma' = \sigma \end{array} \right. :$$

by previous remark, $\gamma(\sigma)$ acts trivially on every $\Delta(\alpha)$ with $\alpha \neq \sigma$, while if $\sigma \in S \cap \Sigma_\ell$ it exchanges D_σ^+ and D_σ^-.

Remark 2.3.10. By the combinatorial description of the equivariant automorphism group we get the following characterizations:

- H is self-normalizing if and only if $\Sigma_\ell = \varnothing$;
- H is spherically closed if and only if $\Sigma_\ell \subset S$;
- H is strict if and only if $S \cap \Sigma = \varnothing$ and $\Sigma_\ell = \varnothing$.

In particular, if $S \cap \Sigma = \varnothing$, then H is self-normalizing if and only if it is spherically closed if and only if it is strict.

Example 2.3.11 (Wonderful model varieties). Besides the wonderful completions of adjoint symmetric spaces, another remarkable class of strict wonderful varieties is that of *wonderful model varieties*, introduced by D. Luna in [Lu 07]. A quasi-affine homogeneous space G/H is called a *model variety* for G if its coordinate ring $\Bbbk[G/H]$ is a model of the representations of Gin the sense of [BGG 76], i.e. if it contains each irreducible representation of G exactly once. For instance this is the case if $H = U$ is a maximal unipotent subgroup of G. If G is a classical group, several examples of model varieties for G were given in [GZ 84] and [GZ 85].

It follows by Theorem 1.1.9 that model varieties are spherical. If G is a connected and semisimple group, in [Lu 07] it has been given a classification of the model varieties for G by means of wonderful varieties. More precisely, there is introduced a wonderful variety M_G^{mod} whose orbits parametrize the model varieties for G in the following way: every orbit of M_G^{mod} is of the shape $G/N_G(H)$, where G/H is a model variety for G, and this correspondence gives a bijection up to isomorphism with the class of model varieties for G.

The spherical system of M_G^{mod} is described as follows. Denote

$$\begin{aligned} R^{\mathrm{mod}} &= \{\alpha+\beta \,:\, \alpha,\beta \in S \text{ are non-orthogonal }\} \\ S^{\mathrm{ev}} &= \{\alpha \in S \,:\, \alpha^\vee \text{ is even on } S\} \end{aligned}$$

and denote by G_α the simple factor of G whose Dynkin diagram contains the vertex corresponding to a given $\alpha \in S$: notice that if $\alpha \in S^{\mathrm{ev}}$ then $G(\alpha)$ is either simply connected and isomorphic to $\mathrm{Spin}(2r+1)$ or it is adjoint and isomorphic to $\mathrm{SO}(2r+1)$. Then the spherical system of M_G^{mod} is $\mathscr{S}_G^{\mathrm{mod}} = (\Sigma_G^{\mathrm{mod}}, \varnothing, \varnothing)$ where

$$\Sigma_G^{\mathrm{mod}} = R^{\mathrm{mod}} \cup \{2\alpha \,:\, \alpha \in S^{\mathrm{ev}} \text{ and } G(\alpha) \text{ is adjoint }\}.$$

Since it possesses no simple spherical roots and no loose spherical roots, it follows that M_G^{mod} is a strict wonderful variety. We will call *wonderful model variety* any localization of M_G^{mod}.

2.4 Morphisms between wonderful varieties

Let M be a wonderful variety with base point x_0 and set $H = \mathrm{Stab}(x_0)$. Set $\mathscr{S} = (\Sigma, S^p, \mathbf{A})$ its spherical system and $\Delta = \Delta(G/H)$ its set of colors.

Definition 2.4.1. A subset $\Delta^* \subset \Delta$ is called *distinguished* if there exists $\delta \in \mathbb{N}_{>0}[\Delta^*]$ such that $c(\delta,\sigma) \geqslant 0$ for every $\sigma \in \Sigma$.

Let $\Delta^* \subset \Delta$ be a subset; then the condition to be distinguished is equivalent to

$$\mathbb{N}_{>0}[\rho(\Delta^*)] \cap (-\mathcal{V}_{G/H}) \neq \varnothing.$$

Consider the smallest face $\mathcal{F} \subset \mathcal{V}_{G/H}$ such that $\mathbb{N}_{>0}\rho(\Delta^*) \cap (-\mathcal{F}) \neq \varnothing$ and denote $N(\Delta^*)$ the cone generated by $\rho(\Delta^*)$ together with $\mathcal{F}$; then $\big(N(\Delta^*), \Delta^*\big)$ is a colored subspace for G/H which intersects the invariant valuation cone $\mathcal{V}_{G/H}$ in a face.

Lemma 2.4.2 ([Lu 01] Lemma 3.3.1)**.** *A subset $\Delta^* \subset \Delta$ is distinguished if and only if there exists a (uniquely defined) subspace $N(\Delta^*) \subset (\Lambda^\vee_{G/H})_{\mathbf{Q}}$ which satisfies the following conditions:*

(DS1) *The pair $\big(N(\Delta^*), \Delta^*\big)$ is a colored subspace for G/H.*

(DS2) *The intersection $N(\Delta^*) \cap \mathcal{V}_{G/H}$ is a face of $\mathcal{V}_{G/H}$.*

Proposition 2.4.3 ([Lu 01] Prop. 3.3.2)**.** *The application which associate to $H' \supset H$ the set $\Delta_\phi \subset \Delta$ of colors which map dominantly on G/H' via the projection $\phi : G/H \to G/H'$ induces an inclusion-preserving bijection as follows:*

$$\left\{ \ \Delta^* \subset \Delta \ \textit{distinguished} \ \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} H' \subset G \ \textit{sober} \ : \\ H \subset H' \ \textit{and} \ H'/H \ \textit{connected} \end{array} \right\}$$

Proof. Let's show that if $H' \supset H$ is a sober subgroup and if $\phi : G/H \to G/H'$ is the projection, then the subset of colors

$$\Delta_\phi = \{D \in \Delta \,:\, \overline{\phi(D)} = G/H'\}$$

is distinguished. Suppose that H'/H is connected. By Theorem 1.6.3 the pair $(\mathcal{C}_\phi, \Delta_\phi)$ is a colored subspace for G/H and the valuation cone $\mathcal{V}_{G/H'}$ is the quotient cone of $\mathcal{V}_{G/H}$ by $\mathcal{C}_\phi$. Thus the claim follows since the condition that $\mathcal{V}_{G/H'}$ is strictly convex (i.e. that H' is a sober subgroup) is equivalent to the fact that $\mathcal{C}_\phi \cap \mathcal{V}_{G/H}$ is a face.

If H'/H is not connected, then the claim follows by considering $H'' = H(H')^\circ$, where $(H')^\circ$ denotes the identity component of H'. Indeed $\phi : G/H \to G/H'$ factors through G/H'' and, if $\phi'' : G/H \to G/H''$ is the projection, then $\Delta_\phi = \Delta_{\phi''}$, $\mathcal{C}_\phi = \mathcal{C}_{\phi''}$ and $\mathcal{V}_{G/H'} = \mathcal{V}_{G/H''}$.

Suppose conversely that $\Delta^* \subset \Delta$ is a distinguished subset. Then by Lemma 2.4.2 together with Theorem 1.6.3 there exists a unique spherical subgroup $H' \supset H$ with H'/H connected such that $\Delta^* = \Delta_\phi$. Since $\mathcal{V}_{G/H'}$ is the quotient of $\mathcal{V}_{G/H}$ by $N(\Delta^*)$ and since by **(DS2)** the intersection $N(\Delta^*) \cap \mathcal{V}_{G/H}$ is a face of $\mathcal{V}_{G/H}$, it follows that $\mathcal{V}_{G/H'}$ is strictly convex, i.e. H' is a sober subgroup. □

If $H' \supset H$ is a sober subgroup such that H'/H is connected, denote M' the canonical embedding of $M(G/H')$ and denote $\Delta' = \Delta(G/H')$. Set $(M')^\circ = M' \smallsetminus \bigcup_{D \in \Delta'} D$ and consider the projection $\phi : M \to M'$: then

$$\phi^{-1}((M')^\circ) = M \smallsetminus \bigcup_{\Delta \smallsetminus \Delta_\phi} D.$$

Since the fibers of ϕ are complete and connected, it follows that

$$\Bbbk[(M')^\circ] = \Bbbk[\phi^{-1}((M')^\circ)].$$

Considering the B-semiinvariant functions, we get then the identification of semigroups

$$\Omega(M') = -{}^{\mathbb{N}\Sigma}\!/_{\Delta_\phi}$$

where

$${}^{\mathbb{N}\Sigma}\!/_{\Delta_\phi} = \{\sigma \in \mathbb{N}\Sigma \,:\, c(D,\sigma) = 0,\ \forall D \in \Delta_\phi\}\,.$$

Therefore M' is smooth if and only if the semigroup ${}^{\mathbb{N}\Sigma}\!/_{\Delta_\phi}$ is free. In [Lu 01, Cor. 5.6.2] it was proved that, in case G is of type A, then such semigroup is necessarily free; although this was claimed in general in [Lu 07], a general proof appeared only recently in [Bra 09].

Theorem 2.4.4 ([Bra 09] Thm. 3.3.1)**.** *If $H' \subset H$ is a sober subgroup such that H'/H is connected, then H' is wonderful.*

Actually [Bra 09, Theorem 3.3.1] is a combinatorial version of previous theorem, which deals with abstract spherical systems and which stems from their classification.

Combining together Theorem 1.6.3, Proposition 2.4.3 and Theorem 2.4.4, we get the following theorem.

Theorem 2.4.5. *There is an inclusion-preserving bijection as follows*

$$\left\{ \Delta^* \subset \Delta \text{ distinguished} \right\} \longleftrightarrow \left\{ \begin{array}{c} H' \subset G \text{ wonderful } : \\ H \subset H' \text{ and } H'/H \text{ connected} \end{array} \right\}$$

Moreover, if $H' \supset H$ is a wonderful subgroup with H'/H connected and if $\Delta^* \subset \Delta$ is the corresponding distinguished subset, then

i) the projection $G/H \to G/H'$ identifies $\Delta(G/H')$ with $\Delta \smallsetminus \Delta^*$;

ii) the spherical system of the wonderful completion of G/H' is

$$\mathscr{S}/\Delta^* = (\Sigma/\Delta^*, S^p/\Delta^*, \mathbf{A}/\Delta^*),$$

defined as follows:

- *Σ/Δ^* is the set of indecomposable elements of the (free) semigroup $\mathbb{N}\Sigma/_{\Delta^*}$;*
- *$S^p/\Delta^* = S^p \cup \{\alpha \in S \,:\, \Delta(\alpha) \subset \Delta^*\}$;*
- *$\mathbf{A}/\Delta^* = \bigcup_{\alpha \in S \cap \Sigma/\Delta^*} \mathbf{A}(\alpha)$, and the pairing is obtained by restriction.*

In the notations of previous theorem, the wonderful completion of G/H' is denoted M/Δ^* and it is called the *quotient wonderful variety* of M by Δ^*, while $\mathscr{S}/\Delta^*$ is called the *quotient spherical system* of $\mathscr{S}$ by Δ^*.

2.5 Faithful divisors

Let M be a spherically closed wonderful variety; fix a base point x_0 and set $H = \mathrm{Stab}(x_0)$. Denote $\mathscr{S} = (\Sigma, S^p, \mathbf{A})$ its spherical system and denote Δ its set of colors. Let $\delta = \sum_{D \in \Delta} n(\delta, D)D \in \mathbb{N}\Delta$ be a divisor generated by global sections; define its *support*

$$\mathrm{Supp}_\Delta(\delta) = \{D \in \Delta \,:\, n(\delta, D) > 0\}$$

and denote $V_\delta = V(\omega(\delta))^*$.

Lemma 2.5.1. *Let M be a wonderful variety and let $\delta \in \mathbb{N}\Delta$ be a divisor generated by global sections; consider the associated morphism $\phi_\delta : M \to \mathbb{P}(V_\delta)$. Then the correspondence of Theorem 2.4.5 gives an inclusion-preserving bijection as follows*

$$\left\{ \begin{array}{c} \Delta^* \subset \Delta \text{ distinguished } : \\ \Delta^* \cap \mathrm{Supp}_\Delta(\delta) = \varnothing \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} H' \subset G \text{ wonderful } : \\ H \subset H' \subset \mathrm{Stab}(\phi_\delta(x_0)) \\ \text{and } H'/H \text{ connected} \end{array} \right\}$$

Proof. Let $H' \supset H$ be a wonderful subgroup with H'/H connected and set $\Delta^* \subset \Delta$ the corresponding distinguished subset. If M' is the wonderful completion of G/H', then the projection $G/H \to G/H'$ extends to a morphism $M \to M'$ and pullback identifies $\mathrm{Pic}(M')$ with the submodule $\mathbb{Z}[\Delta \smallsetminus \Delta^*] \subset \mathbb{Z}\Delta = \mathrm{Pic}(M)$. Thus the map $M \to \mathbb{P}(V_\delta)$ factors through a map $M' \to \mathbb{P}(V_\delta)$ if and only if $\mathrm{Supp}_\Delta(\delta) \subset \Delta \smallsetminus \Delta^*$. □

Definition 2.5.2. A divisor generated by global sections $\delta = \sum n(\delta, D)D \in \mathbb{N}\Delta$ is called *faithful* if it satisfies the following conditions:

(FD1) Every non-empty distinguished subset of Δ intersects $\mathrm{Supp}_\Delta(\delta)$.

(FD2) If $\alpha \in \Sigma_\ell$ is a loose spherical root, then $n(\delta, D_\alpha^+) \neq n(\delta, D_\alpha^-)$.

Proposition 2.5.3 ([BL 08] Prop. 2.4.3)**.** *Let M be a spherically closed wonderful variety and let $\delta \in \mathbb{N}\Delta$. Then the associated morphism $\phi_\delta : M \to \mathbb{P}(V_\delta)$ restricts to an embedding of the open orbit if and only if δ is faithful.*

Proof. Fix $v_0 \in V$ a representative of the line $\phi_\delta(x_0)$. Recall the restriction $\psi : \mathrm{Pic}(M) \to \mathcal{X}(H)$: then $v_0 \in V_{\psi(\delta)}^{(H)}$.

Suppose that $H = \mathrm{Stab}[v_0]$; then **(FD1)** holds by Lemma 2.5.1. Suppose by absurd that **(FD2)** fails and let $\alpha \in \Sigma_\ell \subset S \cap \Sigma$ be a loose spherical root such that $n(\delta, D_\alpha^+) = n(\delta, D_\alpha^-)$. If $\gamma(\alpha) \in \mathrm{Aut}_G(M) = N_G(H)/H$ is the corresponding automorphism, then $\gamma(\alpha)$ exchanges D_α^+ and D_α^- and fixes every other color $D \in \Delta \smallsetminus \Delta(\alpha)$: therefore $\gamma(\alpha)$ fixes δ. The action of $\mathrm{Aut}_G(M)$ on $\mathrm{Pic}(M) = \mathbb{Z}\Delta \simeq \mathcal{X}(B) \times_{\mathcal{X}(B\cap H)} \mathcal{X}(H)$ is defined extending by linearity the right action of $N_G(H)/H$ on Δ, i.e. by the action of $N_G(H)$ on $\mathcal{X}(H)$. Therefore, if $g \in N_G(H)$ is a representative of $\gamma(\alpha)$, then $\psi^g = \psi(\delta)$, i.e. g moves the line $[v_0]$ in a line where H acts by the same character. By Theorem 1.1.9 such a line is unique, thus $g \in H = \mathrm{Stab}[v_0]$ which is absurd.

Suppose conversely that δ is a faithful divisor. By **(FD1)** together with Lemma 2.5.1 it follows that $\dim H = \dim \mathrm{Stab}[v_0]$, therefore by Theorem 1.3.5 we get $H \subset \mathrm{Stab}[v_0] \subset N_G(H)$. Suppose by absurd that there exists $g \in \mathrm{Stab}[v_0] \smallsetminus H$. Then $\psi(\delta)^g = \psi(\delta)$ and the equivariant automorphism corresponding to the coset gH fixes δ: therefore by **(FD2)** we get that every color $D \in \mathrm{Supp}_\Delta(\delta)$ is fixed by g. On the other hand, since H is spherically closed, every element in $N_G(H) \smallsetminus H$ acts non-trivially on Δ. Hence there exists $\alpha \in S$ such that g moves $D \in \Delta(\alpha)$. Therefore $\alpha \in \Sigma_\ell \subset S \cap \Sigma$ and $\Delta(\alpha) = \{D, D\cdot g\}$: it follows $n(\delta, D) = n(\delta, D \cdot g) = 0$, which contradicts **(FD2)**. □

Corollary 2.5.4. *Let M be a wonderful variety and let $\delta \in \mathbb{N}\Delta$ be a divisor generated by global sections; suppose that every distinguished subset of Δ intersects $\mathrm{Supp}_\Delta(\delta)$ and set*

$$\Sigma(\delta) = \left\{\alpha \in \Sigma_\ell \,:\, \alpha \notin S \text{ or } n(\delta, D_\alpha^+) = n(\delta, D_\alpha^-)\right\}.$$

Consider the morphism $\phi_\delta : M \to \mathbb{P}(V_\delta)$; then the spherical system of $\mathrm{Stab}(\phi_\delta(x_0))$ is $\mathscr{S}' = (\Sigma', S^p, \mathbf{A}')$, where

$$\Sigma' = (\Sigma \smallsetminus \Sigma(\delta)) \cup 2\Sigma(\delta) \qquad \text{and} \qquad \mathbf{A}' = \bigcup_{\alpha \in S \cap \Sigma'} \mathbf{A}(\alpha).$$

Proof. Denote $\Gamma_\delta \subset \mathrm{Aut}_G(M)$ the subgroup generated by the elements $\gamma(\sigma)$, with $\sigma \in \Sigma(\delta)$, and consider the quotient variety M/Γ_δ: by Remark 2.3.9 it is a wonderful variety with spherical system $\mathscr{S}'$. Denote $H_\delta \supset H$ the generic stabilizer of M/Γ_δ: reasoning as in the first part of the proof of Proposition 2.5.3 it follows that H_δ fixes $\phi_\delta(x_0)$, thus M/Γ_δ is a spherically closed wonderful variety endowed with a faithful divisor whose associated characters are the same of δ and the claim follows by previous proposition. □

In the hypotheses of previous corollary, the assumption that every distinguished subset of colors intersects $\mathrm{Supp}_\Delta(\delta)$ (which is equivalent to assume that H and

$\mathrm{Stab}(\phi_\delta(x_0))$ have the same dimension) involves no loss of generality: we can always reduce to that case considering, instead of M, the quotient wonderful variety $M/\Delta(\delta)$, where $\Delta(\delta) \subset \Delta$ is the maximal distinguished subset which does not intersect $\mathrm{Supp}_\Delta(\delta)$.

2.6 Abstract spherical systems and Luna diagrams

We give here the definition due to D. Luna of spherical system as an abstract combinatorial object.

Definition 2.6.1. A *spherical system* is a triplet $\mathscr{S} = (\Sigma, S^p, \mathbf{A})$ where

- Σ is a subset of $\Sigma(G)$ without proportional elements;
- S^p is a subset of S;
- $\mathbf{A}$ is a finite abstract set together with a map $c : \mathbf{A} \times \Sigma \to \mathbb{Z}$

satisfying the following properties:

(S) S^p is compatible with all $\sigma \in \Sigma$;

(A1) For all $D \in \mathbf{A}$ and $\sigma \in \Sigma$, it holds $c(D, \sigma) \leqslant 1$ and $c(D, \sigma) = 1$ implies $\sigma \in S \cap \Sigma$

(A2) For all $\alpha \in S \cap \Sigma$, the set $\mathbf{A}(\alpha) = \{D \in \mathbf{A} : c(D, \alpha) = 1\}$ has cardinality 2, and if $\mathbf{A}(\alpha) = \{D^+_\alpha, D^-_\alpha\}$, then $c(D^+_\alpha, \sigma) + c(D^-_\alpha, \sigma) = \langle \alpha^\vee, \sigma \rangle$ for all $\sigma \in \Sigma$;

(A3) $\mathbf{A}$ is the union of the $\mathbf{A}(\alpha)$'s, for $\alpha \in S \cap \Sigma$;

(Σ1) If $2\alpha \in \Sigma \cap 2S$, then $\langle \alpha^\vee, \sigma \rangle$ is a non-positive even integer for all $\sigma \in \Sigma \smallsetminus \{2\alpha\}$;

(Σ1) If $\alpha + \beta \in \Sigma$ with $\alpha, \beta \in S$ and α orthogonal to β, then $\langle \alpha^\vee, \sigma \rangle = \langle \beta^\vee, \sigma \rangle$ for all $\sigma \in \Sigma$.

If M is a wonderful variety, then the triple $\mathscr{S}_M = (\Sigma_M, S^p_M, \mathbf{A}_M)$ of Definition 2.1.9 is a spherical system according previous definition.

Conjecture 2.6.2 ([Lu 01])**.** *Wonderful varieties are classified by spherical systems.*

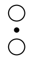
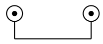
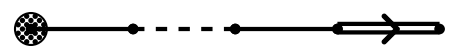
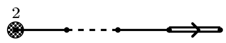
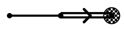
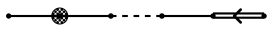
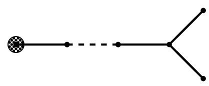
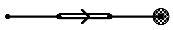
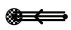
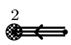
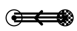
While the "uniqueness part" of the conjecture has been proved by I. Losev in [Lo 09], the "existence part" has been checked directly in many cases by P. Bravi, S. Cupit-Foutou, D. Luna and G. Pezzini (see [Bra 07], [BC 10], [BL 08], [BPe 05], [BPe 09], [Lu 01]) and recently a general proof which avoids a case-by-case approach has been proposed by S. Cupit-Foutou in [Cu 09].

In the classification of spherical G-varieties, the classification of wonderful G_{ad}-varieties takes a prominent role: indeed the classification of the latter implies the whole classication of spherical varieties (see [Lu 01, Theorem 3]).

A very useful tool to represent graphically a spherical system starting from the Dynkin diagram of G are *Luna diagrams*, introduced by D. Luna in [Lu 01]. We now briefly explain how to attach such a diagram to a spherical system; for further details and examples we refer to [BL 08].

Let $\mathscr{S} = (\Sigma, S^p, \mathbf{A})$ be a spherical system.

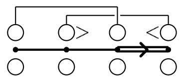
Table 2.2. Diagrams of spherical roots.

