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Dottorato in Matematica, Roma «La Sapienza» (2010).

<http://www.bdim.eu/item?id=tesi_2010_FoschiRachele_1>

SAPIENZA
UNIVERSITÀ DI ROMA

Evolution of conditional dependence of residual lifetimes

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Ph.D. Thesis in Mathematics

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ACADEMIC YEAR 2008 - 2009
JUNE 2010

Contents

Introduction

Multivariate systems constitute a very widespread subject, studied from different points of view: mechanical, analytical, stochastic. By multivariate systems, we mean models consisting of a certain number of objects or units, $D_1, \ldots, D_n$, connected one another in some way. In the stochastic approach, a multivariate system consists in n random variables (= r.v.'s) $T_1, \ldots, T_n$, having a certain joint distribution

$$F(x_1, \ldots, x_n) = \mathbb{P}(T_1 < x_1, \ldots, T_n < x_n),$$

marginal laws $G_1, \ldots, G_n$ and a certain dependence structure among them. The dependence structure reflects the connections existing among the different units. The study of dependence arises in several applied fields. In Reliability Theory, the dependence among the components of a random vector represents the dependence among the components of a system; in a financial frame, the variables may represent the prices of the different assets in a portfolio; in an insurance context, the dependence can be studied, for example, among life-lengths of insured people or among the claims of other kinds of insurance.

In view of these possible applications, we will study in particular non-negative r.v.'s.

The analysis of dependence is a classical subject in applied probability. Recently, studying dependence in risky situations became of particular interest, when, for example, a firm in a market invoices a too small gain or a component of an electric circuit is functioning since an exceedingly long time.

Such events are "risky" in the sense that they are close to the "catastrophic" events of the default of the firm or of the failure of the component, respectively. We call such events, when a random quantity falls below a small threshold or exceeds a large one, *extreme events*.

The interest in studying dependence, when an extreme event occurs, finds a first motivation in Risk Theory: when we study, for example, the optimal diversification in the composition of a portfolio or of another system, we have to take into account the dependence among the different units, that can heavily affect the diversification. A change of the dependence structure, due to the observation of an extreme event, forces us to change our policy in

the choice of the portfolio composition. However, there are several examples in practical situations, where changes of the dependence structure, due to the occurrence of an extreme event, have to be taken into account. In fact, a decision or a strategy to be adopted varies on the basis of the dependence structure of the system.
In view of a mathematical tractability of the model, it is often assumed the hypothesis of independence among r.v.'s, considering independence a good approximation of the actual dependence structure of the model. Due to the afore-mentioned changes, the dependence structure might move too much away from being well approximated by independence. Thus, keeping the assumption of independence might be heavily misleading, when a decision has to be taken also on the basis of the dependence structure.

The phenomenon of the change of dependence, given the occurrence of an extreme event, is known as *tail dependence* (see e.g. [26, 58, 59, 72] and references therein).

Thus, in studying tail dependence, we are conditioning on extreme events. A further step is considering that defaults of firms or failures of units may happen within the time of observation.

A default can be seen as the limit of an extreme event; considering again the examples above, a firm may invoice a small gain and, when it becomes 0, the firm fails, or a component may work for a long time, but, if the time goes to infinity, the component defaults within the time of observation.

When we deal with $T_1, \dots, T_n$, non-negative random variables, it is natural to interpret them as waiting times to n (stochastically dependent) top events, such as the failures of components $D_1, \dots, D_n$ operating in a same system or the defaults of firms in a same market (of course, several different interpretations of interest are possible). A system of this kind, where $T_1, \dots, T_n$ may be lifetimes or times to default, is called a *survival model*. Heuristically speaking, the conditioning events, we typically consider, will be of the kind

$$\{T_1 = t_1, \dots, T_k = t_k, T_{k+1} > t, \dots, T_n > t\}.$$

We will speak of *longitudinal observations of lifetimes*. This kind of events represents the failure at times $t_1, \dots, t_k$ of the units $D_1, \dots, D_k$ respectively and the survival of the other ones till time t.

The possibility of defaults has been considered in studying the evolution of the joint distribution of $(T_1, \dots, T_n)$ or of the survival function, but its implications on the evolution of dependence have not been taken into account.

When the default of a unit causes a decrease in the joint survival probability of residual lifetimes of the remaining units, we speak of *default contagion* (see in particular [72] and references cited therein).

The phenomenon of default contagion is defined by means of stochastic comparisons of probability distributions.

Under a different language, specific notions related with this phenomenon have already been introduced and studied, in the past, within the frame of Reliability Theory (see e.g. [6, 83]). Such notions were formulated in terms of different stochastic orderings and are related to corresponding notions of dependence (see [57, 78]). In this frame, inequalities between conditional probabilities are studied, giving origin to properties of dependence or also of ageing. To this purpose, we notice that stochastic orders constitute the natural language for our analysis (for a complete treatment, see [82]). We will use, in this thesis, both univariate and multivariate notions of stochastic orders, we recall and briefly discuss in Chapter 1.

Tail dependence and *Default contagion* emerge then as two subjects of interest in multivariate survival models and are often related with the analysis of non-negative random variables.

It can be of interest studying the changes in the dependence structure, due to the entire flow of information and not only to extreme events. The approach of considering the conditioning event as a level of information about the environment provides the natural setting for those applications in which different information levels are considered and compared, as in the problems considered here.

Both tail dependence and default contagion are included in the more general subject of the evolution of dependence.

However, in the literature, tail dependence and default contagion are studied separately and by means of different tools. In particular, tail dependence is studied without considering the possibility of defaults; on the other hand, default contagion is described as concerning the evolution of survival functions only at certain random times (*default times*).

One of our purposes is providing a unifying setting for simultaneously studying the two phenomena. In this context, we aim at studying implications or connections existing between default contagion and evolution of dependence. We give results in the particular case $T_1, \ldots, T_n$ are conditionally independent and identically distributed (= conditionally i.i.d.).

Very often, notions of stochastic dependence are conveniently described in terms of copulas (see e.g. [57, 78]).

As a mathematical object, a n-variate copula C can be seen as the restriction to $[0,1]^n$ of a n-variate distribution function with uniform margins. Therefore it obviously satisfies some boundary and positivity conditions (see Section 1.3 below).

In accordance with Sklar's Theorem ([84]), one defines the *connecting copula* associated with a probability distribution F as

$$C(u_1, \ldots, u_n) = F\left(G_1^{-1}(u_1), \ldots, G_n^{-1}(u_n)\right).$$

In view of this definition, the connecting copula captures the dependence of a random vector apart from its marginal behaviour (see in particular Theorem 1.3.6).

Copulas are a natural and consistent way of describing dependence; in fact, they are implicitly contained in every multivariate distribution and do not depend on the marginal distributions of the random vector. This fact has the advantage of permitting us to decompose the study in two different parts: the marginal part and the dependence structure, where copula theory may be applied. For these reasons, the use of copulas has grown in popularity in the past years and now copulas are a well-established tool for working with multivariate probability distributions.

When we deal with non-negative variables, however, it is more natural describing a multivariate model in terms of the joint survival function

$$\overline{F}(x_1, \ldots, x_n) = \mathbb{P}(T_1 > x_1, \ldots, T_n > x_n).$$

The univariate margins will be denoted by $\overline{G}_1, \ldots, \overline{G}_n$. In this case, the dependence structure of the model can be conveniently described by its *survival copula*

$$\hat{C}(u_1, \ldots, u_n) = \overline{F}\left(\overline{G}_1^{-1}(u_1), \ldots, \overline{G}_n^{-1}(u_n)\right).$$

The analysis of time evolution of survival copulas under longitudinal observations of lifetimes is an interesting subject in several fields of applied probability and this thesis is devoted to develop and broaden this topic. This thesis has been motivated by a study of the evolution of survival copulas in both the cases characterized by occurrence or non-occurrence of defaults.

Let us then start with a brief presentation of our results.

We recall the definition of some dependence properties and then provide conditions for their preservation on the family of survival copulas $\{\hat{C}_t\}_{t\geq 0}$, where

$$\hat{C}_t(u_1, \ldots, u_n) = \overline{F}_t\left((\overline{G}_t^{(1)})^{-1}(u_1), \ldots, (\overline{G}_t^{(n)})^{-1}(u_n)\right)$$

is the copula of the r.v.'s $T_1 - t, \ldots, T_n - t$ conditionally on the observation $T_1 > t, \ldots, T_n > t$. It can be shown that

$$\hat{C}_t(u_1, \ldots, u_n) = \frac{\hat{C}[\overline{G}_1((\overline{G}_t^{(1)})^{-1}(u_1) + t), \ldots, \overline{G}_n((\overline{G}_t^{(n)})^{-1}(u_n) + t)]}{\hat{C}(\overline{G}_1(t), \ldots, \overline{G}_n(t))}. \quad (1)$$

The study of the behaviour of dependence, and in particular of the asymptotic behaviour, can be conducted from two different points of view: the first one aims at computing the limit copula of the family ([59, 26]); the second one aims at describing the limit behaviour from a qualitative point of view ([35]).

In such a qualitative approach, the notion of *hyper-property* emerged in [35] as a natural concept. Hyper-dependence properties have been studied for the family of survival copulas $\{\hat{C}_t\}_{t\geq 0}$ or, analogously, for the transformed family $\{C_z\}_{z\in(0,1]}$.

A hyper-dependence property of a copula C corresponds, by definition, to a dependence property of the whole family of threshold copulas $\{C_z\}_{z\in(0,1]}$, associated with C. We have studied relationships among different hyper-properties and among properties and hyper-properties.

We study hyper-properties only in the case of non-occurrence of defaults. Another topic we develop is the monotonicity in t of the family $\{\hat{C}_t\}_{t\geq 0}$, with respect to some partial order on the set of copulas, we will specify later on. Monotonicity of a family of threshold copulas is linked to tail dependence. We provide conditions for increasingness or decreasingness of the map $t \to \hat{C}_t$ in a unified frame, fitting well both to the case of occurrence of defaults and to the case of non-occurrence.

A further formalization is needed, in order to consider the occurrence of defaults among the units.

With the vector $(T_1, \ldots, T_n)$, we associate the counting process (with a finite number of jumps)

$$N_t = \sum_{i=1}^{n} \mathbf{1}_{\{T_i \leq t\}}$$

(see e.g [22, 62] for a systematic treatment of counting processes and point processes). The jump times of $\{N_t\}_{t\geq 0}$ are the order statistics $T_{(1)}, \ldots, T_{(n)}$ of $(T_1, \ldots, T_n)$. We assume $\{N_t\}_{t\geq 0}$ to be a simple counting process, meaning that two jumps occur at the same time with probability 0. This is in particular implied by the hypothesis that the joint distribution of lifetimes is absolutely continuous. Therefore, we will assume $(T_1, \ldots, T_n)$ to admit a joint density function f.
The internal history of $\{N_t\}_{t\geq 0}$, that is the filtration generated by the process, is denoted by $\{\mathcal{F}_t\}_{t\geq 0}$.

We also assume, from this point on, $T_1, \ldots, T_n$ to be exchangeable. For a basic mathematical treatment about exchangeability, see e.g. [1, 29, 31]. For what concerns in particular exchangeability of non-negative random variables, see also [86]. The main reason for such an assumption is that it allows us to analyze conceptual points related to the role of information in the analysis of dependence, picking off those aspects concerning the specific units of the system. Also the fact that we are actually interested in studying the order statistics justifies the exchangeability assumption. In fact, given any random vector $(T_1, \ldots, T_n)$, it is always possible to trace back to an exchangeable one $(\widetilde{T}_1, \ldots, \widetilde{T}_n)$ having the same order statistics $T_{(1)}, \ldots, T_{(n)}$ (see e.g. [52, 86]).

The exchangeability assumption embodies the idea that the units $D_1, \ldots,$ D_n are similar one another, at least as far as our state of information is concerned: we expect $D_1, \ldots,$ D_n have different performances and hence that $T_1, \dots, T_n$ take different values, but we have no reason to suppose they follow different probability laws.

Conditionally on the observed history $\mathcal{F}_t$, it is useful for our purposes to consider the ordered residual lifetimes of the units surviving at time t, i.e. the random variables

$$(T_{(k+1)} - t, \dots, T_{(n)} - t) | \mathcal{F}_t.$$

Notice that assuming $(T_1, \dots, T_n)$ absolutely continuous makes easier the analysis of the conditional distributions.

In order to deal, at any t, with exchangeable random variables, we define the following vector $\mathbf{X}_t \equiv (X_t^1, \dots, X_t^{n-k})$ of *exchangeable residual lifetimes.*

Definition 0.0.1. *The exchangeable residual lifetimes of* $(T_1, \dots, T_n)$ *at time* t *are the exchangeable random variables* $X_t^1, \dots, X_t^{n-k}$ *admitting* $(T_{(k+1)} - t, \dots, T_{(n)} - t) | \mathcal{F}_t$ *as order statistics, where we are assuming* $k = N_t$.

Concerning the distribution of $(X_t^1, \dots, X_t^{n-k})$, we put

$$\overline{F}_t \left(x_1, \dots, x_{n-k} \right) = \mathbb{P} \left(X_t^1 > x_1, \dots, X_t^{n-k} > x_{n-k} | \mathcal{F}_t \right); \tag{2}$$

$$\overline{G}_t \left(x \right) = \overline{F}_t \left(x, 0, \dots, 0 \right) = \mathbb{P} \left(X_t^1 > x | \mathcal{F}_t \right).$$

The *survival copula* of the random vector $\left(X_t^1, \dots, X_t^{n-k} \right)$ is

$$\hat{C}_t(u_1, \dots, u_{n-k}) = \overline{F}_t \{ \overline{G}_t^{-1} \left(u_1 \right), \dots, \overline{G}_t^{-1} \left(u_{n-k} \right) \}. \tag{3}$$

In the literature, several papers have considered the evolution of the family of survival functions $\{\overline{F}_t\}_{t \geq 0}$, both when the observed history consisted in survivals only and in presence of defaults (see e.g. [4, 5, 6, 7, 9, 19, 26, 30, 79]).

As a special aspect of our analysis, on the contrary, we are interested in studying the evolution of dependence among residual lifetimes.

To our purposes, we have to consider the following different processes:

1. the point process, with a finite number of points, $T_1, \dots, T_n$;
2. the associated counting process $N_t = \sum_{i=1}^{n} \mathbf{1}_{\{T_i \leq t\}}$;
3. the copula-valued process $\{\hat{C}_t\}_{t \geq 0}$.

We separately consider the case of non-occurrence or occurrence of defaults (see [49] and [50] respectively). In each of these cases, we give an explicit expression for $\hat{C}_t$. However, in presence of defaults, we provide this formula and some results only in the particular case when $T_1, \ldots, T_n$ are conditionally i.i.d. given a random vector $\mathbf{\Theta} \equiv (\Theta_1, \ldots, \Theta_d)$, we will suppose to have a probability density π.

Conditional independence describes the situation when the stochastic dependence among random variables (and, in particular, among lifetimes) is just created by the influence of common factors $(\Theta_1, \ldots, \Theta_d)$ that are not directly observable. For example, in the field of reliability, the unobservable factors often have the meaning of *environmental conditions*. In the field of financial risk, this type of situation is related with the so-called *information-induced dependence* (see e.g. [72]).

It is important to notice that different problems related with the analysis of dependence take a very special and expressive form, under the assumption that dependence is just produced by conditional independence. This circumstance especially holds for what concerns the problem of evolution of dependence, under defaults and survivals. Since we already assumed $T_1, \ldots, T_n$ to be exchangeable, the hypothesis of conditional independence implies the observable lifetimes are conditionally i.i.d. given $\mathbf{\Theta}$.

In such a situation, the use of a Bayesian approach turns out as a completely natural one and the evolution of the conditional density of $\mathbf{\Theta}$, given observed histories, becomes a central object of interest.

We recall that one of our aims amounts in giving conditions for monotonicity of the family $\{\hat{C}_t\}_{t\geq 0}$, both in presence and in absence of defaults. Our study and, in particular, the comparison between the conditions that we found in the two cases, are developed for a conditionally i.i.d. model.

In this analysis, it turns out that the behavior of copulas at default times triggers properties of default contagion. In this case, we are able to provide some specific results, about the fact that a decrease of the dependence at a default time implies default contagion (see [50]).

We mostly restrict our analysis of evolution to the evolution of the family of the bivariate margins $\hat{C}_t^{(2)}$'s of the survival copulas. In the literature also, the analysis of the bivariate case is preferred to the multivariate one: dependence properties are easier to define and to check and the copula can be represented as a surface in the space (see e.g. [24, 78]). Furthermore, the fact that $T_1, \ldots, T_n$ are exchangeable implies that the dependence structure between two given variables is the same for all the pairs of variables and that, in some sense, the behaviour of (T_1, T_2) (e.g. positive dependence) is representative of behaviour of dependence in higher dimension. Furthermore, because of failures, survival copula's dimension progressively diminishes, until only two surviving units remain.
Since, from now on, we will speak about bivariate copulas, for notational simplicity, we will write $\hat{C}$ in place of $\hat{C}^{(2)}$.

As to the monotonicity of the family $\{\hat{C}_t\}_{t\geq 0}$, we provide conditions for increase of dependence, as time elapses, with respect to the *concor-*

dance order. We say that C_2 is *more concordant* than C_1 ($C_1 \preceq_{PQD} C_2$) if $C_1(u,v) \leq C_2(u,v)$ for any $u, v \in [0,1]$. The family $\{\hat{C}_t\}_{t\geq 0}$ being increasing in the concordance order means of course that the dependence among residual lifetimes becomes stronger and stronger in time.
This fact is of interest, for example, in fields connected with Risk Theory, where decision criteria are also based on certain assumptions on the dependence structure.

The following results will be restated and proved in Chapter 2 of this thesis.

A first condition for monotonicity of the family of survival copulas (see Proposition 2.2.4) requires the positivity of $\hat{C}(z,z)$, for any $z \in (0,1]$, and the existence everywhere of the partial derivatives of $\hat{C}$. This condition can be used only in absence of failures or within the intervals $(T_{(k)}, T_{(k+1)})$ between two defaults times. In fact, as shown in [50, Example 1], at default times, the map $t \mapsto \hat{C}_t$ in general is not continuous and, therefore, it cannot be differentiable, as, on the contrary, the proposition requires.

In the particular case when $T_1, \ldots, T_n$ are conditionally i.i.d. given $\mathbf{\Theta}$, we are able to provide a condition for monotonicity of $t \mapsto \hat{C}_t$ both between two defaults and at default times. In this case we can enrich our analysis, because of the particular form of the survival functions $\overline{F}$, $\overline{F}_t$, $\overline{G}$, $\overline{G}_t$, that can be expressed in terms of the conditional univariate survival function $\overline{G}(s|\theta) = \mathbb{P}(T_1 > s|\mathbf{\Theta} = \theta)$ or of its density $g(s|\theta)$. A key role in our results is played by the behaviour of the conditional hazard rate $r(s|\theta) = \dfrac{g(s|\theta)}{\overline{G}(s|\theta)}$ and of the posterior density of $\mathbf{\Theta}$ conditionally on $\mathcal{F}_t$, π_t. In fact, monotonicity properties of $t \to \hat{C}_t$ can be obtained from monotonicity properties of $t \to \pi_t$ with respect to some stochastic order. On their turn, monotonicity properties of $t \to \pi_t$ can be traced back to monotonicity properties of $\theta \to r(t|\theta)$.

The proofs of these results (see Propositions 2.3.6 and 2.3.7) are analogous each other and are based on properties of univariate and multivariate stochastic orderings. When Θ is a scalar random variable, these results straightly follow by characterizations of univariate stochastic orders (namely, usual stochastic order and likelihood ratio order, see e.g. [82]). In this thesis, we will show how these results may be extended to the case when $\mathbf{\Theta}$ is a random vector. Such an extension is possible under suitable conditions of positive dependence among the components of $\mathbf{\Theta}$.

As we have seen till now, copulas play a key role in our analysis. In our discussion, it is also of interest considering transformations of copulas.

In the topics we are studying, the following transformation acting on bivariate copulas is particularly relevant: given a copula C and an increasing bijection $\psi : [0,1] \to [0,1]$, we consider the function $C_\psi\colon [0,1]^2 \to [0,1]$

defined by

$$C_\psi(u,v) = \psi(C(\psi^{-1}(u), \psi^{-1}(v))). \tag{4}$$

Such a transformation has been considered several times in the literature, under different names, like *distortion* or *transformation* of a copula by means of ψ. It originated from the study of distorted probability distribution functions (especially, power distortions) and has been considered by several authors, like [21, 51, 46, 53, 43, 65, 64, 74, 25, 3].

Incidentally, we notice that transformations of copulas constitute a method for constructing new copulas. The growing importance of copulas in describing dependence in statistical models gave rise to several methods for generating new classes of such functions. The final goal of these investigations is to obtain more flexible families of multivariate distribution functions, having a variety of interesting properties, like tail dependencies, asymmetries, wide range of association. For a more detailed overview of methods for constructing copulas, see e.g. [78].

We point out that C_ψ is not necessarily a copula. It is immediate to see that C_ψ satisfies boundary conditions for a copula, is increasing in each variable and continuous, but it does not satisfy, generally, the positivity on the rectangles. For this reason, it is said that C_ψ is in general a semi-copula ([21, 43]).

We recall (see e.g. [78]) that a (n-variate) copula is a function $C : [0,1]^n \mapsto [0,1]$, satisfying the boundary conditions $C(1,\ldots,1,u_i,1,\ldots,1) = u_i$ for any $u_i \in [0,1]$ and $C(u_1,\ldots,u_{i-1},0,u_{i+1},\ldots,u_n) = 0$ for any $u_1,\ldots,u_{i-1}$, $u_{i+1},\ldots,u_n \in [0,1]$ and positive on the n-rectangles. These facts in particular imply that C is increasing in each variable.

A semi-copula is a function $S : [0,1]^n \mapsto [0,1]$, increasing in any variable, satisfying the same boundary conditions of a copula, but not necessarily positive on the n-rectangles.

Semi-copulas are of theoretical interest in that they play a role, analogous to the one of the connecting copulas, when, in place of a probability measure, we are considering a *capacity*.

Concerning distortions, we study in particular how tail dependence changes under such transformations. To such a purpose, it is however necessary that C_ψ is still a copula. We provide conditions on ψ under which, for a fixed copula C, C_ψ is still a copula (see also [34]).

In particular, we prove that, if ψ preserves the *total positivity* (see e.g. [78]) of C, C_ψ is guaranteed to be a copula. After that, we will be in a position to focus on how tail dependence is affected by such a transformation.

When C_ψ is no more a copula, distortions are still relevant objects, from a theoretical point of view, as we mentioned before, and also in some practical applications. In Reliability Theory, [21, 20] studied this kind of transformation in order to introduce the bivariate ageing function

$B : [0,1] \times [0,1] \to [0,1]$ defined by

$$B(u, v) = \exp\left\{-\overline{G}^{-1}\left(\overline{F}\left(-\log u, -\log v\right)\right)\right\}. \tag{5}$$

The ageing function B can be obtained as a distortion of the survival copula $\hat{C}$:

$$B(u, v) = \exp\left[-\overline{G}^{-1}\{\hat{C}\left(\overline{G}(-\log u), \overline{G}\left(-\log v\right)\right)\}\right]. \tag{6}$$

In general, B is a semi-copula. However, it turns out to be a copula in several cases of interest. A relevant feature of B is that it describes the family of the level curves of $\overline{F}$ and it permits to give a representation of $\overline{F}$ in terms of the pair $\left(\overline{G}, B\right)$.

The semi-copula B is used for the definition of bivariate notions of ageing. Notions of ageing are introduced to compare conditional survival probabilities for residual lifetimes. For details about ageing, we refer e.g. to [14]. The function B can be also used in different settings (see e.g. [45, 76]).

We devote also part of our study to the analysis of ageing functions and ageing properties, under the hypothesis of the exchangeability of the lifetimes.

Exchangeability is in particular necessary for the definition of Schur-concavity (or Schur-convexity) properties of a distribution (for a more detailed discussion on Schur-convexity and other convexity properties, see [70]). Such a property plays an important role in modelling ageing (see [21, 86]).

Ageing properties expressed in terms of ageing functions can be reformulated and studied also in the multivariate case (see [36]).

The fact, that the ageing function B can be obtained as a distortion of the survival copula, is useful to analyze some relations existing among univariate ageing, bivariate ageing and stochastic dependence (see [21]).

As a natural consequence of the introduction of the family $\{\overline{F}_t\}_{t\geq 0}$ and as a continuation of the study of the evolution of dependence, it will be of interest to study the evolution of ageing, by means of the family $\{B_t\}_{t\geq 0}$, where

$$B_t(u_1, \ldots, u_n) = \exp\left\{-\overline{G}_t^{-1}\left(\overline{F}_t\left(-\log u_1, \ldots, -\log u_n\right)\right)\right\}. \tag{7}$$

The study of B and $\{B_t\}_{t\geq 0}$ points out many interesting analogies between evolution of dependence and evolution of ageing. The analysis carried out in [49] suggested us to deepen the formalization of such analogies and to look for their motivation in the abstract structure of the families $\{\hat{C}_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ (see [48]).

To this aim, it will be natural to use the language of semigroups.

Let $\mathcal{U}$ be an arbitrary set, $\oplus$ a binary operation on $\mathcal{U}$ and $1_\oplus$ the neutral element for $\oplus$. If $\mathcal{U}$ is closed with respect to $\oplus$ and $\oplus$ is associative, $(\mathcal{U}, \oplus, 1_\oplus)$ is said a (unitary) *semigroup*.

Let now $\mathcal{T}$ be an arbitrary set. An *action* of $\mathcal{U}$ on $\mathcal{T}$ is a transformation

$$\Phi : \mathcal{T} \times \mathcal{U} \to \mathcal{T}$$

such that

(i) for any $\zeta \in \mathcal{T}$, $\Phi(\zeta, 1_{\oplus}) = \zeta$;

(ii) for any $t,\ s \in \mathcal{U}$, $\Phi(\Phi(\zeta, t), s) = \Phi(\zeta, t \oplus s)$.

The set

$$\mathcal{O}_{\Phi}(\zeta) = \Phi(\zeta, \mathcal{U}) = \{\zeta' \in \mathcal{T} :\ \exists s \in \mathcal{U} :\ \Phi(\zeta, s) = \zeta'\}$$

is the *orbit* of ζ under the action Φ. In most of our applications, we will consider the semigroup $(\mathcal{U}, \oplus, 1_{\oplus})$ coinciding with $(\mathbb{R}_{+}, +, 0)$ and $\mathcal{T}$ coinciding with the family of all the semi-copulas, $\mathcal{S}$, or with the family of all the copulas, $\mathcal{C}$.

We define here, in accordance with such an algebraic setting, the general notion of *hyper-property.*
Let S be a semi-copula, $\mathbf{P}$ a property of semi-copulas and $\mathcal{P}$ the class of all the semi-copulas satisfying $\mathbf{P}$; furthermore, let $\mathcal{O}_{\Phi}(S)$ be the orbit of S under the action Φ. We say that S is hyper$_{\Phi}$-$\mathbf{P}$ if $\mathcal{O}_{\Phi}(S) \subseteq \mathcal{P}$. We denote by hyper$_{\Phi}$-$\mathcal{P}$ the class of all the semi-copulas satisfying hyper$_{\Phi}$-$\mathbf{P}$.

A classic topic in the theory of copulas is the analysis of relationships existing among different properties of dependence. For our purposes, we prefer to describe such properties as classes of copulas associated with the different dependence properties.

Here, we aim at studying also relationships existing among the classes of semi-copulas describing dependence properties or hyper-dependence properties.

Our axiomatic investigation can be also of theoretical interest. The general background, provided by the formalization of hyper-properties, allows us to extend the notion of hyper-property from dependence to other kinds of properties.

In this context, we study the notion of hyper-property regards to the family $\{\hat{C}_t\}_{t \geq 0}$. As proved in [49], in fact, the family $\{\hat{C}_t\}_{t \geq 0}$ is the orbit of $\hat{C}$, under the semigroup action defined by a certain transformation Φ_{dep}. In such a case, we just reobtain hyper-dependence properties.
Analogously, we prove that also the family of ageing functions is the orbit of the ageing function B, under a different semigroup action Φ_{ag}. Thus, we extend the notion of hyper-property to the family $\{B_t\}_{t \geq 0}$. In this case, we speak of *hyper-ageing* properties.

The interest in this common background for the two families $\{\hat{C}_t\}_{t \geq 0}$ and $\{B_t\}_{t \geq 0}$ lies in that it explains some special analogies existing between ageing and dependence. More precisely, these analogies concern the structure of the implications among dependence properties and the structure of implications among ageing properties.

The thesis is organized as follows.

Chapter 1 is devoted to collect the needed mathematical background. To this purpose, we recall basic definitions and facts about counting processes, survival models, univariate and multivariate stochastic orders. A more detailed part is devoted to stochastic dependence and copulas. Stochastic orders are of particular interest for us, because we will use them as a basic language to develop our results. Stochastic dependence and copulas constitute the basic subject for the contributions presented in the thesis.

Results about stochastic dependence and its evolution are explained in detail in Chapter 2. In a first part, we mainly focus on conditions for preservation of dependence properties under a progressive observations of survivals; in a second part, we focus on monotonicity (increase or decrease) of dependence.

Chapter 3 is devoted to distortions of probability measures and of copulas. We also focus on semi-copulas and capacities. In particular, we discuss therein conditions on the distortion under which the distorted copulas are still copulas. Such arguments are relevant in the study of tail dependence of distorted probabilities.

Chapter 4 is devoted to the study of ageing properties of a survival model and of their evolution. We detail some relevant aspect of analytical type for $\{B_t\}_{t\geq 0}$ and present some practical interpretation of these conditions. A part of the chapter is devoted to broaden the study of the connections among dependence, univariate and bivariate ageing, along the line of [21]. In the frame of this elaboration, both analogies and structural differences emerge, between $\{B_t\}_{t\geq 0}$ and $\{\hat{C}_t\}_{t\geq 0}$.

Finally, in Chapter 5, on the basis of these analogies, we provide the theoretical frame unifying the treatments of evolution of dependence and of ageing.

We prove that both the families $\{B_t\}_{t\geq 0}$ and $\{\hat{C}_t\}_{t\geq 0}$ are orbits of actions of a semigroup. In this algebraic setting, a hyper-property is represented by a class of semi-copulas, that is closed under the action of a given semigroup of transformations.
We are specifically interested in studying relationships among dependence properties and conditions for preservation in time of some of them. This can be done by studying relationships among the corresponding classes of copulas and their properties of closure under a given semigroup action.

Acknowledgments

I wish to express my gratitude to my supervisor Fabio Spizzichino, for having encouraged me to undertake the research activity and for having followed me during these years.

Furthermore, I would like to thank my co-authors Fabrizio Durante and Peter Sarkoci for many useful discussions we had together and Giovanna Nappo for having carefully read and reviewed my thesis and for interesting discussions we had during and before my PhD.

Chapter 1

Mathematical background

1.1 Counting processes and survival models

A stochastic process is a family of random variables (= r.v.'s)

$$\mathbf{Y} \equiv \{Y_t : \Omega \to \mathbb{R}^d, t \in I \subseteq \mathbb{R}\},$$

all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In particular, a *counting process* (on $\mathbb{R}_+$) is a stochastic process $\mathbf{N} \equiv \{N_t\}_{t\geq 0}$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, with $N_0 = 0$ and $N_t < \infty$ a.s., and whose path are a.s. right-continuous, piecewise constant and have only jump discontinuities, with jumps of size $+1$.

A simple counting process on the half-line can be defined starting from a family of r.v.'s $\{T_n\}_{n\in\mathbb{N}}$, such that, for any $i, j \in \mathbb{N}$, $i \neq j$, $T_i > 0$, $T_i \neq T_j$ and $T_i < \infty$. Then, $\{N_t\}_{t\geq 0}$ is given by

$$N_t = \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i \leq t\}}.$$

It is natural assuming $E(N_t) < \infty$ for any fixed $t \in \mathbb{R}_+$.

With a general stochastic process on the real half-line, it is associated the family of sub-σ-algebras of $\mathcal{F}$, $(\mathcal{F}_t^Y)_{t\geq 0}$. $\mathcal{F}_t^Y = \sigma(Y_s, s \in [0, t])$ is the *internal history* of the process.

In the case of a counting process $\mathbf{N}$, with a finite number of jumps, $\mathcal{F}_t^N$ can be written as

$$\mathcal{F}_t^N = \sigma(T_1 \wedge t, ..., T_n \wedge t, N_t).$$

In fact, except for the instants $T_1, ..., T_n$, the process is constant.

Since the needed information for studying a counting process is given by the observation of the sequence of the jump times, we can study this kind of processes both from the point of view of the finite-dimensional distributions of the jump times and in terms of *compensators* ([22, 8]).

Definition 1.1.1. *The compensator of the process $\{N_t\}_{t\geq 0}$, with respect to (w.r.t.) $(\mathcal{F}_t^N)_{t\geq 0}$, is the random measure A such that the process $\{A_t\}_{t\geq 0}$, $A_t = A((0,t])$, is predictable and, for any predictable and non-negative process $\{H_t\}_{t\geq 0}$,*

$$E\left[\int_0^\infty H_t dN_t\right] = E\left[\int_0^\infty H_t dA_t\right].$$

Such a measure exists and is unique.

The compensator can be heuristically interpreted as the local expected value of the process **N**, conditionally on its strict past, that is

$$A(dt) = E[N(dt)|\mathcal{F}_{t^-}] = \mathbb{P}(N(dt) = 1|\mathcal{F}_{t^-}).$$

The compensator is a more "regular" object than the process itself and its treatment is connected to the theory of martingales. In fact, the compensator can be also defined as the only predictable process **A**, such that $\mathbf{N} - \mathbf{A} = \mathbf{M}$, where **M** is a zero-mean martingale. In terms of information, **M** can be interpreted as the innovation process, containing that information on **N**, that cannot be inferred from the strict past. From a statistical point of view, instead, it plays the role of a noise, collecting those aspects that cannot be estimated (analogously to residua in regression problems).

In the absolutely continuous case, the compensator can be written in the form $A_t = \int_0^t \lambda_s ds$. Its density, λ_s, is the *stochastic intensity* of the process, that fits well to express the jump times "concentration" in a certain subset of $\mathbb{R}_+$. In the case of a counting process, it can be expressed as

$$\lambda_t = \lim_{\delta\to 0^+} \mathbb{P}(N_{t+\delta} - N_t \geq 1|\mathcal{F}_t^N).$$

The *jump intensities* of each jump time T_i are given instead by

$$\lambda_i(t) = \lim_{\delta\to 0^+} \frac{1}{\delta}\mathbb{P}(T_i < t+\delta|\mathcal{F}_t^N).$$

The stochastic intensity of the process can be written in terms of the jump intensities:

$$\lambda_t = \sum_{i=1}^n \lambda_i(t)\mathbf{1}_{\{T_i>t\}}.$$

The form of the intensities is simpler in particular cases, for example, when the jump times are exchangeable. In this case, we can write them in the following form:

$$\begin{aligned}
\lambda_i^{(n-k)}(t) &= \lim_{\delta\to 0^+} \frac{1}{\delta}\mathbb{P}(T_i < t+\delta|\mathcal{F}_t^N) \\
&= \lim_{\delta\to 0^+} \frac{1}{\delta(n-k)}\mathbb{P}(T_{(k+1)} \leq t+\delta|\mathcal{F}_t^N)
\end{aligned}$$

or equivalently (see [86])

$$\lambda_i^{(n-k)}(t) = \frac{\displaystyle\int_t^\infty \cdots \int_t^\infty f(t_1, ..., t_k, t, \xi_1, ..., \xi_{n-k-1}) d\xi_1 \cdots d\xi_{n-k-1}}{\displaystyle\int_t^\infty \cdots \int_t^\infty f(t_1, ..., t_k, \xi_1, ..., \xi_{n-k}) d\xi_1 \cdots d\xi_{n-k}}.$$

The intensity of a counting process can be also used to represent the stochastic dependence among the r.v.'s $T_1, ..., T_n$. In particular, they are specially suitable in expressing *dynamical* notions of dependence, in that, by definition, they take into account the conditioning on the past.

1.2 Exchangeability

The condition of *exchangeability* of $T_1, \ldots, T_n$ formalizes the situation of *indifference* relative to $T_1, \ldots, T_n$.

We mainly use here terminology and concepts which are specific for the treatment of non-negative random quantities.

Definition 1.2.1. *$T_1, \ldots, T_n$ are exchangeable if they have the same joint distribution of $T_{\sigma_1}, \ldots, T_{\sigma_n}$, for any permutation $\{\sigma_1, \ldots, \sigma_n\}$ of the indexes $\{1, \ldots, n\}$.*

In other words, $T_1, \ldots, T_n$ are exchangeable if their joint distribution function $F(x_1, \ldots, x_n)$ is symmetric in all the variables.

As said in the Introduction, in this thesis, we mostly shall be dealing with absolutely continuous distributions, for which

$$F(x_1, \ldots, x_n) = \int_0^{x_1} \cdots \int_0^{x_n} f(\xi_1, \ldots, \xi_n) d\xi_1 \cdots d\xi_n$$

holds, where f denotes the joint probability density

$$f(x_1, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F(x_1, \ldots, x_n).$$

As can be easily checked, $T_1, \ldots, T_n$ are exchangeable if and only if f is symmetric in all the variables as well.

We notice that, if $T_1, \ldots, T_n$ are exchangeable, $T_{i_1}, \ldots, T_{i_j}$ also are exchangeable, with $J = \{i_1, \ldots, i_j\}$ any subset of $\{1, \ldots, n\}$.

Their joint distribution function of $T_{i_1}, \ldots, T_{i_j}$ depends only on $j = |J|$ and not on the particular choice of the set J:

$$F^{(j)}(x_1, \ldots, x_j) = \lim_{x_{j+1}, \ldots, x_n \to +\infty} F(x_1, \ldots, x_n). \tag{1.1}$$

Furthermore, it is still obviously symmetric in all the variables.

Definition 1.2.2. *An exchangeable distribution function $F(x_1, \ldots, x_n)$ is N-extendible ($N > n$) if we find an exchangeable N-dimensional distribution function $F(x_1, \ldots, x_N)$ such that $F(x_1, \ldots, x_n)$ is the n-dimensional marginal distribution of $F(x_1, \ldots, x_N)$.*
$F(x_1, \ldots, x_n)$ is infinitely extendible if it is N-extendible for any $N > n$.

For an exchangeable distribution function $F(x_1, \ldots, x_n)$, any j-dimensional margin of its is obviously n-extendible.

Let $\rho \in [-1, 1]$ be the *correlation coefficient* of two r.v.'s X, Y:

$$\rho = \rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}. \tag{1.2}$$

Remark 1.2.3. *If $T_1, \ldots, T_n$ are such that (for $1 \le i \neq j \le n$) $Var(T_i) = \sigma^2$ and $Cov(T_i, T_j) = \sigma^2 \rho$. Then, the following holds:*

$$0 \le Var\left(\sum_{i=1}^{n} T_i\right) = n\sigma^2 + n(n-1)\sigma^2\rho$$

and therefore $\rho > -\dfrac{1}{n-1}$. Thus infinite extendibility implies positive correlation. In fact, if $\rho < 0$, then the distribution of $T_1, \ldots, T_n$ cannot be N-extendible for $N > 1 + \dfrac{1}{\rho}$.

