Conjecturing in Dynamic Geometry:
A Model for Conjecture-generation through Maintaining Dragging

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CONJECTURING IN DYNAMIC GEOMETRY: A MODEL FOR CONJECTURE-GENERATION THROUGH MAINTAINING DRAGGING

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ABSTRACT

CONJECTURING IN DYNAMIC GEOMETRY: A MODEL FOR CONJECTURE-GENERATION THROUGH MAINTAINING DRAGGING

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The purpose of this research is to study aspects of the impact of Dynamic Geometry Systems (DGS) in the process of producing conjectures in Euclidean geometry. Previous research has identified and classified a set of dragging schemes spontaneously used by students. Building on these findings, the study focuses on cognitive processes that arise in correspondence to particular dragging modalities in Cabri. Specifically, we have conceived a model describing what seems to occur during a process of conjecture-generation that involves the use of a particular dragging modality, described in the literature as *dummy locus dragging*. In order to accomplish this goal, we preliminarily introduced participants to specific dragging modalities, re-elaborated with a didactic aim from those present in the literature. In particular dummy locus dragging was re-elaborated into what we introduced as *maintaining dragging (MD)*. This study aimed at developing and testing our model of conjecture-generation through MD by analyzing dynamic explorations of open problems in a DGS. The general experimental design was articulated in two phases, an introductory lesson on dragging modalities and interview sessions in which students were asked to solve *conjecturing-open problems*. Subjects were high school students in Italian “licei scientifici”, a total of 31. Data collected included: audio and video recordings, screenshots of the students’ explorations, transcriptions of the task-based interviews, and the students' work on paper that was produced during the interviews. The study shows appropriateness of the model, which we refer to as the *MD-conjecturing Model*. Furthermore the study shed light onto a relationship between abductive processes and use of MD, and motivated the introduction of the notion of *instrumented abduction*. The study has implications for the design of activities based on the use of maintaining dragging with the educational objective of introducing students to conjecturing and proving in geometry.

CHAPTER I

CONTEXTUALIZATION OF THE STUDY WITHIN THE LITERATURE

In this chapter we contextualize our study within the literature, describing how it is situated within the educational issue of conjecturing and proving in Geometry, and in particular how a dynamic geometry system (DGS) might contribute to mathematics teaching and learning in this field. Dragging is a characterizing feature of a DGS, therefore we focus especially on how it has been studied in the literature. We then introduce a general version of the research questions we set out to investigate, and the main goals of the study.

1.1 Contextualization of the Research Problem

This study is situated in the educational context of conjecturing and proving, and in particular on how a DGS may contribute to the conjecturing phase of open problem activities. Therefore in this section we present literature on conjecture-generation and the use of open problems activities in this educational context. Moreover we discuss the role of technology in mathematics education and in particular that of computer-based learning through a DGS. We then look at how a DGS seems to impact conjecture-generation in Geometry.

1.1.1 Conjecture-generation and Open Problems

Research has shown that when a theorem is introduced as a ready-made object, the need for justification is totally absent. Furthermore students do not seem to be naturally inclined to prove theorems given to them as statements that are easy to believe. In particular, studies suggest that surprise, contradiction and uncertainty might be key elements in promoting a feeling of necessity to prove (Hadas, Hershkowitz, & Schwarz, 2000; Goldenberg, Cuoco & Mark, 1998). The terminology “open problem” (Silver, 1995) refers to a problem (or question) stated in a form that does not reveal its solution (or answer). When an open problem is assigned, the solver not only has to find hypotheses justifying a fact, but also has to look for a fact to be justified. In other words open problems can be used to foster conjecture-generation.

In this section we describe studies that suggest how the process of developing a conjecture to prove can be beneficial for the subsequent production of a proof. Then we discuss how open problems can be used to foster conjecture-generation.

Conjecturing and Proving. Literature reveals a debate concerning the relationships between argumentation, conjecture and proof. First it is useful to define argumentation and conjecture (as proof has already been discussed) in the context of open problem investigations. Argumentation can be viewed from a structural point of view, or from a functional point of view (Pedemonte, 2007a). Within discourse, the role of argumentation is to provide a rational justification for a claim (Hanna, 1991; Hoyles & Healy, 1999). In this sense proof can be considered as a particular argumentation in mathematics (Pedemonte, 2007b). In parallel with the definition of theorem (Mariotti et al., 1997; Mariotti, 2000), conjecture can be defined as a triplet (Pedemonte, 2007b): a statement, an argumentation, and a system of conceptions (Balacheff, 2000; Balacheff & Margolinas, 2005).

While argumentation is the process leading to the development of a conjecture, the proof is a subsequent product (Pedemonte, 2003, 2007b). Passing from the development of a conjecture to the construction of a proof is a delicate process. Some authors have underlined that there is a cognitive and epistemological gap between argumentation and proof (Duval, 1995), while others stress the existence of a continuity. This continuity is referred to as "cognitive unity", a notion introduced by Boero, Garuti, and Mariotti, who described it as follows:

> During the production of the conjecture, the student progressively works out his/her statement through an intense argumentative activity functionally intermingled with the justification of the plausibility of his/her choices: during the subsequent proving stage, the student links up with his process in a coherent way, organizing some of the justifications ("arguments") produced during the construction of the statement according to a logical chain (Boero, Garuti & Mariotti, 1996, p.113).

In other words, cognitive unity is established when there is continuity between the argumentative activity that occurs during the conjecturing stage, and the process of formal justification that occurs during the proving stage.

Pedemonte (2003) has developed hypotheses about what kinds of reasoning lead to rupture or cognitive unity between the phase of experimentation-argumentation-conjecturing, versus the phase of proving. By using Toulmin's model (1958) to study and compare the content and the structures of argumentations and of proofs, she has been able to anticipate occasions in which cognitive unity occurs, and cases in which there will be rupture.

<u>Open problems</u>. In the context of open problems students are faced with a situation in which there are no precise instructions, but rather they are left free to explore

the situation and make their own conclusions. More precisely, in Geometry, open problems have been characterized in the following way.

> The statement is short, and does not suggest any particular solution method or the solution itself. It usually consists of a simple description of a configuration and a generic request for a statement about relationships between elements of the configuration or properties of the configuration.
> The questions are expressed in the form “which configuration does...assume when...?” “which relationship can you find between...?” “What kind of figure can...be transformed into?". These requests are different from traditional closed expressions such as "prove that...", which present students with an already established result. (Mogetta et al., 1999, pp. 91-92)

In some of the previous research, the production of conjectures is an explicit request in the text of an open problem (for example, Boero et al., 1996a, 2007; Arzarello et al., 2002; Olivero, 2001, 2002). When this is the case, we will use the terminology *conjecturing open problem*, to distinguish it from other types of open problems.

When a conjecturing open problem is assigned, the solution involves elaborating a conditional relationship between some premise and a certain fact. This relationship may be expressed by means of a conditional statement relating a premise and a conclusion. Such conditional statement constitutes the formulation of the conjecture. Moreover, as research points out (Boero et al., 1996b, pp. 113-114) the process of producing a conjecture may be accompanied by an active recourse to argumentation supporting the acceptability of the conjecture according to the solver’s system of conceptions. Assuming this perspective, the production of a conjecture can be related to the production of a theorem, conceived as the system of statement, proof and theory (Mariotti et al., 1997; Mariotti, 2000). Since research has shown that there can be an opposition between argumentation (leading to the development of a conjecture) and proof (Pedemonte, 2007b; Duval, 1996, 1998, 2006), a distinction must be made between the conjecturing stage and the proving stage of an open problem activity.

Often the conjecturing stage requires the generation of conditionality after a mental and/or physical exploration of the problem situation (Mariotti et al., 1997). As research has pointed out, this process seems to require the “crystallization” of a statement from a “dynamic” exploration of a problem to a “static” conditional expression, through the focus on a “temporal section” (Boero et al., 1999; Boero et al., 2007).

> …the conditionality of the statement can be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process that will be subsequently detached from it and then “crystallized” from a logic point of view (“if…then…”). (Boero et al., 1996a, p. 121)

This involves the identification, within a dynamic experience, of the two components of a (static) conditional statement: a “condition” that will become the premise and a “fact” that will become the conclusion. Searching for a “condition” is frequently referred to during the explorations as finding “when” (Arzarello, 2000, 2001) something happens. Therefore the term “when” becomes particularly significant because it is an element that makes explicit an attempt of linking the world of experience, embedded in real time, to the crystallized formal world of Euclidean Geometry, organized through conditionality.

1.1.2 The Contribution of Dynamic Geometry Systems

Mathematics education supervisors and leaders have been encouraging the use of technology in the classroom (Noss & Hoyles, 1996; NCTM, 2000, 2006; De Villiers, 2004; Mariotti, 2005) to foster mathematical habits of mind (Cuoco, 2008). NCTM’s document *Principles and Standards for School Mathematics* (2000) states: “Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.” (p. 11). A means through which the use of technology is implemented is computer-based learning.

Computer-based learning in the mathematics classroom involves the following specific form of interaction between the learner and the computer. The student's interaction with the computer requires a process of interpretation, which is typical of the mathematical activity. This is described in Balacheff & Kaput (1996, p. 470): "The interaction between a learner and a computer is based on a symbolic interpretation and computation of the learner's input, and the feedback of the environment is provided in the proper register allowing its reading as a mathematical phenomenon." This is one of the reasons why computer-based learning in the mathematics classroom is potentially very powerful. Noss and Hoyles (1996) like to think of the computer as a window that should be looked through to understand the process of meaning-making, because it allows (or forces) all of its users to communicate in the language of the software being used, or of the "microworld" described by the software. In other words, the computer is a channel through which communication can happen and a window through which this can be seen. In Section 1.2.1 we will illustrate how a DGS can be conceived as a microworld.

Computer-based learning can be useful not only for observing a student's mathematical activity, but also for developing approaches for making conjectures and solving problems in different mathematical fields (NCTM, 2000). In the next section we will introduce issues that arise within a particular type of computer-based learning, that is working in a DGS, which is the context in which our study is situated.

<u>Computer-based learning in the context of a DGS.</u> Technology can be integrated into the teaching and learning of Geometry through particular software programs referred to as Dynamic Geometry Systems (DGS). Several studies in the teaching and learning of Euclidean Geometry (for example, Choi-Koh, 1999; Mariotti, 2000; Christou, Mousoulides, Pittalis & Pitta-Pantazi, 2004; De Villiers, 2004) have shown that a DGS

can foster the learners' constructions and ways of thinking, and it can help students overcome some cognitive difficulties that they encounter with conjecturing and proving (for example, Noss & Hoyles, 1996; Mariotti, 2000).

In particular, studies show that a DGS can be motivational for students, because they gain a better understanding and visual grasp of the mathematics they are investigating (Garry, 1997). Students find the feedback they get from a DGS to be efficient and exciting, and they describe computer learning as an alternative style of working which they enjoy (Ruthven & Hennessy, 2002). Moreover, a DGS can be used to overcome some of the difficulties encountered when approaching proof in Geometry, by providing visual feedback and supporting the construction of situations in which "what if" questions can be asked and explored (DeVilliers, 1997, 1998).

In a DGS, it is common for students to use the visual feedback to be convinced of a conjectured attribute on a whole class of objects. Such feedback comes through the use of the dragging function (Mariotti, 2001; Herrera, Sanchez, 2006). Although some teachers are reluctant to use a DGS in the classroom, because they believe that a DGS may prevent students from understanding the need and function of proof (Yerushalmy, Chazan & Gordon, 1993), studies have shown that activities which provide opportunities for the creation of uncertainties (Goldenberg, Cuoco & Mark, 1998; Hadas, Hershkowitz & Schwarz, 2000) lead students to feel the necessity of elaborating a proof.

De Villiers (1997, 1998) illustrated how a DGS provides the perfect situation for asking 'What if?' questions. These questions enrich investigations, because they lead students to generalizations and discoveries. In this sense, he claims that the search for proof becomes an intellectual challenge, stemming from the need to understand *why*. 'What if?' questions are also typical of problems that 'go somewhere' mathematically.

Another reason why several studies in the teaching and learning of Geometry, like those conducted by Choi-Koh (1999), by Mariotti (2000), by Christou, Mousoulides, Pittalis, & Pitta-Pantazi (2004), or by De Villiers (2004), support the use of a DGS in the classroom, is that dynamic geometry systems mediate the interaction between teacher and students. The studies listed above and other studies have shown that a DGS can foster the learners' constructions and ways of thinking, by making tangible the dialogue between learners and their constructions (Noss & Hoyles, 1996; Mariotti, 2000). This can occur, because when using a DGS, students can generate and examine objects on the computer screen and have a common referent for their discussion (NCTM, 2000). For example, research on students "playing with Cabri" has shown that the students find themselves constantly using proper terminology for the objects that they need. This helps them achieve a correct idea, for example, of concepts like "ray", "polygon", "perpendicular", or "parallel" that are otherwise not always immediately understood (Brigaglia & Indovina, 2003).

Finally, a DGS can be used for the exploration of open problems. Research has shown that a DGS impacts students' approach to investigating open problems in Euclidean Geometry, contributing particularly to students' reasoning during the conjecturing phase of open problem activities (for example, Goldenberg, 1993, 1998; De Villiers, 1998, 2004; Laborde, 2000, 2001; Mariotti, 2000a, 2000b, 2001, 2003,2005; Arzarello, 1998a, 1998b, 2002; Olivero, 1999, 2002). The DGS's contribution to the investigation of open problems is based in *dragging*, because it allows the solver to be guided and supported by interacting with the software, as described by Laborde and Laborde:

> ... the changes in the solving process brought by the dynamic possibilities of Cabri come from an active and reasoning visualisation, from what we call an interactive process between inductive and deductive reasoning (Laborde & Laborde, 1991, p. 185).

This brings us to a central feature of explorations in a DGS, that distinguishes this environment from any other. The central feature, upon which our study is founded, is *dragging*. Dragging, and the dynamism it induces on DGS objects, is a distinguishing feature of a DGS in particular with respect to the static domain of Euclidean Geometry. In the following section we describe aspects of the relationship between a DGS, built to incorporate aspects of the Theory of Euclidean Geometry (TEG) and the domain of Euclidean Geometry itself. Then we present a review of crucial research on various aspects of dragging in a DGS.

1.2 Dragging in the Literature

We find it important to underline the relationship that can exist between a DGS built to incorporate particular aspects of the Theory of Euclidean Geometry (TEG) and Euclidean Geometry itself. We will underline this relationship by introducing the conception of a DGS as a "microworld" (Papert, 1980) and describing how aspects of dynamic explorations within a DGS can be put in relationship with the TEG. The most delicate aspect of the transition between the two domains has to do with *dragging* and how it can mediate meanings between the two domains. For example, Lopez-Real and Leung see dragging as a conceptual tool in the following way:

> It seems that dragging in DGE can open up some kind of semantic space (meaning potential) for mathematical concept formation in which dragging modalities (strategies) are temporal-dynamic semiotic mediation instruments that can create mathematical meanings, that is, a window to enter into a new semiotic environment of how geometry can be re-presented (re-shaped). (Lopez-Real & Leung, 2005, p. 666).

1.2.1 A DGS as a Microworld

A fundamental concept, when speaking about a DGS, is that of "microworld"

(Papert, 1980; Noss & Hoyles, 1996; Mariotti, 2006). A concise and eloquent description of the concept of microworld is contained in Balacheff & Kaput (1996, p. 471):

> A microworld consists of the following interrelated essential features: a set of primitive objects, and rules expressing the ways the operations can be performed and associated, which is the usual structure in the *formal system* in the mathematical sense; a *domain of phenomenology* that relates objects and actions on the underlying objects to phenomena at the ‘surface of the screen’. This domain of phenomenology determines the type of feedback the microworld produces as a consequence of user actions and decisions. (emphasis in original).

A microworld can be built to resemble a mathematical world, such as Euclidean Geometry. This is the case of a DGS like Cabri, which contains “objects” such as points, lines, circles, and ways to “manipulate” the objects. These “objects” are made to mathematically resemble a set of objects from a mathematical world (the world of Euclidean Geometry in the case of the DGS used in this study). In other words, the “objects” included offer the opportunity for the user to experiment directly with the “mathematical objects” (Mariotti, 2006), because the logical reasoning behind the objects in the microworld is designed to be the same as that behind the real mathematical objects that they represent. This feature is a key aspect of microworlds in mathematics education, because a DGS that embodies the domain of Euclidean Geometry is not the only kind of microworld that can be created. For example, mathematicians and programmers have constructed microworlds that represent non-Euclidean geometries (Noss & Hoyles, 1996), or other systems of axioms in the field of algebra or analysis.

1.2.2 The Spatio-graphical Field and Theoretical Field in Cabri

The relationship between the physical world (generally referred to as Space) and the theoretical domain is complex. Traditionally the move from observation to theory is considered “natural,” but the complexity of these connections correspond to the complexity of teaching and learning, and they are embodied by the contradiction in curricula, which separates the geometry of observation and the geometry of proof (Mariotti, 1993). In particular, geometric concepts are related to spatial properties of reality, i.e. they are strictly related to images. On the other hand a geometric concept is an active element of thought (Piaget & Inhelder, 1966), which is symbolic from the beginning, and the associated image becomes more and more secondary. Therefore a geometric figure has a spatio-geometric component (which will be referred to here as *figural*), and a theoretical component (which will be referred to here as *conceptual*). The theoretical component is the domain of relations and operations on the object, as well as judgments about it (Laborde, 2002).

A similar distinction as that brought forth by the spatio-graphical field and theoretical field is the distinction between *figural* and *conceptual* components of an activity within a DGS. These notions are developed by Mariotti (2006) from Fischbein's notion of *figural concept* (Fischbein, 1993). Fischbein describes how Geometry deals with mental entities (the so-called geometrical figures), which possess simultaneously conceptual and figural characters.

> A geometrical sphere, for instance, is an abstract ideal, formally determinable entity, like every genuine concept. At the same time, it possesses figural properties, first of all a certain shape. The ideality, the absolute perfection of a geometrical sphere cannot be found in reality. In this symbiosis between concept and figure, as it is revealed in geometrical entities, it is the image component which stimulates new directions of thought, but there are the logical, conceptual constraints which control the formal rigor of the process" (Fischbein, 1993).

The *figural component* of an activity is its connection to the physical world, its concreteness, and the empirical approaches that a student may take when working on it. On the other hand, the *conceptual component* of an activity is its connection to the theoretical world in which it is situated. In a DGS like Cabri this theoretical world is the Theory of Euclidean Geometry (TEG), with its definitions, axioms and theorems.

Research has shown (Bartolini Bussi, 1993) that when dealing with a geometrical problem, students need to relate the spatio-graphical field to the theoretical field and vice versa in a dialectic process, alternating "*experimental moves*" (based on actions on the mechanism and visual experiments) and "*logical moves*" (including the production of statements deduced from other statements accepted as valid). Cabri embodies both a theoretical world, the world of Euclidean Geometry, and a spatio-graphical world, its phenomenological domain, characterized by being mechanical and manipulative. Therefore, if used appropriately, Cabri can foster an interconnected dialectic between the two fields, by providing diagrams whose behavior is controlled by the theory. Furthermore, as stated by Laborde,

> the computer not only enlarges the scope of both possible experimentation and visualization but modifies the nature of the feedback. The feedback is visual on the surface, but it is controlled by the theory underlying the environment (Laborde, 2002).

However, we must not make the mistake of "collapsing" the TEG upon the phenomenology of a DGS, interpreting as "geometrical" everything that occurs in the DGS. For example, studies have shed light onto misleading aspects of DGS that are intrinsically linked to being software programs (for example, Noss et al., 1994; Hölzl, 2001; Strässer, 2001). We will discuss other aspects of a DGS, related to dragging, that contribute to highlighting the gap between the phenomenology of a DGS and the TEG in Chapter 3 and, more extensively, in Chapter 7.

1.2.3 Dragging: a General Overview

Different aspects of the potential of dynamic geometry systems (DGSs) have been widely documented (for example, Laborde 1995; Mariotti 1997, 2002; Noss & Hoyles 1996; Olivero 2002; Hollebrands, 2007). Our study focuses in particular on exploratory activities in which the goal is to produce conjectures, and the main contribution of the DGS to this type of exploration is the possibility it offers the solver to use the dragging function. Dragging is a characterizing feature of dynamic geometry that allows direct manipulation of the figure on the screen (Laborde & Strässer, 1990), inducing transformations which can be visualized as movement of these figures. This way, exploring a figure in dynamic geometry can become a search for interesting properties and relationships between these properties perceived as invariants. The identification of such invariants lies at the heart of a dynamic exploration (Laborde, 2005; Laborde et al., 2006; Hölzl, 1996; Arzarello et al. 1998a, 1998b, 2002; Olivero 2002; Healy & Hoyles 2001; Baccaglini-Frank et al., 2009).

In this section we will introduce some element from the literature on "dragging", following its evolution, situated within the more general empirical research on use of DGSs in the classroom. Gawlick (2002) highlights three stages of such research, that are: (1) research concerning the exploration of the various capabilities of a DGS; (2) research on the students' interaction with the software and their construction of knowledge with respect to the mathematical structures aimed at; (3) research on the use of DGS in the classroom, that investigates both students' uses of dynamic geometry with respect to specific mathematical tasks, and the role of the teacher in the construction of mathematical meanings from situated experiences within the DGS.

<u>The First Stage</u>. During the first stage studies focused on potentials of dynamic geometry, situating their considerations in the perspective of a DGS as a microworld (see also Section 1.2.1). Early studies describe how within a DGS, like Cabri-Géomètre (Laborde & Bellemain, 1993-1998), the following two features have impact on the learning of geometry: 1) "geometrical knowledge" is embedded in Cabri-Géomètre, and

the behavior of the software is controlled by a theory comprising primitives and the drag mode; 2) theoretical concepts are reified and can be handled as material entities (Laborde & Laborde, 1995, p. 243). A fundamental characterizing feature of a DGS is that figures can be constructed starting from a set of basic elements from which other objects are constructed according to a set of given properties describing the dependency relations between them, and base (or basic) points of the figures can be dragged on the screen. During the dragging process the properties according to which the construction was made are maintained, and these may be perceived as invariants.

Various studies address (or contain, even if they are not explicitly focused on it) the use of the drag mode (see, for example, Laborde & Strässer, 1990; Laborde & Laborde, 1991; Laborde, 1992; Noss et al., 1994; Healy et al., 1994; Goldenberg & Cuoco, 1998; Hölzl, 1996, 2001). Most of these studies have underlined the potential of dragging with respect to validating a geometrical construction. For example, Healy, Hölzl, Hoyles and Noss (1994) elaborated the idea that a figure might or might not be "mess up-able", that revealed to be quite powerful for students. Later, Healy (2000) introduced the notions of *robust construction* and *soft construction* to explain students' different ways of interacting with a DGS as they identified and induced geometrical properties on the figures. Although many studies focused on the potentials of dynamic geometry, research also took into consideration some pitfalls (Balacheff, 1993; Healy & Hoyles, 2001), leading to reflection upon different ways in which dragging might affect the learning of Geometry (Hölzl, 1996). This leads to the second stage of research, characterized by a constructivist approach aimed at analyzing students' construction of knowledge as they interacted with the microworld.

The Second Stage. Research became focused on the knowledge constructed by students during technology-based experiences with respect to potential mathematical ideas the technology-based experiences might have contained or been aimed at. Noss and Hoyles's (1996) studies show that the knowledge students would construct during such experiences was tightly linked to the specific environment it was developed within. As mentioned above, Hölzl (1996) described how a DGS may subtly interfere with the intended understanding of Geometry, leading for example to the perception of "false invariants." These are properties that look like invariants of a dynamic figure even though they are not explicitly added as properties during the construction steps nor are they consequences of them. These invariants arise from *how* the software is programmed. As a consequence, the drag mode is not "heuristically neutral" (Hölzl, 1996, p.171). Researchers and educators thus became aware that this and other features of a DGS may change the students' working style (Healy & Hoyles, 2001) and even their conception of Geometry (Balacheff, 1993; Hölzl, 1996). This explains how dynamism cannot be conceived per se as a didactical advantage (Hölzl, 1999), but instead as a non-neutral feature of dynamic geometry to be used consciously.

The awareness of non-neutrality of a DGS, in particular due to the dragging feature within it, recently led to hypothesizing the possibility of introducing a new "grammar" through which statements constructed via dynamic explorations may be expressed (for example, Lopez-Real & Leung, 2006, p.666). This of course re-opens the issue of potentially conceiving a new theory built upon the "axioms" to be defined.

The Third Stage. More recently research has been concerned with implementation of DGS within classroom settings. Among these studies many are focused on cognitive aspects of the *students' use of dragging* during explorations. These studies provide different ways of analyzing and different interpretations of students' activity. Some of them reveal students' difficulties in being aware of the different status of elements comprising a dynamic figure. For instance, Talmon and Yerushalmy (2004) shed light on the complexity of grasping and controlling hierarchical dependency induced on the elements of a figure by the construction steps. The consciousness of the fact that the dragging process may reveal a relationship between geometric properties embedded in the Cabri-figure directs the way of transforming and observing the screen image.

Other studies, that are particularly significant with respect to our research, concern the description of different dragging modalities spontaneously used by students during an open problem exploration (for example, Arzarello et al., 1998a, 1998b, 2002; Olivero, 2002; Leung, 2003, 2008; Lopez-Real & Leung, 2006). Because of their significance for the study presented in this dissertation, they will be presented separately at the end of this Section.

Another group of studies focuses on classroom activities that make use of a DGS. Some of them overcame the conception of a DGS as a "visual amplifier" and explore how its role in fostering the construction of mathematical meanings. In particular, researchers have started investigating how the use of dragging during a dynamic exploration can be interpreted in terms of logical dependency (Mariotti, 2006, 2010; Laborde, 2003; Gousseau-Coutat, 2003). "Feeling motion dependency", which can be interpreted in terms of logical dependency within the mathematical context is a key feature in the development of conjectures originating from the investigation of open problems in a DGS. The solver has to be capable of transforming perceptual data into a conditional relationship between what will become premise and conclusion of the statement of a conjecture (Mariotti, 2006).

In the context of these studies, other interesting aspects, differently related to the management of classroom activities, have been taken into account: the *design of the tasks* as a DGS in the classroom to make mathematical meanings emerge (for example Laborde, 2001, 2003; Gousseau-Coutat, 2003; Healy, 2004; Restrepo, 2009); and the *role of the teacher* in organizing and orchestrating the activities and discussions (for example, Bartolini Bussi & Mariotti, 1999, 2008; Mariotti, 2002) .

As far as the design of the task in concerned, the work by Gousseau-Coutat (2003) is particularly significant for our study. The teaching experiment implemented in a middle school classroom is aimed at fostering the understanding of conditionality by introducing soft constructions. This mediates the distinction between premise and conclusion in a conditional statement.

Let us give an example, consider the following task (Laborde, 2005, p.32-33): “Construct any quadrilateral ABCD , its diagonals and the midpoint of each diagonal. Drag any vertex A, B, C or D so that the midpoints are coinciding.” The essence of the task consists in imposing a condition by dragging (here the coincidence of midpoints) and consequently inducing a visible change on the figure (here it becomes a parallelogram). A teaching experiment developed by Restrepo (2008) stemmed from a similar assumption: fostering students’ awareness of relative dependency in a DGS with the aim of clarifying the distinction between “drawing and figure” (Laborde & Capponi, 1994).

<u>Arzarello et al.’s Cognitive Analysis of Dragging</u>. In the late 90’s a team of researchers, Federica Olivero, Ferdinando Arzarello, Domingo Paola, and Ornella Robutti, analyzed subjects’ spontaneous development of dragging modalities during investigations of open problems in dynamic geometry. The investigations centered upon

the use of dragging from a cognitive point of view, focusing on the way dragging may affect students' reasoning process. This led to a classification (Arzarello et al., 2002; Olivero, 2002) of different dragging modalities that students might use in solving problems, which have been referred to as the "dragging schemes". These dragging schemes can be described as particular ways of dragging points of a dynamic figure on the screen, that is particular uses of the dragging tool, exploited by the user in order to accomplish a task (or sub-task). The classification of the dragging modalities can be summarized as follows:

- **Wandering dragging:** moving the basic points on the screen randomly, without a plan, in order to discover interesting configurations or regularities in the figures.
- **Bound dragging:** moving a semi-draggable point (it is already linked to an object).
- **Guided dragging:** dragging the basic points of a figure in order to give it a particular shape.
- ***Dummy Locus* (or *lieu muet*) dragging**: moving a basic point so that the figure keeps a discovered property; that means you are following a hidden path (*lieu muet*), even without being aware of this.
- **Line dragging:** drawing new points on the ones that keep the regularity of the figure.
- **Linked dragging:** linking a point to an object and moving it onto that object.
- **Dragging test:** moving dragable or semi-dragable points in order to see whether the figure keeps the initial properties. If so, then the figure passes the test; if not, then the figure was not constructed according to the geometric properties you wanted it to have.

Students showed different uses of dragging according to the different aims that direct the solution process: exploring the configuration looking for regularities, making conjectures, testing and validating conjectures, justifying conjectures. The research studies carried out by Olivero, Arzarello, Paola, and Robutti (Olivero, 2000; Arzarello, et al., 1998a, 1998b, 2002) consider expert solvers' production of conjectures and propose a theoretical model describing the whole process developing from the dynamic exploration to the formulation of the conjecture and to its validation. The model is based on the theoretical distinction between "ascending" and "descending" control (Saada-Robert, 1989; Gallo, 1994) and hypothesizes the emergence of abduction when a passage from "ascending control" to "descending control" occurs.

Ascending control. This is the modality according to which the solver 'reads' the figure in order to make conjectures. The stream of thought goes from the figure to the theory, in that the solver tries and finds the bits of theory related to the situation he is confronted with. This modality relates to explorations of the given situation.

Abduction (Peirce, 1960; Magnani, 1997). In the model, abduction means choosing 'which rule this is the case of', that is the subject browses his theoretical knowledge in order to find the piece of theory that suits this particular situation. Explorations are transformed into conjectures.

Descending control (Gallo, 1994). This modality occurs when a conjecture has already been produced and the subject seeks for a validation. S/he refers to the theory in order to justify what he has previously 'read' in the figure and validates his conjectures.

The model assumes that abduction plays an essential role in the process of transition from ascending to descending control, that is from exploring to conjecturing and then to proving. Abduction guides the transition, in that it is the moment in which the conjectures are produced and expressed in a conditional form "if…then". Moreover, Arzarello et al.'s studies suggest that the abduction occurs in correspondence to use of dummy locus dragging. However the model presented above does not allow to gain detailed insight into this delicate transition point that the study refers to. We will illustrate how Arzarello et al. analyzed students' explorations through this model and their "dragging schemes" in Chapter 2, as we elaborate the elements of the theoretical background of our study.

1.3 Research Questions (General) and Goals of This Study

Building on the work of Olivero and Arzarello (Olivero, 1999; Arzarello et al, 1998a, 1998b), we have conceived a model for a cognitive process that can occur during the conjecturing stage of open problem investigations in a DGS. We will introduce this model in Chapter 2 as part of our theoretical background. The contextualization of the problem has led to the following general research questions:

1. During the conjecturing phase of an open problem in a DGS, what forms of reasoning are used and how?
2. Is it possible to associate particular forms of reasoning to particular uses of the dragging tool? If so, how can the association be described?
3. Is it possible to describe a somewhat "general" process leading to the formulation of a conjecture when the solver uses the dragging tool in particular ways? If so how might this process be described?

In Chapter 2 we will introduce the theoretical background we chose and elaborated for our study. Once we have described the constructs we use, we will present the detailed research questions we set out to investigate during this study.