Type of σ	**Diagram of** σ	**Shape of** σ
$\mathsf{A}_1^{\mathrm{I}}$		α_1
$\mathsf{A}_1^{\mathrm{II}}$		$2\alpha_1$
$\mathsf{A}_1 \times \mathsf{A}_1$		$\alpha_1 + \alpha_1'$
A_r, $r \geqslant 2$		$\alpha_1 + \ldots + \alpha_r$
$\mathsf{B}_r^{\mathrm{I}}$, $r \geqslant 2$		$\alpha_1 + \ldots + \alpha_r$
$\mathsf{B}_r^{\mathrm{II}}$, $r \geqslant 2$		$2\alpha_1 + \ldots + 2\alpha_r$
$\mathsf{B}_3^{\mathrm{III}}$		$\alpha_1 + 2\alpha_2 + 3\alpha_3$
C_r, $r \geqslant 3$		$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{r-1} + \alpha_r$
D_r, $r \geqslant 3$		$2\alpha_1 + \ldots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$
F_4		$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
$\mathsf{G}_2^{\mathrm{I}}$		$2\alpha_1 + \alpha_2$
$\mathsf{G}_2^{\mathrm{II}}$		$4\alpha_1 + 2\alpha_2$
$\mathsf{G}_2^{\mathrm{III}}$		$\alpha_1 + \alpha_2$

- Following Table 2.2, represent the spherical roots $\sigma \in \Sigma$ on the Dynkin diagram of G.
- Draw a white circle around the simple roots $\alpha \in S^b = S \smallsetminus (\Sigma \cup \frac{1}{2}\Sigma \cup S^p)$ which do not already possess a black circle around. In this way, the set S^p coincides with the set of simple roots without any circle around, above or below.
- If $\alpha \in S \cap \Sigma$, interpret the circles drawn above and below the corresponding vertex of the Dynkin diagram as the elements of $\mathbf{A}(\alpha)$. Denote D_α^+ the element corresponding to the circle above the vertex and D_α^- that one corresponding to the circle below the vertex: then we may assume that $c(D_\alpha^+, \sigma) \in \{-1, 0, 1\}$ for every $\sigma \in \Sigma$. Join by a line two circles if they correspond to the same element $D \in \mathbf{A}$. Finally, if $\sigma \in \Sigma$ is such that $c(D_\alpha^+, \sigma) = -1$ with $\langle \alpha^\vee, \sigma \rangle \neq 0$, draw an arrow starting from D_α^+ and pointing toward σ.

Once the diagram is drawn, the restricted pairing $c : \mathbf{A} \times \Sigma \to \mathbb{Z}$ can be recovered thanks to Axiom (A2).

Example 2.6.3. Consider the Luna diagram

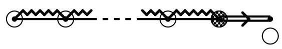
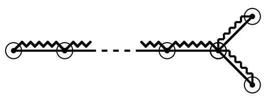
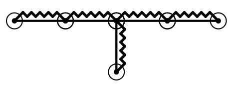
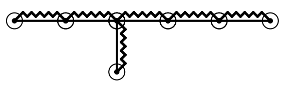
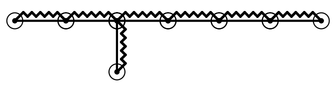
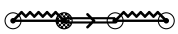
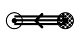


It represents the spherical system for the group of type Spin(9) given by $\mathscr{S} = (S, \varnothing, \mathbf{A})$ where $\mathbf{A}$ is described by following table (for simplicity we write D_i instead of D_{α_i}).

	α_1	α_2	α_3	α_4
$D_1^+ = D_3^+$	1	0	1	0
D_1^-	1	-1	-1	0
$D_2^+ = D_4^+$	0	1	-1	1
D_2^-	-1	1	0	-1
D_3^-	-1	-1	1	-1
D_4^-	0	-1	-1	1

As an example, following their combinatorial description given in Example 2.3.11, in Table 2.3 we draw the Luna diagrams of the wonderful model varieties M_G^{mod} where G is any simple group.

Table 2.3. Luna diagrams of wonderful model varieties.

Type of G	**Diagram of** M_G^{mod}
A_r	
$\mathrm{Spin}(2r+1)$	
$\mathrm{SO}(2r+1)$	
C_r	
D_r	
E_6	
E_7	
E_8	
F_4	
G_2	

Chapter 3

Spherical orbit closures in simple projective spaces

Throughout this chapter, G will denote a simply connected semisimple algebraic group over an algebraically closed field of characteristic zero.

3.1 The variety X_δ and its normalization $\widetilde{X}_\delta$

If λ is a dominant weight, define its *support* as

$$\mathrm{Supp}(\lambda) = \{\alpha \in S \,:\, \langle \lambda, \alpha^\vee \rangle \neq 0\}.$$

Denote by $\mathcal{L}_\lambda$ the line bundle on G/B whose T-weight in the B^--fixed point is λ: then $\Gamma(G/B, \mathcal{L}_\lambda) \simeq V(\lambda)$ is an irreducible G-module of highest weight λ.

If λ, μ are dominant weights and $n \in \mathbb{N}$, the multiplication of sections defines maps as follows:

$$m_{\lambda,\mu} : V(\lambda) \times V(\mu) \to V(\lambda + \mu) \quad \text{and} \quad m^n_\lambda : V(\lambda) \to V(n\lambda).$$

We will denote $m_{\lambda,\mu}(v, w)$ by vw and $m^n_\lambda(v)$ by v^n. Since G/B is irreducible, $m_{\lambda,\mu}$ and m^n_λ induce the following maps at the level of projective spaces:

$$\psi_{\lambda,\mu} : \mathbb{P}(V(\lambda)) \times \mathbb{P}(V(\mu)) \to \mathbb{P}(V(\lambda + \mu)) \quad \text{and} \quad \psi^n_\lambda : \mathbb{P}(V(\lambda)) \to \mathbb{P}(V(n\lambda)).$$

Lemma 3.1.1 ([BGMR 10] Lemma 1)**.** *Let λ, μ be dominant weights.*

i) If $\mathrm{Supp}(\lambda) \cap \mathrm{Supp}(\mu) = \varnothing$, then the map

$$\psi_{\lambda,\mu} \colon \mathbb{P}(V(\lambda)) \times \mathbb{P}(V(\mu)) \to \mathbb{P}(V(\lambda + \mu))$$

is a closed embedding.

ii) For any $n > 0$, the map $\psi^n_\lambda : \mathbb{P}(V(\lambda)) \to \mathbb{P}(V(n\lambda))$ is a closed embedding.

Let M be a wonderful variety with base point x_0 and set $H = \mathrm{Stab}(x_0)$; set $\mathscr{S} = (\Sigma, S^p, \mathbf{A})$ its spherical system and $\Delta = \Delta(G/H)$ its set of colors. Recall the restrictions $\omega : \mathrm{Pic}(M) \to \mathcal{X}(B)$ and $\psi : \mathrm{Pic}(M) \to \mathcal{X}(H)$. If $\delta \in \mathbb{N}\Delta$ is a divisor generated by global sections, denote $V_\delta = V(\omega(\delta))^*$ and let $v_\delta \in (V_\delta)^{(H)}_{\psi(\delta)}$. Define

$X_\delta = \overline{G[v_\delta]} \subset \mathbb{P}(V_\delta)$: equivalently, $X_\delta = \phi_\delta(M)$ is the image of M via the morphism associated to δ.

Let $\Delta^\square = \{D_1, \ldots, D_m\} \subset \Delta$ be a subset such that $\Delta(\alpha) \not\subset \Delta^\square$ for all $\alpha \in S \cap \Sigma$. Recall the restriction to the open orbit $\psi : \mathrm{Pic}(M) \to \mathcal{X}(H)$ and set $\chi_i = \psi(D_i) \in \mathcal{X}(H)$. Fix vectors $v_i \in (V_{D_i})^{(H)}_{\chi_i}$ (uniquely defined up to a scalar factor) and define

$$X_{\Delta^\square} = \overline{G([v_1], \ldots, [v_m])} \subset \mathbb{P}(V_{D_1}) \times \ldots \times \mathbb{P}(V_{D_m}).$$

Thanks to the assumtion on $\Delta^\square$, by Theorem 2.2.3 together with Lemma 3.1.1 we get an embedding

$$X_{\Delta^\square} \subset \mathbb{P}(V_{D_1}) \times \ldots \times \mathbb{P}(V_{D_m}) \hookrightarrow \mathbb{P}(V_{D_1+\ldots+D_m}).$$

Denote $\chi = \chi_1 + \ldots + \chi_m = \psi(D_1 + \ldots + D_m)$. Since $v_1 \cdots v_m \in (V_{D_1+\ldots+D_m})^{(H)}_\chi$ and since such a line is unique, it follows that the image of the base point of $X_{\Delta^\square}$ is the base point of $X_{D_1+\ldots+D_m}$. Thus the embedding above induces an isomorphism of G-varieties

$$X_{\Delta^\square} \simeq X_{D_1+\ldots+D_m}.$$

Similarly, Lemma 3.1.1 shows that for every $n \in \mathbb{N}$ and $D \in \Delta$ there is an isomorphism of G-varieties $X_{nD} \simeq X_D$. Combinining these remarks together we get the following lemma.

Lemma 3.1.2. *Let $\delta \in \mathbb{N}\Delta$ and suppose that $\Delta(\alpha) \not\subset \mathrm{Supp}_\Delta(\delta)$ for every $\alpha \in S \cap \Sigma$. Then $X_\delta \simeq X_{\mathrm{Supp}_\Delta(\delta)}$.*

Proposition 3.1.3. *Suppose that M is a strict wonderful variety and let $\delta, \delta' \in \mathbb{N}\Delta$. Then there exists a G-equivariant morphism $X_\delta \to X_{\delta'}$ if and only if $\mathrm{Supp}_\Delta(\delta') \subset \mathrm{Supp}_\Delta(\delta)$. In particular X_δ and $X_{\delta'}$ are G-equivariantly isomorphic if and only if $\mathrm{Supp}_\Delta(\delta) = \mathrm{Supp}_\Delta(\delta')$.*

Proof. Since a strict wonderful variety has no simple spherical roots, by previous lemma it follows $X_\delta \simeq X_{\mathrm{Supp}_\Delta(\delta)}$ and $X_{\delta'} \simeq X_{\mathrm{Supp}_\Delta(\delta')}$.

By Theorem 2.2.3 the restriction to the closed orbit $\omega : \mathrm{Pic}(M) \to \mathcal{X}(B)$ is injective. In particular by Theorem 2.2.1 this implies that H fixes at most one line in any simple G-module.

Suppose that $\mathrm{Supp}_\Delta(\delta') \subset \mathrm{Supp}_\Delta(\delta)$ and consider the projection

$$\prod_{D \in \mathrm{Supp}_\Delta(\delta)} \mathbb{P}(V_D) \longrightarrow \prod_{D \in \mathrm{Supp}_\Delta(\delta')} \mathbb{P}(V_D) :$$

since every $\mathbb{P}(V_D)$ contain a unique H-fixed point, it follows that the image of the base point of X_δ is the base point of $X_{\delta'}$: thus the restriction to X_δ induces a G-equivariant morphism $X_\delta \to X_{\delta'}$.

Suppose conversely that X_δ dominates $X_{\delta'}$ and write $\delta = \sum_\Delta n(\delta, D)D$ and $\delta' = \sum_\Delta n(\delta', D)D$. Notice that, if $H' \supset H$ is a wonderful subgroup of G and if Σ' is the associated set of spherical roots, then

$$-\Sigma' \subset \mathcal{V}^\vee_{G/H'} \cap \Lambda_{G/H'} \subset \mathcal{V}^\vee_{G/H} \cap \Lambda_{G/H} = -\mathbb{N}\Sigma :$$

since $\Sigma \cap S = \varnothing$, Theorem 1.3.1 shows then $\Sigma' \cap S = \varnothing$ as well.

Denote $H_\delta = \mathrm{Stab}(\phi_\delta(x_0))$ and $H_{\delta'} = \mathrm{Stab}(\phi_{\delta'}(x_0))$ the stabilizers of the base points of X_δ and of X'_δ and consider the projections

$$G/H \longrightarrow G/H_\delta \longrightarrow G/H_{\delta'}.$$

Denote $\Delta(\delta) \subset \Delta(\delta')$ the sets of colors of G/H which map dominantly on X_Δ and on $X_{\delta'}$ respectively. Consider the subgroups $H^*_\delta \subset H_\delta$ and $H^*_{\delta'} \subset H_{\delta'}$ associated as in Theorem 2.4.5 to the distinguished subsets $\Delta(\delta)$ and $\Delta(\delta')$: previous remark together with Corollary 2.5.4 shows then that H_δ is the spherical closure of H^*_δ and $H_{\delta'}$ is the spherical closure of $H^*_{\delta'}$. Therefore we may identify the respective sets of colors and combining with Theorem 2.4.5 we get the following identifications:

$$\Delta(X_\delta) = \Delta(G/H_\delta) = \Delta(G/H^*_\delta) = \Delta(G/H) \smallsetminus \Delta(\delta),$$
$$\Delta(X_{\delta'}) = \Delta(G/H_{\delta'}) = \Delta(G/H^*_{\delta'}) = \Delta(G/H) \smallsetminus \Delta(\delta').$$

Under the above identifications, Theorem 1.4.9 shows that

$$\Delta_Y(X_\delta) \cap \Delta(X_{\delta'}) \subset \Delta_{Y'}(X_{\delta'}).$$

Following, Section 2.2, we may write

$$f_\delta = \prod_{D \in \Delta(G/H_\delta)} f_D^{n(\delta,D)} \qquad \text{and} \qquad f_{\delta'} = \prod_{D \in \Delta(G/H_{\delta'})} f_D^{n(\delta,D)}.$$

If $Y \subset X_\delta$ and $Y' \subset X_{\delta'}$ are the closed orbits, Theorem 1.2.2 shows then the identifications

$$\mathrm{Supp}_\Delta(\delta) = \Delta(X_\delta) \smallsetminus \Delta_Y(X_\delta) \qquad \text{and} \qquad \mathrm{Supp}_\Delta(\delta') = \Delta(X_{\delta'}) \smallsetminus \Delta_{Y'}(X_{\delta'}).$$

Combining all previous identifications in Δ it follows

$$\mathrm{Supp}_\Delta(\delta') = \Delta(X_{\delta'}) \smallsetminus \Delta_{Y'}(X_{\delta'}) \subset \Delta(X_\delta) \smallsetminus \Delta_Y(X_\delta) = \mathrm{Supp}_\Delta(\delta). \qquad \square$$

As will be shown by Corollary 3.4.4, previous proposition is false if M is not strict.

From now on we will assume that M is spherically closed and that $\delta \in \mathbb{N}\Delta$ is a faithful divisor. Set $p : \widetilde{X}_\delta \to X_\delta$ the normalization, which is bijective on the G-orbits by Proposition 1.5.1, and consider the commutative diagram

$$\begin{tikzcd} M \arrow[r, "\tilde{\phi}_\delta"] \arrow[dr, "\phi_\delta"'] & \widetilde{X}_\delta \arrow[d, "p"] \\ & X_\delta \subset \mathbb{P}(V_\delta) \end{tikzcd}$$

Consider the ring

$$\widetilde{A}(\delta) = \bigoplus_{n \in \mathbb{N}} \Gamma(M, \mathcal{O}(n\delta))$$

and consider its subring $A(\delta) \subset \widetilde{A}(\delta)$ generated by V^*_δ: then $A(\delta)$ is the projective coordinate ring of X_δ. Since ϕ_δ is birational and since it factors through $\mathrm{Proj}\, \widetilde{A}(\delta)$, it follows that $\mathrm{Proj}\, \widetilde{A}(\delta)$ and X_δ are birational. Moreover, since M is smooth, $\mathrm{Proj}\, \widetilde{A}(\delta)$ is a normal variety, while following proposition shows that $\widetilde{A}(\delta)$ is integral over $A(\delta)$: therefore $\widetilde{A}(\delta)$ is the projective coordinate ring of $\widetilde{X}_\delta$.

Proposition 3.1.4 ([CCM 06], Prop. 2.1)**.** *Let $\delta \in \mathbb{N}\Delta$. Then $\widetilde{A}(\delta)$ is integral over $A(\delta)$.*

Proof. Let $\mathcal{S}(\mathcal{O}(\delta))$ be the symmetric algebra sheaf constructed over $\mathcal{O}(\delta)$ and let $L = \operatorname{Spec} \mathcal{S}(\mathcal{O}(\delta))$ be the total space of $\mathcal{O}(-\delta)$. Denote by $\overline{L}$ the total space of the tautological bundle on $\mathbb{P}(V_\delta)$: by construction, we have a pullback diagram as follows

$$\begin{tikzcd} L \arrow[r, "\bar{i}"] \arrow[d] & \overline{L} \arrow[d] \\ X_\delta \arrow[r, "i"'] & \mathbb{P}(V_\delta) \end{tikzcd}$$

where i is the natural inclusion. By definition, $\widetilde{A}(\delta) = \Gamma(L, \mathcal{O}_L) = \Gamma(\overline{L}, \bar{i}_*\mathcal{O}_L)$ and the image of the natural morphism $\Gamma(\overline{L}, \mathcal{O}_{\overline{L}}) \to \Gamma(\overline{L}, \bar{i}_*\mathcal{O}_L)$ is the subring $A(\delta)$. Since $\bar{i}$ is projective, $\bar{i}_*\mathcal{O}_L$ is a coherent sheaf on $\overline{L}$: therefore $\widetilde{A}(\delta)$ is a finite $\Gamma(\overline{L}, \mathcal{O}_{\overline{L}})$-module, or equivalently a finite $A(\delta)$-module. □

If $Z \subset X_\delta$ is an orbit, set $Z' = p^{-1}(Z) \subset \widetilde{X}_\delta$ the corresponding orbit. Denote $Z_B \subset Z$ and $Z'_B \subset Z'$ the B-open orbits and fix base points $z_0 \in Z_B$ and $z'_0 \in Z'_B$ so that we have isomorphisms

$$Z' \simeq G/K', \qquad\qquad Z \simeq G/K$$

with $K' \subset K$ a subgroup of finite index. Denote $Y \subset X_\delta$ the closed orbit. Since parabolic subgroups are self-normalizing, Y and $p^{-1}(Y)$ are isomorphic; from now on we will denote both of them with the same letter Y.

Write $\delta = \sum_\Delta n(D,\delta)D$. If $\eta \in (V_\delta^*)^{(B)}$ and if $v_0 \in V_\delta^{(H)}$ is such that $[v_0] = \phi_\delta(x_0)$, set $f_\delta(g) = \langle \eta, g v_0 \rangle \in \Bbbk[G]^{(B\times H)}$; up to a scalar factor, it holds the equality

$$f_\delta = \prod_{D\in\Delta} f_D^{n(\delta,D)}.$$

Then by Proposition 1.5.2 and Theorem 1.2.2 we get

$$\Delta_Y(X_\delta) = \Delta_Y(\widetilde{X}_\delta) = \Delta \smallsetminus \mathrm{Supp}_\Delta(\delta).$$

Since $\widetilde{X}_\delta$ is complete, by Theorem 1.4.9 the cone $\mathcal{C}(\widetilde{X}_\delta)$ contains the G-invariant valuation cone $\mathcal{V}_{G/H}$: therefore $\mathcal{C}(X_\delta) = \mathcal{C}(\widetilde{X}_\delta)$ is the cone generated by $\mathcal{V}_{G/H}$ together with $\rho(\Delta \smallsetminus \mathrm{Supp}_\Delta(\delta))$.

3.2 Orbits in X_δ and in $\widetilde{X}_\delta$

If $W \subset M$ is an orbit, in the following $\delta_W \in \mathrm{Pic}(\overline{W})$ will denote the pullback of $\delta \in \mathrm{Pic}(M)$. Notice that if $\alpha \in S$ then

$$\begin{aligned} \mathrm{Supp}_\Delta(\delta) \cap \Delta(\alpha) \neq \varnothing \quad &\Longleftrightarrow \alpha \notin S^p_Y \\ &\Longleftrightarrow \mathrm{Supp}_{\Delta(W)}(\delta_W) \cap \Delta(W)(\alpha) \neq \varnothing \quad \forall\, W \subset M \end{aligned}$$

where S^p_Y denotes the set of simple roots associated to the closed orbit $Y \subset X_\delta$.

Proposition 3.2.1. *Let $G/K \simeq Z \subset X_\delta$ be an orbit and let $G/K' \simeq Z' = p^{-1}(Z)$; let $G/K_W \simeq W \subset M$ be any orbit which maps on Z and choose the stabilizers so that $K_W \subset K' \subset K$. Then K' is the maximal subgroup such that*

$$K_W \subset K' \subset K \qquad \text{and} \qquad K'/K_W \text{ is connected.}$$

In particular, $Z \simeq Z'$ if and only if K/K_W is connected.

Proof. Set $K^* = K_W K^\circ$ the maximal subgroup of K containing K_W such that K^*/K_W is connected. Since $K_W \subset K'$ and since $K^\circ = (K')^\circ$, by Theorem 1.3.5 iii) we get that $K^* \subset K'$ is a normal subgroup; thus by Lemma 1.5.8 it follows that $K^* = K'$ if and only if $\Lambda_{G/K^*} = \Lambda_{Z'}$.

Consider the inclusions $\Lambda_Z \subset \Lambda_{Z'} \subset \Lambda_W \subset \Lambda_{G/H}$: since Λ_W is saturated in $\Lambda_{G/H}$, Proposition 1.5.9 shows that $\Lambda_{Z'}$ is the saturation of Λ_Z in Λ_W. On the other hand, by Theorem 1.6.3 it follows that Λ_{G/K^*} is saturated in Λ_W: since $[\Lambda_{G/K^*} : \Lambda_Z] = [K : K^*] < \infty$, we get the equality $\Lambda_{G/K^*} = \Lambda_{Z'}$. □

Combining previous proposition together with Theorem 2.4.4 and Corollary 2.5.4 we get the following corollary.

Corollary 3.2.2. *Let $G/K \simeq Z \subset X_\delta$ be an orbit and let $p^{-1}(Z) \simeq G/K'$ with $K' \subset K$. Then K' is a wonderful subgroup and the the associated wonderful variety is the quotient wonderful variety $\overline{W}/\Delta(\delta_W)$, where $\Delta(\delta_W) \subset \Delta(W)$ is the maximal distinguished subset not intersecting* $\mathrm{Supp}_{\Delta(W)}(\delta_W)$. *If moreover M is strict, then K is the spherical closure of K'.*

If $Z \subset X_\delta$ is an orbit and if $Z' \subset \widetilde{X}_\delta$ is the corresponding orbit, denote $\Sigma_Z, \Sigma_{Z'} \subset \mathbb{N}\Sigma$ the sets of spherical roots of the respective wonderful completions. By Corollary 2.5.4 there exists a bijection between Σ_Z and $\Sigma_{Z'}$, which associates to $\gamma \in \Sigma_Z$ the unique $\gamma' \in \Sigma_{Z'}$ which is proportional to γ: more precisely, if $\gamma \neq \gamma'$, then $\gamma = 2\gamma'$.

Consider a spherical root $\sigma \in \Sigma(G)$ such that $2\sigma \in \Sigma(G)$: following Table 2.1, such a root either is a simple root, or it is of type $\mathsf{B}^{\mathrm{I}}_r$ or it is of type $\mathsf{G}^{\mathrm{I}}_2$. If $Z \subset X_\delta$ is any orbit and if $Z' \subset \widetilde{X}_\delta$ is the corresponding orbit, define $\Sigma(\delta_{Z'}) \subset \Sigma_{Z'}$ to be the subset of spherical roots which have to be doubled to get the spherical roots of Z.