The most obvious example of an infinitely extendible distribution function corresponds to $T_1, \ldots, T_n$ being independent and identically distributed (= i.i.d.); thus

$$F(x_1, \ldots, x_n) = G(x_1) \cdots G(x_n).$$

A further step may consist in considering $T_1, \ldots, T_n$ being independent and identically distributed, conditionally on a r.v. or on a random vector $\mathbf{\Theta}$, taking values in $\mathbb{R}^d$, with a distribution Π. Thus

$$F(x_1, \ldots, x_n | \mathbf{\Theta} = \theta) = G(x_1|\theta) \cdots G(x_n|\theta)$$

and

$$F(x_1, \ldots, x_n) = \int_{\mathbb{R}^d} G(x_1|\theta) \cdots G(x_n|\theta) d\Pi(\theta).$$

The following de Finetti theorem states that all the possible situations of infinite extendibility are of these two types.

Theorem 1.2.4. *The distribution of exchangeable r.v.'s $T_1, \ldots, T_n$ is infinitely extendible if and only if $T_1, \ldots, T_n$ are i.i.d. or conditionally i.i.d..*

This result can be formulated and proven in a number of different ways. Complete proofs can be found in [31, 55, 29].

1.3 Dependence concepts and copulas

Besides intensities, specific and widespread tools to describe stochastic dependence are *copulas* ([78]).

Definition 1.3.1. *For $n \geq 2$, a* copula

$$C : [0,1]^n \mapsto [0,1]$$

is a distribution function on $[0,1]^n$ with uniform univariate margins on $[0,1]$.

An equivalent, more analytical-type definition can be given. To this aim, we preliminarily state the following

Definition 1.3.2. *Let $K : A_1 \times \cdots \times A_n \subseteq \mathbb{R}^n \to \mathbb{R}$ and $[\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a n-rectangle contained in $A_1 \times \cdots \times A_n$.*

- *The K-*volume *of $[\mathbf{a}, \mathbf{b}]$ is defined as*

  $$V_K([\mathbf{a}, \mathbf{b}]) = \Delta_{\mathbf{a}}^{\mathbf{b}} K(\mathbf{t}) = \Delta_{a_n}^{b_n} \cdots \Delta_{a_1}^{b_1} K(\mathbf{t}),$$

  where

  $$\Delta_{a_i}^{b_i} = K(t_1, \ldots, t_{i-1}, b_i, t_{i+1}, \ldots, t_n) - K(t_1, \ldots, t_{i-1}, a_i, t_{i+1}, \ldots, t_n).$$

- *K is said to be n-*increasing *if, for any n-rectangle $[\mathbf{a}, \mathbf{b}] \subseteq A_1 \times \cdots \times A_n$,*

  $$V_K([\mathbf{a}, \mathbf{b}]) \geq 0.$$

Definition 1.3.3. *A (n-variate) copula is a function $C : [0,1]^n \mapsto [0,1]$ such that*

1. *for any $u_i \in [0,1]$, $C(1, ..., 1, u_i, 1, ..., 1) = u_i$;*
2. *for any $u_1, ..., u_{i-1}, u_{i+1}, ..., u_n \in [0,1]$,*

   $$C(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_n) = 0;$$

3. *C is n-increasing.*

These conditions in particular imply that C is increasing in each variable.

Theorem 1.3.4 (Sklar). *Given a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ with continuous distribution function F and margins $G_1, \ldots, G_n$, there exists a unique copula C satisfying*

$$F(x_1, \ldots, x_n) = C(G_1(x_1), \ldots, G_n(x_n)) \tag{1.3}$$

and given by

$$C(u_1, \dots, u_n) = F(G_1^{-1}(u_1), \dots, G_n^{-1}(u_n)) \tag{1.4}$$

where $G_i^{-1}(u) = \inf\{x \in \mathbb{R} | G_i(x) \geq u\}$, *for any* $0 \leq u \leq 1,\ i = 1, \dots, n$.

Conversely, given a copula C *and* $G_1, \dots, G_n$ *univariate distribution functions, a function* F*, defined by Eq.* (1.3)*, is a* n*-dimensional distribution function, with univariate margins* $G_1, \dots, G_n$.

The copula associated with the probability distribution F by means of Eq.s (1.3) and (1.4), is called *connecting copula*.

Sklar Theorem constitutes the motivation for calling copulas *dependence structures*, capturing dependence properties, that are scale invariant. In fact, as it can be seen from Eq. (1.3), C combines the marginal distributions into their joint distribution function and permit to distinguish in the model the properties due to the dependence structure from those originated by the marginal behaviour.
This fact allows us to construct different multivariate models by combining the same margins by means of different copulas or by fixing the dependence structure (i.e. the copula) and letting the margins vary.

1.3.1 Analytical properties of copulas

Deterministic dependence among variables is represented by the fact that any variable can be expressed as a function of the other ones. Such dependence can be positive (monotonicity) or negative (counter-monotonicity). For example, in the bivariate case, X, Y are such that $X = g(Y)$ for a suitable function g; if g is increasing, then X, Y are monotonic, if g is decreasing, X, Y are counter-monotonic. Monotonicity and counter-monotonicity represent the upper and lower bounds respectively for all the dependence structures. Monotonicity and counter-monotonicity are described by the copula

$$M(u_1, \dots, u_n) = \min(u_1, \dots, u_n)$$

and

$$W(u_1, \dots, u_n) = \max(0, u_1 + \cdots + u_n - n + 1)$$

respectively. The last one is a copula only for $n = 2$.

Theorem 1.3.5 (Fréchet-Hoeffding bounds). *For any copula* C,

$$W(u_1, \dots, u_n) \leq C(u_1, \dots, u_n) \leq M(u_1, \dots, u_n),$$

for all $u_1, \dots, u_n \in [0, 1]$.

Invariance under increasing transformations of the arguments, or of the margins, is essential to legitimate the use of copulas to model dependence. In fact, transforming the margins allows us to make copulas comparable.

Theorem 1.3.6. *Let $(X_1, \ldots, X_n)$ be a random vector with copula C and $\gamma_1, \ldots, \gamma_n,\ \ \gamma_i : \mathbb{R} \to \mathbb{R},\ i = 1, \ldots, n$, be strictly increasing transformations. Then, the transformed vector $(\gamma_1(X_1), \ldots, \gamma_n(X_n))$ admits the copula C as well.*

1.3.2 Dependence properties

After monotonicity and counter-monotonicity, the easiest structure of dependence in n-variate models is given by independence:

$$F(x_1, \ldots, x_n) = G_1(x_1) \cdots G_n(x_n).$$

Models with independent components admit the connecting copula

$$C(u_1, \ldots, u_n) = \Pi_n(u_1, \ldots, u_n) = u_1 \cdots u_n.$$

A first generalization of this dependence structure may be given by the conditional independence. Conditional independence is relevant in that it models interesting practical situations, as concerns the dependence structure of a system. The most common interpretation of conditional independence is that the units $D_1, \ldots, D_n$ (whose lifetimes are represented by $T_1, \ldots, T_n$) are similar and there is no physical interaction among them. However, their lifetimes are influenced by the values taken by one or more physical quantities $(\Theta_1, \ldots, \Theta_d) = \boldsymbol{\Theta}$, representing environmental conditions. Suppose that the actual value of $\boldsymbol{\Theta}$ is unobservable: we assess a probability distribution on it. This probability is continuously updated in view of the progressive observation of survivals or failures of the units $D_1, \ldots, D_n$.

If we want the variables to be both exchangeable and conditionally independent, we have necessarily to impose them to be conditionally independent and identically distributed (= conditionally i.i.d.).

A particular case of conditionally i.i.d. models, is represented by survival (or connecting) Archimedean copulas.

Definition 1.3.7. *Let $\phi : (0, 1] \to [0, +\infty)$ be a n-completely monotonic decreasing function, such that $\phi(1) = 0$.*

$$\hat{C}(u_1, \ldots, u_n) = \phi^{-1}\left(\sum_{i=1}^{n} \phi(u_i)\right)$$

is a Archimedean copula with generator ϕ.

Archimedean models are also handy from a mathematical point of view, since the study of dependence properties of an Archimedean copula of any dimension can be traced back to the study of properties of ϕ.

Associated with different copulas C, we obtain different dependence structures.

As it happens in the transition from independence to conditional independence, we can build different models, presenting given dependence structures conditionally on a random variable (or vector). Such models are called *factor models*. Using the fact that mixtures of copulas are still copulas, the copula of a factor model can be written in the form

$$C(u_1, \ldots, u_n) = \int_{\mathbb{R}} C_\theta(u_1, \ldots, u_n) d\Pi(\theta),$$

where Π is the distribution of Θ.

We point out that the "conditional dependence" of this kind of models, is not the one we mainly refer to in the thesis, where, by "conditional dependence", we mean the dependence conditional on longitudinal (or *dynamic*) observations of the variables representing the model and not on the observation of a parameter.

Speaking of construction of new copulas, in the bivariate case, a geometric method we will use in the following, mainly for building examples, is the ordinal sum construction.

Definition 1.3.8. *Let $\{J_i\}$, $J_i = [a_i, b_i]$, denote a countable partition of $[0,1]$ and let $\{C_i\}$ be a collection of copulas with the same indexing as $\{J_i\}$. Then the ordinal sum of $\{C_i\}$ with respect to $\{J_i\}$ is the copula C given by*

$$C(u,v) = \begin{cases} a_i + (b_i - a_i) C_i \left(\frac{u-a_i}{b_i-a_i}, \frac{v-a_i}{b_i-a_i} \right) & (u,v) \in J_i^2, \\ M(u,v) & \textit{otherwise.} \end{cases}$$

Heuristically, C is obtained by "pasting" suitably scaled copies of C_i over the squares J_i^2.

The most part of our discussions on dependence properties, in particular the ones about their preservation, will be carried out in the bivariate case.

We recall the following definition:

Definition 1.3.9. *Given two intervals A and B in $\mathbb{R}$, a function $K \colon A \times B \to \mathbb{R}$ is said to be* totally positive of order 2 *(shortly, TP_2) if, for any $x_1, x_2 \in A,\ y_1, y_2 \in B$ such that $x_1 \leq x_2,\ y_1 \leq y_2$,*

$$K(x_1, y_1) K(x_2, y_2) \geq K(x_1, y_2) K(x_2, y_1). \tag{1.5}$$

Analogously, $K \colon A \times B \to \mathbb{R}$ is said to be reverse regular of order 2 *(shortly, RR_2) if* (1.5) *holds with the reverse inequality sign.*

For more details, see [60, 61].

In the thesis, we consider the following properties of positive dependence (see e.g. [78]):

Definition 1.3.10. *Let F be the bivariate distribution function of (X, Y), with margins G_X, G_Y. The following properties* **P** *can be stated both as properties of F ("F is* **P***") and of the random pair (X, Y) ("(X, Y) is* **P***").*

- *F is PQD (Positively Quadrant Dependent) if, for every $(x, y) \in \mathbb{R}^2$, $F(x, y) \geq G_X(x)G_Y(y)$.*
- *F is LTD($Y \mid X$) (Y is Left Tail Decreasing in X) if $\dfrac{F(x,y)}{G_X(x)}$ is decreasing in x.*
  Analogously, F is LTD($X \mid Y$) if, $\dfrac{F(x,y)}{G_Y(y)}$ is decreasing in y.
- *F is SI($Y \mid X$) (Y is Stochastically Increasing in X) if, for any y, $\mathbb{P}(Y > y|X = x)$ is increasing in x.*
  Analogously F is SI($X \mid Y$) if, for any x, $\mathbb{P}(X > x|Y = y)$ is increasing in y.
- *F is LCSD (Left Corner Set Decreasing) if it is TP_2;*
- *F is PLR (Positively Likelihood Ratio dependent) if it is absolutely continuous and its density f is TP_2.*

Dependence properties of a pair (or of a vector) of r.v.'s are characterized by their invariance under increasing transformations of the variables. Because of Theorem 1.3.6, dependence properties can thus be defined in terms of the connecting copula of the variables.

Definition 1.3.11. *Let $C \in \mathcal{C}$ be the connecting copula of the random pair (U, V).*

- *C is PQD if, for every $(u, v) \in [0, 1]^2$, $C(u, v) \geq uv$.*
- *C is LTD($V \mid U$) if $u \mapsto \dfrac{C(u,v)}{u}$ is decreasing on $[0, 1]$.*
  Analogously, C is LTD($U \mid V$) if, $v \mapsto \dfrac{C(u,v)}{v}$ is decreasing on $[0, 1]$.
- *C is SI($V \mid U$) if, for any fixed $v \in [0, 1]$, $C(u, v)$ is concave in u.*
  Analogously C is SI($U \mid V$) if, for any fixed $u \in [0, 1]$, $C(u, v)$ is concave in v.
- *C is TP_2 if it satisfies (1.5).*
- *C is PLR if C is absolutely continuous and its density*
  $c(u, v) = \dfrac{\partial^2 C}{\partial u \partial v}(u, v)$ satisfies (1.5).

The following chain of implications holds (and none of the converse implications is satisfied):

$$\text{PLR} \quad \Rightarrow \quad \text{TP}_2 \quad \Rightarrow \quad \text{LTD}(U \mid V) \text{ (or LTD}(V \mid U)) \quad \Rightarrow \quad \text{PQD};$$

$$\text{PLR} \quad \Rightarrow \quad \text{SI}(U \mid V) \text{ (or SI}(V \mid U)) \quad \Rightarrow \quad \text{LTD}(U \mid V) \text{ (or LTD}(V \mid U)).$$

If U, V are exchangeable, C is symmetric and therefore LTD$(U \mid V)$ and SI$(U \mid V)$ are equivalent to LTD$(V \mid U)$ and SI$(V \mid U)$ respectively. In this case, we simply say that C is LTD or SI.

Let us also introduce the following notation: we denote by $\mathcal{P}_{PQD}$, $\mathcal{P}_{LTD}$, $\mathcal{P}_{TP_2}$, $\mathcal{P}_{SI}$, $\mathcal{P}_{PLR}$ the classes of copulas that are, respectively, PQD, LTD, TP$_2$, SI, PLR.

The corresponding negative properties of dependence, namely NQD, RTI, RR$_2$, SD, NLR, can be obtained by simply reverting the signs in the inequalities in Definition 1.3.11.

An important part of the study of dependence relates to the extreme values, that is to the amount of dependence in the upper-quadrant or lower-quadrant tail of a (bivariate) distribution. Such a tail behaviour of the dependence is described by the following objects. Given two continuous random variables X and Y whose distribution functions are G_X and G_Y, respectively, the *upper tail dependence coefficient* λ_U of (X, Y) is defined by

$$\lambda_U = \lim_{t \to 1^-} \mathbb{P}\left(Y > G_Y^{-1}(t) \mid X > G_X^{-1}(t)\right) \tag{1.6}$$

and the *lower tail dependence coefficient* λ_L by

$$\lambda_L = \lim_{t \to 0^+} \mathbb{P}\left(Y \le G_Y^{-1}(t) \mid X \le G_X^{-1}(t)\right), \tag{1.7}$$

provided that the above limits exist in $[0, 1]$. These two coefficients can be expressed in terms of the copula associated with (X, Y).

Proposition 1.3.12. *Let X and Y be continuous random variables with copula C. If λ_U and λ_L defined by* (1.6) *and* (1.7) *exist and take values in* $[0, 1]$, *then*

$$\lambda_L = \lim_{x \to 0^+} \frac{C(x, x)}{x}, \tag{1.8}$$

$$\lambda_U = \lim_{x \to 1^-} \frac{1 - 2x + C(x, x)}{1 - x} = 2 - \lim_{x \to 1^-} \frac{1 - C(x, x)}{1 - x}. \tag{1.9}$$

Moreover, if λ_U and λ_L exist and are finite, then

$$\lambda_L = \delta_C'(0^+) \quad \textit{and} \quad \lambda_U = 2 - \delta_C'(1^-), \tag{1.10}$$

where $\delta_C \colon [0, 1] \to [0, 1]$ given by $\delta_C(x) = C(x, x)$ is the diagonal section of C.

1.3.3 Dependence in survival models

When we deal with $T_1, \dots, T_n$, non-negative random variables, it is natural to interpret them as waiting times to n (stochastically dependent) top events, such as the failures of components $D_1, \dots, D_n$ operating in a same system or the defaults of firms in a same market, even if different interpretations are possible. A system of this kind, where $T_1, \dots, T_n$ may be lifetimes or times to default is called a *survival model.*

The model is the individuated by the joint survival function of its components,

$$\overline{F}(x_1, \dots, x_n) = \mathbb{P}(T_1 > x_1, \dots, T_n > x_n),$$

with $|J|$-dimensional margins

$$\overline{F}^{(J)}(x_1, \dots, x_{|J|}) = \mathbb{P}(T_{i_1} > x_1, \dots, T_{i_{|J|}} > x_{|J|}), \tag{1.11}$$

for J any subset of $\{1, \dots, n\}$, and, in particular, with univariate margins $\overline{G}_1, \dots, \overline{G}_n$.

We will mostly discuss the case when $T_1, \dots, T_n$ are exchangeable. As said in Section 1.2, in this case, for any $j \in \{1, \dots, n\}$, the j-dimensional margins only depend on the cardinality of the set J and Eq. (1.11) can be simplified as follows:

$$\overline{F}^{(j)}(x_1, \dots, x_j) = \mathbb{P}(T_1 > x_1, \dots, T_j > x_j).$$

In particular, for any $i = 1, \dots, n$, $\overline{G}_i = \overline{G}$.

As we choose survival functions in place of distributions to describe survival models, we use the *survival copula*

$$\hat{C}(u_1, ..., u_n) = \overline{F}\left(\overline{G}_1^{-1}(u_1), ..., \overline{G}_n^{-1}(u_n)\right), \tag{1.12}$$

that is the connecting copula associated to $\overline{F}$, to model dependence among lifetimes. The survival copula can be expressed in terms of connecting copula:

$$\hat{C}(u_1, ..., u_n) = \sum_{k=0}^{n} \left((-1)^k \sum_{i_1, \dots, i_n} C(1, ..., 1, 1 - u_{i_1}, 1, ..., 1, 1 - u_{i_k}, 1, ..., 1) \right).$$

1.3.4 Semi-copulas and capacities

Definition 1.3.13. *A function $S : [0,1]^n \mapsto [0,1]$, increasing in any variable and such that,*

- *for any $u_i \in [0,1]$, $S(1, \dots, 1, u_i, 1, \dots, 1) = u_i$,*
- *for any $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n \in [0,1]$,*

$$S(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_n) = 0,$$

is called a semi-copula*.*

A semi-copula satisfies all the properties of a copula, except for the n-increasingness.

Similarly, it can be defined an analog of a probability measure, that is not additive.

Definition 1.3.14. *Let Ω be a non-empty set and $\mathcal{A}$ a σ-algebra in the power set 2^Ω.*
A mapping $\nu : \mathcal{A} \mapsto [0, 1]$ is called a capacity*, if it is*

- *monotonic, i.e. $\nu(A) \leq \nu(B)$ for all $A \subseteq B$;*
- *normalized, i.e. $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$.*

In what follows, we will consider capacities defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n_+)$.

In analogy with the case of probability measures, with each capacity ν we can formally associate its survival function $\overline{F}_\nu : \mathbb{R}^n_+ \mapsto [0, 1]$, defined by

$$\overline{F}_\nu(x_1, \dots, x_n) = \nu((x_1, +\infty) \times \cdots \times (x_n, +\infty)).$$

Any $\overline{F}_\nu$ associated with a capacity ν satisfies:

(a) $\overline{F}_\nu$ is decreasing in each argument;

(b) $\overline{F}_\nu(0, \dots, 0) = 1$;

(c) $\overline{F}_\nu(x_1, \dots, x_n) \to 0$ when $\max(x_1, \dots, x_n) \to +\infty$.

If ν is a probability measures, then $\overline{F}_\nu$ is the usual probability survival function. However, in general, $\overline{F}_\nu$ does not satisfy the n-increasing property. Each function $\overline{F}_\nu$ satisfying (a)-(c) can be used to construct a capacity ν whose survival function is $\overline{F}_\nu$; but, contrarily to the case of probability measures, such a capacity is not uniquely determined by $\overline{F}_\nu$.

An analog of the first part of Sklar's Theorem holds (see [45]):

Theorem 1.3.15. *Let ν be a capacity on the space $(\mathbb{R}^n_+, \mathbf{B}(\mathbb{R}^n_+))$, $\overline{F}_\nu$ the associated survival function and $\overline{G}_1, \dots, \overline{G}_n$ its margins.*
Suppose that $\overline{G}_1, \dots, \overline{G}_n$ are continuous and strictly decreasing.
Then there exists a unique semi-copula $S_\nu : [0, 1]^n \to [0, 1]$ such that, for every $(x_1, \dots, x_n) \in \mathbb{R}^n_+$,

$$\overline{F}_\nu(x_1, \dots, x_n) = S_\nu(\overline{G}_1(x_1), \dots, \overline{G}_n(x_n)).$$

Within the set of the capacities, distorted probabilities are of particular interest for us.

Definition 1.3.16. *A capacity* ν *is a* distorted probability, *if there exists a probability measure* $\mathbb{P}$ *on* $\mathcal{A}$ *and an increasing bijection* $\psi : [0,1] \to [0,1]$ *such that* $\nu = \psi \circ \mathbb{P}$.

Proposition 1.3.17. *Let* $\mathbb{P}$ *be a probability measure on* $(\mathbb{R}_+^n, \mathbf{B}(\mathbb{R}_+^n))$, $\overline{F}$ *the associated survival function and* $\overline{G}_1, \ldots, \overline{G}_n$ *its margins. Let* $\psi : [0,1] \to [0,1]$ *be an increasing bijection. Then* $D = \psi \circ \overline{F}$ *is the survival function associated with the probability measure* $\psi \circ \mathbb{P}$. *Moreover, if* $\overline{G}_1, \ldots, \overline{G}_n$ *are continuous and strictly decreasing and* $\hat{C}$ *is the connecting copula of* $\overline{F}$, *then*

$$S(u_1, \ldots, u_n) = \psi(C(\psi^{-1}(u_1), \ldots, \psi^{-1}(u_n)))$$

is the formal copula associated with D.

1.4 Univariate and multivariate stochastic orders

1.4.1 Univariate stochastic orders

Let X, Y be two r.v.'s with survival functions $\overline{G}_X$ and $\overline{G}_Y$. We suppose they also admit probability densities g_X and g_Y respectively.

Definition 1.4.1. X *is smaller than* Y *in the usual stochastic order (in short* $X \leq_{st} Y$ *or* $g_X \leq_{st} g_Y$*) if for any* $t \geq 0$,

$$\overline{G}_X(t) \leq \overline{G}_Y(t).$$

Definition 1.4.2. *The hazard rate of a r.v.* X *is* $r_X(t) = \dfrac{g_X(t)}{\overline{G}_X(t)}$.

Definition 1.4.3. X *is smaller than* Y *in the hazard rate order (in short* $X \leq_{hr} Y$ *or* $g_X \leq_{hr} g_Y$*) if for any* $t \geq 0$,

$$r_X(t) \geq r_Y(t).$$

Definition 1.4.4. X *is smaller than* Y *in the likelihood ratio order (in short* $X \leq_{lr} Y$ *or* $g_X \leq_{lr} g_Y$*) if*

$$\frac{g_X(t)}{g_Y(t)} \downarrow t. \tag{1.13}$$

Theorem 1.4.5. $X \leq_{lr} Y \;\Rightarrow\; X \leq_{hr} Y \;\Rightarrow\; X \leq_{st} Y$.

None of the converse implications holds (see e.g. [86] for counterexamples).

For the usual stochastic orderings, the following characterization holds.

Theorem 1.4.6. $X \leq_{st} Y$ *if and only if, for any decreasing function* $\rho : \mathbb{R} \to \mathbb{R}$,

$$\int \rho(\theta) g_X(\theta) d\theta \geq \int \rho(\theta) g_Y(\theta) d\theta. \tag{1.14}$$

For a proof, see [82, 86].

The previous theorem also holds when X, Y are not absolutely continuous, in the following form:

$X \leq_{st} Y$ if and only if, for any decreasing function $\rho : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[\rho(X)] \geq \mathbb{E}[\rho(Y)].$$

The following characterization of the hazard rate order also provides a way to extend the definition of this order to variables that are not necessarily absolutely continuous.

Theorem 1.4.7. *$X \leq_{hr} Y$ if and only if $\dfrac{\overline{G}_Y(t)}{\overline{G}_X(t)}$ is increasing in t.*

1.4.2 Multivariate stochastic orders

Let $\mathbf{X}$, $\mathbf{Y}$ be two random vectors with joint survival functions $\overline{F}_{\mathbf{X}}$, $\overline{F}_{\mathbf{Y}}$ and probability densities $f_{\mathbf{X}}$, $f_{\mathbf{Y}}$ respectively.

On $\mathbb{R}^d$ we consider the usual componentwise partial order, defined as follows: let $\mathbf{x} = (x_1, \ldots, x_d)$, $\mathbf{y} = (y_1, \ldots, y_d)$ be two vectors in $\mathbb{R}^d$; then we write $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for $i = 1, \ldots, d$.

Definition 1.4.8. *A set $B \subseteq \mathbb{R}^d$ is called an* upper set *if $\mathbf{y} \in B$ whenever $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} \in B$.*

Definition 1.4.9. *$\mathbf{X}$ is smaller than $\mathbf{Y}$ in the usual stochastic order (in short $\mathbf{X} \leq_{st} \mathbf{Y}$ or $f_{\mathbf{X}} \leq_{st} f_{\mathbf{Y}}$) if $\mathbb{P}(\mathbf{X} \in B) \leq \mathbb{P}(\mathbf{Y} \in B)$ for any upper set $B \subseteq \mathbb{R}^d$.*

Definition 1.4.10. *$\mathbf{X}$ is smaller than $\mathbf{Y}$ in the upper orthant order (in short $\mathbf{X} \leq_{uo} \mathbf{Y}$ or $f_{\mathbf{X}} \leq_{uo} f_{\mathbf{Y}}$) if $\overline{F}_{\mathbf{X}}(\mathbf{t}) \leq \overline{F}_{\mathbf{Y}}(\mathbf{t})$ for any $\mathbf{t} \in \mathbb{R}^d$.*

$\mathbf{X} \leq_{st} \mathbf{Y}$ implies $\mathbf{X} \leq_{uo} \mathbf{Y}$, as it is immediately seen by letting the upper set B to be a d-rectangle. We may say, in a sense that will be clearer in short, that the upper orthant order is a weak notion of usual stochastic order.

A characterization of the usual stochastic order, analogous to the one in the one-dimensional case, is given by the condition

$$\int_{\mathbb{R}^d} \rho(\mathbf{t}) f_{\mathbf{X}}(\mathbf{t}) d\mathbf{t} \geq \int_{\mathbb{R}^d} \rho(\mathbf{t}) f_{\mathbf{Y}}(\mathbf{t}) d\mathbf{t} \tag{1.15}$$

for any decreasing function $\rho : \mathbb{R}^d \to \mathbb{R}$.

In the following, we use the symbols $\vee$ and $\wedge$:

$$\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, \ldots, x_d \vee y_d),$$

where $x_i \vee y_i \equiv \max\{x_i, y_i\}$; analogously

$$\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, \ldots, x_d \wedge y_d),$$

where $x_i \wedge y_i \equiv \min\{x_i, y_i\}$.

Definition 1.4.11. $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *in the multivariate likelihood ratio order (in short* $\mathbf{X} \leq_{lr} \mathbf{Y}$ *or* $f_{\mathbf{X}} \leq_{lr} f_{\mathbf{Y}}$*) if*

$$f_{\mathbf{X}}(\mathbf{x}) f_{\mathbf{Y}}(\mathbf{y}) \leq f_{\mathbf{X}}(\mathbf{x} \wedge \mathbf{y}) f_{\mathbf{Y}}(\mathbf{x} \vee \mathbf{y})$$

for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$*.*

$\mathbf{X} \leq_{lr} \mathbf{Y}$ implies the so-called *weak likelihood ratio* order. The latter notion is defined by imposing a condition that is analog to the characterization of the likelihood ratio order in the univariate case:

Definition 1.4.12. *We say that* $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *in the weak likelihood ratio order (in short* $\mathbf{X} \leq_{wlr} \mathbf{Y}$ *or* $f_{\mathbf{X}} \leq_{wlr} f_{\mathbf{Y}}$*), if*

$$\frac{f_{\mathbf{X}}(\mathbf{t})}{f_{\mathbf{Y}}(\mathbf{t})} \downarrow \mathbf{t}. \tag{1.16}$$

We recall that $\mathbf{X} \leq_{lr} \mathbf{Y}$ implies $\mathbf{X} \leq_{st} \mathbf{Y}$ and $\mathbf{X} \leq_{wlr} \mathbf{Y}$ implies $\mathbf{X} \leq_{uo} \mathbf{Y}$, but $\mathbf{X} \leq_{wlr} \mathbf{Y}$ does not imply $\mathbf{X} \leq_{st} \mathbf{Y}$.

Definition 1.4.13. *We say that a function* $f : \mathbb{R}^d \to \mathbb{R}$ *is MTP*$_2$ *if*

$$f(\mathbf{x}) f(\mathbf{y}) \leq f(\mathbf{x} \wedge \mathbf{y}) f(\mathbf{x} \vee \mathbf{y})$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$*.*

The Multivariate Total Positivity of order 2 can be interpreted as a strong notion of positive dependence, in analogy with the bivariate case of the TP$_2$ (see also [69] for an application of this notion to the study of interacting particles systems).
The following lemma shows how such a strong positive dependence property can be exploited to trace back a weak stochastic order to its strong version.

Lemma 1.4.14 ([67]). *If* $\dfrac{f_{\mathbf{Y}}(\mathbf{t})}{f_{\mathbf{X}}(\mathbf{t})} \uparrow \mathbf{t}$ *and* $f_{\mathbf{X}}$ *is MTP*$_2$*, then* $f_{\mathbf{X}} \leq_{lr} f_{\mathbf{Y}}$*.*

1.4.3 Stochastic monotonicity

We are often interested in comparing different distributions which arise as conditional distributions for a same random variable (or vector) X, given different observed events.

In this respect, we can obtain concepts of *stochastic monotonicity* corresponding in a natural way to the different notions of stochastic orderings.

Let Z be again a random variable or vector, taking values in a domain $\mathcal{Z} \subseteq \mathbb{R}^d$. We are still under the assumption of absolute continuity for the variables. We can consider then the conditional distribution and density of X given events of the kind $\{Z = z\}$. Let $\overline{G}_X(\cdot|z)$ denotes the conditional survival function.

Let $\preceq$ be a given partial ordering defined on $\mathcal{Z}$ and $\leq_*$ a fixed stochastic order.

Definition 1.4.15. *X is* stochastically increasing *in Z in the $\leq_*$ order, with respect to $\preceq$, if*

$$z' \preceq z'' \;\Rightarrow\; \overline{G}_X(\cdot|z') \leq_* \overline{G}_X(\cdot|z'').$$

When the partial order $\preceq$ on $\mathcal{Z}$ is not specified, we implicitly refer to the componentwise order on $\mathbb{R}^d$.

1.4.4 Dependence orderings

Dependence concepts induce *dependence orderings*, that are partial orderings on the family of copulas (or semi-copulas). However, they can be also interpreted as partial orderings on random vectors.

The most common dependence order is the *concordance order*, also called *PQD order.*

Definition 1.4.16. *Let C_1, C_2 be two copulas. We say that C_2 is more PQD than C_1 ($C_1 \preceq_{PQD} C_2$) if, for any $u, v \in [0, 1]$,*

$$C_1(u, v) \leq C_2(u, v).$$

Let (X_1, Y_1), (X_2, Y_2) be two random vectors and $\hat{C}_1$, $\hat{C}_2$ the survival copulas of (X_1, Y_1) and (X_2, Y_2) respectively.

Definition 1.4.17. (X_2, Y_2) *is* more concordant *or* more PQD *than* (X_1, Y_1) *if*

$$\hat{C}_1 \preceq_{PQD} \hat{C}_2.$$

Remark 1.4.18. *Even if the dependence meaning is lost, the partial order defined in Definition 1.4.16 can be formally extended to the set of semi-copulas.*

Any dependence concept can induce the corresponding dependence ordering: more LTD, more TP_2, more SI, more PLR. For definitions and examples, see e.g. [68].

Chapter 2

Dependence and evolution of dependence

This chapter is devoted to the study of the evolution of dependence, both in absence and in presence of defaults.

The analysis developed in absence of defaults can be seen as a more general case than tail dependence. As said in the Introduction, tail dependence concerns how dependence evolves after we observed extreme events, while by "evolution" we mean the changes in the dependence structure, due to the entire flow of information and not only to extreme events.

The analysis of the evolution in presence of defaults is inspired by the study of default contagion, phenomenon consisting in a decrease of the joint survival function of the $n-k$ surviving units after the k-th failure occurs, at the k-th default time.

The chapter is organized as follows. In Section 1, we provide conditions for preservation of dependence properties along the family of lower threshold copulas $\{C_z\}_{z\in(0,1]}$ of a pair of non-exchangeable r.v.'s, that, without loss of generality, are supposed to be uniformly distributed on $[0,1]$. By a change of parametrization, from $\{C_z\}_{z\in(0,1]}$, it is possible to obtain a family of upper threshold copulas $\{\hat{C}_t\}_{t\geq 0}$, coinciding with the family of survival copulas. Since we are considering a monotonic change of parametrization, the results of Section 1 can be applied to the family $\{\hat{C}_t\}_{t\geq 0}$, considered in Section 2. In order to operate such a change of parametrization and to give it a practical meaning, we assume the variables to be exchangeable. We then study some aspects of the monotonicity behaviour of the family $\{\hat{C}_t\}_{t\geq 0}$, both not considering the possibility of defaults and at default times. More precisely, we find out conditions for stochastic dependence being decreasing at default times and progressively increasing between two subsequent default times.

2.1 Threshold copulas and positive dependence

In this section we consider a pair of two continuous random variables X, Y, with joint distribution function $F(x,y) = \mathbb{P}(X \leq x, Y \leq y)$.

Starting from a notion of positive dependence $\mathbf{P}$ and from the family of the lower threshold copulas C_z associated to a bivariate distribution having copula C, we define different notions of positive dependence for C, reflecting the dependence properties of the copulas C_z for some z.

As first, we analyze some structural aspects of lower threshold copulas and of the given definitions. Furthermore we analyze the specific cases arising from relevant special choices of $\mathbf{P}$ (e.g., PQD, LTD, TP_2, PLR). Our analysis, in particular, allows us to present a number of examples and counter-examples, which can be useful in the study of tail dependence for a bivariate distribution.

For every real z such that $\mathbb{P}(X \leq z, Y \leq z) > 0$, we consider the new distribution function

$$F_z(x,y) = \mathbb{P}(X \leq x, Y \leq y | X \leq z, Y \leq z) . \tag{2.1}$$

As will be made clearer in Subsection 2.1.1, by Theorem 1.3.6, we are allowed, without any loss of generality, to consider, in place of (X,Y), a pair of random variables U, V uniformly distributed over $[0,1]$ and with joint distribution determined by the copula C. We can then replace F with C. Furthermore we consider, for $0 < z \leq 1$, the copula C_z defined as the connecting copula associated with F_z. The copulas C_z $(0 < z \leq 1)$ are called *lower threshold copulas* associated with C. In Subsection 2.1.1 we detail some structural aspects, relevant for our analysis, concerning C_z and the relations between C_z and C.

We can now explain the purposes of this section.
Let $\mathbf{P}$ be a positive dependence property. The condition "C_z satisfies $\mathbf{P}$" (for some z) can actually be interpreted as a condition on C. Now, let Λ be an interval of $(0,1]$.

Definition 2.1.1. *We say that C is $\langle \mathbf{P}; \Lambda \rangle$ if C_z is $\mathbf{P}$ for every $z \in \Lambda$. In particular, we say that C is hyper-$\mathbf{P}$ if C is $\langle \mathbf{P}; (0,1] \rangle$.*

An hyper-$\mathbf{P}$ property may be thought of as a property of positive dependence, of its own. As a main purpose of this section, we are interested in comparing the properties $\mathbf{P}$, $\langle \mathbf{P}; \Lambda \rangle$ and hyper-$\mathbf{P}$.

Our study is carried out in Section 2.1.2; therein we in fact analyze basic aspects of the Definition 2.1.1 and derive some conclusions that can be of interest for research about tail dependence. In particular we are interested to see cases where hyper- $\mathbf{P}$ does coincide with $\mathbf{P}$ or where hyper-$\mathbf{P}$ coincides with some other known property, stronger than $\mathbf{P}$. We specifically analyze here the properties $\langle \mathbf{P}; \Lambda \rangle$ and hyper-$\mathbf{P}$, for $\mathbf{P}$ = PQD, LTD, TP_2, PLR, thus providing a better comprehension of Definition 2.1.1 along with some useful examples and counter-examples.

For a better comparison with the literature about tail dependence, in this section we prefer to express our results in terms of distribution functions, connecting copulas, and lower threshold copulas. In other parts of the thesis, where we mostly refer to applied fields such as reliability, survival analysis, interacting defaults, X, Y are typically non-negative variables and, for given $\overline{F}(x, y) = \mathbb{P}(X > x, Y > y)$, we consider

$$\overline{F}_t(x, y) = \mathbb{P}(X > t + x, Y > t + y | X > t, Y > t),$$

in place of (2.1). For every $t \geq 0$ such that $\mathbb{P}(X > t, Y > t) > 0$, $\overline{F}_t$ is then the survival function of $(X - t, Y - t)$ conditional on the fact that $(X > t, Y > t)$. Denoting by $(X_t,\ Y_t)$ the random pair whose survival function coincides with $\overline{F}_t$, X_t, Y_t are then interpreted as *residual lifetimes.* The evolution of the dependence among X_t, Y_t can be studied in terms of the *upper threshold copulas*, that are the *survival copulas* of the $\overline{F}_t$'s. The results about lower threshold copulas can be equivalently reformulated for upper threshold copulas by means of a simple transformation.

2.1.1 Threshold copulas

Let U, V be two random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and uniformly distributed on $[0, 1]$ with joint distribution function the copula C. For every $z \in (0, 1]$, suppose that $C(z, z) > 0$. We are interested in the conditional distribution F_z of (U, V) given that $U \leq z$ and $V \leq z$. For every $x, y \in [0, z]$, we have:

$$F_z(x, y) = \mathbb{P}(U \leq x, V \leq y \mid U \leq z, V \leq z) = \frac{C(x, y)}{C(z, z)}.$$

The univariate marginal distribution functions G_z e H_z are given, for every $x \in [0, z]$, by

$$G_z(x) = \mathbb{P}(U \leq x, V \leq z \mid U \leq z, V \leq z) = \frac{C(x, z)}{C(z, z)}.$$

and

$$H_z(x) = \mathbb{P}(U \leq z, V \leq x \mid U \leq z, V \leq z) = \frac{C(z, x)}{C(z, z)}.$$

The connecting copula associated with F_z is obtained, for every u, v in $[0, 1]$, from

$$C_z(u, v) = \frac{C(h_z^{-1}(uh_z(z)), k_z^{-1}(vk_z(z)))}{C(z, z)}, \tag{2.2}$$

where, for any fixed $z \in (0, 1]$, $h_z : [0, 1] \to [0, z]$, $h_z(u) = C(u, z)$, is the *horizontal section* of C at the level z, and

$$h_z^{-1}(u) = \sup\{z' \in [0, 1] \mid C(z', z) \leq u\}$$

is its pseudo–inverse; analogously $k_z : [0,1] \to [0,z]$, $k_z(u) = C(z,u)$, is the *vertical section* of C at the level z, and k_z^{-1} is its pseudo–inverse. In particular, we suppose that, for any fixed $z \in (0,1]$, h_z and k_z are strictly increasing on $[0,z]$ and, therefore, h_z^{-1} and k_z^{-1} are their respective standard inverse functions on $[0, C(z,z)]$. Obviously, $h_z(z) = k_z(z) = C(z,z)$ for every $z \in [0,1]$.