Through this qualitative study we potentially seek validation and refinement (if the initial model seems to be valid) of the model, through a spiraling process of experimentation and revision. The final goal is to give a detailed description of some cognitive processes related to conjecturing when particular dragging modalities are adopted in dynamic geometry, thus providing a base for further research and for the development of new curricular activities. In particular we proposed to:

- describe a “general” process of conjecture-generation associated with particular uses of the dragging tool;
- gain insight into cognitive aspects of this process of conjecture-generation, describing potential difficulties that might arise for the solvers;
- and specifically investigate whether there is a relationship between abductive reasoning and specific dragging modalities.

CHAPTER II

THEORETICAL BACKGROUND

This chapter contains descriptions of the concepts and tools that other researchers have developed and that we will make use of in this study. Moreover, we elaborated particular theoretical constructs introduced by other researchers, so that they would become appropriate tools for this study. Our theoretical background takes into consideration and elaborates on the notion of “dragging” within a phenomenological perspective (Section 2.1), basic aspects of the “instrumental approach” (Section 2.2), and the notion of “abduction” (Section 2.3). In Section 2.4 we present the first version of our model together with a hypothesis on introducing solvers to particular ways of dragging. Then we present the dragging modalities we have elaborated from those present in the literature, to introduce to solvers (Section 2.5). This theoretical background allows us to present our more detailed research questions in Section 2.6.

2.1 Dragging Modalities in Our Theoretical Background

When analyzing what has changed in the geometry scenario with the advent of DGSs we can notice a transition from the traditional graphic environment made of paper and pencil, and the classical construction tools like the ruler and compass, to a virtual graphic space, made of a computer screen, graphical tools that are available within a given software environment and a particular mode, the dragging mode, that allows the transformation of images on the screen, giving the effect of “dragging them”. (Mariotti, 2010). The dragging tool can be activated by the user, through the mouse. It can determine the motion of different objects in fundamentally two ways: direct motion, and indirect motion.

The *direct motion* of a basic element (for instance a point) represents the variation of this element in the plane - or within a specific geometrical domain, a line, a segment, a circle when “point on an object” is activated. The *indirect motion* of an element can occur after a construction has been accomplished. In this case dragging the base points, those from which the construction originates, will determine the motion of the new elements obtained through the construction. Therefore, use of dragging can allow the user to feel “motion dependency”, which can be interpreted in terms of logical dependency within the geometrical context (Mariotti, 2010). In this section we will analyze dragging from this phenomenological perspective.

In particular we will discuss how dragging can be used to perceive invariants (Section 2.1.1), and highlight a distinction, described by Mariotti (2010), into two fundamental uses of dragging in a DGS: dragging to test a construction (Section 2.1.2) and dragging to produce a conditional statement (Section 2.1.3). We focus on this second use of dragging and in section 2.1.4 we use an example to analyze some differences in conjecture-generation in a DGS with respect to the paper-and pencil environment, induced by dragging. In particular this kind of dragging can be further separated into exploring the consequences of a certain set of premises (Section 2.1.4.1), and into finding the premise of a conditional statement (Section 2.1.4.2). This corresponds to identifying *under which conditions a given configuration takes on a certain property* (as in Arzarello et al., 2002; Olivero, 2002). Our study focuses particularly on this use of dragging.

2.1.1 Dragging and Perceiving Invariants

The dragging mode allows the transformation of images on the screen by producing a sequence of new images. Each image is reconstructed after the user's choice of a new position for a specific point s/he is dragging, by clicking on it and moving the mouse. The high number of images in this sequence and the speed at which they are produced on the screen give a visual effect of continuity, analogous to what is seen in a movie. The changes in the image on the screen are perceived in contrast to what simultaneously remains invariant, and this constitutes the base of the perception of "movement of the image" (Mariotti, 2010).

In general, and this is the case in a DGS like Cabri, the *invariants* are determined both by the geometrical relations defined by the commands used to accomplish the construction, and by the relationship of dependence between the original relations of the construction and those that are derived as a consequence within the theory of Euclidean Geometry (Laborde & Strässer, 1990). All these invariants appear simultaneously as the *dynamic-figure* is acted upon, and therefore "moves". However there is an a-symmetry between the types of invariants, which is fundamental for conceiving logical dependency within the DGS. Specifically, the a-symmetry leads to a distinction in *direct invariants*, corresponding to geometrical properties defined during the construction of the dynamic-figure, and *indirect invariants*, corresponding to geometrical properties that are consequences of the construction. Perceiving and interpreting invariants is a complex task for a non-expert geometry student. This has been observed and discussed in different studies (Talmon & Yerushalmy, 2004; Restrepo, 2008; Baccaglini-Frank et al., 2009; Mariotti, 2010).

We highlight a distinction, proposed by Mariotti (2010), between two fundamental uses of dragging in a DGS. The distinction aims at describing two situations that correspond to two different specific goals a user might have in mind when using dragging.

- Use of dragging to test whether an accomplished construction is correct, that is dragging that corresponds to check a given goal (for example, if the goal was to construct a square, dragging is used to check the correctness of the construction);
- Use of dragging to formulate a conjecture: given a certain construction the goal is to produce a conditional statement that expresses the logical dependency between properties that can be perceived through dragging the configuration.

2.1.2 Dragging to Test a Construction

In this case perceiving the figure globally will allow the identification of the invariants necessary in order to recognize the correctness of the construction, with reference to a particular definition or characterizing property. The reason such invariants are present, as a direct effect of the construction commands or as a consequence of such commands, may not be important to the user. Instead the evaluation of the correctness of the construction will occur at a global level and it will occur in relation to a system of expectations that the solver will have with respect to the final construction. Let us consider the following type of activity to be carried out within a DGS.

Construct a square. Does your figure correctly represent a square? Why?

This type of activity has been widely described and discussed in various studies (for example, Strässer, 2001, pp. 327-329), so we will not analyze particular solutions here. Instead we will discuss how dragging can be used during an activity of this sort. First we need to consider what it means for a dynamic-figure to “correctly represent” a square. This means to create an object that somehow “incorporates” the conceptual properties and characteristics of the geometric shape (NCTM, 2000), so a correct construction should lead to a dynamic-figure that has such properties as invariants. Such a construction will lead to a dynamic-figure that is “unmess-up-able” (Healy et al., 1994) or “robust” (Healy, 2000), that is, when any of its base points are dragged in any way, the figure remains a square. In this sense a construction that, for example, incorporates the properties (1) angle in A right, (2) angle in B right, (3) segment AD congruent to AB, (4) segment BC congruent to AB will be un-mess-up-able, because these properties are also sufficient for obtaining a robust square. On the other hand, a construction that does not have the sufficient properties for being a square will get deformed if some of its base points are dragged. We will refer to a property that may be induced, but that is not robust, as “soft”, in accordance to the terminology introduced by Healy (2000).

This was an example of how dragging can be used to test a construction. Of course such a task may become a subtask during a more complex activity, or a sub-goal developed by a solver who’s aim is to solve a more complex problem. We will now discuss aspects of the use of dragging in conjecture-generation, that constitute a basis for the present study.

2.1.3 Dragging to Produce a Conditional Statement

The use of Cabri in the generation of conjectures is based on the interpretation of the dragging function in terms of logical control. In other words, the subject has to be capable of transforming perceptual data into a conditional relationship between a premise and a conclusion. The consciousness of the fact that the dragging process may reveal a relationship between geometric properties embedded in the Cabri-figure directs the way of transforming and observing the screen image (Talmon & Yerushalmy, 2004). At the same time, that consciousness is needed to exploit some of the tools offered by the software, like the 'locus of points' or 'point on object'. Such a consciousness is strictly related to the possibility of exploiting the heuristic potential of a DGS.

Dragging for conjecture-generation clearly presents a higher complexity as compared to dragging to test a construction, since it involves not only observing the figure globally and recognizing characterizing properties but also analyzing and decomposing the elements of the figure and the properties they have in order to "see" relationships between such properties. In other words, when the goal is to generate the statement of a conjecture, the interpretation of perceived invariants in terms of a geometric statement is based on the interpretation of dragging in terms of relationships between properties of a figure, and more specifically in terms of invariance during dragging of such relationships between properties of elements of the figure (Mariotti, 2010).

2.1.4 Some Differences in Conjecture-generation in a DGS with Respect to the Paper-and-pencil Environment, Induced by Dragging

Let us consider the following construction, and use it as an example to introduce particular aspects of conjecture-generation in a DGS when the dragging tool is used. *ABCD is a quadrilateral in which D is chosen on the parallel line to AB through C, and the perpendicular bisectors of AB and CD are constructed.*

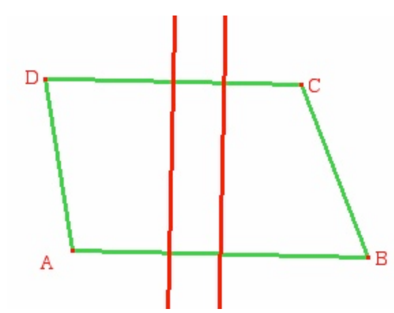


Figure 2.1.4.1: ABCD as a result of the construction described in the example above.

In a paper-and-pencil environment geometrical properties are static and "at the same level" with respect to the solver's perception. The perpendicular bisectors appear to be parallel and segments AB *and* CD appear to be parallel. It is up to the conjecturer to introduce a logical dependence between the properties s/he perceives. If we think about the figure the solver is making conjectures on (so a mental construction of the solver) as a figural concept (Fischbein, 1993; Mariotti, 1995, p. 112), we may consider its figural components and its conceptual components. It is under the conceptual control that the solver may imagine certain properties as logically dependent upon others. In this case "AB parallel to CD" *implies* "perpendicular bisectors parallel". Furthermore, in the paper-and-pencil environment, no element of the figure is privileged with respect to others, and reasoning on a specific unique drawing that represents a class of figures requires a high harmonization between the figural component and the conceptual component.

On the other hand, in a DGS, properties can be perceived as invariants with respect to dragging. In this example, the *constructed* parallelism and perpendicularity are conserved during dragging, but also the parallelism between the two perpendicular bisectors. The leap in complexity is constituted by becoming aware of the hierarchy induced on the properties of the construction and on the fact that such a hierarchy corresponds to logical relationships between the properties of the "geometric figure". Therefore the figural component that the solver deals with may profit from a dynamic representation. The distinction between direct and indirect movement may be interpreted in terms of a logical dependence of one property upon another of a certain figure. This distinction can, in the example we are considering, lead to the following conjecture: "if two sides are parallel, then the corresponding perpendicular bisectors are parallel." The interpretation of dragging in terms of conservation of the relationship between invariants corresponds to a logical control over the generality of the relationship between properties of a given figure. We use this idea to develop our hypothesis on how the sensation of “causality” may occur through dragging in a DGS, as part of the cognitive model we describe in Chapter 4.

Another factor that needs to be taken into account when describing dragging in conjecture-generation is that a dynamic figure depends on its base points, and the figure’s possible movements depend on the steps of the construction that induce corresponding invariant properties of the figure. This constitutes an essential aspect of the “being dynamic” of a Cabri-figure. In the example above, A, B, and C are base-points of the dynamic-figure with two degrees of freedom. Therefore they can be dragged to any place on the screen, while D can only be dragged along the parallel line to AB through C. Dependent elements of a construction, like the perpendicular bisectors in our example, cannot be directly acted upon. The basic and constructed elements of a figure are determined by the steps of the construction, and their different status determines how the dynamic-figure will behave during dragging. However, it is up to the solver to translate “these steps” into geometrical properties, reach other derived properties through deductive reasoning, and discover new properties that are logically linked to one another. Therefore we have shown how in a DGS the analysis of the status of the different elements of a figure, and first of all of points, can support the solver in determining and checking properties of figures and relationships between them. However the solver still is completely responsible for the non-trivial task of making sense of what s/he experiences, and s/he may encounter various difficulties along the way. In fact this task of sense-making is neither simple nor spontaneous, and it may take a considerable amount of time and training for the solver to be able to conquer it.

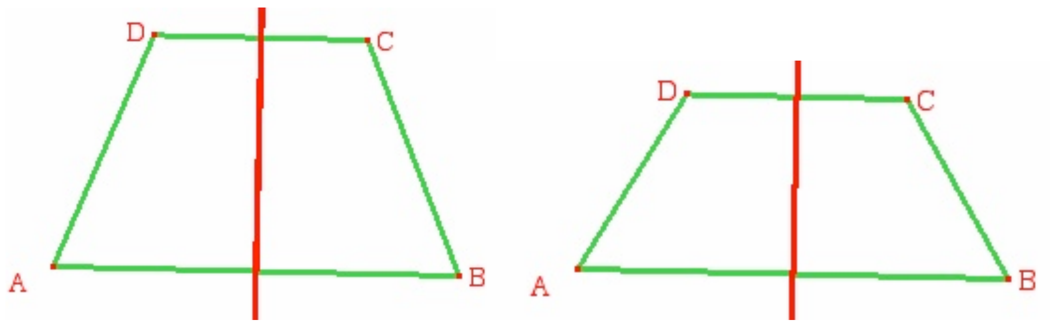
Moreover, when “exploring” a figure in a paper-and-pencil environment the solver may perform mental experiments on the figure and help him/herself by re-drawing the figure after having imposed a desired change. In order to do this the solver must keep track of the conceptual components of the figure and make sure that these are all present in the new drawing. Typically the solver will produce the drawing of a “transformed” figure in a position that is “quite a bit” different from the original configuration. On the other hand, in a DGS, deformations can be performed “continuously” and each new figure will automatically exhibit all the properties according to which the original figure was constructed. In this manner the solver does not have to keep track of all the conceptual components and reconstruct the figure after each move. Instead s/he can observe change and invariance through small perturbations of the figure, that is, dragging a base point “only a little” to explore the figure. This allows a different type of exploration that involves a “dialogue” with the software: while in the paper-and-pencil environment the “moves” have to be conceived mostly in the solvers’ head before s/he represents them on the paper, in a DGS the solver may use a trial-and-error technique using “small moves” and “continuous dragging”.

This difference may be particularly evident in explorations that involve the search for conditions under which a certain property is verified by a figure, as we will describe in Section 2.1.4.2. In a paper-and-pencil environment the solver may have to represent a sequence of images, each of which is “quite deformed” with respect to the previous one, and each image represents the previous figure after one of its elements have changed their position (and consequently all the elements that depend on this first one, since the conceptual properties of the figure must remain unvaried). In a DGS the sequence appears to be continuous and it is obtained by clicking on a base point of the initial dynamic-figure and dragging it along the screen. A potential regularity in the movement of the dragged-base-point may become evident to the solver at this point. Vice versa, in the paper-and-pencil environment the solver will have had to conceive the property corresponding to such regularity before redrawing the figure, in order to produce the discrete sequence of images.

2.1.4.1 Dragging to Find The *Conclusion* of a Conjecture. A key element for the interpretation of the Cabri-figure resides in the relationship between the properties defined during the construction of the figure, through the commands used and the properties that are consequences of these. A conjecture can emerge from the observation of the link between the properties that have been constructed and the properties that can be observed, but that have not been directly constructed, and that can be unexpected. This link can be interpreted as a conditional relationship expressed by a statement in which the constructed properties constitute the premise, while the "new" invariant properties observed constitute the conclusion. Naturally all this is referred to as the conservation of invariants with respect to dragging of any base point. In mathematical terms, this is equivalent to *exploring the consequences of a certain set of premises*. The premises are represented by the set of properties established by the commands used during the construction of the dynamic figure.

Although exploring the consequences of a certain set of premises has been the main focus of many studies in the literature, it is possible to use dragging for generating conjectures in a different way that involves the induction of *soft invariants* (Laborde, 2005). This corresponds to identifying *under which conditions a given configuration takes on a certain property* (as in Arzarello et al., 2002; Olivero, 2002). Our study focuses particularly on this use of dragging.

2.1.4.2 Dragging to Find the *Premise* of a Conjecture. Base points may be dragged in particular ways, for example in order to induce soft properties on a dynamic-figure (Laborde, 2005). In the example we have been analyzing above (presented in Section 2.1.4) it is possible to try to induce a soft invariant like "coinciding perpendicular bisectors".



a) b)

Figures 2.1.4.2a - b: The figures show the effect of dragging ABCD's base point C while trying to maintain the coincidence of the perpendicular bisectors.

In terms of invariants, identifying *under which conditions a given configuration takes on a certain property* means establishing the invariance of a particular property with respect to a particular movement, that is inducing a soft invariant. The special movement corresponds to the figure's assuming a specific condition. This way of dragging was initially described as *dummy locus dragging* (Arzarello et al., 2002). The model we are going to introduce aims at describing aspects of a process of conjecture-generation that seem to occur when this type of dragging is used.

2.2 The Instrumental Approach

In Chapter 1 we have discussed how the guiding role of dragging has been described in the previous literature and specifically by Arzarello et al., who introduced the classification of specific ways of dragging (Arzarello et al., 2002), and by Leung, who has provided an interpretation of dragging through the lens of variation (Leung, 2008). In the previous section we described aspects of dragging that are relevant to our theoretical framework. Although Arzarello's classification is not explicitly framed in the instrumentation approach, it is possible to consider dragging after the instrumentation approach (Vérillon & Rabardel, 1995; Rabardel & Samurçay, 2001), as has been done fruitfully by other researchers (for example, Leung & Lopez-Real, 2006; Leung, 2008; Strässer, 2009). Under the lens of the instrumental approach, dragging may be interpreted as an explorative tool that can support the task of conjecture-generation, and the use of which may be acquired through a process of *instrumental genesis* (Rabardel & Samurçay, 2001; Rabardel, 2002). This process occurs when an individual is confronted with a task and, having an artifact at his/her disposal, s/he develops specific *utilization schemes*. In this section we will introduce the notions of artifact, instrument, and utilization scheme, developed within the instrumental approach, (Section 2.2.1) and how we use them to interpret dragging (Section 2.2.2).

2.2.1 Artifacts, Instruments, and Utilization Schemes

The instrumental approach has been developed as a perspective that puts forward a psychological conceptualization of artifacts as instruments, with the aim of making "the conceptualization equally pertinent in ergonomics and in didactics" (Rabardel, 2002, p. 18). Rabardel conceives

> the instrument in the essence of its constituting relation: the subject's use of the artifact as a means he/she associates with his/her action. The point of view adopted will be that in which machines, technical objects, symbolic objects and systems, i.e. artifacts, will be considered as material or symbolic instruments. (Rabardel, 2002, p. 18)

In particular the instrumental approach may be used as a perspective through which to look at human-computer interactions. The instrumental approach "was developed so the user could have a view of the system in which people, machines, tasks and materials are seen as interconnected in a terminology founded in the realm of tasks significant to the user" (Rabardel, 2002, p.7).

Within this perspective a cognitive model is outlined that describes the integration of tools in different activities. The model introduces a crucial distinction between the tool itself (called artifact) and the combination of this tool and the modalities of its use to solve problems. We could give a very concise overview of the model as follows. The model assumes that for each subject the use of an artifact gives rise to a mental construction, called *instrument*, that denotes the psychological construct of the user: "a whole incorporating an artifact (or a fraction of an artifact) and one or more utilization schemes" (Rabardel, 2002, p.65). The user develops procedures and rules of actions when using the artifact and so s/he constructs *utilization schemes*, during a process of *instrumental genesis*. Our study focuses on describing a possible utilization scheme for the artifact "dragging" with respect to the task of generating a conjecture, but it will not take into consideration the process of development of the utilization schemes (that is the process of instrumental genesis). Therefore in the following we are going to focus our discussion on the notions of artifact, instrument, and utilization scheme, as developed within an instrumental approach.

<u>Artifact.</u> Different approaches aimed at analyzing human interaction with objects (we have called these artifacts until now) have referred to these "objects" of interaction as "technical objects", "material objects", or "artifacts". The psychological definition of the notion of instrument – that used within Rabardel's instrumental approach – replaced the term "technical object" with "Fabricated Material Object (FMO)" (Rabardel & Vérillon 1985), to be able to examine the technical object from other points of view than that of

the technique itself (Rabardel, 2002, p. 38-39). The terminology FMO was then replaced by the shorter, lighter, and more neutral word *artifact*:

> The term “fabricated material object” was chosen to allow the most neutral possible name and avoid anticipating the analysis perspective to be adopted. This undertaking seems even more essential today given the issues at stake in technocentric and anthropocentric design. But we feel the term fabricated material object, a heavy circumlocution, should now be replaced by that of artifact. This word is almost synonymous and its usage is fairly widespread, particularly in the field of human sciences (Rabardel, 2002, p.39).

The notion of artifact does not specify a particular type of relation to the object, nor is it necessarily a material object; it is “the thing liable to be used and elaborated so as to participate in finalized activities” (p. 39). Within the instrumental approach the artifact is analyzed in light of its functions, as a means of action, “placed in a finalized activity from the viewpoint of the person using it” (p. 41).

Finally, a central issue is the relations between human activity and artifacts, which can be analyzed, thanks to the notion of artifact described above, along two lines: design activities, to gain a better understanding of the mechanisms and processes by which artifacts are designed to provide designers with real aids that must integrate the activity rather than hinder or even prevent it; and usage and utilization activities, analyzing and understanding what these activities are from the perspective of the users themselves (Rabardel, 2002). Our study is situated within the second line, that of usage and utilization activities, since it aims to construct a model for particular utilization activities for the artifact “dragging” with respect to the task of generating a conjecture. Future studies that might arise from our findings should investigate the line of “design activities” for which our study can only provide some experimental hypotheses to be further elaborated and tested. We will now describe the notion of utilization scheme.

Utilization Scheme. When a person uses an artifact to accomplish a task, s/he structures the activity and actions in relatively structured ways. These have been referred to as *utilization schemes* (Rabardel & Vérillon, 1985). This notion makes use of the Piagetian construct of *action scheme*, “a structured group of generalizable features of the action that allows the same action to be repeated or applied to new contents” (Rabardel, 2002, p.65). Moreover, according to Piaget a scheme is

> a means that allows the subject to assimilate the situations and objects with which he/she is confronted They are the structures that prolong biological organization and share with the latter an assimilating capacity to incorporate an external reality into the subject’s organization cycle: everything that meets a need is liable to be assimilated. (Rabardel, 2002, p.70).

While Piagetian psychology was centered on logical structures, Vergnaud put forward a theory on conceptual fields, placing his reflection within cognitive psychology. He describes behavior organizing schemes, in which the subjects' knowledge in act (i.e. the cognitive elements that allow the subject's action to be operational) can be recognized. In particular, for Vergnaud (Vergnaud, 1990) a scheme comprises

- anticipations of the goal the subject is aiming for, and of the potential intermediate steps in this process;
- rules of action, like “if...then,” which allow the generation of a sequence of actions;
- inferences (reasoning) that allow the subject to calculate rules and anticipations based on information and the operational invariants system he/she disposes of;
- operational invariant, that allow the subject to recognize the elements pertinent to the situation, and to collect information on the situation being analyzed.

The instrumental approach makes use, in particular, of the notion of operational invariant, which “allows us to identify the characteristics of situations that subjects truly take into consideration. These may be familiar situations for which operational invariants are already constituted, or situations in which their elaboration is underway" (Rabardel, 2002, p.79-80).

Instrument. Finally, Rabardel conceives *instrument* from a psychological point of view, as a mixed entity, "a whole incorporating an artifact (or a fraction of an artifact) and one or more utilization schemes" (Rabardel, 2002, p.65). The instrumental approach sees the instrument as one of the poles engaged in instrument utilization situations. These are: the subject (e.g.user, operator, worker, agent), the instrument (e.g. tool, machine, system, utensil, product), the object towards which the action, aided by the instrument, is directed (e.g. matter, reality, object of the activity). The model of Instrumented-mediated Activity Situations (IAS) describes this situation (Rabardel & Vérillon, 1985) bringing out the multiplicity and complexity of relations and interactions between the different poles.

As in previous literature, within the instrumental approach, an instrument is conceived as an intermediary entity, "a medium term, or even an intermediary world between two entities: the subject, actor, user of the instrument and the object of the action" (Rabardel, 2002, p.63). Moreover,

> The instrument's intermediary position makes it the mediator of relations between subject and object. It constitutes an intermediary world whose main feature is being adapted to both subject and object. This adaptation is in terms of material as well as cognitive and semiotic properties in line with the type of activity in which the instrument is inserted or is destined to be inserted. (Rabardel, 2002, p.63).

The mediation may be of an epistemic nature – from the object to the subject, here the instrument is a means allowing knowledge of the object – or of a pragmatic nature – from the subject to the object, here the instrument is a means of a transforming action directed towards the object. Moreover, since instruments are conceived as a means of action, depending on the type of action, they may be *material instruments* (for a transformation of a material object with a hand-held tool), *cognitive tools* (for cognitive decision making, for example in a situation of managing a dynamic environment), *psychological tools* (for the management of one's own activity), or *semiotic tools* (for a semiotic interaction with a semiotic objector with others).

Because of the goals of this study we are particularly interested in instruments conceived within the instrumental approach, as cognitive tools. In studies that may be developed from ours, as consequences and continuations of our research, the notions of semiotic instrument and of psychological tool may become essential elements of the new theoretical frameworks. For now we will concentrate on the notion of cognitive tool, as a last element of this part of our framework.

Cognitive Tool. Some aspects developed in the instrumental approach can be recognized in Norman's notion of *cognitive artifact* (Norman, 1991). In particular he analyzes approaches to activity distinguishing several dimensions of influence of artifacts on the distribution of actions in time (precomputation), the distribution of actions among people (distributed cognition) and the changes in actions required by individuals in order to perform the activity. Moreover Norman suggests distinguishing between passive artifacts such as books and active artifacts such as computers, and he focuses on analyzing the object's influence on the tasks the user is facing (Norman, 1991).

Within Norman's perspective, activity is taken into consideration within a triadic model, similar to the IAS initially developed by Vérillon and Rabardel (1985). The triadic model is composed of a person, a task, and a cognitive artifact. Norman defines a cognitive artifact as "a device designed to maintain, display, or operate upon information in order to serve a representational function" (Norman, 1991). In particular, according to

Norman, a cognitive artifact has the role of changing the nature of the task performed by the person. Moreover, he conceives a cognitive artifact as something that expands and enhances the cognitive capabilities of its user.

The notion of cognitive tool is developed within the instrumental approach which goes beyond an approach in terms of tasks, taking into consideration the activity as well (Rabardel, 2002). In this sense a cognitive tool is a concept similar to Norman's notion of cognitive artifact, but enriched with an activity-centered perspective. We will consider the artifact "dragging" and describe how it can be conceived as a cognitive tool, and more in general as an instrument.

2.2.2 Dragging within the Instrumental Approach

In this study we consider dragging to be an artifact and place a user in the context of solving a problem, in particular of generating a conjecture (task). The solver can associate to the *dragging artifact* a variety of utilization schemes in order to accomplish the task of generating a conjecture, thus obtaining an *instrument*. We would like to highlight how the notion of "dragging schemes" developed from the original definition by Arzarello et al. (Arzarello et al., 2002) and how we will make use of it in this study. The terminology "dragging schemes" was used for the first time by Arzarello and his colleagues who gave an *a posteriori* description and classification of expert solvers' uses of the dragging mode, from a cognitive point of view. We described Arzarello et al.'s classification in Chapter 1, together with other dragging modalities, also referred to as "dragging stratagies" that have been identified by Leung and other researchers (Lopez-Real & Leung, 2006; Leung et al., 2006; Leung, 2008).

In this study we propose to describe, in further depth with respect to the previous research, cognitive processes associated to particular ways of dragging. We make a distinction between “ways of dragging” or “dragging modalities/strategies” and “dragging schemes” to separate what might be observed externally as a particular way of dragging from the description of a utilization scheme (an internal mental construct of the solver) associated to a particular way of dragging. In this sense our model proposes the description of a potential utilization scheme associated to the dragging modality *dummy locus dragging* or *lieu muet dragging*, as previous literature has described it (Arzarello et al., 2002; Olivero, 2002).

We mentioned how in the research that led Arzarello et al. to the cognitive description of the dragging modalities were determined *after* the observation of the solvers’ exploration. On the contrary, in order to study how the expert use of specific dragging modalities may influence the generation of conjectures, in this study we decided to introduce students to such modalities in order to observe the use that might be made of them. In Section 2.5 we will describe the dragging modalities we adapted from previous research and introduced to students through appropriate in-class activities that we will describe in the Chapter 3. Here, as far as the theoretical frame is concerned, we note that we conceived introducing our dragging modalities to the participants of the study as providing them with a cognitive tool, that might enhance their capabilities with respect to the task of conjecture-generation in a DGS. Moreover, we interpreted the dragging modalities as a potential instrument in the following sense. If the solvers developed appropriate utilization schemes – and in particular a utilization scheme associated to *dummy locus dragging* that we intend to describe through a specific model.

2.3 Abduction

In this section we describe the notions of abduction that we chose as theoretical tools for this study. A goal of this research is to unravel a possible relationship between particular dragging modalities and abduction that previous research has hypothesized (Arzarello et al., 2002; Olivero, 1999, 2002). Therefore we will consider the notion of abduction introduced by Peirce (1960), which was used by Arzarello, Micheletti, Olivero, Paola, and Robutti (Arzarello et al., 1998; Olivero, 2002; Arzarello et al., 2002) to analyze solvers' development of conjectures when their "dragging schemes" were being used. This is the notion we initially used to conceive our first hypothetical model. We will then highlight some problematic issues of this notion when analyzing abduction in conjecture generation, and how we therefore enriched our framework with another conception of "abduction", Magnani's more recent description, which is more in line with Peirce's description of abduction in the second phase of his work.

Other researchers have studied various uses of abduction in mathematics education (for example, Simon, 1996; Cifarelli, 1999, 2000; Reid, 2003; Ferrando, 2006), using different approaches with respect to that of Peirce. In particular, Cifarelli has studied relationships between abductive approaches and problem-solving strategies. The purpose of his work was to clarify the processes through which learners construct new knowledge in mathematical problem solving situations. He focused particularly on instances where the learner's emerging abductions or hypotheses help to facilitate novel solution activity (Cifarelli, 1999). The basic idea is that an abductive approach may serve to organize, reorganize and transform problem solvers' actions. Specifically, Cifarelli analyzed how the cognitive activity of "within-solution posing, in which one reformulates a problem as it is being solved" (Silver & Cai, 1996, p.523) may aid the solver to consider "hypothesis-based" questions and situations (Silver & Cai, 1996, p.529), and may aid the solvers to abduce novel ideas about problems during the solution process (Cifarelli, 1997, 1998, 1999, 2000). Although conjecture-generation in open problem situations may be seen as a form of problem solving, we do not analyze abduction with respect to solvers' reformulation of the problem they are solving, as in Cifarelli's studies, so our perspective is different with respect to that described above.

After presenting the example of analysis using Peirce's first conception of abduction (Section 2.3.1), we will present our considerations on abduction in conjecture generation that led to our use of the more general notion of abduction introduced by Magnani, along the lines of Peirce's later conception (Section 2.3.2). Moreover we found it useful to consider the distinction between "selective" and "creative" abduction and Hoffmann's distinction of abduction into six types (Hoffmann, 2007), together with Magnani's notion of manipulative abduction (Magnani, 2001).

2.3.1 Arzarello et al.'s Use of Abduction as a Tool of Analysis

Our study is grounded within the research of Arzarello, Micheletti, Olivero, Paola, and Robutti (Arzarello et al., 1998; Arzarello et al., 2002; Olivero, 2002), that made use of the following notion of abduction developed by Peirce.