Lemma 3.2.3. *An orbit $Z \subset X_\delta$ is not isomorphic to its corresponding orbit $Z' \subset \widetilde{X}_\delta$ if and only if Z possesses a spherical root γ of the shape $\gamma = 2\sigma_1 + \ldots + 2\sigma_k$, where $\sigma_1, \ldots, \sigma_k \in \Sigma$ are pairwise distinct elements (and where $\gamma' = \sigma_1 + \ldots + \sigma_k \in \Sigma_{Z'}$).*

Proof. By Corollary 2.5.4, Z and Z' are not isomorphic if and only if $\Sigma(\delta_{Z'}) \neq \varnothing$; suppose $\gamma' \in \Sigma(\delta_{Z'})$. By Proposition 3.2.1 the wonderful completion of Z' is the quotient of a wonderful subvariety $M' \subset M$. If Σ' is the set of spherical roots of M', we can write $\gamma' = a_1\sigma_1 + \ldots + a_k\sigma_k$ with $\sigma_1, \ldots, \sigma_k \in \Sigma'$.

Since $2\gamma' \in \Sigma(G)$, by the discussion preceeding the lemma γ' is either a simple root, or it is of type $\mathsf{B}^{\mathrm{I}}_r$ or it is of type $\mathsf{G}^{\mathrm{I}}_2$. If γ' is a simple root or if it is of type $\mathsf{B}^{\mathrm{I}}_r$ then it follows immediately that every a_i is equal to one. Suppose instead that γ' is of type $\mathsf{G}^{\mathrm{I}}_2$; in order to show the thesis it is enough to consider the case where M' is a wonderful variety whose spherical roots are all supported on a subset $S' = \{\alpha_1, \alpha_2\} \subset S$ of type G_2. An easy computation shows that, if $\Sigma' = S'$ and if Δ^* is any distinguished subset of colors of M', then the quotient M'/Δ^* never possesses $2\alpha_1 + \alpha_2$ as a spherical root. Therefore, if $\gamma' = 2\alpha_1 + \alpha_2$, it must be either $\Sigma' = \{2\alpha_1 + \alpha_2\}$ or $\Sigma' = \{\alpha_1, \alpha_1 + \alpha_2\}$ and the claim follows. □

As exemplified in the following sections (Example 3.3.5 and Example 3.4.2), Proposition 3.2.1 together with Corollary 2.5.4 allow to compute explicitly the set of orbits of X_δ and that of $\widetilde{X}_\delta$ in terms of their spherical systems. This is further simplified by the following proposition, which shows that, given an orbit $Z \subset X_\delta$, there exists a minimal orbit $W_Z \subset M$ mapping on Z. If $\gamma = \sum_{\sigma \in \Sigma} n_\sigma \sigma \in \Sigma_Z$, define

$$\mathrm{Supp}_\Sigma(\gamma) = \{\sigma \in \Sigma \,:\, n_\sigma \neq 0\}$$

its support over Σ; define

$$\Sigma(Z) = \bigcup_{\gamma \in \Sigma_Z} \mathrm{Supp}_\Sigma(\gamma).$$

Proposition 3.2.4. *Let $Z \subset X_\delta$ be an orbit and let $W_Z \subset M$ the orbit whose closure has $\Sigma(Z)$ as set of spherical roots. Then W_Z maps on Z and and every other orbit which maps on Z contains W_Z in its closure.*

Proof. Let $W \subset M$ be an orbit mapping on Z and let $\Sigma_W \subset \Sigma$ be the associated set of spherical roots. Since $\phi_\delta(W) = Z$, we get $\Sigma_Z \subset \mathbb{N}\Sigma_W$: this shows $\Sigma(Z) \subset \Sigma_W$, i.e. $W_Z \subset \overline{W}$. In order to prove the equality $\phi_\delta(W_Z) = Z$, notice that

$$\Lambda_{\phi_\delta(W_Z)} = \Lambda_{W_Z} \cap \Lambda_Z = \Lambda_Z :$$

since $\phi_\delta(W_Z) \subset \overline{Z}$ by the first part of the proof, this implies the claim. □

Unlike the symmetric case (see [Maf 09]), in the general spherical case there does not need to exist a maximal orbit in M mapping on a fixed orbit $Z \subset X_\delta$: for instance this is shown by Example 3.3.5 and by Example 3.4.2.

Since $\Sigma(Z)$ depends only on Σ_Z (or equivalently on $\Sigma_{Z'}$), we get the following corollaries.

Corollary 3.2.5. *Two orbits $W_1, W_2 \subset M$ map to the same orbit in X_δ if and only if*

$$\Sigma_{W_1} \big/ \Delta(\delta_{W_1}) = \Sigma_{W_2} \big/ \Delta(\delta_{W_2}),$$

where δ_{W_i} is the pullback of δ to $\overline{W_i}$ and where $\Delta(\delta_{W_i})$ is the maximal distinguished subset of colors of W_i not intersecting the support of δ_{W_i}.

Corollary 3.2.6. *Two orbits in X_δ (resp. in $\widetilde{X}_\delta$) have different sets of spherical roots; in particular two orbits in X_δ (resp. in $\widetilde{X}_\delta$) are never isomorphic.*

If $S \cap \Sigma = \varnothing$, Corollary 2.2.4 shows that the restriction map to the closed orbit $\omega : \mathrm{Pic}(M) \to \mathcal{X}(B)$ is injective: this means that the generic stabilizer H never fixes two different lines in the same simple module. However, if $S \cap \Sigma \neq \varnothing$, it could happen that a simple module $\mathbb{P}(V)$ contains two different orbits both isomorphic to the open orbit G/H: previous corollary shows then that there does not exist any spherical orbit in $\mathbb{P}(V)$ containing both of them in its closure. For instance, this occurs in the following example.

Example 3.2.7. Consider $M = \mathbb{P}^1 \times \mathbb{P}^1$, which is a wonderful variety for $G = \mathrm{SL}(2)$, and fix the base point $([1,0],[0,1])$ so that the generic stabilizer is the maximal torus T of diagonal matrices. Consider the simple module $V = \Bbbk[x,y]_5$ formed by the homogeneous polynomials of degree 5: then $G[x^4y]$ and $G[x^3y^2]$ are distinct orbits in $\mathbb{P}(V)$ both isomorphic to the open orbit G/T.

3.3 Bijectivity in the strict case

Suppose that M is strict. The following is a stronger version of Lemma 3.2.3.

Lemma 3.3.1. *Let M be a strict wonderful variety and let δ be a faithful divisor on it. Let $Z \subset X_\delta$ be an orbit, then $Z \not\simeq Z'$ if and only if there exists a spherical root $\gamma \in \Sigma_Z$ of type $\mathsf{B}_r^{\mathrm{II}}$ and a spherical root $\sigma \in \mathrm{Supp}_\Sigma(\gamma)$ of type $\mathsf{B}_2^{\mathrm{I}}$.*

Proof. By Lemma 3.2.3, we may assume that Z' possesses a spherical root γ of type $\mathsf{B}_r^{\mathrm{I}}$ or of type $\mathsf{G}_2^{\mathrm{I}}$. Since $S \cap \Sigma = \varnothing$, it is uniquely determined a spherical root $\sigma \in \mathrm{Supp}_\Sigma(\gamma)$ which is of type $\mathsf{B}_s^{\mathrm{I}}$ (with $2 \leqslant s \leqslant r$) in the first case and of type $\mathsf{G}_2^{\mathrm{I}}$ in the second case. Since M is strict, by Remark 2.3.10 the latter cannot happen; thus we are in the first case.

Suppose that $s > 2$ and $2\gamma \in \Sigma_Z$; let $\beta \in S$ be the short root in the support of σ. Since M is strict, Remark 2.3.10 shows that β moves a color $D_\beta \in \Delta$, while $s > 2$ implies $c(D_\beta, \tau) \geqslant 0$ for every $\tau \in \Sigma$: therefore $\{D_\beta\}$ is distinguished and by the faithfulness of δ we get $D_\beta \in \mathrm{Supp}_\Delta(\delta)$, which implies $\beta \notin S_Y^p$. But this is a contradiction since $2\gamma \in \Sigma_Z$ implies $\beta \in S_Z^p \subset S_Y^p$. $\square$

If $\sigma \in \Sigma$ is a spherical root of type $\mathsf{B}_2^{\mathrm{I}}$, write $\sigma = \alpha_\sigma^\sharp + \alpha_\sigma^\flat$, where $\alpha_\sigma^\sharp, \alpha_\sigma^\flat \in S$ are respectively the long simple root and the short simple root in the support of σ. Since M is strict, Remark 2.3.10 shows that both $\alpha_\sigma^\sharp$ and $\alpha_\sigma^\flat$ move exactly one color; set $\Delta(\alpha_\sigma^\sharp) = \{D^\sharp(\sigma)\}$ and $\Delta(\alpha_\sigma^\flat) = \{D^\flat(\sigma)\}$.

Lemma 3.3.2. *Let M be a strict wonderful variety and let δ be a faithful divisor on it; let $\sigma \in \Sigma$ be a spherical root of type $\mathsf{B}_2^{\mathrm{I}}$.*

i) If $D^\flat(\sigma) \in \mathrm{Supp}_\Delta(\delta)$, then no orbit $Z \subset X_\delta$ possesses a spherical root $\gamma \in \Sigma_Z$ of type $\mathsf{B}_r^{\mathrm{II}}$ with $\sigma \in \mathrm{Supp}_\Sigma(\gamma)$.

ii) If $\mathrm{Supp}_\Delta(\delta) \cap \{D^\sharp(\sigma), D^\flat(\sigma)\} = \{D^\sharp(\sigma)\}$, then there exists an orbit $Z \subset X_\delta$ such that $2\sigma \in \Sigma_Z$; in particular $Z \not\simeq Z'$ and the normalization $p : \widetilde{X}_\delta \to X_\delta$ is not bijective.

Proof. i). If $Z \subset X_\delta$ possesses a spherical root γ of type $\mathsf{B}_r^{\mathrm{II}}$ supported on σ, then $\alpha_\sigma^\flat \in S_Z^p \subset S_Y^p$. But this is a contradiction since following the remark at the beginning of Section 3.2 $D^\flat(\sigma) \in \mathrm{Supp}_\Delta(\delta)$ implies $\alpha_\sigma^\flat \notin S_Y^p$.

ii). Consider the rank one orbit $W \subset M$ whose unique spherical root is σ. If $\Delta(W)(\alpha_\sigma^\flat) = \{'D^\flat(\sigma)\}$ and $\Delta(W)(\alpha_\sigma^\sharp) = \{'D^\sharp(\sigma)\}$, by Remark 2.2.5 we get

$$\mathrm{Supp}_{\Delta(W)}(\delta_W) \cap \{'D^\sharp(\sigma), 'D^\flat(\sigma)\} = \{'D^\sharp(\sigma)\}.$$

Set $Z = \phi_\delta(W)$ and $Z' = p^{-1}(Z)$; set $\Delta(\delta_W) \subset \Delta(W)$ the maximal distinguished subset not intersecting the support of δ_W. Since $c('D^\flat(\sigma), \sigma) = 0$ and since $'D^\sharp(\sigma)$ is the unique color $D \in \Delta(W)$ such that $c(D, \sigma) > 0$, we get

$$'D^\flat(\sigma) \in \Delta(\delta_W) = \{D \in \Delta(W) \,:\, c(D,\sigma) = 0\} \smallsetminus \mathrm{Supp}_{\Delta(W)}(\delta_W).$$

By Corollary 3.2.2, this shows $\Sigma_{Z'} = \{\sigma\}$. On the other hand $\Delta(Z')(\alpha^\flat_\sigma) = \varnothing$, thus $\sigma \in \Sigma_Z$ is a loose spherical root and by Remark 2.3.10 it follows that Z' is not spherically closed, which implies the claim by Proposition 2.3.3. ☐

Corollary 3.3.3. *i) If M is a wonderful adjoint symmetric variety and if δ is a faithful divisor on it, then the normalization $p : \widetilde{X}_\delta \to X_\delta$ is bijective.*

ii) Suppose that the Dynkin diagram of G is simply laced. If M is any strict wonderful variety for G and if $\delta \in \mathrm{Pic}(M)$ is any faithful divisor, then the normalization $p : \widetilde{X}_\delta \to X_\delta$ is bijective.

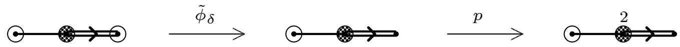
iii) If $D^\flat(\sigma) \in \mathrm{Supp}_\Delta(\delta)$ for every $\sigma \in \Sigma$ of type $\mathsf{B}^{\mathrm{I}}_2$, then the normalization morphism $p : \widetilde{X}_\delta \to X_\delta$ is bijective.

Proof. By the classification of symmetric varieties (see for instance [Ti 06, Table 5.2]), we deduce that a wonderful adjoint symmetric variety never possesses a spherical root of type $\mathsf{B}^{\mathrm{I}}_2$. Since such varieties are strict (see Example 2.1.5), all of the claims above follow by previous lemmas. ☐

Another proof of Corollary 3.3.3 i) was given in [Maf 09]. Following examples show some cases wherein the conditions of Lemma 3.3.1 are fulfilled:

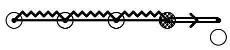
Example 3.3.4. Consider the wonderful model variety M of Spin(7), whose spherical system is expressed by the Luna diagram

Then the divisor $\delta = D_{\alpha_2}$ is faithful. Consider the codimension one orbit $W \subset M$ having spherical root $\alpha_2 + \alpha_3$; following Proposition 3.2.1 and Corollary 2.5.4, we get the following sequence of Luna diagrams



where the first one represents the orbit $W \subset M$, the second one represents the orbit $\tilde{\phi}_\delta(W) \subset \widetilde{X}_\delta$ and the third one represents the orbit $\phi_\delta(W) \subset X_\delta$.

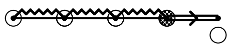
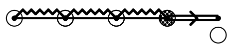
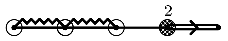
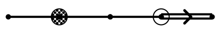
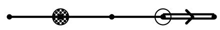
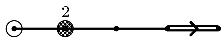
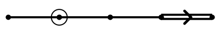
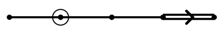
Example 3.3.5. Consider the wonderful model variety M of SO(11), whose spherical system is expressed by the Luna diagram



Then the divisor $\delta = D_{\alpha_2}$ is faithful. See Table 1 for a full list of the orbits in X_δ and in $\widetilde{X}_\delta$ (for simplicity, in the table orbits in M are described by giving a subset of its spherical root index set).

As illustrated above, examples of strict wonderful varieties possessing a faithful divisor δ such that the normalization $p : \widetilde{X}_\delta \to X_\delta$ is not bijective arise from the context of wonderful model varieties (see Example 2.3.11). As will be shown in the following, the case of a general strict wonderful variety substantially follows from this special case.

Table 3.1. Example 3.3.5 , $\delta = D_{\alpha_2}$.

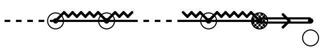
Maximal Orbits	**Minimal Orbit**	**Orbit in** $\widetilde{X}_\delta$	**Orbit in** X_δ	$\Sigma(\delta_{Z'})$
$\{1,2,3,4,5\}$	$\{1,2,3,4,5\}$			$\varnothing$
$\{1,2,3,4\}$	$\{1,2,3,4\}$			$\varnothing$
$\{1,2,4,5\}$	$\{1,2,4\}$			$\{\alpha_4+\alpha_5\}$
$\{1,2,3,5\}$	$\{1,2\}$			$\varnothing$
$\{2,3,4,5\}$	$\{2,4\}$			$\left\{\sum_{i=2}^5 \alpha_i\right\}$
$\{1,3,4,5\}$ $\{2,3,5\}$	$\varnothing$			$\varnothing$

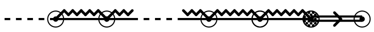
Consider a strict wonderful variety M and let δ be a faithful divisor on it. Let $\sigma \in \Sigma$ be a spherical root of type $\mathsf{B}_2^{\mathrm{I}}$ and set $\Gamma(\sigma)$ the connected component of the Dynkin diagram of G where σ is supported. If $\Gamma(\sigma)$ is of type B or C, number the simple roots in $\Gamma(\sigma)$ which are not in S^p starting from the extreme of the diagram which contains the double link.

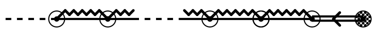
If $\{D_\sigma^\flat, D_\sigma^\sharp\}$ contains a distinguished subset, then by Lemma 3.3.2 we get that there is no orbit $Z \subset X_\delta$ possessing a spherical root γ of type $\mathsf{B}_r^{\mathrm{II}}$ with $\sigma \in \mathrm{Supp}_\Sigma(\gamma)$ if and only if $D_\sigma^\flat \in \mathrm{Supp}_\Delta(\delta)$. For instance, this is the case if one of the following conditions is fulfilled:

- $\Gamma(\sigma)$ is of type B or C and σ is the unique spherical root supported on α_2;
- $\Gamma(\sigma)$ is of type C and $2\alpha_2 \in \Sigma$.

Suppose that $\{D_\sigma^\flat, D_\sigma^\sharp\}$ does not contain any distinguished subset. If $\Gamma(\sigma) \neq \mathsf{F}_4$, then there exists $\tau \in \Sigma$ supported on α_2 different both from σ and from $2\alpha_2$; by a case-by-case check, it turns out that either τ has support of type A_2 or $\Gamma(\sigma)$ is of type C and τ has support of type $\mathsf{A}_1 \times \mathsf{A}_1$. Thus the Luna diagram of M in $\Gamma(\sigma)$ has one of the following shapes:

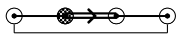
(B1) 

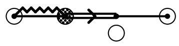
(B2) 

(C1) 

(C2) 

(F1) 

(F2) 

(F3) 

Suppose that we are not in case $\mathsf{C2}$ and that $\Gamma(\sigma) \neq \mathsf{F}_4$: then we are substantially reduced to the case of a wonderful model variety. Let $m(\sigma) \geqslant 3$ be the first integer such that the simple root $\alpha_{m(\sigma)}$ occurs in the support of one and only one spherical root with support of type A_2. For $1 \leqslant k \leqslant m(\sigma)$, set $\Delta(\alpha_k) = \{D_k\}$. Set $\Delta(\sigma) = \{D_1, \ldots, D_{m(\sigma)}\}$ and define $\Delta(\sigma)^{\mathrm{even}}, \Delta(\sigma)^{\mathrm{odd}} \subset \Delta(\sigma)$ as the subsets whose element index is respectively even and odd.

Lemma 3.3.6. *Let M be a strict wonderful variety possessing a spherical root σ of type $\mathsf{B}_2^{\mathrm{I}}$ such that the Luna diagram of M in $\Gamma(\sigma)$ is of type $\mathsf{B1}$ and let δ be a faithful divisor on M. Then there does not exist any orbit $Z \subset X_\delta$ possessing a spherical root γ of type $\mathsf{B}_r^{\mathrm{II}}$ with $\sigma \in \mathrm{Supp}_\Sigma(\gamma)$ if and only if $D_1 \in \mathrm{Supp}_\Delta(\delta)$ or the following conditions are both fulfilled:*

i) $\mathrm{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\mathrm{even}} = \varnothing$*;*

ii) If M possesses a spherical root supported on $\alpha_{m(\sigma)+1}$, then $m(\sigma)$ is odd.

Proof. By Lemma 3.3.2 we may assume that $\mathrm{Supp}_\Delta(\delta) \cap \{D_1, D_2\} = \varnothing$. Notice that $\Delta(\sigma) \smallsetminus \{D_{m(\sigma)}\}$ is distinguished and that conversely any distinguished subset which intersects $\Delta(\sigma)$ contains $\Delta(\sigma) \smallsetminus \{D_{m(\sigma)}\}$. Number the $m(\sigma)$ spherical roots supported on $\{\alpha_1, \ldots, \alpha_{m(\sigma)}\}$ from the right to the left: set $\sigma_1 = 2\alpha_1$ and, if $2 \leqslant i \leqslant m(\sigma)$, set $\sigma_i = \alpha_{i-1} + \alpha_i$.

If $W \subset M$ is an orbit, denote $\Sigma' \subset \Sigma$ its set of spherical roots and Δ' its set of colors; for $1 \leqslant i \leqslant m(\sigma)$ set $\Delta'(\alpha_i) = \{D'_i\}$ and set $\Delta'(\sigma) = \{D'_1, \ldots, D'_{m(\sigma)}\}$. Denote $q : \mathrm{Pic}(M) \to \mathrm{Pic}(\overline{W})$ the pullback map and notice that q induces a bijection between $\Delta(\sigma)$ and $\Delta'(\sigma)$: indeed following Remark 2.2.5 we get $q(D_i) = D'_i$ for every $1 < i \leqslant m(\sigma)$, while

$$q(D_1) = \begin{cases} D'_1 & \text{if } 2\alpha_1 \in \Sigma' \\ 2D'_1 & \text{if } 2\alpha_1 \notin \Sigma' \end{cases} :$$

therefore, if $i \leqslant m(\sigma)$, δ is supported on D_i if and only if $\delta_W = q(\delta)$ is supported on D'_i.

($\Longrightarrow$) Consider the codimension one orbit W whose set of spherical roots is $\Sigma' = \Sigma \smallsetminus \{\sigma_3\}$; set $Z = \phi_\delta(W)$ and $Z' = p^{-1}(Z)$. Denote $\Delta^* \subset \Delta'$ the maximal distinguished subset of colors which does not intersect the support of δ_W; since $D'_1 \notin \mathrm{Supp}_{\Delta'}(\delta_W)$ and since it is non-negative against any spherical root, we get $D'_1 \in \Delta^*$.

Suppose that i) or ii) fails. Notice that, in order to show that $Z \not\simeq Z'$, it is enough to show that $D'_2 \notin \Delta^*$. Indeed, on one hand by Proposition 3.2.1 together with Lemma 2.5.1 this implies $\sigma \in \Lambda_{Z'}$: in fact $c(D',\sigma) = 0$ for every $D' \in \Delta' \smallsetminus \{D'_2, D'_3\}$ and $D'_2 \notin \Delta^*$ implies $D'_3 \notin \Delta^*$. On the other hand, since $D'_1 \in \Delta^*$, we get $\Delta(Z')(\alpha_1) = \Delta(Z)(\alpha_1) = \varnothing$: since Z is spherically closed, by Remark 2.3.10 this implies that $\sigma \notin \Lambda_Z$. Therefore, if $D'_2 \notin \Delta^*$, then we get $\sigma \in \Lambda_{Z'} \smallsetminus \Lambda_Z$ and $2\sigma \in \Sigma_Z$ and $Z \not\simeq Z'$.

Suppose first that i) fails and that $D'_2 \in \Delta^*$. Then it must be either $\Delta'(\sigma)^{\mathrm{even}} \subset \Delta^*$ or $\Delta'(\sigma) \smallsetminus \{D'_{m(\sigma)}\} \subset \Delta^*$: this follows by considering the conditions defining a distinguished subset only for $\sigma_1, \sigma_2, \sigma_4, \ldots, \sigma_{m(\sigma)}$ and noticing that the minimal subsets with this property which contain D'_2 are $\{D'_1\} \cup \Delta'(\sigma)^{\mathrm{even}}$ and, in case $m(\sigma)$ is even, $\Delta'(\sigma) \smallsetminus \{D'_{m(\sigma)}\}$. Since we are supposing that i) fails, the first case is not possible, while the second case is not possible because δ is faithful: therefore $D'_2 \in \Delta^*$ and $Z \not\simeq Z'$.