Remark 2.1.2. *Eq.* (2.2) *shows that, for any z, C_z only depends on the restriction of C on $[0,z]^2$.*

The copulas C_z defined by (2.2) are called *lower threshold copulas* associated with C.

In the sequel, we will denote by $\widetilde{\mathcal{C}}$ the class of all copulas C satisfying our assumptions, i.e. $C(z,z) > 0$ for every $z \in (0,1]$, and C has horizontal and vertical sections (at a fixed $t \in (0,z]$) strictly increasing on $[0,z]$. In particular, every $C \in \widetilde{\mathcal{C}}$ generates the family of copulas $\{C_z\}_{z\in(0,1]}$. We notice that $C_1 = C$.

Remark 2.1.3. *Given a copula C, the left–residuum of C is the function $R_C^l : [0,1]^2 \to [0,1]$ defined by $R_C^l(x,y) = \sup\{z' \in [0,1] \mid C(z',x) \leq y\}$ and the right–residuum of C is the function $R_C^r : [0,1]^2 \to [0,1]$ defined by $R_C^r(x,y) = \sup\{z' \in [0,1] \mid C(x,z') \leq y\}$. These two functions have been proved to be useful in multivalued logic. Here, it is important to note that, by using the results of [40], both R_C^l and R_C^r are continuous in each argument with $R_C^l(z,u) = h_z^{-1}(u)$ and $R_C^r(z,u) = k_z^{-1}(u)$.*

Remark 2.1.4. *In previous papers, the lower and upper threshold copulas are called lower and upper tail dependence copulas. Here, we prefer to adopt a different terminology following [72, section 7.6.3], also in order to avoid confusion with the (different) notion of tail copula presented in [47].*

Before investigating the evolution of the dependence along the family $\{C_z\}_{z\in(0,1]}$, for what follows, it is important to note that, with z spanning $(0,1]$, the family has "no jumps", in the sense that the copulas C_{z_0} and C_{z_1} are close each other with respect to the L^∞–norm for sufficiently close z_0 and z_1, as stated in the following result.

Proposition 2.1.5. *Let $C \in \widetilde{\mathcal{C}}$. The mapping $\Psi : (0,1] \to \widetilde{\mathcal{C}}$, $z \mapsto C_z$, is continuous, in the sense that, for every $(u,v) \in [0,1]^2$ and for every $z_0 \in (0,1]$, $C_z(u,v)$ converges to $C_{z_0}(u,v)$ when z tends to z_0.*

Proof. We have to prove that, for all $u,v \in [0,1]$, for every fixed z_0 in $(0,1]$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that, if $|z - z_0| < \delta$, then $|C_z(u,v) - C_{z_0}(u,v)| < \varepsilon$.

First, for fixed $z_0 \in (0,1]$ and $(u,v) \in [0,1]^2$, consider that

$$\begin{aligned}|C_z(u,v) - C_{z_0}(u,v)| &= \left|\frac{C(x,y)}{C(z,z)} - \frac{C(x_0,y_0)}{C(z_0,z_0)}\right| \\ &= \frac{|C(x,y)C(z_0,z_0) - C(x_0,y_0)C(z,z)|}{C(z,z)C(z_0,z_0)},\end{aligned}$$

where

$x = h_z^{-1}(uh_z(z))$, $y = k_z^{-1}(vk_z(z))$, $x_0 = h_{z_0}^{-1}(uh_{z_0}(z_0))$, $y_0 = k_{z_0}^{-1}(vk_{z_0}(z_0))$.

For $|z - z_0| < \delta$, set $\alpha(z) = \dfrac{1}{C(z,z)C(z_0,z_0)}$ and $\overline{\alpha} = \sup\limits_{z:|z-z_0|<\delta} \alpha(z)$.
We have:

$$\begin{aligned}&|C(x,y)C(z_0,z_0) - C(x_0,y_0)C(z,z)| \\ \leq\ &|C(x,y)C(z_0,z_0) - C(x,y)C(z,z)| + |C(x,y)C(z,z) - C(x_0,y_0)C(z,z)| \\ =\ &C(x,y)|C(z_0,z_0) - C(z,z)| + C(z,z)|C(x,y) - C(x_0,y_0)|.\end{aligned}$$

Since a copula is a Lipschitz function (with constant 1),

$$|C(z_0,z_0) - C(z,z)| \leq 2|z - z_0| < 2\delta.$$

Analogously,

$$|C(x,y) - C(x_0,y_0)| \leq |x - x_0| + |y - y_0|.$$

In order to estimate $|x - x_0|$ and $|y - y_0|$, we notice that $h_z^{-1}(w)$ and $k_z^{-1}(w)$ are alternative notations for the left- and right- residua of C, $R^l = R_C^l$ and $R^r = R_C^r$ respectively. Then

$$\begin{aligned}&|x - x_0| = |R^l(z, uC(z,z)) - R^l(z_0, uC(z_0,z_0))| \\ \leq\ &|R^l(z, uC(z,z)) - R^l(z_0, uC(z,z))| + |R^l(z_0, uC(z,z)) - R^l(z_0, uC(z_0,z_0))|.\end{aligned}$$

R^l being continuous in each argument means that, for all $\eta_1,\ \eta_2 > 0$, there exist $\delta_1,\ \delta_2 > 0$ such that

i) for $|z - z_0| < \delta_1$, $|R^l(z,\cdot) - R^l(z_0,\cdot)| < \eta_1$;

ii) for $|w - w_0| < \delta_2$, $|R^l(\cdot,w) - R^l(\cdot,w_0)| < \eta_2$.

Therefore, for $u|C(z,z) - C(z_0,z_0)| \leq 2u|z - z_0| < \delta_2$,

$$|R^l(z_0, uC(z,z)) - R^l(z_0, uC(z_0,z_0))| \leq \eta_1.$$

Since $u|C(z,z) - C(z_0,z_0)| \leq 2u|z - z_0| < 2u\delta \leq 2\delta$, it is sufficient to take $\delta_2 = 2\delta$.

By decreasingness of R^l in the first variable,

$$|R^l(z,uC(z,z)) - R^l(z_0,uC(z,z))|$$
$$= \begin{cases} R^l(z,uC(z,z)) - R^l(z_0,uC(z,z)), & z \in (z_0-\delta, z_0), \\ R^l(z_0,uC(z,z)) - R^l(z,uC(z,z)), & z \in (z_0, z_0+\delta). \end{cases}$$

By increasingness of R^l in the second variable,

$R^l(z,uC(z,z)) - R^l(z_0,uC(z,z)) \leq R^l(z,uC(z_0,z_0)) - R^l(z_0,uC(z_0-\delta,z_0-\delta)) \leq$
$|R^l(z,uC(z_0,z_0)) - R^l(z_0,uC(z_0,z_0))| + |R^l(z_0,uC(z_0,z_0)) - R^l(z_0,uC(z_0-\delta,z_0-\delta))|$.
Again, by continuity of R^l in both the variables, for all $\bar{\eta}_1,\ \bar{\eta}_2 > 0$, there exist $\bar{\delta}_1,\ \bar{\delta}_2 > 0$ such that $|z-z_0| < \bar{\delta}_1$ and $|uC(z_0,z_0) - uC(z_0-\delta,z_0-\delta))| < \bar{\delta}_2$ respectively imply

$$|R^l(z,uC(z_0,z_0)) - R^l(z_0,uC(z_0,z_0))| < \bar{\eta}_1$$

and

$$|R^l(z_0,uC(z_0,z_0)) - R^l(z_0,uC(z_0-\delta,z_0-\delta))| < \bar{\eta}_2.$$

Since $u|C(z_0,z_0) - C(z_0-\delta,z_0-\delta)| < 2u\delta \leq 2\delta$, it is sufficient again to take $\bar{\delta}_2 = 2\delta$ and $\bar{\delta}_1 = \delta$.

Analogously

$R^l(z_0,uC(z,z)) - R^l(z,uC(z,z)) \leq R^l(z_0,uC(z_0+\delta,z_0+\delta)) - R^l(z,uC(z_0,z_0)) \leq$
$|R^l(z_0,uC(z_0+\delta,z_0+\delta)) - R^l(z_0,uC(z_0,z_0))| + |R^l(z_0,uC(z_0,z_0)) - R^l(z,uC(z_0,z_0))|$.
By continuity of R^l in the both variables, again, for all $\underline{\eta}_1,\ \underline{\eta}_2 > 0$, there exist $\underline{\delta}_1,\ \underline{\delta}_2 > 0$ such that, for

$$|z-z_0| < \underline{\delta}_1 \text{ and } |uC(z_0,z_0) - uC(z_0+\delta,z_0+\delta))| < \underline{\delta}_2,$$
$$|R^l(z_0,uC(z_0+\delta,z_0+\delta)) - R^l(z_0,uC(z_0,z_0))| < \underline{\eta}_1$$

and

$$|R^l(z_0,uC(z_0,z_0)) - R^l(z,uC(z_0,z_0))| < \underline{\eta}_2.$$

Again, it is sufficient to take $\underline{\delta}_1 = \delta$ and $\underline{\delta}_2 = 2\delta$.

Therefore

$$|R^l(z,uC(z,z)) - R^l(z_0,uC(z,z))| < \max(\bar{\eta}_1^l + \bar{\eta}_2^l, \underline{\eta}_1^l + \underline{\eta}_2^l) \equiv \eta^l.$$

Thus, for all η^l, there exists a $\delta_\eta > 0$ such that, for

$$|z-z_0| < \delta_\eta,\ |x-x_0| < \eta^l,$$

where it has to be

$$\delta_\eta = \min(\underline{\delta}_1, \bar{\delta}_1, \underline{\delta}_2, \bar{\delta}_2) = \delta.$$

By the same arguments applied to R^r_C, we have also that, for $|z-z_0| < \delta$, $|y-y_0| < \eta^r$. Thus, by putting $\eta = \eta^l + \eta^r$,

$$|C_z(u,v) - C_{z_0}(u,v)| \leq \overline{\alpha}(2\delta C(x,y) + \eta C(z,z)) \leq \overline{\alpha}(2\delta + \eta).$$

Choosing $\delta < \dfrac{\varepsilon}{2\overline{\alpha}} - \dfrac{\eta}{2}$, the proof is concluded. $\square$

In order to state some properties about the family $\{C_z\}_{z\in(0,1]}$, it is hence important to consider whether the absolute continuity is preserved by every C_z.

Proposition 2.1.6. *Let C be a copula having non-zero first derivatives almost everywhere on $[0,1]^2$. If C is absolutely continuous, then C_z is absolutely continuous for every $z \in (0,1]$.*

Proof. For every $z \in (0,1]$, the second mixed derivative of C_z is given by:

$$\partial^2_{12} C_z(u,v) = \frac{C(z,z)\partial^2_{12}C(h_z^{-1}(uh_z(z)), k_z^{-1}(vk_z(z)))}{\partial_1 C(h_z^{-1}(uh_z(z)), z) \cdot \partial_2 C(z, k_z^{-1}(vk_z(z)))}. \tag{2.3}$$

We have to check the equality

$$C_z(u,v) = \int_0^u \int_0^v \partial^2_{12} C_z(\gamma,\theta)\, \mathrm{d}\,\gamma \mathrm{d}\,\theta. \tag{2.4}$$

Applying (2.2) and (2.4), we obtain

$$\frac{C(x,y)}{C(z,z)} = \int_0^u \int_0^v \frac{C(z,z)\partial^2_{12}C(h_z^{-1}(\gamma h_z(z)), k_z^{-1}(\theta k_z(z)))}{\partial_1 C(h_z^{-1}(\gamma h_z(z)), z)\partial_2 C(z, k_z^{-1}(\theta k_z(z)))}\, \mathrm{d}\,\gamma \mathrm{d}\,\theta.$$

Changing variables by means of

$$\xi = h_z^{-1}(\gamma h_z(z)), \quad \eta = k_z^{-1}(\theta k_z(z)), \tag{2.5}$$

we obtain

$$C(x,y) = \int_0^x \int_0^y \partial^2_{12} C(\xi,\eta) \mathrm{d}\,\xi \mathrm{d}\,\eta,$$

which is the desired assertion. □

2.1.2 Lower threshold copulas and generated dependence properties

Consider $C \in \widetilde{\mathcal{C}}$ and let $\{C_z\}_{z\in(0,1]}$ be the family of corresponding lower threshold copulas.

As mentioned before, in this subsection we aim at comparing, for a given dependence property $\mathbf{P}$, the properties $\mathbf{P}$, hyper-$\mathbf{P}$, and $\langle \mathbf{P}; \Lambda \rangle$. Preliminarily, we point out some basic aspects of Definition 2.1.1. Then, for a number of relevant notions of dependence $\mathbf{P}$, we specifically analyze the properties $\langle \mathbf{P}; \Lambda \rangle$, hyper-$\mathbf{P}$, and relations among them.

Remark 2.1.7. *Let Λ be an arbitrary set of $(0,1]$, $\overline{\Lambda}$ being the closure of Λ and consider the two conditions $\langle \mathbf{P}; \Lambda \rangle$, $\langle \mathbf{P}; \overline{\Lambda} \rangle$. As an immediate consequence of the continuity property, proved in Proposition 2.1.5, these two conditions coincide. We may then argue that, in considering the property $\langle \mathbf{P}; \Lambda \rangle$, we can limit attention to the cases when Λ is a closed subset of $(0,1]$, i.e., an interval or a union of disjoint intervals.*

Actually we are mainly interested in subsets Λ of the form $(0, \lambda]$, for some constant $\lambda \in (0, 1]$.

We notice that the condition $\langle \mathbf{P}; (0, \lambda] \rangle$ for a copula C has the following immediate meaning: it means that C may not satisfy $\mathbf{P}$ and that the property $\mathbf{P}$ possibly holds for all C_z, with z being below a given λ. This notion can be of interest in the study of tail dependence. In fact we are typically interested in proving that C_z satisfies a dependence property $\mathbf{P}$ in the limit for $z \to 0$. Thus, proving that C is $\langle \mathbf{P}; (0, \lambda] \rangle$ guarantees the above condition without explicitly computing the limit $\lim_{z \to 0} C_z$.
A recent result about contagion and connections with tail dependence is given by [39].

As a consequence of the formula (2.2) (see Remark 2.1.2), we can state that $\langle \mathbf{P}; \Lambda \rangle$ only depends on the behaviour of C on Λ^2 (see Propositions 2.1.8, 2.1.12, 2.1.13, 2.1.16).

Generally, the property $\langle \mathbf{P}; \Lambda \rangle$ does not imply $\mathbf{P}$ for C (see Example 2.1.11) nor does $\mathbf{P}$ imply $\langle \mathbf{P}; \Lambda \rangle$ (see Example 2.1.9).

As far as the property hyper-$\mathbf{P}$ is concerned, we notice that hyper-$\mathbf{P}$ implies $\mathbf{P}$, just by definition, since $C_1 = C$.

Specifically, we pass to analyze the properties $\langle \mathbf{P}; \Lambda \rangle$ and hyper-$\mathbf{P}$ for relevant notions of dependence recalled below (see also [63, 78] for a more complete overview).

From now on, in this section, we assume $\Lambda = (0, \lambda]$ be an interval of $(0, 1]$ and $C \in \widetilde{\mathcal{C}}$.

Proposition 2.1.8. *A copula C is $\langle \mathrm{PQD}; \Lambda \rangle$ if and only if C satisfies*

$$C(x, y)C(z, z) \geq C(x, z)C(z, y) \tag{2.6}$$

for every $x, y, z \in \Lambda$, $x \leq z$ and $y \leq z$.

Proof. By definition, C is $\langle \mathrm{PQD}; \Lambda \rangle$ means that $C_z(u, v) \geq uv$ for all $z \leq \lambda$, that is $C\left(h_z^{-1}(uh_z(z)), k_z^{-1}(vk_z(z))\right) \geq uvC(z, z)$.
By putting $x = h_z^{-1}(uh_z(z)),\ y = k_z^{-1}(vk_z(z))$, we get

$$C(x, y) \geq \frac{h_z(x)}{h_z(z)} \frac{k_z(y)}{k_z(z)} h_z(z),$$

that is inequality (2.6). □

According to the general Definition 2.1.1, the copulas $\Pi(u, v) = uv$ and $M(u, v) = \min(u, v)$ are hyper-PQD.

It is also easy to show that, if C is TP_2, then it is *hyper*-PQD. Moreover, as stated in general before, if C is *hyper*-PQD, then it is PQD. The converse implication is false, as the following example shows.

Example 2.1.9. *Let C be the copula given by*

$$C(u,v) = \min\left(u, v, \frac{u^2+v^2}{2}\right).$$

Then C is PQD, but not $\langle \mathrm{PQD}; \Lambda \rangle$ for $\Lambda = \left(0, \frac{3}{5}\right]$. In fact, by considering $x = y = \frac{1}{2}$, we obtain that

$$C(x,y)C(z,z) = \frac{36}{100}\frac{25}{100} < \left(\frac{61}{200}\right)^2 = C(x,z)C(z,y).$$

Therefore, $C(u,v) \geq uv$ on $[0,1]^2$, but $C_{\frac{3}{5}}(u,v) < uv$ for some $(u,v) \in [0,1]^2$. Actually, in view of Proposition 2.1.5, for every z belonging to a neighbourhood of $\frac{3}{5}$, C_z is not PQD.

It is interesting to note that, for an Archimedean copula $C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v))$ [78], the notions of TP_2 and *hyper*-PQD coincide. Furthermore, these conditions are equivalent to $z \mapsto \varphi(e^{-z})$ being convex: the proof can be derived directly from [2, section 4.5].

It is known and immediately follows by Definition 1.3.11, that upper and lower bound of a PQD copula are respectively $M(u,v) = \min(u,v)$ and $\Pi(u,v) = uv$. Since, in general, a PQD copula is not *hyper*-PQD, the lower bound for C is not preserved for C_z. Even in this case, we can obtain for C a better lower bound than $W(u,v) = \max(u+v-1, 0)$.

Proposition 2.1.10. *If C is PQD, then, for every $z \in (0,1]$, we have that*

$$C_z(u,v) \geq \max(uvz^3, \max(u+v-1, 0)). \tag{2.7}$$

Proof. Since C is PQD, $uv \leq C(u,v) \leq \min(u,v)$.
In particular, $uz \leq h_z(u) \leq \min(u,z)$ and, consequently,

$$u \leq h_z^{-1}(u) \leq \min\left(\frac{u}{z}, z\right) \quad \text{for all } u \in [0,z].$$

Since the fact that an analogous inequality can be proved for k_z, we obtain

$$C_z(u,v) \geq \frac{C(h_z^{-1}(uz^2), k_z^{-1}(vz^2))}{z} \geq \frac{h_z^{-1}(uz^2) \cdot k_z^{-1}(vz^2)}{z} \geq \ uvz^3,$$

and hence (2.7) holds. □

Now, note that if C satisfies $\langle \mathrm{PQD}, (0,\lambda] \rangle$ for a given $\lambda < 1$, then C needs not be PQD.

Example 2.1.11. *Consider, for example, the copula C given by*

$$C(u,v) = \begin{cases} \dfrac{uv}{\lambda}, & (u,v) \in [0,\lambda]^2, \\ \lambda + (1-\lambda)C'\left(\dfrac{u-\lambda}{1-\lambda}, \dfrac{v-\lambda}{1-\lambda}\right), & (u,v) \in [\lambda,1]^2, \\ \min(u,v), & \text{otherwise}, \end{cases}$$

where $C'(u,v) = uv[1-(1-u)(1-v)]$ is a copula that is not PQD, and hence nor C is PQD. Actually, C is an ordinal sum of the copulas Π and C' with respect to the partition $([0,\lambda],[\lambda,1])$ (see e.g. [78] or previous chapter). Now, compute $C_z(u,v)$ for any $z \in (0,\lambda]$. We obtain

$$C_z(u,v) = \frac{\lambda}{z^2} \cdot C(uz,vz) = uv.$$

It follows that C_z is PQD for any $z \leq \lambda$, even if C is not PQD.

Now, let us consider the other dependence properties, LTD, TP_2, PLR. Similarly to Proposition 2.1.8, the following result holds for the LTD property.

Proposition 2.1.12. *C is $\langle \mathrm{LTD}(V \mid U); \Lambda \rangle$ if and only if*

$$C(x,y)C(x',z) \geq C(x,z)C(x',y) \tag{2.8}$$

for all $z \in \Lambda$, $x,x',y \in [0,z]$ such that $x \leq x'$.

Proof. Since C is $\langle \mathrm{LTD}(V \mid U); \Lambda \rangle$, then, for any $u \leq u'$,

$$\frac{C(h_z^{-1}(uC(z,z)), k_z^{-1}(vC(z,z)))}{u} \geq \frac{C(h_z^{-1}(u'C(z,z)), k_z^{-1}(vC(z,z)))}{u'},$$

that is

$$\frac{C(z,z)}{C(x,z)}C(x,y) \geq C(x',y)\frac{C(z,z)}{C(x',z)}$$

for all $x,x',y \leq z$ such that $x \leq x'$. Thus, we obtain (2.8). □

An analogous condition holds and can be proved in the same way for $\langle \mathrm{LTD}(U \mid V); \Lambda \rangle$.

Examples of *hyper*-$\mathrm{LTD}(V \mid U)$ copulas are $\Pi(u,v) = uv$ and $M(u,v) = \min(u,v)$. Since they are symmetric with respect to u and v, they also are *hyper*-$\mathrm{LTD}(U \mid V)$.

As to TP_2, we find some differences from the other dependence properties taken into account until now. As we can easily deduce from the following proposition, $\langle \mathrm{TP}_2; \Lambda \rangle$ is a weaker property than TP_2. We can build a copula satisfying $\langle \mathrm{TP}_2; \Lambda \rangle$, but not TP_2 (see Example 5.2.9 below).

Proposition 2.1.13. *A copula C is $\langle \mathrm{TP}_2; \Lambda \rangle$ if and only if C satisfies* (1.5) *on* Λ^2.

Proof. By definition, C is $\langle \mathrm{TP}_2; \Lambda \rangle$ if and only if, for every $z \in \Lambda$, C_z is TP_2, i.e.

$$C_z(u, v) C_z(u', v') \geq C_z(u, v') C_z(u', v),$$

for all $u \leq u'$ and $v \leq v'$. Writing explicitly the multiplicands and making suitable substitutions, we obtain that, for every $x, x', y, y' \in [0, z]$, $x \leq x'$, $y \leq y'$, and for every $z \in \Lambda = (0, \lambda]$,

$$C(x, y) C(x', y') \geq C(x, y') C(x', y),$$

which is the desired assertion. □

In particular, C is *hyper*-TP_2 if and only if C is TP_2, as the following corollary states.

Corollary 2.1.14. *Let $C \in \widetilde{\mathcal{C}}$ be the connecting copula of the random pair (U, V). Then the following statements are equivalent:*

(a) C is TP_2;

(b) C is hyper-TP_2;

(c) C is hyper-LTD$(V \mid U)$ and hyper-LTD$(U \mid V)$.

It is easy to show that both *hyper*-LTD$(V \mid U)$ and *hyper*-LTD$(U \mid V)$) separately imply that C is PQD.

We note that, in order to obtain for C the stronger property TP_2, both *hyper*-LTD$(V \mid U)$ and *hyper*-LTD$(U \mid V)$ have to be satisfied. Statement (c) can be simplified if we have a condition that guarantees

$$LTD(V \mid U) \Leftrightarrow LTD(U \mid V) \tag{2.9}$$

(and therefore *hyper*-LTD$(V \mid U) \Leftrightarrow$ *hyper*-LTD$(U \mid V)$). In this case we simply say that C is LTD (*hyper*-LTD). A sufficient condition for (2.9) is C being exchangeable.

Corollary 2.1.15. *Let $C \in \widetilde{\mathcal{C}}$ be the connecting copula of the exchangeable random pair (U, V). Then the following statements are equivalent:*

(a) C is $\langle \mathrm{LTD}(V \mid U); \Lambda \rangle$;

(b) C is $\langle \mathrm{LTD}(U \mid V); \Lambda \rangle$;

(c) C satisfies (1.5) *on* Λ^2.

In particular, C hyper-LTD$(V \mid U)$ (resp. hyper-LTD$(U \mid V)$) is equivalent to C being TP_2.

Finally, we consider the PLR property, which can be introduced only for a copula C that is *absolutely continuous* on $[0,1]^2$, i.e.

$$C(u,v) = \int_0^u \int_0^v \partial_{12}^2 C(\gamma,\theta)\,\mathrm{d}\,\gamma \mathrm{d}\,\theta,$$

where $\partial_{12}^2 C$ denotes the second mixed derivative of C.

Proposition 2.1.16. *C is $\langle \mathrm{PLR}; \Lambda \rangle$ if and only if $\partial_{12}^2 C$ satisfies* (1.5) *on* Λ^2.

Proof. By definition, C is $\langle \mathrm{PLR}; \Lambda \rangle$ if and only if C_z is PLR for every $z \in \Lambda$, i.e.

$$\frac{\partial^2 C_z(u,v)}{\partial u \partial v} \frac{\partial^2 C_z(u',v')}{\partial u \partial v} \geq \frac{\partial^2 C_z(u',v)}{\partial u \partial v} \frac{\partial^2 C_z(u,v')}{\partial u \partial v}$$

for every $u, u', v, v' \in [0,1]$, $u \leq u'$ and $v \leq v'$. Using Eq. (2.3), after some simplification, we obtain the desired assertion. □

Analogously to TP_2, it may be proven that C being PLR is equivalent to C being *hyper*-PLR.

The relationships among considered dependence properties and related *hyper*-dependence properties are summarized here.

$$\begin{array}{ccccccc}
\langle \mathrm{PLR}; \Lambda \rangle & \Rightarrow & \langle \mathrm{TP}_2; \Lambda \rangle & \Rightarrow & \langle \mathrm{LTD}(V \mid U); \Lambda \rangle \ (\text{or } \langle \mathrm{LTD}(V \mid U); \Lambda \rangle) & \Rightarrow & \langle \mathrm{PQD}; \Lambda \rangle \\
\Uparrow & & \Uparrow & & \not\Uparrow \not\Downarrow & & \not\Uparrow \not\Downarrow \\
\mathrm{PLR} & \Rightarrow & \mathrm{TP}_2 & \Rightarrow & \mathrm{LTD}(U \mid V) \ (\text{or } \mathrm{LTD}(V \mid U)) & \Rightarrow & \mathrm{PQD} \\
\Updownarrow & & \Updownarrow & & \Uparrow & & \Uparrow \\
\textit{hyper}\text{-PLR} & \Rightarrow & \textit{hyper}\text{-TP}_2 & \Rightarrow & \textit{hyper}\text{-LTD}(U \mid V) \ (\text{or } \textit{hyper}\text{-LTD}(V \mid U)) & \Rightarrow & \textit{hyper}\text{-PQD} \\
 & & & & \Downarrow & & \Downarrow \\
 & & & & \langle \mathrm{LTD}(V \mid U); \Lambda \rangle \ (\text{or } \langle \mathrm{LTD}(V \mid U); \Lambda \rangle) & \Rightarrow & \langle \mathrm{PQD}; \Lambda \rangle
\end{array}$$

2.2 Semigroups of semi-copulas and evolution of dependence at increase of age

Let $\overline{F}(x_1, \ldots, x_n) = \mathbb{P}(T_1 > x_1, \ldots, T_n > x_n)$ be a n-variate joint survival function, with univariate margins $\overline{G}_i(x) = \mathbb{P}(T > x)$, $i = 1, \ldots, n$.

Such a $\overline{F}(x_1, \ldots, x_n)$ identifies a survival model M, consisting of n units $D_1, \ldots, D_n$, whose lifetimes are n random variables $T_1, \ldots, T_n$.

The dependence structure of M is described by suitable analytical properties of the survival copula $\hat{C}(u_1, \ldots, u_n) = \overline{F}\left\{\overline{G}_1^{-1}(u_1), \ldots \overline{G}_n^{-1}(u_n)\right\}$.

We recall that, first of all, $\overline{G}_1, \ldots, \overline{G}_n$ have to be continuous, in order to admit a unique copula. This implies $\overline{F}$ to be continuous as well. We also assume $\overline{G}_1, \ldots, \overline{G}_n$ to be strictly monotonic, so that they are invertible, and $\overline{F}$ to be strictly decreasing in each variable. Recalling that the n-variables functions we deal with are symmetric, we could limit ourselves to assume

that, for any fixed $t \geq 0$, $\overline{F}(\cdot, t, \ldots, t)$ is strictly decreasing on $[0, t]$. This assumption is sufficient to imply $\hat{C}(\cdot, z, \ldots, z)$ being strictly increasing on $[0, z]$, for any $z \in [0, 1]$.
Furthermore, we recall, we are in the absolutely continuous case, so that both $\overline{F}$ and $\overline{G}_1, \ldots, \overline{G}_n$ admit density.

An item of general interest is the model M_t, obtained from M by conditioning on survival at time $t > 0$ of all the units, that is, conditional on the event $\{T_1 > t, \ldots, T_n > t\}$. The model M_t can be seen as an evolution at time t of the model $M \equiv M_0$. Furthermore, M_t is formally analogous to M and it is described by the same kind of functions: a conditional joint survival function

$$\begin{aligned} \overline{F}_t\left(x_1, \ldots, x_n\right) &= \mathbb{P}\left(T_1 > t + x_1, \ldots, T_n > t + x_n | T_1 > t, \ldots, T_n > t\right) \\ &= \frac{\overline{F}\ \left(x_1 + t, \ldots, x_n + t\right)}{\overline{F}\ \left(t, \ldots, t\right)}, \end{aligned}$$

with univariate margins

$$\overline{G}_t^{(i)}\left(x\right) \equiv \mathbb{P}\left(T_i > t + x | T_1 > t, \ldots, T_n > t\right), \quad i = 1, \ldots, n.$$

As a natural consequence of the introduction of the families $\{\overline{F}_t\}_{t\geq 0}$ and $\{\overline{G}_t^{(i)}\}_{t\geq 0}$, it is interesting to study the evolution in time of the family of copulas $\{\hat{C}_t\}_{t\geq 0}$, where

$$\hat{C}_t(u_1, \ldots, u_n) \equiv \overline{F}_t\left\{(\overline{G}_t^{(1)})^{-1}\left(u_1\right), \ldots (\overline{G}_t^{(n)})^{-1}\left(u_n\right)\right\}. \tag{2.10}$$

Under the above-mentioned continuity and monotonicity assumptions on $\overline{F}$ and $\overline{G}_1, \ldots, \overline{G}_n$, it is easy to check that we can write $\hat{C}_t$ in terms of $\hat{C}$:

$$\hat{C}_t(u_1, \ldots, u_n) = \frac{\hat{C}[\overline{G}_1((\overline{G}_t^{(1)})^{-1}(u_1) + t), \ldots, \overline{G}_n((\overline{G}_t^{(n)})^{-1}(u_n) + t)]}{\hat{C}(\overline{G}_1(t), \ldots, \overline{G}_n(t))}. \tag{2.11}$$

However, we are mainly interested in the case when $T_1, \ldots, T_n$ are exchangeable.

As already discussed in Section 1.2, all the notation can be simplified as follows: the joint survival function $\overline{F}$ is symmetric and therefore for any $i = 1, \ldots, n$, $\overline{G}_i = \overline{G}$. Consequently, the survival copula is symmetric as well and, in view of Eq. (1.4), it can be written as

$$C(u_1, \ldots, u_n) = F(G^{-1}(u_1), \ldots, G^{-1}(u_n)). \tag{2.12}$$

Since conditioning is symmetric in all the variables, the vector

$$(T_1 - t, \ldots, T_n - t) | T_1 > t, \ldots, T_n > t$$

of the *residual lifetimes* of $D_1, \ldots, D_n$ at time t is exchangeable as well.

Proposition 2.2.1. *Let* $(T_1, \ldots, T_n)$ *an exchangeable vector. Then, for any* $t \geq 0$, $(T_1 - t, \ldots, T_n - t) | T_1 > t, \ldots, T_n > t$ *is exchangeable.*

Proof. Let σ be a permutation of n elements:

$$\sigma(1, \ldots, n) = (\sigma_1, \ldots, \sigma_n) .$$

By hypothesis, for any σ,

$$\overline{F}(x_1, \ldots, x_n) = \overline{F}(x_{\sigma_1}, \ldots, x_{\sigma_n}) . \tag{2.13}$$

Since we can write $\overline{F}_t$ in terms of $\overline{F}$, we have that, for any permutation σ,

$$\overline{F}_t(x_1, \ldots, x_n) = \frac{\overline{F}\ (x_1 + t, \ldots, x_n + t)}{\overline{F}\ (t, \ldots, t)} = \frac{\overline{F}\ (\sigma(x_1 + t, \ldots, x_n + t))}{\overline{F}\ (t, \ldots, t)}$$

$$= \frac{\overline{F}\ (x_{\sigma_1} + t, \ldots, x_{\sigma_n} + t)}{\overline{F}\ (t, \ldots, t)} = \overline{F}_t(x_{\sigma_1}, \ldots, x_{\sigma_n}) .$$

□

Therefore, for any $t \geq 0$, the conditional joint survival function $\overline{F}_t$ is symmetric and its margins are given, for any $i = 1, \ldots, n$, by

$$\overline{G}_t^{(i)}(x) = \overline{G}_t(x) = \frac{\overline{F}\ (x + t, t, \ldots, t)}{\overline{F}\ (t, \ldots, t)}.$$

Again, we write $\hat{C}_t$ in terms of $\hat{C}$:

$$\hat{C}_t(u_1, \ldots, u_n) = \frac{\hat{C}[\overline{G}(\overline{G}_t^{-1}(u_1) + t), \ldots, \overline{G}(\overline{G}_t^{-1}(u_n) + t)]}{\hat{C}(\overline{G}(t), \ldots, \overline{G}(t))}. \tag{2.14}$$

From now on, in this section, we focus on the bivariate case. We consider a pair of exchangeable lifetimes X, Y and the family of the conditional survival functions $\overline{F}_t(x, y)$ of $(X - t, Y - t)$ given $(X > t, Y > t)$.

Survival functions and consequently survival copulas

$$\hat{C}_t(u, v) = \frac{\hat{C}[\overline{G}(\overline{G}_t^{-1}(u) + t), \overline{G}(\overline{G}_t^{-1}(v) + t)]}{\hat{C}(\overline{G}(t), \overline{G}(t))}. \tag{2.15}$$

are more natural tools when we deal with non-negative random variables, in applied fields as Reliability or Finance.

It is possible to obtain the family of survival copulas $\{\hat{C}_t\}_{t \geq 0}$ from the one of lower threshold copulas $\{C_z\}_{z \in (0,1]}$, defined by (2.2) (applied to the survival copula $\hat{C}$), by adopting for the family a different parametrization. In this case, the change of parametrization is $t = \overline{G}^{\ -1}(z)$.

Such a change of parameterization is made possible by

- the strict monotonicity of the margin $\overline{G}$;
- the exchangeability of the variables, that is by the fact that the margins are equal.

Under these hypotheses, all the results of the previous section can be extended to the family of survival copulas.

The parametrization in $t \in [0, +\infty)$ has the disadvantage to depend on the margin $\overline{G}$; however, it is more proper to state and prove the following result.

Let be $r, s \geq 0$, $\hat{C}_r$ the survival copula obtained by applying (2.15) to $\hat{C}$, for $t = r$, and $(\hat{C}_r)_s$ the survival copula obtained by applying again (2.15), this time to $\hat{C}_r$, for $t = s$.

Proposition 2.2.2. *Using the above notation, for any $r, s \geq 0$,*

$$(\hat{C}_r)_s = \hat{C}_{r+s}.$$

Proof. For fixed $r > 0$ we consider the survival model with joint survival function $\overline{F}_r$ and margin $\overline{G}_r$.
We now notice the semigroup property of the families $\{\overline{F}_t\}_{t\geq 0}$ and $\{\overline{G}_t\}_{t\geq 0}$. We can associate to the new model the families $\{(\overline{F}_r)_s\}_{s\geq 0}$ and $\{(\overline{G}_r)_s\}_{s\geq 0}$, such that, for any $t, s \geq 0$,

$$(\overline{F}_r)_s = \overline{F}_{r+s} \text{ and } (\overline{G}_r)_s = \overline{G}_{r+s}.$$

By definition, the survival copula of the model $\overline{F}_r$ is

$$\hat{C}_r(u, v) \equiv \overline{F}_r\left(\overline{G}_r^{-1}(u), \overline{G}_r^{-1}(v)\right).$$

We have to prove that

$$(\hat{C}_r)_s = \hat{C}_{r+s} \qquad \forall r, s \geq 0.$$

By applying (2.15) to $(\hat{C}_r)_s$, we obtain

$$(\hat{C}_r)_s(u, v) = \frac{\hat{C}_r\left[\overline{G}_r\left(\overline{G}_{r+s}^{-1}(u) + s\right), \overline{G}_r\left(\overline{G}_{r+s}^{-1}(v) + s\right)\right]}{\hat{C}_r(\overline{G}_r(s), \overline{G}_r(s))}. \tag{2.16}$$

By applying again (2.15) to $\hat{C}_r$ in Eq. (2.16),

$$(\hat{C}_r)_s(u, v) = \frac{\hat{C}\left[\overline{G}\left(\overline{G}_{r+s}^{-1}(u) + s + r\right), \overline{G}\left(\overline{G}_{r+s}^{-1}(v) + s + r\right)\right]}{\hat{C}(\overline{G}(r+s), \overline{G}(r+s))}. \tag{2.17}$$

The thesis follows by pointing out that the right-hand side of the last equation coincides with $\hat{C}_{r+s}(u, v)$. $\square$

As mentioned in the Introduction, one purpose of ours is to analyze increase or decrease of dependence between residual lifetimes, with respect to the PQD order (see Definition 1.4.16).

Considering $X_2 = X_1 - t$ and $Y_2 = Y_1 - t$, monotonicity of the mapping $t \mapsto \hat{C}_t$, with respect to the "more PQD" order, has the following meaning: if $t \mapsto \hat{C}_t$ is increasing in t, the residual lifetimes will be more and more dependent as age increases.

We are then interested in describing analytical conditions for such monotonicity properties.

To this purpose, we suppose $t \mapsto \hat{C}_t(u, v)$ differentiable for any $u, v \in (0, 1)$ (see below).

Since we are dealing with exchangeable variables and hence with symmetric survival functions and survival copulas, we can write

$$\nabla \hat{C}(z, z) \cdot (1, 1) = 2 \frac{\partial}{\partial x_1} \hat{C}(z, z) = 2 \frac{\partial}{\partial x_2} \hat{C}(z, z).$$

Remark 2.2.3. *The differentiability of $t \mapsto \hat{C}_t(u, v)$ for any $u, v \in (0, 1)$ is guaranteed by the following conditions on $\hat{C}$:*

- $\hat{C}(t, t) > 0$ *for any* $t \in (0, 1]$,
- $\frac{\partial}{\partial x_1} \hat{C}(u, v)$, $\frac{\partial}{\partial x_2} \hat{C}(u, v)$ *exist and are strictly positive for any* $(u, v) \in (0, 1)^2$.