> According to Peirce, of the three logic operations, namely deduction, induction, abduction (or hypothesis), the last is the only one "which introduces any new idea; induction does nothing but determine a value, and deduction merely evolves the necessary consequences of a pure hypothesis. Deduction proves that something must be; induction shows that something actually is operative; abduction merely suggests that something may be." (CP, 5.171). Abduction looks at facts and look for a theory to explain them, but it can only say a "might be", because it has a probabilistic nature. The general form of an abduction is:
>
> a fact A is observed
> if C was true, then A would certainly be true
> So, it is reasonable to assume C is true.
>
> An example illustrates this concept. Suppose I know that a certain bag is full of white beans. Consider the following sentences: A) these beans are white; B) the beans in that bag are white; C) these beans are from that bag. A deduction is a concatenation of the form: B and C, hence A; an induction would be: A and C,

hence B; an abduction is: A and B, hence C (Peirce called hypothesis the abduction). (Peirce, 1960, p.372).

In this section we will show an example of how this notion of abduction was used in these analyses. Our goal was to "zoom into" cognitive processes that occur in correspondence to what Arzarello et al. had described as "the most delicate cognitive point" of the conjecture generation and that Arzarello et al. characterized by the presence of an abduction (Arzarello et al., 1998, p. 30). In doing so, we found that the conception of abduction described above did not seem to always provide insight. Therefore we enriched our framework with the definition of abduction presented by Magnani (2001), which is also more in line with the conception that Peirce reached in the second phase of his thinking. We use this second conception of abduction more as a frame of reference to discuss the general nature of a process than as a tool of analysis, as we will describe in detail in Chapter 6.

In the following paragraph is an example of subjects' spontaneous use of dragging for investigating a given task. The analysis shows how the notion of abduction is used to look at the exploration, and it puts the subjects' use of dragging modalities in relationship to changes in cognitive levels of investigation.

Task: *Let ABCD be a quadrilateral. Consider the bisectors of its internal angles and their intersection points H, K, L, M of pairwise consecutive bisectors. Drag ABCD, considering all its different configurations: what happens to the quadrilateral HKLM? What kind of figure does it become?*

Episode 1: They use guided dragging in order to get different shapes of ABCD. Ascending control is guiding their experiments, as their aim is to get some conjectures about the configuration. The last step allows them to see a degenerate case: HKLM disappears into one point.
Episode 2: Now a regularity is discovered; so they use ***dummy locus*** **dragging**. They drag ABCD so to keep the property they have just found out. They are still

> in the stream of **ascending control**, as they are exploring the situation, but now they have a plan in their mind: they look for some common properties to all those figures which make HKLM one point.
> Episode 3: Even if the locus is not explicitly recognized by the students, it is this kind of dragging that allows them to discover some regularity of the figures. Here they make an abduction, because they select 'which rule it is the case of': this is the case of circumscribed quadrilaterals. Referring to the example by Peirce, one can say that: A is “the sum of two opposite sides equals the sum of the other two”, B is “a quadrilateral is circumscribed to a circle if and only if the sum of two opposite sides equals the sum of the other two”, i.e. something you know while C is “these quadrilateral are circumscribed”. Their reasoning is: A & B, then C. Once they have selected the right geometric property, they can 'conclude' that this is the case of circumscribed quadrilaterals. The conditional form is virtually present: its ingredients are all alive, but their relationships are still reversed, with respect to the conditional form; the direction after which the subjects see things is still in the stream of the exploration through dragging, the control of the meaning is ascending, namely they are looking at what they have explored in the previous episodes in an abductive way. The direction of control now changes: here students use the construction modality (and the consequent dragging test) to check the hypothesis formulated through abduction and at the end they write down a sentence in which the way of looking at figures has been reversed. By *dummy locus* dragging, they have seen that when the intersection points are kept to coincide the quadrilateral is always circumscribed to a circle. Now they formulate the conjecture in a logical way, which reverses the stream of thought: if the quadrilateral is circumscribed then the points coincide.
> Episode 4: At the end they check their conjecture. Now they are using the dragging test and their actions show descending control. (Arzarello et al., 2002).

We highlight Arzarello et al.’s analysis of the abduction:

> Referring to the example by Peirce, one can say that: A is “the sum of two opposite sides equals the sum of the other two”, B is “a quadrilateral is circumscribed to a circle if and only if the sum of two opposite sides equals the sum of the other two”, i.e. something you know while C is “these quadrilateral are circumscribed”. Their reasoning is: A & B, then C.

After introducing elements that we used to enrich our framework with respect to the notion of abduction, in the next section we will re-analyze the exploration described above. This way we hope to show how we inherited the conception of abduction present in Arzarello et al.’s framework and enriched it with elements that help gain further insight into abduction in conjecture-generation.

2.3.2 Abduction in the Formulation of Conjectures

Let us consider the first definition of abduction given by Peirce.

> a fact A is observed
> if C was true, then A would certainly be true
> So, it is reasonable to assume C is true. (Peirce, CP 5.189)

Using Peirce's conception of abduction described above, we needed to establish what the *product* of the abduction was in the case of conjecture-generation. Is it what Peirce called the "abductive hypothesis" (C with respect to the definition above)? or is it the "rule" (B with respect to the definition above), which can be a conditional statement containing the abductive hypothesis itself? Peirce discussed this issue in the following terms: "The hypothesis cannot be admitted, even as a hypothesis, unless it be supposed that it would account for the facts or some of them." Therefore A can be abductively conjectured only when its entire content is already present in the "rule" 'If A were true, C would be a matter of course'" (CP 5.189), which shows how the phenomenon would be produced, come about, or result in case the abductive hypothesis A were true. An abduction may "consist in making the observed facts natural chance results, as the kinetical theory of gases explain facts; or it may render the fact necessary" (CP 7.220).

We therefore elaborated our framework taking into consideration another description of abduction. Starting from a later characterization provided by Peirce, that is abduction as "the process of forming an explanatory hypothesis" (Peirce, CP 5.171), Magnani proposed the following conception of abduction:

> the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated. (Magnani, 2001, pp. 17-18).

While using Peirce's first definition illustrated by the example of the bag of beans, the product is the abductive hypothesis, a fact (these beans are from that bag), while choosing Magnani's conception of abductive process, we may consider the product to be the *conditional link* between the hypothesis and the observation (if these beans are from that bag, then they are white, what Peirce called "rule"). The conditional link is by all means an "explanatory hypothesis" in Magnani's words, developed to explain a situation as a whole. In the context of dynamic geometry, in the process we studied, this rule arises from capturing the logical dependence of two (or more) invariants. When solvers explore an open problem situation in dynamic geometry and are asked to formulate conjectures on a certain geometrical object, they frequently notice invariants, that is, properties of the figure that remain constant during the dragging of a point. Through a conjecture the students try to logically link two (or more) of such geometrical invariants, finding an "explanatory hypothesis" for the observed phenomenon.

Therefore the solver's perceiving one invariant can lead to the observation of another, and to the idea that this second one might explain the first. The final product of the abductive process, in this case, is the statement of a geometrical conjecture. If we describe the process as a whole, from the initial random dragging of points to the formulation of a conjecture, and therefore consider the final *conjecture* to be the final product of the process, Magnani's description seems to be appropriate. In fact in an open problem what is required as an answer is a statement expressing the conditional link between the hypothesis and the observation. If instead we "zoom in" and focus on the steps at which the students find a second invariant (that seems to be invariant when the first invariant is maintained), and (implicitly) link it to the first, stating, for example: "This property is true [the property is the second invariant]" we claim that an abduction has occurred and that the statement "This property is true [the property is the second invariant]" is Peirce's abductive hypothesis. We note that this statement is not a simple observation of another invariant of the figure (which can also occur), but instead a tentative explanation (not yet in the form of a conjecture) of why the first invariant is maintained. This can be seen in various protocols when the students express themselves using phrases like: “Because/since/every time (Italian: “poiché, ogni qualvolta”) this property is true [the property is the second invariant], this property is true [the property is the first invariant]”; or like: “In order that (Italian: “affinchè”)/so that (Italian: “perché”) this property is true [the property is the first invariant], this property is/has to be true [the property is the second invariant].”

A second issue we took into consideration in analyzing possible abductions was the fact that the formulation of a conjecture requires *generating* the rule itself, and this may occur in different ways.

Selective and Creative Abduction. Generation of the *rule* in the abduction may occur through different modalities. Let us start by considering Peirce’s bean example, again, which seems similar to examples that may be found in Eco (1983; Meyer, 2010), such as: I see smoke, I know that when there is smoke there is a fire, so there is a fire. Notice that “I know that when there is smoke there is fire” is analogous to “the beans in that bag are white” in that these are rules that come from a knowledge set that a particular person assumes to be true. In these cases one is finding the rule in one’s “bag of already-known rules” that fits the initial observation (fire or white beans). On the other hand, especially when generating a conjecture, the rule introduced by the solver may not belong to his/her “bag of already-known rules”.

We may phrase the question as whether abduction is the logic of *constructing* a hypothesis, or the logic of *selecting* a hypothesis from among many possible ones. Peirce analyzed this issue, as well, and seemed to treat the logic of constructing a hypothesis versus that of selecting a hypothesis as the same question. In fact in some of his writings he maintains: “Abduction consists in studying facts and devising a theory to explain them” (CP 5.145); “Abduction is the process of forming an explanatory hypothesis” (CP 5.171); or abduction “consists in examining a mass of facts and in allowing these facts to suggest a theory” (CP 8.209). However in other writings he regards abduction as “the process of choosing a hypothesis” (CP 7.219). We found it useful to consider Meyer’s description of two general patterns of abduction, based on Peirce’s description of abduction as: “The surprising fact, C, is observed; But if A were true, C would be a matter of course. Hence, there is reason to suspect that A is true” (Peirce, CP 5.189). Meyer describes two general patterns of abduction, that can be represented as follows.

$$\begin{array}{ll} \text{result:} & R(x_0) \\ \hline \text{rule:} & \forall i: C(x_i) \Rightarrow R(x_i) \\ \text{case:} & C(x_0) \end{array} \qquad \begin{array}{ll} \text{result:} & R(x_0) \\ \text{rule:} & \forall i: C(x_i) \Rightarrow R(x_i) \\ \hline \text{case:} & C(x_0) \end{array}$$

Figure 2.3.2.1: Two general patterns of abduction.

The first case represents the cognitive ‘flash of genius’, while the second represents abduction as a process of making a hypothesis plausible (Meyer, 2008, p. 2). The first form of abduction – when a new rule emerges – has been described as “creative” by Eco (1983, p.207). On the other hand Eco describes “undercoded“ or “overcoded” abductions as those in which the explanation of given facts occurs through already-known rules. Thus the generation of one discovery can imply a) a new case (all kinds of abduction), b) the relationship between the observed facts and the associated or the generated rule (all kinds of abduction) and c) a new rule (by a creative abduction). As these aspects can only be hypothetical at first place, they have to be verified in the next step (Meyer, 2008). Magnani refers to these two forms of abduction as “selective” and “creative”: selective abduction is a process through which the right explanatory hypothesis is found from a given set of possible explanations, while creative abduction is a process which generates the (right) explanatory hypothesis (Magnani, 2001).

Moreover, Hoffmann, viewing abduction as the generation of a new idea (Hoffmann, 2007), considered two issues that we found relevant with respect to the analyses we needed to make. These issues are: (1) whether the ideas we introduce in the abduction is only new for us as individuals or new for our civilization, or not new at all; (2) whether the idea is the result of a reification, that is something that can be represented by a singular concept, or by a symbol, or a new *perspective* on the same data as produced by a theoretical transformation (Hoffmann, 2007, p. 4). Based on Peirce's work, Hoffmann proposed a distinction of six types of abduction based on combining the different issues.

For our research it was important to focus on the solver's perspective. Therefore we did not need to take into consideration whether the idea was already part of the culture's knowledge or not. We simply considered whether the idea was new or not for the solver, and used this distinction to characterize selective versus creative abductions. We interpreted Hoffmann's theoretical transformation, as a movement between different contexts, for example, a change of the theory used to explain certain facts. This can change aspects of the explanation, such as the systems of representation used, the types of arguments used, and the domain of their validity. We therefore created a template, adapted from Hoffmann's table (2007, p. 4), aimed at classifying different types of abduction. The template is displayed the following Table (2.3.2.2).

	"idea" based on reification within a <u>single context</u>	*"idea" based on combining different perspectives on data (passing between <u>different contexts</u>)*
the explanation is possible by referring to an idea already in the solver's mind (<u>selective abduction</u>)		
the explanation is possible by referring to an idea that is new for the solver (<u>creative abduction</u>)		

Table 2.3.2.2: Our template for the analysis of abduction, adapted from Hoffmann's table.

Finally, we added a last notion to our theoretical framework, that of Magnani's manipulative abduction (Magnani, 2001, 2004):

> Manipulative abduction happens when we are thinking *through* doing and not only, in a pragmatic sense, about doing. It refers to an extra-theoretical behavior that aims at creating communicable accounts of new experiences to integrate them into previously existing systems of experimental and linguistic (theoretical) practices. Gooding (1990) refers to this kind of concrete manipulative reasoning when he illustrates the role in science of the so-called "construals" that embody tacit inferences in procedures that are often apparatus and machine based. The embodiment is of course an expert manipulation of objects in a highly constrained experimental environment, and is directed by abductive movements that imply the strategic application of old and new *templates* of behavior mainly connected with extratheoretical components, for instance emotional, esthetical, ethical, and economic. (Magnani, 2004, p.2).

We then used the framework we constructed to re-analyze some of Arzarello et al.'s data, before using it for analyzing our own data. In the next section we will show an example of what our re-analysis of Arzarello et al.'s data led to. As will be discussed in Chapter 6 the interpretation of our results within this framework finally led us to conceiving a new form of abduction that we described as *instrumented abduction*.

Analysis of Arzarello et al.'s example through our new framework. In the example of Arzarello et al.'s cognitive analysis of dragging we highlighted their description of the abduction:

> Referring to the example by Peirce, one can say that: A is "the sum of two opposite sides equals the sum of the other two", B is "a quadrilateral is circumscribed to a circle if and only if the sum of two opposite sides equals the sum of the other two", i.e. something you know while C is "these quadrilateral are circumscribed". Their reasoning is: A & B, then C. (Arzarello et al., 2002)

With respect to Meyer's description of the two patterns of abduction, we can classify this abduction differently, depending on whether the solvers knew the rule "a quadrilateral is circumscribed to a circle if and only if the sum of two opposite sides equals the sum of the other two" from their theoretical knowledge or not. In the following diagram the two possible classifications are explained.

Selective abduction	Creative abduction
Result: The sum of two opposite sides is equal to the sum of the other two. _________________________ **Rule**: A quadrilateral can be circumscribed if and only if the sum of two opposite sides is congruent to the sum of the other two **Case**: These quadrilaterals are circumscribable	**Result**: The sum of two opposite sides is equal to the sum of the other two. **Rule**: A quadrilateral can be circumscribed if and only if the sum of two opposite sides is congruent to the sum of the other two _________________________ **Case**: These quadrilaterals are circumscribable

Figure 2.3.2.3: Our template for the analysis of abduction, adapted from Hoffmann's table.

Moreover, using the template introduced above, we may place this abduction in one of the cells of the first column of our table. This is the case because the idea resides entirely in the domain of the Theory of Euclidean Geometry (TEG), that is, in a single context.

	"idea" based on reification within a <u>single context</u>	*"idea" based on combining different perspectives on data (passing between <u>different contexts</u>)*
the explanation is possible by referring to an idea already in the solver's mind (<u>selective abduction</u>)	Example: the abduction described in Arzarello et al's analysis (selective form)	
the explanation is possible by referring to an idea that is new for the solver (<u>creative abduction</u>)	Example: the abduction described in Arzarello et al's analysis (creative form)	

Table 2.3.2.4: Placement of examples in the literature within our template for the analysis of abduction.

However, if we continue analyzing the exploration according to our conception of abduction in conjecture-generation, we can observe a second inference that we would classify as an abduction. In particular, it seems to be the "invention of a rule".

- The solvers observe a first fact: the internal quadrilateral "collapses" in these cases;
- they observe a second fact: the sum of two opposite sides is equal to the sum of the other two. The two facts occur simultaneously.
- The solvers introduce a rule (they did not know): if a quadrilateral has the sum of two opposite sides congruent to the sum of the other two, the quadrilateral formed by the intersections of the internal bisectors collapses.

If we look at the process of conjecture-generation as a whole, leading to the statement of a conjecture as the final product, the process could be illustrated as follows.

rule: if a quadrilateral has the sum of two opposite sides congruent to the sum of the other two ⇒ the quadrilateral of the intersections of the internal angle bisectors collapses

Abduction:

observed fact: The sum of two opposite sides is equal to the sum of the other two.

____

rule (from TEG): A quadrilateral can be circumscribed if and only if the sum of two opposite sides is congruent to the sum of the other two

hypothesis: These quadrilaterals are circumscribable

conjecture: If the quadrilaterals are circumscribable ⇒ the quadrilateral of the intersections of the internal angle bisectors collapses

Figure 2.3.2.5: Description of the process of conjecture-generation as a whole.

If we were to place the conjecture, as the product of an abduction in our table, it would go in the cell that represents a creative abduction in which there is a passage between contexts.

	"idea" based on reification within a single context	*"idea" based on combining different perspectives on data (passing between different contexts)*
the explanation is possible by referring to an idea already in the solver's mind (selective abduction)	Example: the abduction described in Arzarello et al's analysis	
the explanation is possible by referring to an idea that is new for the solver (creative abduction)		Example: the product of the process described in our re-analysis of Arzarello et al.'s example

Table 2.3.2.6: Placement of the abductions described above within our template.

The framework we elaborated with respect to the notion of abduction helped us analyze this delicate process in the context of conjecture-generation. In this context the framework was enlightening because it allowed us to unravel aspects of a particular abductive process involved in conjecture-generation when *dummy locus dragging* is used by the solver. In particular, this framework led us to a new conception of the form of abduction used in the complex process analyzed in our study, that of instrumented abduction (Chapter 6).

2.4 The Initial Version of the Model

This section presents our first ideas for a model that could potentially describe a process of conjecture-generation when *dummy locus dragging* is used by the solver. In order to test the validity of these initial ideas, we tried to use them to analyze descriptions of students' work contained in the research by Olivero, Arzarello, Paola, and Robutti (Olivero, 2000; Arzarello, et al., 2002). This led to an initial version of the model that we describe here together with an example of how it can be used to analyze a hypothetical exploration of one of the activities we developed for the study. Although there are similarities between the analysis of the exploration we present here and Arzarello et al.'s examples of their cognitive analysis of dragging, a distinguishing feature of our research is that it does not attempt to *classify* students' activity but instead to describe cognitive processes involved in a process of conjecture-generation that are associated to particular ways of dragging. In particular we concentrate on the potential abductive reasoning that may occur in relation to certain dragging modalities, with particular focus on the details of cognitive processes related to *dummy locus dragging* that may occur during this conjecturing stage. We noticed how an elaboration of our model could be complementary to Olivero's work, since it could evolve into a refined description of a process, which takes place and is present in Olivero's episodes. While Olivero focused on students' different uses of the dragging tool during the development of a conjecture, our model focuses on the mental process that might take place in relation to the use of such dragging modalities (especially of dummy locus dragging).

From the perspective of the instrumental approach, our model attempts to describe a utilization scheme for a particular way of dragging, dummy locus dragging. Moreover, a difference with respect to previous research is that we preliminarily *introduce* solvers to certain dragging modalities, "giving them as an artifact to be used in solving geometrical open problems". This allows us to study a particular utilization scheme associated to the artifact and constructed by the solvers with respect to the general task of conjecture-generation. The decision of introducing certain dragging schemes to the solvers brought us to reason upon which dragging modalities to introduce, and how to introduce them.

In the following sections we will describe the framework within which we constructed our model, discuss our hypothesis on what introducing certain dragging modalities would lead to, introduce the first version of our model, and finally describe the dragging modalities we decided to introduce and the terminology we used to introduce them.

2.4.1 Constructing the Model and Our Hypothesis on Introducing Dragging Modalities

A goal of this study was to describe, from a cognitive point of view, a process of conjecture-generation when a particular dragging modality was used by the solver. The construction of a model describing such process, if possible, seemed to be the best way of finding answers to accomplish this. In this section we will describe our rationale with respect to this decision, and then explain our hypothesis on how the introduction of certain dragging modalities would facilitate our study.

The idea of constructing a model of the structure of thought, or cognitive model, can be found in Piaget's introductory chapter to *The Child's Conception of the World* (1929). Referring to Piaget's work, Ginsburg (1981) describes how the investigation of activities of the mathematical mind should have three aims: "the *discovery* of cognitive activities (structures, processes, thought patterns, etc.), the *identification* of cognitive activities, and the evaluation of levels of *competence*" (p. 5). As cognitive activities are *discovered*, a model may be constructed (and successively refined as more is discovered), then such model becomes functional to identifying the cognitive activities when they occur since it provides a lens through which these can be seen and discussed. Moreover, the model may be used by an external observer/researcher/teacher to evaluate the solver's level of competence in tasks that involve cognitive processes described by the model.

Since we wanted to "zoom into" certain cognitive aspects of the process of conjecture-generation we aimed to describe, and these aspects were related to the use of *dummy locus dragging* which in the literature was described as a dragging modality spontaneously but rarely used by students (Arzarello et al., 2002), we conceived a hypothesis that might allow us to observe more occurrences of this dragging modality. In a way our hypothesis would hopefully lead to an "unnatural" experimental setting in which we would be able to observe many more occurrences of our desired phenomenon than in a "natural setting". Of course this hypotheses about introducing particular dragging modalities not only has consequences with respect to potentially observed phenomena, but also, and more importantly, it has potential didactical consequences that we will discuss within this thesis, in particular in Chapter 7.

In order to introduce our model, let us start with an example of solvers' use of dragging modalities in conjecture-generation, described by Olivero, Arzarello, Paola, and Robutti (Olivero, 2000; Arzarello, et al., 2002).

Task: *Construct two points (A, B) and a third point C so that the angle ACB is 60 degrees. Are there other choices of C for which this is possible? Make a conjecture*.

> You can start to drag C (wandering dragging). You notice that there are other places on the screen in which the angle ACB is 60 degrees, so you start to drag trying to maintain this property (guided dragging). You start to "see" a path along which you can drag C and maintain the property, so you stay along it (lieu muet) ... You might decide to mark a few points along the path in order to visualize the path more explicitly (line dragging). The path looks like two arcs of circles through A, B. Now you make a conjecture: "If C is on the greater arc of the two circles through A, B (as drawn below), then the angle ACB is 60 degrees." To draw the circles and test your conjecture you need to know more about how to draw the circles (Olivero, 2000).

Based on Arzarello et al.'s analyses, similar to the one above, and on some preliminary observations we carried out, we developed a schematic description, through four steps, of what might occur during the conjecturing stage as a solver approaches an open problem in the Cabri environment, not having been introduced to the dragging modalities.

Step 1: experimentation with transformational reasoning and discovery of different dragging strategies

↓

Step 2: conscious use of different dragging strategies to further investigate (in particular *dummy locus* dragging)

↓

Step 3: abduction using the *path*

↓

Step 4: formulation of a conjecture (through an inversion of the abduction)

Figure 2.4.1.1: Our first schematic description of solvers' explorations.

Our hypothesis about introducing solvers to particular dragging modalities (especially *dummy locus dragging*) is the following:

> *By introducing subjects to the dragging strategies during activities before the assigned open problems, step 2 can be directly induced. That is, the types of reasoning that occur in subjects who spontaneously become familiar with the dragging strategies are analogous to those of subjects who have been given the dragging strategies "a priori".*

Preliminary observations and a pilot study seemed to confirm our hypothesis, and the initial version of the model we will describe below. Therefore we developed our study upon this framework. We will now introduce our initial model and provide an example of how it could potentially be used as a tool of analysis.

2.4.2 Our Initial Model and an Example

We built our initial model making the hypothesis that after being introduced to particular dragging modalities, in particular *dummy locus dragging*, solvers would proceed more or less as described in the steps introduced in the previous section. In particular, the initial version of our model is described below.

Step 1: conscious use of different dragging strategies to investigate the situation after *wandering dragging*, in particular *dummy locus dragging* to maintain a geometrical property of the figure (we name such property *intentionally induced invariant*, or III), and use of the *trace tool*.

Step 2: consciousness of the *locus* that appears through *dummy locus dragging* this marks a shift in control from ascending to descending – and description of a second invariance (*invariant observed during dragging*, or IOD).

Step 3: hypothesis of a conditional link between the *intentionally induced invariant* and the *invariant observed during dragging*, to explain the situation. Other forms of dragging may be performed: *line dragging*, *linked dragging*, and the *dragging test*.

Step 4: the formulation of a conjecture of the form 'if IOD then III' emerge as product of abduction.

We used this first version of the model for preliminary observations of solvers' conjecturing process and on hypothetical explorations. These seemed to show that the model was indeed appropriate. Below is an example of a hypothetical exploration analyzed through our initial model.

The activity is one of the activities we developed for the study. We introduce these activities in Chapter 3.

Activity: Draw three points A, M, K, then construct point B as the symmetric image of A with respect to M, and point C as the symmetric image of A with respect to K. Construct point D as the symmetric image of B with respect to K. Drag M and make conjectures about ABCD. Then try to prove your conjectures.

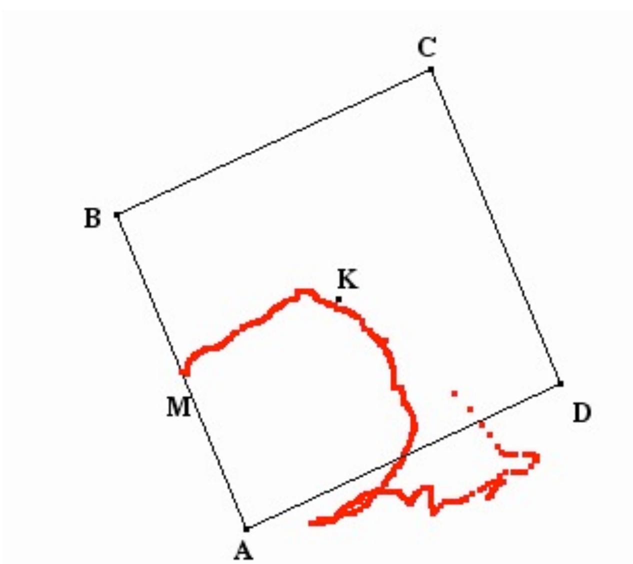


Figure 2.4.2.1: Dragging with the trace tool can help a student notice a *locus* (or *lieu*).

A Response: Through *wandering dragging* solvers may notice that ABCD can become different types of parallelograms. In particular, they might notice that in some cases ABCD seems to be a rectangle (they can choose this as the III). With the intention of maintaining this property as an invariant, solvers might mark some configurations of M for which this seems to be true, and through the trace tool, try to drag maintaining the property, as shown in Figure 2.4.2.1. This can lead to noticing some regularity (IOD) in the movement of M, which might lead to awareness of an object along which to drag (the circle of diameter AK, potentially not yet recognized as "a circle"). At this point, when such awareness arises, we can speak of *path* with respect to the regularity of the movement of M.

If solvers recognize the path to be a familiar geometrical object, like in this case, they might be inclined to constructing it, as shown in Figure 2.4.2.2, and dragging along it (*line dragging*), or even linking the free point to it (*linked dragging*) and performing a *dragging test*. Through this abductive process, as an attempt at explaining the experienced situation, as Magnani (2001) describes, solvers may hypothesize a conditional link between the III and IOD. At this point the abduction leads to a hypothesis of the form 'if IOD then III', and therefore to a conjecture like the following: "If M is on the circle of diameter AK, then ABCD is a rectangle," or (if they discover or derive a property of the base-points which is equivalent to M lying on the circle): "If AKM is a right triangle, ABCD is a rectangle."

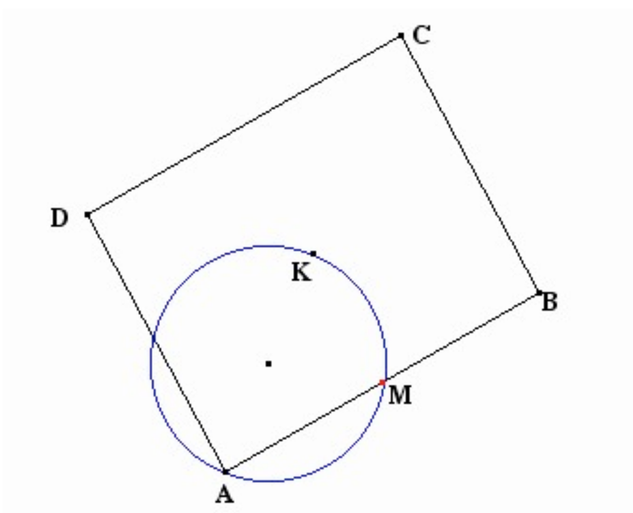


Figure 2.4.2.2: M is being dragged along the path (*line dragging*).

In the case of the first conjecture, here is how we hypothesize the abduction (*creative abduction*) might go.

- III: ABCD is a rectangle.
- IOD: when M dragged along the *path*, fact A seems to be true. The path is a known geometric figure*:* the circle of diameter AK.
- Product of the abduction: If point M lies on the circle of diameter AK, ABCD is a rectangle.

This product of the abduction coincides with a formulation of a conjecture. However, solvers might also perform a second abduction (this time a *direct abduction*) linking the property "M belongs to the circle" to a property of the base-points of the construction. In this case this may lead to a formulation of the conjecture like: "If the triangle AMK is a right triangle (with ∠AMK as the right angle), ABCD is a rectangle." In this case the further elaboration of the geometrical properties recognized in the path will have led to a key idea (Raman, 2003) of a possible proof.

<u>The Notion of Path.</u> The example of Olivero's analysis (2000) we introduced in Section 2.4.1 contained a reference to how *dummy locus* dragging involves dragging along a hidden *path*. Such a path can then be made explicit and it can be used for line dragging, linked dragging, and the dragging test. We focused on the role of such path, imagining that some pre-conceived notion of it may guide the solvers' production of the conjecture, and though it may play a role in the abduction, it may no longer appear in the formulation of the conjecture. However at this preliminary phase of elaboration of the model we did not explicitly define the term *path*, since we wanted to reach a definition as a potential result of the study. Here path refers to "what can be made explicit" through the trace mark, and we used the terms in informal discussions as a synonym of

"trajectory" as used by Arzarello et al. (2002), "set of points with a property", or "locus of validity", as used by Leung and Lopez-Real (2002). In this section, we will try to introduce our initial considerations on the concept of *path* and its significance for the model.

One of the dragging schemes, *dummy locus dragging*, involves dragging a point with the intention of maintaining a given property of the figure (which becomes the III). Some regularity may appear during this dragging stage, leading to the discovery of particular constraints that the dragged point has to respect (that can be expressed in the IOD). Because of their origin from dragging, such constraints may be interpreted as the property of the point to move on a particular trajectory (to belong to a particular figure). In mathematical terms, we can speak of a hidden locus (*dummy locus*). However we note that it does not necessarily coincide with the mathematical notion of locus, which would be the set of *all* points of the plane that guarantee verification of the III when the base point is chosen from within the set. Instead the path may be a proper subset of the locus of points with the characterizing property. The path can be made explicit by the trace tool, through which it appears on the screen. During *dummy locus draggin*g the solver notices regularities of the point's movement and conceptualizes them as leading to an explicit object. We refer to this object as a *path* when the solver gains consciousness of it, as it is generated through dragging, and consciousness of its property that if the dragged point is on it, a geometrical property of the Cabri-figure is maintained. In this sense a *path* is the reification (Sfard, 1991) of an abstract idea, similar to that of *locus*, that can be used to "control the figure", in a "descending control" mode (Arzarello et al., 2002). Zooming into Step 2, above, we observe that this is the point of the process in which the notion of *path* arises, and we can add a step to indicate the (potential) geometric interpretation of the *path*, in order to (potentially, after Step 3) perform *line dragging*, *linked dragging*, and the *dragging test along such path*.