Suppose now that ii) fails and that $D'_2 \in \Delta^*$: thus $m(\sigma)$ is even and there exists a spherical root σ' supported on $\alpha_{m(\sigma)+1}$. Set $m_1 := m(\sigma)$ and notice that σ' has necessarily support of type A. Set $m_2 > m_1 + 1$ the first integer such that α_{m_2} occurs in the support of exactly one spherical root with support of type A and, proceeding similarly, define a sequence

$$m_1 < m_2 < \ldots < m_k$$

until no spherical root is supported on α_{m_k+1}. If $1 \leqslant j \leqslant m_k$, set $\Delta(\alpha_j) = \{D_j\}$ and $\Delta'(\alpha_j) = \{D'_j\}$; if $1 \leqslant i \leqslant k$, set

$$\Delta_i = \bigcup_{t=m_{i-1}+1}^{m_i} \Delta(\alpha_t), \qquad \Delta'_i = \bigcup_{t=m_{i-1}+1}^{m_i} \Delta'(\alpha_t)$$

(where $m_0 := 0$). Set moreover $\Delta_i^{\mathrm{even}} \subset \Delta_i$ and $(\Delta'_i)^{\mathrm{even}} \subset \Delta'_i$ the subsets whose element index t is even. Define $k_0 \in \{1, \ldots, k\}$ the first integer such that m_{k_0} is odd or define $k_0 = k$ otherwise. Then it is easy to show that $D'_2 \in \Delta^*$ if and only if $\Delta^* \cap \Delta'_i = (\Delta'_i)^{\mathrm{even}}$ for every $i \leqslant k_0$, which is impossible by following remark. Indeed notice that $\Delta_{k_0}^{\mathrm{even}} \subset \Delta$ is distinguished, therefore since δ is faithful, it must be $\mathrm{Supp}_\Delta(\delta) \cap \Delta_{k_0}^{\mathrm{even}} \neq \varnothing$, which implies $\mathrm{Supp}_{\Delta(W)}(\delta_W) \cap (\Delta')_{k_0}^{\mathrm{even}} \neq \varnothing$. Therefore if ii) fails it must be $D'_2 \notin \Delta^*$ and we get $Z \not\simeq Z'$.

($\Longleftarrow$) Set $M' \subset M$ the G-stable prime divisor associated to the spherical root σ_1 and set $W \subset M'$ the open orbit. If $Z \subset X_\delta$ is an orbit possessing a spherical root γ of type $\mathsf{B}_r^{\mathrm{II}}$ with $\sigma \in \mathrm{Supp}_\Sigma(\gamma)$, then

$$\sigma_1 \notin \Sigma(Z) = \bigcup_{\gamma \in \Sigma_Z} \mathrm{Supp}_\Sigma(\gamma):$$

in fact no spherical root supported on α_1 is compatible with γ. Therefore by Proposition 3.2.4 such an orbit is necessarily contained in $\phi_\delta(M')$ and, in order to prove the claim, it is enough to show that it is true for any orbit which is contained in $\phi_\delta(M')$. Set $\Delta^* \subset \Delta'$ the maximal distinguished subset which does not intersect $\mathrm{Supp}_{\Delta(W)}(\delta_W)$.

Suppose that both i) and ii) hold. Then $\Delta'(\sigma)^{\mathrm{even}}$ is distinguished and by i) it follows that $\Delta'(\sigma)^{\mathrm{even}} \subset \Delta^*$. Notice that $\Delta^* \cap \Delta'(\sigma)^{\mathrm{odd}} = \varnothing$: indeed otherwise it should be $\Delta'(\sigma) \setminus \{D'_1, D'_{m(\sigma)}\} \subset \Delta^*$, which contradicts the faithfulness of δ since $\Delta(\sigma) \setminus \{D_{m(\sigma)}\} \subset \Delta$ is distinguished and $D_1 \notin \mathrm{Supp}_\Delta(\delta)$ by assumption.

Therefore $\Delta^* \cap \Delta'(\sigma) = \Delta'(\sigma)^{\mathrm{even}}$ and we get $\sigma \notin \Sigma(\phi_\delta(W))$: indeed, since $D'_3 \notin \Delta^*$, it follows that $\alpha_3 \notin S^p_{\phi_\delta(W)}$, therefore a spherical root $\gamma \in \Sigma_{\phi_\delta(W)}$ with support of type B_r is necessarily a multiple of σ, and this cannot happen since $c(D'_2, \sigma) = 1$. To conclude, it is enough to notice that, if $Z \subset \phi_\delta(M')$ is any orbit, then $\Sigma(Z) \subset \Sigma(\phi_\delta(W))$. $\square$

Corollary 3.3.7. *Let M be a strict wonderful variety possessing a spherical root σ of type $\mathsf{B}^{\mathrm{I}}_2$ such that the Luna diagram of M in $\Gamma(\sigma)$ is of type* B2 *and let δ be a faithful divisor on M. Then there does not exist any orbit $Z \subset X_\delta$ possessing a spherical root γ of type $\mathsf{B}^{\mathrm{II}}_r$ with $\sigma \in \mathrm{Supp}_\Sigma(\gamma)$ if and only if $D_1 \in \mathrm{Supp}_\Delta(\delta)$.*

Proof. Let M' be the wonderful variety whose spherical system is the same one of M with one further spherical root $2\alpha_1$: then M is identified with a G-stable prime divisor of M' and the Luna diagram of M' in $\Gamma(\sigma)$ is of the type considered in previous lemma. Denote Σ' and Δ' respectively the set of spherical roots and the set of colors of M'; observe that the pullback map $q : \mathrm{Pic}(M') \to \mathrm{Pic}(M)$ induces an isomorphism between the sublattices generated by $\Delta \setminus \{D_{\alpha_1}\}$ and $\Delta' \setminus \{D'_{\alpha_1}\}$. If $D_1 \in \mathrm{Supp}_\Delta(\delta)$ then the claim follows by Lemma 3.3.2; thus we may assume $D_1 \notin \mathrm{Supp}_\Delta(\delta)$ and we may identify δ with a divisor δ' on M' which is still faithful.

If $Z \subset \phi_{\delta'}(M')$ is an orbit possessing a spherical root γ of type $\mathsf{B}^{\mathrm{II}}_r$ with $\sigma \in \mathrm{Supp}_\Sigma(\gamma)$, then $2\alpha_1 \notin \Sigma'(Z)$ and by Proposition 3.2.4 we get $Z \subset X_\delta = \phi_{\delta'}(M)$: therefore such an orbit exists in X_δ if and only if it exists in $\phi_{\delta'}(M')$ and we can apply previous lemma. In order to get the claim it is enough to notice that if condition ii) of Lemma 3.3.6 holds, then (in the notations of that lemma) $\Delta(\sigma)^{\mathrm{even}} = q(\Delta'(\sigma)^{\mathrm{even}}) \subset \Delta$ is distinguished: thus $\mathrm{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\mathrm{even}} \neq \varnothing$ and consequently i) fails. $\square$

If they are defined, set

$$e_\sigma(\delta) = \min\{k \leqslant m(\sigma) \ : \ D_k \in \mathrm{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\mathrm{even}}\},$$

$$o_\sigma(\delta) = \min\{k \leqslant m(\sigma) \ : \ D_k \in \mathrm{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\mathrm{odd}}\}$$

or set $e_\sigma(\delta) = +\infty$ (resp. $o_\sigma(\delta) = +\infty$) otherwise.

Lemma 3.3.8. *Let M be a strict wonderful variety possessing a spherical root σ of type $\mathsf{B}^{\mathrm{I}}_2$ such that the Luna diagram of M in $\Gamma(\sigma)$ is of type* C1 *and let δ be a faithful divisor on M. Then there does not exist any orbit $Z \subset X_\delta$ possessing 2σ as a spherical root if and only if $o_\sigma(\delta) \geqslant e_\sigma(\delta) - 1$.*

Proof. Notice that if $m(\sigma)$ is even then $\Delta(\sigma)^{\mathrm{odd}}$ is distinguished, while if $m(\sigma)$ is odd then $\Delta(\sigma)^{\mathrm{even}}$ is distinguished: thus at least one between $e_\sigma(\delta)$ and $o_\sigma(\delta)$ is finite. By Lemma 3.3.2, we may assume $\min\{e_\sigma(\delta), o_\sigma(\delta)\} > 2$. Number the $m(\sigma) - 1$ spherical roots supported on $\{\alpha_1, \ldots, \alpha_{m(\sigma)}\}$ from the right to left: if $i < m(\sigma)$, set $\sigma_i = \alpha_i + \alpha_{i+1}$.

If $W \subset M$ is an orbit , denote $\Sigma' \subset \Sigma$ its set of spherical roots and Δ' its set of colors; for $1 \leqslant i \leqslant m(\sigma)$ set $\Delta'(\alpha_i) = \{D'_i\}$ and set $\Delta'(\sigma) = \{D'_1, \ldots, D'_{m(\sigma)}\}$. Denote $q : \mathrm{Pic}(M) \to \mathrm{Pic}(\overline{W})$ the pullback map and observe that q induces a bijection between $\Delta(\sigma)$ and $\Delta'(\sigma)$: by Remark 2.2.5 it follows $q(D_i) = D'_i$ for every $i \leqslant m(\sigma)$, therefore δ is supported on D_i if and only if $\delta_W = q(\delta)$ is supported on D'_i.

($\Longrightarrow$) Suppose that $o_\sigma(\delta) < e_\sigma(\delta) - 1$. In particular this implies $o_\sigma(\delta) < m(\sigma)$: indeed by the remark at the beginning of the proof if $m(\sigma)$ is odd then $e_\sigma(\delta) < m(\sigma)$, while if $m(\sigma)$ is even then $o_\sigma(\delta) < m(\sigma)$.

Consider the orbit $W \subset M$ whose spherical roots are $\sigma_1, \ldots, \sigma_{o_\sigma(\delta)}$, set $Z = \phi_\delta(W)$ and $Z' = p^{-1}(Z)$. Then the maximal distinguished subset of Δ' which does not intersect the support of δ_W is

$$\Delta^* = \Delta' \smallsetminus \Big(\Delta'(\sigma)^{\mathrm{odd}}_{\leqslant o_\sigma(\delta)+2} \cup \mathrm{Supp}_{\Delta'}(\delta_W)\Big),$$

which by hypothesis contains $\Delta'(\sigma)^{\mathrm{even}}_{\leqslant o_\sigma(\delta)+1}$ (where the notations are the obvious ones); thus $\Delta^* \cap \{D'_1, D'_2, D'_3\} = \{D'_2\}$. Since $c(D', \sigma) = 0$ for every $D' \in \Delta' \smallsetminus \{D'_1, D'_3\}$, by Proposition 3.2.1 together with Lemma 2.5.1 we get $\sigma \in \Lambda_{Z'}$. On the other hand, $D'_2 \in \Delta^*$ implies $\Delta(Z)(\alpha_2) = \varnothing$: since Z is spherically closed, we get then $\sigma \notin \Sigma_Z$ and $2\sigma \in \Sigma_Z$.

($\Longleftarrow$) Suppose that $o_\sigma(\delta) \geqslant e_\sigma(\delta) - 1$. Fix an orbit $W \subset M$, set $Z = \phi_\delta(W)$ and $Z' = p^{-1}(Z)$. We may assume that $\sigma \in \Sigma'$, since otherwise there is nothing to prove. Set $\Delta^* \subset \Delta'$ the maximal distinguished subset which does not intersect the support of δ_W and notice that $2\sigma \in \Sigma_Z$ if and only if $\Delta^* \cap \{D'_1, D'_2, D'_3\} = \{D'_2\}$. Such condition does not hold if $\sigma_2 \notin \Sigma'$ or if $\sigma_3 \notin \Sigma'$, since then it would be $D'_1 \in \Delta^*$: thus we may assume that $\Sigma' \supset \{\sigma_1, \sigma_2, \sigma_3\}$.

Set $k < m(\sigma)$ the maximum such that $\sigma_i \in \Sigma'$ for every $i \leqslant k$. By considering the conditions defining a distinguished set only for $\sigma_1, \ldots, \sigma_k$ it follows that, if $D'_2 \in \Delta^*$, then either $\Delta'(\sigma)_{\leqslant k} \subset \Delta^*$ or $\Delta'(\sigma)^{\mathrm{even}}_{\leqslant k+1} \subset \Delta^*$. If we are in the first case, then we are done; suppose we are in the second case. Then it must be $e_\sigma(\delta) > k+1$ and, by the hypothesis, we get $o_\sigma(\delta) > k$. Since it is distinguished and it does not intersect the support of δ_W, we get then $\Delta'(\sigma)_{\leqslant k} \subset \Delta^*$: therefore the condition $\Delta^* \cap \{D'_1, D'_2, D'_3\} = \{D'_2\}$ is not satisfied whenever $o_\sigma(\delta) \geqslant e_\sigma(\delta) - 1$ and the claim follows. □

Combining together Lemma 3.3.6, Corollary 3.3.7 and Lemma 3.3.8, we get the following theorem (the cases wherein the Luna diagram of M in $\Gamma(\sigma)$ is of type $\mathsf{C2}$, $\mathsf{F1}$, $\mathsf{F2}$ or $\mathsf{F3}$ are easily treated directly).

Theorem 3.3.9. *Let M be a strict wonderful variety and let δ be a faithful divisor on it. Then the normalization $p : \widetilde{X}_\delta \to X_\delta$ is bijective if and only if the following conditions are fulfilled, for every spherical root $\sigma \in \Sigma$ of type $\mathsf{B}_2^{\mathrm{I}}$:*

i) If the Luna diagram of M in $\Gamma(\sigma)$ is of type $\mathsf{B1}$, then $D^\flat(\sigma) \in \mathrm{Supp}_\Delta(\delta)$ or the following conditions are both satisfied:

- $\mathrm{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\mathrm{even}} = \varnothing$;
- *If M possesses a spherical root supported on $\alpha_{m(\sigma)+1}$, then $m(\sigma)$ is odd.*

ii) If the Luna diagram of M in $\Gamma(\sigma)$ is of type $\mathsf{B2}$, then $D^\flat(\sigma) \in \mathrm{Supp}_\Delta(\delta)$.

iii) If the Luna diagram of M in $\Gamma(\sigma)$ is of type $\mathsf{C1}$, then $o_\sigma(\delta) \geqslant e_\sigma(\delta) - 1$.

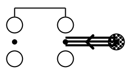
iv) Otherwise, if $D^\sharp(\sigma) \in \mathrm{Supp}_\Delta(\delta)$, then $D^\flat(\sigma) \in \mathrm{Supp}_\Delta(\delta)$ as well.

3.4 Bijectivity in the non-strict case

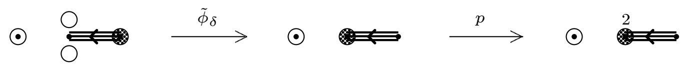
In this section we briefly consider the non-strict case giving some sufficient conditions of bijectivity and non-bijectivity of the normalization map.

Suppose that M is not strict and let $\delta = \sum_\Delta n(\delta, D)D$ be a faithful divisor on M, suppose that $Z \subset X_\delta$ is an orbit such that $\Sigma(\delta_Z)$ contains a non-simple spherical root γ. Following examples show that, unlike from the strict case (see Lemma 3.3.1), γ may be as well of type $\mathsf{G}_2^{\mathrm{I}}$ and, in case γ is of type $\mathsf{B}_r^{\mathrm{I}}$, then it does not necessarily come from a spherical root of type $\mathsf{B}_2^{\mathrm{I}}$.

Example 3.4.1. Consider the wonderful variety M whose spherical system is expressed by the Luna diagram

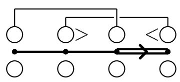


Then the divisor $\delta = D^+_{\alpha_1}$ is faithful. Consider the codimension one orbit $W \subset M$ whose spherical roots are α_2 and $\alpha_2 + \alpha_3$; following Proposition 3.2.1 and Corollary 2.5.4, we get the sequence of Luna diagrams



where the first one represents the orbit $W \subset M$, the second one the orbit $\tilde{\phi}_\delta(W) \subset \widetilde{X}_\delta$ and the third one the orbit $\phi_\delta(W) \subset X_\delta$.

Example 3.4.2. Consider the wonderful variety M whose spherical system is expressed by the Luna diagram



Then the divisor $\delta = D^+_{\alpha_1}$ is faithful. See Table 2 for a full list of the orbits in $\widetilde{X}_\delta$ and in X_δ (for simplicity, in the table orbits in M are described by giving a subset of its spherical root index set).

Lemma 3.4.3. *Suppose that M is a spherically closed wonderful variety and let $\delta = \sum_\Delta n(\delta, D)D$ be a faithful divisor on it; let $\alpha \in S \cap \Sigma$.*

i) If $Z \subset X_\delta$ is an orbit such that $2\alpha \in \Sigma_Z$, then $n(\delta, D^+_\alpha) = n(\delta, D^-_\alpha)$.

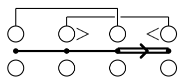
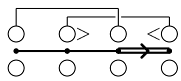
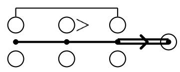
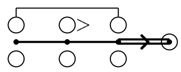
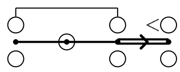
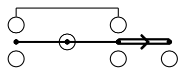
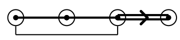
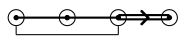
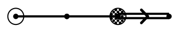
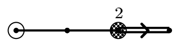
ii) If $n(\delta, D^+_\alpha) = n(\delta, D^-_\alpha)$ is non-zero, then there exists an orbit $Z \subset X_\delta$ such that $2\alpha \in \Sigma_Z$.

Proof. Suppose that $W \subset M$ is an orbit with set of spherical roots $\Sigma' \subset \Sigma$ and set of colors Δ'. If $\alpha \in S \cap \Sigma'$, set $\Delta'(\alpha) = \{'D^+_\alpha, 'D^-_\alpha\}$; then by the description of the pullback map $q : \mathrm{Pic}(M) \to \mathrm{Pic}(\overline{W})$ given in Remark 2.2.5 if $\delta_W = q(\delta)$ it follows that

$$n(\delta_W, 'D^+_\alpha) = n(\delta, D^+_\alpha), \qquad n(\delta_W, 'D^-_\alpha) = n(\delta, D^-_\alpha).$$

i). Let $Z \subset X_\delta$ be an orbit possessing 2α as a spherical root; let $Z' = p^{-1}(Z)$ and let $W \subset M$ be an orbit which maps on Z. Then by Proposition 1.5.9 we get that $\alpha \in \Sigma_{Z'}$. By Proposition 3.2.2 together with Theorem 2.4.5 we may identify

Table 3.2. Example 3.4.2, $\delta = D^+_{\alpha_1}$.

Maximal Orbits	**Minimal Orbit**	**Orbit in $\widetilde{X}_\delta$**	**Orbit in X_δ**	$\Sigma(\delta_{Z'})$
$\{1,2,3,4\}$	$\{1,2,3,4\}$			$\varnothing$
$\{1,2,3\}$	$\{1,2,3\}$			$\varnothing$
$\{1,3,4\}$	$\{1,3,4\}$			$\{\alpha_4\}$
$\{2,3,4\}$	$\{2,3,4\}$			$\varnothing$
$\{1,3\}$	$\{1,3\}$			$\varnothing$
$\{3,4\}$	$\{3,4\}$			$\{\alpha_3+\alpha_4\}$
$\{1,2,4\}$ $\{2,3\}$	$\varnothing$			$\varnothing$

$\Delta(Z')(\alpha)$ with $\Delta(W)(\alpha)$. Corollary 2.5.4 shows then $n(\delta_W, {}'D^+_\alpha) = n(\delta_W, {}'D^-_\alpha)$ and by the remark at the beginning of the proof this implies the thesis.

ii). Consider the rank one orbit W whose unique spherical root is α, set $Z = \phi_\delta(W)$ and $Z' = p^{-1}(Z)$. Then $\alpha \in \Sigma_{Z'}$ is a loose spherical root and by the remark at the beginning of the proof we get $n(\delta_{Z'}, {}'D^+_\alpha) = n(\delta_{Z'}, {}'D^-_\alpha)$, where $\delta_{Z'}$ is the pullback of a hyperplane section of $\overline{Z}$ and where by Proposition 3.2.2 together with Theorem 2.4.5 $\Delta(Z')(\alpha)$ is identified with $\Delta(W)(\alpha)$. Then by Corollary 2.5.4 we get that $2\alpha \in \Sigma_Z$. $\square$

Suppose that $\alpha \in S \cap \Sigma$. As shown by Example 3.4.2, if $n(\delta, D^+_\alpha) = n(\delta, D^-_\alpha) = 0$, then it may not exist any orbit $Z \subset X_\delta$ possessing 2α as a spherical root; conversely, if there exists such an orbit, it may be as well $n(\delta, D^+_\alpha) = n(\delta, D^-_\alpha) = 0$.

As a corollary of previous lemma, we get the following sufficient conditions.

Corollary 3.4.4. *Suppose that M is a spherically closed wonderful variety and let $\delta = \sum_\Delta n(\delta, D)D$ be a faithful divisor on it.*

i) If there exists $\alpha \in S \cap \Sigma$ such that $n(\delta, D^+_\alpha) = n(\delta, D^-_\alpha)$ is non-zero, then the normalization $p : \widetilde{X}_\delta \to X_\delta$ is not bijective.

ii) If the Dynkin diagram of G is simply laced and if $n(\delta, D_\alpha^+) \neq n(\delta, D_\alpha^-)$ for every $\alpha \in S \cap \Sigma$, then the normalization $p : \widetilde{X}_\delta \to X_\delta$ is bijective.

Reasoning as in Lemma 3.3.2 and in Corollary 3.3.3, other sufficient conditions of bijectivity can be obtained imposing further conditions on the support of δ on the multiple links of the Dynkin diagram of G and on the simple spherical roots of M.

Chapter 4

Simple linear compactifications of semisimple adjoint groups

Otherwise differently stated, throughout this section G will denote a simply connected semisimple algebraic group over an algebraically closed field of characteristic zero.

4.1 The varieties X_λ and X_Π

Consider the $G \times G$-variety

$$X_\lambda = \overline{(G \times G)[\mathrm{Id}]} \subset \mathbb{P}(\mathrm{End}(V(\lambda)^*),$$

which is a simple compactification of a quotient of the adjoint group G_{ad}; denote $\widetilde{X}_\lambda \to X_\lambda$ its normalization. Define the *support* of λ as the set

$$\mathrm{Supp}(\lambda) = \{\alpha \in S \,:\, \langle \lambda, \alpha^\vee \rangle \neq 0\} :$$

by Proposition 3.1.3, there exists a $G \times G$-equivariant surjective morphim $X_\lambda \to X_{\lambda'}$ if and only if $\mathrm{Supp}(\lambda) \supset \mathrm{Supp}(\lambda')$; in particular X_λ and $X_{\lambda'}$ are $G \times G$-equivariantly isomorphic if and only if $\mathrm{Supp}(\lambda) = \mathrm{Supp}(\lambda')$.