We will denote by $\frac{\partial R}{\partial x_i}(z, w)$ the partial derivative of the residuum $R(x_1, x_2)$ with respect to the i-th variable computed in $(x_1, x_2) = (z, w)$.

As we can expect in view of Eq. (2.15), under the hypothesis that $t \mapsto \hat{C}_t(u, v)$ is differentiable for any $u, v \in (0, 1)$, the monotonicity properties of $t \mapsto \hat{C}_t$ can be characterized in terms of $\hat{C}$ and its sections. We have in fact

Proposition 2.2.4. *Let $t \mapsto \hat{C}_t(u, v)$ be differentiable for any $u, v \in (0, 1)$. The map $t \mapsto \hat{C}_t$ is increasing if and only if, for any $u, v \in [0, 1]$,*

$$2 \left[\hat{C}(u, v) - \nabla \hat{C}(u, v) \cdot \left(\frac{\partial R(1, u)}{\partial x_1}, \frac{\partial R(1, v)}{\partial x_1} \right) \right. \geq 0. \quad (2.18)$$

Proof. In view of the semigroup property of $\{\hat{C}_t\}_{t \geq 0}$, it is sufficient to study the sign of the derivative of $\hat{C}_t$ w.r.t. t for $t = 0$. Since the change of parameter given by $z = \overline{G}(t)$ is strictly decreasing, instead of differentiating Eq. (2.15) w.r.t. t, we can differentiate w.r.t. z the simpler Eq. (2.2), that, in the case of exchangeable variables, can be further simplified as

$$C_z(u, v) = \frac{\hat{C}(h_z^{-1}(u h_z(z)), h_z^{-1}(v h_z(z)))}{\hat{C}(z, z)}.$$

We recall that $h_z^{-1}(w)$ can be alternatively denoted by $R(z,w)$. Since we are in the exchangeable case, $R^l_{\hat{C}} = R^r_{\hat{C}} = R$.

Thus we need to check that

$$\left.\frac{\partial}{\partial z} C_z(u,v)\right|_{z=1} \leq 0.$$

To this purpose, we now compute the partial derivative $\dfrac{\partial}{\partial z} C_z(u,v)$.

$$\begin{aligned}
\frac{\partial}{\partial z} C_z(u,v) \quad = \quad & \frac{1}{\hat{C}(z,z)^2}\Bigg\{\hat{C}(z,z)\nabla\hat{C}(R(z,u\hat{C}(z,z)),R(z,v\hat{C}(z,z))) \\
& \cdot\left(\frac{dR}{dz}(z,u\hat{C}(z,z)),\frac{dR}{dz}(z,v\hat{C}(z,z))\right) \\
- \quad & \hat{C}\left(R(z,u\hat{C}(z,z)),R(z,v\hat{C}(z,z))\right)\nabla\hat{C}(z,z)\cdot(1,1)\Bigg\},
\end{aligned}$$

where

$$\frac{dR}{dz}(z,u\hat{C}(z,z)) = \frac{\partial R}{\partial x_1}(z,u\hat{C}(z,z)) + u\frac{\partial R}{\partial x_2}(z,u\hat{C}(z,z))\left[\nabla\hat{C}(z,z)\cdot(1,1)\right].$$

Since $[\hat{C}(z,z)]^2$ is positive for any $z>0$,

$$\frac{\partial}{\partial z} C_z(u,v) \leq 0$$

if and only if

$$\begin{aligned}
&\hat{C}(z,z)\nabla\hat{C}(R(z,u\hat{C}(z,z)),R(z,v\hat{C}(z,z)))\cdot \\
&\quad\cdot\left(\begin{array}{c}
\frac{\partial R}{\partial x_1}(z,u\hat{C}(z,z)) + 2u\frac{\partial R}{\partial x_2}(z,u\hat{C}(z,z))\frac{\partial\hat{C}}{\partial x_1}(z,z) \\
\frac{\partial R}{\partial x_1}(z,v\hat{C}(z,z)) + 2v\frac{\partial R}{\partial x_2}(z,v\hat{C}(z,z))\frac{\partial\hat{C}}{\partial x_1}(z,z)
\end{array}\right) \\
&\quad -2\hat{C}\left(R(z,u\hat{C}(z,z)),R(z,v\hat{C}(z,z))\right)\frac{\partial}{\partial x_1}\hat{C}(z,z) \leq 0. \qquad (2.19)
\end{aligned}$$

We recall that, by definition of copula, $\hat{C}(1,1)=1$ and $\dfrac{\partial}{\partial x_1}\hat{C}(1,1) = 1$; furthermore, by definition of residuum, $R(1,w)=w$ and, consequently, $\dfrac{\partial R}{\partial x_2}(1,w) = 1$. By putting $z=\overline{G}(0)=1$ in Eq. (2.19), we obtain

$$\nabla\hat{C}(u,v)\cdot\left(\frac{\partial R}{\partial x_1}(1,u)+2u,\frac{\partial R}{\partial x_1}(1,v)+2v\right) - 2\hat{C}(u,v)\frac{\partial\hat{C}}{\partial x_1}(1,1) \leq 0.$$

$\square$

Remark 2.2.5. *A sufficient condition for $z \mapsto C_z$ being decreasing is*

$$\nabla \hat{C}(u, v) \cdot \left(\frac{\partial R}{\partial x_1}(1, u) + 2u, \frac{\partial R}{\partial x_1}(1, v) + 2v \right) \leq 0.$$

In fact, since $\hat{C}(z, z)$ is increasing in z, it is sufficient to impose the numerator of (2.15) *decreasing in z.*

As far as notions of dependence are concerned, the approach of defining potentially new conditions of dependence has been developed more systematically in the previous section.

Even by changing parametrization, the same conditions for preservation of dependence properties hold, as the following proposition state.
We start by considering again the notion of PQD.

Proposition 2.2.6. *The condition $\hat{C}_t$ "PQD for all $t \geq 0$" ($\hat{C}_t$ hyper-PQD) is equivalent to*

$$\hat{C}(u, u)\hat{C}(u'', v'') \geq \hat{C}(u, v'')\hat{C}(u, u''), \tag{2.20}$$

for any u, u'', v'' such that $0 \leq u'' \leq u \leq 1,\ 0 \leq v'' \leq u \leq 1$.

Proposition 2.2.7. *The condition "$\hat{C}_t$ LTD for all $t \geq 0$" ($\hat{C}_t$ hyper-LTD) is equivalent to $\hat{C}$ being TP_2.*

Proposition 2.2.8. *$\hat{C}_t$ TP_2 for all $t \geq 0$ ($\hat{C}_t$ hyper-TP_2) is equivalent to $\hat{C}$ TP_2.*

For sake of completeness, we report here also the proofs obtained in the particular frame $\hat{C}$ has the meaning of a survival copula.

Proof (Prop. 2.2.6). It is well known (and immediate to check) that the survival copula of a bivariate survival function $\overline{F}$ is PQD if and only if the survival function itself is PQD, that is

$$\overline{F}(x, y) \geq \overline{G}(x) \cdot \overline{G}(y).$$

Thus, the condition $\hat{C}_t$ PQD for all $t \geq 0$ is equivalent to $\overline{F}_t$ being PQD for all $t \geq 0$, that is

$$\frac{\overline{F}\,(t + x, t + y)}{\overline{F}\,(t, t)} \geq \frac{\overline{F}\,(t + x, t)}{\overline{F}\,(t, t)} \frac{\overline{F}\,(t, t + y)}{\overline{F}\,(t, t)} \tag{2.21}$$

for any $t, x, y \geq 0$.

Eq. (2.21) can also be written in terms of the survival copula, as

$$\hat{C}\left(\overline{G}(t), \overline{G}(t)\right) \hat{C}\left(\overline{G}(t + x), \overline{G}(t + y)\right) \geq \hat{C}\left(\overline{G}(t + x), \overline{G}(t)\right) \hat{C}\left(\overline{G}(t), \overline{G}(t + y)\right).$$

By the arbitrariness in the choice of $t, x, y \geq 0$, the proof can be completed by letting

$$u = \overline{G}(t),\ u'' = \overline{G}(t + x),\ v'' = \ \overline{G}(t + y).$$

$\square$

Proof (Prop. 2.2.7). It is also well known (and, again, immediate to check) that the survival copula of $\overline{F}$ is LTD if and only if

$$\frac{\overline{F}(x, y)}{\overline{G}(x)} \text{ non-decreasing in } x, \text{for any } y \geq 0.$$

In view of (4.3), $\hat{C}_t$ LTD for all $t \geq 0$ means

$$\frac{\overline{F}\left(t + x'', t + y\right)}{\overline{F}\left(t + x'', t\right)} \geq \frac{\overline{F}\left(t + x', t + y\right)}{\overline{F}\left(t + x', t\right)}$$

for any $t, x', x'', y \geq 0$, with $x'' > x'$.

By the arbitrariness of t, x', y and the condition $x'' > x'$, we can easily see that the above inequality is equivalent to the TP_2 property of $\overline{F}$. $\square$

Proof (Prop. 2.2.8). It is known (see e.g. [77]) that, for any fixed $t \geq 0$,

$$\overline{F}_t\ TP_2 \ \Leftrightarrow\ \hat{C}_t\ TP_2.$$

But, since

$$\overline{F}\ TP_2 \ \Rightarrow\ \overline{F}_t\ TP_2 \ \ \forall t \geq 0,$$

as straightly follows by the definitions of $\overline{F}_t$ and TP_2, it is sufficient $\hat{C}\ TP_2$ to conclude that

$$\hat{C}_t\ TP_2 \ \ \forall t \geq 0.$$

$\square$

Remark 2.2.9. *We have already observed, in the general case, that $\hat{C}_t$ PQD for all $t \geq 0$ is not enough to get $\hat{C}\ TP_2$. However, we can still express $\hat{C}\ TP_2$ as a PQD-condition on models of residual lifetimes. Consider the family $\{\overline{F}_{a,b}\}_{a,b \geq 0}$ of joint survival functions,*

$$\overline{F}_{a,b}(x, y) = \mathbb{P}(X - a > x, Y - b > y | X > a, Y > b) = \frac{\overline{F}(x + a, y + b)}{\overline{F}(a, b)}$$

(so that, with this notation, $\overline{F}_t(x, y) = \overline{F}_{t,t}(x, y)$) and the corresponding families

$$\left\{\overline{G}^{(X)}_{a,b}\right\}_{a,b \geq 0},\ \left\{\overline{G}^{(Y)}_{a,b}\right\}_{a,b \geq 0},\ \left\{\hat{C}_{a,b}\right\}_{a,b \geq 0}.$$

It can be proved that the following equivalences holds

$$\hat{C}_{a,b}\ PQD\ \Leftrightarrow\ \overline{F}_{a,b}\ PQD\ \ and\ \ \hat{C}_{a,b}\ TP_2\ \Leftrightarrow\ \overline{F}_{a,b}\ TP_2;$$

moreover,

$$\overline{F}_{a,b}\ PQD\ \forall\ a,b\ \Leftrightarrow\ \overline{F}_{a,b}\ TP_2\ \forall\ a,b\ \Leftrightarrow\ \overline{F}\ TP_2.$$

Analogous results to those above can be easily formulated for the negative dependence properties NQD, LTI, RR_2, corresponding to PQD, LTD, TP_2 respectively.

2.3 Dynamics of dependence properties for lifetimes influenced by unobservable environmental factors

In the previous sections of this chapter we studied the evolution of stochastic dependence properties of $\overline{F}_t$, as t increases, without the occurrence of failures. In this section we take into account this eventuality.

Roughly speaking, we condition on an event of the form

$$E_t = \{T_{(1)} = t_1, \dots, T_{(k)} = t_k, T_{(k+1)} > t\},$$

where $0 < t_1 < \cdots < t_k < t$. It is useful for our purposes to consider the ordered residual lifetimes of the units surviving at time t, i.e. the random variables

$$(T_{(k+1)} - t, \dots, T_{(n)} - t)|\mathcal{F}_t.$$

From now on, even if not specified, by this writing, we implicitly mean that $k = N_t$.

In order to deal, at any t, with exchangeable random variables, we define the following vector $\mathbf{X}_t \equiv (X_t^1, \dots, X_t^{n-k})$ of *exchangeable residual lifetimes.*

Definition 2.3.1. *The exchangeable residual lifetimes of* $(T_1, \dots, T_n)$ *at time t are the exchangeable random variables* $X_t^1, \dots, X_t^{n-k}$ *admitting* $(T_{(k+1)} - t, \dots, T_{(n)} - t)|\mathcal{F}_t$ *as order statistics.*

Concerning the distribution of $(X_t^1, \dots, X_t^{n-k})$, it is given by

$$\overline{F}_t\,(x_1, ..., x_{n-k}) =$$

$$\mathbb{P}\left(X_t^1 > x_1, ..., X_t^{n-k} > x_{n-k}|T_{(1)} = t_1, ..., T_{(k)} = t_k, T_{(k+1)} > t\right) =$$

$$\frac{\displaystyle\int_{x_1+t}^{\infty}\cdots\int_{x_{n-k}+t}^{\infty} f(\xi_1,...,\xi_{n-k},t_1,...,t_k)d\xi_1\cdots d\xi_{n-k}}{\underbrace{\displaystyle\int_{t}^{\infty}\cdots\int_{t}^{\infty}}_{n-k\ times} f(\xi_1,...,\xi_{n-k},t_1,...,t_k)d\xi_1\cdots d\xi_{n-k}}.$$

At jump times, that is, if $T_{(k+1)} = t = t_{k+1}$, the distribution of $(X_t^1, \dots, X_t^{n-k-1})$, it is given by $\overline{F}_{T_{(k+1)}}(x_1, ..., x_{n-k-1}) =$

$$\frac{\displaystyle\int_{x_1+t_{k+1}}^{\infty}\cdots\int_{x_{n-k-1}+t_{k+1}}^{\infty} f(\xi_1,...,\xi_{n-k-1},t_1,...,t_{k+1})d\xi_1\cdots d\xi_{n-k-1}}{\underbrace{\displaystyle\int_{t_{k+1}}^{\infty}\cdots\int_{t_{k+1}}^{\infty}}_{n-k-1\ times} f(\xi_1,...,\xi_{n-k},t_1,...,t_{k+1})d\xi_1\cdots d\xi_{n-k-1}}.$$

The j-dimensional margins $\overline{F}_t^{(j)}$, $t \in [T_{(k)}, T_{(k+1)})$, $j \le n-k$, are given by

$\overline{F}_t^{(j)}(x_1, ..., x_j) =$

$$\mathbb{P}\left(X_t^1 > x_1, ..., X_t^j > x_j, X_t^{n-k-j} > 0, ..., X_t^{n-k} > 0 | T_{(1)} = t_1, ..., T_{(k)} = t_k, T_{(k+1)} > t\right)$$

$$= \frac{\displaystyle\int_{x_1+t}^{\infty}\cdots\int_{x_j+t}^{\infty}\overbrace{\int_{t}^{\infty}\cdots\int_{t}^{\infty}}^{n-k-j\ times} f(\xi_1,...,\xi_{n-k},t_1,...,t_k)d\xi_1\cdots d\xi_{n-k}}{\underbrace{\displaystyle\int_{t}^{\infty}\cdots\int_{t}^{\infty}}_{n-k\ times} f(\xi_1,...,\xi_{n-k},t_1,...,t_k)d\xi_1\cdots d\xi_{n-k}}.$$

In particular, since (for reasons that will be clearer in the following) we are interested in dependence of pairs of r.v.'s, we explicitly write the bi-dimensional margins, at any time t or at jump times:

$\overline{F}_t^{(2)}(x_1, x_2) =$

$$\mathbb{P}\left(X_t^1 > x_1, X_t^2 > x_2, X_t^3 > 0, ..., X_t^{n-k} > 0 | T_{(1)} = t_1, ..., T_{(k)} = t_k, T_{(k+1)} > t\right)$$

$$= \frac{\displaystyle\int_{x_1+t}^{\infty}\int_{x_2+t}^{\infty}\overbrace{\int_{t}^{\infty}\cdots\int_{t}^{\infty}}^{n-k-2\ times} f(\xi_1,...,\xi_{n-k},t_1,...,t_k)d\xi_1\cdots d\xi_{n-k}}{\underbrace{\displaystyle\int_{t}^{\infty}\cdots\int_{t}^{\infty}}_{n-k\ times} f(\xi_1,...,\xi_{n-k},t_1,...,t_k)d\xi_1\cdots d\xi_{n-k}}. \tag{2.22}$$

$$\overline{F}^{(2)}_{T_{(k+1)}}(x_1,x_2) =$$

$$\frac{\int_{x_1+t_{k+1}}^{\infty}\int_{x_2+t_{k+1}}^{\infty}\overbrace{\int_{t_{k+1}}^{\infty}\cdots\int_{t_{k+1}}^{\infty}}^{n-k-3\ times} f(\xi_1,...,\xi_{n-k-1},t_1,...,t_{k+1})d\xi_1\cdots d\xi_{n-k-1}}{\underbrace{\int_{t_{k+1}}^{\infty}\cdots\int_{t_{k+1}}^{\infty}}_{n-k-1\ times} f(\xi_1,...,\xi_{n-k},t_1,...,t_{k+1})d\xi_1\cdots d\xi_{n-k-1}}.$$

In this section, we provide specific results for the case when $T_1,\ldots,T_n$ are conditional independent given a random vector $\mathbf{\Theta} = (\Theta_1,\ldots,\Theta_d)$. As done before, we consider the dependence between two variables, and thus we study the evolution of the family $\{\hat{C}_t^{(2)}\}_{t\geq 0}$, where $\hat{C}_t^{(2)}$ is the bivariate margin of the survival copula in (2.14).
Our results will be based on some notions of multivariate stochastic ordering, we have discussed in Chapter 1. We study some aspects of the monotonicity behaviour of the family $\{\hat{C}_t^{(2)}\}_{t\geq 0}$, both at default times and between two subsequent default times. More precisely, we find out conditions for stochastic dependence being decreasing at default times and progressively increasing between two subsequent default times. On this purpose we will present a preliminary result (see Proposition 2.3.3) that relates monotonicity properties of the posterior densities of $\mathbf{\Theta}$ to properties of hazard rates of $T_1,\ldots,T_n$ given $\mathbf{\Theta}$.

In order to adopt the same notation of the previous sections as to the bivariate copula and since, where not specified otherwise, we will discuss the behaviour of the bivariate margin of the copula, from now on, we will write $\hat{C}_t$ instead of $\hat{C}_t^{(2)}$.

2.3.1 Family of survival functions and survival copulas in the conditionally i.i.d. case

Let $\mathbf{\Theta}$ be a random vector with joint density π_0. We denote by $\theta = (\theta_1,\ldots,\theta_d)$ its realization. Let $T_1,\ldots,T_n$ be conditionally i.i.d. given $\mathbf{\Theta}$, with conditional survival function $\overline{G}(x|\theta)$ and conditional density $g(x|\theta)$. For our purposes, our first step is the computation of the survival copula $\hat{C}_t$. To this aim, we adapt the expression for $\overline{F}$ to the present case of conditional independence and identical distribution:

$$\overline{F}(x_1,\ldots,x_n) = \int_{\mathbb{R}^d}\overline{G}(x_1|\theta)\cdots\overline{G}(x_n|\theta)\pi_0(\theta)d\theta =$$

$$\int_{\mathbb{R}^d}\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} g(\xi_1|\theta)\cdots g(\xi_n|\theta)d\xi_1\cdots d\xi_n\pi_0(\theta)d\theta.$$

In particular, the one-dimensional *predictive* survival function and probability density respectively become

$$\overline{G}(x) = \int_{\mathbb{R}^d} \overline{G}(x|\theta)\pi_0(\theta)d\theta,$$

$$g(x) = \int_{\mathbb{R}^d} g(x|\theta)\pi_0(\theta)d\theta.$$

We denote respectively by $r(x)$ and $r(x|\theta)$ the *predictive* hazard rate of each T_i and the *conditional* hazard rate, i.e.

$$r(x) = \frac{g(x)}{\overline{G}(x)}, \quad r(x|\theta) = \frac{g(x|\theta)}{\overline{G}(x|\theta)}.$$

In terms of the conditional univariate survival functions $\overline{G}(x|\theta)$, we want to write down, for any $x,\ 2 \leq j \leq n-k$, the joint survival function $\overline{F}_t$.

In this respect it is important to notice that, if $T_1, \ldots, T_n$ are conditionally i.i.d. given $\boldsymbol{\Theta}$, it can be shown that $X_t^1, \ldots, X_t^{n-k}$ are conditionally i.i.d. given $\boldsymbol{\Theta}$ as well. In particular, each X_t^i has conditional univariate survival function $\overline{G}_t(x|\theta)$.

We can now proceed with the computation of $\overline{F}_t$. For any t, conditionally on the history $\mathcal{F}_t$, $\boldsymbol{\Theta}$ admits density

$$\pi_t(\theta) \equiv \pi(\theta|\mathcal{F}_t).$$

If the observation up to t is

$$E_t = \{T_{(1)} = t_1, \ldots, T_{(k)} = t_k, T_{(k+1)} > t\},$$

by applying the Bayes' formula, we can write, for $t \in (T_{(k)}, T_{(k+1)})$,

$$\pi_t(\theta) \propto [\overline{G}(t|\theta)]^{n-k} g(t_1|\theta) \cdots g(t_k|\theta)\pi_0(\theta); \tag{2.23}$$

analogously

$$\pi_{T_{(k)}}(\theta) \propto [\overline{G}(t_k|\theta)]^{n-k} g(t_1|\theta) \cdots g(t_k|\theta)\pi_0(\theta). \tag{2.24}$$

We point out that, since

$$\pi_{T_{(k)}^-}(\theta) \propto [\overline{G}(t_k|\theta)]^{n-k+1} g(t_1|\theta) \cdots g(t_{k-1}|\theta)\pi_0(\theta), \tag{2.25}$$

it is possible to write

$$\pi_{T_{(k)}}(\theta) \propto \pi_{T_{(k)}^-}(\theta)\frac{g(t_k|\theta)}{\overline{G}(t_k|\theta)} = \pi_{T_{(k)}^-}(\theta) r(t_k|\theta). \tag{2.26}$$

Furthermore

$$\overline{F}_t(x_1, \ldots, x_{n-k}) = \int_{\mathbb{R}^d} \frac{\overline{G}(x_1 + t|\theta) \cdots \overline{G}(x_{n-k} + t|\theta)}{[\overline{G}(t|\theta)]^{n-k}} \pi_t(\theta)d\theta. \tag{2.27}$$

Thus the univariate margin becomes

$$\overline{G}_t(x) = \overline{F}_t(x, 0, \ldots, 0) = \int_{\mathbb{R}^d} \frac{\overline{G}(x+t|\theta)}{\overline{G}(t|\theta)} \pi_t(\theta) d\theta. \tag{2.28}$$

We are now in a position to write, for any t, the survival copula $\hat{C}_t$. By combining (2.27) and (2.28), we obtain

$$\hat{C}_t(u_1, \ldots, u_{n-k}) = \int_{\mathbb{R}^d} \frac{\overline{G}(\overline{G}_t^{-1}(u_1)+t|\theta)}{\overline{G}(t|\theta)} \cdots \frac{\overline{G}(\overline{G}_t^{-1}(u_{n-k})+t|\theta)}{\overline{G}(t|\theta)} \pi_t(\theta) d\theta. \tag{2.29}$$

In particular, we will study the evolution of the bivariate copulas, obtained as the bivariate margin of the copula in the previous formula and therefore given by

$$\hat{C}_t(u, v) = \int_{\mathbb{R}^d} \frac{\overline{G}(\overline{G}_t^{-1}(u)+t|\theta)}{\overline{G}(t|\theta)} \frac{\overline{G}(\overline{G}_t^{-1}(v)+t|\theta)}{\overline{G}(t|\theta)} \pi_t(\theta) d\theta. \tag{2.30}$$

Remark 2.3.2. *As to the univariate survival function, it is interesting to notice the difference between $\overline{G}_t(x)$ in* (2.28) *and*

$$\overline{H}_t(x) \equiv \mathbb{P}(X > t + x | X > t) = \frac{\overline{G}(x+t)}{\overline{G}(t)} :$$

$\overline{G}_t(x)$ is the univariate survival function conditional on the history up to t of all the variables in the model, while $\overline{H}_t(x)$ is the univariate survival function of one variable, conditional on the survival at t of only that variable.
On the other hand, since conditionally on $\mathbf{\Theta}$ the variables are independent, we can notice that, given $\mathbf{\Theta}$, conditioning on $\{X > t\}$ is equivalent to conditioning on $\mathcal{F}_t$. Therefore the two univariate conditional survival functions $\overline{G}_t(x|\theta)$ and $\overline{H}_t(x|\theta)$ do coincide:

$$\overline{G}_t(x|\theta) \equiv \mathbb{P}(X > t + x | \mathcal{F}_t, \theta) = \mathbb{P}(X > t + x | X > t, \theta) = \overline{H}_t(x|\theta).$$

2.3.2 Monotonicity properties of survival copulas

In order to analyze the evolution of dependence when t elapses, we consider, in particular, the family of survival copulas $\hat{C}_t(u, v)$, defined in (2.30).

We aim at obtaining conditions for monotonicity properties of $\left\{\hat{C}_t\right\}_{t\geq 0}$.

It is natural to split the analysis of $\left\{\hat{C}_t\right\}_{t\geq 0}$ into two different stages, namely:

a) at default times $T_{(k)}$'s

or

b) between two of them, i.e. within the intervals $(T_{(k)}, T_{(k+1)})$, for $k = 0, \ldots, n-2$.

Under the hypothesis that $T_1, \ldots, T_n$ are conditionally i.i.d., the survival copulas are given by rather explicit expressions. Therefore, in the following, we can study the monotonicity of the process $\left\{\hat{C}_t\right\}_{t\geq 0}$ by means of direct comparisons.

In both the two cases a) and b), monotonicity properties of $t \to \hat{C}_t$ will be easily achieved by imposing suitable monotonicity assumptions on the conditional hazard rate $r(t|\theta)$.

More precisely, monotonicity properties of $t \to \hat{C}_t$ can be obtained from monotonicity properties of $t \to \pi_t$, as stated in Propositions 2.3.6 and 2.3.7 below. On their turn, monotonicity properties of $t \to \pi_t$ can be traced back to monotonicity properties of $\theta \to r(t|\theta)$, as illustrated in the following Proposition 2.3.3. We also remark that this proposition has some connections with the notion of default contagion, as we will see below.

Proposition 2.3.3. *The following statements are equivalent:*

(a) $r(t|\theta) \uparrow \theta$;

(b) $\pi_{T_{(k)}} \geq_{wlr} \pi_{T^-_{(k)}}$ *a.s.;*

(c) $\pi_{t'} \geq_{wlr} \pi_{t''}$, *for any* $t' < t''$, $t', t'' \in [T_{(k)}, T_{(k+1)})$, $k = 0, \ldots, n-2$.

Proof. The implication (a) $\Rightarrow$ (b) is obvious by taking into account Eq. (1.16) and the identity $\dfrac{\pi_{T_{(k)}}(\theta)}{\pi_{T^-_{(k)}}(\theta)} = r(T_{(k)}|\theta)$.

Conversely, by statement (b),

$$\frac{\pi_{T_{(k)}}(\theta)}{\pi_{T^-_{(k)}}(\theta)} = r(T_{(k)}|\theta) \ \uparrow \theta \ \ a.s..$$

Since, for any $t > 0$, $g(t) > 0$, $r(T_{(k)}|\theta) \uparrow \theta$ a.s. implies $r(t|\theta) \uparrow \theta$ for any $t > 0$.

In order to prove that (a) implies (c), we notice that

$$r(t|\theta) \uparrow \theta \ \Leftrightarrow \ \int_{t'}^{t''} r(t|\theta)dt \uparrow \theta \ \ \forall\ t' < t''.$$

Since $\overline{G}(x|\theta) = \exp\left\{-\int_0^x r(t|\theta)dt\right\}$,

$$\frac{\overline{G}(t''|\theta)}{\overline{G}(t'|\theta)} = \exp\left\{-\int_{t'}^{t''} r(t|\theta)dt\right\} \downarrow \theta$$

and the same holds for $\dfrac{\pi_{t''}(\theta)}{\pi_{t'}(\theta)} = \left(\dfrac{\overline{G}(t''|\theta)}{\overline{G}(t'|\theta)}\right)^{n-k}$, with $t', t'' \in [T_{(k)}, T_{(k+1)})$.

Therefore, by definition of weak likelihood ratio order, $\pi_{t''} \leq_{wlr} \pi_{t'}$.

Conversely, by statement (c),

$$\int_{t'}^{t''} r(t|\theta)dt \uparrow \theta$$

for any $t' < t''$, $t', t'' \in [T_{(k)}, T_{(k+1)})$, $k = 0, \dots, n-2$.
Thus, given $a, b \in \mathbb{R}_+$, $a < b$,

$$\int_a^b r(t|\theta)dt =$$

$$\int_a^{T_{(i)}} r(t|\theta)dt + \int_{T_{(i)}}^{T_{(i+1)}} r(t|\theta)dt + \cdots + \int_{T_{(j-1)}}^{T_{(j)}} r(t|\theta)dt + \int_{T_{(j)}}^{b} r(t|\theta)dt \uparrow \theta.$$

□

We notice that, if Θ is a scalar r.v., the weak likelihood ratio order in Proposition 2.3.3 is equivalent to the likelihood ratio order. In this case, we have then sufficient arguments to prove the following results about monotonicity of $\{\hat{C}_t\}_{t\geq 0}$.

Proposition 2.3.4. *Let $r(t|\theta)$ be monotonic w.r.t. θ and $\dfrac{\overline{G}\left(\overline{G}_t^{-1}(u) + t|\theta\right)}{\overline{G}\,(t|\theta)}$ increasing w.r.t. t. Then $t \mapsto \hat{C}_t$ is increasing in the interval between two jumps.*

Proof. Assume, for instance, $r(t|\theta)$ increasing in θ. The proof when $r(t|\theta)$ is decreasing in θ is analogous.

In view of Eq. (2.29), $t \mapsto \hat{C}_t$ being increasing is equivalent to

$$\int_{\mathbb{R}} \frac{\overline{G}(\overline{G}_{t'}^{-1}(u) + t'|\theta)}{\overline{G}(t'|\theta)} \cdot \frac{\overline{G}(\overline{G}_{t'}^{-1}(v) + t'|\theta)}{\overline{G}(t'|\theta)} \pi_{t'}(\theta)d\theta \leq$$

$$\int_{\mathbb{R}} \frac{\overline{G}(\overline{G}_{t''}^{-1}(u) + t''|\theta)}{\overline{G}(t''|\theta)} \cdot \frac{\overline{G}(\overline{G}_{t''}^{-1}(v) + t''|\theta)}{\overline{G}(t''|\theta)} \pi_{t''}(\theta)d\theta \tag{2.31}$$

for any $t', t'' \in [T_{(k)}, T_{(k+1)})$, $t' \leq t''$. By Proposition 2.3.3, $\theta \to r(t|\theta)$ being increasing implies $\pi_{t'} \geq_{lr} \pi_{t''}$. By (1.15), Eq. (2.31) would be easily obtained under the condition $\dfrac{\overline{G}(\overline{G}_t^{-1}(u) + t|\theta)}{\overline{G}(t|\theta)}$ being decreasing w.r.t. θ. This fact too follows by Proposition 2.3.3.

Furthermore, in order to guarantee (2.31), we also need that $\dfrac{\overline{G}(\overline{G}_t^{-1}(u) + t|\theta)}{\overline{G}(t|\theta)}$ is increasing w.r.t. t. □

Proposition 2.3.5. *Let $r(t|\theta)$ be a monotonic function of θ. Then $\hat{C}_{T_{(k)}} \preceq_{PQD} \hat{C}_{T^-_{(k)}}$ a.s..*

Proof. Assume $\theta \to r(t|\theta)$ to be increasing. By Proposition 2.3.3, $\pi_{T_{(k)}}(\theta) \geq_{lr} \pi_{T^-_{(k)}}(\theta)$ a.s.. On the other hand, since $r(t|\theta) \uparrow \theta$ is equivalent to $\dfrac{\overline{G}(x+t|\theta)}{\overline{G}(t|\theta)} \downarrow \theta$, $\dfrac{\overline{G}(\overline{G}_t^{-1}(u)+t|\theta)}{\overline{G}(t|\theta)}$ is non-increasing in θ. Thus, if, furthermore, the inequality

$$\frac{\overline{G}(\overline{G}^{-1}_{T_{(k)}}(u)+T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \leq \frac{\overline{G}(\overline{G}^{-1}_{T^-_{(k)}}(u)+T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \tag{2.32}$$

holds, by applying (1.15), we would obtain the thesis, that is

$$\int_{\mathbb{R}} \frac{\overline{G}(\overline{G}^{-1}_{T_{(k)}}(u)+T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \cdot \frac{\overline{G}(\overline{G}^{-1}_{T_{(k)}}(v)+T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \pi_{T_{(k)}}(\theta)d\theta \leq$$

$$\int_{\mathbb{R}} \frac{\overline{G}(\overline{G}^{-1}_{T^-_{(k)}}(u)+T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \cdot \frac{\overline{G}(\overline{G}^{-1}_{T^-_{(k)}}(v)+T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \pi_{T^-_{(k)}}(\theta)d\theta.$$

The inequality (2.32) is guaranteed by $\overline{G}^{-1}_{T_{(k)}}(u) \geq \overline{G}^{-1}_{T^-_{(k)}}(u)$ for any $u \in [0,1]$, that is equivalent to $\overline{G}_{T_{(k)}}(x) \leq \overline{G}_{T^-_{(k)}}(x)$ for any $x \in \mathbb{R}$. Recalling that

$$\overline{G}_{T_{(k)}}(x) = \int_{\mathbb{R}} \frac{\overline{G}(x+T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \pi_{T_{(k)}}(\theta)d\theta,$$

$$\overline{G}_{T^-_{(k)}}(x) = \int_{\mathbb{R}} \frac{\overline{G}(x+T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \pi_{T^-_{(k)}}(\theta)d\theta,$$

$\overline{G}_{T_{(k)}}(x) \leq \overline{G}_{T^-_{(k)}}(x)$ follows by condition (1.15). Thus we can conclude that, for any $u, v \in [0,1]$, $\hat{C}_{T_{(k)}}(u,v) \leq \hat{C}_{T^-_{(k)}}(u,v)$. $\square$

When $\boldsymbol{\Theta}$ is a random vector, in order to obtain the same theses of Propositions 2.3.4 and 2.3.5, we need the assumption π_t MTP$_2$ for any t. As a matter of fact, this hypothesis is rather strong. Sufficient conditions for it can be formulated in terms of $r(t|\theta)$ and $\overline{G}(t|\theta)$ by taking into account the expression (2.23) (see also [83]).

However, $t \to \pi_t$ monotonic in the weak likelihood ratio order (as in Proposition 2.3.3) and π_t MTP$_2$ for any t guarantee $t \to \pi_t$ being monotonic in the likelihood ratio order (see Lemma 1.4.14). The latter condition implies the monotonicity of $t \to \pi_t$ with respect to the usual stochastic order.

We are now in a position to state and prove the following results about the monotonicity of the family $\left\{\hat{C}_t\right\}_{t\geq 0}$ of the survival copulas.

Proposition 2.3.6. *Let $r(t|\theta)$ be monotonic w.r.t. θ, π_t MTP$_2$ for any t and*

$$\frac{\overline{G}\left(\overline{G}_t^{-1}(u)+t|\theta\right)}{\overline{G}\,(t|\theta)} \text{ increasing w.r.t. } t.$$

Then $t \mapsto \hat{C}_t$ is increasing in the interval between two jumps.

Proof. Assume, for instance, $r(t|\theta)$ increasing in θ. The proof when $r(t|\theta)$ is decreasing in θ is analogous. Let us denote

$$\rho(\theta, t) = \frac{\overline{G}(\overline{G}_t^{-1}(u)+t|\theta)}{\overline{G}(t|\theta)} \cdot \frac{\overline{G}(\overline{G}_t^{-1}(v)+t|\theta)}{\overline{G}(t|\theta)}. \tag{2.33}$$

In view of Eq. (2.30), the condition $t \mapsto \hat{C}_t$ being increasing reads as

$$\int_{\mathbb{R}^d} \rho(\theta, t')\pi_{t'}(\theta)d\theta \leq \int_{\mathbb{R}^d} \rho(\theta, t'')\pi_{t''}(\theta)d\theta \tag{2.34}$$

for any $t' \leq t''$. Thus, we want to prove Eq. (2.34). By Proposition 2.3.3, $\theta \to r(t|\theta)$ increasing implies $\pi_{t'} \geq_{wlr} \pi_{t''}$. Since π_t is MTP$_2$ for any t, by Lemma 1.4.14, $\pi_{t'} \geq_{lr} \pi_{t''}$ and, therefore, in particular, it will be $\pi_{t'} \geq_{st} \pi_{t''}$. By Proposition 2.3.3, $\theta \to r(t|\theta)$ increasing implies $\theta \to \rho(\theta, t)$ decreasing. Therefore, by Eq. (1.15), $\pi_{t'} \geq_{st} \pi_{t''}$ implies

$$\int_{\mathbb{R}^d} \rho(\theta, t')\pi_{t'}(\theta)d\theta \leq \int_{\mathbb{R}^d} \rho(\theta, t')\pi_{t''}(\theta)d\theta.$$

Since, by hypothesis, $t \to \rho(\theta, t)$ is increasing,

$$\int_{\mathbb{R}^d} \rho(\theta, t')\pi_{t''}(\theta)d\theta \leq \int_{\mathbb{R}^d} \rho(\theta, t'')\pi_{t''}(\theta)d\theta$$

and therefore

$$\int_{\mathbb{R}^d} \rho(\theta, t')\pi_{t'}(\theta)d\theta \leq \int_{\mathbb{R}^d} \rho(\theta, t'')\pi_{t''}(\theta)d\theta,$$

that is $\hat{C}_{t'} \preceq_{PQD} \hat{C}_{t''}$ for any $t' < t''$, $t', t'' \in [T_{(k)}, T_{(k+1)})$. $\square$

Proposition 2.3.7. *Let $r(t|\theta)$ be a monotonic function of θ and π_t MTP$_2$ for any t. Then*

$$\hat{C}_{T_{(k)}} \preceq_{PQD} \hat{C}_{T_{(k)}^-} \text{ a.s.}. \tag{2.35}$$

Proof. Let us define, as in Eq. (2.33),

$$\rho(\theta, T_{(k)}) = \frac{\overline{G}(\overline{G}_{T_{(k)}}^{-1}(u)+T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \cdot \frac{\overline{G}(\overline{G}_{T_{(k)}}^{-1}(v)+T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)}$$

and

$$\rho(\theta, T^-_{(k)}) = \frac{\overline{G}(\overline{G}^{-1}_{T^-_{(k)}}(u) + T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \cdot \frac{\overline{G}(\overline{G}^{-1}_{T^-_{(k)}}(v) + T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)}.$$

We want to prove that

$$\int_{\mathbb{R}^d} \rho(\theta, T_{(k)}) \pi_{T_{(k)}}(\theta) d\theta \leq \int_{\mathbb{R}^d} \rho(\theta, T^-_{(k)}) \pi_{T^-_{(k)}}(\theta) d\theta \quad a.s..$$

Assume $\theta \to r(t|\theta)$ to be increasing. The proof when $\theta \to r(t|\theta)$ is decreasing is analogous.
By Proposition 2.3.3, $\pi_{T_{(k)}} \geq_{wlr} \pi_{T^-_{(k)}}$ a.s. and then, by Lemma 1.4.14, $\pi_{T_{(k)}} \geq_{lr} \pi_{T^-_{(k)}}$ holds.
On the other hand, $r(t|\theta) \uparrow \theta$ implies $\dfrac{\overline{G}(x+t|\theta)}{\overline{G}(t|\theta)} \downarrow \theta$ and therefore $\theta \to \rho(\theta, t)$ decreasing. We recall that

$$\overline{G}_{T_{(k)}}(x) = \int_{\mathbb{R}^d} \frac{\overline{G}(x + T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \pi_{T_{(k)}}(\theta) d\theta,$$

$$\overline{G}_{T^-_{(k)}}(x) = \int_{\mathbb{R}^d} \frac{\overline{G}(x + T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \pi_{T^-_{(k)}}(\theta) d\theta.$$

Since $\pi_{T_{(k)}} \geq_{lr} \pi_{T^-_{(k)}}$ a.s. implies $\pi_{T_{(k)}} \geq_{st} \pi_{T^-_{(k)}}$ a.s., by condition (1.15), it follows that $\overline{G}_{T_{(k)}}(x) \leq \overline{G}_{T^-_{(k)}}(x)$. This implies

$$\frac{\overline{G}(\overline{G}^{-1}_{T_{(k)}}(u) + T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)} \leq \frac{\overline{G}(\overline{G}^{-1}_{T^-_{(k)}}(u) + T_{(k)}|\theta)}{\overline{G}(T_{(k)}|\theta)}, \tag{2.36}$$

that is $\rho(\theta, T_{(k)}) \leq \rho(\theta, T^-_{(k)})$. Hence

$$\int_{\mathbb{R}^d} \rho(\theta, T_{(k)}) \pi_{T_{(k)}}(\theta) d\theta \leq \int_{\mathbb{R}^d} \rho(\theta, T_{(k)}) \pi_{T^-_{(k)}}(\theta) d\theta \leq \int_{\mathbb{R}^d} \rho(\theta, T^-_{(k)}) \pi_{T^-_{(k)}}(\theta) d\theta.$$

Thus we can conclude that, for any u, v in $[0,1]$, $\hat{C}_{T_{(k)}}(u,v) \leq \hat{C}_{T^-_{(k)}}(u,v)$ a.s., that is the thesis. □

It can be easily shown that the statement (a) in Proposition 2.3.3 implies a condition of default contagion, in the following sense:

$$\overline{F}^{(n-k)}_{T^-_{(k)}}(x_1, \dots, x_{n-k}) \geq \overline{F}^{(n-k)}_{T_{(k)}}(x_1, \dots, x_{n-k}), \quad \forall\, x_1, \dots, x_{n-k} \geq 0. \tag{2.37}$$

It is then interesting to notice that the assumption guaranteeing default contagion also implies, under our conditions, a jump downward of the copulas of the surviving units, at default times, as the following proposition states.