We expected the *path* to play an important role in relation to the abductive processes that originate a conjectures in a DGS. In particular, we advanced the hypothesis that recognizing a *path* might be necessary to foster the formulation of a conjecture. although it may no longer explicitly appear in the formulation of the conjecture.

2.5 Dragging Modalities to Be Introduced in the Classroom

The hypothesis on the effect of introducing particular ways of dragging implied, at a theoretical level, an explicit distinction between *dragging schemes* and *dragging modalities* in order to be consistent with an instrumental approach (Vérillon & Rabardel, 1995; Rabardel & Samurçay, 2001; Rabardel, 2002) to dragging. In Section 2.2.2 we made a distinction between "ways of dragging" or "dragging modalities" and "dragging schemes" to separate what might be observed externally as a particular way of dragging from the description of a utilization scheme (an internal mental construct of the solver) associated to a particular way of dragging. Moreover, at a practical level, our hypothesis implied an elaboration of specific dragging modalities to be introduced. In this section we will describe the dragging modalities we chose to introduce solvers to, and the terminology we elaborated to do so.

"Our" Dragging Modalities. In order to determine the dragging modalities to be introduced to students, we elaborated Arzarello et al.'s findings (Arzarello et al., 2002). We considered dragging modalities that seemed particularly significant for the investigation for the solution of open problems and that could be easily introduced as tools to solve them. The four modalities we elaborated are described below:

- wandering/random dragging (Italian: "trascinamento libero"): randomly dragging a base point on the screen, looking for interesting configurations or regularities of the Cabri-figure;
- maintaining dragging (Italian: "trascinamento di mantenimento"): dragging a base point so that the Cabri-figure maintains a certain property;
- dragging with trace activated (Italian: "trascinamento con traccia"): dragging a base point with the trace activated;
- dragging test (Italian: "test di trascinamento"): dragging free or semi-free points to see whether the constructed figure maintains the desired properties. In this mode it can be useful to make a new construction or redefine a point on an object to test a formulated conjecture.

We described *wandering dragging* to the students as randomly dragging a base point on the screen. However we made it explicit that this mode could be used to look for interesting configurations or irregularities of the Cabri-figure. In this sense this scheme is a sort of fusion between *wandering dragging* and *guided dragging,* described by Arzarello et al. (2002). Once a particularly interesting potential property of a figure is detected, the user can use *maintaining dragging* (MD) to try to drag a base point and maintain the interesting property observed. For example, the solver may notice that a certain quadrilateral, part of the Cabri-figure, can "become" a square, and thus attempt to drag a base point trying to maintain the quadrilateral a square. In other words, *maintaining dragging* (MD) involves the recognition of a particular configuration as interesting, and the user's attempt to induce the particular property to become an invariant during dragging. Using Healy's terminology (2000) such invariant is a *soft* invariant.

We chose this terminology as opposed to *dummy locus dragging* (Arzarello et al., 2002) because we did not want the name to suggest any particular mathematical idea (for instance that of locus) to the students. Moreover, our definition of *maintaining dragging* differs slightly from what in literature has been referred to as *dummy locus dragging*. In literature this dragging modality is described as "wandering dragging that has found its path", a *dummy locus* that is not yet visible to the subject (Arzarello et al., 2002, p. 68). Instead, we consider *maintaining dragging* to be the mode in which a base point is dragged with the specific intention of the user to maintain a particular property.

With *dragging with trace activated* we intend any form of dragging after the trace function has been activated on one or more points of the figure. This tool arises from the combination of two Cabri functions, "dragging" plus "trace", which together constitute a *new* global tool that can be used in the process of conjecture-generation. Combining maintaining dragging with the trace activated on the selected base point can be particularly useful during certain processes of conjecture-generation. Although during the introductory lessons we did not explicitly specify particular points to activate the trace on, we only proposed to activate it on the base point selected to be dragged.

Finally the *dragging test* refers to a test that a figure can be put through in order to verify whether it has been properly constructed or not (Olivero, 2002; Laborde, 2005). The *dragging test* after having reconstructed the figure we were investigating, adding a new property (by construction) to it that we had hypothesized might induce the original interesting soft invariant to become a robust invariant. Thus the dragging test was applied to test whether the originally desired property was actually maintained during dragging. An expert might say we were using the dragging test to test a conjecture, even if the statement of such conjecture might not have been explicit at that point. In this sense the dragging test we introduced was slightly different from the one introduced in literature. We introduced the dragging test in a broader way, without constraining the properties to be observed during the test to necessarily being robust (Healy, 2000). In fact we consider the *dragging test* to be the dragging mode in which a base point is dragged with the intention of observing two invariant properties (which may be soft) simultaneously. We view this dragging mode as distinct from *maintaining dragging* because in this mode the two invariants that the user intends to observe have already been explicitly identified.

2.6 The (Specific) Research Questions

Given the theoretical framework we developed and presented in this Chapter, we now introduce the specific research questions we set out to investigate through our study.

1. What relationship do the forms of reasoning used by solvers during the conjecturing stage of an open problem in a DGS, have with the ways in which solvers use the dragging tool?
2. When a solver engages in the activities proposed in this study within a DGS there seems to be a common *process* used to generate conjectures through use of *maintaining dragging (MD)*.
    a. Does our model describe this process adequately?
    b. How does the model describe the dragging scheme and how can we refine the description?
    c. What insight into the process of conjecture-generation can be gained when using our model as a tool of analysis for solvers' explorations?

d. What is the role of the *path* within this model? Moreover is the path, as a part of the model, a useful tool of analysis?

e. How does the model highlight abductive processes involved in conjecture-generation when MD is used?

3. In cases where students do not use MD, is it possible to outline how they might develop effective use of MD?

CHAPTER III

METHODOLOGY

In Chapter 1 and Chapter 2 we described the literature in the field with respect to the initial problem we were interested in, and developed a theoretical background for this study from existing theoretical constructs elaborated in other research studies. This allowed us to reach a detailed set of research questions focusing on forms of reasoning and associated dragging modalities potentially used by solvers during the conjecturing stage of an open problem in a DGS. In particular we wanted to focus on a possible *process* of conjecture-generation that might be common to various solvers who use particular dragging modalities, “zooming into” solvers’ use of *maintaining dragging,* and relating it to some cognitive processes involved. We decided to do so by constructing and refining a model that describes a specific processes of conjecture-generation that may be carried out when the solver uses maintaining dragging. We described our initial model in our theoretical background (Chapter 2) of this study. In this chapter we will describe our methodological choices for the study.

In particular, in Section 3.1 we will discuss our choice of methodology for the study, briefly introducing the methodological tools of clinical interviews and teaching experiments, and explain the rationale for our choice. In this section we will also illustrate the experimental design of our study. Then, in Section 3.2 , we will explain how our data were collected, describing in detail how we made use of the methodological tools we chose. This section includes a description of the introductory lesson, how we modified it after the pilot study, and how we carried out the semi-structured clinical interviews in the pilot study and in the final study. Finally we provide an a priori analysis of an activity proposed during the interviews. In Section 3.3 we describe the data collected and how they were analyzed, focusing on the outcomes of the different ways in which they were analyzed.

3.1 Choosing a Methodology

Our study aims at investigating and describing particular cognitive processes related to dragging and involved in conjecture-generation in dynamic geometry. The study achieves this goal by elaborating a model through which such cognitive processes can be described and analyzed. Therefore the study has an empirical and qualitative nature. In particular, there are two aspects of the study that influenced our choice of the methodology to utilize. First we needed to be able to observe solvers during open-problem activities in dynamic geometry that involved the development of conjectures. We needed to also be able to interact with the solver in cases in which external observation did not give sufficient insight. This motivated our choice of using clinical interviews.

Second, we were particularly interested in cognitive processes associated with a specific way of dragging, maintaining dragging, and we knew from previous research that this was not usually spontaneously used. Therefore we wanted to be able to "provoke" explorations in which this way of dragging occurred. To this end we developed an introductory teaching intervention during which a researcher worked within a classroom, introducing four "ways of dragging". The solvers for the interviews were then chosen from within the classrooms in which the ways of dragging had been introduced. This teaching intervention exhibits characteristics of a very brief teaching experiment, however we prefer to not define it as such for reasons we will explain in the next section. Instead we will refer to this teaching intervention as the "introductory lesson". In the next section we will briefly introduce the methodological tools of clinical interviews and teaching experiments, explaining why we chose them for our study. Then in Section 3.1.2 we will describe the experimental design of our study.

3.1.1 Clinical Interviews and Teaching Experiments

The clinical interview is a research methodology that has its roots in Piaget's *méthode clinique* (Piaget, 1929), which was developed as "a flexible method of questioning intended to explore the richness of children's thought, to capture its fundamental activities, and to establish the child's cognitive competence" (Ginsburg, 1981, p. 4). This methodology aimed at developing a theory to explain the individual cognitions of children and that also takes into account the social context in which learning takes place, recognizing the fundamental role of language and the importance of clarification of meaning as researchers ask questions and pose problems (Hunting, 1997). In this sense it is possible to find some common roots between the *méthode clinique* and in the Vygotskian *teaching experiment* (Hunting, 1997, p. 146). Further analogies can be seen in a common aim of the two methods, that of building and testing theory about mathematics learning and teaching "searching for explanatory patterns and principles, anomalies and alternative ways of conceptualizing problems in the field" (Hunting, 1997, p.146). Moreover, both methods aim at investigating what might go on in children's heads and how it might go on, by constructing models relative to the child's goals-directed mathematical activity (Steffe, 1991).

However the teaching experiment differs from the clinical interview. We will briefly discuss some fundamental differences between these methodologies that reside at the levels of (1) the time over which they are carried out, (2) the types of interactions they take into consideration, and (3) their design. With respect to the issue of time, the teaching experiment "is directed toward understanding the progress students make over extended periods" (Steffe & Thompson, 2000), while the clinical interview is aimed at describing what might be going on in a child's mind *at the time* of the interview. As for the types of interaction involved, a teaching experiment takes into consideration interaction between the teacher and the students, and between students. Moreover, in a teaching experiment, the interactions (at least some of them) are aimed at supporting learning. Instead, a clinical interview can be described as "a one-to-one encounter between an interviewer, who has a particular research agenda, and a subject" (diSessa, 2007). The focus, in the case of a clinical interview is shifted towards the interviewee's words and actions, instead of on his/her interaction with the interviewer. The interviewer's role could be described as that of an "active observer": his/her aim is to "see" what is in the interviewee's mind, but since there is no direct access, s/he must ask appropriate questions and "pry" at the interviewee's words and actions to test the model s/he is using to interpret such words and actions.

We now come to the issue of the design of a teaching experiment with respect to that of a clinical interview. A teaching experiment "involves experimentation with the ways and means of influencing students' mathematical knowledge" (Steffe & Thompson, 2000). Thus it is designed to investigate and support students' learning, potentially describing learning trajectories and elaborating tools to help the teacher foster them. The learning process that the teaching experiment aims to investigate therefore plays a fundamental role in the design. Moreover the role of the teacher is also built into the design and studied explicitly. A teaching experiment is not typically limited to a series of problematic situations presented to students who are then asked to engage in solving them while being observed.

On the other hand, during a clinical interview,

> The interviewer proposes usually problematic situations or issues to think about and the interviewee is encouraged to engage these as best he/she can. The focal issue may be a problem to solve, something to explain, or merely something to think about. An interviewer may encourage the subject to talk aloud while thinking and to use whatever materials may be at hand to explore the issue or explain his/her thinking. (diSessa, 2007, p. 525).

While in a teaching experiment a goal may be to affect students' learning through intervention, during a clinical interview the interviewer may attempt to perform minimal intervention, in order to least affect the solvers' performance (Steffe & Thompson, 2000; diSessa, 2007). Instead, the interviewer tries to make inferences, constructing and testing a model portraying a cognitive structure to represent what might be in the solver's mind (Ginsburg, 1981). The inferences are made on solvers' behavior, which includes physical actions and language (Steffe & Thompson, 2000; Hunting, 1997; diSessa, 2007) used during the dynamic explorations. The inferences the researcher continually makes postulate possible meanings that lie behind students' language and actions (Steffe & Thompson, 2000, p. 277), and s/he does this in a flexible way, adapting the inquiry to the solver's responses (Ginsburg, 1981; diSessa, 2007). Such inferences, together with the observations of the solver's behavior, can allow the description of the solver's "goal-directed action patterns", taking "action" to refer to mental as well as physical action (Steffe, 1991, p.179). Important inferences can also be made from the analysis of "essential mistakes" (Steffe & Thompson, 2000), since "essential mistakes provide stability in a dynamic living model of students' mathematics" (Steffe & Thompson, 2000, p.278).

If on the one hand the interviewer wants to observe as much as possible and interfere as little as possible with the interviewee's cognitive processes, in order to test and modify his/her inferences, the interviewer needs to interact with the interviewee. Thus s/he can develop a set of questions and prompts ahead of time, to use at specific moments of the interview, and that interfere as minimally as possible with the interviewee's thought process. In developing our questions and prompts, our underlying assumption was that human knowledge and activity patterns are "generative" (diSessa, 2007), that is

> People learn much of the time, and a significant part of the knowledge that they have will be directed toward generating new knowledge and new ways of behaving. Generativity may show in short-term adaptation to a particular problem or even to a particular prompt from the interviewer…(diSessa, 2007, p. 530).

The clinical interview is designed to investigate the structure of thought by reaching a "clear description of mind" (Ginsburg, 1981), and it is particularly appropriate for studying specific cognitive processes (Cohen & Manion, 1994). Our main goal as researchers was to construct, refine and test a cognitive model describing processes that might go on in the mind of a solver engaging in a particular kind of open problems. Therefore we chose the clinical interview as the main methodological tool for our study.

Finally, our model may be seen as describing a *utilization scheme* (Rabardel, P., & Samurçay, R., 2001; Vérillon, P., & Rabardel, P.,1995), as described in Chapter 2, associated to the artifact "maintaining dragging". Since a scheme is a mental construct, it cannot be accessed directly, but only inferred through the activity the solver engages in and that can be observed. Furthermore a scheme is difficult to "put into words", but it can emerge from the search for invariant organizations of a determined activity (Bourmaud, 2006). In particular a scheme may be inferred from: regularities in the solver's behavior, the existence of a choice among different possible ones, the transformation of the situation knowing the effect of such activity on the situation, and from how the activity is carried out (Zanarelli, 2003; Bourmaud, 2006). In this sense our model aims at describing a scheme by analyzing an *invariant* *organization* of the activity of conjecture-generation when maintaining dragging is used. In order to make inferences and construct and refine our model, we developed various questions and prompts to use during the interviews if the solver exhibited certain behaviors. Therefore we refer to our interviews as *semi-structured clinical interviews*. We will describe these questions and prompts in Section 3.2.2.

As mentioned in the introductory paragraph to this section, an issue we needed to deal with was the fact that according to previous research, maintaining dragging was not usually spontaneously used. Therefore in order to be able to "provoke" explorations in which this way of dragging occurred, we developed an introductory teaching intervention, introducing four "ways of dragging". The solvers for the interviews were then chosen from within the classrooms in which the ways of dragging had been introduced. This teaching intervention exhibits characteristics of a very brief teaching experiment, however we prefer to not define it as such for reasons we will explain in the next section. Instead we will refer to this teaching intervention as the *introductory lesson*, and we will describe it in more detail in Section 3.2.1.

3.1.2 The Experimental Design of the Study

We first conceived a preliminary model to test and refine during a pilot study, using clinical interviews (Ginsburg, 1981; Steffe, 1991; Hunting, 1997; diSessa, 2007) based on open-problem-activity tasks (Goldin, 2000), as we will describe in Section 3.2. Before conducting the clinical interviews with the participants, we had them take part in an introductory lesson during which they were introduced to the four ways of dragging we had elaborated (these are described in Chapter 2). We used every interview to test and refine our model and prompts. This "spiraling process" has been successfully used by other researchers in qualitative studies that involve the construction and successive re-elaborations of a theoretical framework and/or of a model (Hadas, Hershkowitz, & Schwarz, 2000; Steffe & Thompson, 2000). Once the model was sufficiently refined, we used it as a tool of analysis through which to interpret the data obtained.

The DGS we chose to use is “Cabri-Geometry II Plus,” developed by Laborde and Bellemain (1993-1998). Both the pilot study and the final study were structured in the following general way. Solvers were students from three Italian high schools (*licei scientifici*) between the ages of 15 and 18, who had been using Cabri in the classroom for at least one year prior to this study: 9 (3 single students and 3 pairs) students for the pilot study and 22 (11 pairs) for the final study. First solvers were introduced to the dragging schemes during an introductory lesson that took place during their regular school hours. Then we conducted the semi-structured clinical interviews with the solvers. Between the pilot study and the final study we applied the necessary modifications to the activities proposed during the interviews, to the research questions, to our cognitive model, and to the prompts to be used during the interviews.

3.2 How Data Were Collected

As described above, we first had our participants take part in an introductory lesson in which they would become familiar with the four ways of dragging we were interested in studying. In particular our aim was to help students become somewhat comfortable with maintaining dragging, which they do not tend to use spontaneously, according to previous research. In Section 3.2.1 we will describe this introductory lesson and how we modified it after the pilot study. The rest of this section is dedicated to the characteristics of the semi-structured clinical interviews we carried out (Section 3.2.2) with a particular focus on how we prepared for the interviews and on how we conducted the interviews, and on a description of the open-problem activities we used during the interviews (Section 3.2.2).

<u>3.2.1 The Introductory Lesson</u>

The lesson was focused on the dragging schemes: as students explored, they were asked to drag points in particular ways and describe their observations and perceptions (for example, how they moved their hand while dragging) with respect to a particular configuration. Students were asked to share their ideas with the whole class, in a discussion guided by the instructor who gave names to specific "ways of dragging" while the students explained how they used them. While exploring with the four dragging modalities during the introductory lessons, the *dragging with trace activated* scheme was only activated on the base point being dragged. No reference to the formulation of a conjecture was made, nor were any indications for using the dragging schemes given at this point. Students were told that these ways of dragging "may be useful for exploring figures in dynamic geometry", but that they were free to do whatever they felt worked best for them during the interviews. The teaching intervention had the limited aim of introducing students to different ways of dragging and to new terminology which (we hoped) they could use during the interviews. The only "teaching" that occurred had to do with the ways dragging, not with a particular process of conjecture-generation. This was important because our goal was to test whether our model was appropriate for describing the *scheme* developed by students in correspondence to the ways of dragging and to maintaining dragging in particular.

During the introductory lessons the interviewer/instructor explained how she was interested in understanding a thought process and how solvers could help her achieve this goal by speaking out loud and explaining as much as they could to her aloud. She also explained that any time she would ask "why?" it did not mean that the solver was wrong (Hunting, 1997; diSessa, 2007), but that she was seeking for an explanation with the aim of understanding the solvers' thought process, thus valuing any clarification the interviewee might be able to provide and refraining from any type of judgment (Hunting, 1997; Ginsburg, 1981).

After the pilot study we revised the introductory lesson, and decided to add a part aimed at helping students overcome some difficulties related to the control of the different status of objects of Cabri-figures. Therefore the lesson was carried out over two class periods. The first lesson was developed around recognition of base points and dependent points of a Cabri-figure that originated from a step-by-step construction the students were asked to make.

In the final study the intervention consisted of two one-hour lessons with the following goals and activities.

Goals of Lesson 1
• to distinguish between base points (in general, objects) and dependent points (in general, objects) of a Cabri figure that originated from a step-by-step construction (given explicitly); • to experience how different Cabri figures that can represent "a parallelogram" (robustly) can originate from different step-by-step constructions and thus have different base points (in general, objects) and dependent points (in general, objects); • and to experience the different behaviors of such Cabri figures when their base points are dragged.

Goals of Lesson 2
• to explore a Cabri-figure that originated from a given step-by-step construction by dragging its base points;

- to experience (physically) and describe different ways of dragging base points of a Cabri-figure;
- to learn names for four "ways of dragging": *wandering dragging*, *maintaining dragging*, *dragging with trace activated*, *dragging test*;
- to attempt to formulate conjectures on the Cabri figure being explored through dragging, but with no guidance from the instructor.

3.2.2 The Semi-Structured Clinical Interviews

As described in Section 3.2.2 and in Section 3.5, the activities proposed were open-ended tasks (we will discuss our specific open-problem activities in Section 3.3.3. This form of activity, being unstructured and open-ended, is designed to give the solver the opportunity to display his/her "natural inclination" (Piaget, 1929), and it seems to be optimal for providing a window into solvers' thinking by maximizing the opportunity for observation and reflection upon their thought process (Hunting, 1997; Ginsburg, 1981). Moreover this type of activity allows detailed follow-up-questions (Hunting, 1997), which are appropriate for testing cognitive models. In the following paragraphs we will describe the interviewer's preparation for the interviews and how they were conducted.

Preparation for the interviews is fundamental in obtaining significant data (Hunting, 1997). As interviewers, we kept in mind our developing model, but were aware of not knowing whether it was appropriate or not. Therefore we were open to different interpretations of the solvers' activity while formulating questions on-line and off-line (Ackermann, 1995; diSessa, 2007), that is during the interviews and between one interview and the other. While the materials provided to the solvers (the Cabri environment, paper, a pen) and the activities were the same, the interviewer's prompts and questions would depend on the solvers' responses. Typical requests to a solver were to explain an action, to describe what s/he was looking at or trying to accomplish, or to provide clarification or elaboration of a statement s/he made (diSessa, 2007). However subsequent prompts and requests would be formulated using the solvers' language, in an attempt to make confirm an interpretation or test an alternative one (Ginsburg, 1981).

Moreover, we elaborated some questions and prompts that we would use when a solver seemed to "get stuck". We were aware of the fact that certain prompts might change the solver's processes of thoughts and actions, however we wanted to be able to observe certain types of explorations even if they did not occur spontaneously. Furthermore we were aware that solvers could make remarkable progress with basic assistance (Hunting, 1997), and that they can adapt to a particular problem or prompt by generating new knowledge (diSessa, 2007). Therefore we also analyzed students' responses to our prompts, searching for potential recurring behaviors that might further shed light onto the process described by our model. We kept track of the different types of questions and interventions we chose to use during the interviews, and whether they would be asked in recurring sequences. These sequences were then analyzed as a second level of findings (Chapter 6).

We will now describe the questions and prompts we prepared for the interviews, refining them after the pilot study. We took into consideration different difficulties that solvers had encountered during the pilot study and tried to present the solvers with new tasks (more or less) implicitly related to the original tasks. We would choose to lead solvers to a different interpretation of the activity when the solvers did not seem to be making sense of the interviewer's inquiries (diSessa, 2007). We also developed a series of prompts that could be used interchangeably when the solvers seemed to be experiencing a particular difficulty related to maintaining dragging. Our use of the prompts also depended on the solvers' responses.

The questions and prompts we prepared were the following:

- *So how can you construct a …[the type of quadrilateral the solver had been exploring]…that passes the dragging test and that follows the steps of the initial construction?*

  The idea behind this intervention is to lead the solvers to further explore the interesting configuration, and generate new conjectures. Moreover we expected it to help them become aware of the different status of objects of the construction and look for "constructable properties" to add to the steps of the construction that will induce the desired type of quadrilateral robustly.
- *Are there other ways to obtain a robust…[the type of quadrilateral the solver had been exploring]?*
- *So how about trying maintaining dragging, do you remember? Like what you tried in class.*

  or

  You mentioned the property that made ABCD a… Can you try to maintain that?

  or

  Is it not possible to maintain that property? Can you tell me why not?

  With these questions we would try to foster the use of maintaining dragging by asking the solvers explicitly, in cases in which they had not used it previously.
- *Ok, I know it's difficult, but can you ask your partner to help tell you where to move the point?*

  We used this prompt if a solver was experiencing difficulties performing maintaining dragging.

- *Do you remember that we used dragging with the trace activated in class? Do you want to try that here?*

  We used this question if the solvers were not able to describe regularities in the movement of the dragged base point during maintaining dragging.

- *Ok, so this …[object in the geometric description of the path (GDP)]…moves as you drag. Can you try to describe one that does not move?*

  We used this prompt in cases in which the solvers had reached a GDP that was not P-invariant (if P was the base point dragged) and they were experiencing difficulties performing a robust dragging test.

- *So can you give me a conjecture now?*

  or

  How about a conjecture that describes what you have done till now?

  We used these prompts if solvers would not provide a conjecture after an exploration, in particular one that might have involved the use of maintaining dragging.

Finally, in preparing for the interviews, we took into consideration the issue of length of each interview (Hunting, 1997; diSessa, 2007). After the pilot study we decided that the ideal time to optimize the collection of significant data with the participants of this study and the type of activities used was one hour and thirty minutes per pair of students.

<u>Conducting the interviews</u>. With respect to diSessa's description of the clinical interview methodology we introduced in Section 3.1, the interviews we conducted in our final study differed in that we worked with pairs of students instead of one-on-one. In the pilot study we experimented with both types of settings, but it became clear that students seemed to share their thoughts more openly when interacting

principally with a fellow student, as opposed to only with the interviewer. This finding is in line with what has been found in other studies (for example, Clements, 2000; Hadas, Hershkowitz, Schwarz, 2000; Schoenfeld, 1983). Moreover, a fundamental characteristic of clinical interviews is putting the interviewee at ease (Ginsburg, 1981; Steffe, 1991; Hunting, 1997; diSessa, 2007), and peer interaction seems to foster this (diSessa, 2007, p. 551).

When conducting the interviews a fundamental goal was to pose questions that appeared to be sensible inquiries to the interviewee (diSessa, 2007, pp. 527-528). The questions posed in our activities came from "the context" of the introductory lessons, however a goal of the interviewer during the open-problem-activity sessions was to uncover the solvers' understanding of the task and capture the sense the interviewees were making of the problems by asking them to help her see their ideas (Ginsburg, 1981; diSessa, 2007). The interviewer would have this secondary goal in mind when constructing hypotheses and responding on-the-fly (Ackermann, 1995), or choosing which prompt to use during the interviews. Moreover, the interviewer would try to be flexible in formulating hypotheses on the solvers' behavior (Ginsburg, 1981) and in using the language of the solvers by repeating and rephrasing statements that they made (Hunting, 1997; Ginsburg, 1981).

Another aspect we considered when conducting the interviews was the "redistribution of authority and responsibility" (diSessa, 2007). The interviewer would ask questions and prompt the interviewees, but she would not be the "holder of knowledge" nor judge the solvers' responses. However the role of the interviewer was asymmetric since the interviewees had met her for the first time during the introductory lessons in which she was the instructor. This was another reason why being the "observer" of peer interactions was more functional to the study than a one-on-one interaction with a single solver. This way the interviewer could remove herself from the "action" in the exploration. This was accomplished also by physically standing (or sitting) behind the solvers during the final study, and intervening only to ask for clarification or to suggest prompts from the guiding sequence. Moreover the interviewer explicitly stated that there were no "right or wrong" answers. She would repeat this whenever solvers seemed to be looking for confirmation or asking whether a particular comment or answer was "right?". When, during the interviews, solvers would hesitate after being asked "why?" they had said or done something, the interviewer would explicitly repeat what she had explained during the introductory lessons, that is that her intention was not to point out anything "right or wrong" but instead to understand what the solver was thinking.

3.2.3 Open-Problem Activities for the Interviews: Step-by-step Construction Problems

As described in Chapter 2, the terminology "open problem" (Arsac et al., 1988; Silver, 1995) refers to a problem or question stated in a form that does not reveal its solution or answer. In the context of open problems students are faced with a situation in which there are no precise instructions, but rather they are left free to explore the situation and make their own conclusions. In other words, when an open problem is assigned, the solution consists in elaborating a conditional relationship between some premise and a certain fact. Often the solving process requires the generation of conditionality after a mental and/or physical exploration of the problem situation (Mariotti et al., 1997). In some of the previous research, the production of conjectures is an explicit request in the text of an open problem (for example, Boero, 1996a, 2007; Arzarello, 2002; Olivero, 2001, 2002). When conjectures are explicitly requested in the text of the problem, we will use the terminology *conjecturing open problem*, to distinguish it from other types of open problems.

The dynamic nature of the exploration of open problem situations becomes particularly evident in the context of a DGS, where the figures can actually be explored dynamically through the dragging mode. This makes DGSs an ideal environment for posing conjecturing open problems and for observing and investigating processes of generation of conjectures. In a DGS, a conjecturing open problem typically takes the form of a generic request for a statement about relationships between elements of the configuration or between properties of the configuration. The questions are expressed in the form “which configuration does… assume when…?” “Which relationship can you find between…?” “What kind of figure can… be transformed into?” (Olivero, 2001). For example, a conjecturing open problem ask: “Construct two points (A, B) and a third point C so that the angle ACB is 60 degrees. Are there other choices of C for which this is possible? Make a conjecture.”

To explore the validity of our model we constructed conjecturing open problems, characterized by a sequence of steps, which students are asked to follow, leading to the construction of a dynamic figure, followed by an open question explicitly asking for a conjecture. We will refer to this type of conjecturing open problems as *step-by-step construction problems*. We constructed these step-by-step construction problems so that explorations in which solvers would search for invariants using maintaining dragging would be fruitful, that is it would be possible to make a path explicit using the trace mark and observe an invariant during dragging (IOD). We developed and used the following step-by-step construction problems for the study.

Problem 1

- Draw three points: A, M, K.
- Construct point B as the symmetric image of A with respect to M
- and C as the symmetric image of A with respect to K.
- Construct the parallel line *l* to BC through A.
- Construct the perpendicular to *l* through C,
- and construct D as the point of intersection of these two lines.
- Consider the quadrilateral ABCD.

Make conjectures on the types of quadrilaterals it can become, trying to describe *all the ways* in which it can become a particular type of quadrilateral.

Problem 2

- Draw a point P
- and a line *r* through P.
- Construct the perpendicular to *r* through P
- and construct a point C on this line.
- Construct the symmetric image of C with respect to P and call it A.
- *r* separates the plane in two semi-planes. Choose a point D on the semi-plane that contains A.
- Construct the line through D and P.
- Construct the circle with center in C and radius CP.
- Let B be the second intersection of the circle with the line through D and P.
- Consider the quadrilateral ABCD.

Make conjectures on the types of quadrilaterals it can become, trying to describe *all the ways* in which it can become a particular type of quadrilateral.

Problem 3

- Draw three points: A, M, K.
- Construct point B as the symmetric image of A with respect to M
- and C as the symmetric image of A with respect to K.
- Construct D as the symmetric image of B with respect to K.
- Consider the quadrilateral ABCD.

Make conjectures on the types of quadrilaterals it can become, trying to describe *all the ways* in which it can become a particular type of quadrilateral.

Problem 4

- Draw three points: A, B, C.
- Construct the parallel line *l* to AC through B,
- and the perpendicular line to *l* through C.
- Construct D as the intersection of these two lines.
- Consider the quadrilateral ABCD.

Make conjectures on the types of quadrilaterals it can become, trying to describe *all the ways* in which it can become a particular type of quadrilateral.

<u>A-priori Analysis of Problem 4</u>. We developed the step-by-step construction problems for the study so that the use of maintaining dragging on certain (if not all) base points of each dynamic-figure would potentially lead to the discovery of a new invariant, an IOD. In this section we will analyze Problem 4, as an example, to show how we thought they might be explored.

Problem 4

- Draw three points: A, B, C.
- Construct the parallel line *l* to AC through B,

- and the perpendicular line to *l* through C.
- Construct D as the intersection of these two lines.
- Consider the quadrilateral ABCD.

Make conjectures on the types of quadrilaterals it can become, trying to describe *all the ways* in which it can become a particular type of quadrilateral.