Suppose that λ is regular, i.e. that $\mathrm{Supp}(\lambda) = S$: then $X_\lambda = M$ is the wonderful compactification of the adjoint group G_{ad} (see Example 2.1.5). The closed orbit of M is isomorphic to $G/B \times G/B$ and the restriction of line bundles induces an homomorphism

$$\omega : \mathrm{Pic}(M) \longrightarrow \mathcal{X}(B) \times \mathcal{X}(B)$$

which is injective and identifies $\mathrm{Pic}(M)$ with the sublattice $\{(\lambda, \lambda^*) \,:\, \lambda \in \mathcal{X}(B)\}$. Therefore $\mathrm{Pic}(M)$ is identified with $\mathcal{X}(B)$ and we will denote $\mathcal{L}_\lambda \in \mathrm{Pic}(M)$ the line bundle whose image is (λ, λ^*). Via the map ω, the spherical roots of M are identified with the simple roots of G, while the colors of M are identified with the fundamental dominant weights of G. In particular, a line bundle $\mathcal{L}_\lambda$ is generated by its sections if and only if $\lambda \in \mathcal{X}(B)^+$.

By Theorem 2.2.2, it holds the following description of $\Gamma(M, \mathcal{L}_\lambda)$ as a $G \times G$-module:

$$\Gamma(M, \mathcal{L}_\lambda) \simeq \bigoplus_{\mu \in \mathcal{X}(B)^+ \,:\, \mu \leqslant \lambda} \mathrm{End}(V(\mu)).$$

Following Section 3.1, if $\lambda \in \mathcal{X}(B)^+$ denote

$$\widetilde{A}(\lambda) = \bigoplus_{n\in\mathbb{N}} \Gamma(M, \mathcal{L}_{n\lambda})$$

and denote $A(\lambda) \subset \widetilde{A}(\lambda)$ the subalgebra generated by $\mathrm{End}(V(\lambda)) \subset \Gamma(M, \mathcal{L}_\lambda)$; consider the natural gradings on $\widetilde{A}(\lambda)$ and $A(\lambda)$ respectively defined by $\widetilde{A}_n(\lambda) = \Gamma(M, \mathcal{L}_{n\lambda})$ and $A_n(\lambda) = \widetilde{A}_n(\lambda) \cap A(\lambda)$. Then

$$\widetilde{X}_\lambda = \mathrm{Proj}\, \widetilde{A}(\lambda) \qquad \text{and} \qquad X_\lambda = \mathrm{Proj}\, A(\lambda).$$

A set of dominant weights $\Pi \subset \mathcal{X}(B)^+$ is said to be *simple* if it possesses a unique maximal element with respect to the dominance order. If Π is such a set, consider the variety

$$X_\Pi = \overline{(G\times G)[\mathrm{Id}_\Pi]} \subset \mathbb{P}\Big(\bigoplus_{\nu\in\Pi} \mathrm{End}(V(\nu))^*\Big),$$

where $\mathrm{Id}_\Pi = (\mathrm{Id}_\nu)_{\nu\in\Pi}$. If $\Pi = \{\mu_1, \ldots, \mu_m\}$, sometimes we will denote X_Π simply by $X_{\mu_1,\ldots,\mu_m}$.

Suppose that $\Pi \subset \mathcal{X}(B)^+$ is simple and denote $\lambda \in \Pi$ the maximal element. By the description of the space of sections of $\mathcal{L}_\lambda$, it follows $\bigoplus_{\nu\in\Pi} \mathrm{End}(V(\nu)) \subset \Gamma(M, \mathcal{L}_\lambda)$: thus we get

$$\widetilde{X}_\lambda lra X_\Pi \longrightarrow X_\lambda$$

and X_Π is a simple variety with the same normalization of X_λ. As in the case $\Pi = \{\lambda\}$, denote $A(\Pi) = \bigoplus_{n\in\mathbb{N}} A_n(\Pi)$ the projective coordinate ring of X_Π, namely the subalgebra of $\widetilde{A}(\lambda)$ generated by $\bigoplus_{\nu\in\Pi} \mathrm{End}(V(\nu))$. Notice that every simple linear compactification of a quotient of G_{ad} arises in this way.

Denote $\phi_\lambda \in \mathrm{End}(V_\lambda)$ a highest weight vector and consider the $B\times B^-$-stable affine open subsets $X^\circ_\lambda \subset X_\lambda$ and $X^\circ_\Pi \subset X_\Pi$ defined by the non-vanishing of ϕ_λ; then we get

$$\Bbbk[X^\circ_\Pi] = \left\{ \frac{\phi}{\phi^n_\lambda} \,:\, \phi \in A_n(\Pi) \right\} \supset \left\{ \frac{\phi}{\phi^n_\lambda} \,:\, \phi \in A_n(\lambda) \right\} = \Bbbk[X^\circ_\lambda]$$

Previous rings are not $G\times G$-module. However, since they are the coordinate ring of an open subset of a $G\times G$-variety, they are $\mathfrak{g}\oplus\mathfrak{g}$-modules.

Lemma 4.1.1. *Suppose that $\Pi \subset \mathcal{X}(B)^+$ is simple with maximal element λ. Then, as a $\mathfrak{g}\oplus\mathfrak{g}$-algebra, the coordinate ring $\Bbbk[X^\circ_\Pi]$ is generated by $\Bbbk[X^\circ_\lambda]$ together with the set $\{\phi_\mu/\phi_\lambda\}_{\mu\in\Pi}$.*

Proof. Since the projective coordinate ring $A(\Pi)$ is generated in degree one by $\bigoplus_{\mu\in\Pi} \mathrm{End}(V(\mu))$, it follows that $\Bbbk[X^\circ_\Pi]$ is generated as an algebra by its subset

$$B(\Pi) = \left\{ \frac{\phi}{\phi_\lambda} \,:\, \phi \in \bigoplus_{\mu\in\Pi} \mathrm{End}(V(\mu)) \right\}.$$

Using the action of the Lie algebra $\mathfrak{g}\oplus\mathfrak{g}$, let's show that $B(\Pi)$ is contained in the $\mathfrak{g}\oplus\mathfrak{g}$-subalgebra $\overline{B}(\Pi) \subset \Bbbk[X^\circ_\Pi]$ generated by $\Bbbk[X^\circ_\lambda]$ together with $\{\phi_\mu/\phi_\lambda\}_{\mu\in\Pi}$.

Suppose indeed that α is a simple root and that $\phi/\phi_\lambda \in \overline{B}(\Pi)$; then $f_\alpha(\phi)/\phi_\lambda \in \overline{B}(\Pi)$ as well since

$$\frac{f_\alpha(\phi)}{\phi_\lambda} = f_\alpha\left(\frac{\phi}{\phi_\lambda}\right) + \frac{\phi}{\phi_\lambda} \cdot \frac{f_\alpha(\phi_\lambda)}{\phi_\lambda}. \qquad \square$$

Definition 4.1.2. Suppose that $\Pi, \Pi' \subset \mathcal{X}(B)^+$ are simple with maximal elements respectively λ and λ'. Then Π and Π' are called *equivalent* and we write $\Pi \sim \Pi'$ if $\mathrm{Supp}(\lambda) = \mathrm{Supp}(\lambda')$ and if there exists a bijection $\mu \mapsto \mu'$ between $\Pi \smallsetminus \lambda$ and $\Pi' \smallsetminus \lambda'$ such that $\lambda' - \mu' = \lambda - \mu$ for every $\mu \in \Pi \smallsetminus \{\lambda\}$.

It follows by Lemma 4.1.1 that if Π and Π' are equivalent, then $X_\Pi \simeq X_{\Pi'}$.
Given $\lambda, \mu \in \mathcal{X}(B)^+$, consider the multiplication map

$$m_{\lambda,\mu} : \Gamma(M, \mathcal{L}_\lambda) \times \Gamma(M, \mathcal{L}_\mu) \to \Gamma(M, \mathcal{L}_{\lambda+\mu}).$$

As in [Ka 02] or in [DC 04], it is possible to identify sections of a line bundle on M with functions on G and use the description of the multiplication of matrix coefficients. Recall that as a $G \times G$-module it holds the decomposition

$$\Bbbk[G] = \bigoplus_{\lambda \in \mathcal{X}(B)^+} \mathrm{End}(V(\lambda)) \simeq \bigoplus_{\lambda \in \mathcal{X}(B)^+} V(\lambda)^* \otimes V(\lambda).$$

More explicitly if V is a G-module, define $c_V : V^* \otimes V \to \Bbbk[G]$ as usual by $c_V(\psi \otimes v)(g) = \langle \psi, gv \rangle$. If we multiply functions in $\Bbbk[G]$ of this type then we get

$$c_V(\psi \otimes v) \cdot c_W(\chi \otimes w) = c_{V\otimes W}\big((\psi \otimes \chi) \otimes (v \otimes w)\big) :$$

in particular we get that the image of the multiplication $\mathrm{End}(V(\lambda)) \otimes \mathrm{End}(V(\mu)) \to \Bbbk[G]$ is the sum of all $\mathrm{End}(V(\nu))$ with $V(\nu) \subset V(\lambda) \otimes V(\mu)$.

As a consequence we get the following description of the multiplication map.

Lemma 4.1.3 ([Ka 02] Lemma 3.1, [DC 04] Lemma 3.4)**.** *Let $\lambda' \leqslant \lambda$ and $\mu' \leqslant \mu$ be dominant weights. Then the image of $\mathrm{End}(V(\lambda')) \otimes \mathrm{End}(V(\mu')) \subset \Gamma(M, \mathcal{L}_\lambda) \otimes \Gamma(M, \mathcal{L}_\mu)$ in $\Gamma(M, \mathcal{L}_{\lambda+\mu})$ via the multiplication map $m_{\lambda,\mu}$ is*

$$\bigoplus_{V(\nu) \subset V(\lambda') \otimes V(\mu')} \mathrm{End}(V(\nu))$$

Recall the Parthasarathy-Ranga Rao-Varadarajan conjecture, proved independently by S. Kumar [Ku 88] and O. Mathieu [Mat 89].

Theorem 4.1.4 (PRV Conjecture)**.** *Let $\lambda, \mu \in \mathcal{X}(B)^+$ be dominant weights and let $\nu \leqslant \lambda + \mu$ be a dominant weight of the shape $\nu = w\lambda + w'\mu$, with $w, w' \in W$. Then $V(\nu) \subset V(\lambda) \otimes V(\mu)$.*

If λ is a dominant weight, denote $\Pi(\lambda)$ the set of weights occurring in the simple G-module $V(\lambda)$ and denote

$$\Pi^+(\lambda) = \Pi(\lambda) \cap \mathcal{X}(B)^+ = \{\mu \in \mathcal{X}(B)^+ \,:\, \mu \leqslant \lambda\}.$$

Using previous theorem together with the description of Lemma 4.1.3, S. S. Kannan proved the surjectivity of $m_{\lambda,\mu}$.

Theorem 4.1.5 ([Ka 02] Cor. 3.3)**.** *Let $\lambda, \mu \in \mathcal{X}(B)^+$ be dominant weights. Then the multiplication map*

$$m_{\lambda,\mu} : \Gamma(M, \mathcal{L}_\lambda) \times \Gamma(M, \mathcal{L}_\mu) \longrightarrow \Gamma(M, \mathcal{L}_{\lambda+\mu}).$$

is surjective.

Proof. Thanks to Theorem 2.2.2 together with Lemma 4.1.3, it is enough to show that, given any dominant weight $\nu \leqslant \lambda + \mu$, there exist dominant weights $\lambda' \leqslant \lambda$ and $\mu' \leqslant \mu$ such that $V(\nu) \subset V(\lambda') \otimes V(\mu')$.

Let $v_\lambda \in V(\lambda)$ and $v_\mu \in V(\mu)$ be highest weights vectors: then $v_\lambda \otimes v_\mu \in V(\lambda) \otimes V(\mu)$ is a highest weight vector of weight $\lambda+\mu$, it follows that $V(\lambda+\mu) \subset V(\lambda) \otimes V(\mu)$.

Suppose that $\nu \leqslant \lambda + \mu$ is a dominant weight. Then ν occurs as weight in $V(\lambda + \mu)$ and by previous remark we may write $\nu = \lambda'' + \mu''$ with $\lambda'' \in \Pi(\lambda)$ and $\mu'' \in \Pi(\mu)$. If W is the Weyl group of G w.r.t. T, then $\Pi(\lambda)$ and $\Pi(\mu)$ are W-stable, take $w, w' \in W$ such that $w\lambda'' \in \Pi^+(\lambda)$ and $w'\mu'' \in \Pi^+(\mu)$ and set $\lambda' = w\lambda''$ and $\mu' = w'\mu''$. Then the PRV conjecture implies $V(\nu) \subset V(\lambda') \otimes V(\mu')$. □

With completely different techniques, previous theorem was later generalized by R. Chirivì and A. Maffei to the case of an arbitrary symmetric adjoint wonderful variety in [CM 04].

Let $\lambda \in \mathcal{X}(B)^+$ and consider the ring $\widetilde{A}(\lambda) = \bigoplus_{n \in \mathbb{N}} \Gamma(M, \mathcal{L}_{n\lambda})$: by the surjectivity of the multiplication, it follows that $\widetilde{A}(\lambda)$ is generated in degree one. Therefore, by the description of $\Gamma(M, \mathcal{L}_\lambda)$ as a $G \times G$-module it follows that

$$\widetilde{X}_\lambda \simeq X_{\Pi^+(\lambda)} \subset \mathbb{P}\Big(\bigoplus_{\mu \in \mathcal{X}(B)^+ \, : \, \mu \leqslant \lambda} \mathrm{End}(V(\mu))^* \Big).$$

As a consequence of the description of the multiplication given in Lemma 4.1.3 we get the following characterization of morphisms between simple compactifications with the same closed orbit.

Lemma 4.1.6. *Let Π, Π' be simple subsets with the same maximal element λ. Then there exists a $G \times G$-equivariant surjective morphism $X_\Pi \to X_{\Pi'}$ if and only if for every $\nu \in \Pi'$ there exist $\mu_1, \ldots, \mu_m \in \Pi$ together with non-negative integers $k_0, k_1, \ldots, k_m$ such that*

$$V\big(\nu + (n-1)\lambda\big) \subset V(\mu_1)^{\otimes k_1} \otimes \ldots \otimes V(\mu_m)^{\otimes k_m} \otimes V(\lambda)^{\otimes k_0},$$

where $n = k_0 + \ldots + k_m$.

Proof. Consider the $B \times B^-$-stable affine open subsets $X^\circ_\Pi \subset X_\Pi$ and $X^\circ_{\Pi'} \subset X_{\Pi'}$: since they intersect the closed orbit, they intersect every orbit, therefore there exists a $G \times G$-equivariant morphism $X_\Pi \to X_{\Pi'}$ if and only if there exists a $B \times B^-$-equivariant morphism $X^\circ_\Pi \to X^\circ_{\Pi'}$.

Let $\nu \in \Pi'$ and consider the $B \times B^-$-semiinvariant function $\phi_\nu / \phi_\lambda \in \Bbbk[X^\circ_{\Pi'}]$: then $\phi_\nu / \phi_\lambda \in \Bbbk[X^\circ_\Pi]$ if and only if there exists $n \in \mathbb{N}$ such that

$$\phi_\nu \phi_\lambda^{n-1} \in A_n(\Pi) = \Big(\bigoplus_{\mu \in \Pi} \mathrm{End}(V(\mu)) \Big)^n.$$

Since such a function is $B \times B^-$-semiinvariant of weight $(\nu+(n-1)\lambda, \nu^*+(n-1)\lambda^*)$, by Lemma 4.1.3 this is equivalent to the inclusion

$$V(\nu+(n-1)\lambda) \subset \Big(\bigoplus_{\mu\in\Pi} V(\mu)\Big)^{\otimes n}.$$

Then the claim follows by Lemma 4.1.1. $\square$

If Π is a simple subset with maximal element λ, denote

$$\Omega(\Pi) = \Big\{\nu - n\lambda \,:\, V(\nu) \subset \Big(\bigoplus_{\mu\in\Pi} V(\mu)\Big)^{\otimes n}\Big\}:$$

it is a semigroup and by previous lemma it is the image of

$$\mathbb{k}[X_\Pi^\circ]^{(B\times B^-)}\big/\mathbb{k}^* \subset \mathcal{X}(B)\times\mathcal{X}(B)$$

in $\mathcal{X}(B)$ via the projection on the first factor. If $\Pi = \{\mu_1, \dots, \mu_m\}$, sometimes we will denote $\Omega(\Pi)$ simply by $\Omega(\mu_1, \dots, \mu_m)$.

The following is a restatement of Lemma 4.1.6 in terms of the semigroups defined above.

Lemma 4.1.7. *Let Π, Π' be simple subsets with the same maximal element λ. Then there exists a $G \times G$-equivariant surjective morphism $X_\Pi \to X_{\Pi'}$ if and only if $\Omega(\Pi') \subset \Omega(\Pi)$ if and only if $\nu - \lambda \in \Omega(\Pi)$ for all $\nu \in \Pi'$.*

Definition 4.1.8. A dominant weight $\mu \leqslant \lambda$ is called *trivial* if it satisfies one the following equivalent conditions:

i) $\mathrm{Hom}\Big(\mathrm{End}(V(\mu)), \Gamma(X_\lambda, \mathcal{L}_\lambda)\Big) \neq 0$, where $\mathcal{L}_\lambda \in \mathrm{Pic}(X_\lambda)$ denotes the restriction of the hyperplane bundle on $\mathbb{P}(\mathrm{End}(V(\lambda))$.

ii) There exists a $G \times G$-equivariant isomorphism $X_{\lambda,\mu} \simeq X_\lambda$.

iii) $\Omega(\lambda, \mu) = \Omega(\lambda)$.

iv) There exists $n \in \mathbb{N}$ such that $V(\mu + (n-1)\lambda) \subset V(\lambda)^{\otimes n}$.

v) $\mu - \lambda \in \Omega(\lambda)$.

Corollary 4.1.9. *Let $\mu \leqslant \lambda$ and $\mu' \leqslant \lambda'$ be non-trivial weights with* $\mathrm{Supp}(\lambda) = \mathrm{Supp}(\lambda')$*; if $X_{\lambda,\mu}$ dominates $X_{\lambda',\mu'}$, then*

$$\lambda' - \mu' \geqslant \lambda - \mu.$$

In particular, $X_{\lambda,\mu} \simeq X_{\lambda',\mu'}$ if and only if $\{\lambda, \mu\}$ and $\{\lambda', \mu'\}$ are equivalent.

Proof. By Lemma 4.1.7 it is enough to notice that if $\mathrm{Supp}(\lambda) = \mathrm{Supp}(\lambda')$ and $\Omega(\lambda', \mu') \subset \Omega(\lambda, \mu)$, then $V(\mu' - \lambda' + n\lambda) \subset V(\mu)^{\otimes k} \otimes V(\lambda)^{\otimes n-k}$ implies

$$\lambda' - \mu' \geqslant k(\lambda - \mu) \geqslant \lambda - \mu.$$

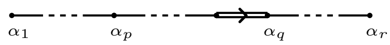
$\square$

Suppose that Π is simple with maximal element λ. Then, by the isomorphism $\widetilde{X}_\lambda \simeq X_{\Pi^+(\lambda)}$, previous lemma yields as well a criterion of normality for X_Π in terms of tensor product inclusions: X_Π is normal if and only if, for every dominant weight $\nu \leqslant \lambda$, there exists $n \in \mathbb{N}$ such that

$$V(\nu + (n-1)\lambda) \subset \Big(\bigoplus_{\mu \in \Pi} V(\mu) \Big)^{\otimes n}.$$

Together with P. Bravi, A. Maffei and A. Ruzzi, in [BGMR 10] we exploited such criterion to give a necessary and sufficient combinatorial condition for X_Π to be normal.

Definition 4.1.10. If $S' \subset S$ is a non-simply laced connected component, order the simple roots in $S' = \{\alpha_1, \dots, \alpha_r\}$ starting from the extreme of the Dynkin diagram of S' which contains a long root and denote α_q the first short root in S'. If λ is a dominant weight such that $\alpha_q \notin \mathrm{Supp}(\lambda)$ and such that $\mathrm{Supp}(\lambda) \cap S'$ contains a long root, denote α_p the last long root which occurs in $\mathrm{Supp}(\lambda) \cap S'$: for instance, if S' is not of type G_2, then the numbering is as follows



The *little brother* of λ with respect to S' is the dominant weight

$$\lambda^{\mathrm{lb}}_{S'} = \lambda - \sum_{i=p}^{q} \alpha_i = \begin{cases} \lambda - \omega_1 + \omega_2 & \text{if } G \text{ is of type } \mathsf{G}_2 \\ \lambda + \omega_{p-1} - \omega_p + \omega_{q+1} & \text{otherwise} \end{cases}$$

where ω_i is the fundamental weight associated to α_i if $1 \leqslant i \leqslant r$, while $\omega_0 = \omega_{r+1} = 0$. The set of the little brothers of λ will be denoted by $\mathrm{LB}(\lambda)$, while if S is connected and non-simply laced set $\lambda^{\mathrm{lb}} = \lambda^{\mathrm{lb}}_S$.

Theorem 4.1.11 ([BGMR 10] Thm. 12)**.** *Suppose that $\Pi \subset \mathcal{X}(B)^+$ is simple with maximal element λ. Then the variety X_Π is normal if and only if $\Pi \supset \mathrm{LB}(\lambda)$. In particular, X_λ is normal if and only if λ satisfies the following condition:*

($\star$) *For every non-simply laced connected component $S' \subset S$, if $\mathrm{Supp}(\lambda) \cap S'$ contains a long root, then it contains also the short root which is adjacent to a long simple root.*

We conclude this section with some results on tensor product decompositions which will be useful in the following. If $\nu = \sum n_\alpha \alpha \in \mathbb{Z}S$, recall its *support over S* defined as follows

$$\mathrm{Supp}_S(\nu) = \{\alpha \in S \,:\, n_\alpha \neq 0\}.$$

Lemma 4.1.12 ([BGMR 10] Lemma 6)**.** *Let λ, μ, ν be dominant weights and let $S' \subset S$ be such that $\mathrm{Supp}_S(\lambda + \mu - \nu) \subset S'$. Let $L \subset G$ be the standard Levi subgroup associated to S' and, if $\pi \in \mathcal{X}(B)^+$, denote by $V_L(\pi)$ the simple L-module of highest weight π. Then*

$$V(\nu) \subset V(\lambda) \otimes V(\mu) \iff V_L(\nu) \subset V_L(\lambda) \otimes V_L(\mu).$$

Proof. If $\mathfrak{a}$ is any Lie algebra, denote $\mathfrak{U}(\mathfrak{a})$ the corresponding universal enveloping algebra.