Proposition 2.3.8. *$\hat{C}_{T_{(k)}} \preceq_{PQD} \hat{C}_{T^-_{(k)}}$ implies default contagion in the sense of (2.37).*

The analysis of a simple and very well known model allows us to show now an example of application of both Proposition 2.3.6 and 2.3.7 in the case $\theta \in \mathbb{R}_+$.

Example 2.3.9. *Consider $\overline{G}(t|\theta) = e^{-\theta t}$, so that $r(t|\theta) = \theta$. The conditional hazard rate $r(t|\theta)$ is constant w.r.t. t and increasing w.r.t. θ and the assumptions of Proposition 2.3.6 and Proposition 2.3.7 are satisfied. In particular, if the a priori distribution of Θ is Gamma(α_0, β_0), then, at any time $t > 0$, the conditional distribution of Θ is again Gamma and $\hat{C}_t$ is a Clayton copula. More precisely,*

$$\pi_t(\theta) = \frac{\beta_t^{\alpha_t}}{\Gamma(\alpha_t)} \theta^{\alpha_t - 1} e^{-\beta_t \theta},$$

with $\alpha_t = \alpha_0 + N_t$, $\beta_t = \beta_0 + \sum_{k=1}^n \min(T_{(k)}, t)$, and

$$\hat{C}_t(u, v) = \left(u^{-\frac{1}{\alpha_t}} + v^{-\frac{1}{\alpha_t}} - 1 \right)^{-\alpha_t}.$$

We obtain that $t \to \hat{C}_t$ remains constant between two subsequent default times and makes a jump downward at instants of default.

2.3.3 Concluding remarks

In this last passage, we present some remarks and comments about the results we have obtained in this section. As we have already said, our main results are contained in the Propositions 2.3.6 and 2.3.7.

Proposition 2.3.6 states that, if the conditional hazard rate $r(t|\theta)$ is monotonic in θ and if the components of $\mathbf{\Theta}$ satisfies a suitable positive dependence property, namely MTP_2, dependence among residual lifetimes continuously increases at the increase of survival time.

In particular, since residual lifetimes become more and more dependent, Proposition 2.3.6 gives conditions for the phenomenon of tail dependence. The latter circumstance means that extremal events are more dependent than non-extremal ones. In other words, we can intuitively expect that, eventually, failures will occur each close to the other ones.

Proposition 2.3.7 states instead that, under the same conditions on $r(t|\theta)$ and on $\mathbf{\Theta}$, inequality (2.35) holds, i.e. the dependence among residual lifetimes discontinuously decreases when a failure occurs.

Notice that (2.35), i.e. $\hat{C}_{T_{(k)}} \preceq_{PQD} \hat{C}_{T^-_{(k)}}$, and default contagion are two phenomena referring to the behaviour of residual lifetimes at the instants of defaults. Actually, they appear to be two different phenomena; they are however related in some way. More precisely, it can be shown that $r(t|\theta)$

monotonic in θ also implies default contagion. Still remaining in the present case of conditionally i.i.d. observations, we can however say more: it can be proved that $\hat{C}_{T_{(k)}} \preceq_{PQD} \hat{C}_{T^-_{(k)}}$ implies default contagion (see Proposition 2.3.8).

Let us now come to comment on some technical assumptions in Propositions 2.3.6 and 2.3.7.

Notice that, even if the likelihood ratio order is a very well known and most used notion, in Proposition 2.3.3, it is sufficient requiring the weak likelihood ratio order. In fact, as it happens in the proofs of the propositions referring to the univariate case, we actually use the condition $\dfrac{\pi_t(\theta)}{\pi_s(\theta)}$ being monotonic in θ. When θ is univariate, such a condition corresponds just to a characterization of the likelihood ratio order; in the multivariate case, it gives the definition of weak likelihood ratio order. The point is that, in order to prove Propositions 2.3.6 and 2.3.7, we need monotonicity of the family $\{\pi_t\}_{t\geq 0}$ with respect to the usual stochastic order. In the multivariate case, the usual stochastic order is implied by the likelihood ratio order, but not by the weak likelihood ratio order. In order to retrieve the monotonicity of the family $\{\pi_t\}_{t\geq 0}$ with respect to the usual stochastic order, it is necessary imposing, further, the condition that $\boldsymbol{\Theta}$ is MTP_2.

As a matter of fact, in the multivariate case, we distinguish between weak and strong notions of a same stochastic ordering, whereas, in the univariate case, the two notions do coincide. Adding the condition $\boldsymbol{\Theta}$ MTP_2 allows us to obtain, from the weaker notion of stochastic order, the stronger one.

More in general, one could argue that MTP_2, representing a strong notion of dependence, allows us to treat the random vector $\boldsymbol{\Theta}$ like a scalar random variable. On its turn, this fact makes it possible to automatically extend many results, valid for the case of a univariate non-observable factor, to the multivariate case.

Chapter 3

Distorted copulas: constructions and tail dependence

In this chapter, we examine under which conditions on an increasing bijection ψ of $[0,1]$, the *distortion* $C_\psi \colon [0,1]^2 \to [0,1]$,

$$C_\psi(x,y) = \psi(C(\psi^{-1}(x)), \psi^{-1}(y)), \tag{3.1}$$

of a given copula C, is still a copula. In particular, when the copula C is totally positive of order 2, we give a sufficient condition on ψ which ensures that any distortion of C by means of ψ is still a TP_2 copula. The presented results allow us to introduce in a more flexible way families of copulas exhibiting different behaviour in the tails.

The study of distortions is of general interest since they can be used for generating, in a flexible way, new families of copulas. In order to do this, we need conditions on ψ warranting that, for a given C, C_ψ is still a copula. Moreover, it would be also of interest to investigate how some dependence properties of C change or are preserved under these distortions.

In this chapter, we revisit the distortion of copulas from a new perspective, which includes the results obtained in the literature. In sections 3.2 and 3.2.1, we mainly focus on some statistical aspects of copulas. In particular, in section 3.2, the TP_2 property is studied, both as a dependence property and for its implications on the preservation of the 2-increasing property under distortion. In section 3.2.1, dependence on the tails of the copula is investigated. More precisely, we analyze how the tail dependence properties are modified under distortions. These results will show the usefulness of distortion for generating flexible statistical models. The presented results may be seen in the frame of the more general case of distorted probabilities. Section 3.2.2 is devoted to some comments clarifying this possible extension.

3.1 Revisiting distortions of copulas

A large part of literature about distortions mainly focuses on the determination of the class of all increasing bijections ψ such that the mapping given by Eq. (3.1) transforms any copula C into another copula. Here we are rather interested in the (generally, larger) class of all increasing bijections ψ such that, for a *given* copula C, C_ψ is still a copula.

We call an *order isomorphism* any increasing bijective transformation of $[0,1]$. The set of all order isomorphisms is denoted by $\mathcal{I}$. If ψ is an order isomorphism and C a mapping from $[0,1]^2$ to $[0,1]$, then we call *ψ-transform* of C, or *ψ-distortion* of C, the mapping $C_\psi\colon [0,1]^2 \to [0,1]$ defined by (3.1).

It is obvious that all the ψ-distortions of a copula C are such that C_ψ is increasing in each variable and $C_\psi(x,1) = C_\psi(1,x) = x$ for every $x \in [0,1]$; but not necessarily C_ψ satisfies the 2-increasing property. Thus C_ψ's are (continuous) *semi-copulas*, but not necessarily copulas (see [18, 21, 43, 44]). From another point of view, any of such distortions can be considered as the semi-copula associated with a suitable distorted probability (as clarified by [45]).

Given $C \in \mathcal{C}$, let $\mathcal{I}(C)$ be the set of all order isomorphisms which, being applied to C, give rise to another copula. Precisely

$$\mathcal{I}(C) = \{\psi \in \mathcal{I} \,|\, C_\psi \in \mathcal{C}\}.$$

By the literature cited above, it is well known that, regardless of C, the set $\mathcal{I}(C)$ contains $\mathcal{I}_{cx}$, the set of all convex bijections on $[0,1]$. However, as a key observation stimulating this investigation, for a *fixed* copula C, the inclusion of $\mathcal{I}_{cx}$ into $\mathcal{I}(C)$ may be strict. For example, it is well known that $\mathcal{I}(M) = \mathcal{I}$, while $\mathcal{I}(W) = \mathcal{I}_{cx}$ (see, for example, [74]).

Example 3.1.1. *If C is an Archimedean copula with an additive generator $f\colon [0,1] \to [0,+\infty[$, i.e.*

$$C(x,y) = f^{(-1)}(f(x) + f(y)), \tag{3.2}$$

then $\mathcal{I}(C)$ consists of all $\psi \in \mathcal{I}$ such that $f \circ \psi^{-1}$ is convex.

Example 3.1.2. *If C is an extreme value copula (see, e.g., [78]), then $\mathcal{I}(C)$ also contains all the power functions $\psi(t) = t^\alpha$ for every $\alpha > 0$.*

These facts spur us to investigate whether, fixed a copula C, it is possible to construct the set $\mathcal{I}(C)$ or, at least, to find its elements that are not convex. As we will see at the end of the section, these results will be useful for constructing families of copulas starting with some fixed C.

For this end, we need some preliminary considerations.

Lemma 3.1.3. *[70, Proposition 4.B.2] Let A be an interval of $\mathbb{R}$ and let $f\colon A \to \mathbb{R}$. If f is convex and increasing, then, for every $a_1, a_2, a_3, a_4 \in A$ such that*

$$a_1 \le \min(a_2, a_3) \le \max(a_2, a_3) \le a_4,$$

and $a_1 + a_4 \ge a_2 + a_3$, we have

$$f(a_1) + f(a_4) \ge f(a_2) + f(a_3).$$

Then, we have to consider the following definition on $\mathcal{I}$, which can be obtained from [54].

Definition 3.1.4. *Let φ, ψ be in $\mathcal{I}$. We say that φ is* less convex *than ψ (and we write $\varphi \le_{\mathrm{C}} \psi$) if $\psi \circ \varphi^{-1}$ is convex.*

Let us denote by I an arbitrary set and by Id_I the identity function on the set I.
It can be shown quite easily, that the relation $\le_{\mathrm{C}}$ is an ordering on $\mathcal{I}$, i.e. it is reflexive, transitive and antisymmetric. Moreover, we have that $\varphi \le_{\mathrm{C}} \mathrm{Id}_{[0,1]}$ if and only if φ is concave, and, analogously, $\mathrm{Id}_{[0,1]} \le_{\mathrm{C}} \varphi$ if and only if φ is convex, where $\mathrm{Id}_{[0,1]}$ is the identity function on $[0,1]$. More results about the convex ordering among probability distribution functions can be derived from [23], where different assumptions are given on the functions φ and ψ (see also [73]), and from [75] and [82], where this order is considered under different names (e.g., likelihood ratio order).

Here we would like just to stress that, by using the previous definition, we have that $\varphi \le_{\mathrm{C}} \psi$ if and only if $\varphi \circ \psi^{-1}$ is concave, i.e., for all $c, d \in [0,1]$ where $c < d$ and for every $y \in]c, d]$

$$\frac{\varphi \circ \psi^{-1}(y) - \varphi \circ \psi^{-1}(c)}{y - c} \ge \frac{\varphi \circ \psi^{-1}(d) - \varphi \circ \psi^{-1}(c)}{d - c},$$

which is equivalent to the fact that

$$\frac{\varphi(x) - \varphi(a)}{\varphi(b) - \varphi(a)} \ge \frac{\psi(x) - \psi(a)}{\psi(b) - \psi(a)} \tag{3.3}$$

holds for all $a, b \in [0,1]$ where $a < b$ and for every $x \in]a, b]$.

The following theorem is the main result of this section. It states a closure property of the sets $\mathcal{I}(C)$ with respect to the afore-mentioned convex ordering. It will allow us to single out further (non convex) isomorphisms belonging to $\mathcal{I}(C)$, once we have found one.

Theorem 3.1.5. *Let $\varphi, \psi \in \mathcal{I}$ such that $\varphi \le_{\mathrm{C}} \psi$. If φ is an element in $\mathcal{I}(C)$, then so is ψ.*

Proof. First note that, in order to prove that $\psi \in \mathcal{I}(C)$, it is enough to verify the 2-increasingness of C_ψ, as the border conditions for a copula are trivially preserved.

If $R = [x_1, x_2] \times [y_1, y_2]$ is a rectangle of $[0,1]^2$ and ψ is an order isomorphism, we denote $R^{\psi^{-1}} = [\psi^{-1}(x_1), \psi^{-1}(x_2)] \times [\psi^{-1}(y_1), \psi^{-1}(y_2)]$. Now, let C be a copula. We denote by $C^R_{i,j} = C(x_i, y_j)$, where $i, j \in \{1, 2\}$, the value that C assumes on the vertex (x_i, y_j) of the rectangle R. We introduce the set

$$\mathrm{Rq}(C) = \big\{(C^R_{1,1}, C^R_{1,2}, C^R_{2,1}, C^R_{2,2}) \in [0,1]^4 \ \mid \ R \text{ is a rectangle } \big\},$$

that consists of all quadruples of values of C in the vertices of all possible rectangles of $[0,1]^2$. For every $\psi \in \mathcal{I}$, it is not difficult to prove that

$$\mathrm{Rq}(C) = \big\{(C^{R^{\psi^{-1}}}_{1,1}, C^{R^{\psi^{-1}}}_{1,2}, C^{R^{\psi^{-1}}}_{2,1}, C^{R^{\psi^{-1}}}_{2,2}) \in [0,1]^4 \ \mid \ R \text{ is a rectangle } \big\},$$

where, for $i, j \in \{1, 2\}$, $C^{R^{\psi^{-1}}}_{i,j} = C(\psi^{-1}(x_i), \psi^{-1}(y_j))$ denotes the value that C assumes on the vertex $(\psi^{-1}(x_i), \psi^{-1}(y_j))$ of the rectangle $R^{\psi^{-1}}$. Therefore, in order to prove that $\psi \in \mathcal{I}$ is also an element of $\mathcal{I}(C)$, we only have to prove that, for every $(a, b, c, d) \in \mathrm{Rq}(C)$,

$$\psi(a) - \psi(b) - \psi(c) + \psi(d) \geq 0.$$

Now, let (a, b, c, d) be in $\mathrm{Rq}(C)$ and let $\varphi \in \mathcal{I}(C)$. Thanks to the monotonicity of φ, we have

$$\varphi(a) \leq \min(\varphi(b), \varphi(c)) \leq \max(\varphi(b), \varphi(c)) \leq \varphi(d).$$

Moreover, since $\varphi \in \mathcal{I}(C)$,

$$\varphi(a) - \varphi(b) - \varphi(c) + \varphi(d) \geq 0.$$

From Lemma 3.1.3, this inequality is preserved when the following affine mapping $T_{a,d}$ is applied to each term:

$$T_{a,d} \colon [\varphi(a), \varphi(d)] \to [\psi(a), \psi(d)],\ x \mapsto \frac{x - \varphi(a)}{\varphi(d) - \varphi(a)} \left(\psi(d) - \psi(a)\right) + \psi(a).$$

Therefore, we obtain that

$$\big(T_{a,d} \circ \varphi\big)(a) - \big(T_{a,d} \circ \varphi\big)(b) - \big(T_{a,d} \circ \varphi\big)(c) + \big(T_{a,d} \circ \varphi\big)(d) \geq 0.$$

We can check easily that $\big(T_{a,d} \circ \varphi\big)(a) = \psi(a)$ and $\big(T_{a,d} \circ \varphi\big)(d) = \psi(d)$. Moreover, by assumption $\varphi \leq_{\mathrm{C}} \psi$ and, in particular, by (3.3), we have

$$\big(T_{a,d} \circ \varphi\big)(b) \geq \psi(b) \quad \text{and} \quad \big(T_{a,d} \circ \varphi\big)(c) \geq \psi(c).$$

Therefore also

$$\psi(a) - \psi(b) - \psi(c) + \psi(d) \geq 0,$$

which is the desired assertion. □

To put Theorem 3.1.5 differently, every $\mathcal{I}(C)$ is an *upper set* with respect to convex ordering $\leq_{\mathrm{C}}$.

As a consequence of Theorem 3.1.5, we may obtain the following result, already known in the literature.

Corollary 3.1.6. *For every $C \in \mathcal{C}$, $\mathcal{I}_{cx} \subseteq \mathcal{I}(C)$.*

Proof. Let C be a copula. Clearly $\mathrm{Id}_{[0,1]} \in \mathcal{I}(C)$. Moreover, by the definition of the relation $\leq_{\mathrm{C}}$ and Theorem 3.1.5 it follows easily that $\mathrm{Id}_{[0,1]} \leq_{\mathrm{C}} \psi$ if and only if ψ is convex. □

The following theorem shows a further procedure to obtain members of $\mathcal{I}(C)$, starting with two other members belonging to this class. The result is based on the closure property of sets $\mathcal{I}(C)$ with respect to convex combinations.

Theorem 3.1.7. *Let C be a copula. If φ, ψ are members of $\mathcal{I}(C)$, then so is $\alpha\varphi + (1-\alpha)\psi$ for any $\alpha \in [0,1]$.*

Proof. The assumption $\varphi, \psi \in \mathcal{I}(C)$ is equivalent to

$$\begin{aligned} \varphi(C^R_{1,1}) - \varphi(C^R_{1,2}) - \varphi(C^R_{2,1}) + \varphi(C^R_{2,2}) &\geq 0, \\ \psi(C^R_{1,1}) - \psi(C^R_{1,2}) - \psi(C^R_{2,1}) + \psi(C^R_{2,2}) &\geq 0, \end{aligned}$$

for every rectangle R. Multiplying the first inequality by α, the second one by $(1-\alpha)$ and adding them up yields the inequality

$$\varrho(C^R_{1,1}) - \varrho(C^R_{1,2}) - \varrho(C^R_{2,1}) + \varrho(C^R_{2,2}) \geq 0,$$

where $\varrho = \alpha\varphi + (1-\alpha)\psi$. Since this inequality also holds for any rectangle R, we have $\varrho \in \mathcal{I}(C)$. □

Summarizing, given a copula C and $\psi \in \mathcal{I}(C)$, Theorems 3.1.5 and 3.1.7 might suggest two methods for constructing other elements in $\mathcal{I}(C)$:

- take all $\varphi \in \mathcal{I}$ such that $\psi \leq_{\mathrm{C}} \varphi$,
- take all convex combinations between ψ and such a φ or any convex $\varphi \in \mathcal{I}$.

Both these methods may be applied for constructing families of copulas, based on C, by means of suitable ψ-transforms as above. In particular, they are "relevant" (i.e. do not produce just convex isomorphisms) when ψ or φ is not convex. In the next, we will see a way to obtain a possibly not convex $\psi \in \mathcal{I}(C)$.

3.2 The TP_2 property under distortions

This section aims at analyzing how copulas satisfying TP_2 property can be transformed into other TP_2 copulas by means of suitable distortions. This fact is relevant from a statistical point of view: it does mean that, starting with a given element in $\mathcal{C}$, we can construct a family of copulas sharing a strong dependence property. Furthermore, from a more formal point of view, the preservation of the TP_2 property guarantees the preservation of the 2-increasing property as well.

Thus, starting with a copula C satisfying the TP_2 property, we would like to find conditions on the transformation ψ such that C_ψ is also TP_2. Preliminarily, we state the following result.

Lemma 3.2.1. *Let $C\colon [0,1]^2 \to [0,1]$ be increasing in each variable. If C is TP_2, then it is 2-increasing.*

Proof. Suppose that C is TP_2. Then, for every x_1, x_2, y_1, y_2 in $[0,1]$, $x_1 \leq x_2$ and $y_1 \leq y_2$, we have that

$$C(x_1, y_1)C(x_2, y_2) \geq C(x_1, y_2)C(x_2, y_1).$$

Because the logarithm function is strictly increasing, it follows that

$$\log C(x_1, y_1) + \log C(x_2, y_2) \geq \log C(x_1, y_2) + \log C(x_2, y_1), \tag{3.4}$$

i.e. $\log C$ is 2-increasing. Now, we recall that if H is a 2-increasing and monotonic function and ϕ is convex and increasing, then $\phi \circ H$ is monotonic and 2-increasing (see [70, page 151]). In particular, by applying this result to the exponential function and to the 2-increasing and monotonic function $\log \circ C$, we obtain that $C = \exp(\log C)$ is 2-increasing, which is the desired assertion. $\square$

Theorem 3.2.2. *Let C be a TP_2 copula. Let $\psi \in \mathcal{I}$. If $\psi \circ \exp\colon (-\infty, 0] \to [0,1]$ is log-convex, then*

(i) C_ψ is TP_2;

(ii) $\psi \in \mathcal{I}(C)$.

Proof. In order to prove (i), given a rectangle $R = [x_1, x_2] \times [y_1, y_2]$ of $[0,1]^2$ we set

$$a_{ij} = C(\psi^{-1}(x_i), \psi^{-1}(y_j)).$$

By definition, every $a_{ij} \in [0,1]$ and

$$a_{11} \leq \min(a_{12}, a_{21}) \leq \max(a_{12}, a_{21}) \leq a_{22}.$$

Since C is TP_2, it follows that $a_{11}a_{22} \geq a_{12}a_{21}$, which implies

$$\log a_{11} + \log a_{22} \geq \log a_{12} + \log a_{21}. \tag{3.5}$$

Now, by assumption $\gamma\colon (-\infty, 0] \to (-\infty, 0]$, $\gamma(t) = \log(\psi(e^t))$ is convex and increasing, and, by applying Lemma 3.1.3 to each term of inequality (3.5), we obtain

$$\log(\psi(a_{11})) + \log(\psi(a_{22})) \geq \log(\psi(a_{12})) + \log(\psi(a_{21})),$$

which, in its turn, implies

$$\psi(a_{11})\psi(a_{22}) \geq \psi(a_{12})\psi(a_{21}). \tag{3.6}$$

Thus, C_ψ is TP_2. Finally, from Lemma 3.2.1 it follows that C_ψ is also 2-increasing and, since C_ψ satisfies also the border conditions for a copula, C_ψ is a copula. □

Note that the condition $\log \circ \psi \circ \exp$ convex is sometimes referred to as *geometric convexity* of ψ (compare with [81]).

Remark 3.2.3. *Log-convexity also plays a key role in characterizing univariate ageing notions. For example, we say that a survival distribution function $\overline{F}$ is Decreasing Failure Rate (DFR) when it is log-convex (compare, e.g., with [68]). This means that we might construct isomorphisms ψ that satisfy the assumptions of Theorem 3.2.2 just by taking $\psi = \overline{F} \circ (-\log)$, where $\overline{F}$ is a suitable DFR survival distribution. By using this fact, we can obtain the following examples corresponding to (versions of) Weibull, Gompertz and Lomax survival distribution functions:*

- $\psi(t) = \exp\left(-(-\log(t))^\alpha\right)$, *where* $\alpha \in]0, 1]$*;*
- $\psi(t) = \exp\left(-\frac{\alpha^{-\log(t)}-1}{\log(\alpha)}\right)$, *where* $\alpha \in]0, 1]$*;*
- $\psi(t) = (1 - \log(t))^{-\alpha}$, *where* $\alpha > 0$.

For a fixed C, the sufficient condition of Theorem 3.2.2 need not be necessary. In fact, the copula M is TP_2 and it coincides with any of its transformation M_ψ, apart from the properties of $\psi \in \mathcal{I}$. However, for the class of strict Archimedean copulas, which can be obtained as a distortion of the copula Π, we have the following characterization.

Corollary 3.2.4. *Let $\psi \in \mathcal{I}$. Then Π_ψ is a TP_2 copula if and only if $\psi \circ \exp$ is log-convex.*

Proof. Suppose that Π_ψ is a TP_2 copula. Then, $\Pi_\psi = C$ is a strict Archimedean copula additively generated by $f(t) = -\log(\psi^{-1}(t))$. Now, it is known from Proposition 6.1 by [21] (compare also with [12]) that C is TP_2 if and only if f^{-1} is log-convex, which is equivalent to $t \mapsto \log(\psi(e^{-t}))$ convex on $[0, +\infty)$, that is $\psi \circ \exp$ log-convex on $(-\infty, 0]$. □

Now, a remarkable fact is derived from Theorem 3.2.2 by considering that every power function $\psi \in \mathcal{I}$ has the property that $\psi \circ \exp$ is log-convex.

Corollary 3.2.5. *Let C be a TP_2 copula and $\psi \in \mathcal{I}$, $\psi(t) = t^\alpha$ for $\alpha > 0$. Then C_ψ is a TP_2 copula.*

This observation is of a great importance since such transformations allow us to construct several parametric families of copulas starting with a known copula C that is TP_2.

Example 3.2.6. *For every $\theta \in [-1, 1]$, let us consider the* Farlie-Gumbel-Morgenstern family *of copulas whose elements are given by*

$$C_\theta(x, y) = xy + \theta xy(1 - x)(1 - y). \tag{3.7}$$

Every C_θ is TP_2 when $\theta \in [0, 1]$. Let us consider $\psi(t) = t^\alpha$ for every $\alpha > 0$ and the transformed copulas $(C_\theta)_\psi$, that we denote by $C_{\alpha,\theta}$. We have that

$$C_{\alpha,\theta}(x, y) = xy[1 + \theta(1 - x^{\frac{1}{\alpha}})(1 - y^{\frac{1}{\alpha}})]^\alpha \tag{3.8}$$

defines a family of copulas that are TP_2 when $\theta \in [0, 1]$ (compare also with [10]).

Example 3.2.7. *Let C be a* semilinear copula *(compare with [33, 41]), i.e. there exists $f : [0, 1] \to [0, 1]$ that is strictly increasing and continuous with $\dfrac{f(t)}{t}$ decreasing on $]0, 1]$ and $f(1) = 1$, such that*

$$C(x, y) = \min(x, y)f(\max(x, y)).$$

Let us consider $\psi(t) = t^\alpha$ for every $\alpha > 0$. Since C is TP_2, then C_ψ is a copula for every $\alpha > 0$. Actually, the distorted copula C_ψ is again a semilinear copula generated by $g(t) = f^\alpha(t^{1/\alpha})$.

3.2.1 Tail dependence coefficients under distortions

A possible reason for adding new parameters to already known copulas lies in producing families that exhibit some more flexible properties. In particular, copulas with different tail behaviour are usually required for building stochastic models for estimating extreme and risky events [72, 57, 80]. In this section, we show how the distortions of a given copula C may modify the tail behaviour of C, as measured by its tail dependence coefficients. Specifically, we state formulas linking the original tail dependence coefficients of C and the ones obtained from some distortion of C.

Here we state our results related to the tail dependence coefficients of the distorted copula.

Proposition 3.2.8. *Let C be a copula such that $\lambda_L(C)$ exists and is finite. Let $\psi \in \mathcal{I}(C)$. If, for some $\alpha > 0$,*

$$\lim_{t\to 0^+} \frac{\psi(t)}{t^\alpha} = b \in (0, +\infty).$$

Then $\lambda_L(C_\psi) = (\lambda_L(C))^\alpha$.

Proof. Let C be a copula with diagonal section δ_C. If $\delta_C = 0$ on $[0, \varepsilon]$ for a small $\varepsilon > 0$, then we easily obtain that $\lambda_L(C_\psi) = 0 = (\lambda_L(C))^\alpha$. Otherwise, suppose that δ_C is strictly increasing on $[0, \varepsilon]$ for some small $\varepsilon > 0$. Taking into account Eq. (1.8), the lower tail dependence coefficient for C_ψ can be expressed as

$$\lambda_L(C_\psi) = \lim_{x\to 0^+} \frac{\psi(C(x,x))}{\psi(x)} = \lim_{x\to 0^+} \left(\frac{\psi(C(x,x))}{[C(x,x)]^\alpha} \cdot \frac{[C(x,x)]^\alpha}{x^\alpha} \cdot \frac{x^\alpha}{\psi(x)} \right).$$

Therefore, from the given assumptions, $\lambda_L(C_\psi) = [\lambda_L(C)]^\alpha$. □

Proposition 3.2.9. *Let C be a copula such that $\lambda_U(C)$ exists and is finite. Let $\psi \in \mathcal{I}(C)$. If, for some $\alpha > 0$,*

$$\lim_{t\to 1^-} \frac{1-\psi(t)}{(1-t)^\alpha} = b \in (0, +\infty),$$

then $\lambda_U(C_\psi) = 2 - (2 - \lambda_U(C))^\alpha$.

Proof. Let C be a copula with diagonal section δ_C. Then, δ_C is strictly increasing on $[1-\varepsilon, 1]$ for some small $\varepsilon > 0$.

By taking into account (1.10), the upper tail dependence coefficient of C_ψ can be expressed as

$$\lambda_U(C_\psi) = 2 - \lim_{x\to 1^-} \frac{1-\psi(C(x,x))}{1-\psi(x)} = 2 - \lim_{x\to 1^-} \frac{1-\psi(\delta_C(x))}{1-\psi(x)}.$$

If, for some $\alpha > 0$, $\lim_{t\to 1^-} \frac{1-\psi(t)}{(1-t)^\alpha} = b \in (0, +\infty)$, then

$$\lim_{x\to 1^-} \frac{1-\psi(\delta_C(x))}{1-\psi(x)} = \lim_{x\to 1^-} \left(\frac{1-\psi(\delta_C(x))}{(1-\delta_C(x))^\alpha} \cdot \frac{(1-\delta_C(x))^\alpha}{(1-x)^\alpha} \cdot \frac{(1-x)^\alpha}{1-\psi(x)} \right).$$

Therefore, $\lambda_U(C_\psi) = 2 - (2 - \lambda_U(C))^\alpha$. □

Example 3.2.10. *For every $\theta \in [-1, 1]$, let us consider the* Farlie-Gumbel-Morgenstern *family $\{C_\theta\}$ of copulas whose elements are given by* (3.7)*. If we consider $\widetilde{\psi} \in \mathcal{I}$, then $\widetilde{\psi}(t) = e^{-(-\log t)^\alpha}$ for every $\alpha \in (0, 1)$. It can be showed that $\widetilde{\psi}$ satisfies the assumptions of Theorem 3.2.2 and Proposition 3.2.9. Thus, $(C_\theta)_{\widetilde{\psi}}$ is a modification of the Farlie-Gumbel-Morgenstern copula having, additionally, upper tail dependence coefficient equal to $2 - 2^\alpha$.*

For the case of strict Archimedean copulas, the above results can be stated in the following simpler forms.

Corollary 3.2.11. *Let $C(x,y) = f^{-1}(f(x)+f(y))$ be an Archimedean copula additively generated by f.*

(i) If $te^{\alpha f(t)} \stackrel{t\to 0^+}{\longrightarrow} b \in (0,+\infty)$ for some $\alpha > 0$, then $\lambda_L(C) = 0$.

(ii) If $\dfrac{1-t}{(1-e^{-f(t)})^{\alpha}} \stackrel{t\to 1^-}{\longrightarrow} b \in (0,+\infty)$ for some $\alpha > 0$, then $\lambda_U(C) = 2 - 2^{\alpha}$.

Proof. The proof follows easily by the Propositions 3.2.8 and 3.2.9 and by the fact that any Archimedean copula $C(x,y) = f^{-1}(f(x)+f(y))$ can be represented as a distortion Π_ψ of the independence copula Π (that has upper and lower tail dependence coefficient equal to 0) with $\psi = f^{-1}\circ(-\log)$. □

More general results about the tail dependence coefficients of an Archimedean copulas are given by [59, 58] and [27, 28] in terms of regularly varying properties of the additive generator.

3.2.2 Concluding remarks

In previous sections of this chapter, we have obtained several results concerning the so-called distortions of a given copula. Here, we would like to clarify how these results may be reformulated and reinterpreted in the more general context of bivariate distribution functions.

We start by a simple observation. Let us consider a bivariate continuous distribution function F and an order isomorphism $\psi \in \mathcal{I}$. Suppose that $F_\psi = \psi\circ F$ is a distribution function. If $\mathbb{P}_F$ is the probability measure generated by F on the Borel sets of $\mathbb{R}^2$, then F_ψ is simply the distribution function of the probability measure $\psi\circ\mathbb{P}_F$. Generalized measures obtained by means of the distortion of a probability measure are usually known as *distorted probabilities* [32].

Now, let C be the copula of F. It is quite easy to prove that the copula of F_ψ is simply C_ψ (see, e.g., [43]). Moreover, the following result can be also stated.

Proposition 3.2.12. *Let us consider a bivariate continuous distribution function F and an order isomorphism $\psi \in \mathcal{I}$. Let C be the copula of F and let $\mathbb{P}_F$ be the probability measure induced by F on $\mathbb{R}^2$. The following statements are equivalent:*

- *$F_\psi(x,y) = \psi(F(x,y))$ is a distribution function,*
- *$\psi \in \mathcal{I}(C)$.*

Thus, investigations about distorted copulas can be as well applied to the cases of distortions of probabilities and distribution functions.

Finally, notice that, under suitable assumptions, the distortions F_ψ obtained from any $\psi \in \mathcal{I}(C)$ have a quite distinguished property: they describe all the bivariate distribution functions having the same level sets as F (see, for example, [45, 76]). Interesting statistical motivations for defining distribution functions by means of level sets have been examined by [10].

Chapter 4

Ageing and its evolution

This chapter is devoted to the study of the ageing properties of a survival model and of their evolution. We recall the notions of univariate ageing and then we introduce notions of bivariate and multivariate ageing, that can be expressed in terms of a suitable semi-copula B, called *ageing function*.

4.1 Univariate and bivariate ageing properties

First, we recall the following notions of univariate ageing for a survival function $\overline{G}$.

Definition 4.1.1. *We say that*

- $\overline{G}$ *is* New Better than Used *(shortly, NBU) if and only if for all* $x, y \in \mathbb{R}_+$, $\overline{G}(x+y) \leq \overline{G}(x)\overline{G}(y)$;
- $\overline{G}$ *is* Increasing Failure Rate *(shortly, IFR) if and only if* $\overline{G}$ *is log-concave (that is,* $\log \circ \overline{G}$ *is concave).*

We say that these notions describe *positive ageing*. The corresponding negative notions are obtained by reverting the sign in the inequalities:

Definition 4.1.2. *We say that*

- $\overline{G}$ *is* New Worse than Used *(shortly, NWU) if and only if for all* $x, y \in \mathbb{R}_+$, $\overline{G}(x+y) \geq \overline{G}(x)\overline{G}(y)$;
- $\overline{G}$ *is* Decreasing Failure Rate *(shortly, DFR) if and only if* $\overline{G}$ *is log-convex.*

Since we are considering absolutely continuous r.v.'s, IFR and DFR are equivalent to the hazard rate $r(t)$ being increasing and decreasing respectively.
It is also immediate to check that IFR implies NBU and DFR implies NWU.

For a detailed discussion about univariate ageing, see [14].

Correspondingly to these two properties, there exist different notions of bivariate ageing; we have considered the following extensions. In the sequel, $\overline{F}$ will denote the joint survival function of the random vector (X, Y).

Definition 4.1.3. *We say that*

- $\overline{F}$ *is (bivariate) NBU if* $\overline{F}(x, y) \geq \overline{G}(x + y)$ $\forall x, y \in \mathbb{R}_+$*;*
- $\overline{F}$ *is (bivariate) IFR if* $\overline{F}(x+\tau, y-\tau) \geq \overline{F}(x, y)$ $\forall\ x < y,\ 0 \leq \tau \leq y-x$*.*

We remark that the notion of bivariate NBU can be alternatively expressed in terms of the stochastic comparison

$$\mathbb{P}(X > x + \tau | X > x) \leq \mathbb{P}(Y > \tau | X > x),\ \forall x, \tau \in \mathbb{R}_+.$$

This inequality makes more explicit the following interpretation of the bivariate NBU: the "new" component Y is "better" than the "used" component X, in the sense that the survival probability of Y is higher.

As to the notion of bivariate IFR, it corresponds to $\overline{F}$ being Schur-concave.