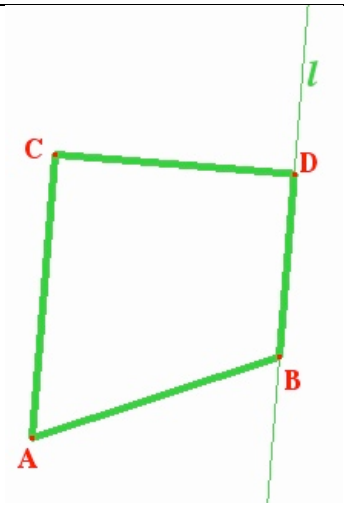


Figure 3.3.3.1: the quadrilateral ABCD as a result of the step-by-step construction.

From the steps of the construction, immediate conclusions are:

1) the angles ACB and CBD are congruent because BD is parallel to AC;

2) the angle ACD is right, because CD is perpendicular to *l*, which is parallel to AC;

3) the triangle BCD is right, and therefore inscribed in a semicircle with diameter BC;

4) ABCD is a right trapezoid.

The presence of two right angles implies that the only quadrilaterals it may be possible to explore are right trapezoids, rectangles, and squares.

There are three base points, A, B, C, that can be dragged to explore other possible configurations. Dragging any of these base points it is possible to obtain a rectangle. The GDPs for each of these base points when maintaining the property “ABCD rectangle” are:

- for A, the circle with diameter BC;
- for B, the perpendicular line to AC through A;

- for C, the perpendicular line to AB through A.

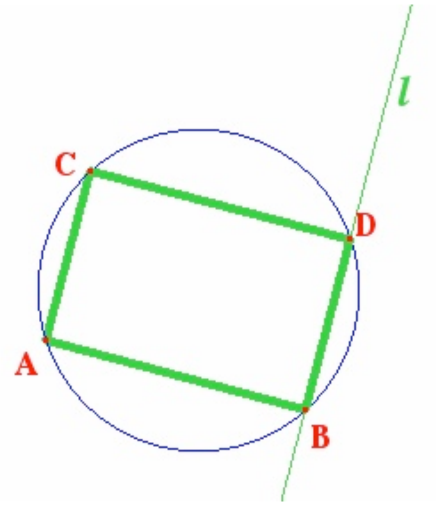


Figure 3.3.3.2: if A is dragged maintaining ABCD rectangle, a GDP is the circle with diameter BC.

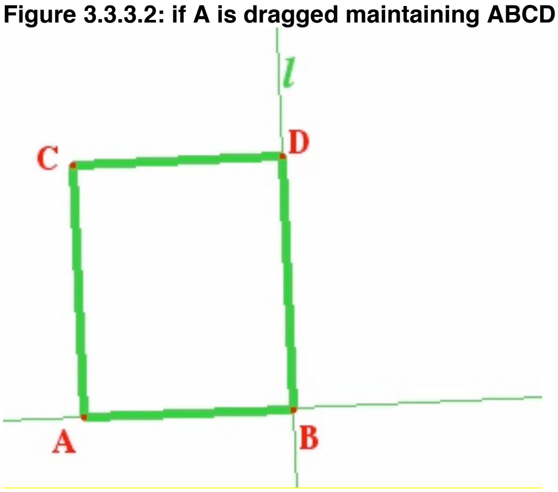


Figure 3.3.3.3: if B is dragged maintaining ABCD rectangle, a GDP is the perpendicular line to AC through A.

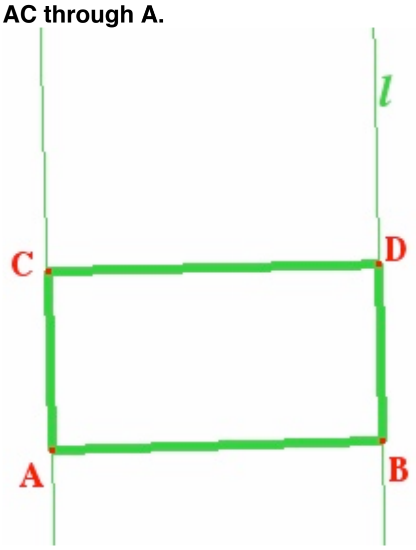


Figure 3.3.3.4: if C is dragged maintaining ABCD rectangle, a GDP is the perpendicular line to AB through A.

These GDPs are invariant with respect to the base point dragged to determine them, and they do not depend in any way from the base point being dragged to determine them. Therefore it is possible to redefine the dragged base point upon each of them to obtain a robust rectangle.

- Once A is redefined on the circle, the angle CAB is a robust right angle, because inscribed in a semicircle. Therefore three of ABCD's angles are right, which implies that they are all right and ABCD is a robust rectangle.
- Once B is redefined on the perpendicular line to AC through A, the angle CAB is a robust right angle and therefore we have the same conclusion as in the previous case.
- Once C is redefined on the perpendicular line to AB through A, the angle CAB is a robust right angle and therefore we have the same conclusion as in the previous case.

The only other possible configuration to explore is "ABCD square". This configuration can be obtained again dragging any of the base points, but it may not be maintained during dragging. A square may be obtained in the following ways:

- positioning A on one of the intersections of the circle with diameter CB with the perpendicular bisector of BC;
- positioning B on one of the intersections of the circle with radius AC and center in A with the perpendicular line to AC through A;
- positioning C on one intersections of the circle with radius AB and center in A with the perpendicular line to AB through A.

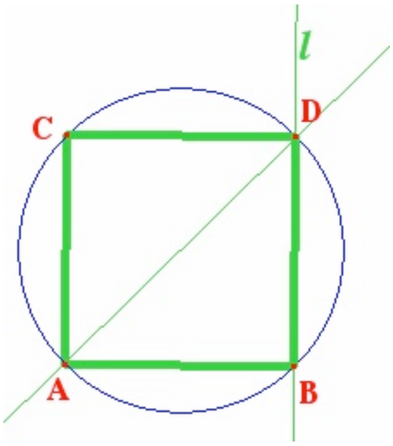


Figure 3.3.3.5: One of the two positions of A to obtain a square.

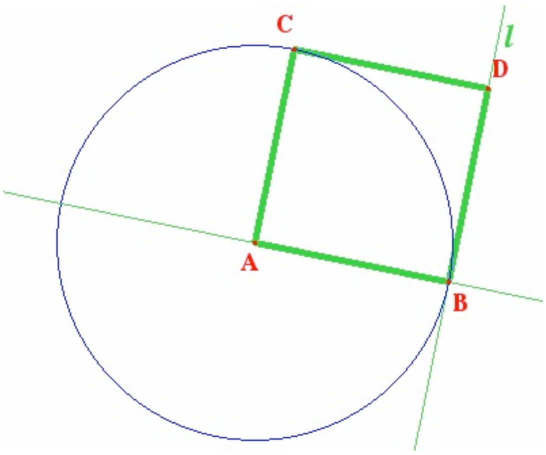


Figure 3.3.3.6: One of the two positions of B to obtain a square.

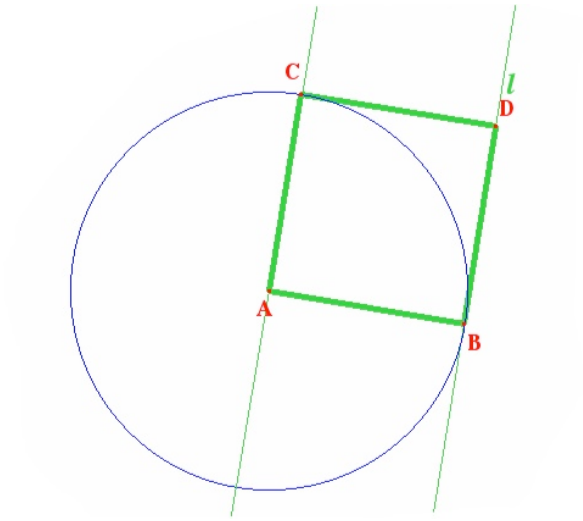


Figure 3.3.3.7: One of the two positions of C to obtain a square.

3.3 The Collected Data and How They Were Analyzed

The data collected included: audio and video tapes and transcriptions of the introductory lessons; Cabri-files worked on by the instructor and the students during the classroom activities; audio and video tapes, screenshots of the students' explorations taken at 1-second intervals with screen-capturing software that would run in the background while the students were working in Cabri; transcriptions of the task-based interviews, and the students' work on paper that was produced during the interviews.

We analyzed the data collected through different filters. At one level, we looked at how solvers used the dragging tool during the process of conjecture-generation, searching for recurring behaviors, and trying to link such behaviors to the forms of reasoning that might be involved. In particular, we used the data to confirm and refine our model by looking for and trying to describe an invariant behavior corresponding to the use of maintaining dragging. The final model, as presented in Chapter 4 is the outcome of such analysis. Throughout this chapter we also highlight the aspects of the model that were added and refined during the study.

A second level of analysis consisted in using the model itself as a tool of analysis of the data generated from the interviews. We interpreted solvers' behaviors through the lens of the model, using it in particular to gain insight into difficulties that solvers seemed to be facing. These difficulties that solvers encountered can be considered "essential mistakes", using the terminology of Steffe and Thompson (2000). In our case we considered "essential mistakes" the solvers' difficulties and behaviors that deviated from the model that seemed to "fit" for other solvers. This second type of analysis allowed us to also advance hypotheses on specific sources of difficulties, which we describe in Chapter 5.

We then used a third filter, that of the model (Chapter 4) together with the factors that seemed to contribute to solvers' difficulties (Chapter 5) to further elaborate and refine our conception of "expert behavior" with respect to maintaining dragging. This third level of analysis allowed us, in particular, to develop the notion of *path* and highlight its significance, and to “capture” the abductive process involved in conjecture-generation when maintaining dragging is used, according to the description provided by our model. Finally, the analysis through this lens of solvers’ responses to our prompts and of the order in which the prompts were given, led to some insight into a possible process through which solvers would become “experts”. We present the findings of this third level of analysis in Chapter 6.

CHAPTER IV

THE MD-CONJECTURING MODEL

Our main goal was to interpret and describe cognitive processes leading to the formulation of a conjecture, when certain dragging schemes are used. In particular we wanted to zoom into the crucial point described in Arzarello et al.'s model (Arzarello et al., 2002), in which *dummy locus dragging* seemed to be used by the solvers. Therefore we focused specifically on developing a new model describing a way in which *maintaining dragging* (MD) may be used to generate conjectures when exploring a step-by-step open problem. In this chapter we present our model describing "expert use" of MD in the process of conjecture-generation. We therefore refer to our model as the MD-conjecturing Model.

The MD-conjecturing Model consists of a series of phases characterized by specific tasks that the solver accomplishes, and described through novel key concepts and the relationships between them. These concepts and relationships seem to be the main ingredients that come into play during the conjecturing process and that are elaborated into the final conjecture (considered as the product of this process). Our initial hypothetical model includes the following notions and the relationships between them: intentionally induced invariant (III), invariant observed during dragging (IOD), path, geometric description of the path (GDP), conditional link (CL).

During the study we collected data in order to see whether our model could be suitable to describe the process we were interested in. Analyzing the data through the lens of the model led to a refinement and later to a redefinition of the model. We would like to show examples of how we used the initial model to analyze transcripts, and how certain analyses led to refinements of the model itself. Therefore in this chapter we first introduce the initial model through a simulated exploration in section 4.1; then sections 4.2, 4.3, 4.4, and 4.5 introduce the phases of the model, leading to the formulation of a conjecture and characterized by the presence of specific elements and their mutual relationships. In each of these sections the phase will be described and then exemplified through students' transcripts, analyzed through the lens of the model. In addition, where refinements of the model took place we will have such refinements emerge from students' transcripts. The new notions and processes that emerged from the analyses include: (basic and derived) construction-invariant, point-invariant, basic property, minimum basic property (section 4.2). As mentioned above, the episodes from the transcripts of students' interviews presented in this chapter represent cases that have been classified as "experts' use" of the schemes, that is the use made by solvers for whom maintaining dragging has become an acquired instrument with respect to the task of producing a conjecture.

Finally, the analysis of the transcripts led us to notice the centrality of invariants, of which different types are described in the model, and solvers' perception of them during the explorations. This led to a new conception of the model, which we present in section 4.6. Here we re-describe the model through the particular types of invariants the solver may treat throughout a dynamic exploration.

4.1 Introduction of the Model through a Simulated Exploration

In this section we will show how the initial elements of the model come into play in an example of a hypothetical exploration based on a step-by-step-construction problem. We briefly recall that we defined a step-by-step construction problem as a sequence of steps, which students are asked to follow, leading to the construction of a dynamic figure, followed by an open question explicitly asking for a conjecture (for the definition see Section 3.2.3). We will use Problem 2 introduced in Section 3.2.2.

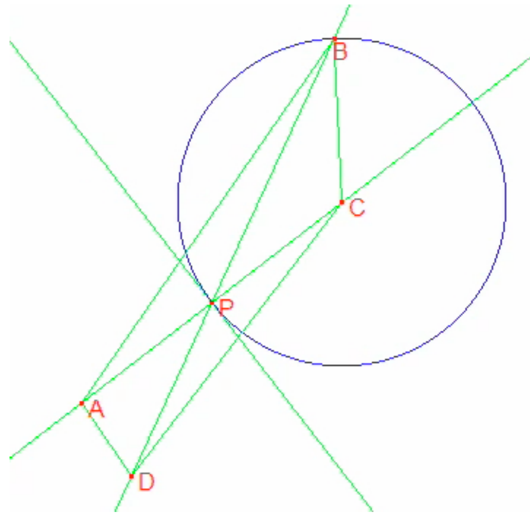


Figure 4.1.1 ABCD as a result of the step-by-step construction.

- Draw a point P
- and a line *r* through P.
- Construct the perpendicular line to *r* through P
- and choose a point C on it.
- Construct a symmetric point to C with respect to P and call it A.
- On the semi-plane identified by *r* containing A, draw a point D.
- Construct the line through D and P.
- Construct the circle with center in C and radius CP.
- Let B be the second intersection of the line through P and D with the circle.
- Consider the quadrilateral ABCD.

Make conjectures on the types of quadrilaterals that it can become, describing all the possible ways it can become a certain quadrilateral. Write your conjectures and then prove them.

Let us assume we decide to start dragging the base point D. While dragging, we see that the quadrilateral ABCD may become something that “looks like” a parallelogram. We can try to use maintaining dragging to move point D while trying to keep ABCD a parallelogram.

With respect to the maintaining dragging scheme, we say that the property "ABCD parallelogram" that we are inducing is called an intentionally induced invariant (III). While we drag we can look for some regularity to emerge from the movement of the point we are dragging (D in this case). We are looking for what the model refers to as an invariant observed during dragging (IOD). In order to make the transition from a regularity to an invariant (which can then be interpreted as a geometric property), we can look for a *path*, or a set of points along which we can drag our base point in order to maintain (indirectly) the intentionally induced invariant (III).

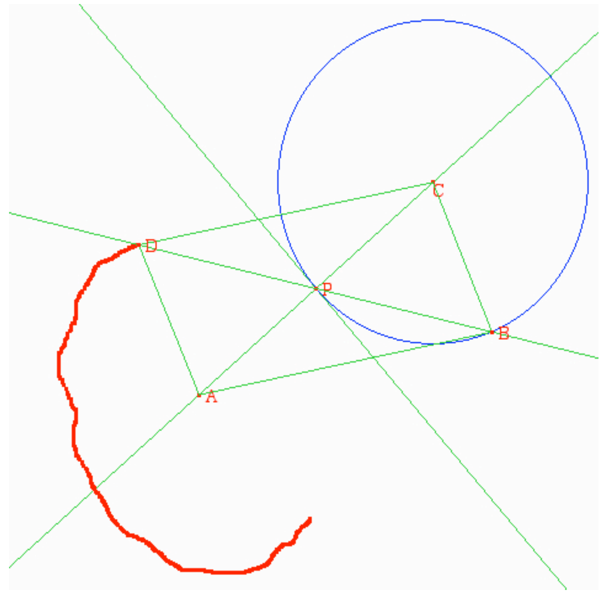


Figure 4.1.2 Effect of maintaining dragging with the trace activated on the dragged base point.

Then we can try to give a geometric description of the path (GDP) thus potentially obtaining a new geometrical property that can be applied to the figure. Maintaining dragging with the trace activated can help us make the path explicit, and help us conceive a geometric description of the path (GDP).

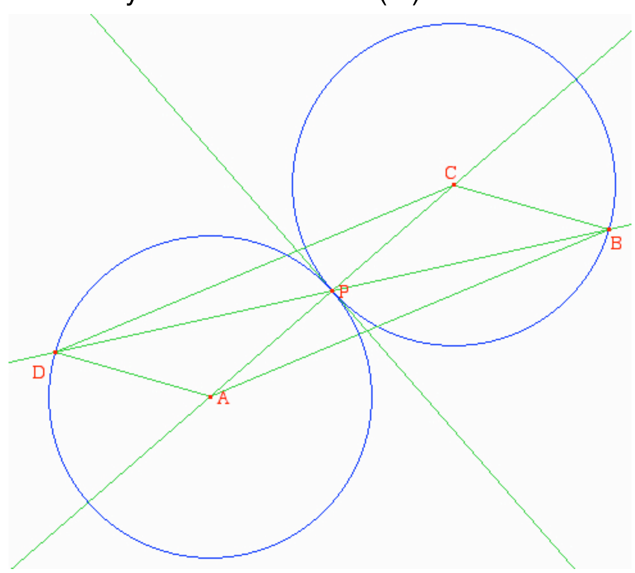


Figure 4.1.3: A GDP has been constructed and a dragging test may now be performed.

In this case we could interpret the trace as something like a circle, providing an argumentation like: "as we go down we have to also move over and move like B moves on the circle" to reach a description of the path as a symmetric circle to the existing one.

Continuing the dragging of D we can keep on checking our geometric description of the path (GDP) by looking at the two (assumed) invariants (D on path and ABCD parallelogram) occur simultaneously.

This is already a soft version of the dragging test. We refine our geometric description of the path (GDP) until we reach a constructible one, like a symmetric circle with respect to the one in the steps of the construction, with center in A, and radius PA. At this point we can perform a more convincing, but still soft dragging test by dragging D along the constructed circle and making sure the two invariants occurred simultaneously. We may even want to link D to the circle and obtain two robust invariants that now we can observe that now we can observe occurring simultaneously when dragging any base point of the construction. The properties we concentrate on now during this robust dragging test are: D on the circle and ABCD parallelogram.

We can now formulate a conjecture taking our invariant observed during dragging (IOD) as the premise and our intentionally induced invariant (III) as the conclusion of this statement. Since the invariant observed during dragging (IOD) was "D belongs to the constructed circle[1]" and the intentionally induced invariant (III) was "ABCD parallelogram" we obtain the following conjecture*: If D belongs to the circle centered in A with radius AP, then ABCD is a parallelogram.*

We might prefer to describe the IOD in a more "static" way, for example, by noticing that "D belongs to the constructed circle" implies "AD congruent to AP", and vice versa. In this case we could decide to substitute the premise expressed in the original conjecture with the new one "AD congruent to AP", obtaining a new conjecture: *If AD is congruent to AP, then ABCD is a parallelogram.*

---

[1] Notice the transition from "D is dragged along the circle" to this crystallized form. We will discuss this in more detail in Chapter 7.

In the remainder of this section and in Section 4.2, in order to become accustomed to the terminology, we will continue to write each element of the MD-conjecturing Model completely, including the abbreviation in parentheses. From section 4.3 on, we will only use the abbreviations.

We can describe the exploration as a sequence of tasks – or sub-tasks of the main task of generating a conjecture – to be accomplished during the process of conjecture-generation when the maintaining dragging scheme is used.

- Task 1: Determine a configuration to be explored by inducing it as a (soft) intentionally induced invariant (III): through wandering dragging the solver can look for interesting configurations and conceive them as potential invariants to be intentionally induced.
- Task 2: Look for a condition that makes the intentionally induced invariant (III) visually verified through maintaining dragging. This can occur through
    - a geometric interpretation of the movement of the dragged base point
    - or a geometric interpretation of the trace.

  The "condition" may be considered the movement of the dragged base point along a *path* which can also acquire a geometrical description (GDP). The belonging of the dragged base point to a path with a geometric description determines the (IOD). When the two invariants are observed simultaneously, the solver will have direct control over the invariant observed during dragging (IOD) and indirect control over the intentionally induced invariant (III). This may guide the conception of a conditional link (CL) between the two invariants.
- Task 3: Verify the conditional link (CL) through the dragging test. This requires

the accomplishment of at least some of the following subtasks:

- representing the invariant observed during dragging (IOD) through a construction of the proposed geometric description of the path (GDP);
- performing soft dragging test by dragging the base point along the constructed geometric description of the path (GDP);
- performing a robust dragging test by providing (and constructing) a geometric description of the path (GDP) that is not dependent upon the dragged base point and redefine the base point on it in order to have a robust invariant, then perform the dragging test.

The table below contains the key elements of the model, the abbreviations used to denote them, and their definitions.

Intentionally Induced Invariant (III)	Property (or configuration) that the solver chooses to try to maintain
Path	Set of points with the following property: if the dragged-base-point coincides with any of these points then the intentionally induced invariant (III) is (visually) verified
Geometric Description of the Path (GDP)	Geometric characterization of the path
Invariant Observed During Dragging (IOD)	Property (or configuration) that seems to be maintained by the Cabri-figure while an intentionally induced invariant (III) is being induced through maintaining dragging
Conditional Link (CL)	(implicit) logical connection between the invariant observed during dragging (IOD) and the intentionally induced invariant (III)
Conjecture	(explicit) statement with a premise and a conclusion that expresses the conditional link (CL) explicitly.

Table 4.1.4 Key elements of the MD-conjecturing Model

4.2 Intentionally Induced Invariant (III)

As shown in the previous section, the first task described by our model is the determination of an interesting configuration to explore. An “interesting configuration” in this case is a configuration in which the solver recognizes a particular property that s/he conceives as potentially invariant with respect to some kind of movement. In this case the solver may become interested in “when” the Cabri-figure maintains a certain property, for example “when it becomes a particular type of geometrical figure”. In other words, the solver begins to search for “the conditions under which” the interesting property is obtained and maintained, that is conditions under which the property becomes an invariant with respect to movement. To accomplish this the solver may decide to apply the *maintaining dragging scheme*. We therefore define the intentionally induced invariant (III) as

> *a property (or configuration) that the solver finds interesting and chooses to try to maintain during dragging.*

After the intentionally induced invariant (III) has been chosen, the solver will concentrate on maintaining it, visually, while dragging a base point of the Cabri-figure. This means that at this point the intentionally induced invariant (III) is a soft property[1] of the Cabri-figure, and therefore maintaining it approximately while continuously dragging a base point may not be a simple task, if it is possible at all, which also depends heavily on the manual skills of the solver.

Moreover, we refer to the intentionally induced invariant (III) as an *indirect invariant*, in that it can only be controlled indirectly, through the dragging of a base point.

[1] We recall that a “soft property” is a geometrical property that a Cabri-figure may assume if the solver positions its base points appropriately, but it is not a property that can be derived from the steps of the construction and therefore it will not automatically be maintained by Cabri during dragging. The terminology “soft” and “robust” properties was introduced by Healy (2000) and discussed in Chapter 2.

In other words, it is a property that is indirectly related to these base points, and in particular to the one being dragged, and can be maintained by dragging a base point in a way that is not immediately accessible or obvious. Applying the maintaining dragging scheme in this manner guides the search of “the conditions” for which the intentionally induced invariant (III) can be maintained, and these conditions are immediately controllable by the solver, in that they are described through the solver’s interpretation of the movement *s/he* is imposing directly on the base point. In this sense, such “conditions” can be interpreted as the premise of the statement of the future conjecture. While the property maintained as an intentionally induced invariant (III) is already expressed geometrically, the invariant observed during dragging is first perceived through haptic and visual sensations of movement. Therefore these “conditions” that emerge need to be re-elaborated into what will become the premise of the conjecture through a non-trivial process. This motivated our separate introduction of the definition of geometrical description of the path (GDP) and invariant observed during dragging (IOD).

The first excerpt below illustrates how a student, J, decides to explore “when” the quadrilateral considered is a parallelogram, and how he induces this property as an invariant using maintaining dragging.

Excerpt 4.2.1. This excerpt is from a student’s work on Problem 2 and it illustrates the initiation of maintaining dragging: an intent to explore “when” ABCD is a parallelogram, as a property to induce as an invariant during dragging.

Episode	**Brief Analysis**
[1] J: So... [2] J: parallelogram... [3] J: When is it parallelogram?	The student J chooses the property “parallelogram” ([2]) as his intentionally induced

[4] J: Well, ok, more or less...[dragging P] [5] I: Are you trying to make one or to maintain it a parallelogram? [6] J: To maintain it. ...[he switches to dragging a different base point] [7] J: Here...maybe	invariant (III) and tries to maintain it ([6]) first dragging the base point P ([6]) and then the base point D ([7]).
[8] J: Oh dear! [9] J: Somewhere over there, anyway... [10] J: hmmm	J recognizes that there will be other good positions "over there" ([9]).

Table 4.2.1: Analysis of Excerpt 4.2.1

J seems to have conceived the property "ABCD parallelogram" as a potential invariant to intentionally induce (III), because he seems to be focusing on it with respect to movement. In particular he seems to conceive it as a potential III with respect to the movement of different base points (he switches from dragging P to dragging D). Further evidence that he has conceived the property with respect to movement (and thus an III as described by our model) is that J recognizes that there will be other good positions "over there" ([9]). Overall J's manual skills seem good and allow him to coordinate hand movement with observation of the intentionally induced invariant (III) ([4]). This will help J make the transition to the perception of an invariant observed during dragging (IOD).

The example we just saw in Excerpt 4.2.1 was an example of a behavior which appeared to be perfectly coherent with what our initial model described. Now we would like to describe what various students' transcripts showed as a recurring behavior that occurred before the identification and induction of an intentionally induced invariant (III). These observations led to an enrichment of our initial model, which we will describe through examples below.

4.2.1 A Preliminary Phase

Frequently, the first part of each exploration was characterized by a use of wandering dragging, during which solvers' attention is caught by properties that are invariant for random dragging of the base point being considered, and potentially for random dragging of the other base points as well. These invariants appear to be "robust" (Healy, 2000) or "un-mess-up-able" (Healy et al., 1994), and they seem to capture students' attention before other properties that are not "always verified", or "soft" invariants (Healy, 2000). This is interesting because different behaviors that can precede the use of maintaining dragging emerge. In this sense we speak of a "preliminary phase".

In order to have an appropriate terminology to describe students' recurring behaviors when encountering robust invariants, we coined the notions of "construction-invariant" and of "point-invariant" (Baccaglini-Frank et al., 2009). We will discuss each of these notions in the paragraphs below and give examples of excerpts which led to their emergence. Moreover, the investigation of these robust invariants can culminate in a first conjecture, which frequently makes use of a characterizing property of the type of quadrilateral being investigated (we refer to these ad "basic conjectures" and provide a definition and discussion in Chapter 5).

4.2.1.1 Construction-invariants. During this preliminary phase, solvers frequently use wandering dragging to move the various base points of the construction. During this phase of the exploration, the solver may notice *construction-invariants*, that is,

> *geometrical properties of the figure which are true for any choice of the base points.*

Typically, construction-invariants are described by the solver as “things that are *always* true”, indicating generality with respect to the step-by-step construction. In particular, the solver may recognize the geometrical figure s/he is asked to consider as “always being” a specific type of geometrical figure. In the initial example in section 4.1, ABCD in general is not any specific type of quadrilateral, however, for example, the property “PA congruent to PC” is a construction-invariant, and thus students might refer to it as being “always” true for any movement of the base points.

The solver may give an argumentation as to why s/he thinks the property is “always” maintained by the considered figure, and in doing this, s/he will link back to the description of the step-by-step construction. During this process the steps are “translated” into mathematical properties which become the premise of a possible first conjecture. These mathematical properties are linked to the construction-invariant, which will become the conclusion of the possible conjecture, as the “reasons” why it is true. The argumentation may proceed deductively, using theorems from Euclidean geometry, from the reinterpretation of the steps as conditions for the interesting property that was perceived.

It is interesting how although various construction-invariants can be perceived, the construction-invariant that is typically featured in a conjecture is not explicitly expressed in any of the construction steps. Therefore, it seems useful to make a distinction between construction-invariants that are explicitly expressed by the steps of the construction – we will call these *basic construction-invariants* – versus construction-invariants that can be derived from the steps of the construction through deductive arguments – we will call these *derived construction-invariants*. The excerpts below show two examples of how students perceive what we have defined derived construction-invariants, before they even start dragging.

Excerpt 4.2.2. This excerpt is from two students' work on Problem 1, and it shows how the property "ABCD is a right trapezoid" is perceived as a derived construction-invariant.

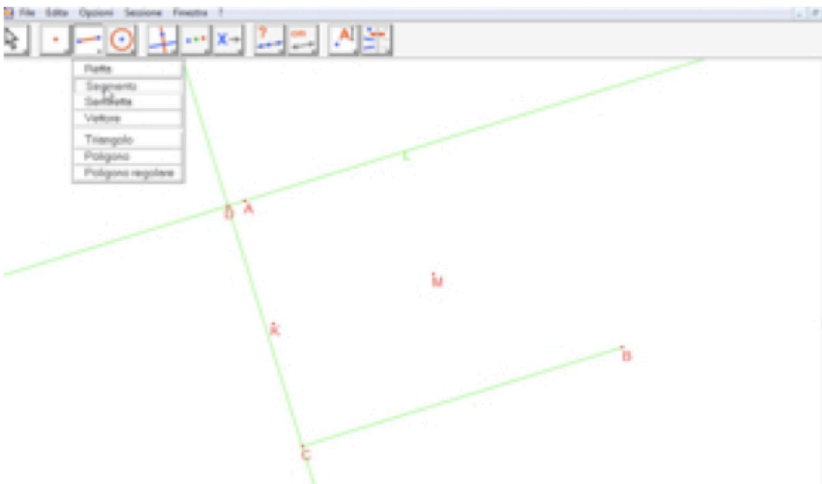


Figure 4.2.2: A screenshot of V&R's exploration

Episode	**Brief Analysis**
[1] V: Always a trapezoid...because it's constructed so that...	V perceives a construction-invariant. Recurring use of the word "always".
[2] V:...at least when...	
[3] R: also always...	
[4] V: ....it becomes a parallelogram	The justification of why ABCD not only appears to be a right trapezoid, but it actually "is always" such a figure seems to be: line *l* is constructed as parallel to segment BC, and CD is constructed as perpendicular to *l*, and thus to BC.
[5] R: ...always a right trapezoid, because this is perpendicular to...	
[6] V: to...	
[7] R: ..the base	
…	
[8] I: Ok.	
[9] V: This one here is perpendicular to this	V makes the deduction explicit. As she

one, and so since both are [10] R: and so... [11] V: these two here parallel, therefore... [12] R: Right, and so ok. [13] I: ok. [Written conjecture: “The quadrilateral ABCD is always a trapezoid, because two bases are parallel. It is also a right trapezoid, because DC ⊥ to CB.”]	says “this one” ([9]) she points to DC and DA. She deduces that since DC is perpendicular to *l* (construction step), and *l* is parallel to BC (construction step), a theorem guarantees that CD will also be perpendicular to BC.

Table 4.2.2: Analysis of Excerpt 4.2.2

The episode occurs before any dragging, immediately after the steps of the construction are complete. We interpret this as evidence that the solvers perceive the property “ABCD right trapezoid” as a derived construction-invariant, because the behavior indicates that they have perceived the property at a theoretical level: the students seem to interpret the steps of the construction as premises to start their deductive reasoning from. We think the solvers have perceived the property at a theoretical level because of R’s (and later of V’s) argumentation. R feels the need to justify the claim that ABCD is a right trapezoid, referring to a fact that he has derived from the steps of the construction.