Suppose that $V_L(\nu) \subset V_L(\lambda) \otimes V_L(\mu)$; fix maximal vectors $v_\lambda \in V_L(\lambda)$ and $v_\mu \in V_L(\mu)$ for the Borel subgroup $B \cap L \subset L$ and fix $p \in \mathfrak{U}(\mathfrak{l} \cap \mathfrak{u}^-) \otimes \mathfrak{U}(\mathfrak{l} \cap \mathfrak{u}^-)$ such that $p\,(v_\lambda \otimes v_\mu) \in V_L(\lambda) \otimes V_L(\mu)$ is a maximal vector of weight ν. Since $V_L(\lambda) \otimes V_L(\mu) \subset V(\lambda) \otimes V(\mu)$, we only need to prove that $p\,(v_\lambda \otimes v_\mu)$ is a maximal vector for B too. If $\alpha \in S'$ then we have $e_\alpha p\,(v_\lambda \otimes v_\mu) = 0$ by hypothesis. On the other hand, if $\alpha \in S \smallsetminus S'$, notice that e_α commutes with p, since by its definition p is supported only on the f_α's with $\alpha \in S'$. Since $v_\lambda \otimes v_\mu$ is a maximal vector for B, then we get

$$e_\alpha p\,(v_\lambda \otimes v_\mu) = p\, e_\alpha (v_\lambda \otimes v_\mu) = 0;$$

thus $p\,(v_\lambda \otimes v_\mu)$ generates a simple G-module of highest weight ν.

Assume conversely that $V(\nu) \subset V(\lambda) \otimes V(\mu)$ and fix $p \in \mathfrak{U}(\mathfrak{u}^-) \otimes \mathfrak{U}(\mathfrak{u}^-)$ such that $p\,(v_\lambda \otimes v_\mu) \in V(\lambda) \otimes V(\mu)$ is a maximal vector of weight ν. Since $\mathrm{Supp}_S(\lambda+\mu-\nu) \subset S'$, we may assume that the only f_α's appearing in p are those with $\alpha \in S'$; therefore $p\,(v_\lambda \otimes v_\mu) \in V_L(\lambda) \otimes V_L(\mu)$ and it generates a simple L-module of highest weight ν. □

Corollary 4.1.13. *Let $\mu \leqslant \lambda$ be dominant weights and suppose that* $\mathrm{Supp}_S(\lambda - \mu)$ *is simply laced regarded as a subset of the vertices of the Dynkin diagram of G. Then $\mu \leqslant \lambda$ is trivial.*

Proof. By Theorem 4.1.11 applied to the semisimple part of L, there exists $n \in \mathbb{N}$ such that $V_L(\mu + (n-1)\lambda) \subset V_L(\lambda)^{\otimes n}$: by previou lemma, this implies $V(\mu + (n-1)\lambda) \subset V(\lambda)^{\otimes n}$ and $\mu \leqslant \lambda$ is trivial. □

Another useful lemma is the following.

Lemma 4.1.14 ([BGMR 10] Lemma 7)**.** *Fix $\lambda, \mu, \nu \in \mathcal{X}(B)^+$ such that $V(\nu) \subset V(\lambda) \otimes V(\mu)$. Then, for any $\nu' \in \mathcal{X}(B)^+$, it also holds*

$$V(\nu + \nu') \subset V(\lambda + \nu') \otimes V(\mu).$$

Proof. If $\pi, \pi' \in \mathcal{X}(B)^+$, recall the multiplication $V(\pi) \otimes V(\pi') \to V(\pi + \pi')$ defined by identifying $V(\pi)$ and $V(\pi')$ with the global sections of the associated line bundle on the flag variety G/B. We will denote the image of a tensor $v \otimes w$ by vw.

Fix a maximal vector $v_{\nu'} \in V(\nu')$ and consider the U-equivariant map

$$\begin{array}{cccc} \phi: & V(\lambda) \otimes V(\mu) & \longrightarrow & V(\lambda + \nu') \otimes V(\mu) \\ & w_1 \otimes w_2 & \longmapsto & w_1 v_{\nu'} \otimes w_2 \end{array}$$

The claim follows since, if $v_\nu \in V(\lambda) \otimes V(\mu)$ is a U-invariant vector of weight ν, then $\phi(v_\nu) \in V(\lambda + \nu') \otimes V(\mu)$ is a U-invariant vector of weight $\nu + \nu'$. □

Corollary 4.1.15. *Let $\nu \leqslant \mu \leqslant \lambda$ be dominant weights such that* $\mathrm{Supp}(\lambda) \cap \mathrm{Supp}_S(\mu - \nu) \neq \varnothing$ *and suppose moreover that $\mu - \nu$ is the highest long root of the root subsystem generated by* $\mathrm{Supp}_S(\mu - \nu)$*. Then $V(\nu + \lambda) \subset V(\mu) \otimes V(\lambda)$.*

Proof. Denote L the Levi subgroup associated to $\mathrm{Supp}_S(\mu - \nu)$ and denote $\mathfrak{g}$ its Lie algebra. Consider $\mu - \nu$: by the assumption on $\mu - \nu$, we have an isomorphism of $\mathfrak{l}$-modules $V_L(\mu - \nu) \simeq \mathfrak{l}$. Therefore the $\mathfrak{l}$-action induces a surjective morphism

$$V_L(\mu - \nu) \otimes V_L(\lambda) \longrightarrow V_L(\lambda)$$

which is non-zero by the assumption on λ: hence we get an inclusion $V_L(\lambda) \subset V_L(\mu - \nu) \otimes V_L(\lambda)$. By Lemma 4.1.14 this implies $V_L(\nu + \lambda) \subset V_L(\mu) \otimes V_L(\lambda)$, thus the claim follows by Lemma 4.1.12. □

4.2 The odd orthogonal case: the coordinate ring of X_λ

We now describe some more explicit results about tensor products decompositions which we will use to describe the semigroup $\Omega(\lambda)$ in the case $G = \mathrm{Spin}(2r + 1)$. Unless otherwise stated, we use the numbering of simple roots and fundamental weights of Bourbaki [Bo 75].

Lemma 4.2.1. *Suppose that* $G = \mathrm{Spin}(2r + 1)$.

i) For every $1 \leqslant i \leqslant r$ *it holds*

$$V(\omega_i + \omega_r) \subset V(\omega_i) \otimes V(\omega_1 + \omega_r).$$

ii) For every $1 \leqslant i < r$ *it holds*

$$V(\omega_i) \subset V(\omega_i) \otimes V(2\omega_1).$$

iii) For every $1 \leqslant i \leqslant j < r - 1$ *they hold*

$$V(\omega_i + \omega_j) \subset V(\omega_i) \otimes V(\omega_1 + \omega_{j+1});$$
$$V(\omega_i + \omega_{r-1}) \subset V(\omega_i) \otimes V(\omega_1 + 2\omega_r).$$

Proof. The claims can be easily shown with the generalized Littlewood-Richardson rule [Na 93]. Consider indeed the following semi-standard B-tableau of shape ω_i:

$$T_1 = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \vdots \\ \hline i \\ \hline 0 \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \vdots \\ \hline i \\ \hline \overline{1} \\ \hline \end{array} \qquad T_3 = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \vdots \\ \hline i \\ \hline \overline{j+1} \\ \hline \end{array}$$

If λ is a dominant weight, denote $Y(\lambda)$ the generalized Young diagram of shape λ. Then the claims are a consequence of following remarks:

i) $Y(\omega_1 + \omega_r) + T_1$ is a generalized Young diagram of shape $\omega_i + \omega_r$,

ii) $Y(2\omega_1) + T_2$ is a generalized Young diagram of shape ω_i, *iii)* If $j < r-1$ then $Y(\omega_1 + \omega_{j+1}) + T_3$ is a generalized Young diagram of shape $\omega_i + \omega_j$, while if $j = r-1$ then $Y(\omega_1 + 2\omega_r) + T_3$ is a generalized Young diagram of shape $\omega_i + \omega_j$. □

We are going now to prove the following combinatorial characterization of trivial weights.

Theorem 4.2.2. *Suppose that $G = \mathrm{Spin}(2r+1)$ and let $\mu \leqslant \lambda$ be dominant weights. Let q and l be the maximal integers such that $\alpha_q \in \mathrm{Supp}(\lambda)$ and $\alpha_l \in \mathrm{Supp}(\mu)$ and write $\lambda - \mu = \sum_{i=1}^r a_i \alpha_i$. Then $\mu \leqslant \lambda$ is trivial if and only if a_r is even or $a_r > 2\min\{r-l, r-q\}$.*

In the notations of the theorem, notice that $a_i = a_r$ for all $i \geqslant \max\{q, l\}$. Therefore we may restate the theorem as follows, without referring to the particular weight λ but only to its support. Given a weight ν we denote

$$\mathrm{Supp}(\nu)^- = \{\alpha \in S \,:\, \langle \nu, \alpha^\vee \rangle < 0\}.$$

Theorem 4.2.3. *Suppose that $G = \mathrm{Spin}(2r+1)$ and let λ be a dominant weight, denote $q < r$ the maximal integer such that $\alpha_q \in \mathrm{Supp}(\lambda)$. Then it holds the following description:*

$$\Omega(\lambda) = \left\{ \nu = -\sum_{i=1}^r a_i \alpha_i \in -\mathbb{N}S \,:\, \begin{array}{c} \mathrm{Supp}(\nu)^- \subset \mathrm{Supp}(\lambda) \textit{ and} \\ a_r \textit{ is even } \textit{ or } a_r > 2\min\{r - l(\nu), r-q\} \end{array} \right\}$$

where $l(\nu) \leqslant r$ denotes the maximum such that $a_{l(\nu)-1} \neq a_{l(\nu)} = a_{l(\nu)+1}$.

If $\alpha_r \in \mathrm{Supp}(\lambda)$, then the theorem is equivalent to the normality of X_λ (see Theorem 4.1.11). Therefore we will assume that $\alpha_r \notin \mathrm{Supp}(\lambda)$.

Exploiting the Schur-Weyl duality, following proposition clarifies the condition in the theorem.

Proposition 4.2.4. *If $\mu \leqslant n\omega_1$ is a dominant weight, denote $n\omega_1 - \mu = \sum_{i=1}^r a_i \alpha_i$. Denote α_l the last simple root in $\mathrm{Supp}(\mu)$ or set $l = 0$ if $\mu = 0$. Then $V(\mu) \not\subset V(\omega_1)^{\otimes n}$ if and only if $a_r < 2(r-l)$ is odd.*

Proof. Regard $\mathrm{SO}(2r+1) \subset \mathrm{GL}(2r+1)$ and denote $\mathfrak{h} \subset \widetilde{\mathfrak{h}}$ the respective Cartan subalgebras of diagonal matrices. Denote $\varepsilon_1, \ldots, \varepsilon_{2r+1}$ the basis of $\widetilde{\mathfrak{h}}^*$ defined by $\varepsilon_i(A) = a_i$, where $A = \mathrm{diag}(a_1, \ldots, a_{2r+1}) \in \widetilde{\mathfrak{h}}$; if $\lambda = \sum_{i=1}^r \lambda_i \varepsilon_i$ is a weight denote $|\lambda| = \sum_{i=1}^r \lambda_i$. With respect to this basis μ is expressed as follows

$$\mu = (n - a_1)\varepsilon_1 + \sum_{i=2}^r (a_{i-1} - a_i)\varepsilon_i.$$

By Schur-Weyl duality for orthogonal groups (see [GW 09, Appendix F]) it follows that $V(\mu) \subset V(\omega_1)^{\otimes n}$ if and only if μ extends to a dominant weight $\lambda = \sum_{i=1}^{2r+1} \lambda_i \varepsilon_i \in \widetilde{\mathfrak{h}}^*$ such that

$$\begin{cases} |\lambda| \leqslant n \\ |\lambda| \equiv n \mod 2 \\ \lambda_1^t + \lambda_2^t \leqslant 2r+1 \end{cases}$$

where $\lambda^t = (\lambda^t_1, \ldots, \lambda^t_{\lambda_1})$ is the transposed of $\lambda = (\lambda_1, \ldots, \lambda_r)$ regarded as a partition. If λ is such a weight, then either $\lambda_i = 0$ for $i > r$ or

$$\lambda = \sum_{i=1}^{\lambda^t_2} \lambda_i \varepsilon_i + \sum_{i=\lambda^t_2+1}^{\lambda^t_1} \varepsilon_i$$

with $\lambda_i \geqslant 2$ for $i \leqslant \lambda^t_2$.

Suppose that λ is of the first kind: then a_r is even since $a_r = n - |\mu| \equiv 0 \mod 2$. Suppose conversely that λ is of the second kind: then $\lambda^t_1 + \lambda^t_2 \leqslant 2r + 1$ implies $l = 2r + 1 - \lambda^t_1$ and we get $a_r > 2(r - l)$ since

$$a_r = n - |\mu| = n - |\lambda| + 2(\lambda^t_1 - r - 1) + 1 = n - |\lambda| + 2(r - l) + 1.$$

Suppose conversely that a_r is even or that $a_r > 2(r - l)$, let's show that $V(\mu) \subset V(\omega_1)^{\otimes n}$. Define $\lambda \in \widetilde{\mathfrak{h}}^*$ as follows

$$\lambda = \begin{cases} \sum_{i=1}^{l} \mu_i \varepsilon_i & \text{if } a_r \text{ is even} \\ \sum_{i=1}^{l} \mu_i \varepsilon_i + \sum_{i=l+1}^{2r-l+1} \varepsilon_i & \text{if } a_r > 2(r-l) \text{ is odd} \end{cases}$$

Then λ satisfies the conditions given by Schur-Weyl duality and we get the claim. □

Lemma 4.2.5. *Let λ, π be dominant weights and suppose that $\lambda \leqslant \pi$ is trivial. Then*

$$\Bbbk[X^\circ_\lambda] \subset \Bbbk[X^\circ_\pi]_{(\phi_\lambda/\phi_\pi)}.$$

In particular $\Omega(\lambda) \subset \Omega(\pi)_{\pi-\lambda}$, where the latter denotes the semigroup generated by $\Omega(\pi)$ together with $\pi - \lambda$.

Proof. Since $\lambda - \pi \in \Omega(\pi)$, it follows that $X_\pi \simeq X_{\pi,\lambda}$. Therefore X_π is endowed with an ample line bundle with a $B \times B^-$-semiinvariant section s_λ of weight (λ, λ^*), which generates a module isomorphic to $\mathrm{End}(V(\lambda))$. Correspondingly, we get a rational application $X_\pi \dashrightarrow X_\lambda$ which is regular in the affine set

$$(X^\circ_\pi)_{(\phi_\lambda/\phi_\pi)} = X_\pi \smallsetminus \bigcup_{\alpha \in \mathrm{Supp}(\lambda+\pi)} D_\alpha,$$

where D_α is the color associated to α. Thus we get $\Bbbk[X^\circ_\lambda] \subset \Bbbk[X^\circ_\pi]_{(\phi_\lambda/\phi_\pi)}$. □

Corollary 4.2.6. *Let λ be a dominant weight such that $\alpha_r \notin \mathrm{Supp}(\lambda)$. Suppose that $\mu \leqslant \lambda$ is trivial and denote $\lambda - \mu = \sum a_i \alpha_i$. If α_q and α_l are the last simple roots respectively in $\mathrm{Supp}(\lambda)$ and in $\mathrm{Supp}(\mu)$, then either a_r is even or $a_r > 2 \min\{r - l, r - q\}$.*

Proof. Suppose that $\lambda = n\omega_1$ and suppose that $V(\mu + (m-1)n\omega_1) \subset V(n\omega_1)^{\otimes m}$. Then we get $V(\mu + (m-1)n\omega_1) \subset V(\omega_1) \otimes mn$ and by previous proposition it follows that $\lambda - \mu = mn\omega 1 - \mu - (m-1)n\omega 1$ satisfies the condition.

Suppose now that $\alpha_r \notin \lambda$: then there exists $n > 0$ such that $\lambda \leqslant n\omega_1$ with $\mathrm{Supp}_S(n\omega_1 - \lambda) \subset \{\alpha_1, \ldots, \alpha_{q-1}\}$, where α_q is the last simple root which occurs in $\mathrm{Supp}(\lambda)$. Since $\alpha_r \notin \mathrm{Supp}_S(n\omega_1 - \lambda)$, it follows that $\lambda \leqslant n\omega_1$ is trivial: by Lemma

4.2.5 we get then $\mu - \lambda \in \Omega(\lambda) \subset \Omega(\omega_1)_{n\omega_1 - \lambda}$. Therefore there exist $k, m \in \mathbb{N}$ and a trivial weight $\mu' \leqslant m\omega_1$ such that

$$\mu - \lambda = \mu' - m\omega_1 + k(n\omega_1 - \lambda).$$

If $m\omega_1 - \mu' = \sum a'_i\alpha_i$, then by the definition of n it follows $a_i = a'_i$ for all $i \geqslant q$. Therefore by the case treated at the beginning we get that either a_r is even or $a_r > 2\min\{r-1, r-l'\}$, where l' is the maximum such that $\alpha_{l'} \in \mathrm{Supp}(\mu')$. If $\mu' = 0$, then $a_r > 2(r-1) \geqslant 2(r-q)$ and the claim follows. Suppose now $\mu' \neq 0$ and notice that, if $l' > q$, then $l' = l$: therefore $2(r-l') \geqslant 2\min\{r-l, r-q\}$ and the claim follows. □

We now prove in a constructive way that the condition of the theorem is sufficient. Suppose that $\mu \leqslant \lambda$ are dominant weights and denote $\lambda - \mu = \sum_{i=1}^r a_i\alpha_i$, denote α_q and α_l the last simple roots respectively in $\mathrm{Supp}(\lambda)$ and in $\mathrm{Supp}(\mu)$. We will distinguish three different cases:

i) $a_{r-1} \neq a_r$, i.e. $\alpha_r \in \mathrm{Supp}(\mu)$ (Lemma 4.2.7).

ii) $a_{r-1} = a_r$ is even (Lemma 4.2.8).

iii) $a_{r-1} = a_r > 2\min\{r-l, r-q\}$ is odd (Lemma 4.2.9).

Lemma 4.2.7. *Let $\mu \leqslant \lambda$ be dominant weights and suppose that $\alpha_r \notin \mathrm{Supp}(\lambda)$. If $\alpha_r \in \mathrm{Supp}(\mu)$, then $\mu \leqslant \lambda$ is trivial.*

Proof. We proceed by induction on $a_{r-1}a_r$. Suppose that either $a_{r-1} = 0$ or $a_r = 0$: then $\mathrm{Supp}_S(\lambda - \mu)$ has type A and the claim follows by Corollary 4.1.13.

Assume that a_{r-1} and a_r are both non-zero. Denote $p < r-1$ the maximum such that $a_{p-1} = 0$ or set $p = 1$ otherwise and define

$$\mu' = \mu + \sum_{i=p}^{r} a_i\alpha_i = \mu - \omega_{p-1} + \omega_p.$$

Notice that $\mu' \leqslant \lambda$ is dominant: indeed

$$\langle \mu, \alpha_{p-1}^\vee \rangle = \langle \lambda, \alpha_{p-1}^\vee \rangle + a_{p-2} + a_p \geqslant a_p > 0.$$

Moreover by Lemma 4.2.1 i) together with Lemma 4.1.14 it holds $V(\mu + \lambda) \subset V(\mu') \otimes V(\lambda)$, thus we get the inclusion $\Omega(\lambda, \mu) \subset \Omega(\lambda, \mu')$.

Consider $\mu' \leqslant \mu$ and denote $\lambda - \mu = \sum a'_i\alpha_i$: then it still satisfies the hypotheses of the lemma and $a'_{r-1}a'_r < a_{r-1}a_r$, therefore by induction we may assume that it is trivial, i.e. $\Omega(\lambda, \mu') = \Omega(\lambda)$. It follows then $\Omega(\lambda, \mu) = \Omega(\lambda)$ as well, i.e. $\mu \leqslant \lambda$ is trivial. □

Lemma 4.2.8. *Let $\mu \leqslant \lambda$ be dominant weights, suppose that $\alpha_r \notin \mathrm{Supp}(\lambda)$ and denote $\lambda - \mu = \sum_{i=1}^r a_i\alpha_i$. If $a_{r-1} = a_r$ is even then $\mu \leqslant \lambda$ is trivial.*

Proof. Denote α_q the last simple root in $\mathrm{Supp}(\lambda)$. Up to consider the couple $\mu + \omega_q \leqslant \lambda + \omega_q$, which is equivalent to $\mu \leqslant \lambda$, we may assume that $\alpha_q \in \mathrm{Supp}(\mu)$.

We proceed by induction on a_r. Suppose that $a_r = 0$: then $\mathrm{Supp}_S(\lambda - \mu)$ has type A and the claim follows by Corollary 4.1.13.

Suppose $a_{r-1} = a_r \geqslant 2$ and notice that since μ is dominant it follows

$$a_q \geqslant a_{q+1} \geqslant \ldots \geqslant a_{r-1} = a_r \geqslant 2.$$

Denote p the maximum such that $a_{p-1} = 0$ or set $p = 1$ otherwise and define

$$\mu' = \mu + \sum_{i=p}^{q} \alpha_i + 2 \sum_{i=q+1}^{r} a_i\alpha_i = \begin{cases} \mu - \omega_{p-1} + \omega_p - \omega_{q-1} + \omega_q & \text{if } q < r-2 \\ \mu - \omega_{p-1} + \omega_p - \omega_{r-1} + 2\omega_q & \text{if } p = r-1 \end{cases}$$

Notice that $\mu' \leqslant \lambda$ is dominant: indeed $\alpha_q \in \mathrm{Supp}(\mu)$ by the assumption at the beginning of the proof, while

$$\langle \mu, \alpha_{p-1}^\vee \rangle = \langle \lambda, \alpha_{p-1}^\vee \rangle + a_{p-2} + a_p \geqslant a_p > 0.$$

Moreover by Lemma 4.2.1 iii) together with Lemma 4.1.14 it holds $V(\mu + \lambda) \subset V(\mu') \otimes V(\lambda)$, thus we get the inclusion $\Omega(\lambda, \mu) \subset \Omega(\lambda, \mu')$.

Consider $\mu' \leqslant \mu$ and denote $\lambda - \mu = \sum a_i'\alpha_i$: then either $q = r-1$ and $\alpha_r \in \mathrm{Supp}(\mu')$ or $a_{r-1}' = a_r' = a_r - 2$: therefore $\mu' \leqslant \lambda$ is trivial, in the first case by Lemma 4.2.7 and in the second case by inductive hypothesis. Therefore we get $\Omega(\lambda, \mu) \subset \Omega(\lambda, \mu') = \Omega(\lambda)$ and $\mu \leqslant \lambda$ is trivial. □

Lemma 4.2.9. *Suppose that $\mu \leqslant \lambda$ are dominant weights and denote $\lambda - \mu = \sum_{i=1}^r a_i\alpha_i$. Denote α_q and α_l the last simple roots respectively in* $\mathrm{Supp}(\lambda)$ *and in* $\mathrm{Supp}(\mu)$ *and suppose $q < r$. If $a_r \geqslant 2\min\{r-l, r-q\}$ is odd, then $\mu \leqslant \lambda$ is trivial.*

Proof. Up to consider the couple $\mu + \omega_q \leqslant \lambda + \omega_q$, which is equivalent to $\mu \leqslant \lambda$, we may assume that $\alpha_q \in \mathrm{Supp}(\mu)$, i.e. $q \leqslant l$.