Remark 4.1.4. *Schur-concave (or Schur-convex) distributions describe a very particular case of exchangeability. However they are important both from a conceptual and an application-oriented point of view, in that they give rise, in a sense, to a natural generalization of the fundamental case of independent lifetimes with increasing (or decreasing) failure rate.*
In fact, if X, Y *are independent,*

$$\overline{F}(x + \varepsilon, y - \varepsilon) = \overline{G}(x + \varepsilon)\overline{G}(y - \varepsilon);$$

therefore, $\overline{F}$ *being Schur-concave is equivalent to*

$$\frac{\overline{G}(x)}{\overline{G}(x + \varepsilon)} \leq \frac{\overline{G}(y - \varepsilon)}{\overline{G}(y)}.$$

On its turn, this last condition is equivalent to $\overline{G}$ *being log-concave.*

A particular case of Schur-concave and Schur-convex distribution is, on its turn, a Schur-constant distribution function, $\overline{F}(x, y) = \overline{G}(x + y)$. This case constitutes the generalization of the basic case of independent exponential lifetimes.
The exponential distribution, with its memory-less property, is the basic and idealized probability model for standard reliability methods and for univariate ageing analysis. Similarly, Schur-constant distributions can be seen as the idealized models in the setting of multivariate bayesian analysis of lifetimes. They describe a property of indifference to ageing of the units of the model.

4.2 Semigroups of semi-copulas and evolution of dependence at increase of age

In this section, we consider a pair of exchangeable lifetimes X, Y and the families of the conditional survival functions. We analyze some properties of ageing for $\overline{F}_t(x, y)$ and some relations among properties of dependence and properties of ageing.

Let us consider the function $B : [0, 1] \times [0, 1] \to [0, 1]$ defined by

$$B(u, v) = \exp\left\{-\overline{G}^{-1}\left(\overline{F}(-\log u, -\log v)\right)\right\}. \tag{4.1}$$

It is immediate to see that B satisfies boundary conditions for a copula, is increasing in each variable and continuous, but it does not satisfy, generally, the rectangular inequality. For this reason we say that B is generally a semi-copula ([21, 43]). However, it turns out to be a copula in several cases of interest.

The function B can be used to describe certain "bivariate ageing" properties of the pair (X, Y) and has been called "bivariate ageing function".

By imposing appropriate dependence conditions on B, it is possible to characterize some conditions of bivariate ageing for (X, Y).

Besides dependence properties, an important property of ageing functions is the *supermigrativity* (see [21], where the class of supermigrative semi-copulas, $\mathcal{P}_{SM}$, is denoted by $\mathcal{P}_+^{(3)}$, and [37, 36]).

Definition 4.2.1. *We say that a semi-copula S is* supermigrative *(SM) if, for any $0 < s < 1$, $0 \leq v \leq u \leq 1$,*

$$S(us, v) \geq S(u, sv). \tag{4.2}$$

Analogously, we say that S is submigrative *(sM) if inequality* (4.2) *holds, with the reverted sign.*

Thus, as in [19], we give the following characterization:

Proposition 4.2.2. *For a bivariate survival function $\overline{F}$, we say that*

- *$\overline{F}$ is NBU if B is PQD;*
- *$\overline{F}$ is IFR if B is SM.*

In view of this fact, B can be used to analyze some relations existing among univariate ageing, bivariate ageing, and stochastic dependence (see [21]). From a more technical point of view, relevant features of B are that it describes the family of the level curves of $\overline{F}$ and it permits to give a representation of $\overline{F}$ in terms of the pair $(\overline{G}, B)$.

As we have seen also in the previous chapters, an item of general interest is the conditional survival function:

$$\overline{F}_t\,(x,y) = \mathbb{P}\,(X > t+x, Y > t+y | X > t, Y > t)\,, \tag{4.3}$$

for $t > 0$.

As a natural consequence of the introduction of the family $\{\overline{F}_t\}_{t\geq 0}$, it is of interest to study the evolution of the family denoted by $\{B_t\}_{t\geq 0}$, with obvious use of the notation (see also below).

In this section, we study the behaviour of the family $\{B_t\}_{t\geq 0}$ and detail some relevant aspect of analytical type for $\{B_t\}_{t\geq 0}$. We mention some practical interpretation of these analytical conditions, in particular about increase of ageing.
Furthermore, we point out both analogies and structural differences between $\{B_t\}_{t\geq 0}$ and $\{\hat{C}_t\}_{t\geq 0}$.

A further passage is devoted to discuss specific results, along the lines indicated in [19, 21], related with evolution of dependence and bivariate ageing.

4.2.1 Some basic facts

We begin this subsection with some remarks on arguments contained in [19, 21].

Remark 4.2.3. *Notice that, if $\overline{G}(x) = e^{-x}$, i.e. if $\overline{G}(-\log u) = u$, then $B = \hat{C}$ and thus B is certainly a copula. More generally, we also observe that, if $\overline{G}(-\log u)$ is concave, then B is copula. This fact follows by the general method of transforming copulas by means of distortions (see Chapter 3 or [34]), that is by transformations of the kind*

$$C_\phi(u,v) = \phi^{-1}\big(C(\phi(u),\phi(v))\big),$$

with $\phi : [0,1] \to [0,1]$, ϕ bijective and concave (see e.g. [43, 65, 74, 77]).

Let us consider now the joint law of the residual lifetimes $(X-t, Y-t)$, conditional on the observation of the survival data $\{X > t, Y > t\}$, i.e. $\overline{F}_t\,(x,y)$.

We are interested in studying, in this section, the ageing function of $\overline{F}_t$, for any $t \in \mathbb{R}_+$. For this reason we need that $\overline{G}_t\,(x)$ is continuous and strictly decreasing on $\mathbb{R}_+$ in each variable. This is guaranteed by our assumption that $\overline{F}(x,y)$ is a continuous and strictly decreasing on $\mathbb{R}_+$. This assumption is also equivalent to $B(u,v)$ being strictly increasing in u, for all $v \in (0,1]$, a hypothesis we will use later.
For $0 \leq u \leq 1$, $0 \leq v \leq 1$, we put

$$B_t(u,v) \equiv \exp\left[-\overline{G}_t^{-1}\{\overline{F}_t\,(-\log u, -\log v)\}\right]. \tag{4.4}$$

Remark 4.2.4. *For $t = 0$, B_t coincides with B as given in formula* (4.1).

The structure of the relation between B_t and B is radically different from the one binding $\hat{C}_t$ and $\hat{C}$ (see Chapter 2). In fact we obtain it in an implicit form, as follows by [19, Lemma 12]: $B_t\ (u,v)$ is such that

$$B\left(ue^{-t}, ve^{-t}\right) = B\left(B_t(u,v)e^{-t}, e^{-t}\right); \tag{4.5}$$

actually $B_t\ (u,v)$ is the unique solution σ of the equation

$$B\left(ue^{-t}, ve^{-t}\right) = B\left(\sigma e^{-t}, e^{-t}\right).$$

In view of Eq. (4.5) and under our continuity and monotonicity assumptions in this chapter, it is possible to obtain an explicit expression for B_t in terms of B, as the following corollary states.

Corollary 4.2.5. *Let $b_z : [0,1] \to [0,z]$, $b_z(u) = B(u,z)$, be the section of B at level z.*

$$B_t(u,v) = e^t b_{e^{-t}}^{-1}(B\left(ue^{-t}, ve^{-t}\right)).$$

Proof. Since, for any fixed $z \in [0,1]$, $B(\cdot, z)$ is strictly increasing, the inverse of any horizontal section of B is well defined and we just apply it to Eq. (4.5). □

Remark 4.2.6. *For any $t > 0$, B_t only depends on B and, more precisely, on the restriction of B on the square $[0, e^{-t}]^2$.*

However, for some purposes, it reveals to be clearer maintaining (4.5) in an implicit form.

Proposition 4.2.7. *$\{B_t\}_{t\geq 0}$ is a semigroup, i.e.*

$$\left(B_s\right)_r = \left(B_r\right)_s = B_{r+s} \qquad \forall t, s \geq 0.$$

Proof. In order to prove that $(B_r)_s = B_{r+s}$, in view of Eq. (4.5), we only have to check that

$$B_s(ue^{-r}, ve^{-r}) = B_s\left(B_{r+s}(u,v)e^{-r}, e^{-r}\right). \tag{4.6}$$

The latter is in fact the analog of the relation (4.5), with B replaced by B_s. On the other hand, by letting $t = r + s$ in Eq. (4.5), we can also write

$$B(ue^{-r-s}, ve^{-r-s}) = B\left(B_{r+s}(u,v)e^{-r-s}, e^{-r-s}\right). \tag{4.7}$$

Again using (4.5), for the left-hand side member of (4.7) we have

$$B(ue^{-r-s}, ve^{-r-s}) = B\left(B_s(ue^{-r}, ve^{-r})e^{-s}, e^{-s}\right); \tag{4.8}$$

similarly, for the right-hand side member of (4.7),

$$B\left(B_{r+s}(u,v)e^{-r-s},e^{-r-s}\right)=B\left(B_s(B_{r+s}(u,v)e^{-r},e^{-r})e^{-s},e^{-s}\right). \quad (4.9)$$

Now, Eq. (4.5) shows that the left-hand side of (4.8) and the left-hand side of (4.9) are equal. Thus the right-hand sides of (4.8) and of (4.9) coincide. From the equality

$$B\left(B_s(ue^{-r},ve^{-r})e^{-s},e^{-s}\right)=B\left(B_s(B_{r+s}(u,v)e^{-r},e^{-r})e^{-s},e^{-s}\right),$$

Eq. (4.6) follows, since B is strictly increasing in each variable. □

Remark 4.2.8. *We thought it is useful to present two independent proofs of Proposition 4.2.7 and Proposition 2.2.2. However, one could also obtain each proposition from the other one, by taking into account Eq.* (4.11) *(see below).*
We notice that the proof of Proposition 2.2.2 does not require that $\hat{C}$ is strictly increasing in each variable, while the equivalent condition B strictly increasing in each variable is needed for the proof of Proposition 4.2.7.

One purpose of ours is to analyze increase or decrease of ageing in time.

Proposition 4.2.9. *Let $t\mapsto B_t(u,v)$ be differentiable for any $u,v\in(0,1)$. The map $t\mapsto B_t$ is increasing in the PQD order (see Definition 1.4.16) if*

$$(u,v)\cdot\nabla B(u,v)\le (B(u,v),1)\cdot\nabla B(B(u,v),1). \quad (4.10)$$

Proof. We have to compute the partial derivative of $B_t(u,v)$ w.r.t. t. Differentiating Eq. (4.5), we obtain

$$-ue^{-t}\frac{\partial}{\partial x_1}B(ue^{-t},ve^{-t})-ve^{-t}\frac{\partial}{\partial x_2}B(ue^{-t},ve^{-t})=$$

$$e^{-t}\left[\frac{\partial}{\partial t}B_t(u,v)-B_t(u,v)\right]\frac{\partial}{\partial x_1}B\left(B_t(u,v)e^{-t},e^{-t}\right)-e^{-t}\frac{\partial}{\partial x_2}B\left(B_t(u,v)e^{-t},e^{-t}\right).$$

Again, in view of the semigroup property of $\{B_t\}_{t\ge0}$, we can restrict ourselves to study its sign only for a fixed t, e.g., for $t=0$. We have

$$-(u,v)\cdot\nabla B(u,v)=$$

$$\frac{\partial}{\partial t}B_t(u,v)\frac{\partial}{\partial x_1}B\left(B(u,v),1\right)-B(u,v)\frac{\partial}{\partial x_1}B\left(B(u,v),1\right)-\frac{\partial}{\partial x_2}B\left(B(u,v),1\right).$$

Hence

$$\frac{\partial}{\partial t}B_t(u,v)=B(u,v)+\frac{\frac{\partial}{\partial x_2}B\left(B(u,v),1\right)-(u,v)\cdot\nabla B(u,v)}{\frac{\partial}{\partial x_1}B\left(B(u,v),1\right)}.$$

Since it is immediate by the definition of semi-copula that $\dfrac{\partial}{\partial x_1}B(u,v) \geq 0$ for any $u, v \in [0,1]$, $\dfrac{\partial}{\partial t}B_t(u,v) \geq 0$ when (4.10) holds. $\square$

Concerning the condition $t \mapsto B_t$ increasing, we point out an aspect of the inequality $B_1 \preceq_{PQD} B_2$. This inequality can be equivalently expressed in terms of the level sets of the corresponding survival functions $\overline{F}_1$, $\overline{F}_2$. For $z \in [0,1]$, let

$$L_z^{(\overline{F})} \equiv \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, \overline{F}(x,y) \geq z\}.$$

It can be easily shown that $B_1 \preceq_{PQD} B_2$ if and only if, for any $z \in [0,1]$,

$$L_{\overline{G}_1(z)}^{(\overline{F}_1)} \subseteq L_{\overline{G}_2(z)}^{(\overline{F}_2)}.$$

Remark 4.2.10. *The differentiability of $t \mapsto B_t$ is guaranteed by the only existence and strictly positivity of $\dfrac{\partial}{\partial x_1}B(u,v)$, $\dfrac{\partial}{\partial x_2}B(u,v)$ on the square $(0,1]^2$.*

4.2.2 Some properties of ageing and dependence and their relation

We start this subsection by analyzing some relations between dependence and ageing properties along the same line of [21]. We recall that, as a consequence of Eq. (2.12), for $n = 2$, and Eq. (4.1), the following relations between B and $\hat{C}$ hold:

$$B(u,v) = \exp\left[-\overline{G}^{-1}\{\hat{C}\left(\overline{G}(-\log u), \overline{G}\left(-\log v\right)\right)\}\right]; \qquad (4.11)$$

$$\hat{C}(u,v) = \overline{G}\left\{-\log B\left(e^{-\overline{G}^{-1}(u)}, e^{-\overline{G}^{-1}(v)}\right)\right\}. \qquad (4.12)$$

Remark 4.2.11. *$\hat{C}$ is strictly increasing in each variable if and only if B is strictly increasing in each variable.*
$\hat{C}$ is strictly increasing in each variable if and only if $\hat{C}_t$ is strictly increasing in each variable u and v.
Hence B is strictly increasing in each variable if and only if B_t is strictly increasing in each variable u and v.

The following Proposition is analogous to some consequences of Propositions 5.2, 5.3, 5.4 of [21]. We however deal here with the concept of TP_2, that was not considered there; our proof is direct and independent of the results of [21].

Proposition 4.2.12.

1. $\hat{C}$ TP_2, $\overline{G}$ $IFR \;\Rightarrow\; B$ TP_2.

2. B TP_2, $\hat{C}$ $RR_2 \;\Rightarrow\; \overline{G}$ IFR.

3. B TP_2, $\overline{G}$ $DFR \Rightarrow\; \hat{C}$ TP_2

Proof. For simplicity sake let

$$x = -\log u,\ x' = -\log u',\ y = -\log v,\ y' = -\log v'$$

and

$$\alpha_{11} = \hat{C}(\overline{G}(x'), \overline{G}(y')), \quad \alpha_{12} = \hat{C}(\overline{G}(x'), \overline{G}(y)),$$
$$\alpha_{21} = \hat{C}(\overline{G}(x), \overline{G}(y')), \quad \alpha_{22} = \hat{C}(\overline{G}(x), \overline{G}(y)),$$

where $x' < x$ and $y' < y$. Thus we have

$$\alpha_{22} < \alpha_{12},\ \alpha_{21} < \alpha_{11}$$

and

$$-\log\alpha_{22} > -\log\alpha_{12},\ -\log\alpha_{21} > -\log\alpha_{11}.$$

1. In view of the adopted notation, the assumption $\hat{C}$ TP_2 becomes

$$\alpha_{11}\alpha_{22} \geq \alpha_{12}\alpha_{21}$$

or, equivalently,

$$\log\alpha_{11} - \log\alpha_{12} \geq \log\alpha_{21} - \log\alpha_{22}.$$

Furthermore, since $\overline{G}$ is IFR,

$$D^{-1}(x) = \overline{G}^{-1}(e^{-x})$$

is concave and increasing in x.
Thus, applying $D^{-1}(\cdot)$ to $-\log\alpha_{ij},\ i,j = 1,2$, we obtain
$D^{-1}(-\log\alpha_{12}) - D^{-1}(-\log\alpha_{11}) \geq$

$$\geq D^{-1}(-\log\alpha_{22}) - D^{-1}(-\log\alpha_{21})$$

and hence

$$\overline{G}^{-1}(\alpha_{11}) + \overline{G}^{-1}(\alpha_{22}) \leq \overline{G}^{-1}(\alpha_{12}) + \overline{G}^{-1}(\alpha_{21}). \tag{4.13}$$

This is equivalent to B TP_2, in fact we can rewrite (4.13) as

$$-\overline{G}^{-1}(\alpha_{11}) - \overline{G}^{-1}(\alpha_{22}) \geq -\overline{G}^{-1}(\alpha_{12}) - \overline{G}^{-1}(\alpha_{21}).$$

By applying the exponential to both the members, we obtain

$$e^{-\overline{G}^{-1}(\alpha_{11})}e^{-\overline{G}^{-1}(\alpha_{22})} \geq e^{-\overline{G}^{-1}(\alpha_{12})}e^{-\overline{G}^{-1}(\alpha_{21})},$$

that is

$$B(u,v)B(u',v') \geq B(u,v')B(u',v).$$

2. By the assumption $\hat{C}$ RR_2

$$\alpha_{11}\alpha_{22} \leq \alpha_{12}\alpha_{21}.$$

Thus, by putting

$$u_{ij} = -\log \alpha_{ij}, \quad i, j = 1, 2,$$

we can write

$$u_{22} - u_{21} \geq u_{12} - u_{11}, \tag{4.14}$$

where $u_{22} > u_{21}$, $u_{12} > u_{11}$. Furthermore, since by hypothesis B is TP_2, Eq. (4.13) holds, or, equivalently,

$$D^{-1}(u_{12}) - D^{-1}(u_{11}) \geq D^{-1}(u_{22}) - D^{-1}(u_{21}). \tag{4.15}$$

By (4.14) and since $D^{-1}(x)$ is increasing in x, this last inequality holds only if $D^{-1}(x)$ is concave in x, that is $\overline{G}$ is IFR.

3. We have to prove now that

$$\alpha_{11}\alpha_{22} \geq \alpha_{12}\alpha_{21},$$

that is equivalent to

$$u_{22} - u_{21} \leq u_{12} - u_{11}.$$

Since $\overline{G}$ is DFR, $D^{-1}(x)$ is increasing and convex in x. Thus, if, against the thesis, we had

$$u_{22} - u_{21} > u_{12} - u_{11},$$

we would have consequently

$$D^{-1}(u_{12}) - D^{-1}(u_{11}) < D^{-1}(u_{22}) - D^{-1}(u_{21}),$$

that would contradict Eq. (4.15) and therefore the hypothesis that B is TP_2.

□

In the following, we want to show that, also as far as B is concerned, we can find a family of semi-copulas $\mathcal{P}$ such that $B \in \mathcal{P}$ is equivalent to $B_t \in \mathcal{P}$ for all $t \geq 0$. To this purpose we compare conditions of the type $B \in \mathcal{P}$ and $\{B_t \in \mathcal{P}, \ \forall t \geq 0\}$. As already mentioned, for suitable families $\mathcal{P}$, the condition $B \in \mathcal{P}$ describes a property of bivariate ageing for $\overline{F}$. In particular, we recall that the conditions $B \in \mathcal{P}_{SM}$ and $B \in \mathcal{P}_{PQD}$ can be seen as bivariate notions of IFR and NBU respectively (see e.g. [21]). In this respect, it is useful to point out the following facts:

Lemma 4.2.13. *(see [21]) $B \in \mathcal{P}_{SM}$ is equivalent to $\overline{F}$ being Schur-concave.*

It can be easily checked that

Proposition 4.2.14. *$B \in \mathcal{P}_{SM}$ is equivalent to*

$$B_t \in \mathcal{P}_{SM} \ \forall \ t \geq 0.$$

Lemma 4.2.15. *(see [19]) $B \in \mathcal{P}_{SM}$ is equivalent to $B_t \in \mathcal{P}_{PQD}$ for all $t \geq 0$.*

In view of the afore-mentioned "bivariate ageing" interpretations of the conditions $B \in \mathcal{P}_{SM}$ and $B \in \mathcal{P}_{PQD}$, we can read Lemma 4.2.13 as follows: $\overline{F}_t$ bivariate NBU for all $t \geq 0$ is equivalent to $\overline{F}$ being bivariate IFR.

For a fixed univariate survival function $\overline{H}(x)$, consider now

$$\overline{H}_t(x) = \mathbb{P}(X > x + t | X > t) = \frac{\overline{H}(x+t)}{\overline{H}(t)}.$$

The following chain of equivalences is very well known and easy to check:

$$\overline{H} \ IFR \ \Leftrightarrow \ \overline{H}_t \ IFR \ \Leftrightarrow \ \overline{H}_t \ NBU \quad \forall t \geq 0.$$

Remark 4.2.16. *Let us consider the family of the Archimedean semi-copulas $\{A_t\}_{t\geq 0}$ associated to the survival functions $\overline{H}_t$'s,*

$$A_t(u,v) = \overline{H}_t \left[\overline{H}_t^{-1}(u) + \overline{H}_t^{-1}(v) \right].$$

Following the arguments in [11] and [21], we can say that A_t describes (univariate) ageing properties of $\overline{H}_t$, in the sense that positive ageing properties of $\overline{H}_t$ correspond to negative dependence properties of A_t.

The equivalence

$$\overline{H} \ IFR \ \Leftrightarrow \ \overline{H}_t \ NBU \ \forall t \geq 0$$

can be written in the form

$$A \in \mathcal{P}_{sM} \Leftrightarrow \ A_t \ NQD \ \forall t \geq 0. \tag{4.16}$$

We notice that, as a straight consequence of the semigroup property of $\{\overline{H}_t\}_{t\geq 0}$, $\{A_t\}_{t\geq 0}$ too is a semigroup and (4.16) *can be analogously proven to Lemma 4.2.13.*

Concerning the TP_2 property for B, we can see that B_t TP_2 for all $t \geq 0$ is not implied by B TP_2.

On the other hand, the property B_t TP_2 for all $t \geq 0$ is actually also a stronger condition than B being SM. In fact, as an immediate consequence of Lemma 4.2.15 and of the fact that $TP_2 \Rightarrow PQD$, we have

Corollary 4.2.17. $B_t\ TP_2\ \ \forall t \geq 0 \Rightarrow\ B \in \mathcal{P}_{SM}$.

As we have seen from Proposition 4.2.12, the link between ageing and dependence properties is not immediate, in the sense that we cannot derive ageing properties from dependence ones only, nor viceversa: we need a further condition on univariate ageing.

By combining Proposition 4.2.12 with Corollary 4.2.17 and Proposition 2.2.8, we obtain a link between SM property for B and TP_2 property for $\hat{C}$.

Corollary 4.2.18. $\hat{C}\ TP_2,\ \overline{G}_t\ IFR\ \Rightarrow\ B\ SM$.

4.3 Ageing functions and multivariate notions of NBU and IFR

For $n \geq 2$, let $\mathbf{T} = (T_1, \ldots, T_n)$ be a vector of exchangeable continuous lifetimes with joint survival function $\overline{F}$ and margins $\overline{G}$. As in the bivariate case, for such models, we want to study some properties of multivariate ageing of $\overline{F}$ that are described by means of the multivariate ageing function $B_{\overline{F}}$, which is a useful tool for describing the level curves of $\overline{F}$.
Since, from now on, when we refer to a B, we suppose it is the ageing function associated to some given $\overline{F}$, we drop $\overline{F}$ from the notation.

Specifically, the attention is devoted on notions that generalize the univariate concepts of New Better than Used and Increasing Failure Rate. These multivariate notions are satisfied by random vectors whose components are conditionally i.i.d., having univariate conditional survival function that is New Better than Used (respectively, Increasing Failure Rate). Furthermore, they also have an interpretation in terms of comparisons among conditional survival functions of residual lifetimes, given a same history of observed survivals.

As it is well known, several approaches have been proposed in the literature to define properties of multivariate ageing that could be considered as natural extensions of the univariate ageing notions.

For such models, we aim at studying some properties of multivariate ageing of $\overline{F}$. More in particular, we will consider properties that are described by means of the *multivariate ageing function* $B_{\overline{F}}$ or, simply, $B\colon [0,1]^n \to [0,1]$, given by

$$B_{\overline{F}}(\mathbf{u}) = \exp\left(-\overline{G}^{\,-1}(\overline{F}(-\log(u_1), \ldots, -\log(u_n)))\right). \tag{4.17}$$

Studying time evolution of ageing properties corresponds to studying the evolution in time of the family of semi-copulas $\{B_t\}_{t\geq 0}$, where

$$B_t(u_1, \ldots, u_n) \equiv \exp\left[-\overline{G}_t^{-1}\{\overline{F}_t\,(-\log u_1, \ldots, -\log u_n)\}\right]. \tag{4.18}$$

As to the ageing function, the relation between B_t and the generator of the family $B_0 \equiv B$ is provided by the following Lemma, consisting in an extension of [19, Lemma 12].

Lemma 4.3.1. *Let $\overline{F}$ strictly decreasing in each variable. Then*

$$B\left(u_1 e^{-t}, \ldots, u_n e^{-t}\right) = B\left(B_t(u_1, \ldots, u_n)e^{-t}, e^{-t}, \ldots, e^{-t}\right). \tag{4.19}$$

Proof. By definition of $\overline{F}_t$ and B, we have

$$\begin{aligned} \overline{F}_t(-\log u_1, \ldots, -\log u_n) &= \frac{\overline{F}(t - \log u_1, \ldots, t - \log u_n)}{\overline{F}(t, \ldots, t)} \\ &= \frac{\overline{G}(-\log B(u_1 e^{-t}, \ldots, u_n e^{-t}))}{\overline{F}(t, \ldots, t)}. \end{aligned}$$

On the other hand,

$$\overline{F}_t(-\log u_1, \ldots, -\log u_n) = \overline{G}_t(-\log B_t(u_1, \ldots, u_n)).$$

Now

$$\overline{G}_t(-\log B_t(u_1, \ldots, u_n)) = \frac{\overline{G}(-\log B(B_t(u_1, \ldots, u_n)e^{-t}, e^{-t}, \ldots, e^{-t}))}{\overline{F}(t, \ldots, t)};$$

therefore

$$\overline{G}(-\log B(B_t(u_1, \ldots, u_n)e^{-t}, e^{-t}, \ldots, e^{-t})) = \overline{G}(-\log B(u_1 e^{-t}, \ldots, u_n e^{-t})).$$

By the strict increasingness of $\overline{G}$, it follows

$$B(B_t(u_1, \ldots, u_n)e^{-t}, e^{-t}, \ldots, e^{-t}) = B(u_1 e^{-t}, \ldots, u_n e^{-t}).$$

□

The following corollary provides an explicit expression for B_t in terms of B. Let $b_z : [0,1] \to [0,z]$, $b_z(u) = B(u, z, \ldots, z)$, be the section of B at level z.

Corollary 4.3.2.

$$B_t(u_1, \ldots, u_n) = e^t b_{e^{-t}}^{-1}\left(B\left(u_1 e^{-t}, \ldots, u_n e^{-t}\right)\right). \tag{4.20}$$

Proof. Since, for any fixed $z \in [0,1]$, $B(\cdot, z, \ldots, z)$ is strictly increasing, the inverse of any section of B is well defined and we just apply it to Eq. (4.19). □

As discussed in some previous papers (see [18, 19, 21, 45, 49]), B can be used for investigating some notions of multivariate ageing. Specifically, in [18] (see also [21]), it has been argued that notions of multivariate ageing based on B can be defined by means of the following scheme:

(i) Consider a univariate ageing notion **P** (e.g., NBU, IFR).

(ii) Take the joint survival function $\overline{F}$ of n i.i.d. lifetimes and prove results of the following type: each lifetime has the property **P** if and only if B has the property $\widetilde{\mathbf{P}}$.

(iii) Define a multivariate ageing notion as follows: any exchangeable survival function $\overline{F}$ is multivariate-**P** if B has the property $\widetilde{\mathbf{P}}$.

Actually, in [18, 21] the above analysis has been developed for the case $n = 2$, where it is also shown that, for notions of this type, the relations among univariate ageing, multivariate ageing and dependence properties of $\overline{F}$ can be easily analyzed. In this section, we aim at pointing out features and differences that arise in the extension of this study to the multivariate case, making it worth of a further analysis.

Specifically, we concentrate our attention on notions that generalize the univariate concepts of NBU and IFR. As we will show, the multivariate notions to be introduced are satisfied by random vectors whose components are conditionally i.i.d. having NBU (respectively, IFR) univariate conditional survival function. This circumstance has been considered as a natural requirement for Bayesian notions of multivariate ageing (see e.g. [13]). Moreover, it implies the usual assumption that multivariate extensions of some univariate ageing property **P** should be satisfied by vectors of i.i.d. lifetimes of type **P**.

Furthermore, these notions also have an interpretation in terms of the comparisons among conditional survival functions of residual lifetimes, given a same history of observed survivals, another interesting property of Bayesian ageing according to [16].

This section is organized as follows. Subsection 4.3.1 contains basic definitions and properties of multivariate ageing functions. Some subclasses of such functions are also introduced and the relations among them discussed. Subsection 4.3.2 presents some definitions of multivariate ageing extending notions of NBU and IFR. Their properties are discussed in detail. Finally, subsection 4.3.3 is devoted to a short discussion about given definitions and results.

4.3.1 Multivariate ageing function: definitions and properties

We start this subsection introducing some useful notations and definitions.

Through this section, we will often formulate our results referring to one of the following assumptions:

Assumption 1 (exchangeable case).
We consider an exchangeable random vector $\mathbf{T} = (T_1, \ldots, T_n)$ $(n \geq 2)$ of

continuous lifetimes with joint survival function $\overline{F}\colon \mathbb{R}_+^n \to [0,1]$ and univariate survival margins equal to $\overline{G}$. We recall, we are supposing $\overline{G}$ to be strictly decreasing on $\mathbb{R}_+$ with $\overline{G}(0) = 1$ and $\overline{G}(+\infty) = 0$.

Assumption 2 (i.i.d. case).
Under Assumption 1, we suppose in addition that $T_1, \dots, T_n$ are independent.

We recall that, when the components of $\mathbf{T}$ are independent, the survival copula $\hat{C}(\mathbf{u}) = \Pi_n(\mathbf{u}) = u_1 \cdots u_n$; in this case, we will also denote $\overline{F}$ by means of the symbol $\overline{F}_{\Pi,\overline{G}}$. In this case, the multivariate ageing function B, given by (4.17), can be written

$$B_\Pi(\mathbf{u}) = \exp\left(-\overline{G}^{\,-1}\left(\overline{G}(-\log(u_1)) \cdots \overline{G}(-\log(u_n))\right)\right). \tag{4.21}$$

In the sequel, we use the term "multivariate ageing function" to denote any continuous semi-copula $B\colon [0,1]^n \to [0,1]$, that can be obtained from some survival function $\overline{F}$ by means of (4.17). Note that every copula $\hat{C}$ is a multivariate ageing function, since it can be obtained as the multivariate ageing function of a survival function $\overline{F}$ having copula $\hat{C}$ and univariate survival margin $\overline{G}(t) = \exp(-t)$.

Within the family of the multivariate ageing functions, we define the following classes; as we will see, these classes will be used to express our multivariate ageing notions.

Definition 4.3.3. *Let B be a multivariate ageing function. We say that:*

(A1) $B \in \mathcal{P}_{PLOD}$ *if and only if for every* $\mathbf{u} \in [0,1]^n$

$$B(u_1, \dots, u_n) \geq \Pi_n(u_1, \dots, u_n). \tag{4.22}$$

(A2) $B \in \mathcal{P}_{PPLOD}$ *if and only if for all* $i, j \in \{1, \dots, n\}$, $i \neq j$, *and for every* $\mathbf{u} \in [0,1]^n$,

$$B(u_1, \dots, u_i, \dots, u_j, \dots, u_n) \geq B(u_1, \dots, u_i u_j, \dots, 1, \dots, u_n). \tag{4.23}$$

(A3) $B \in \mathcal{P}_{SM}$ *if and only if for all* $i, j \in \{1, \dots, n\}$, $i \neq j$, *for all* $u_i, u_j \in [0,1]$, $u_i \geq u_j$, *and for every* $s \in (0,1)$,

$$B(u_1, \dots, u_i s, \dots, u_j, \dots, u_n) \geq B(u_1, \dots, u_i, \dots, u_j s, \dots, u_n). \tag{4.24}$$

The corresponding classes $\mathcal{P}_{NLOD}$, $\mathcal{P}_{PNLOD}$, $\mathcal{P}_{sM}$ are defined by reversing the inequality signs in (4.22), (4.23) *and* (4.24), *respectively.*

The property of (4.22) is a pointwise comparison between the multivariate ageing function B and the copula Π_n. In particular, copulas satisfying (4.22) are called *positive lower orthant dependent* (shortly, PLOD, see [57]). Properties expressed in (4.23) and (4.24) are essentially inequalities related to the bivariate sections of B. In particular, (4.24) defines the *supermigrativity* (compare with [37]), that, in the multivariate case, consists of the supermigrativity of all the bivariate sections of B. Eq. (4.23) is one of the weaker forms of (4.24), obtained by letting in it $u_i = 1$ and $s = \frac{1}{u_j}$, or it can also be seen as a pairwise PLOD. Therefore, $\mathcal{P}_{SM} \subseteq \mathcal{P}_{PPLOD}$; but the converse inclusion is not true, as shown in Example 4.3.6 below.

Furthermore, $\mathcal{P}_{PPLOD} \subseteq \mathcal{P}_{PLOD}$. In fact, by iteratively applying (4.23), we obtain that, for every $\mathbf{u} \in [0,1]^n$,

$$
\begin{aligned}
&B(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_k, \ldots, u_n) \geq B(u_1, \ldots, u_i u_j, \ldots, 1, \ldots, u_k, \ldots, u_n) \\
&\geq B(u_1, \ldots, u_i u_j u_k, \ldots, 1, \ldots, 1, \ldots, u_n) \geq \cdots \geq B(1, \ldots, u_1 \cdots u_n, \ldots, 1) \\
&= u_1 \cdots u_n.
\end{aligned}
$$

Since a multivariate ageing function B is such that $B(\mathbf{u}) = u_i$ for any $\mathbf{u} \in [0,1]^n$ having all the components equal to 1 except possibly for the i-th one, $B \in \mathcal{P}_{PPLOD}$ is equivalent to $B \in \mathcal{P}_{PLOD}$ for the case $n = 2$. However, in the n-dimensional case, $n \geq 3$, $\mathcal{P}_{PPLOD}$ is strictly included in $\mathcal{P}_{PLOD}$, as it will be shown in Example 4.3.5.

In the following example, we consider the case of the so-called TTE models (see [15, 86]). These models can be characterized as those multivariate survival functions admitting an Archimedean survival copula.

Example 4.3.4. *Let B be a multivariate ageing function that can be written in the form:*

$$
B(\mathbf{u}) = \psi^{-1}\left(\sum_{i=1}^{n} \psi(u_i)\right) \tag{4.25}
$$

for some strictly decreasing $\psi\colon [0,1] \to \mathbb{R}_+$ such that $\psi(0) = +\infty$ and $\psi(1) = 0$. This ψ is usually called additive generator *of B. Such a B belongs to the class of the n-*dimensional strict triangular norms *(see [66]). In particular, B is also a copula (usually called* strict Archimedean copula*), when ψ^{-1} is n-completely monotone (see [71]). Now, for a semi-copula B of type* (4.25) *the following statements can be proved :*

- *(i) $B \in \mathcal{P}_{PLOD}$ if and only if $B \in \mathcal{P}_{PPLOD}$, and this happens when $\psi(uv) \leq \psi(u) + \psi(v)$ for all $u, v \in [0,1]$;*
- *(ii) $B \in \mathcal{P}_{SM}$ if and only if ψ^{-1} is log–convex (see [21, 37]).*

Notice that the multivariate ageing functions B_Π of (4.21) *are of the form* (4.25) *with $\psi(t) = -\log(\overline{G}(-\log(t)))$.*

We conclude this part by providing some examples clarifying the relations among the above mentioned classes.

Example 4.3.5. *Let $f\colon [0,1] \to [0,1]$ be the function given by*

$$f(t) = \begin{cases} et, & t \in [0, e^{-2}], \\ e^{-1}, & t \in]e^{-2}, e^{-1}], \\ t & t \in]e^{-1}, 1]. \end{cases}$$

Let $C\colon [0,1]^3 \to [0,1]$ be given by $C(u_1, u_2, u_3) = u_{(1)} f(u_{(2)}) f(u_{(3)})$, where $u_{(1)}, u_{(2)}, u_{(3)}$ denote the components of $\mathbf{u}$ rearranged in increasing order. Since $f(1) = 1$, f is increasing, and $f(t)/t$ is decreasing on $]0,1]$, it follows that C is a copula (see [42, Theorem 3]). Actually, C is the survival copula of a random vector (X_1, X_2, X_3) having the stochastic representation $X_i = \max(Y_i, Z)$ $(i = 1, 2, 3)$, where Y_1, Y_2, Y_3, Z are independent lifetimes. Roughly speaking, C is the survival copula of a random vector of independent lifetimes (Y_1, Y_2, Y_3) affected by a common shock Z (see also [38]).

It follows from [42] that C belongs to $\mathcal{P}_{PLOD}$. However, $C \notin \mathcal{P}_{PPLOD}$. In fact, by taking $u_1 = e^{-\frac{5}{2}}$, $u_2 = e^{-\frac{3}{2}}$ and $u_3 = e^{-\frac{1}{2}}$, we have that $C(u_1, u_2, u_3) = e^{-4} < e^{-\frac{7}{2}} = C(u_1, u_2 u_3, 1)$.

Example 4.3.6. *Let B be the multivariate ageing function of type (4.25), where $\psi\colon [0,1] \to \mathbb{R}_+$ is given by*

$$\psi(t) = \begin{cases} -\log(t), & t \in]0, e^{-2-\varepsilon}], \\ -\frac{\varepsilon}{1+\varepsilon}(\log(t) + 1) + 2, & t \in]e^{-2-\varepsilon}, e^{-1}], \\ -2\log(t), & t \in]e^{-1}, 1], \end{cases}$$

with $\varepsilon \in]0,1[$. Now, let us consider $g\colon \mathbb{R}_+ \to \mathbb{R}_+$, $g(t) = \psi(\exp(-t))$, given by

$$g(t) = \begin{cases} 2t, & t \in [0,1], \\ \dfrac{\varepsilon}{1+\varepsilon}(t-1) + 2, & t \in (1, 2+\varepsilon], \\ t, & t \in (2+\varepsilon, +\infty[. \end{cases}$$

Now, g is not concave and, hence, ψ^{-1} is not log–convex. Thus, in view of Example 4.3.4, $B \notin \mathcal{P}_{SM}$. However, it can be shown that $\psi(uv) \le \psi(u) + \psi(v)$ for all $u, v \in [0,1]$. From Example 4.3.4, it follows that $B \in \mathcal{P}_{PPLOD}$.

4.3.2 Multivariate ageing notions of NBU and IFR for exchangeable random variables

In this subsection, we consider the families $\mathcal{P}_{PLOD}, \mathcal{P}_{PPLOD}, \mathcal{P}_{SM}$ to define notions of positive ageing in terms of the multivariate ageing function B. Notice that, since negative properties can be introduced and studied in a similar way, they will not be considered in detail.

As stated in the introduction, we aim at extending an ageing notion from the univariate case to the n-dimensional case ($n \geq 2$), following the line of [18]. To this end, we link univariate ageing notions to analytical properties of a multivariate ageing function. More precisely, we link properties of a survival function $\overline{G}$ to properties of the multivariate ageing function B_Π, which is associated with n i.i.d. lifetimes whose marginal survival function is $\overline{G}$. The following result can be given.