Further evidence to support our claim that the property is being perceived as a derived-construction-invariant comes from the use of the word “always [Italian: “sempre”]. In this excerpt “always” seems to refer to a fact that the solver assumes will be true no matter what (no matter how or which points one drags, in this case): “always a trapezoid” ([1]), “always a right trapezoid” ([5]).

Finally, the formulation of the written conjecture shows how properties in steps of the construction have become the premise (they follow the "because"), while the perceived derived-construction-invariant has become the conclusion of the statement. However, this formulation shows traces of steps of the argumentation, and it is written in a form that contains the conclusion "ABCD is a trapezoid", moreover a "right trapezoid", before the premise "the bases are parallel and DC ⊥ to CB". The facts in the premise are still used as justifications in the argumentation and have not been completely elaborated into an "if…then statement". We will deal with the formulation of the conjecture in section 4.5 of this chapter. What seems to be important for this section is to highlight the dominant role of the derived-construction-invariant within the conjecture. This seems to strengthen our claim that construction-invariants interest solvers in this preliminary phase of the explorations, because they seem to be discoveries, worth spending a conjecture to highlight.

Excerpt 4.2.3. This excerpt is from two students' work on Problem 3, and it is an example of how two students perceived a derived construction-invariant without dragging any of the base points.

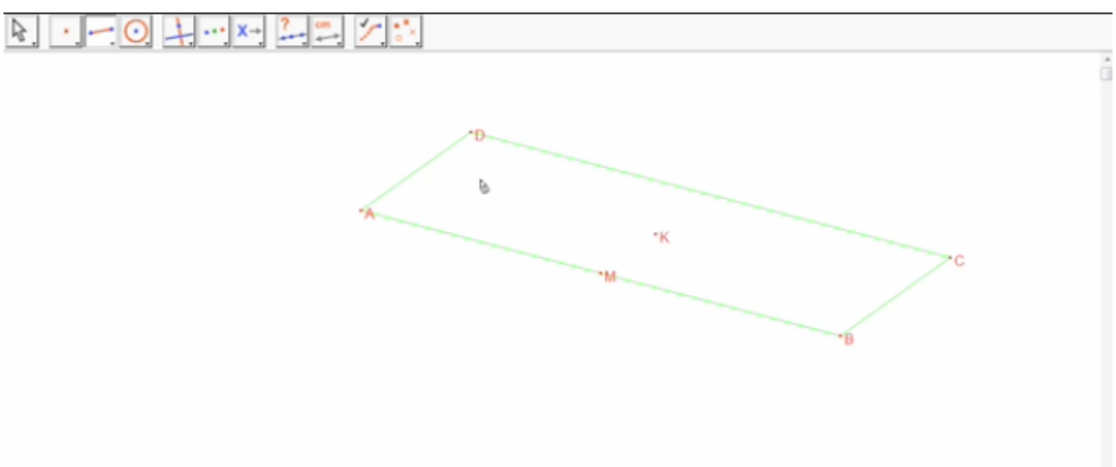


Figure 4.2.3: A screenshot of Ste & Sim's exploration.

Episode	Brief Analysis
[1] Ste: Make conjectures on the types of quadrilaterals..[rereading the assignment]...ok, good.	Ste re-reads the task.
[2] Sim: AM equals MB, ... [3] Sim: BK...equals KD, no? [4] Ste: ehm, wait [5] Sim: CK equals KA...it's always a parallelogram, therefore. [6] Sim: Because the diagonals intersect [7] Sim&Ste: at their midpoints. [8] Sim: Ok. [Written conjecture: "ABCD is always a parallelogram."]	Sim immediately refers to the steps of the construction to explain why ABCD is "always" a parallelogram ([5]). The explanation occurs through a deduction, using the theorem: "if the diagonals of a quadrilateral intersect at their midpoints, the quadrilateral is a parallelogram", together with the fact that BK equals KD ([3]) and CK equals KA ([5]) from the properties contained in the steps of the construction.

Table 4.2.3: Analysis of Excerpt 4.2.3

The students seem to immediately perceive the property "ABCD parallelogram" a derived construction-invariant, because they seem to immediately interpret the property at a theoretical level as in Excerpt 4.2.2, without needing to move the figure at all to check generality. In fact during the entire excerpt the students do not move the figure. Instead they seem to recognize a familiar type of quadrilateral, a parallelogram, and recognize it as significant in order to respond to the question they read in the task ([1]). This strengthens our claim that construction-invariants interest solvers in this preliminary phase of the explorations.

Further evidence that derived construction-invariants seem to be perceived as "discoveries" is that in the final formulation of the written conjecture ("ABCD is always a parallelogram") the premise (that consists of the steps of the construction) is implicit. The only reference to the premise can be seen in the use of "always" which seems to link the Cabri-figure to the steps of the construction which the students used as arguments to prove the statement. It seems as if the perception of a construction invariant as a property of the figure which is "always" true overpowers the need to write a proper mathematical "if…then statement". Although the students perceive various construction invariants (AM congruent to MB, BK congruent to KD, CK congruent to KA), the construction invariant that is featured in the conjecture (ABCD parallelogram) is not explicit from any of the construction steps. It seems likely that the solvers choose to make a conjecture having this invariant in it because part of the task is to find which types of quadrilaterals ABCD can become. In any case, our distinction between basic construction-invariants and derived construction-invariants seems to be insightful for describing such behaviors.

4.2.1.2 Point-invariants. When solvers investigate invariant properties of a Cabri-figure, they may be deceived by properties that seem robust invariants when a certain base point is dragged, but that are not robust invariants when a different base point is dragged. We therefore conceived a *point-invariant* as

> *a geometrical property that is true for a particular choice of a base-point of the construction, while the other base-points are fixed.*

If the particular base-point considered is P, we will call such invariant a *P-invariant*.
In the excerpt below we will show how two students can perceive a point-invariant and how the notions of point-invariant and (basic and derived) construction-invariant can be a useful tool of analysis.

Excerpt 4.2.4. This excerpt is from two students' work on Problem 1. It shows how two students notice and describe a point-invariant. The name of the solver who is holding the mouse is in bold letters.

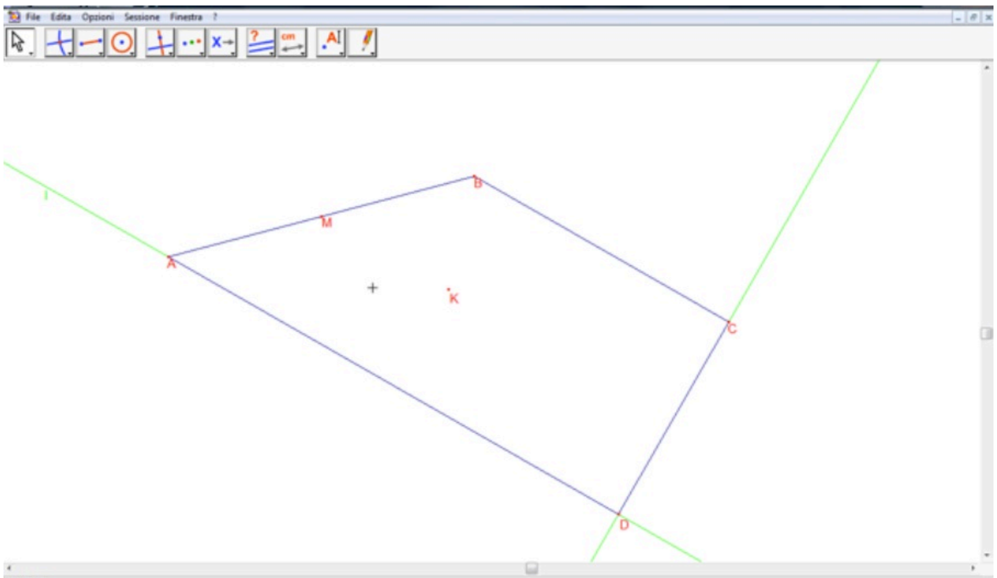


Figure 4.2.4: A screenshot of Ale & Pie's exploration

Episode	**Brief Analysis**
[1] **Pie**: the segment BC...if it varies what does it depend on?	The solvers become interested in segment BC and how it varies ([1])
[2] **Pie**: So, point B is the symmetric image of A...	or is "fixed" ([3]), while dragging the
[3] Ale: I think that the segment [pointing to BC] is fixed.	base point A.
[4] **Pie**: ...and C is the symmetric image of A with respect to K. Therefore if I vary A, C varies too.	Ale perceives the length of segment BC as fixed.
[5] **Pie**: because...they are...I mean A has influence over both B and C.	Pie perceives the dependence of BC on A because, as he says, B and C
[6] Ale: But the distance between B and C always	are both symmetric images of A and

stays the same. [7] **Pie**: Here there is basically AK and KC are the same and AM and BM are always the same. [8] Ale: Yes, try to move it? [referring to point A] [9] **Pie**: yes. [10] Ale: Hmm... [11] I: What are you looking at? [12] Ale: No, nothing, just that...I wanted to...now we can also put that the distance between B and C always stays the same...in any case it does not vary. [conjecture: As the exploration continues, Ale's idea is overcome before the students write a conjecture. Instead Pie focuses on the derived construction-invariant "MK parallel to BC" and writes the conjecture: "The segment MK is parallel to BC."]	therefore varying only A will make them both vary ([2], [4], [5]). Ale interrupts insisting on the invariance of the length of BC ([6]). Pie attempts to describe the behavior of the figure while dragging A. Ale insists on wanting to see the invariance of BC during dragging. Pie seems to agree with Ale's observation on the length of BC, but seems less convinced. Ale strongly states once again his perception of the length of BC being invariant.

Table 4.2.4: Analysis of Excerpt 4.2.4

We propose this excerpt as an example in which our new terminology with respect to invariants seems useful for analyzing the solvers' behavior. Such terminology allows us, for example, to interpret Ale's insistence on the length of BC being constant as his perception of such invariant as a (derived) construction-invariant. We believe this because he uses "always" in [6] and in [12], and strengthens his claim by adding that "in any case it does not vary" ([12]).

Moreover, our terminology allows us to explain Pie's behavior, in this excerpt and in the continuation of the exploration before the formulation of the written conjecture, as his correctly interpreting the length of BC as an A-invariant. We claim this because later, after this episode but before writing a conjecture, Pie will show that the length of BC is not a construction-invariant, by dragging a different base point, and showing that it varies. Moreover, during this excerpt Pie seems to focus more on explaining *why* the invariance might be the case ([7]), referring to point A frequently in his interventions. This seems to show that Pie seems more inclined to correctly perceive the property "length BC constant" as an A-invariant. However, faced with Ale's strong belief in "length BC" as a derived construction-invariant, for the moment Pie seems to accept it as such. The notions of construction-invariant and point-invariant have revealed themselves to be very useful in the analysis of other similar episodes in different solvers' explorations.

4.2.1.3 Basic Properties and Minimum Basic Properties. We also observed a recurring behavior related to the perception and choice of an invariant to induce. When looking for particular types of geometrical figures during wandering dragging, the solver may either notice that the considered figure can become a different (more particular) type of geometrical figure for some positions (or dispositions) of the base points, or s/he may try to make the figure into a particular configuration. In this second case the *guided* component (see Section 2.5 on our introduced dragging modalities) of our notion of wandering dragging becomes evident. In order to accomplish the task of investigating whether a certain type of geometric figure can occur, the solver may choose to substitute the whole figure with a characterizing property that may be easier to induce on the Cabri-figure. For example in the simulated exploration in Section 4.1 the solver could have worked with the property "diagonals intersecting at their midpoints" to investigate the case of the parallelogram. This phenomenon of substitution of a property with one that is considered “easier” to maintain is recurrent, and led us to introduce the following definition of *basic property*:

> *a property immediately taken from a definition or characterization of a type of geometrical figure.*

This property is in a logical relation, in this case a double implication, with the type of geometrical figure the solver is investigating, and it may serve as a bridge during the rest of the exploration. In fact the solver may refer to this property instead of to the type of geometrical figure s/he is exploring, because the conclusion, which describes the type of geometrical figure, is implied by the basic property being true. In particular, the solver may use a basic property of the theoretical geometrical figure s/he is referring to and *apply* it (mentally) to the construction, linking it and comparing it to the premises obtained from the steps described in the step-by-step construction. If part of the basic property is already in the premise, then the solver may “slim down” this basic property to a *minimum basic property*. Such a minimum basic property, together with the hypotheses from the steps of the construction, logically implies the conclusion, which is the type of geometrical figure investigated. For example, in the simulated exploration the solver could have induced “PD congruent to PA” in order to explore the case of the parallelogram. In this case the basic property, which may also be a minimum basic property, becomes the intentionally induced invariant (III) that is used during the maintaining dragging applied to the figure.

Sometimes solvers use basic properties, or minimum basic properties, to make the task of maintaining dragging easier. In this case, the minimum basic property is not conceived until after the maintaining dragging starts with the induction of an intentionally induced invariant (III), like “a type of geometrical figure”. If this happens, the initial intentionally induced invariant (III) is substituted with the minimum basic property which becomes the new intentionally induced invariant (III) that the solver tried to induce.

With respect to the formulation of a conjecture, “the case” of a particular geometric figure that is recognized will become the conclusion of the future conjecture, while the solver proceeds to search for conditions that give such case. The substitution of the whole “case” with a basic property or minimum basic property makes this search easier. However, once conditions are obtained, through the geometric description of the path (GDP) and the invariant observed during dragging (IOD) (see section 4.2 for details), the solver skips over the basic property or minimum basic property directly to the “case” s/he was interested in initially. Thus we also refer to the basic property or minimum basic property as a “bridge property.”

The following excerpts are taken from various students’ work, and they show different occurrences of bridge properties. Excerpt 5 illustrates how solvers notice and make use of a minimum basic property, while Excerpt 6 shows how a minimum basic property is conceived to help the solvers carry out maintaining dragging.

<u>Excerpt 4.2.5.</u> This excerpt is taken from two students’ exploration of Problem 2, and it exemplifies the identification of a basic property, slimmed down to a minimum basic property, which the solvers use to obtain the configuration they are interested in. The name of the solver who is performing the dragging is in bold letters.

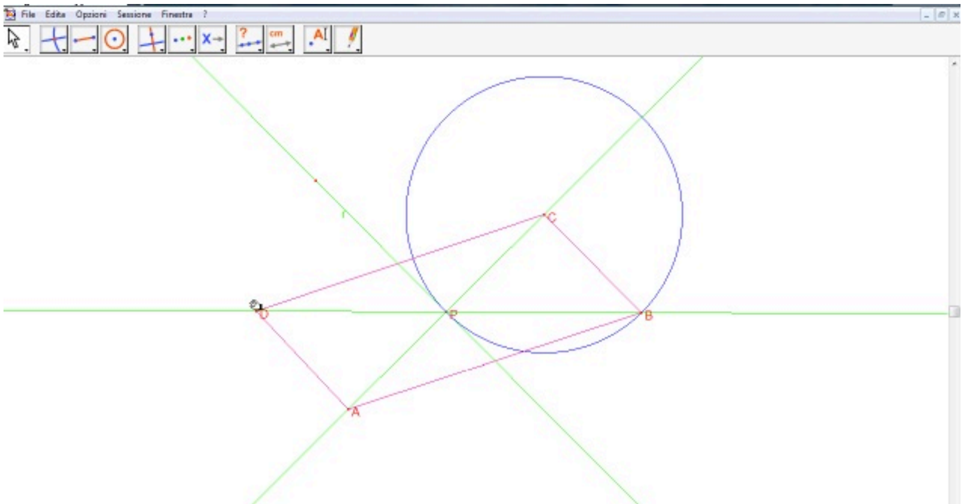


Figure 4.2.5: A screenshot of F & G's exploration

Episode	**Brief Analysis**
[1] **F**: wait, it is a…let's try to for example make it become a parallelogram. [2] G: No… yes, go. [3] **F**: Like this.	F proposes to try to make ABCD a parallelogram ([1]) and seems to be unsure about how to drag the base point D in order to do this.
[4] G: So, for it to be a parallelogram… I think it always is a parallelogram. [5] **F**: Let's try? [6] G: No, no, there, it's a parallelogram… [7] **F**: because like this it's…	F's initial dragging suggests to G, for an instant, that "ABCD parallelogram" might be a construction-invariant (notice the use of "always" in [4]), but then further movement of the base point leads G to quickly discard such hypothesis ([6]).
[8] G: I understand! so, C… we have to have the diagonals that intersect each other at their midpoints, right? [9] **F**: Right.	G conceives a basic property that might help F make ABCD into a parallelogram, and he exclaims: "I understand! [it: ho capito!]" ([8]).

[10] G: And we know that CA is always divided by P. [11] **F**: exactly, so… [12] G: therefore it's enough that PB is equal to PD. [13] **F**: exactly. [14] G: you see that if you do, like, "maintaining dragging"… trying to let them more or less be the same [15] **F**: exactly… well, okay.	G proceeds to "slim down" the basic property making it into a minimum basic property: "it's enough that PB is equal to PD" ([12]). Making use of the fact "CA is always divided by P" ([10]) G concludes that a sufficient condition (notice the "for it to be" [4] and "it's enough that" [12]) for ABCD to be a parallelogram is "PB is equal to PD" ([12]), and therefore proposes this as an intentionally induced invariant (III) ([14]).

Table 4.2.5: Analysis of Excerpt 4.2.5

We think that this excerpt is a good example of how a basic property, with respect to the initial III that has been conceived, can be "slimmed down" to a minimum basic property and used to make maintaining dragging manually easier. Initially F seems to be struggling with maintaining dragging, trying to maintain the III "ABCD parallelogram". Then G notices that this is equivalent to maintaining the quadrilateral's diagonals intersecting at their midpoints (basic property), and "slims down" this property to "PB congruent to PD" (minimum basic property). Moreover he recognized that this is a sufficient condition in order to maintain the basic property and thus the initial III. Therefore the solvers are able to use the property "PB congruent to PD" as a "bridge" to their initial III. in order to proposes He proposes to use the property "diagonals that intersect at their midpoints" as a basic property, in that its being satisfied will definitely imply the desired property to induce (ABCD parallelogram). We speak of a "bridge" property because the minimum basic property acts as a bridge to the III both throughout the maintaining dragging and when the solvers are ready to formulate the conjecture. In fact, in the written conjecture the solvers produce after finding an invariant observed during dragging (IOD) in this exploration, the solvers do not refer to the property “PB equal to PD”, but directly to “ABCD parallelogram” as the conclusion of the statement of their conjecture.

<u>Excerpt 4.2.6.</u> This excerpt is from two students’ work on Problem 2, and it shows how two students make use of a minimum basic property to make the task of maintaining dragging easier to accomplish.

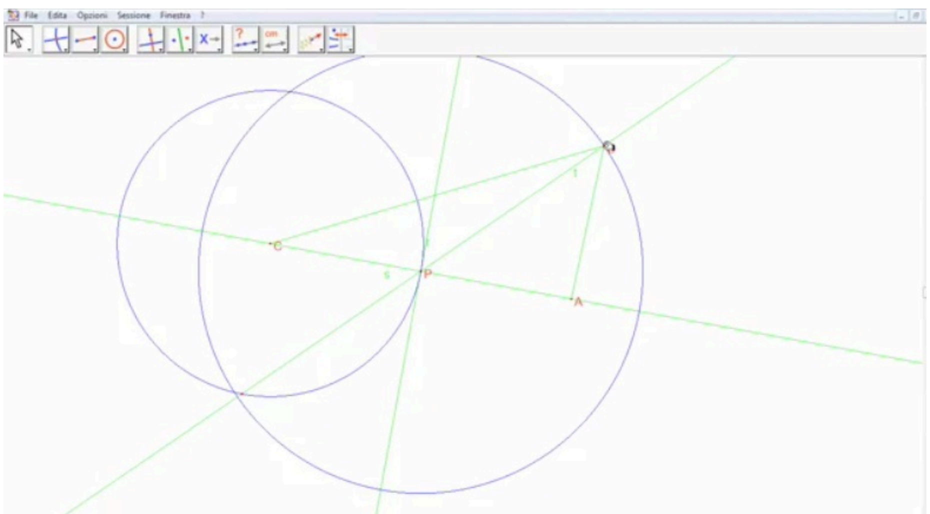


Figure 4.2.6: A screenshot of Giu & Ste's exploration

Before the moment when this excerpt starts the two students have made conjectures about when ABCD is a parallelogram, but they have not been able to drag the base point D maintaining such property. This seems to stimulate Giu to come up with the property “this thing here” (concurrence of the intersection if the two circles and the line PD) that he refers to in [1].

Episode	**Brief Analysis**
[1] Giu: Try to see...so that [It: “in modo che”] this thing here...remains... [concurrence of an	Giu seems to have thought of the minimum basic property ([1]) by using

intersection of the two circles and the line through P and D]. [2] **Ste**: and let's do trace of D. [3] Giu: Actually...I was thinking of the trace...no, you're right because B is always on the circle... [4] **Ste**: what a big idiot!... [5] Giu: and do the trace of D, exactly. [6] **Ste**: So, let's call this one...B so this way it looks nice, there. [7] Giu: At least this way we can refer to them somehow! [8] **Ste**: Exactly. So... [9] Giu: Go, trace...	the basic property "diagonals that intersect at their midpoints", which he refers to earlier during the activity. The minimum basic property arises because of the desire to drag the base point D maintaining ABCD a parallelogram, a property that seems to be too difficult to maintain without a simplification of what to observe during maintaining dragging and maintaining dragging with trace activated ([1], [10]).
[10] Giu: Try to maintain all these things here [pointing to the intersection of the two circles and line PD, where B is marked] [11] **Ste**: It'll be hard... [12] Giu: Try! [13] **Ste**: There...	The basic property "diagonals that intersect at their midpoints" is slimmed down to "PB congruent to PD" and then to "this thing here"/"these things here" ([1]/[10]).
[14] Giu: There, more or less...yes, yes, yes, not too much, there. [written conjecture: "If D $\in$ circle with radius PC and center P, and PD passes through the intersection of the two circles$\Rightarrow$ABCD is a	To maintain seems to be easier for S than maintaining "ABCD parallelogram". In fact he is afraid it will be hard , but then succeeds, with

rectangle."]	support from Giu ([14]).

Table 4.2.6: Analysis of Excerpt 4.2.6

In this excerpt we focus on how the minimum basic property ("this thing here" ([1])) is successfully used to make the task of maintaining dragging easier. In fact we chose to begin this excerpt with the solvers' identification if a property to maintain during dragging instead of their initial III. This property has been reached through the slimming down of the basic property "diagonals intersecting at their midpoints", reduced to "PD congruent to PB", and highlighted by the concurrence of three objects (two circles and a line) in the solvers' figure. We do not focus on the slimming-down process here, but instead on the practical function that this minimum basic property has with respect to the task of maintaining dragging. The solvers then succeed to perform maintaining dragging and perceive an invariant observed during dragging (IOD), which they use as a premise in their final conjecture, in which the conclusion is "ABCD parallelogram".

However in this episode the minimum basic property seems to be solely used as a tool to overcome a manual difficulty, and thus as a "bridge" for maintaining dragging but not for the formulation of the conjecture, which still contains traces of it in its premise. This is why in this excerpt we prefer not to refer to the minimum basic property as a bridge property; its potential of "bridging" seems to be only partially exploited, in particular it does not seem to act as a bridge to the conjecture.

4.3 Invariant Observed During Dragging (IOD), Path, and Geometric Description of the Path (GDP)

According to our model, the exploration process continues with the search for a "way" to maintain a certain property invariant during dragging. This "way" to maintain may be interpreted as a "condition under which" the III is (visually) verified. Thinking about the DGS, which is the domain of phenomenology in which such interpretation of a relationship of conditionality occurs, “conditionality” may be associated to “causality”. That is, the connection between direct and indirect movement of objects can have the effect of leading the solver to link the idea of “cause of an effect” (direct movement “causes” indirect movement) to “condition for…”, and finally to logical dependency (“if…then…”). This may happen because while dragging the base point trying to maintain the III, the solver’s attention can shift to the movement of the dragged base point (and keep shifting back and forth to and from it). The combination of sight and haptic perception may guide the solver’s interpretation of “some regularity” in the movement of the base point. Moreover, an expert will have activated the maintaining dragging scheme with the explicit intention of looking for such regularity. In this case the solver is confident about the fact that dragging continuously the base point considered and maintaining the chosen III *is* possible. The solver may refer to the “dragging continuously” as a unit, as “something,” which can allow him/her to express the regularity of such continuous dragging as what s/he is looking for. We call this “something”, which does not yet have the regularity expressed but that withholds the potential of being expressed through it, a *path*, and provide the following definition:

> *a path is a continuous set of points on the plane with the following property: when the dragged-base-point coincides with any point of the path, the III is visually verified*.

Summarizing, the characteristics of a path are:

- being a continuous set of positions for the dragged-base-point,
- when the dragged-base-point is in any of the positions of the points of the path the III “happens”,
- it has the potential of making some regularity in the movement become explicit.

The possibility of explicitly dealing with the object we define as path seems to be fundamental in expert use of maintaining dragging, and it therefore plays a central role in the cognitive model. We have further developed the notion of path, as a finding of our research, and will focus on such notion in Chapter 6.

Dragging with trace activated is a tool that the user may choose to use in order to have additional guidance in making the potential regularity evident is on the dragged-base-point. This may help the solver describe the regularity s/he was looking for, as s/he may use it to propose a *geometric description of the path* (GDP), that is

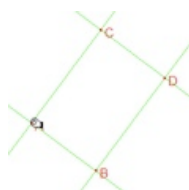
> *a description of the path in terms of a known geometrical object linked to the Cabri-figure.*

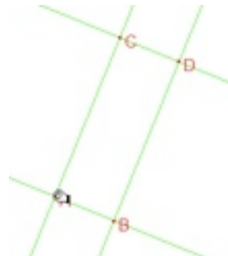
After the activation of the trace, a set of points – linked to a possible regularity in movement – appears on the screen as a trail left by the dragged-base-point. This mark may suggest a precise geometrical object which can be described in relation to the rest of the Cabri-figure. For instance, in our initial example, it may become clear that a GDP is a circle, and more precisely, the circle with center in A and radius AP. From a GDP the solver can reach the property s/he was looking for during maintaining dragging, that is the invariant observed during dragging (IOD). In the simulated exploration in section 4.1, this would be: “D belongs to the circle with center in A and radius AP”.

We will now show how the path, its expression through the geometrical description of the path (GDP), and its elaboration into an invariant observed during dragging (IOD) come into play in various excerpts from the transcripts of some students’ explorations. The first excerpt is an example of two students searching for a GDP using maintaining dragging with the trace activated. Their GDP seems to coincide with how they interpret the mark left by the trace. The second excerpt illustrates how two students seem to have conceived a GDP, which does not seem to be confirmed by the mark left by the trace during maintaining dragging with the trace activated. In fact in this excerpt an initial GDP is rejected thanks to characteristics of the path brought out by the trace. The third excerpt shows how an IOD emerges from a GDP obtained by correctly interpreting the trace, and how the IOD is then constructed by the solvers.

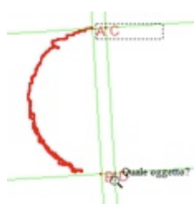
<u>Excerpt 4.3.1</u>

This excerpt is taken from two students' work on Problem 4, and it shows how two solvers, who seem to have conceived a path, reach a GDP which they seem satisfied with. The bold letters refer to the solver who is dragging. Since the excerpt of the transcript is rather long we have divided it into several episodes.

Episode 1	**Brief Analysis**
[1] **Ste**: I have to make it so that the... [2] Giu: B stays [3] **Ste**: that...uh, B remains on the intersection.  [4] Giu: Exactly. [5] **Ste**: which is...I mean I have to drag this, right? [6] I: Maintaining the property rectangle... …	The solvers resort to the bridge property (see section 4.2.1.3) "B on the intersection" ([3]) to make the task of maintaining dragging easier. The solvers have chosen "ABCD is a rectangle" as an III.
Episode 2	**Brief Analysis**
[12] **Ste**: Identical...ta-ta-ta-ta...ta-ta-ta [13] I: Giu, what are you seeing?	While Ste is concentrated on maintaining the III ([12]-[14]), Giu seems to be looking for a GDP, and recognizes a continuous curve

[14] Giu: Uhm, I don't know...I thought it was making a pretty precise curve...but it's hard to ...to understand. We could try to do "trace"  [15] **Ste**: trace! [16] Giu: This way at least we can see if...	("pretty precise curve" [14]) instead of discrete positions. He then wants to better understand ([14]) and "see" ([16]), so he proposes the use of the trace tool ([14]).
Episode 3 [17] **Ste**: Where is it? [18] Giu: Uh, if you ask me... [19] **Ste**: Trace! [they giggle as they search for it in the menus] [20] **Ste**: Trace of A... …	**Brief Analysis** After the trace is activated ([17]-[20]) Ste starts maintaining dragging again.
Episode 4 [28] I: So Ste, what are you looking at to maintain it? [29] **Ste**: Uhm, now I am basically looking at B to do something decent, but...  [30] I: Are you looking to make sure that the line goes through B? [31] **Ste**: Yes, exactly. Otherwise it comes out too sloppy...	**Brief Analysis** Ste is using the property "the line goes through B" as his III ([29], [30]). Both students show the intention of uncovering a path by referring to "it" ([31], [33], [34]).

[32] I: and you, Giu what are you looking at? [33] Giu: That it seems to be a circle... [34] **Ste**: I'm not sure if it is a circ... [35] Giu: It's an arc of a circle, I think the curvature suggests that.	Giu, in particular concentrates on describing the path geometrically and he seems to recognize in the trace a circle ([33]) or an arc of a circle ([35]).
Episode 5 [36] **Ste**: Yes, but.. [37] Giu: But passing through B [38] **Ste**: Ah yes, B [39] Giu: B because it can also become a line [40] **Ste**: Yes, it could be B. [41] **Ste**: I would dare to say with center in C?...no, it seems more, no.  [42] **Ste**: It seemed like [43] Giu: No, the center is more or less over there...in any case inside [44] **Ste**: Hmm [45] Giu: Ok, do half and then more or less you understand it, where it goes through. [46] **Ste**: But C is staying there, so it could be that BC is...is [47] Giu: right! because considering BC a diameter of a circle,	**Brief Analysis** The solvers' attention seems to shift to the mark left on the screen by the trace. Now that a first GDP is given, the solvers try to ameliorate the description by adding properties: "(a circle) passing through B" ([37], [38], [39], [40]), "with center in C" ([41]), with BC as a diameter ([46], [47]). As Ste continues to drag, Giu checks and confirms the suggested properties and tries to justify them providing argumentations based on visual observations, recognition of geometrical properties, and the knowledge of particular theorems ([49], [55]).

[48] **Ste**: Well yes, actually it passes through C also because if then I make it collapse, uh, [49] Giu: Exactly because CB is...consider it a diameter. A...so ABC is a right triangle [50] **Ste**: Aaaaa...because when A [51] Giu: B... [52] **Ste**: because when it comes to the point when...yes, well, anyway, we understand, then it arrives to C. [53] Giu: Yes, because this way, since it is right, [54] **Ste**: and this one here is a diameter [55] Giu: exactly. Since the angle in A is always right, ABC can be inscribed in a semicircle. [56] I: Ok.  [57] Giu: ...which is what is being traced by A. [58] **Ste**: Exactly...very theoretically. [59] **Ste**: Well... [60] **Ste**: I wouldn't call this...aaaa...there [61] **Ste**: No, but it jumps, when it's closer it's easier.	For example, he justifies the property “BC is a diameter” using the theorem “a triangle inscribed in a semicircle is right” ([49]) together with the consideration that the angle in A needs to be right in order to have a rectangle, which Ste agrees with ([55]). Ste seems to have some difficulty dragging as he drags A closer to C, but is able to overcome the manual difficulty.
Episode 6 [62] **Ste**: It surely can look like a circle. [63] Giu: Well, in theory...you can see it goes	**Brief Analysis** Ste continues to drag and both solvers seem to be checking the

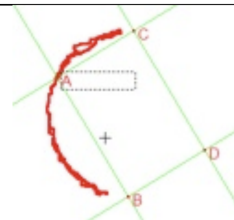
through B and C. [64] I: Ok, are you sure of this? [65] Giu and Ste: Yes. 	proposed GDP, confirming it ([62], [63]) with considerable confidence ([65]).