We proceed by induction on $r - l$, the basis being the case $l = r$ treated in Lemma 4.2.7. Suppose that $l < r$. Then the hypothesis $a_r > 2(r-l)$ toghether with the fact that μ is dominant imply

$$a_q \geqslant \ldots \geqslant a_l = a_{l+1} = \ldots = a_r \geqslant 3.$$

Denote p the maximum such that $a_{p-1} = 0$ or set $p = 1$ otherwise and define

$$\mu' = \mu + \sum_{i=p}^{l} \alpha_i + 2 \sum_{i=l+1}^{r} \alpha_i = \begin{cases} \mu - \omega_{p-1} + \omega_p - \omega_l + \omega_{l+1} & \text{if } l < r-1 \\ \mu - \omega_{p-1} + \omega_p - \omega_{r-1} + 2\omega_r & \text{if } l = r-1 \end{cases}$$

Notice that $\mu' \leqslant \lambda$ is dominant: indeed $\alpha_l \in \mathrm{Supp}(\mu)$ by definition, while

$$\langle \mu, \alpha_{p-1}^\vee \rangle = \langle \lambda, \alpha_{p-1}^\vee \rangle + a_{p-2} + a_p \geqslant a_p > 0.$$

Moreover by Lemma 4.2.1 iii) together with Lemma 4.1.14 it holds $V(\mu + \lambda) \subset V(\mu') \otimes V(\lambda)$, thus we get the inclusion $\Omega(\lambda, \mu) \subset \Omega(\lambda, \mu')$.

Consider $\mu' \leqslant \mu$ and denote $\lambda - \mu = \sum a_i'\alpha_i$. If $a_r' = 1$, then $a_r = 3$ and by $a_r > 2(r-l)$ we get $l = r-1$: thus $\alpha_r \in \mathrm{Supp}(\mu')$ and $\mu' \leqslant \lambda$ is trivial by Lemma 4.2.7. Otherwise $l' = l+1$ and $a_r' = a_r - 2 > 2(r-l-1) = 2(r-l')$: thus $\mu' \leqslant \lambda$ still satisfies the hypothesis of the lemma and it is trivial by the inductive hypothesis. Therefore we get $\Omega(\lambda, \mu) \subset \Omega(\lambda, \mu') = \Omega(\lambda)$ and $\mu \leqslant \lambda$ is trivial. □

Remark 4.2.10. Suppose that $\mathrm{Supp}(\lambda) = \{\alpha_{r-1}\}$. Then, following Theorem 4.2.3, if X is a $G \times G$-variety such that $\widetilde{X}_\lambda \to X \to X_\lambda$, then it must be either $X \simeq \widetilde{X}_\lambda$ or $X \simeq X_\lambda$. Indeed if $\mu = \lambda - \sum_{i=1}^r a_i\alpha_i$ is a dominant weight and if $a_{r-1} = a_r = 1$, then it must be $a_1 = \ldots = a_{r-2} = 0$.

4.3 The odd orthogonal case: morphisms

Suppose that λ is a dominant weight and let Π, Π' be simple subsets such that

$$\begin{tikzcd} \widetilde{X}_\lambda \arrow[r] \arrow[d] & X_{\Pi'} \arrow[d] \\ X_\Pi \arrow[r] & X_\lambda \end{tikzcd}$$

In this section we are going to characterize combinatorially the existence of an equivariant morphism $X_\Pi \to X_{\Pi'}$ which makes the diagram commute (Theorem 4.3.8). In particular, it will follow a combinatorial criterion to establish if two simple subsets give rise to isomorphic compactifications.

4.3.1 Remarks on tensor products decompositions

In order to obtain a combinatorial characterization of morphisms, we need first to describe some explicit results about tensor products decompositions.

Lemma 4.3.1. *Suppose that G is a simple group of type B_r and let μ be a dominant weight.*

i) Let $\nu \leqslant \mu+\omega_1$ be such that $V(\nu) \subset V(\mu)\otimes V(\omega_1)$. Then there exists $1 \leqslant k \leqslant r$ such that either

$$\mu+\omega_1-\nu = \sum_{i=1}^{k-1}\alpha_i + 2\sum_{i=k}^{r}\alpha_i \qquad \text{or} \qquad \mu+\omega_1-\nu = \sum_{i=1}^{k}\alpha_i.$$

ii) Let $\nu \leqslant \mu+n\omega_1$ be such that $V(\nu) \subset V(\mu)\otimes V(\omega_1)^{\otimes n}$; denote $\mu+n\omega_1-\nu = \sum_{i=1}^r a_i\alpha_i$ and set $I = \{i<r \,:\, a_i < a_{i+1}\}$. Then

$$2\sum_{i\in I}(a_{i+1}-a_i) \leqslant a_r.$$

Proof. Let's show i), ii) follows by induction. Denote $\mu = \sum m_i\omega_i$ and $\nu = \sum n_i\omega_i$. By the generalized Littlewood-Richardson rule [Na 93], either $\nu = \mu$ (i.e. $\mu+\omega_1-\nu = \sum_{i=1}^r \alpha_i$) or there exists $1 \leqslant k \leqslant r$ such that $n_i = m_i$ for every $i \notin \{k-1,k\}$ and such that:

$$\text{If } k<r \quad \text{then} \qquad \begin{cases} n_{k-1} = m_{k-1}-1 \\ n_k = m_k+1 \end{cases} \quad \text{or} \quad \begin{cases} n_{k-1} = m_{k-1}+1 \\ n_k = m_k-1 \end{cases}$$

$$\text{If } k=r \quad \text{then} \qquad \begin{cases} n_{k-1} = m_{k-1}-1 \\ n_k = m_k+2 \end{cases} \quad \text{or} \quad \begin{cases} n_{k-1} = m_{k-1}+1 \\ n_k = m_k-2 \end{cases}$$

It follows that

$$\mu+\omega_1-\nu = \begin{cases} \sum_{i=1}^{k-1}\alpha_i & \text{in the first and in the third case} \\ \sum_{i=1}^{k-1}\alpha_i + 2\sum_{i=k}^{r}\alpha_i & \text{in the second and in the fourth case} \end{cases}$$

Thus we get i).

Let's show ii) by induction on n. Suppose $V(\nu) \subset V(\mu') \otimes V(\omega_1)$, with $V(\mu') \subset V(\mu) \otimes V(\omega_1)^{\otimes n-1}$, write $\mu + (n-1)\omega_1 - \mu' = \sum b_i \alpha_i$. Set $I' = \{i < r \,:\, b_i < b_{i+1}\}$, by induction it holds

$$2 \sum_{i \in I'} (b_{i+1} - b_i) \leqslant b_r.$$

Then by i) it follows that either

$$2 \sum_{i \in I} (a_{i+1} - a_i) \leqslant 2 \sum_{i \in I'} (b_{i+1} - b_i) \leqslant b_r \leqslant a_r$$

or

$$2 \sum_{i \in I} (a_{i+1} - a_i) \leqslant 2 \sum_{i \in I'} (b_{i+1} - b_i) + 1 \leqslant b_r + 2 = a_r. \qquad \square$$

Lemma 4.3.2. *Suppose that G is a simple group of type B_r and let μ, ν be dominant weights such that $V(\nu) \subset V(\mu) \otimes V(\omega_r)^{\otimes n}$. If $\mu + \omega_r - \nu = \sum_{i=1}^r a_i \alpha_i$, then*

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_r.$$

Proof. It's enough to consider the case $n = 1$, the general case easily follows by induction. Denote $\mu = \sum m_i \omega_i$ and $\nu = \sum n_i \omega_i$. By the generalized Littlewood-Richardson rule [Na 93], there exists a sequence $(s_1, \ldots, s_r)$ with $s_i \in \{+, -\}$ such that

$$n_i = \begin{cases} m_i + 1 & \text{if } (s_i, s_{i+1}) = (+,-) \\ m_i & \text{if } s_i = s_{i+1} \\ m_i - 1 & \text{if } (s_i, s_{i+1}) = (-,+) \end{cases} \qquad \text{if } i < r$$

and

$$n_r = \begin{cases} m_r + 1 & \text{if } s_r = + \\ m_r - 1 & \text{if } s_r = - \end{cases}$$

On the other hand, we have

$$m_i + (a_{i+1} - a_i) = n_i + (a_i - a_{i-1}) \qquad \text{if } i < r$$

and

$$m_r = n_r - 1 - 2(a_r - a_{r-1}).$$

Let's show by induction on i that $a_i - a_{i-1} \geqslant 0$. Suppose that $i = r$. If $s_r = +$, then it follows $m_r = n_r - 1$ and we get $a_r - a_{r-1} = 0$, while if $s_r = -$, then it follows $m_r = n_r + 1$ and we get $a_r - a_{r-1} = 1$.

Suppose now that $i < r$.

- If $s_i = s_{i+1}$, then $n_i = m_i$ and we get $a_i - a_{i-1} = a_{i+1} - a_i \geqslant 0$ by the inductive hypothesis.

- If $(s_i, s_{i+1}) = (-,+)$, then $n_i = m_i - 1$ and we get $a_i - a_{i-1} = a_{i+1} - a_i + 1 > 0$ by the inductive hypothesis.

- Finally if $(s_i, s_{i+1}) = (+,-)$, then $n_i = m_i + 1$ and we get $a_i - a_{i-1} = a_{i+1} - a_i - 1$. If $s_j = -$ for every $j > i$, then we get $a_{i+1} - a_i = a_r - a_{r-1} = 1$ and the claim follows. Otherwise, if $k > i$ is the minimum such that $s_k = +$, then we get $a_{i+1} - a_i = a_k - a_{k-1} + 1 > 0$ by the inductive hypothesis and the claim follows. $\square$

Lemma 4.3.3. *Suppose that G is a simple group of type B_r and let μ, ν be dominant weights such that $V(\nu) \subset V(\mu) \otimes V(\omega_1)^{\otimes n_1} \otimes V(\omega_r)^{\otimes n_2}$. If $\mu + n_1\omega_1 + n_r\omega_r - \nu = \sum_{i=1}^r a_i\alpha_i$, then*

$$\sum_{i=1}^{r-1} |a_{i+1} - a_i| \leqslant a_1 + a_r.$$

Proof. It follows combining Lemma 4.3.1 and Lemma 4.3.2. □

Proposition 4.3.4. *Suppose that G is a simple group of type B_r and let λ be a dominant weight such that $\langle \lambda, \alpha^\vee \rangle$ is even. Suppose that μ, ν are dominant weights such that $V(\nu) \subset V(\mu) \otimes V(\lambda)^{\otimes n}$ and denote $\mu + n\lambda - \nu = \sum_{i=1}^r a_i\alpha_i$. Then the followings hold:*

i) If $\alpha_p \in \mathrm{Supp}(\lambda)$ is the first simple root, then $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_p$.

ii) If $\alpha_s, \alpha_t \in \mathrm{Supp}(\lambda)$ $(s < t)$ are such that $\alpha_i \notin \mathrm{Supp}(\lambda)$ for every $s < i < t$, then

$$\sum_{i=s}^{t-1} |a_i - a_{i+1}| \leqslant a_s + a_t.$$

iii) If $\alpha_q \in \mathrm{Supp}(\lambda)$ is the last simple root and if $I_q = \{i \geqslant q : a_i < a_{i+1}\}$, then

$$2 \sum_{i \in I_q} (a_{i+1} - a_i) \leqslant a_r.$$

Proof. It's enough to consider the case $n = 1$, the general case easily follows by induction.

i) We may assume that $p > 1$. Let $N > 0$ be such that $\lambda \leqslant N\omega_r$ with $\mathrm{Supp}_S(N\omega_r - \lambda) = \{\alpha_{p+1}, \ldots, \alpha_r\}$. By Theorem 4.1.11, the variety X_{ω_r} is normal; thus $\lambda \leqslant N\omega_r$ is trivial and there exists $m > 0$ such that

$$V(\lambda + (m-1)N\omega_r) \subset V(N\omega_r)^{\otimes m}.$$

By Lemma 4.1.14 applied to $V(\nu) \subset V(\mu) \otimes V(\lambda)$ it follows then

$$V(\nu + (m-1)N\omega_r) \subset V(\mu) \otimes V(\lambda + (m-1)N\omega_r) \subset V(\mu) \otimes V(\omega_r)^{\otimes mN}.$$

Denote $\sum_{i=1}^r b_i\alpha_i = \mu + N\omega_r - \nu$ the difference between the highest weight on the right and the highest weight on the left: then Lemma 4.3.2 shows that $b_1 \leqslant \ldots \leqslant b_r$. Since by the choice of N it follows $b_i = a_i$ for every $i \leqslant p$, the claim follows.

ii) Denote $\lambda_1 = \sum_{i=1}^s \langle \lambda, \alpha_i^\vee \rangle \omega_i$ and $\lambda_2 = \sum_{i=t}^r \langle \lambda, \alpha_i^\vee \rangle \omega_i$. Let $N_1 > 0$ be such that $\lambda_1 \leqslant N_1\omega_1$ with $\mathrm{Supp}_S(N_1\omega_1 - \lambda_1) = \{\alpha_1, \ldots, \alpha_{s-1}\}$ and let $N_2 > 0$ be such that $\lambda_2 \leqslant N_2\omega_r$ with $\mathrm{Supp}_S(N_2\omega_r - \lambda_2) = \{\alpha_{t+1}, \ldots, \alpha_r\}$. By Theorem 4.1.11, the variety $X_{\omega_1 + \omega_r}$ is normal; in particular it follows that $\lambda \leqslant N_1\omega_1 + N_2\omega_r$ is trivial and there exists $m > 0$ such that

$$V(\lambda + (m-1)(N_1\omega_1 + N_2\omega_r)) \subset V(N_1\omega_1 + N_2\omega_r)^{\otimes m}.$$

By Lemma 4.1.14 applied to $V(\nu) \subset V(\mu) \otimes V(\lambda)$ it follows then

$$\begin{aligned} V(\nu + (m-1)(N_1\omega_1 + N_2\omega_r)) \subset V(\mu) \otimes V(\lambda + (m-1)(N_1\omega_1 + N_2\omega_r)) \subset \\ \subset V(\mu) \otimes V(\omega_1)^{\otimes mN_1} \otimes V(\omega_r)^{\otimes mN_2}. \end{aligned}$$

Denote $\sum_{i=1}^r b_i\alpha_i = \mu + m\omega_r - \nu$ the difference between the highest weight on the right and the highest weight on the left: then Lemma 4.3.3 shows

$$\sum_{i=1}^{r-1} |b_i - b_{i+1}| \leqslant b_1 + b_r,$$

which in particular implies

$$\sum_{i=s}^{t-1} |b_i - b_{i+1}| \leqslant b_s + b_t.$$

Since by the choice of N_1 and N_2 it follows that $b_i = a_i$ for every $s \leqslant i \leqslant t$, the claim follows.

iii) We may assume that $q < r$, otherwise there is nothing to prove. Since $\langle \lambda, \alpha^\vee \rangle$ is even, there exists an integer $N > 0$ such that $\lambda \leqslant N\omega_1$ with $\mathrm{Supp}_S(N\omega_1 - \lambda) = \{\alpha_1, \ldots, \alpha_{q-1}\}$. Since $\alpha_r \notin \mathrm{Supp}_S(N\omega_1 - \lambda)$, Proposition 4.2.4 shows that $V(\lambda) \subset V(\omega_1)^{\otimes N}$, hence $V(\nu) \subset V(\mu) \otimes V(\lambda)$ implies

$$V(\nu) \subset V(\mu) \otimes V(\omega_1)^{\otimes N}.$$

Denote $\sum_{i=1}^r b_i\alpha_i = \mu + N\omega_1 - \nu$ the difference between the highest weight on the right and the highest weight on the left: then Lemma 4.3.1 shows that

$$2 \sum_{i \in I_q} (b_{i+1} - b_i) \leqslant b_r.$$

Since by the choice of N it follows that $b_i = a_i$ for every $i \geqslant q$, the claim follows. □

4.3.2 Combinatorial characterization of morphisms

We are now ready to state the main theorem of this section. From now on we will assume that $\alpha_r \notin \mathrm{Supp}(\lambda)$, i.e. that X_λ is not normal.

Definition 4.3.5. Suppose that $G = \mathrm{Spin}(2r+1)$. Suppose that $\nu \leqslant \mu \leqslant \lambda$ are non-trivial weights and set $\mu - \nu = \sum_{i=1}^r a_i\alpha_i$. Then we say that μ and ν are λ-*comparable* and we write $\nu \leqslant_\lambda \mu$ if following conditions are fulfilled:

(λ-C1) If $\alpha_p \in \mathrm{Supp}(\lambda)$ is the first simple root, then $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_p$.

(λ-C2) If $\alpha_s, \alpha_t \in \mathrm{Supp}(\lambda)$ $(s < t)$ are such that $\alpha_i \notin \mathrm{Supp}(\lambda)$ for every $s < i < t$, then

$$\sum_{i=s}^{t-1} |a_i - a_{i+1}| \leqslant a_s + a_t.$$

(λ-C3) If $\alpha_q \in \mathrm{Supp}(\lambda)$ is the last simple root and if $I_q = \{i \geqslant q \,:\, a_i < a_{i+1}\}$, then

$$2 \sum_{i \in I_q} (a_{i+1} - a_i) \leqslant a_r.$$

It is easy to see that $\leqslant_\lambda$ is a partial order on the set of non-trivial weights $\mu \leqslant \lambda$.

Theorem 4.3.6. *Suppose that* $G = \mathrm{Spin}(2r+1)$ *and let* $\Pi \subset \mathcal{X}(B)^+$ *be simple with maximal element* λ. *Then* $\Omega(\Pi) = \bigcup_{i=1}^m \Omega(\lambda, \mu_i)$, *where* $\mu_1, \ldots, \mu_m \in \Pi \smallsetminus \{\lambda\}$ *are the non-trivial elements which are maximal with respect to the partial order* $\leqslant_\lambda$.

Before to prove the theorem, we deduce some corollaries. Extend trivially $\leqslant_\lambda$ to a partial order relation on $\Pi^+(\lambda)$ by setting $\nu \leqslant_\lambda \mu$ if and only if $\mu = \nu$ or $\mu = \lambda$, for all trivial weights $\nu, \mu \in \Pi^+(\lambda)$.

Definition 4.3.7. If $\Pi \subset \mathcal{X}(B)^+$ is a simple subset with maximal element λ, denote

$$\Pi_{\mathrm{red}} = \{\mu \in \Pi \,:\, \mu \text{ is maximal w.r.t. } \leqslant_\lambda\}.$$

If $\Pi = \Pi_{\mathrm{red}}$, then Π is said to be *reduced*.

Theorem 4.3.8. *Suppose that* $G = \mathrm{Spin}(2r+1)$.

i) If $\Pi \subset \mathcal{X}(B)^+$ *is simple, then* $X_\Pi \simeq X_{\Pi_{\mathrm{red}}}$.

ii) Let $\Pi, \Pi' \subset \mathcal{X}(B)^+$ *be simple subsets with the same maximal element* λ. *Then* X_Π *dominates* $X_{\Pi'}$ *if and only if for every* $\mu' \in \Pi'$ *there exists* $\mu \in \Pi$ *such that* $\mu' \leqslant_\lambda \mu$.

In particular we get the following classification of the simple linear compactifications of $\mathrm{SO}(2r+1)$.

Corollary 4.3.9. *Simple linear compactifications of* $\mathrm{SO}(2r+1)$ *are classified by simple reduced subsets* $\Pi \subset \mathcal{X}(B)^+$ *up to equivalence.*

Let's prove now Theorem 4.3.6. First we will prove that, if $\nu \leqslant \lambda$ is a non-trivial weight such that $\Omega(\lambda, \nu) \subset \Omega(\Pi)$, then there exists a non-trivial weight $\mu \in \Pi$ such that $\nu \leqslant_\lambda \mu$ (Proposition 4.3.10). Then by following steps we will prove that if $\nu \leqslant_\lambda \mu$ are non trivial then $\Omega(\lambda, \nu) \subset \Omega(\lambda, \mu)$ (Proposition 4.3.11).

Proposition 4.3.10. *Suppose that* $\Pi \subset \mathcal{X}(B)^+$ *is simple with maximal element* λ *and let* $\nu \leqslant \lambda$ *be non-trivial. If* $\Omega(\lambda, \nu) \subset \Omega(\Pi)$ *then there exists* $\mu \in \Pi$ *such that* $\nu \leqslant_\lambda \mu$.

Proof. Let's treat the case $\Pi = \{\lambda, \mu, \mu'\}$, where $\mu \leqslant \lambda$ and $\mu' \leqslant \lambda$ are non-trivial dominant weights; the general case is analogous. Let $n, k, k' > 0$ with $n \geqslant k + k'$ be such that

$$V(\nu + (n-1)\lambda) \subset V(\mu)^{\otimes k} \otimes V(\mu')^{\otimes k'} \otimes V(\lambda)^{\otimes n-k-k'}. \tag{4.1}$$

Suppose that $k > 0$; then $\nu \leqslant \mu$ and denote $\mu - \nu = \sum_{i=1}^r a_i \alpha_i$. Proceeding as in Proposition 4.3.4, let's show that $\nu \leqslant_\lambda \mu$.

To show **(λ-C1)**, let $\pi = N\omega_r$ be such that $\lambda \leqslant \pi$ with $\mathrm{Supp}_S(\pi - \lambda) = \{\alpha_{p+1}, \ldots, \alpha_r\}$, where $\alpha_p \in \mathrm{Supp}(\lambda)$ is the first simple root. By Theorem 4.1.11, the variety X_π is normal; hence $\mu \leqslant \pi$ and $\mu' \leqslant \pi$ are trivial and there exist $m, m' > 0$ such that

$$V(\mu + (m-1)\pi) \subset V(\pi)^{\otimes m} \qquad \text{and} \qquad V(\mu' + (m'-1)\pi) \subset V(\pi)^{\otimes m'}.$$

By Lemma 4.1.14 applied to (4.1) it follows then

$$\begin{aligned} V(\nu + (n-1)\lambda + (k-1)(m-1)\pi + k'(m'-1)\pi) \subset \\ \subset V(\mu) \otimes V(\mu + (m-1)\pi)^{\otimes k-1} \otimes V(\mu' + (m'-1)\pi)^{\otimes k'} \otimes V(\lambda)^{\otimes n-k-k'} \subset \\ \subset V(\mu) \otimes V(\pi)^{\otimes (k-1)m + k'm'} \otimes V(\lambda)^{\otimes n-k-k'}. \end{aligned}$$

Denote $\sum_{i=1}^r b_i\alpha_i$ the difference between the highest weight on the right and the highest weight on the left: then

$$\sum_{i=1}^r b_i\alpha_i = \mu - \nu + (k+k'-1)(\pi - \lambda)$$

and Proposition 4.3.4 shows

$$b_1 \leqslant \ldots \leqslant b_p.$$

By the construction of π it follows that $b_i = a_i$ for every $i \leqslant p$, thus we get **(λ-C1)**.