Proposition 4.3.7. *Under Assumption 2, the following statements are equivalent:*

(a) $\overline{G}$ *is NBU,*

(b) $B_\Pi \in \mathcal{P}_{PLOD}$*,*

(c) $B_\Pi \in \mathcal{P}_{PPLOD}$.

Proof.
(a)$\Longleftrightarrow$(b): Let $\overline{G}$ be NBU. It can be proved by induction that, for all $x, y \in \mathbb{R}_+$, $\overline{G}(x+y) \leq \overline{G}(x)\overline{G}(y)$ is equivalent to

$$\overline{G}\left(\sum_{i=1}^{n} x_i\right) \leq \prod_{i=1}^{n} \overline{G}(x_i),$$

for any $\mathbf{x} \in \mathbb{R}_+^n$. Setting $x_i = -\log(u_i)$, we obtain that, for all $\mathbf{u} \in [0,1]^n$,

$$\overline{G}(-\log(u_1 \cdots u_n)) \leq \overline{G}(-\log(u_1)) \cdots \overline{G}(-\log(u_n)), \tag{4.26}$$

from which it straightly follows that

$$\exp\left(-\overline{G}^{-1}(\overline{G}(-\log(u_1 \cdots u_n)))\right) \leq \exp\left(-\overline{G}^{-1}(\overline{G}(-\log(u_1)) \cdots \overline{G}(-\log(u_n)))\right),$$

i.e. $B_\Pi \geq \Pi_d$.
(a)$\Longleftrightarrow$(c): Since $\overline{G}$ is NBU, $\overline{G}(-\log(u_i u_j)) \leq \overline{G}(-\log(u_i))\overline{G}(-\log(u_j))$ holds for all $u_i, u_j \in [0,1]$. By multiplying both the sides of the inequality by $\prod_{k \in J} \overline{G}(-\log(u_k))$, where $J = \{1, 2, \ldots, n\} \setminus \{i, j\}$ and $u_k \in [0,1]$ for every $k \in J$, and applying the function $\exp \circ (-\overline{G}^{-1})$ to both members, we obtain

$$B_\Pi(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_n) \geq B_\Pi(u_1, \ldots, u_i u_j, \ldots, 1, \ldots, u_n),$$

that is $B_\Pi \in \mathcal{P}_{PPLOD}$. ☐

Therefore, we can write

$$\mathcal{P}_{PLOD} \cap \{B_\Pi : \overline{G} \text{ is NBU}\} = \mathcal{P}_{PPLOD} \cap \{B_\Pi : \overline{G} \text{ is NBU}\}.$$

Notice that, in general, $\mathcal{P}_{PLOD} \neq \mathcal{P}_{PPLOD}$.

Proposition 4.3.8. *Under Assumption 2, the following statements are equivalent:*

(a) $\overline{G}$ *is IFR,*

(b) $B_\Pi \in \mathcal{P}_{SM}$.

Proof. Let $\overline{G}$ be IFR. As it easily follows, this fact is equivalent to

$$\frac{\overline{G}(x_i+\sigma)}{\overline{G}(x_i)} \geq \frac{\overline{G}(x_j+\sigma)}{\overline{G}(x_j)},$$

for any $x_i, x_j \in \mathbb{R}_+,\ x_i \leq x_j$ and $\sigma \geq 0$. By substituting $x_i = -\log(u_i)$, $x_j = -\log(u_j),\ \sigma = -\log(s)$, we obtain

$$\overline{G}(-\log(u_i s))\overline{G}(-\log(u_j)) \geq \overline{G}(-\log(u_j s))\overline{G}(-\log(u_i)),$$

for any $u_i, u_j \in (0,1],\ u_i \geq u_j$ and $s \in (0,1)$. By multiplying both the sides of the inequality by $\prod_{k\in J} \overline{G}(-\log(u_k))$, where $J = \{1,2,\dots,n\} \setminus \{i,j\}$ and $u_k \in [0,1]$ for every $k \in J$, and applying the function $\exp\circ(-\overline{G}^{-1})$ to both the members, we obtain

$$B_\Pi(u_1,\dots,u_i s,\dots,u_j,\dots,u_n) \geq B_\Pi(u_1,\dots,u_i,\dots,u_j s,\dots,u_n),$$

that is $B_\Pi \in \mathcal{P}_{SM}$. □

The previous proposition is actually a reformulation in terms of the multivariate ageing function of well-known results concerning the joint survival function $\overline{F} = \overline{F}_{\Pi,\overline{G}}$ of i.i.d. lifetimes that are IFR. As noted several times in the literature (see, for example, [15, 13, 86]), such a $\overline{F}$ is *Schur–concave*, i.e. for every $s \geq 0$ and for every $i,j \in \{1,2,\dots,n\}$, $i<j$, the mapping

$$x_i \mapsto \overline{F}(x_1,\dots,x_{i-1},x_i,x_{i+1}\dots,x_{j-1},s-x_i,x_{j+1},\dots,x_n)$$

is decreasing on $\left[\frac{s}{2},+\infty\right]$ (see [70, A.2.b]). This is equivalent to

$$\overline{F}(x_1,\dots,x_i+\tau,\dots,x_j-\tau,\dots,x_n) \geq \overline{F}(x_1,\dots,x_i,\dots,x_j,\dots,x_n) \quad (4.27)$$

for all $i,j \in \{1,2,\dots,n\}$, $i<j$, for every $\mathbf{x} \in \mathbb{R}^n_+$ such that $x_i \leq x_j$, and for every $\tau \in [0, x_j - x_i]$.

Remark 4.3.9. *As noted, B_Π is actually a n-dimensional strict triangular norm additively generated by $\psi = (-\log)\circ\overline{G}\circ(-\log)$. In this context, Propositions 4.3.7 and 4.3.8 can be reinterpreted in the following sense: univariate ageing properties of ψ^{-1}, which is a univariate survival function, reflect on special inequalities holding for the triangular norm generated by ψ. As a consequence, these results can be seen as extensions of the investigations in [12].*

Now, by using Propositions 4.3.7 and 4.3.8 and the scheme presented at the beginning of the present section, we introduce the following notions of multivariate ageing for an exchangeable survival function $\overline{F}$.

Definition 4.3.10. *Under Assumption 1, we say that:*

- $\overline{F}$ *is* B-multivariate-NBU of the first type *(shortly, B-MNBU1) if and only if* $B \in \mathcal{P}_{PLOD}$*;*
- $\overline{F}$ *is* B-multivariate-NBU of the second type *(shortly, B-MNBU2) if and only if* $B \in \mathcal{P}_{PPLOD}$*;*
- $\overline{F}$ *is* B-multivariate-IFR *(shortly, B-MIFR) if and only if* $B \in \mathcal{P}_{SM}$*.*

In order to avoid confusions with other multivariate notions of ageing introduced in the previous literature, we used the prefix "B-" for the notions introduced above. This also serves to underline the fact that all these notions are expressed in terms of the multivariate ageing function B of $\overline{F}$. Now, we would like to underline some properties of these notions.

First, notice that any k–dimensional marginal of $\overline{F}$ $(2 \leq k \leq n-1)$ has the same multivariate ageing property of $\overline{F}$. This point is formalized in the following result.

Proposition 4.3.11. *Suppose that Assumption 1 holds. For every* $2 \leq k \leq n$*, let* $\overline{F}^{(k)}$ *be the* k*–dimensional marginal of* $\overline{F}$*. If* $\overline{F}$ *is B-MNBU1 (respectively, B-MNBU2 or B-MIFR), then* $\overline{F}^{(k)}$ *is B-MNBU1 (respectively, B-MNBU2 or B-MIFR).*

Proof. If $\overline{F}^{(k)} : \mathbb{R}_+^k \to [0,1]$ is the k–dimensional margin of $\overline{F}$ $(2 \leq k \leq n)$, given by

$$\overline{F}^{(k)}(x_1, \dots, x_k) = \overline{F}(x_1, \dots, x_k, 0, \dots, 0),$$

then it follows from (4.17) that

$$B_{\overline{F}^{(k)}}(u_1, \dots, u_k) = B_{\overline{F}}(u_1, \dots, u_k, 1, \dots, 1).$$

Easy calculations show that $B_{\overline{F}^{(k)}}$ is in $\mathcal{P}_{PLOD}$ (respectively, $\mathcal{P}_{PPLOD}$ or $\mathcal{P}_{SM}$), when B is in $\mathcal{P}_{PLOD}$ (respectively, $\mathcal{P}_{PPLOD}$ or $\mathcal{P}_{SM}$), which is the desired assertion. $\square$

The previous definitions of multivariate ageing admit some probabilistic interpretations in terms of conditional survival probabilities for residual lifetimes. Before stating them, we clarify the notation. For every $\mathbf{x} \in \mathbb{R}_+^n$ we denote by $\hat{\mathbf{x}}_i$ the vector of $\mathbb{R}_+^{n-1}$ obtained by depriving $\mathbf{x}$ of its i-th component. Similar agreement will be applied to random vectors.

Proposition 4.3.12. *Under Assumption 1, the following statements hold:*

(a) $\overline{F}$ *is B-MNBU1 if and only if for every* $i \in \{1, 2, \ldots, n\}$, $\mathbf{x} \in \mathbb{R}_+^n$ *and* $\tau > 0$,

$$\begin{aligned} &\mathbb{P}\left(T_1 > x_1, \ldots, T_i > x_i + \tau, \ldots T_n > x_n \mid T_i > x_i\right) \\ &\qquad \geq \mathbb{P}\left(T_i > x_1 + \cdots + x_i + \cdots + x_n + \tau \mid T_i > x_i\right). \end{aligned} \tag{4.28}$$

(b) $\overline{F}$ *is B-MNBU2 if and only if for all* $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, *for every* $\hat{\mathbf{x}}_j \in \mathbb{R}_+^{n-1}$ *and* $\tau > 0$,

$$\mathbb{P}\left(T_j > \tau \mid \hat{\mathbf{T}}_j > \hat{\mathbf{x}}_j\right) \geq \mathbb{P}\left(T_i > \tau + x_i \mid \hat{\mathbf{T}}_j > \hat{\mathbf{x}}_j\right). \tag{4.29}$$

(c) $\overline{F}$ *is B-MIFR if and only if for all* $i, j \in \{1, 2, \ldots, n\}$, *for every* $\mathbf{x} \in \mathbb{R}_+^n$ *such that* $x_i \leq x_j$, *and for every* $\tau > 0$,

$$\mathbb{P}\left(T_i > x_i + \tau \mid \mathbf{T} > \mathbf{x}\right) \geq \mathbb{P}\left(T_j > x_j + \tau \mid \mathbf{T} > \mathbf{x}\right). \tag{4.30}$$

Proof.

(a) By definition, $\overline{F}$ is B-MNBU1 if and only if for every $\mathbf{u} \in [0, 1]^n$,

$$\exp\left(\overline{G}^{-1}\left(\overline{F}(-\log(u_1), \ldots, -\log(u_n))\right)\right) \geq u_1 \cdots u_n.$$

Thus, for every $\mathbf{x} \in \mathbb{R}_+^n$, we have that

$$\overline{F}(x_1, \ldots, x_n) \geq \overline{G}(x_1 + \cdots + x_n), \tag{4.31}$$

which is equivalent to the fact that Eq. (4.28) holds.

(b) Since $\overline{F}$ is B-MNBU2, for all $i, j \in \{1, \ldots, n\}$, $i \neq j$, and for every $\mathbf{u} \in [0, 1]^n$,

$$\begin{aligned} &\exp\left(\overline{G}^{-1}\left(\overline{F}(-\log(u_1), \ldots, -\log(u_i), \ldots, -\log(u_j), \ldots, -\log(u_n))\right)\right) \\ &\geq \exp\left(\overline{G}^{-1}\left(\overline{F}(-\log(u_1), \ldots, -\log(u_i u_j), \ldots, 1, \ldots, -\log(u_n))\right)\right), \end{aligned}$$

that is equivalent to

$$\overline{F}(x_1, \ldots, x_i, \ldots, \tau, \ldots, x_n) \geq \overline{F}(x_1, \ldots, \tau + x_i, \ldots, 0, \ldots, x_n) \tag{4.32}$$

for all $\hat{\mathbf{x}}_i \in \mathbb{R}_+^{n-1}$ and $\tau > 0$. This last condition can be expressed as

$$\begin{aligned} &\mathbb{P}(T_j > \tau \mid T_1 > x_1, \ldots, T_i > x_i, \ldots, T_j > 0, \ldots, T_n > x_n) \\ &\geq \mathbb{P}(T_i > \tau + x_i \mid T_1 > x_1, \ldots, T_i > x_i, \ldots, T_j > 0, \ldots, T_n > x_n), \end{aligned}$$

that is the assertion.

(c) We have just to prove that $\overline{F}$ is B-MIFR if and only if $\overline{F}$ is Schur–concave. Then, the assertion will follow, since the Schur–concavity of $\overline{F}$ is equivalent to the fact that Eq. (4.30) holds (see [85] and [86, Proposition 4.15]).

Now, the equivalence between $\overline{F}$ being B-MIFR and $\overline{F}$ being Schur-concave follows by extending [21, Lemma 4.2] from the bivariate to the n-dimensional case. In detail, $\overline{F}$ is Schur-concave if and only if for all $i, j \in \{1, 2, \ldots, n\}$, $i < j$, for every $\mathbf{x} \in \mathbb{R}_+^n$ such that $x_i \leq x_j$ and for every $\tau \in \left[0, \frac{x_j - x_i}{2}\right]$, inequality (4.27) holds. In terms of B, this is equivalent to

$$B(e^{-x_1}, \ldots, e^{-x_i-\tau}, \ldots, e^{-x_j+\tau}, \ldots, e^{-x_n}) \\ \geq B(e^{-x_1}, \ldots, e^{-x_i}, \ldots, e^{-x_j}, \ldots, e^{-x_n}). \tag{4.33}$$

In other words,

$$B\left(u_1, \ldots, u_i s, \ldots, \tfrac{u_j}{s}, \ldots, u_n\right) \geq B(u_1, \ldots, u_n), \tag{4.34}$$

for all $i, j \in \{1, \ldots, n\}$, $i < j$, for every $\mathbf{u} \in]0, 1]^n$ such that $u_i \geq u_j$ and for every $s \in \left[\frac{u_j}{u_i}, 1\right]$, which is an equivalent way of expressing the fact that $B \in \mathcal{P}_{SM}$.

□

Note that conditions (4.29) and (4.30) can then be expressed as comparisons between residual lifetimes, conditionally on a same history. Specifically, the condition $\overline{F}$ B-MNBU2 is equivalent to

$$\left[T_i \mid \hat{\mathbf{T}}_i > \hat{\mathbf{x}}_i\right] \geq_{st} \left[T_j - x_j \mid \hat{\mathbf{T}}_i > \hat{\mathbf{x}}_i\right], \tag{4.35}$$

for all $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, and for every $\mathbf{x} \in \mathbb{R}_+^n$, where $\geq_{st}$ denotes the univariate usual stochastic order (see [82]). Instead, the fact that $\overline{F}$ is B-MIFR can be expressed as

$$\left[T_i - x_i \mid \mathbf{T} > \mathbf{x}\right] \geq_{st} \left[T_j - x_j \mid \mathbf{T} > \mathbf{x}\right], \tag{4.36}$$

for all $i, j \in \{1, 2, \ldots, n\}$, for every $\mathbf{x} \in \mathbb{R}_+^n$ such that $x_i \leq x_j$.

Comparisons of laws of different lifetimes, conditionally on the same state of information, have been considered in [16, 17] as a way for defining possible notions of multivariate ageing that are appropriate in situations where "the (Bayesian) dependence due to learning about some unobservable quantity cannot be neglected" (see [16]). This approach leads us to notions of multivariate ageing, that are different from other notions where the laws of the same vector of surviving components, conditional on two different states of information, are compared (see, e.g., [68, 82] and the references therein).

Thanks to the probabilistic interpretations given by (4.35) and (4.36), an interesting link between B-MNBU2 and B-MIFR can be proved. Let us consider the vector of the residual lifetimes of $\mathbf{T}$ at time $t > 0$, $\mathbf{X}_t = [\mathbf{T} - \mathbf{t} \mid \mathbf{T} > \mathbf{t}]$, where $\mathbf{t} = (t, \ldots, t)$. Let us denote by $\overline{F}_t : \mathbb{R}_+^n \to [0,1]$ the joint survival function of $\mathbf{X}_t$ and by $B_{\overline{F}_t}$ the corresponding multivariate ageing function. By extending some results related to the bivariate case (see [19, 20, 21, 49]), the following one can be proved.

Proposition 4.3.13. *Under Assumption 1, for every $t \geq 0$ $\overline{F}_t$ is B-MNBU2 if and only if $\overline{F}$ is B-MIFR.*

Proof. $\overline{F}_t$ is B-MNBU2 for every $t \geq 0$ if and only if

$$\overline{F}(x_1{+}t, \ldots, x_i{+}t, \ldots, x_j{+}t, \ldots, x_n{+}t) \geq \overline{F}(x_1{+}t, \ldots, t, \ldots, x_j{+}x_i{+}t, \ldots, x_n{+}t)$$

for every $t \geq 0$, $\mathbf{x} \in \mathbb{R}_+^n$ and $i, j \in \{1, 2, \ldots, n\}$, that is equivalent to the fact that $\overline{F}$ is Schur-concave. □

Note that, if $\overline{F}$ is B-MNBU2, then $\overline{F}_t$ may not be B-MNBU2 for some $t > 0$ (see [48] for an example in the bivariate case). However, for the notion of B-MIFR, we can prove the following result.

Corollary 4.3.14. *Under Assumption 1, if $\overline{F}$ is B-MIFR, then $\overline{F}_t$ is B-MIFR for every $t \geq 0$.*

Proof. From Proposition 4.3.13, if $\overline{F}$ is B-MIFR, then $\overline{F}_{t+s}$ is B-MNBU2 for every $t, s \geq 0$. As a consequence, $\overline{F}_t$ is B-MIFR for every $t \geq 0$. □

Concerning the inequality (4.28), it is not clear whether it can also be expressed as comparisons of lifetimes conditionally on the same history, in a similar way to the inequalities in (4.35) and (4.36). However, it is possible to give it an intuitive interpretation in reliabilistic terms, similarly as done in [21, Example 4.2] for the case $n = 2$.

Remark 4.3.15. *Notice that inequality* (4.29) *implies inequality* (4.28)*; this can be seen from subsection 4.3.1 by using the multivariate ageing function B and the given definitions of B-MNBU1 and B-MNBU2. Actually, as shown in Example* (4.3.4)*, inequalities* (4.29) *and* (4.28) *coincide for TTE models, but not in general. Consider, for instance, a multivariate survival function $\overline{F}$ whose margins are exponential and whose copula is that one of Example 4.3.5.*

The notions of multivariate ageing introduced in Definition 4.3.10 are preserved under mixtures, as specified by the following Proposition.

Proposition 4.3.16. *Let $(\overline{F}_\theta)_{\theta\in\Theta}$ be a family of survival functions satisfying Assumption* 1. *Let λ be a distribution on Θ. Let $\overline{F}$ be the mixture of $(\overline{F}_\theta)_{\theta\in\Theta}$ with respect to λ, given, for every $\mathbf{x} \in \mathbb{R}_+^n$, by*

$$\overline{F}(\mathbf{x}) = \int_\Theta \overline{F}_\theta(\mathbf{x})\mathrm{d}\lambda(\theta).$$

The following statements hold:

(a) *if $\overline{F}_\theta$ is B-MNBU1 for every $\theta \in \Theta$, then $\overline{F}$ is B-MNBU1;*

(b) *if $\overline{F}_\theta$ is B-MNBU2 for every $\theta \in \Theta$, then $\overline{F}$ is B-MNBU2;*

(c) *if $\overline{F}_\theta$ is B-MIFR for every $\theta \in \Theta$, then $\overline{F}$ is B-MIFR.*

Proof. Part (a) follows by considering that every $\overline{F}_\theta$ satisfies (4.31) and hence the mixture $\overline{F}$ satisfies (4.31), which is an equivalent formulation of the B-MNBU1 property for $\overline{F}$. Analogously, part (b) easily follows from the fact that every $\overline{F}_\theta$ satisfies (4.32).

Finally, if every $\overline{F}_\theta$ is B-MIFR, then it is Schur-concave. As a consequence, the mixture $\overline{F}$ is also Schur-concave and therefore B-MIFR (see [70]). □

Consequently, the following interesting result can be easily derived.

Proposition 4.3.17. *Under Assumption 1, suppose that $\overline{F}$ is the survival function of conditionally i.i.d. lifetimes given a common factor Θ with prior distribution λ. Moreover, suppose that $\overline{G}(\cdot \mid \theta)$ is NBU (respectively, IFR). Then $\overline{F}$ is B-MNBU2 (respectively, B-MIFR).*

Thus, the given definition of multivariate ageing has an interesting property: mixtures of i.i.d. lifetimes that are NBU (respectively, IFR) conditionally on the same factor Θ, are also multivariate NBU (respectively, IFR).

Finally, we would like to discuss a possible application of our results in the construction of multivariate stochastic models. To this end, we give the following proposition that extends some results in [21] to the multivariate case.

Proposition 4.3.18. *Under Assumption 1, the following statements hold:*

(a) *if $\hat{C} \in \mathcal{P}_{PLOD}$ and $\overline{G}$ is NBU, then $\overline{F}$ is B-MNBU1;*

(b) *if $\hat{C} \in \mathcal{P}_{PPLOD}$ and $\overline{G}$ is NBU, then $\overline{F}$ is B-MNBU2;*

(c) *if $\hat{C} \in \mathcal{P}_{SM}$ and $\overline{G}$ is IFR, then $\overline{F}$ is B-MIFR.*

Proof.

(a) Let $\hat{C} \in \mathcal{P}_{PLOD}$. Then,, for every $\mathbf{u} \in [0,1]^n$,

$$\exp\left(-\overline{G}^{-1}\left(\hat{C}(\overline{G}(-\log(u_1)), \ldots, \overline{G}(-\log(u_n)))\right)\right) \geq B_\Pi(u_1, \ldots, u_n).$$

By considering Proposition 4.3.7(a) and the analog of Eq. (4.11) in the n-variate case,

$$B(\mathbf{u}) = \exp\left(-\overline{G}^{-1}\left(\hat{C}\left(\overline{G}(-\log(u_1)), \ldots, \overline{G}(-\log(u_n))\right)\right)\right),$$

it follows that $B \in \mathcal{P}_{PLOD}$.

(b) Let $\hat{C} \in \mathcal{P}_{PPLOD}$. Then, for every $\mathbf{u} \in [0,1]^n$,

$$\begin{aligned}
& \overline{G}\left(-\log\left(B\left(e^{-\overline{G}^{-1}(u_1)}, \ldots, e^{-\overline{G}^{-1}(u_i)}, \ldots, e^{-\overline{G}^{-1}(u_j)}, \ldots, e^{-\overline{G}^{-1}(u_n)}\right)\right)\right) \\
\geq\ & \overline{G}\left(-\log\left(B\left(e^{-\overline{G}^{-1}(u_1)}, \ldots, e^{-\overline{G}^{-1}(u_i u_j)}, \ldots, 1, \ldots, e^{-\overline{G}^{-1}(u_n)}\right)\right)\right) \\
\geq\ & \overline{G}\left(-\log\left(B\left(e^{-\overline{G}^{-1}(u_1)}, \ldots, e^{-(\overline{G}^{-1}(u_i)+\overline{G}^{-1}(u_j))}, \ldots, 1, \ldots, e^{-\overline{G}^{-1}(u_n)}\right)\right)\right),
\end{aligned}$$

where the last inequality follows from the fact that $\overline{G}$ is NBU. Setting $x_i = e^{-\overline{G}^{-1}(u_i)}$, it follows that $B \in \mathcal{P}_{PPLOD}$, which is the desired assertion.

(c) Since $\hat{C} \in \mathcal{P}_{SM}$, for every $\mathbf{u} \in [0,1]^n$ such that $u_i \geq u_j$ and for every $s \in (0,1)$,

$$\hat{C}(u_1, \ldots, u_i s, \ldots, u_j, \ldots, u_n) \geq \hat{C}(u_1, \ldots, u_i, \ldots, u_j s, \ldots, u_n).$$

In particular, for every $0 < s_j \leq s_i < 1$,

$$\hat{C}(u_1, \ldots, u_i s_i, \ldots, u_j, \ldots, u_n) \geq \hat{C}(u_1, \ldots, u_i, \ldots, u_j s_j, \ldots, u_n). \tag{4.37}$$

Now, for every $k \in \{1, 2, \ldots, n\}$, set

$$\alpha_k = \overline{G}^{-1}(u_k), \quad s_i = \frac{\overline{G}(\alpha_i + \sigma)}{\overline{G}(\alpha_i)}, \quad s_j = \frac{\overline{G}(\alpha_j + \sigma)}{\overline{G}(\alpha_j)},$$

where $\sigma = \overline{G}^{-1}(u_i s_i) - \overline{G}^{-1}(u_i) = \overline{G}^{-1}(u_j s_j) - \overline{G}^{-1}(u_j)$. Since $\overline{G}$ is IFR, $s_i \geq s_j$. Moreover, from (4.37) we obtain

$$\begin{aligned}
& \hat{C}\left(\overline{G}(\alpha_1), \ldots, \overline{G}(\alpha_i + \sigma), \ldots, \overline{G}(\alpha_j), \ldots, \overline{G}(\alpha_n)\right) \\
& \qquad \geq \hat{C}\left(\overline{G}(\alpha_1), \ldots, \overline{G}(\alpha_i), \ldots, \overline{G}(\alpha_j + \sigma), \ldots, \overline{G}(\alpha_n)\right).
\end{aligned}$$

By applying to both the sides of this inequality the transformation $\exp \circ (-\overline{G}^{-1})$, we have

$$B(x_1, \dots, x_i s', \dots, x_j, \dots, x_n) \geq B(x_1, \dots, x_i, \dots, x_j s', \dots, x_n),$$

for every $\mathbf{x} \in [0,1]^n$ such that $x_i \geq x_j$ and for every $s' \in (0,1)$, that is B is SM.

□

Remark 4.3.19. *As already noted, both the conditions $\hat{C} \in \mathcal{P}_{PPLOD}$ and $\hat{C} \in \mathcal{P}_{SM}$ imply that $\hat{C} \in \mathcal{P}_{PLOD}$, which is considered as a notion of multivariate positive dependence. Thus, roughly speaking, Proposition 4.3.18 suggests that positive univariate ageing and (some kind of) positive dependence play in favour of positive multivariate ageing. However, note that positive multivariate ageing can coexist with several forms of dependence and univariate ageing: this fact was already stressed, for example, in [21].*

4.3.3 Discussion and concluding remarks

In this section, we have presented an extension to the n-dimensional case ($n \geq 2$) of bivariate ageing notions discussed in [18, 21].

An interesting point concerns the extension of the NBU property, that, for the multivariate case $n \geq 3$, can lead us to two different formulations. In fact, this happens when the components of $\overline{F}$ are coupled by a copula $\hat{C}$ outside the Archimedean class (see Example 4.3.4).

Here, it should be considered that, following the scheme (i)–(iii) presented at the beginning of the section, it is possible that several properties of B_Π describe the same bivariate ageing of a joint survival function $\overline{F}_{\Pi,\overline{G}}$ of independent components. In [21], for instance, different notions of bivariate IFR have been discussed. When such situations occur, it is quite natural to consider all these different multivariate notions of a given univariate ageing property $\mathbf{P}$ and select among them those properties of B_Π with most interesting probabilistic meaning.

Also for these reasons, we wanted to stress that the introduced multivariate ageing notions exhibit some interesting (Bayesian) probabilistic properties: they are closed under mixtures and can be characterized in terms of comparisons of conditional survival functions given a same observed history.

Finally, it is of interest also for some statistical purposes that Proposition 4.3.18 can be used when we want to construct, for components judged to be similar, a multivariate survival model, satisfying some kind of ageing conditions. In fact, by using the celebrated Sklar's Theorem [84], such a model can be constructed just by conveniently choosing some univariate survival function $\overline{G}$ (e.g., satisfying NBU or IFR property) and a suitable copula $\hat{C}$ (belonging to some class $\mathcal{P}$); hence, we join them for creating the multivariate survival function $\overline{F} = \hat{C}(\overline{G}, \ldots, \overline{G})$. This procedure hence provides sufficient conditions for multivariate ageing in terms of univariate ageing and stochastic dependence.

Chapter 5

Hyper-properties and semigroups

In previous chapters, evolution of dependence and evolution of ageing for vectors of non-negative random variables have been separately considered and some analogies between the two evolutions emerges. In this concluding section, we present a unified approach, based on semigroup arguments, explaining the origin of such analogies and relations among properties of stochastic dependence and ageing.

In order to describe the limit behaviour of the family of lower threshold copulas $\{C_z\}_{z\in(0,1]}$ from a qualitative point of view, we have introduced, in [35] (see also Chapter 2), the *hyper-properties*. A hyper-property of a copula C corresponds, by definition, to a dependence property of the whole family of threshold copulas associated with C. Hyper-properties can be defined as well for other families of copulas, e.g. for the family of survival copulas $\{\hat{C}_t\}_{t\geq 0}$ (see [49]). In this case, we will call them *hyper-dependence* properties.

In this chapter, we analyze the notion of hyper-property and investigate the algebraic structure of hyper-properties. We find that such a notion is very general, not only concerning dependence properties. In fact, it can be expressed in terms of actions of semigroups and of their orbits. Such a general background allows us to extend the notion of hyper-property from dependence to other kinds of properties. For example, this formulation fits well to the study of ageing properties. To this purpose, we extend the notion of hyper-property from copulas to semi-copulas.

In fact, a suitable semi-copula B is used for describing multivariate ageing properties (see [19, 21, 36]). In parallel with dependence, it is also of interest studying evolution of ageing, introducing a family of semi-copulas $\{B_t\}_{t\geq 0}$ (see [49]).

This common background also allows us to explain some analogies between the structure of the implications among dependence properties and the structure of implications among ageing properties. Some of them have been already considered and pointed out in [49].

The chapter is organized as follows. Section 5.1 is devoted to general results on semigroup actions and their orbits. In this general algebraic setting, a hyper-property is represented by a class of copulas, that is closed under the action of a given semigroup of transformations. In Section 5.2, we define a transformations of a copula $\hat{C}$, $\Phi_{dep} : \mathcal{C} \times \mathbb{R}_+ \to \mathcal{C}$ such that the elements of the family of survival copulas $\{\hat{C}_t\}_{t\geq 0}$ are obtained as $\hat{C}_t = \Phi_{dep}(\hat{C}, t)$. We proved in [49] (see also Chapter 2) that Φ_{dep} is an action of the semigroup $(\mathbb{R}_+, +, 0)$ on $\widetilde{\mathcal{C}}$. Analogously to the transformation Φ_{dep}, a transformation Φ_{ag} is defined, such that $B_t = \Phi_{ag}(B, t)$. It has been proved in [49] (see also Chapter 4) that also $\Phi_{ag} : \mathcal{S} \times \mathbb{R}_+ \to \mathcal{S}$ is an action of the semigroup $(\mathbb{R}_+, +, 0)$, in this case on $\mathcal{S}$. We devote Section 5.2 to the extension of the notion of hyper-property to ageing and to apply results of Section 5.1.
In view of the common algebraic background of dependence and ageing properties, we give there an explanation of their behaviour and of the similarities between the two kinds of properties, that already emerged in [35, 49] (see also the previous chapters). We also enrich their treatment by providing some examples.

5.1 Hyper-properties and semigroups

We want to provide a general approach to the study of the evolution of some properties within a family of copulas. In this section, we study, in an abstract frame, properties of copulas, that are called *hyper-properties*. They are properties of a copula $C \in \mathcal{C}$, but they correspond to properties of the whole family associated with C by a certain transformation $\Phi(\cdot, t)$. Since we will not restrict ourselves to consider dependence properties, we extend the treatment to semi-copulas and families of semi-copulas.

In order to define and to explain what a hyper-property is, it will be convenient to recall some basic notation about semigroups, actions and orbits (see e.g. [56]).

In the following $\mathcal{U}$ will denote an arbitrary set and $\oplus$ a binary operation on $\mathcal{U}$. $(\mathcal{U}, \oplus)$ is a *semigroup* if $\mathcal{U}$ is closed with respect to $\oplus$ and $\oplus$ is associative. If, furthermore, $\mathcal{U}$ contains a neutral element $1_\oplus$ for the operation $\oplus$, $(\mathcal{U}, \oplus, 1_\oplus)$ is said to be a *monoid* or a *unitary semigroup*. Since we will consider here unitary semigroups, from now on we will refer to them simply as *semigroups*.

Let $\mathcal{T}$ be an arbitrary set.

Definition 5.1.1. *An* action *of* $\mathcal{U}$ *on* $\mathcal{T}$ *is a transformation*

$$\Phi : \mathcal{T} \times \mathcal{U} \to \mathcal{T}$$

such that

(i) for any $\zeta \in \mathcal{T}$, $\Phi(\zeta, 1_\oplus) = \zeta$;

(ii) for any $t,\ s \in \mathcal{U}$, $\Phi(\Phi(\zeta, t), s) = \Phi(\zeta, t \oplus s)$.

The set

$$\mathcal{O}_\Phi(\zeta) = \Phi(\zeta, \mathcal{U}) = \{\zeta' \in \mathcal{T} :\ \exists s \in \mathcal{U} :\ \Phi(\zeta, s) = \zeta'\}$$

is the orbit *of* ζ *under the action* Φ.

In most of what follows, we will consider the semigroup $(\mathcal{U}, \oplus, 1_\oplus)$ coinciding with $(\mathbb{R}_+, +, 0)$ and $\mathcal{T}$ coinciding with $\mathcal{S}$ or with $\mathcal{C}$.

Let us proceed now in explaining what hyper-dependence is.
Let $\mathcal{P}$ and $\mathcal{P}'$ two different classes of semi-copulas.
For example, a class $\mathcal{P}$ may consist of all the semi-copulas satisfying a certain dependence property **P** (as in Definition 1.3.11): namely, $\mathcal{P}_{PQD}$, $\mathcal{P}_{LTD}$, $\mathcal{P}_{TP_2}$, $\mathcal{P}_{SI}$ are the classes of semi-copulas that are, respectively, PQD, LTD, TP_2, SI.

Starting from the above notions of dependence, in [35] new properties of copulas have been introduced, called *hyper-dependence* properties. Their peculiarity is to be defined as properties of a copula, in relation to properties of the whole associated family; i.e. we say that $\hat{C}$ is hyper$_{dep}$-**P** if $\hat{C}_t$ is **P** for every $t \geq 0$. We redefine hyper-properties here, in accordance with the above algebraic setting.

Let Φ_{dep} be the transformation such that

$$\hat{C}_t = \Phi_{dep}(\hat{C}, t), \tag{5.1}$$

defined by Eq. (2.14).

Definition 5.1.2. $C \in \mathcal{C}$ *is hyper*$_{dep}$-**P** *if* $\mathcal{O}_{\Phi_{dep}}(C) \subseteq \mathcal{P}$.
Hyper$_{dep}$-$\mathcal{P}$ *is the class of all hyper*$_{dep}$-**P** *copulas.*

More in general, let **P** be a property of semi-copulas, $S \in \mathcal{S}$ and $\mathcal{O}_\Phi(S)$ the orbit of S under the action Φ.

Definition 5.1.3. *We say that* S *is hyper*$_\Phi$-**P** *if* $\mathcal{O}_\Phi(S) \subseteq \mathcal{P}$.
We denote by hyper$_\Phi$-$\mathcal{P}$ *the class of all the hyper*$_\Phi$-**P** *semi-copulas.*

Remark 5.1.4. *The class hyper*$_\Phi$-$\mathcal{P}$ *is contained in* $\mathcal{P}$.

The analysis of hyper-dependence suggests us to extend to classes of copulas associated with properties of dependence or hyper-dependence the study of relationships of the kind in (**??**). In such a case, a more heterogeneous landscape emerges. In particular, we can list the following (non-exhaustive) situations:

i) $\mathcal{P} =$ *hyper*$_\Phi$-$\mathcal{P}$;

ii) $\mathcal{P} \subset \mathcal{P}'$, $hyper_\Phi$-$\mathcal{P} = hyper_\Phi$-$\mathcal{P}'$;

iii) $\mathcal{P} \nsubseteq \mathcal{P}'$, $\mathcal{P}' \nsubseteq \mathcal{P}$, $hyper_\Phi$-$\mathcal{P}' \subset \mathcal{P}$;

iv) $\mathcal{P} \nsubseteq \mathcal{P}'$, $\mathcal{P}' \nsubseteq \mathcal{P}$, $hyper_\Phi$-$\mathcal{P} = hyper_\Phi$-$\mathcal{P}'$.

The formalization in the items i), ii, iii), iv) is suggested by relations that we have noticed among classes of copulas corresponding to dependence or hyper-dependence properties.

Example 5.1.5. *i)* $\mathcal{P}_{TP_2} = hyper_{dep}$-$\mathcal{P}_{TP_2}$ *(see [49, 35]);*

ii) $hyper_{dep}$-$\mathcal{P}_{TP_2} = hyper_{dep}$-$\mathcal{P}_{LTD}$*, but* $\mathcal{P}_{TP_2} \subsetneq \mathcal{P}_{LTD}$ *(see [49, 35]);*

iii) $\mathcal{P}_{TP_2}$ *and* $\mathcal{P}_{SI}$ *are not comparable, but* $hyper_{dep}$-$\mathcal{P}_{SI} \subsetneq \mathcal{P}_{TP_2}$ *(see Proposition 5.2.11 below).*

The cases listed above can be alternatively formulated as

i) $\mathcal{P}$ s.t. $S \in \mathcal{P} \Rightarrow \mathcal{O}_\Phi(S) \subseteq \mathcal{P}$;

ii) $\mathcal{P} \subset \mathcal{P}'$ s.t. $\mathcal{O}_\Phi(S) \subseteq \mathcal{P} \Leftrightarrow \mathcal{O}_\Phi(S) \subseteq \mathcal{P}'$;

iii) $\mathcal{P} \nsubseteq \mathcal{P}'$, $\mathcal{P}' \nsubseteq \mathcal{P}$ s.t. $\mathcal{O}_\Phi(S) \subseteq \mathcal{P}' \Rightarrow S \in \mathcal{P}$;

iv) $\mathcal{P} \nsubseteq \mathcal{P}'$, $\mathcal{P}' \nsubseteq \mathcal{P}$ s.t. $\mathcal{O}_\Phi(S) \subseteq \mathcal{P} \Leftrightarrow \mathcal{O}_\Phi(S) \subseteq \mathcal{P}'$.

The above examples show how our formalism fits well to describe relations among dependence properties and hyper-dependence properties. We have introduced it in order to express other properties of copulas and semi-copulas (for example, ageing properties) in a form that is similar to the one of dependence. Such a formalism will help us in singling out and studying the common bases of different kinds of properties that originate some systematic analogies.

The following propositions will show that the afore-mentioned alternatives, i), ii), iii), iv), are sufficient to obtain all the possible relations involving hyper-properties.

As above, Φ denotes the action of a semigroup.