Table 4.3.1: Analysis of Excerpt 4.3.1

In this Excerpt we can see how the GDP arises and is used to conceive an IOD. In Episode 1 Ste is using a bridge property (see section 4.2.1.3) to simplify the task of performing maintaining dragging, although he still seems to describe it as being difficult throughout this episode. In Episode 2 Giu seems to be searching for a GDP and identifies some regularity in the movement of the dragged-base-point. In particular this suggests that the solvers have conceived a path. Reaching a GDP, however does not seem to be a simple task. Ste's initial hesitation in Episodes 1 and 2 seems to be evidence confirming our idea that the coordination of visual and haptic sensations with an "overall" view of the figure is not easy to achieve, and it may be aided through the trace tool, which Ste immediately proposes to activate.

The solvers seem to help each other by separating tasks: as Ste concentrates on maintaining his bridge property invariant, Giu can focus on recognizing the mark left by the trace as "a pretty precise curve" ([14]). Moreover, there seems to be the intention of looking for something, which we interpret as making the path explicit, that is searching for an IOD though movement of the dragged-base-point along a GDP. In fact the solvers seem partially satisfied when they reach a first GDP in Episode 4. However, throughout Episode 5 they continue to refine their GDP, helping themselves with other geometric properties. These strengthen their argument about the correctness of their suggested refinements, and finally the solvers seem to be satisfied in Episode 6.

The solvers seem to confirm an IOD as they drag A along the circle with diameter BC in Episodes 5 and 6. Overall this Excerpt is a good example of how the GDP can arise and be used to develop the IOD, and therefore complete the search for a condition for the III to be (visually) verified.

Excerpt 4.3.2

This excerpt is taken from two students' work on Problem 2. It shows the solvers' belief in the existence of a path and traces of an implicit idea for the GDP. However the conceived GDP doesn't seem to correspond to what they observe during the maintaining dragging. They want to therefore make the path explicit through activation of the trace, and they use the trace to reject an incorrect GDP. The lines of the transcript are marked by their times relative to the beginning of the excerpt in order to show the development over time of this part of the investigation. In particular we chose not to include parts of the exploration in which the solvers were not investigating "the case of the parallelogram", as they refer to it. The bold refers to the solver who is holding the mouse.

Episode 1	**Brief Analysis**
(0:41) **F**: exactly. [he drags D a bit, in a way that looks like he is trying to maintain the property parallelogram] (0:48) G: you see that if you do, like, maintaining dragging … trying to keep them more or less the same… (0:57) **F**: exactly [murmuring]… well, okay.	F and G decide to use maintaining dragging to investigate "when ABCD is a parallelogram" (intent repeated in (2:41) and (3:05)). In a previous episode they have noticed that the property "ABCD parallelogram" can be substituted with the sufficient property "diagonals of ABCD

…	congruent", a bridge property (0:48).
Episode 2 (2:41) **F:** For the parallelogram, uh, let's try to use "trace" to see if we can see something. G: go, let's try [speaking together with him]…uh, "trace" is over there. … [They have a little trouble activating the trace]	F proposes to activate the trace in order to "see something" (2:41).
Episode 3 (3:05) G: and now what are we doing? Oh yes, for the parallelogram? (3:07) **F**: yes, yes, we are trying to see when it remains a parallelogram. (3:18) G: yes, okay the usual circle comes out. (3:23) **F**: wait, because here…oh dear! where is it going? (3:35) I: What are you looking at as you drag? (3:38) **F**: I am looking at when ABCD is a parallelogram. You try [handing the mouse to G]	**Brief Analysis** G reminds himself what their intention was and seems to be concentrating on the movement of the dragged-base-point, while F, who is dragging, concentrates on maintaining the property "ABCD parallelogram" (3:07). G (too?) quickly proposes a GDP (3:18). It is not clear what "usual" refers to: maybe to a previous investigation. However what F sees does not seem to be the circle he had in mind (maybe the circle centered in P with radius AC) and he appears unhappy and confused when he does not understand

…	"where it is going" (3:23). After repeating his intention of investigating "when ABCD is a parallelogram" (3:38) F hands the mouse to G, asking him to try.
Episode 4	**Brief Analysis**
[G tries dragging some other points looking for the "draggable" ones, and there is a short diversion on "the case of the rectangle". Then G starts dragging point D.] (4:17) F: …turn it. No, it's not necessarily the same circle, because, I don't know at some point I don't know, keep going… by tomorrow… keep going… careful you are making it too long …	F uses what he sees to discuss why his initial idea (involving some circle he never describes explicitly) does not seem correct. He also tries to guide G while he tries to perform maintaining dragging with the trace activated.

Table 4.3.2: Analysis of Excerpt 4.3.2

We consider this Excerpt to contain evidence that the solvers have conceived a path, because when using MD in Episodes 1 and 2, the students express their intention either in a generic way ("to see something") or in a more specific way (to see "when" ABCD is a parallelogram). That is, the solvers seem to want to find a situation or configuration that "happens" simultaneously with the III, because they believe in the existence of "something" that will make the "parallelogram happen". Our model refers to this "something" as *path*.

In Episodes 3 and 4 the GDP plays a fundamental role in the solvers' conceiving and then rejecting a geometrical object along which the dragging may be thought to occur. F’s “same circle” probably refers to the one he has in mind, which didn’t coincide with what he saw in his trace (4:17 and 5:01). This leads to a rejection of the initial GDP. Overall the Excerpt seems to be evidence of the fact that the solvers, F in particular, have conceived a path and that they seem to know that they will need to describe it geometrically. Moreover they seem to be “expecting” it and “looking for it”.

Excerpt 4.3.3

This excerpt is taken from two students’ work on Problem 4. The solvers activate the trace while using maintaining dragging and they are able to reach a GDP and IOD that satisfy them, and proceed to construct the IOD. The bold refers to the solver who is dragging.

Episode 1	**Brief Analysis**
[1] **F**: so... Let's take A. Wait, let's first put A so that it is a nice rectangle. It seemed too good... [2] **F**: “trace”...A. [3] **F**: to maintain the property rectangle. [4] G: you are not maintaining it. …	F wants to perform dragging with trace activated to gain insight into when the property rectangle is maintained ([3]), and starts to perform maintaining dragging, dragging the base-point A.
Episode 2	**Brief Analysis**
[5] G: circle with... [6] **F**: no [7] G: eh, no. [8] **F**: look at C. C doesn't move.	Notice how G, who is not dragging, observes both the property to be maintained ([4]) and the emergence of a GDP ([5] and following). G “sees” a circle

[9] G: I see a kind of circle with... [10] **F**:... with radius CB, and center... [11] G: No, with diameter AD, I see. [12] **F**: Ah, wait I am... [13] G: I see it with diameter AD. like with diameter AD.	([9]), while F (who is dragging) seems to be focusing more on what is and is not moving. He notices that C does not move ([8]), and seems to want to use this point to enhance the GDP that G has started to provide ([5], [9]). G, instead, insists on a GDP as the circle with diameter AD.
Episode 3 [14] **F**: wait, no, let's...uhm... [15] G: with diameter CB instead, that... as a consequence... [16] **F**: I would say that I made it very ugly, but... no, I would say... I would trace CB and its [17] **F**&G together: midpoint [18] G: for the radius [19] **F**: Exactly! [20] G: go! Get rid of... [21] **F**: then the radius [22] G: get rid of the trace. ….	**Brief Analysis** The solvers briefly discuss which GDP to use and by line [17] they both seem to agree on what to construct as the GDP ([17], [18]). The conflict between the GDPs is resolved, as F and G agree on the GDP as the circle with diameter BC passing through B.
Episode 4 [23] G: Ok, go. [24] **F**: okay, so let's draw...yes... no first	Brief Analysis The solvers now construct a circle with center in the midpoint of BC and passing

let's draw [25] G: no, it's enough that you do, I think, midpoint. … [26] **F**: Should we call it? [27] G: Circle, do circle. [28] G: eh, let's choose... [29] **F**: well, I would say B and C because they are the two points that don't move...here...yes, because actually now we take A. [30] G: eh, we did it...cute! [31] **F**: yes, definitely.	through B which is the GDP they have agreed upon. F feels the need to explain again why he prefers the diameter BC to AD, basing his argument of the fact the former points do not move ([29]), as if this gave them a different status (which unfortunately he does not make more explicit than this). To make sure the constructed GDP is a good one, F drags A along it, and the solvers seem to be satisfied ([30], [31]).

Table 4.3.3: Analysis of Excerpt 4.3.3

During the exploration, before the part this excerpt is taken from, the solvers were uncertain whether the interesting configuration "ABCD rectangle" was possible to maintain during dragging or not, and therefore whether a (continuous) path existed or not. The solvers reached the conviction that there is a path by noticing more and more "good positions" for the dragged-base-point. Thus in [1] F demonstrates belief in the existence of a path and he wants to perform dragging with trace activated to gain insight into such path ([3]). G seems to focus on the mark left by the trace and on trying to describe what he sees emerging.

The conflict that emerges between the two GDPs seems to provide evidence that the solvers have conceived a path and are looking for a condition for the III to be maintained. They are looking for such condition as "dragging along some regular path" that they expect to be able to describe geometrically. In particular, we notice how in [16] F refers to "it" as the mark he "made". The mark left by the trace seems to be an "object" for the solvers, and it seems to have the purpose of making something else visible. This something else is the *path*.

We interpret G's not wanting to use the trace any longer ([22]) as evidence that the solvers make use of the trace solely to make the GDP explicit and to simplify the task of providing a GDP, visualizing it through the trace instead of only through the movement of the dragged-base-point. Moreover the solvers want to construct the object representing their GDP and to try dragging along it. Doing this, both students seem to be checking the validity of their final GDP, probably by making sure the III is visually maintained while dragging along it. Thus the GDP allows the solvers to have a good description of the object-to-drag-along, which can then be interpreted as the IOD, which is "A moves on a circle with diameter BC" or "A belongs to a circle with diameter BC". Once the IOD is conceived this checking of the GDP becomes a (soft) dragging test in which a conditional link between the IOD and the III is being confirmed, as we will discuss in section 4.4 of this chapter.

4.4 Putting Together the III and the IOD: the Conditional Link (CL)

At this point of the conjecture-generation process the solver is dealing with two invariants that seem to be occurring simultaneously: the one s/he induced intentionally (III), and the one observed during dragging (IOD). Although the two invariants may be established by now, it is possible that a relationship between them may not have yet been established, or that the solver is not even aware that a link between them exists or should exist. We will describe these situations of difficulty in Chapter 5. A first link between these two invariants may be a link of "mechanical causality", that is a relationship that arises within the phenomenological realm of the DGS. In this realm one invariant, the IOD, is controlled directly, while the III can only be induced indirectly, in a "mechanical" way by acting on the IOD. According to the MD-conjecturing Model, this mechanical causality needs to be interpreted geometrically, as a conditional link (CL), which we define as:

a relationship of logical dependency between two invariants perceived by a solver, and interpreted within the world of geometry.

A first link between the two perceived invariants is given by their simultaneity, and in addition, after discovering the IOD, the solver can directly act on the base point to maintain it, and indirectly feel and observe the maintaining of the III, as a consequence. This may guide the solver to perceive "mechanical causality" within the DGS, and ultimately "conditionality" within the world of geometry. The conditional link will finally be explicitly expressed as a conditional statement, the conjecture, in which the IOD can become the premise and the III the conclusion of the statement.

We suggest that a bridge between the experiential field in the phenomenological domain of a DGS and the formal world of Euclidean geometry may be established through the interpretation that may be summarized briefly as follows:

simultaneity + control → causality within Cabri → conditionality in Geometry

simultaneity + direct control → premise of the final statement

simultaneity + indirect control → conclusion of the final statement.

Expert users of the maintaining dragging may easily interpret - almost unconsciously - the emergence of simultaneous invariants as a conditional link between the corresponding geometrical properties. Therefore the process of conjecture formulation may lead, in a straightforward manner, to a successful outcome, that is the formulation of a conjecture linking two perceived invariants. We will describe this behavior in Chapter 6.

However, in some cases solvers may experience simultaneity and control with respect to two invariants, but not conceive a CL between them (at least not in a way that can be perceived by an external observer). This may be due to an inability of the solver to capture mechanical causality within a DGS, or the inability to make the transition from the world of Cabri and the phenomena that occur within it to Geometry and conditional links between geometrical properties. As a consequence, a conjecture may not be produced. In this case a link between invariants may be perceived in the world of Cabri, but no conditional link seems to be conceived. We will discuss these possibilities in Chapter 5 and Chapter 6.

Once the two invariants are identified, we can observe different manifestations of the solver's belief in a CL between them. These manifestations have to do with different ways of dragging with the intention of checking the link. In order to become more convinced of the existence of a link between these properties, the solver may behave in the following ways. After constructing the object that corresponds to the GDP, s/he may drag the base point approximately along this object with the intention of verifying the simultaneity of the III and IOD. We refer to this kind of dragging check as a soft dragging test. If, instead, s/he constructs the IOD robustly and drags the base point, verifying the simultaneity of the III and IOD, s/he has performed a robust dragging test. In particular, through a robust construction of the GDP and a re-construction of the Cabri-figure with a new property the solver can express the change of both the epistemic and the logical value of the IOD. The new property consists in the dragged-base-point now being linked robustly to the object representing the GDP: such a property is no longer a possibility, but a fact expressing something "true". After the redefinition of the base point, if the III also becomes a robust invariant, the dragging test is passed, establishing a precise logical status of the relationship between the invariants. The robust dragging test is extremely convincing: in fact, constructing one property robustly will have led to the robustness of another. This not only shows the fact that the IOD and III do occur simultaneously, but also that there is a precise conditional relationship between them. At this point the solver can express the link of simultaneity (and potentially of mechanical causality) between the III and the IOD into a conditional link between the corresponding properties, and think something along the lines of: "the robustness of one property implied the robustness of another."

When a solver performs a dragging test in the ways described above, it shows that s/he is aware of a CL. Further evidence is provided by what solvers say, and by their effectiveness in checking the behavior of the various elements they are keeping track of. It is worth remarking that most evidence of the solver's awareness of a CL is indirect, as shown in the excerpts below. In particular, the excerpts provide different examples of evidence of the solvers' awareness of a CL.

In the first excerpt (Excerpt 4.4.1) we will show the smooth emergence of a CL described through the evidence of effective use of checking through a form of the dragging test. The second excerpt (Excerpt 4.4.2) shows how evidence may be provided by the solvers' realization that a particular GDP and dragging along it do not provide a satisfactory IOD, so the solvers make new hypotheses and modify their proposed GDP and IOD. The evidence provided by realizing that a particular GDP and moreover an IOD are not satisfactory can affect the GDP and the IOD themselves. Sometimes the modification is a *generalization of the GDP* (section 4.4.1). We present an example of this in Excerpt 4.4.3.

Excerpt 4.4.1

This excerpt is taken from a student's work on Problem 2, and it shows how the student shifts her attention from the movement of the dragged base point to the III she is maintaining. Even though she has not constructed the circle that represents her GDP, her dragging witnesses that she has established a CL.

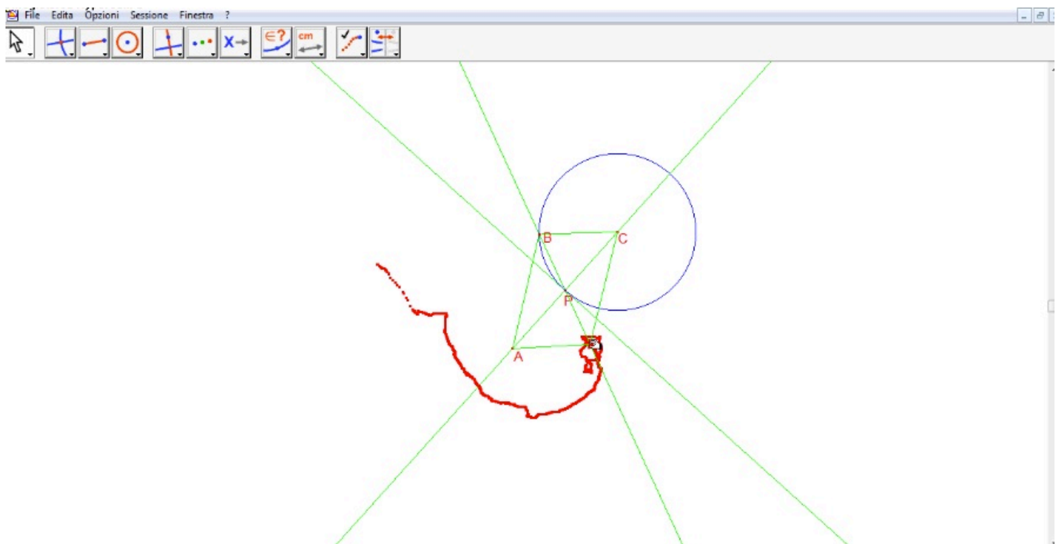


Figure 4.4.1: A screenshot of Isa's exploration.

Episode	**Brief Analysis**
[1] Isa: parallel...here, ok, it collapses...it becomes a line, I mean all the points of all the lines coincide.	As Isa drags D she explains that she is looking at the sides of the quadrilateral and trying to keep them parallel ([1], [5], [6]).
[2] Isa: and over here...ok…no, no, no, no	
[3] Isa: There it collapses...so...	
[4] I: What is it that you are looking at here to do it?	
[5] Isa: I am trying to make a parallelogram, uh, to put two sides parallel.	Isa's III during this application of MD is the "two sides parallel" ([5]). She hesitates

[6] Isa: and so AD and BC. [7] I: uhm. [8] Isa: So now I need to go back a second… no, no, no, no… [9] I: eh, it's hard when you go close… [10] Isa: alright, anyway, here it should...how nice!...be here. [11] Isa: There… [12] I: Let's continue over here… [13] Isa: So, like this...uhm...here it becomes easier...There, more or less [14] I: uhm. [15] Isa: So, let's see to try it. So, if I construct, uh, if I move D on a circle with center in A, and, theoretically, radius AP... [16] I:...hmmm [17] Isa: ...I find that the quadrilateral is a parallelogram, except when, uh, D comes to lie on the line CA.	when the quadrilateral seems to collapse ([1], [3]), and expresses increasing and decreasing levels of difficulty in using maintaining dragging ([9], [13]). Isa seems to have conceived an IOD, which she states explicitly in line [15] ("I move D on a circle with center in A, and, theoretically, radius AP"), because she is able to predict what "should" happen ([10]). This indicates that she is focusing also on the IOD and while she is dragging she is establishing a CL between the IOD and the III. In the conditional statement ([15], [17]), which is her first expression of a conjecture (see section 4.5) the GDP (circle with center in A and radius AP), the IOD (moving D on the circle), and a CL (if IOD then III) are made explicit.

Table 4.4.1: Analysis of Excerpt 4.4.1

Overall this is an example in which the two invariants, the III and the IOD, and the CL between them emerge fluidly, almost as if the process of conjecture generation was occurring "automatically". We will return to this idea later in Chapter 6. Moreover, we use this Excerpt to highlight Isa's use of "when" ([9], [17]). As in other excerpts, the word “when” seems to refer to a time that corresponds to a specific position during the motion, which, according to the solver, corresponds to an exceptional phenomenon. In this case the exceptional phenomenon seems to be the “collapsing” of ABCD – exceptional with respect to the general “being a parallelogram” or even “a quadrilateral”. While in other occasions the word “when” seemed to be used to refer to a phenomenon that occurs over time (a movement for example), here Isa seems to use it to refer to an instant in which something interesting happens.

What the two uses of the word seem to have in common is that they also refer to a second phenomenon noticed by the solver that occurs simultaneously with the first exceptional phenomenon. In this case the second phenomenon is “D comes to lie on line CA” ([17]). Therefore a relationship of simultaneity is established between the two phenomena, expressed in a form such as: “when…occurs, …occurs”. The a-symmetry of the statement establishes an order in the simultaneity, which adds to the word “when” a causal meaning (within the world of the dynamic geometry) in addition to its temporal meaning. This may be the seed that gives origin to a CL that can then become a conjecture.

Excerpt 4.4.2

This excerpt (FS_Ud_F&G_p6_CLparall1 from 9:18 to 12:46) is taken from two students’ work on Problem 2. In the excerpt the students try testing a CL between an IOD they have conceived and the III. The students seems to be aware of a CL between a generic IOD and their III, and through a soft dragging test they reject the initial GDP that had led to the IOD and conceive a new GDP and IOD. They test the new CL with a soft dragging test. The excerpt is taken from the continuation of the exploration shown in Excerpt 4.3.2, thus the lines are labeled with their times relative to the times of that excerpt in order to show the solvers' progression over time. The bold letters refer to the solver who is dragging.

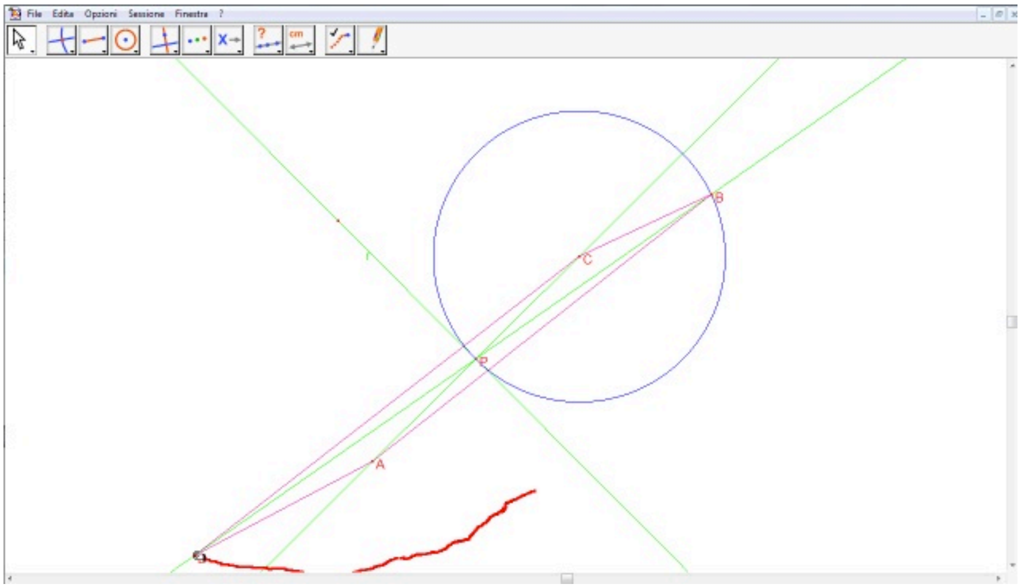


Figure 4.4.2: A screenshot of F & G's exploration

Episode	**Brief Analysis**
(5:01) F: but you see? This one is always longer than that one… it's too long, if you go, let's say, along the circle here, this one is too long. So, maybe it's not necessarily the case that D is on a circle so that [Italian: "in modo che"] ABCD is a parallelogram.	With his argumentation F rejects the proposed GDP, and re-launches the search for an IOD (6:36). In the argumentation a CL emerges between "D on a circle" and "ABCD parallelogram" (5:01).
…	
(6:36) F: exactly. Now there is this problem of the parallelogram in which we can't exactly find when it is.	Restatement of the III.

(6:44) G: eh, uh, we discovered when… (6:50) G: Let's try to think about it without, like…because if when you move this, maintaining always the same distance,…	G seems to conceive "PB congruent to PD as an III."
(7:02) F: because you see, if we then do a kind of circle starting from here, like this, it's good it's good it's good it's good, and then here… see, if I go more or less along a circle that seemed good, instead it's no good… Because, you see, in a certain sense B, at this point the circle (7:24) G: eh, it's linked to the circle (7:25) F: exactly, and so in a certain sense it goes … down along a slope and so… it's no… no good. So, when is it any good? … (8:05) G: because I think if you do like, a circle with center (8:07) **F**: A, you say… (8:09) G: symmetric with respect to this one, you have to make it with center A. (8:10) **F**: uh huh (8:11) G: Do it! (8:13) **F**: with center A and radius AP?	Back to his argumentation (7:02) F tries to explain why a circle seems to be "no good" (he probably still has in mind "his" circle described in the analysis above). Although such circle is never described geometrically, F and G seem to have a similar object in mind. Most importantly the solvers seem to have in mind a CL between the III and a hypothetical IOD that they are still searching for (6:36 and 6:44). As F discusses why the circle he had in mind is no good, F's attention seems to shift to the movement of point B (7:02) and then to the figure as a whole. At this point G has handed the mouse back to F who starts using MD without the trace. Now G proposes a new GDP and F proceeds to construct this geometrical object.

(8:14) G: with center A and radius AP. I, I think…	
(8:20) **F**: let's move D. more or less…	When F and G refer to it looking right (8:24) and to having found it (8:29) it is reasonable to assume that they are verifying a CL.
(8:24) G: it looks right doesn't it?	
(8:27) **F**: yes.	
(8:29) G: Maybe we found it!	

Table 4.4.2: Analysis of Excerpt 4.4.2

Overall this excerpt shows how the solvers seem to already start out their search for a GPD as if they already knew how to reach their IOD from it. They seem to be implicitly assuming that the invariance they will observe will be something like "D is on a …[path to be made explicit through a GDP]". This implicit assumption seems to guide their exploration and make all the pieces fall together in an almost "automatic" way. This phenomenon will be further discussed in Chapter 6, where we will discuss the process underlying expert use of MD for conjecture-generation.

In this particular Excerpt, in spite of the incorrectness of the specific GDP, we can observe F's consciousness of a conditional link between the III and the IOD he had hypothesized at time 5:01. In his rejection of the GDP we can see the CL appear between D being on a circle and ABCD being a parallelogram through his words: "*so that* [Italian: "in modo che"]". In other words, the fact that ABCD is (or will become) a parallelogram is linked to the movement of D along a hypothetical circle, and linked in a way that implies conditionality: the movement is *so that* the particular configuration occurs. Moreover, after the construction of the new GDP (a circle with center in A and radius AP (8:17)), the solvers seem to feel the need to check their idea, and they use a soft dragging test to become convinced that dragging D on the constructed object guarantees the simultaneous appearance of the III. In this case, the soft dragging test seems to have an exploratory nature, and be part of an argumentation that makes use of the dragging tool to convince and give confidence in a certain idea. We will discuss these types of arguments in further detail in Chapter 6.

In the following subsection we will present two excerpts in which verifying the CL through dragging tests leads to a generalization of the preconceived path.

4.4.1 Generalization of the Preconceived Path

As the solver tests the validity of a hypothesized IOD, s/he might realize that the GDP s/he has provided may be “generalized” to a larger set of points. Frequently the GDP that the solver provides is a geometrical figure that s/he may have recognized only a “piece” of. In this case the dragging test can show that the III is actually verified along the “whole” object. In order to verify “the goodness” of a certain GDP, that is to verify that the path is “more than” s/he had initially conceived, the solver needs to be able to concentrate on both the III and the IOD simultaneously (or switch quickly and frequently from one to the other). Therefore, checking the CL may also lead to what we call “generalization of the preconceived path”. We consider this to be a phenomenon that provides further evidence of solvers’ awareness of a CL between the IOD and the III, and of the relationship between the GDP and the IOD.

Excerpt 4.4.3 shows an example of this generalization of a preconceived path through a soft dragging test, however the process may also occur through a robust dragging test, as shown in Excerpt 4.4.4.

Excerpt 4.4.3. This excerpt is from two students’ work on Problem 2. It is the continuation of Excerpt 4.4.2, and it shows how checking a CL can lead to the generalization of a preconceived path. The lines are marked with the times relative to the beginning of Excerpt 4.3.2 (continued in Excerpt 4.4.2).

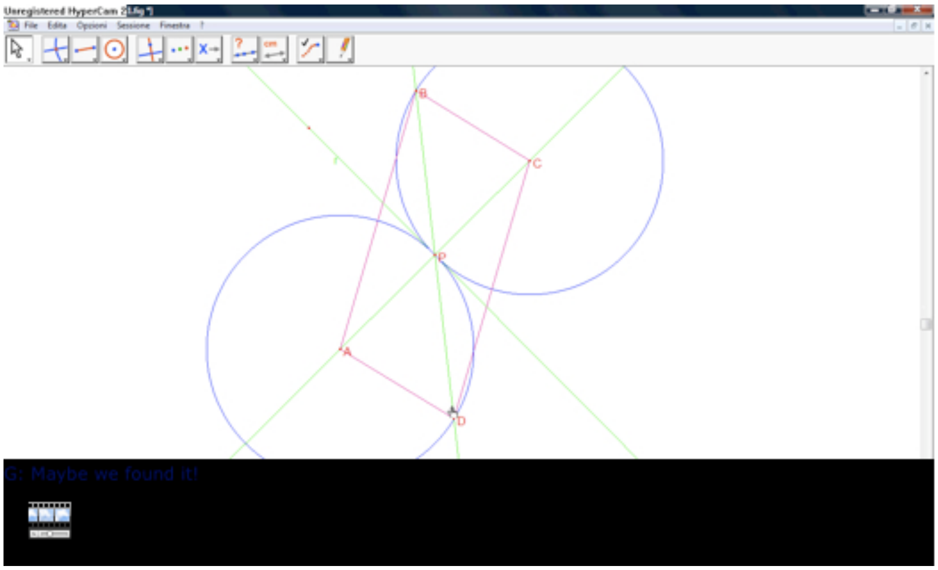


Figure 4.4.3: A screenshot of F & G's exploration

Episode	**Brief Analysis**
(12:50) **F**: but maybe… maybe only along this [pointing to the lower right part of the circle, the region in which he had performed the maintaining dragging] (12:51) G: Let's try to (12:53) **F**: let's try to, right, go the whole way around (12:54) G: like this yes, like this yes, like this yes (12:55) **F**: yes, yes, yes (12:58) G: over here too, I think (13:00) **F**: yes	The solvers have constructed the circle with center in A and radius AP and they seem to conceive only part of it as the path. This can be inferred from F's words in (12:50). The solvers seem to have conceived the IOD as "D belonging to the path" or "D moving along the path" as shown in (12:50), and in (12:51) and (12:53) when the solvers want to "try" to see if it works "the whole way around". As F drags D along the circle the solvers seem to be checking a CL, and when they

(13:01) G: yes. (13:02) **F**: I would definitely say so. (13:03) G: okay we found it. (13:06) **F**: Okay, so that's write that…	reach the upper part of the circle, they seem to be quite satisfied with what they see [(12:58)-(13:02)].

Table 4.4.3: Analysis of Excerpt 4.4.3

We assume that "trying" refers to testing the CL between the IOD (D on the object representing the GDP) and the III (ABCD parallelogram, or PB=PD for G who seemed to be using the bridge property in previous episodes of this exploration). In this sense, the solvers' actions let us infer their conception of a CL. Further evidence that they seem to have conceived a CL is their "checking something" in various instances, as marked by the repeating of "yes" rhythmically while watching D move along the circle (12:55-(13:01). We can infer that as D moves they are checking that the rest of the circle constitutes "good choices" for the dragged base point. That is, positions that guarantee the III to be visually verified.