To show **(λ-C2)**, let $\pi = N_1\omega_1 + N_r\omega_r$ be such that $\lambda \leqslant \pi$ and $\mathrm{Supp}_S(\pi - \lambda) = \{\alpha_1, \ldots, \alpha_{s-1}, \alpha_{t+1}, \ldots, \alpha_r\}$, where $\alpha_s, \alpha_t \in \mathrm{Supp}(\lambda)$ are such that $\alpha_i \notin \mathrm{Supp}(\lambda)$ for every $s < i < t$. By Theorem 4.1.11, the variety X_π is normal; hence $\mu \leqslant \pi$ and $\mu' \leqslant \pi$ are trivial and there exist $m, m' > 0$ such that

$$V(\mu + (m-1)\pi) \subset V(\pi)^{\otimes m} \qquad \text{and} \qquad V(\mu' + (m'-1)\pi) \subset V(\pi)^{\otimes m'}.$$

As in the previous case, by Lemma 4.1.14 applied to (4.1) it follows then

$$\begin{aligned} V(\nu + (n-1)\lambda + (k-1)(m-1)\pi + k'(m'-1)\pi) \subset \\ \subset V(\mu) \otimes V(\pi)^{\otimes (k-1)m + k'm'} \otimes V(\lambda)^{\otimes n-k-k'}. \end{aligned}$$

Denote $\sum_{i=1}^r b_i\alpha_i$ the difference between the highest weight on the right and the highest weight on the left: then

$$\sum_{i=1}^r b_i\alpha_i = \mu - \nu + (k+k'-1)(\pi - \lambda)$$

and Proposition 4.3.4 shows

$$\sum_{i=s}^{t-1} |b_i - b_{i+1}| \leqslant b_s + b_t.$$

By the construction of π it follows that $b_i = a_i$ for every $s \leqslant i \leqslant t$, thus we get **($\lambda$-C2)**.

To show **(λ-C3)**, let $N > 0$ be such that $\lambda \leqslant N\omega_1$ with $\mathrm{Supp}_S(N\omega_1 - \lambda) = \{\alpha_1, \ldots, \alpha_{q-1}\}$, where $\alpha_q \in \mathrm{Supp}(\lambda)$ is the last simple root. Denote $\lambda - \mu = \sum_{i=1}^r m_i\alpha_i$ and $\lambda - \mu' = \sum_{i=1}^r m'_i\alpha_i$. Since μ and μ' are dominant, we get

$$N \geqslant m_1 \geqslant m_2 \geqslant \ldots \geqslant m_r \qquad \text{and} \qquad N \geqslant m'_1 \geqslant m'_2 \geqslant \ldots \geqslant m'_r.$$

Hence $\mu \leqslant (N - m_r)\omega_1$ and $\mu' \leqslant (N - m'_r)\omega_1$ are trivial. Since $\nu \leqslant \mu$, the latter is non-zero. As well, if $k_1 \neq 0$, we may assume that μ' is non-zero. Therefore Proposition 4.2.4 shows

$$V(\mu) \subset V(\omega_1)^{\otimes N - m_r} \qquad \text{and} \qquad V(\mu') \subset V(\omega_1)^{\otimes N - m'_r}$$

and by (4.1) we get

$$V(\nu + (n-1)\lambda) \subset V(\mu) \otimes V(\omega_1)^{\otimes (k-1)(N-m_r)+k'(N-m'_r)} \otimes V(\lambda)^{\otimes n-k-k'}.$$

Denote $\sum_{i=1}^r b_i\alpha_i$ the difference between the highest weight on the right and the highest weight on the left: then

$$\sum_{i=1}^r b_i\alpha_i = \mu - \nu + (k+k'-1)(N\omega_1 - \lambda) - ((k-1)m_r + k'm'_r)\omega_1$$

and Proposition 4.3.4 shows

$$2\sum_{i\in I_q}(a_{i+1}-a_i) = 2\sum_{i\in I_q}(b_{i+1}-b_i) \leqslant b_r \leqslant a_r. \qquad \square$$

Following proposition shows that it holds as well the converse of Proposition 4.3.10: if $\nu \leqslant_\lambda \mu$ then $\Omega(\lambda,\nu) \subset \Omega(\lambda,\mu)$. It follows that there exists an equivariant morphism $X_{\lambda,\mu} \to X_{\lambda,\nu}$ if and only if $\nu \leqslant_\lambda \mu$.

Proposition 4.3.11. *Let λ be a dominant weight and suppose $\nu \leqslant \mu \leqslant \lambda$ are non-trivial weights. If $\nu \leqslant_\lambda \mu$, then $\Omega(\lambda,\nu) \subset \Omega(\lambda,\mu)$.*

If $I \subset S$ is a set of simple roots, define its *border* as follows

$$\partial I = \{\alpha \in S \smallsetminus I \,:\, \exists \beta \in I \text{ s. t. } \langle \alpha, \beta^\vee \rangle \neq 0\}.$$

Define moreover the *closure* of I as $\overline{I} = I \cup \partial I$ and the *interior* of I as $I^\circ = I \smallsetminus \overline{S \smallsetminus I}$.

In following lemmas we will prove previous proposition first assuming $\mathrm{Supp}_S(\mu - \nu) \cap \mathrm{Supp}(\lambda) = \{\alpha_q\}$ and then assuming $\mathrm{Supp}_S(\mu-\nu) \cap \mathrm{Supp}(\lambda)^\circ = \varnothing$. Then we will deduce a general proof.

Notice that up to consider simple subsets equivalent to $\{\lambda,\mu\}$ and to $\{\lambda,\nu\}$, we may assume (and we will assume) that $\mathrm{Supp}(\lambda) \subset \mathrm{Supp}(\mu) \cap \mathrm{Supp}(\nu)$.

Lemma 4.3.12. *Let λ be a dominant weight and suppose $\nu \leqslant \mu \leqslant \lambda$ are non-trivial weights such that $\mathrm{Supp}_S(\mu-\nu) \cap \mathrm{Supp}(\lambda) = \{\alpha_q\}$, where α_q is the last simple root occurring in $\mathrm{Supp}(\lambda)$. If $\nu \leqslant_\lambda \mu$, then $\Omega(\lambda,\nu) \subset \Omega(\lambda,\mu)$.*

Proof. Denote $\mu - \nu = \sum_{i=1}^r a_i\alpha_i$. Since $\mu \leqslant \lambda$ and $\nu \leqslant \lambda$ are non trivial, it follows by Theorem 4.2.2 that $a_{r-1} = a_r$ is an even integer. We will show the claim by induction on the partial order $\leqslant_\lambda$.

Case 1. Suppose that $a_r = 0$. In particular it follows $a_{r-1} = 0$ as well, and **(λ-C1)**, **(λ-C2)** and **(λ-C3)** imply

$$a_1 \leqslant \ldots \leqslant a_{q-1} \leqslant a_q \geqslant a_{q+1} \geqslant \ldots \geqslant a_{r-1} = a_r = 0.$$

Denote $k_0 < q$ the minimum such that $a_{k_0} \neq 0$ and $k_1 > q$ the maximum such that $a_{k_1} \neq 0$. Set

$$\nu' = \nu + \sum_{i=k_0}^{k_1} \alpha_i = \nu - \omega_{k_0-1} + \omega_{k_0} + \omega_{k_1} - \omega_{k_1-1}.$$

Notice that ν' is dominant and $\nu < \nu' \leqslant \mu$: indeed by the definitions of k_0 and k_1 it follows

$$\langle \nu, \alpha^\vee_{k_0-1} \rangle = \langle \mu, \alpha^\vee_{k_0-1} \rangle + a_{k_0} > 0 \qquad \text{and} \qquad \langle \nu, \alpha^\vee_{k_1+1} \rangle = \langle \mu, \alpha^\vee_{k_1+1} \rangle + a_{k_1} > 0.$$

If L is the Levi subgroup associated to the subset of simple roots $\{\alpha_{k_0}, \ldots, \alpha_{k_1}\}$, since $\nu' - \nu$ is the highest long root, Corollary 4.1.15 shows that $V(\nu + \lambda) \subset V(\nu') \otimes V(\lambda)$, thus $\Omega(\nu, \lambda) \subset \Omega(\nu', \lambda)$. Therefore $\nu' \leqslant \lambda$ is non-trivial, moreover it is easily seen that $\nu \leqslant_\lambda \nu' \leqslant_\lambda \mu$. Thereofore by induction on the partial order $\leqslant_\lambda$ we get $\Omega(\lambda, \nu') \subset \Omega(\lambda, \mu)$, which implies the thesis.

Case 2. Suppose that $I_q = \varnothing$ and that $a_r \neq 0$. Then **(λ-C3)** implies

$$a_q \geqslant a_{q+1} \geqslant \ldots \geqslant a_r.$$

Denote $k_0 < q$ the minimum such that $a_{k_0} \neq 0$ and set

$$\nu' = \nu + \sum_{i=k_0}^{q} \alpha_i + 2 \sum_{i=q+1}^{r} \alpha_i = \begin{cases} \nu - \omega_{k_0 - 1} + \omega_{k_0} - \omega_q + \omega_{q+1} & \text{if } q < r-1 \\ \nu - \omega_{k_0 - 1} + \omega_{k_0} - \omega_{r-1} + 2\omega_r & \text{if } q = r \end{cases}$$

Notice that ν' is dominant and $\nu < \nu' \leqslant \mu$: indeed $\alpha_q \in \mathrm{Supp}(\lambda) \subset \mathrm{Supp}(\nu)$ and by the definition of k_0 it follows

$$\langle \nu, \alpha^\vee_{k_0 - 1} \rangle = \langle \mu, \alpha^\vee_{k_0 - 1} \rangle + a_{k_0} > 0.$$

If L is the Levi subgroup associated to the subset of simple roots $\{\alpha_{k_0}, \ldots, \alpha_r\}$, then by Lemma 4.2.1 iii) we get $V_L(2\omega_q) \subset V_L(\omega_{k_0} + \omega_{q+1}) \otimes V_L(\omega_q)$. Lemma 4.1.14 and Lemma 4.1.12 show then $V(\nu + \lambda) \subset V(\nu') \otimes V(\lambda)$, thus $\Omega(\nu, \lambda) \subset \Omega(\nu', \lambda)$ and $\nu' \leqslant \lambda$ is non-trivial. Moreover it is easily seen that $\nu \leqslant_\lambda \nu' \leqslant_\lambda \mu$. Consider $\nu' \leqslant_\lambda \mu$: then either it falls in Case 1 or it still falls in Case 2. Therefore by induction on the partial order $\leqslant_\lambda$ we get then $\Omega(\lambda, \nu') \subset \Omega(\lambda, \mu)$, which implies the thesis.

Case 3. Suppose that $I_q \neq \varnothing$; in particular this implies $a_r \geqslant 2$. Notice that $a_j \neq 0$ for every $j \geqslant q$: indeed otherwise **(λ-C3)** implies $a_r = 0$ since

$$2a_r \leqslant 2 \sum_{i \in I_j} (a_{i+1} - a_i) \leqslant 2 \sum_{i \in I_q} (a_{i+1} - a_i) \leqslant a_r.$$

Let $k_0 < q$ the minimum such that $a_{k_0} \neq 0$ and let $k_1 \in I_q$ be the maximum such that $k_1 - 1 \notin I_q$: therefore either $k_1 = q$ and

$$a_q < \ldots < a_j \geqslant \ldots \geqslant a_r \qquad \text{with } q \leqslant j \leqslant r$$

or

$$a_{k_1 - 1} \geqslant a_{k_1} < \ldots < a_j \geqslant \ldots \geqslant a_r \qquad \text{with } q < k_1 < j \leqslant r.$$

Set

$$\nu' = \nu + \sum_{i=k_0}^{k_1} \alpha_i + 2 \sum_{i=k_1+1}^{r} \alpha_i = \nu - \omega_{k_0 - 1} + \omega_{k_0} - \omega_{k_1} + \omega_{k_1 + 1}.$$

Notice that ν' is dominant and $\nu < \nu' \leqslant \mu$: indeed $\alpha_q \in \mathrm{Supp}(\lambda) \subset \mathrm{Supp}(\nu)$ and by the definition of k_0 it follows

$$\langle \nu, \alpha^\vee_{k_0 - 1} \rangle = \langle \mu, \alpha^\vee_{k_0 - 1} \rangle + a_{k_0} > 0,$$

while by the definition of k_1 it follows that either $\alpha_{k_1} = \alpha_q \in \mathrm{Supp}(\lambda) \subset \mathrm{Supp}(\nu)$ or $k_1 > q$ and

$$\langle \nu, \alpha^\vee_{k_1} \rangle = \langle \mu, \alpha^\vee_{k_1} \rangle + a_{k_1 - 1} + a_{k_1 + 1} - 2a_{k_1} > 0.$$

If L is the Levi subgroup associated to the subset of simple roots $\{\alpha_{k_0}, \ldots, \alpha_r\}$, then by Lemma 4.2.1 iii) we get $V_L(\omega_q + \omega_{k_1}) \subset V_L(\omega_{k_0} + \omega_{k_1+1}) \otimes V_L(\omega_q)$. Lemma 4.1.14 and Lemma 4.1.12 show then $V(\nu+\lambda) \subset V(\nu') \otimes V(\lambda)$, thus $\Omega(\nu, \lambda) \subset \Omega(\nu', \lambda)$ and $\nu' \leqslant \lambda$ is non-trivial.

Notice that $\nu \leqslant_\lambda \nu' \leqslant_\lambda \mu$: indeed, if $\mu - \nu' = \sum_{i=1}^r a'_i \alpha_i$ and $I'_q = \{i \,:\, a'_i < a'_{i+1}\}$, then we get

$$2\sum_{i \in I'_q} (a'_{i+1} - a'_i) = 2\left(\sum_{i \in I_q} (a_{i+1} - a_i)\right) - 2 \leqslant a_r - 2 = a'_r.$$

Therefore by induction on the partial order $\leqslant_\lambda$ we reduce to Case 2 and we get $\Omega(\lambda, \nu') \subset \Omega(\lambda, \mu)$, which implies the thesis. □

Lemma 4.3.13. *Let λ be a dominant weight and suppose $\nu \leqslant \mu \leqslant \lambda$ are non-trivial weights such that $\mathrm{Supp}_S(\mu - \nu) \cap \mathrm{Supp}(\lambda)^\circ = \varnothing$. If $\nu \leqslant_\lambda \mu$, then $\Omega(\lambda, \nu) \subset \Omega(\lambda, \mu)$.*

Proof. Denote $\mu - \nu = \sum_{i=1}^r a_i \alpha_i$ and denote $\mathrm{Supp}(\lambda) = \{\alpha_{i_1}, \ldots, \alpha_{i_n}\}$. Set k_0 the minimum such that $a_{k_0} \neq 0$ and set

$$j = \min\{s \leqslant n \,:\, k_0 \leqslant i_s\}$$

If $j = n$, then $\mathrm{Supp}_S(\mu - \nu) \cap \mathrm{Supp}(\lambda) = \{\alpha_q\}$ and the claim has been proved in Lemma 4.3.12. Suppose $j < n$, by induction on the partial order $\leqslant_\lambda$ we will reduce to the case $j = n$.

Conditions **(λ-C1)** and **(λ-C2)** imply that $a_{k_0} \leqslant a_{k_0+1} \leqslant \ldots \leqslant a_{i_j}$. Set

$$k_1 = \begin{cases} \max\{k < i_{j+1} \,:\, a_k \geqslant a_{k+1}\} & \text{if } a_s \neq 0 \text{ for all } i_j < s < i_{j+1} \\ \min\{k < i_{j+1} \,:\, a_{k+1} = 0\} & \text{otherwise} \end{cases}$$

and set

$$\nu' = \nu + \alpha_{k_0} + \ldots + \alpha_{k_1} = \nu - \omega_{k_0-1} + \omega_{k_0} + \omega_{k_1} - \omega_{k_1-1}.$$

Notice that ν' is dominant and $\nu < \nu' \leqslant \mu$: indeed by the definitions of k_0 and k_1 we get

$$\langle \nu, \alpha^\vee_{k_0-1} \rangle = \langle \mu, \alpha^\vee_{k_0-1} \rangle + a_{k_0} > 0,$$
$$\langle \nu, \alpha^\vee_{k_1+1} \rangle = \langle \mu, \alpha^\vee_{k_1+1} \rangle + a_{k_1} + a_{k_1+2} - 2a_{k_1+1} > 0.$$

If L is the Levi subgroup associated to the subset of simple roots $\{\alpha_{k_0}, \ldots, \alpha_{k_1}\}$, since $\nu' - \nu$ is the highest long root, Corollary 4.1.15 shows that $V(\nu + \lambda) \subset V(\nu') \otimes V(\lambda)$, thus $\Omega(\nu, \lambda) \subset \Omega(\nu', \lambda)$. Therefore $\nu' \leqslant \lambda$ is non-trivial.

Notice that $\nu \leqslant_\lambda \nu' \leqslant_\lambda \mu$: indeed, if $\mu - \nu' = \sum_{i=1}^r a_i \alpha_i$, then **($\lambda$-C1)** and **($\lambda$-C3)** are straightforward, while **(λ-C2)** follows by

$$\sum_{i=i_j}^{i_{j+1}-1} |a'_{i+1} - a'_i| = \sum_{i=i_j}^{i_{j+1}-1} |a_{i+1} - a_i| - 1 \leqslant a_{i_j} + a_{i_{j+1}} - 1 = a'_{i_j} + a'_{i_{j+1}}.$$

Therefore proceeding inductively on the partial order $\leqslant_\lambda$, we reduce to a weight which satisfies the hypotheses of Lemma 4.3.12. □

We are now able to prove Proposition 4.3.11 in full generality.

Proof of Proposition 4.3.11. If $\mu - \nu = \sum a_i \alpha_i$, denote

$$\mu_0 = \nu + \sum_{\alpha_i \in S \smallsetminus \mathrm{Supp}(\lambda)^\circ} a_i \alpha_i.$$

By construction, if $\langle \mu_0, \alpha^\vee \rangle < 0$, then $\alpha \in \mathrm{Supp}(\lambda) \subset \mathrm{Supp}(\nu)$: therefore, μ_0 is dominant and $\nu \leqslant_\lambda \mu_0 \leqslant_\lambda \mu$.

Since $\mathrm{Supp}_S(\mu_0 - \nu) \cap \mathrm{Supp}(\lambda)^\circ = \varnothing$, by Lemma 4.3.13 we get $\Omega(\lambda, \nu) \subset \Omega(\lambda, \mu_0)$.

If $\alpha \in \mathrm{Supp}_S(\mu - \mu_0) \subset \mathrm{Supp}(\lambda)^\circ$, consider $\mu_1 = \mu_0 + \alpha$: up to consider equivalent subsets, we may assume that μ_1 is dominant. Denote L the Levi subgroup whose unique simple root is α. Since it is the highest root of the associated root system, by Lemma 4.1.15 we get $V(\mu_0 + \lambda) \subset V(\mu_1) \otimes V(\lambda)$: in particular this shows the inclusion $\Omega(\lambda, \mu_0) \subset \Omega(\lambda, \mu_1)$. Proceeding inductively root by root we get $\Omega(\lambda, \mu_0) \subset \Omega(\lambda, \mu)$, which implies the claim. □

4.4 The case G_2

Lemma 4.4.1. *Suppose that G is a simple group of type G_2.*

i) $V(\omega_1 + \omega_2) \subset V(2\omega_1) \otimes V(\omega_2)$

ii) $V(4\omega_1 + 6\omega_2) \subset V(3\omega_2)^{\otimes 3}$

Proposition 4.4.2. *Suppose G is a simple group of type G_2 and let λ be a dominant weight with* $\mathrm{Supp}(\lambda) = \{\alpha_2\}$. *Then a dominant weight $\mu \leqslant \lambda$ is trivial if and only if $\lambda - \mu \neq \alpha_1 + \alpha_2$.*

Proof. If $\lambda - \mu = \alpha_1 + \alpha_2$, then $\mu = \lambda^{\mathrm{lb}}$ is the little brother of λ: therefore by Theorem 4.1.11 μ is non-trivial.

Suppose conversely that $\mu \leqslant \lambda$ is a dominant weight and that $\lambda - \mu \neq \alpha_1 + \alpha_2$; let's show that $\mu - \lambda \in \Omega(\lambda)$. By Lemma 4.1.12 we may assume that both a_1 and a_2 are non-zero.

Case 1. Suppose that $a_1 = 1$; then it follows $a_2 > 1$. Since μ is dominant, it follows that $\langle \mu, \alpha_1^\vee \rangle \geqslant 3a_2 - 2 \geqslant 4$ and that $\langle \lambda, \alpha_2^\vee \rangle \geqslant 2a_2 - 1 \geqslant 3$. Denote

$$\mu' = \mu + \alpha_1 + 2\alpha_2 = \mu - 4\omega_1 + 3\omega_2 :$$

then by Lemma 4.4.1 ii) together with Lemma 4.1.14 it follows that $V(\mu + 2\lambda) \subset V(\mu') \otimes V(\lambda)^{\otimes 2}$. Since $\mathrm{Supp}_S(\lambda - \mu') \subset \{\alpha_2\}$, it follows that $\mu' \leqslant \lambda$ is non-trivial and we get the claim.

Case 2. Suppose that $a_1 > 1$. Denote b_1 the maximum integer such that $2b_1 \leqslant a_1$ and denote

$$\mu' = \mu + b_1\omega_1 = \mu + 2b_1\alpha_1 + b_1\alpha_2.$$

Since μ is dominant we get $a_2 > b_1$; therefore $\mu \leqslant \mu' \leqslant \lambda$ and by Lemma 4.4.1 i) together with Lemma 4.1.14 it follows that $V(\mu + \lambda) \subset V(\mu') \otimes V(\lambda)$.

If a_1 is even, then $\mathrm{Supp}_S(\lambda - \mu') = \{\alpha_2\}$ and we are done, while if $a_2 - b_1 > 1$ then the claim follows since $\mu' \leqslant \lambda$ falls in case 1.

Therefore we are reduced to the case $\lambda - \mu' = \alpha_1 + \alpha_2$. Then $a_1 = 2a_2 - 1$ and by $\langle \mu, \alpha_1^\vee \rangle = 3a_2 - 2a_1 \geqslant 0$ we get $\lambda - \mu = 3\alpha_1 + 2\alpha_2 = \omega_2$. Since ω_2 is the highest long root, it follows then $V(\omega_2) \subset V(\omega_2)^{\otimes 2}$ and Lemma 4.1.14 implies $V(\mu + \lambda) \subset V(\lambda)^{\otimes 2}$. □

Corollary 4.4.3. *Suppose G is a simple group of type G_2. Then the adjoint group G_{ad} admits a unique non-normal simple projective embedding, namely X_{ω_2}.*

4.5 The symplectic case: the coordinate ring of X_λ

In the following conjecture we give a description of the coordinate ring $\Bbbk[X_\lambda^\circ]$, where G is a simple group of type C_r. The conjecture has been checked in many cases with the aid of the software [LiE 07].

Conjecture 4.5.1. *Suppose G is a simple group of type C_r. Let λ be a dominant weight and set α_q the last simple root which occurs in* $\mathrm{Supp}(\lambda)$ *with $0 \leqslant q < r$. Let $\mu \leqslant \lambda$ be a dominant weight, write $\lambda - \mu = \sum_{i=1}^r a_i \alpha_i$ and set $a_0 = 0$. Then $\mu \leqslant \lambda$ is non-trivial if and only if*

$$0 < a_r < r - q \qquad \textit{and} \qquad a_{r-a_r-1} < a_{r-a_r} < \ldots < a_{r-1} < 2a_r.$$

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