Proposition 5.1.6. *For* $\mathcal{P} \subset \mathcal{P}'$*, the conditions*

$$\mathcal{P} = hyper_\Phi\text{-}\mathcal{P} \tag{5.2}$$

and

$$hyper_\Phi\text{-}\mathcal{P} = hyper_\Phi\text{-}\mathcal{P}' \tag{5.3}$$

are equivalent to

$$\mathcal{P} = hyper_\Phi\text{-}\mathcal{P}'. \tag{5.4}$$

Proof. If (5.2) and (5.3) hold, it obviously follows (5.4). Conversely, if (5.4) holds, by [35, Prop. 7], (5.2) is satisfied and this automatically implies (5.3). $\square$

The following proposition states, instead, that, if a hyper-property hyper-$\mathbf{P}'$ implies a property $\mathbf{P}$, then hyper-$\mathbf{P}$ and hyper-$\mathbf{P}'$ coincide, even when $\mathbf{P}$ is strictly stronger than $\mathbf{P}'$. In terms of classes $\mathcal{P}, \mathcal{P}'$:

Proposition 5.1.7. *If $\mathcal{P} \subset \mathcal{P}'$, then*

$$hyper_\Phi\text{-}\mathcal{P}' \subset \mathcal{P} \tag{5.5}$$

if and only if

$$hyper_\Phi\text{-}\mathcal{P} = hyper_\Phi\text{-}\mathcal{P}'. \tag{5.6}$$

Proof. Since $\mathcal{P} \subset \mathcal{P}'$, obviously (5.6) implies (5.5).
Conversely, let us suppose (5.5) holds. We have to prove that both the inclusions $hyper_\Phi$-$\mathcal{P} \subset hyper_\Phi$-$\mathcal{P}'$ and $hyper_\Phi$-$\mathcal{P} \supset hyper_\Phi - \mathcal{P}'$ hold.
Since $\mathcal{P} \subset \mathcal{P}'$, $hyper_\Phi$-$\mathcal{P} \subseteq hyper_\Phi$-$\mathcal{P}'$ obviously follows.
By definition of $hyper_\Phi$-$\mathcal{P}'$, $S \in hyper_\Phi$-$\mathcal{P}'$ implies $\mathcal{O}_\Phi(S) \subseteq \mathcal{P}'$.
Let us now consider $S_{t_0} \in \mathcal{S}$ such that $S_{t_0} = \Phi(S, t_0)$, for some $t_0 > 0$. By definition of action, $\mathcal{O}_\Phi(S_{t_0}) \subseteq \mathcal{O}_\Phi(S)$. In fact,

$$\mathcal{O}_\Phi(S_{t_0}) = \{S' \in \mathcal{S} : \exists s \in \mathcal{U} : \Phi(S_{t_0}, s) = S'\} \subseteq \mathcal{P}'.$$

But $\Phi(S_{t_0}, s) = \Phi(S, t_0 \oplus s)$ and, therefore, $\mathcal{O}_\Phi(S_{t_0})$ coincides with the sub-orbit of S, $\Phi(S, t_0 \oplus \mathcal{U})$. Therefore $\mathcal{O}_\Phi(S) \subseteq \mathcal{P}'$ implies $\mathcal{O}_\Phi(S_{t_0}) \subseteq \mathcal{P}'$ for any $t_0 \geq 0$, i.e. that any sub-orbit of S is contained in $\mathcal{P}'$. By (5.5), this fact implies that $S_{t_0} \in \mathcal{P}$. Hence $\mathcal{O}_\Phi(S) \subseteq \mathcal{P}'$ implies $\mathcal{O}_\Phi(S) \subseteq \mathcal{P}$, that is $hyper_\Phi$-$\mathcal{P} \subseteq hyper_\Phi$-$\mathcal{P}'$.

$\square$

Remark 5.1.8. *From implication "(5.5) $\Rightarrow$ (5.6)", it follows that it cannot hold the chain of inclusions*

$$hyper_\Phi\text{-}\mathcal{P} \subset hyper_\Phi\text{-}\mathcal{P}' \subset \mathcal{P} \subset \mathcal{P}',$$

but, given

$$hyper_\Phi\text{-}\mathcal{P}' \subset \mathcal{P} \subset \mathcal{P}',$$

it has necessarily to be

$$hyper_\Phi\text{-}\mathcal{P} = hyper_\Phi\text{-}\mathcal{P}'.$$

An analog of Proposition 5.1.7, when two properties $\mathbf{P}$ and $\mathbf{P}''$ are not comparable is given by the following

Proposition 5.1.9. *If $\mathcal{P} \nsubseteq \mathcal{P}''$, $\mathcal{P}'' \nsubseteq \mathcal{P}$ and $\mathcal{P} = hyper_\Phi$-$\mathcal{P}'$ for some $\mathcal{P}'$, then*

$$\mathcal{P}'' \subset \mathcal{P}' \tag{5.7}$$

is equivalent to

$$hyper_\Phi\text{-}\mathcal{P}'' \subset \mathcal{P}. \tag{5.8}$$

Proof. Obviously (5.7) implies (5.8).
Viceversa: let $S \in hyper_\Phi$-$\mathcal{P}'' \subset \mathcal{P}$. If $S \in \mathcal{P}''$, it has necessarily to be $S \in \mathcal{P}'$. In fact, if not so, it would be $S \in \mathcal{P}''\backslash\mathcal{P}'$ and, therefore, $S \notin hyper_\Phi$-$\mathcal{P}' = \mathcal{P}$, against the hypothesis. □

Some corollaries of Propositions 5.1.7, 5.1.6, 5.1.9 will be obtained below. In order to illustrate their meaning and usefulness, we need to recall and point out the following fact. Since the semigroup acting on $\mathcal{S}$, $(\mathbb{R}_+, +, 0)$, is totally ordered, it is possible to define an orientation on any orbit, corresponding to the "increasing" direction on $\mathbb{R}_+$. Thus an orbit can be seen as a trajectory in the space of semi-copulas.
Starting from any point on it, an orbit can be gone along in both directions: forwards and backwards. More precisely: let us consider the orbit generated by $S \in \mathcal{S}$; any element of the orbit, $D = S_{t_0} = \Phi(S, t_0)$, is individuated by an element $t_0 \in \mathbb{R}_+$. Starting from S_{t_0}, it is possible to go along the orbit both forwards, by taking t increasing on $(t_0, +\infty)$, and backwards, by taking t decreasing on $[0, t_0)$. If, furthermore, we interpret $t \in \mathbb{R}_+$ as a time parameter (as we will do in the following), then "forwards" and "backwards" respectively mean "for future times" and "for past times".

We are now in a position to state some corollaries about the evolution of properties along an orbit. They are related to the preservation of a property on a sub-orbit, but not necessarily on an entire orbit. Their proofs derive from a basic consequence of Φ being an action of a semigroup on $\mathcal{S}$, we have already exploited in proving Proposition 5.1.7: for any $t_0 \in \mathbb{R}_+$, the orbit $\Phi(S_{t_0}, \mathbb{R}_+)$, generated by the element $S_{t_0} = \Phi(S, t_0) \in \mathcal{O}_\Phi(S)$, is a sub-orbit of S itself.

If $\mathcal{P}$ is closed under the action Φ, the property **P** can arise along the orbit, but, from that point forward, it is necessarily preserved. Since we are considering semigroup actions, nothing can be said about their backward behaviour. Under actions of semigroups, the orbits are not a partition of the set: they are allowed to join, but not to separate. Thus, given an element S_{t_0}, it is not possible to univocally reconstruct backwards the orbit it belongs to. We will provide in the following examples of such constructions.

Corollary 5.1.10. *Let $\mathcal{P}, \mathcal{P}' \subseteq \mathcal{S}$ such that $\mathcal{P} = hyper_\Phi$-$\mathcal{P} \subset \mathcal{P}'$.*

1. *Then*

$$S_{t_0} \in \mathcal{P} \text{ for some } t_0 \in \mathcal{U} \;\Rightarrow\; \Phi(S, t_0 \oplus \mathcal{U}) \subseteq \mathcal{P}.$$

2. If, furthermore,

$$hyper_\Phi\text{-}\mathcal{P} = hyper_\Phi\text{-}\mathcal{P}', \tag{5.9}$$

then $S_{t_0} \in \mathcal{P}$ for some $t_0 \in \mathcal{U}$ if and only if

$$\Phi(S, t_0 \oplus \mathcal{U}) = \{S' \in \mathcal{S} : \exists t \in t_0 \oplus \mathcal{U} : \Phi(S, t) = S'\} \subseteq \mathcal{P}'.$$

Proof. We start by proving the second statement. By hypothesis and condition (5.9), $\mathcal{P} = hyper_\Phi\text{-}\mathcal{P}'$ holds and therefore $S_{t_0} \in \mathcal{P}$ is equivalent to $\mathcal{O}_\Phi(S_{t_0}) \subseteq \mathcal{P}'$.

By letting $\mathcal{P}' = \mathcal{P}$ in the statement 2., we obtain 1.. $\square$

The following corollaries can be easily proved starting from Propositions 5.1.7 and 5.1.9 respectively.

Corollary 5.1.11. *Let be $\mathcal{P} \subset \mathcal{P}'$. If $\mathcal{P} = hyper_\Phi\text{-}\mathcal{P}'$, then*

$$\Phi(S, t_0 \oplus \mathcal{U}) \subseteq \mathcal{P}' \text{ for some } t_0 \in \mathcal{U} \Leftrightarrow S_{t_0} \in \mathcal{P}.$$

Corollary 5.1.12. *Let be $\mathcal{P} \nsubseteq \mathcal{P}'$ and $\mathcal{P}' \nsubseteq \mathcal{P}$. If*

$$hyper_\Phi\text{-}\mathcal{P}' \subset \mathcal{P},$$

then

$$\Phi(S, t_0 \oplus \mathcal{U}) \subseteq \mathcal{P}' \text{ for some } t_0 \in \mathcal{U} \Rightarrow S_{t_0} \in \mathcal{P}.$$

Remark 5.1.13. *If instead we consider an action Ψ of a group $(\mathcal{G}, \oplus, 1_\oplus)$ on $\mathcal{S}$, $\mathcal{P}$ would be closed under the action Ψ, both forward and backward. Consequently, if an element of an orbit belongs to $\mathcal{P}$, the whole orbit is contained in $\mathcal{P}$. In other words, $\mathcal{P}$ and $hyper_\Psi\text{-}\mathcal{P}$ always coincide.*
In this case, an analog of Corollary 5.1.10 holds.

Corollary 5.1.14. *Let $\mathcal{P}, \mathcal{P}' \subseteq \mathcal{S}$ be such that $S \in \mathcal{P} \Leftrightarrow \mathcal{O}_\Psi(S) \subseteq \mathcal{P}'$. Then $S_{t_0} \in \mathcal{P}$ for some $t_0 \in \mathcal{G}$ if and only if $\mathcal{O}_\Psi(S) \subseteq \mathcal{P}'$.*

Proof. For any fixed $t_0 \in \mathcal{G}$, by the hypothesis $S_{t_0} \in \mathcal{P} \Leftrightarrow \mathcal{O}_\Psi(S_{t_0}) \subseteq \mathcal{P}'$. By definition of action, $\mathcal{O}_\Psi(S_{t_0}) = \Psi(S, t_0 \oplus \mathcal{G})$. Since $\mathcal{G}$ is a group, the coset $t_0 \oplus \mathcal{G}$ coincides with $\mathcal{G}$, therefore

$$\Psi(S, t_0 \oplus \mathcal{G}) = \Psi(S, \mathcal{G}) = \mathcal{O}_\Psi(S).$$

$\square$

5.2 Hyper-dependence and hyper-ageing properties of semi-copulas

We devote this section to the application of the results of Section 5.1 to dependence and bivariate ageing.

The theoretical frame developed in the previous section allows us both to explain some previously noticed analogies between dependence and bivariate ageing and to obtain new results.

We recall that suitable analytical properties of the survival copula $\hat{C}$ describe properties of dependence of the model, while properties of the ageing function B describe some properties of multivariate ageing. Studying the evolution of these properties as time elapses, corresponds to study the evolution of the family of copulas $\{\hat{C}_t\}_{t\geq 0}$ or of the family of semi-copulas $\{B_t\}_{t\geq 0}$. In particular, we are interested in relationships among these properties and in conditions for preservation in time of some of them.
We point out that relationships among properties of copulas may be seen as relationships among classes of copulas, while preservation in time, for example, of a dependence property $\mathbf{P}$ means that the corresponding class of $\mathbf{P}$ copulas, $\mathcal{P}$, is closed under the action of Φ_{dep}; i.e. $\mathcal{P}$ and *hyper*$_{dep}$-$\mathcal{P}$ coincide.

Φ_{ag} instead is such that

$$B_t = \Phi_{ag}(B, t) \tag{5.10}$$

and it is defined by Eq. (4.20).

Analogously to Definition 5.1.2 of hyper-dependence, we obtain the definition of hyper-ageing properties by substituting, in Definition 5.1.3, the transformation Φ with Φ_{ag}.

Definition 5.2.1. *$S \in \mathcal{S}$ is hyper$_{ag}$-$\mathbf{P}$ if $\mathcal{O}_{\Phi_{ag}}(S) \subseteq \mathcal{P}$.*
Hyper$_{ag}$-$\mathcal{P}$ is the class of all hyper$_{ag}$-$\mathbf{P}$ semi-copulas.

It has been already proved in Chapters 2 and 4 (see [49]), that both Φ_{dep} and Φ_{ag} are actions of the semigroup $(\mathbb{R}_+, +, 0)$ on $\mathcal{C}$ and $\mathcal{S}$ respectively.

Remark 5.2.2. *In defining hyper-properties, we have to specify the action we are referring to. In fact, for a given $\mathbf{P}$, hyper$_\Phi$-$\mathbf{P}$ obviously depends on the orbit of the action Φ and, therefore, on the particular choice of the transformation.*

The following example shows that, for a given property $\mathbf{P}$ of a semi-copula S, hyper$_{dep}$-$\mathbf{P}$ and hyper$_{ag}$-$\mathbf{P}$ are two different properties of S. In other words, since in general $\Phi_{dep}(S) \neq \Phi_{ag}(S)$, the class hyper$_{dep}$-$\mathcal{P}$ does not coincide with hyper$_{ag}$-$\mathcal{P}$.

Example 5.2.3. *Let $S(u, v) = uv[1 + (1-u)(1-v)]$. S is TP_2 and, as proved in [49], $\hat{C}_t = \Phi_{dep}(S, t)$ is TP_2 for any $t \geq 0$.*

On the contrary, $B_t = \Phi_{ag}(S,t)$ is not TP_2 for any $t \geq 0$. In fact, for $t = 1$ and $u' = \frac{1}{5},\ u'' = v' = \frac{1}{2},\ v'' = \frac{3}{5}$,

$$B_t(u'',v'')B_t(u',v') - B_t(u',v'')B_t(u'',v') = -0.2099 < 0.$$

By adopting a notation analogous to [35], we also define properties of a semi-copula corresponding to dependence or ageing properties holding on some subset of its orbit. Let be $\Lambda \subset \mathbb{R}_+$.

Definition 5.2.4. *We say that $C \in \mathcal{C}$ is $\langle \mathbf{P}; \Lambda \rangle_{dep}$ if $\Phi_{dep}(C,t) \in \mathcal{P}$ for any $t \in \Lambda$.*

Remark 5.2.5. *The property $\langle \mathbf{P}; [t_0, +\infty\rangle_{dep}$ for a survival copula $\hat{C}$ means that C may not satisfy $\mathbf{P}$ and that the property $\mathbf{P}$ holds for all $\hat{C}_t$, with t greater than a given t_0. This notion can be of interest in the field of tail dependence. In fact we are typically interested in proving that $\hat{C}_t$ satisfies a dependence property $\mathbf{P}$ in the limit for $t \to +\infty$. Thus, proving that $\hat{C}$ is $\langle \mathbf{P}; [t_0, +\infty\rangle_{dep}$ guarantees the above condition without explicitly computing the limit $\lim_{t\to+\infty} C_t$.*

Analogously

Definition 5.2.6. *$S \in \mathcal{S}$ is $\langle \mathbf{P}; \Lambda \rangle_{ag}$ if $\Phi_{ag}(S,t) \in \mathcal{P}$ for any $t \in \Lambda$.*

The general framework we have provided in the previous section allows us both to formalize results previously obtained and to obtain new ones.

The results we will provide now are a consequence of the statements in Section 5.1, about actions and orbits. We will present them in two different subsections, devoted to the study of the action Φ_{dep} and Φ_{ag} respectively.

We focus our attention on the case $n = 2$.

5.2.1 Dependence and hyper-dependence

It is known (see [35] and [49]) that, for any $C \in \mathcal{C}$,

$$C \text{ is } TP_2 \ \Leftrightarrow\ C \text{ is hyper}_{dep}\text{-}TP_2, \tag{5.11}$$

that is

$$\mathcal{P}_{TP_2} = hyper_{dep}\text{-}\mathcal{P}_{TP_2}.$$

To be precise, in [35], the above equivalence (5.11) has been shown for the family of threshold copulas $\{C_z\}_{z\in(0,1]}$,

$$C_z(u,v) = \frac{C(h_z^{-1}(uC(z,z)), h_z^{-1}(vC(z,z)))}{C(z,z)}. \tag{5.12}$$

In the present case, when the random variables are exchangeable, (5.12) is obtained from Eq. (2.15) by operating a change of parametrization, making the family $\{\hat{C}_t\}_{t\geq 0}$ independent of $\overline{G}$ (see e.g. [49]). Eq. (5.12) is formally analogous to Eq. (2.15); therefore we denote the transformation linking C_z to C by

$$C_z = \widetilde{\Phi}_{dep}(C, z).$$

However, before proceeding with our discussion, we will prove that

1. the same equivalence holds for any other monotonic parametrization of the family;

2. statement (5.11) can be extended to the set of semi-copulas.

In [49] it has been proved that Φ_{dep} is an action of the semigroup $(\mathbb{R}_+, +, 0)$, by considering it acting on the survival copula $\hat{C}$. For a fixed copula $\hat{C}$ and for a fixed marginal survival function $\overline{G}$, $\{\hat{C}_t\}_{t\geq 0}$ is the orbit of $\hat{C}$ under Φ_{dep}.

Sometimes, for example when making the family independent of the margin, it may be convenient considering a different parametrization of the family $\{\hat{C}_t\}_{t\geq 0}$, by means of a strictly monotonic function $\psi : \mathbb{R}_+ \to \psi(\mathbb{R}_+) \subseteq \mathbb{R}$. We obtain the family $\{\hat{C}_z^{(\psi)}\}_{z\in\psi(\mathbb{R}_+)}$, with $\hat{C}_z^{(\psi)} = \hat{C}_{\psi^{-1}(z)}$. By replacing t with $\psi^{-1}(z)$ in Eq. (2.15), a different transformation $\Phi_{dep}^{(\psi)} : \mathcal{C} \times \psi(\mathbb{R}_+) \to \mathcal{C}$ is defined, mapping $(\hat{C}, z)$ in $\hat{C}_z^{(\psi)}$.
In [35], e.g., the family $\{C_z\}_{z\in(0,1]}$ is originated by choosing $\psi = \overline{G}$.

It can be proved that a monotonic change of parametrization leaves unchanged the orbits of $\mathcal{C}$ under Φ_{dep}. Actually, this fact holds more in general, independently of the transformation Φ, as the following proposition states.

Let Φ be an action of $(\mathbb{R}_+, +, 0)$ on $\mathcal{S}$ and define $\Phi^{(\psi)} : \mathcal{S} \times \psi(\mathbb{R}_+) \to \mathcal{S}$, $\Phi^{(\psi)}(S, z) = \Phi(S, \psi^{-1}(z))$.

Proposition 5.2.7. *For any strictly monotonic ψ, $\Phi^{(\psi)}$ is an action of the semigroup $(\psi(\mathbb{R}_+), \oplus, \psi(0))$ on $\mathcal{S}$.*

Furthermore, for any fixed $S \in \mathcal{S}$,

$$\mathcal{O}_{\Phi^{(\psi)}}(S) = \mathcal{O}_{\Phi}(S).$$

Proof. By letting

$$w \oplus z = \psi(\psi^{-1}(w) + \psi^{-1}(z)),$$

$\Phi^{(\psi)}$ satisfies conditions (i) and (ii) in Definition 5.1.1.

Now, for any fixed $S \in \mathcal{S}$,

$$\mathcal{O}_{\Phi^{(\psi)}}(S) = \{S' \in \mathcal{S} : \exists z \in \psi(\mathbb{R}_+) : \Phi^{(\psi)}(S, z) = S'\}.$$

Since ψ is strictly monotonic,

$$\mathcal{O}_{\Phi^{(\psi)}}(S) = \{S' \in \mathcal{S} : \exists t \in \mathbb{R}_+ : \Phi^{(\psi)}(S, \psi(t)) = S'\}.$$

Now, by definition, $\Phi^{(\psi)}(S, \psi(t)) = \Phi(S, t)$. Thus, for any $S \in \mathcal{S}$,

$$\mathcal{O}_{\Phi^{(\psi)}}(S) = \mathcal{O}_{\Phi}(S).$$

$\square$

Even without its statistical meaning, the transformation Φ_{dep} can be formally extended to the set of semi-copulas.

It can be proven that the equivalence between $\mathcal{P}_{TP_2}$ and hyper$_{dep}$-$\mathcal{P}_{TP_2}$ still holds for semi-copulas, as the following proposition states.

Proposition 5.2.8. *Let be $S \in \mathcal{S}$. $S \in \mathcal{P}_{TP_2}$ if and only if $S \in$ hyper$_{dep}$-$\mathcal{P}_{TP_2}$.*

Proof. By [34, Lemma 3.1], $\mathcal{P}_{TP_2} \cap \mathcal{C} = \mathcal{P}_{TP_2} \cap \mathcal{S}$. Therefore, if $S \in \mathcal{S}$ is TP_2, then $S \in \mathcal{C}$.
Conversely, if $S \in \mathcal{S}$ is not TP_2, it cannot be hyper$_{dep}$-TP_2, because this fact would trivially imply that S is TP_2, against the hypothesis.
Therefore the equivalence (5.11) holds. $\square$

Since $\mathcal{P}_{TP_2}$ is closed under Φ_{dep}, by Corollary 5.1.10, it follows that, if $t_0 \geq 0$ exists, such that $\hat{C}_{t_0}$ is TP$_2$, then $\hat{C}$ is $\langle TP_2; [t_0, +\infty)\rangle_{dep}$.

Since in this case the parameter $t \in \mathbb{R}_+$ represents a time, we can say that the action Φ_{dep} preserves the TP$_2$ property "in the future", but not "in the past". In other words, an orbit can enter the class $\mathcal{P}_{TP_2}$ of TP$_2$ copulas, but it cannot go out. The following example shows that, if $\hat{C}_{t_0}$ is TP$_2$, $\hat{C}$ has not necessarily to be $\langle TP_2; [0, t_0)\rangle_{dep}$.

Example 5.2.9. *Let $z_0 \in (0, 1)$ and*

$$\hat{C}(u, v) = \begin{cases} uv, & u, v \in [0, z_0], \\ z_0 + (1 - z_0) W\left(\frac{u - z_0}{1 - z_0}, \frac{v - z_0}{1 - z_0}\right), & u, v \in (z_0, 1], \\ \min(u, v), & \textit{otherwise}, \end{cases}$$

where $W(u, v) = \max(u + v - 1, 0)$ is the lower Frechet bound.
We recall that, for any $t \geq 0$, $\hat{C}_t$ only depends on the behaviour of $\hat{C}$ on the square $[0, z]^2$, $z = \overline{G}(t)$. Thus $\hat{C}_{t_0}$ is TP$_2$, for $t_0 = \overline{G}^{-1}(z_0)$, and $\hat{C}_t$ continues to be TP$_2$ for $t > t_0$. For $t < t_0$ instead, $\hat{C}_t$ is not even PQD. In fact, if we consider $z_0 = \frac{1}{2}$ and $z_1 = \frac{3}{4}$: $C(z_1, z_1) = \frac{1}{2} < \frac{9}{16} = z_1^2$. Therefore, for $z_0 = \frac{1}{2}$, $\hat{C}_t$ is not PQD at least for $t \leq \overline{G}^{-1}(\frac{3}{4})$.

Remark 5.2.10. *By Proposition 5.1.7, an orbit cannot enter $\mathcal{P}_{TP_2}$ from $\mathcal{P}_{LTD}$, but it has to pass $\mathcal{P}_{LTD}$ by, directly passing from $\mathcal{P}_{PQD}$ to $\mathcal{P}_{TP_2}$.*

In fact, let us suppose $\hat{C}_{t_0} \in \mathcal{P}_{TP_2}$ for some $t_0 > 0$ and $\hat{C}_t \in \mathcal{P}_{LTD} \setminus \mathcal{P}_{TP_2}$ for any $t \in [0, t_0)$. Since $\mathcal{P}_{TP_2} \subset \mathcal{P}_{LTD}$, $\hat{C}$ would belong to hyper$_{dep}$-$\mathcal{P}_{LTD}$. But hyper$_{dep}$-$\mathcal{P}_{LTD}$ coincides with hyper$_{dep}$-$\mathcal{P}_{TP_2}$, that, on its turn, coincides with $\mathcal{P}_{TP_2}$, against the hypothesis $\hat{C} \in \mathcal{P}_{LTD} \setminus \mathcal{P}_{TP_2}$.

Roughly speaking Corollary 5.1.10 implies that if we observe at a certain time a strong dependence between the variables, the structure of the dependence will not change. This preservation is not warranted instead by weaker dependence notions: the PQD property, for example, is not necessarily preserved in time (for a more detailed discussion about this topic, see [35]).

Since TP_2 is equivalent to hyper$_{dep}$-TP_2, for $t_0 > 0$, $\langle TP_2; [t_0, +\infty)\rangle_{dep}$ is weaker than TP_2. Thus, we expect that a weaker property than hyper$_{dep}$-LTD is sufficient to guarantee $\langle TP_2; [t_0, +\infty)\rangle_{dep}$. In fact, by Corollary 5.1.11, it follows that $\hat{C}$ being $\langle LTD; [t_0, +\infty)\rangle_{dep}$ is equivalent to $\hat{C}$ being $\langle TP_2; [t_0, +\infty)\rangle_{dep}$.

A further dependence property, we have not considered in previous works, is SI. Without making explicit computations, the following inclusion can be proved.

Proposition 5.2.11. *Hyper$_{dep}$-$\mathcal{P}_{SI} \subset \mathcal{P}_{TP_2}$.*

Proof. Since $\mathcal{P}_{SI} \subset \mathcal{P}_{LTD}$, *hyper$_{dep}$*-$\mathcal{P}_{SI} \subset$ *hyper$_{dep}$*-$\mathcal{P}_{LTD}$. But, as we have said before, *hyper$_{dep}$*-$\mathcal{P}_{LTD} = \mathcal{P}_{TP_2}$ and, therefore, *hyper$_{dep}$*-$\mathcal{P}_{SI} \subset \mathcal{P}_{TP_2}$. □

By Corollary 5.1.12, it follows

Corollary 5.2.12. *If $t_0 \geq 0$ exists, such that $\hat{C}$ is $\langle SI; [t_0, +\infty)\rangle_{dep}$, then $\hat{C}$ is $\langle TP_2; \Lambda\rangle_{dep}$.*

We summarize the implications among the considered properties in the following table, similar to the one in [35]:

$$\begin{array}{ccc}
\langle \mathrm{TP}_2; \Lambda\rangle_{dep} & \Rightarrow & \\
\not\Uparrow\ \not\Downarrow & & \langle \mathrm{LTD}; \Lambda\rangle_{dep} \\
\langle \mathrm{SI}; \Lambda\rangle_{dep} & \Rightarrow & \\
\not\Uparrow\ \not\Downarrow & & \not\Uparrow\ \not\Downarrow \\
\mathrm{SI} & \Rightarrow & \\
\not\Uparrow\ \not\Downarrow & & \mathrm{LTD} \\
\mathrm{TP}_2 & \Rightarrow & \\
\Updownarrow & & \Uparrow \\
hyper_{dep}\text{-}\mathrm{TP}_2 & \Leftrightarrow & \\
\Uparrow\ \not\Downarrow & & hyper_{dep}\text{-}\mathrm{LTD} \\
hyper_{dep}\text{-}\mathrm{SI} & \Rightarrow &
\end{array} \tag{5.13}$$

For $\Lambda = [t_0, +\infty),\ t_0 > 0$, the only different relationships are:

$$\begin{array}{ccc}
\langle \mathrm{TP}_2; \Lambda\rangle_{dep} & \Leftrightarrow & \\
\Uparrow\ \not\Downarrow & & \langle \mathrm{LTD}; \Lambda\rangle_{dep} \\
\langle \mathrm{SI}; \Lambda\rangle_{dep} & \Rightarrow &
\end{array} \tag{5.14}$$

Remark 5.2.13. *In the Archimedean case, while the classes $\mathcal{P}_{PQD}$, $\mathcal{P}_{LTD}$ and $\mathcal{P}_{TP_2}$ continue to be different, hyper-$\mathcal{P}_{PQD}$, hyper-$\mathcal{P}_{LTD}$ and hyper-$\mathcal{P}_{TP_2}$ coincide. By Proposition 5.1.7, it means that, for example, if $\hat{C} \notin \mathcal{P}_{TP_2}$, its orbit can enter the class at a certain time t_0. However, by an analogous reasoning to the one in Remark 5.2.10, before t_0, $\hat{C}_t$ has to be not PQD.*

5.2.2 Ageing and hyper-ageing

We mainly focus our discussion on of B's properties to supermigrativity and to its connections with PQD and TP_2. In fact, the scheme of the implications among PQD, SM and TP_2, investigated in [49], reflects the ones in (5.13) and (5.14).

In [19] it was proved that $hyper_{ag}$-$\mathcal{P}_{PQD} = \mathcal{P}_{SM}$. In view of the semigroup structure of $\{B_t\}_{t\geq 0}$, it follows that $\mathcal{P}_{SM} = hyper_{ag}$-$\mathcal{P}_{SM}$. Thus, by Corollary 5.1.10, we have:

Corollary 5.2.14. *If $t_0 \geq 0$ exists, such that B_{t_0} is SM, then B is $\langle SM; [t_0, +\infty)\rangle_{ag}$.*

As for TP_2 property for the survival copulas, we can say that the property SM for the ageing function can arise at some time t_0, but, once it has appeared, it is necessarily preserved for future times. This analogy holds because $\mathcal{P}_{SM} = hyper_{ag}$-$\mathcal{P}_{SM}$ so like $\mathcal{P}_{TP_2} = hyper_{dep}$-$\mathcal{P}_{TP_2}$, that is, both $\mathcal{P}_{TP_2}$ and $\mathcal{P}_{SM}$ are closed under some semigroup actions. Thus, the orbits of an ageing function under Φ_{ag} can enter $\mathcal{P}_{SM}$, but they cannot go out.

Analogously to Example 5.2.9,

Example 5.2.15. *We consider, for $z_0 \in (0,1)$, the ordinal sum*

$$B(u,v) = \begin{cases} uv, & u,v \in [0,z_0], \\ z_0 + (1-z_0)W\left(\frac{u-z_0}{1-z_0}, \frac{v-z_0}{1-z_0}\right), & u,v \in (z_0,1], \\ \min(u,v), & \textit{otherwise.} \end{cases}$$

B_t is SM for $t \geq -\log z_0$, but, for $t < -\log z_0$, B_t is not even PQD.

If $\inf \Lambda > 0$, $\langle SM; \Lambda\rangle_{ag}$ is a weaker property than SM and we find a weaker property than hyper_{ag}-PQD implying it.

Corollary 5.2.16. *If $t_0 \geq 0$ exists, such that B is $\langle PQD; [t_0, +\infty)\rangle_{ag}$, then B is $\langle SM; [t_0, +\infty)\rangle_{ag}$.*

This implication only holds for intervals of the kind $\Lambda = [t_0, +\infty)$, for any $t_0 \geq 0$. For a general interval $\Lambda = [t_0, t_1]$, $0 \leq t_0 \leq t_1$, $\langle PQD; \Lambda\rangle_{ag}$ does not necessarily imply any more $\langle SM; \Lambda\rangle_{ag}$, as the following example shows.

Example 5.2.17. *Let us consider, for $z_0 \in (0,1)$,*

$$B(u,v) = \begin{cases} \min\left(u, v, \frac{u^2+v^2}{2}\right) & u, v \in [0, z_0], \\ z_0 + \frac{(u-z_0)(v-z_0)}{1-z_0}, & u, v \in (z_0, 1], \\ \min(u,v), & \textit{otherwise.} \end{cases}$$

B is at least $\langle PQD; [0, -\log z_0]\rangle_{ag}$, but B is not SM. In fact B does not satisfy Eq. (4.2)*, for $u = s = \frac{1}{2}$, $v = \frac{1}{4}$, $z_0 \in [\frac{1}{4}, \frac{1}{2}]$.*
$B \notin \mathcal{P}_{SM}$ implies that B cannot be $\langle SM; \Lambda\rangle_{ag}$ for Λ containing 0. Therefore, for a general $\Lambda \subseteq \mathbb{R}_+$, B being $\langle PQD; \Lambda\rangle_{ag}$ does not imply B being $\langle SM; \Lambda\rangle_{ag}$.

Another relation is provided by the following proposition:

Proposition 5.2.18 ([49]). *hyper$_{ag}$-$\mathcal{P}_{TP_2} \subset \mathcal{P}_{SM}$.*

By Corollary 5.1.12, it follows:

Corollary 5.2.19. *If $t_0 \geq 0$ exists, such that B is $\langle TP_2; [t_0, +\infty)\rangle_{ag}$, then B is $\langle SM; [t_0, +\infty)\rangle_{ag}$.*

We summarize here the implications discussed in the present paragraph:

$$\begin{array}{ccccc} hyper_{ag}\text{-}TP_2 & \Rightarrow & TP_2 & \not\Rightarrow \not\Leftarrow & \langle TP_2; \Lambda\rangle_{ag} \\ \Downarrow & & \not\Uparrow \not\Downarrow & & \not\Downarrow \\ hyper_{ag}\text{-}SM & \Leftrightarrow & SM & \Rightarrow \not\Leftarrow & \langle SM; \Lambda\rangle_{ag} \\ \Updownarrow & & \Downarrow & & \Downarrow \not\Uparrow \\ hyper_{ag}\text{-}PQD & \Rightarrow & PQD & \not\Rightarrow \not\Leftarrow & \langle PQD; \Lambda\rangle_{ag} \end{array}$$

If $\Lambda = [t_0, +\infty)$, the only different implications in the table are:

$$\langle TP_2; \Lambda\rangle_{ag} \Rightarrow \langle SM; \Lambda\rangle_{ag} \Leftrightarrow \langle PQD; \Lambda\rangle_{ag}.$$

5.3 Conclusions

We consider the families $\{\hat{C}_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$, used to describe dependence and ageing of a model.

Starting from previous works, in [49] it has been hinted at the fact that the obtained results are based on the semigroup structure of the two families. In this paper, we develop this analysis, by studying in detail the consequences of such a semigroup structure.

In this frame, we prosecuted here the study of the concept of hyper-property, introduced in [35]. We presented an algebraic approach to this investigation and found that the notion of hyper-property is something general, not only related to dependence properties. Thus, we extended the study of hyper-properties to ageing too. We obtain the results in Section 5.2 by means of general propositions, without making explicit computations.

The common algebraic structure of $\{\hat{C}_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ allows us to explain some systematic analogies between dependence and ageing properties and between the structures of relations existing among them.

We notice in fact a parallelism between the properties TP_2 for $\hat{C}$ and SM for B, SI for $\hat{C}$ and TP_2 for B, LTD for $\hat{C}$ and PQD for B. The explanation of the behavioural similarities lies in that each pair of classes of semi-copulas (corresponding to the afore-mentioned properties) has the same properties with respect to the two different actions Φ_{dep} and Φ_{ag}.
For example, from both LTD and PQD, under these two different actions, we obtain two known properties of semi-copulas:

$$\mathcal{P}_{TP_2} \subset \mathcal{P}_{LTD},\ \mathcal{P}_{SM} \subset \mathcal{P}_{PQD};$$

$$\mathcal{P}_{TP_2} = hyper_{dep}\text{-}\mathcal{P}_{LTD},\ \mathcal{P}_{SM} = hyper_{ag}\text{-}\mathcal{P}_{PQD}.$$

The fact that $\mathcal{P}_{TP_2}$ is closed under Φ_{dep} and $\mathcal{P}_{SM}$ is closed under Φ_{ag}, so like other analogies discussed in Section 5.2, can be seen as a consequence of the above-mentioned analogy between LTD and PQD, due to the fact that both Φ_{dep} and Φ_{ag} are actions of a semigroup on the set of semi-copulas.

Conclusions

In this thesis we developed the subject of the time evolution of dependence among residual lifetimes. In particular, among the topics studied in the related literature, there are tail dependence and default contagion. The first one concerns the asymptotical evolution of dependence; the second consists in a stochastic order condition on the survival function immediately before and immediately after a default.

One of our aims was investigating connections among: evolution of dependence between two subsequent default times, evolution of dependence at default times and default contagion. We fulfil this aim by providing a unified frame for the study of evolution of dependence. In particular, connections have been studied in detail in the case when the lifetimes are conditionally i.i.d..

Some aspects related to the extension to more general cases can be the subject of further insights, both as to the analysis of conditions for preservation of dependence properties and as to the monotonicity property of the family of threshold copulas with respect to some orders.

From an application-oriented point of view, it would be also interesting to study models where the units are divided into groups. The units are exchangeable within any group, but not globally.

Default contagion and tail dependence are very topical subjects. In particular, the last recent financial crisis has high-lit the importance of studying connections among:

- evolution of dependence (in normal conditions),
- dependence conditionally on extreme events
- dependence conditionally on defaults.

In this context, our study tells us, intuitively speaking, that, if the conditional hazard rates of the "countries" are monotonic functions of some underlying factors, dependence increases under extreme events and discontinuously decreases when a default occurs. Notice however that, along with the decrease of dependence among the not defaulted "countries", we also find a decrease of their joint survival probability.

Actually, the scenario we deal with nowadays in finance is not exactly the same we considered in our treatment, where the default times are conditionally i.i.d..

In fact, the dependence among countries is not only "information induced", that is due to our level of information about some common environmental factors, but it is also due to "physical" interaction (loans, investments, etc).

Furthermore, it is also realistic considering the case when the countries are not exchangeable: some countries are economically stronger than others or their production is linked to a specific sector... in short, they do not play a same role in the market.

Some other aspects of the connection between tail dependence and default contagion arises however also in fields different from Finance.

In the above, we mentioned connections among different kinds of dependence or, in other words, among dependencies in different situations. Another purpose of ours is studying links between dependence and ageing.

Ageing is mainly used in Reliability Theory, while it is not so a widespread topic in Risk Theory and Finance. Actually, one can expect that the same concepts are applicable in such fields.

In order to give a unified frame for the study of evolution of dependence and of ageing, we provided a semigroup approach. Such an approach is based on the fact that both the families describing evolution of dependence and of ageing respectively are proved to be orbits of the survival copula and of the ageing function under different actions of a semigroup.

This theoretical frame is useful to study ageing, in parallel with dependence, and also to explain analogies existing between evolution of dependence and evolution of ageing.

We developed here this approach in the case conditional on the observation of survivals only. However, such an approach could be extended to the analysis of evolution of both dependence and ageing in presence of defaults. This would be interesting both from a mathematical and from an application-oriented point of view.

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