When F and G finally exclaim: "I would definitely say so" (13:03) and "okay we found it" (13:06), they seem to be confirming their hypothesis for what the generalized GDP and the IOD might be. This confirmation comes from a very careful check that dragging along the "whole" circle (IOD) guaranteed that ABCD remained a parallelogram (III), and thus that it was correct to conceive the CL as existing between these two properties. The two solvers then have no trouble in immediately making the transition to the formulation of the conjecture (even in a written form!) as if it were "automatic" from what they experienced. Such automaticity will be further discussed in Chapter 6.

Excerpt 4.4.4. This excerpt (FS_Ud_FG_p1 between time 37:27 and 38:50) is taken from two students' work on Problem 4. It is a continuation of Excerpt 4.3.3, and it shows the process of generalization of a preconceived path through a robust dragging test.

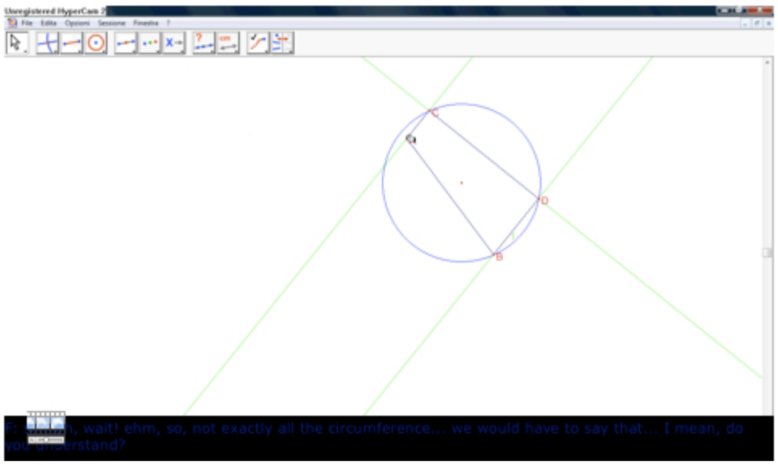


Figure 4.4.4: A screenshot of F and G's exploration.

Episode	**Brief Analysis**
[1] **F**: so...ah, wait! ehm, so, not exactly all the circle... we would have to say that... I mean, do you understand? [2] G: No… [3] **F**: It's not exactly on all of the circle. [4] G: No, it is you who...[unclear] [5] **F**: No, no...wait... here is good. It's good, it's good, [6] G: it's good... [7] **F**: more or less... it's good [8] G: come on...[in a low voice] [9] **F**: let's try... how do we...eh, let's link A to the circle, so we can see well. How do	As in Excerpt 4.4.3, the solvers have conceived a path that F thinks is only partially described ([1], [3]) by the circle they have drawn (with diameter BC), because as he manually follows (soft dragging test) the circle ([4]-[7]) he is unsure of the acceptability of this GDP when he approaches points C and B (previous part of exploration and here in [5], [7]). G, however, seems convinced (but not strongly) that it is a F “who is not” dragging properly ([4]) and he says he does not understand what F is referring to

you do link? Wait, wait... … [10] G: "redefinition of an object" [11] **F**: let's take A... [12] G: "point on an object" [13] **F**: wait, let's move A off from there..."redefinition of an object"..."this point" let's do point... [14] G: No, "point on an object" [15] **F**: Point on an object that is on this circle. [16] G: There. [17] **F**: there, now we can see it well. [18] **F**: Good, here there are no problems... [20] G: always! [21] **F**: Yes, it was I who was... [22] G: yes... [23] **F**: Yes, it is always, always. [24] F: So, write…	([2]). To get over this indecision F proposes to link A to the circle (and thus perform a *robust dragging test*), and does this ([9]-[15]) in order to "see it well" ([17]). G seems not to be surprised at seeing that the III is maintained on what he had conceived as the entire GDP ([20]), and F realizes it was he who was not being precise while dragging ([21]), and agrees that the property rectangle (III) is maintained along the whole circle ([23]).

Table 4.4.4: Analysis of Excerpt 4.4.4

This passage can be read as F's generalization of what he initially conceives as the GDP (only some of the circle), and as a verification, for both solvers, of the CL between the IOD and III. Now the CL between "when it is a rectangle (III)" (expressed previously in the exploration), that led to "A on the circle" (IOD), has been verified, and the solvers seem to immediately (almost "automatically") make the transition to the expression of a conjecture in "static" logic terms, right after line [24]. We will discuss this in the next section of this chapter.

<u>4.4.2 Concluding Remarks</u>

In this section we have presented excerpts that seem to provide evidence that the solvers have conceived a conditional link between two invariant properties discovered during the exploration. The evidence of such conceptualization is necessarily indirect, and expressed through different behaviors that we could observe. In particular, we showed how evidence of a CL may be interpreted as an effective use of checking the IOD and the III simultaneously through a form of the dragging test, as occurs smoothly in Excerpt 4.4.1. We provide a different form of evidence of the conception of a CL in Excerpt 4.4.2, by showing how the solvers' realize that a particular GDP and dragging along it do not provide a satisfactory IOD and so they make new hypotheses and modify their proposed GDP and IOD. The last two excerpts show how having conceived a CL allows the solvers to perform a soft (Excerpt 4.4.3) or robust (Excerpt 4.4.4) dragging test which can lead to a generalization of a GDP and IOD.

Although evidence of the conception of a *conditional link* (CL) is necessarily indirect, the notion seems to be a useful one, because it sheds light onto the process of conjecture-generation when MD is used. Such process can be seen as a slow adjustment and falling-into-place, during the exploration, of all the pieces and the relations between them that we have described in the MD-conjecturing Model. Once everything has been put into place, the *conjecture*, an explicit statement given by the solvers (in written or oral form), can be formulated. We describe this last step in the next section, and we remark here that unlike the parts of the model that we have described until now, the conjecture is the only element of our model that can be accessed directly.

In fact, as we will describe in the next section, in the context of the MD-conjecturing Model, we have decided to consider the *conjecture* to be only the explicit (oral or written) statement through which students directly describe the CL between the invariants they have observed. We will give our working definition of “conjecture” in the following section.

Before closing the section we would like to discuss the transition from a CL to a conjecture. We have defined the CL to be “a relationship of logical dependency” which the solver has conceived but not yet expressed. Of course, as discussed above, the notion seems to be quite useful, but it cannot be accessed directly by an external observer, since it is part of the knowledge the solver is developing and using to carry out the process of conjecture formulation. We have decided to describe the CL as a relationship of logical dependency, or conditionality, however it is possible that the solver needs to first conceive “causality”. We could imagine a “causal link” to be what a solver can conceive when s/he interprets the experience still within the Cabri world, or the phenomenological world more in general, dominated by time. The point at which a “causal link” becomes a “conditional link” is not clear, and, as before, if it occurs, such a transition can only be seen indirectly by the observer. All we can see directly is the outcome of the process of interpretation of Cabri-phenomena as geometrical objects and logical relationships between them, that being, what we refer to as *conjecture*. In the next section we will show how the conjecture can be formulated in a variety of acceptable ways. However it is when the final conjecture is expressed as a “static” form that we have direct evidence of the transition having occurred completely and successfully, a transition that has occurred through the establishment of the CL, and in which simultaneity and the solver’s control seem to play an important role.

Previous research has described a similar transition as a “crystallization” of a statement from a “dynamic” exploration of a problem to a “static” logic expression, through the focus on a “temporal section” (Boero et al., 1999; Boero et al., 2007) of the exploration, also described in Boero et al. (1999) as a PGC1 (“a time section in a *dynamic exploration* of the problem situation: during the exploration one identifies a configuration inside which B happens, then the analysis of that configuration suggests the premise A, hence “if A, then B”). We are not sure this description completely illustrates the case in our model. However we think that this PGC does describe rather accurately some students’ other behaviors that we have observed, that do not involve use of the maintaining dragging scheme (MDS) in a DGS. We will discuss the MD-conjecturing Model with respect to Boero et al.’s PGCs in Chapter 7.

4.5 Formulating and Testing the Conjecture

Once the solver has reached a GDP and interpreted dragging along it as an invariant, the IOD, and once the solver has conceived a CL between the IOD and the III, s/he may want to test the appropriateness of the IOD through a soft dragging test, as described in the previous section. For expert users “testing the appropriateness of the IOD” means visually verifying that in fact the direct movement of the dragged-base-point along a specific GDP does have the effect of preserving the III. This can be thought of as the IOD “causing” the III, or that “it is a condition under which” the III is visually verified. In other words this dragging test is verifying the CL that the solver has hypothesized between the two invariants that s/he is trying to induce. We have described how in the analyses of students’ transcripts the CL can only be captured indirectly, through different manifestations (like different forms of the dragging test) that allow us to infer its existence. In a way, we could consider the CL developed between the IOD and the III as the seed of a conjecture, or as a non-explicit conjecture. However, for clarity of the model and of the analyses that can be obtained through it, we prefer to separate what solvers express explicitly from what can be inferred from their behaviors. In particular, we separate the moment in which the solvers explicitly formulate their conjectures in an oral and/or written form. Therefore, in our model, we consider a *conjecture* to be

> *an explicit statement, that can be written or oral, of the CL, conceived by the solver, between the IOD and the III.*

If we refer to the conjecture as stated above, we clearly have an element that can be accessed directly. Moreover, it is the only element of our model that we have direct access to, since all the elements we introduced until now can only be perceived indirectly in the analyses, through students' words, actions, and, in general, behaviors.

Our data shows that solvers' conjectures are not all expressed analogously. In the excerpts below we will show the different types of formulations that various solvers used, and that we considered consistent with respect to our model. With "consistent" we mean that the conjecture seemed to correctly express a CL between the perceived invariants, and to yield a proper conceptualization of the premise and the conclusion of the conjecture. Such consistency could also be captured through how solvers addressed premises and conclusions when they attempted to prove their conjectures, however we do not analyze proofs in this study.

In the following analyses, we will also highlight how the same solvers may express their conjecture in successively more geometrical ways. For example, some solvers state a conjecture orally before writing it down, and the two formulations of the conjecture differ; other solvers give different oral formulations before reaching the one they choose to write down. This allows us to capture aspects of the non-trivial transition that solvers using the MDS make from the dynamic Cabri environment to the static world of traditional Euclidean geometry.

Moreover, we have noticed that many solvers choose to perform a robust dragging test after they have formulated a conjecture, as a corroborating test of their conjecture. This type of dragging test seems to be more efficient than the soft dragging test for checking both the IOD and the III at the same time. We advance the hypothesis that the consciousness of the reconstruction of the Cabri-figure with a new property given by the IOD (now constructed robustly), or the construction of the GDP and redefinition of the dragged base point upon it, cognitively replaces the role of the solver's "direct control" over one of the invariants. The CL between the IOD and the III could be verified by consciously controlling the IOD and watching the III occur simultaneously, as a consequence. As we described in section 4.4, the recognition of "a condition" and "a consequence" may occur through the type of control exercised over each invariant: simultaneity + direct control leads to "a condition", while simultaneity + indirect control leads to "a consequence". During the transition to the conjecture, these then need to be interpreted as a premise and a conclusion. In this sense, when solvers check their conjecture with a robust dragging test, the direct (or indirect) control is substituted by knowing that the figure has been reconstructed with a specific added property (and not with another). Simultaneity can then be perceived in a stronger way than before, since, if the conjecture is valid, the IOD and the III will occur simultaneously for the dragging of any base point. Therefore the solver can check the separation the two invariants into premise and conclusion of the conjecture.

Some solvers choose to test their conjectures in a different way, without dragging. Although this was not introduced during the introductory lessons on the dragging modalities, some solvers are aware of the command that we refer to as "ask

Cabri". This is a command that allows the user to select pairs of objects of a Cabri-figure and click on an icon (the 8th item on the top command bar in Cabri-Géomètre II Plus) and select a question, like "parallel?", which opens a window that contains the software's answer to the question, such as "the two objects are parallel". We did not expect students to use this command, however we witnessed different occasions in which they did in the following way, after they had formulated a conjecture. The solvers would robustly construct the property expressed in the premise of the conjecture, and then consider a property that characterized the type of quadrilateral described in the conclusion of the conjecture. They would use this property to ask Cabri, through the appropriate tool in one of the menus in the toolbar, whether such property was verified. In this sense we consider this behavior as another way of checking a conjecture, other than using the dragging test. One of the excerpts we present in this section shows an example of such behavior. For clarity, we have divided the excerpts and examples we present into the following subsections: transition to the conjecture (subsection 4.5.1), various formulations of conjectures (subsection 4.5.2), and testing the conjecture (subsection 4.5.3).

4.5.1 Transition to the Conjecture

In Section 4.4 we discussed the conditional link (CL) that the solver conceives between two invariants of the Cabri-figure s/he is exploring. When the CL is made explicit, it can contain traces of the dynamic exploration that gave origin to it. Sometimes the process of making the CL explicit can be difficult for the solver to carry out, and it may not lead to a conjecture that is coherent with our model. In the following subsection we consider the case in which the transition from a CL to a conjecture is successful. Such a transition occurs internally, in the solver's mind, so once again we can only gather evidence of the process indirectly through what can be observed and inferred through students' behavior, words and actions.

In this subsection we provide examples that seem to yield evidence of the transition from the conception of a CL to the formulation of a conjecture, as anticipated at the end of Section 4.4. The first excerpt shows an example of how the transition can occur smoothly without any apparent difficulties. The second excerpt is an example of a slightly less smooth transition: the solvers first orally give a concise narrative of their exploration, and then they provide a written conjecture.

Excerpt 4.5.1.1. The excerpt below shows a smooth transition from the construction of a GDP to the formulation of a written conjecture. The two students seem satisfied with their proposed GDP, through which they have characterized their IOD. Without apparent difficulties, the students express their conjecture orally in a static form, and immediately write it down. The excerpt is the continuation of the exploration carried out by the students of Excerpt 1 in section 4.3 of this chapter, in which they had used maintaining dragging to induce "ABCD parallelogram" as an III and search for an IOD.

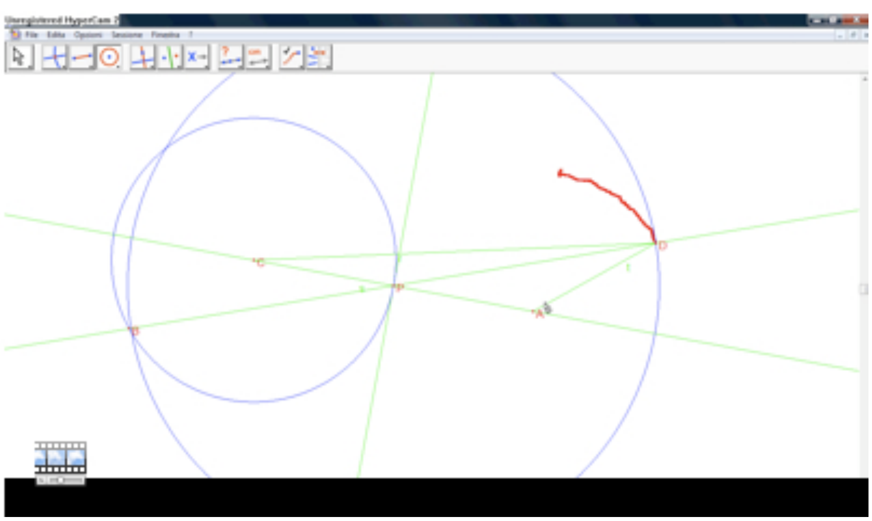


Figure 4.5.1.1: A screenshot of the exploration

Episode 1	Brief Analysis
[1] Giu: So I was thinking something like this [as he constructs the circle centered in A with radius AP]. Let's see if it goes, let's see if it goes, let's see if it goes... [2] Ste: yes [3] Giu: Yesss!! [4] Ste&Giu: Yes, nice! [laughing] … [5] Ste: If CP is equal to PA, say by definition … [10] Giu: We have lots of things. [11] Ste: We could say…construct two circles…these two. [12] I: Well, one you have already [13] Giu: It's enough to say that PA has to always be the same as AD, because if they are the same, it [D] has to necessarily be on the same circle, because they are two radii. …	After having conceived a GDP, the circle with center in A and radius AP, and an IOD (D moving along the circle described as the GDP) Giu proposes to construct the object representing the GDP while leaving the trace visible on the screen ([1]). Both solvers seem to be quite satisfied ([2]-[4]) in seeing how the construction nicely fit the trace. The solvers seem to be discussing ([5]-[13]) what a sufficient condition in order to have "D on the same circle" ([13]) might be. They seem to find a condition that implies their IOD (D on the circle): the congruence of two segments, PA and AD which become radii of this circle.
Episode 2	**Brief Analysis**
[14] I: Let's write. [15] Giu: So let's say: if A…so we already have that this is equal to this, that this is	Giu proposes the conjecture "If PA equals PD…ABCD is a parallelogram" ([15]-[17]).

equal to this, so if PA equals AD… [16] Ste: everything we said… [17] Giu: also…well, ABCD is a parallelogram. [They write: "ABCD is a parallelogram if PA is equal to AD"]	In the written conjecture the conclusion precedes the premise.

Table 4.5.1.1: Analysis of Excerpt 4.5.1.1

Although the solvers seem to approve of the proposed GDP and of the IOD as "D belonging to this object" when they construct the circle that represents their GDP, the solvers do not use this IOD directly in the conjecture. Instead they seem to elaborate their findings in lines [5]-[13] and express the premise as "PA equals AD" ([15]). The argumentation that leads to an oral conjecture seems to go back along some of the steps that led to the construction of the second circle ([11]). In order to reach their written conjecture, the solvers seem to re-elaborate what they have observed during the exploration, starting from some properties of the construction, and searching for a sufficient condition (or a chain of such conditions) ultimately implying the III.

The only difference between the conjecture proposed in lines 15-17 and the one they write, is the order of the premise and conclusion, which is reversed in the written conjecture. This could indicate a desire to focus on "the case of the parallelogram" which is what they had explored until then. However the condition that generates this case is still clearly marked by the "if" that makes it the premise of the written statement.

<u>Excerpt 4.5.1.2</u>. This excerpt contains another example of transition, this time from an oral expression of a conjecture in a narrative and dynamic form, to a different

written one. The excerpt is taken from two students' work on Problem 2. The episode starts with Rai's response to the interviewer's insistent request for a conjecture.

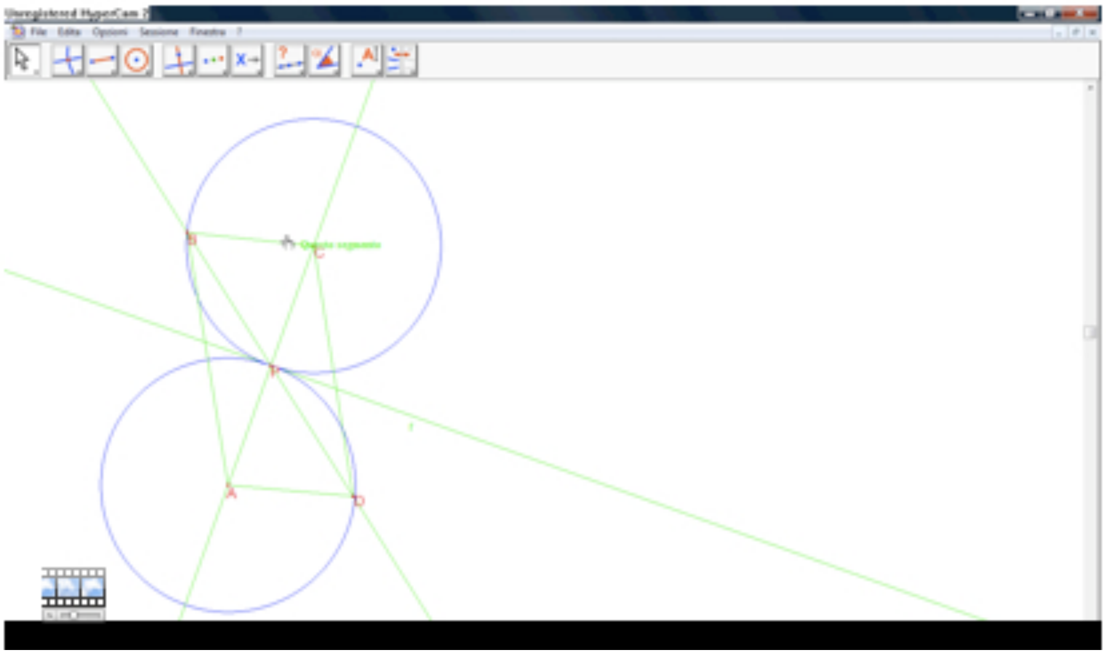


Figure 4.5.1.2: A screenshot of the exploration

Episode	**Brief Analysis**
[1] Rai: Ok. [2] Rai: That if, uhm, we want to maintain, uhm...PB equal and symmetric to PD, [3] Rai: ...we can draw a circle with center in A and through P. [4] Lo: So if D belongs to the circle with center in A [5] Rai: and through P [6] Lo: and through P, then the polygon ABCD is a parallelogram.	Rai seems to be considering this ([2]) as his III. Rai seems to propose a geometric condition that realizes the invariance of the III. He seems to be trying to express geometrically what he had observed during the exploration, while Lo ([4]-[6]) gives an oral conjecture, trying to restate Rai's idea.

[They write: "If D belongs to the circle through P and with center in A, ABCD is a parallelogram."] [7] Lo: I would say that this is what we said.	The solvers seem to agree upon such reformulation, choosing to write it down as their conjecture.

Table 4.5.1.2: Analysis of Excerpt 4.5.1.2

The first statement ([2]-[3]) is expressed in an "if…then…" form, however it is still embedded in the experience of dragging in the Cabri environment, expressing a geometric condition that they used to obtain the invariance of the III. Lo seems to re-elaborate on Rai's description and make the transition to a geometrical description of the IOD ([4]). In the first formulation, Rai's "premise" contains reference to wanting to "maintain" a property, which is they key to the formulation of the written conjecture. This in fact occurs instantly after the expression of the oral one, in lines [4]-[6], before the solvers decide to move on to a different case.

Concluding Remarks. As mentioned above, the transition to a conjecture is not a trivial process and it can constitute a difficulty for some students. In the analyses above we tried to capture evidence of the transition from inferences we made based on the comparison between certain oral or written statements of the solvers. We have observed that frequently, even after having expressed a conjecture, solvers feel the need to perform a dragging test to become more convinced of the goodness of their conjecture, or to check their conjecture once it has been formulated. In subsection 4.5.3 we provide examples of such behavior, however first, in the following subsection, we will present different ways in which the statements of the generated-conjectures were formulated by different solvers.

4.5.2 Various Formulations of Conjectures

We discussed how conjectures are an explicit expression of a conceived CL between invariants. Thus different conjectures may be expressing the same CL. This allows us to talk about *classes* of conjectures, each class expressing a given CL between given invariants. For example, there may be differences between an oral and a written expression of what the same solver sees as “the same conjecture”. There may also be differences among conjectures in a same class expressed by one or the other solver, when they are working in pairs and discussing what to write down as their final conjecture. In some cases there is negotiation or an argumentation in favor of a particular formulation, but frequently some expressions are used interchangeably, which we take as indirect evidence of the reference to the same CL.

In the subsections below, we identify three characteristics that different conjectures from the same class may present: conjectures that contain traces of the dynamic exploration (subsection 4.5.2.1), “transitional conjectures” in which “when” and “if” seem to be used interchangeably (subsection 4.5.2.2), and conjectures that do not contain traces of the dynamic exploration (subsection 4.5.2.3).

4.5.2.1 Conjectures with Traces of the Dynamic Exploration. In this subsection we describe some conjectures that contain dynamic aspects. That is, they contain terms that seem to refer to the Cabri world, like “move”, “stay on”, “remain”, and so on. In the section we consider various formulations of conjectures in which the solvers seem to have correctly conceived the premise and the conclusion. Evidence of such conceptualization is necessarily indirect. However strong evidence of the correct conceptualization can be found in cases in which the solvers transition fluidly to proof, making explicit the distinction between premise and conclusion of their conjecture.

Although such fluid transition is not necessarily present for all proper conceptualizations of the CL, the cases in which it *is* present definitely suggest that the CL has been conceived coherently with respect to our model. Therefore in this section we use presence of this fluid transition as a criterion for selecting examples of different formulations of conjectures of the same class.

First we show an example in which the solver states in her written conjecture that a point "stays on" a certain intersection point. The second example shows a formulation of an oral conjecture in which the solvers use the words "moves" and "remains". Finally we present an excerpt from two students' exploration, in which they formulate a conjecture on a derived-construction invariant as a statement with strong dynamic aspects.

This example is taken from a student's exploration of Problem 2.

Written conjecture (4.5.2.1.1):

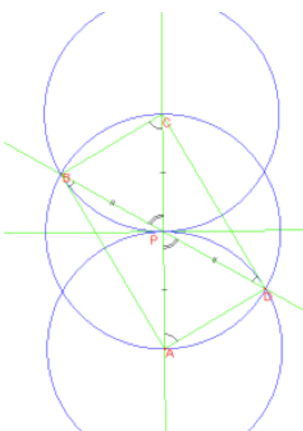


Figure 4.5.2.1.1 Screenshot of the solvers' exploration when they expressed a conjecture.

"If D stays on the point of intersection of the circle with radius AP (center A) and the circle with radius PA (center P), then ABCD is a rectangle." [Italian: "Se D sta nel punto d'intersezione tra la circonferenza di raggio AP (centro A) e circonferenza di raggio PA (centro P), allora ABCD è un rettangolo."]

Notice how the expression "D stays on" seems to indicate a strong link to the dynamic exploration. However the conjecture is expressed as an "if…then…" statement and the solver showed no difficulty in recognizing the premises she was to begin with when constructing a proof.

The following example (PS_Fin_ValeRic_p3_c4) is taken from a student's exploration of Problem 3. The student gives the following oral conjecture (4.5.2.1.2):

> *"If M moves along a line through M and perpendicular to segment MK, then the figure remains a rectangle." [Italian: "Se M si muove su una retta per M e perpendicolare al segmento MK, allora la figura rimane un rettangolo."]*

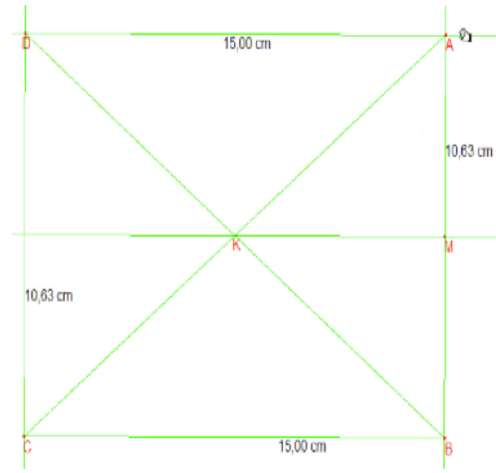


Figure 4.5.2.1.2 Screenshot of the solvers' exploration when they expressed a conjecture.

The dynamic aspects of this oral conjecture are evident in the expressions "move on" and "remains". The solver is expressing the IOD in a dynamic form, as movement along the GDP, and the III is also expressed dynamically as ABCD *remaining* a rectangle. Time is still present in the formulation of this conjecture that seems to summarize the exploration experience. However the premise and conclusion have been clearly separated and the CL correctly established, as the complete "if…then…" statement indicates.

<u>Excerpt 4.5.2.1.3.</u> This excerpt is taken from two students' exploration of Problem 1. The bold letters refer to the solver who is performing the dragging.

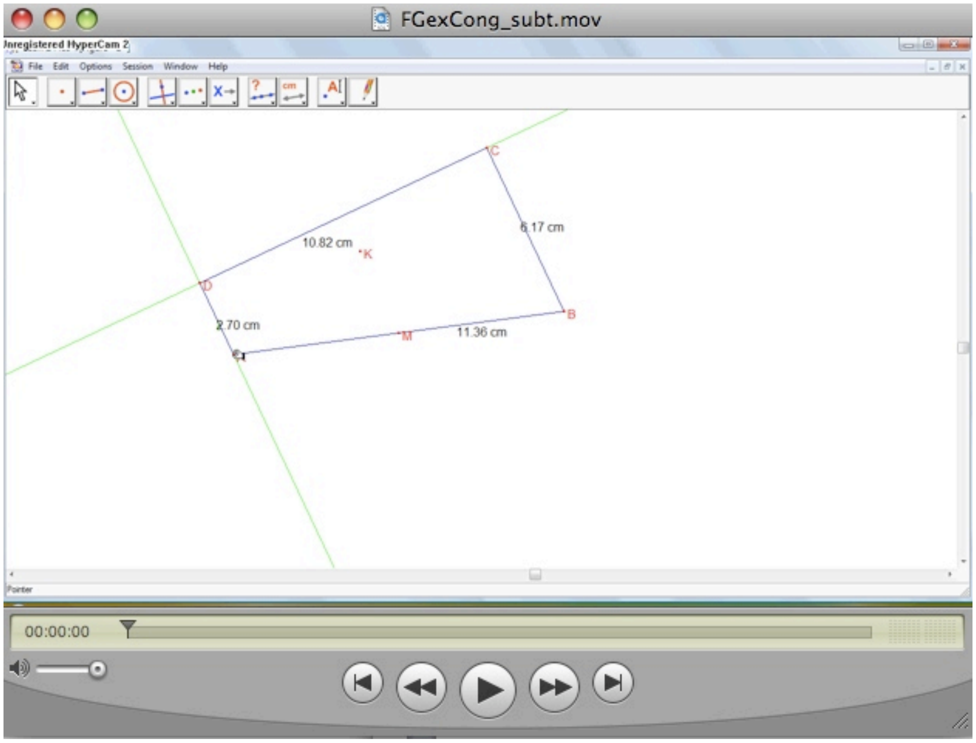


Figure 4.5.2.1.3 Screenshot of the solvers' exploration

Episode 1	**Brief Analysis**
[1] **F**: Good, so we can say, about the quadrilateral, that as A varies, uh, we always obtain a trapezoid. [2] G: a right trapezoid. [3] I: Ok. [4] **F**: a right trapezoid. [5] I: Ok, a trapezoid that is also a right trapezoid. [6] **F**: Yes, a right trapezoid.	The solvers have been dragging the base point A and have conceived the property "ABCD is a right trapezoid" ([1], [2], [4], [5], [6]) as an invariant. A first conjecture is stated by F in line [1]: "as A varies, uh, we always obtain a trapezoid"
Episode 2	**Brief Analysis**
[7] **F**: [writing] moving A...	A second conjecture, in the same

[8] **F**: ...a right [9] **F**: trapezoid. [10] **F**: Yes. [11] **F**: Ok. [12] I: So this is a conjecture... [13] **F**: Yes. [14] I: on the quadrilateral? [15] F&G: Yes. They write: “Moving A freely we always get a right trapezoid.” It: “Muovendo A liberamente otteniamo sempre un trapezio rettangolo.”	equivalence class as the first, is: “Moving A freely we always get a right trapezoid.” Both conjectures contain dynamic elements: a reference to *movement* in the premises, and a temporal reference, “always” in the conclusions.
Episode 3 [16] G: Let's prove this one right away. [17] F: So, in the hypotheses we have CD is perpendicular to AD. [18] I: Yes	**Brief Analysis** We have evidence that the conjectures refer to the same CL because F immediately starts the proving process by stating the premises explicitly ([17] and following omitted from the excerpt).

Table 4.5.2.1.3: Analysis of Excerpt 4.5.2.1.3

4.5.2.2 Use of “when” and “if” as synonyms. As mentioned above, we can only infer that the terms “when” and “if” are sometimes actually used as synonyms, but some behaviors seem to favor such interpretation. In particular, we consider the words to be used as synonyms in at least three situations. (1) When the conjecture is expressed using “when” but when the solvers start proving the conjecture they use the condition expressed through the “when” as the premise, and the other part of the statement as the conclusion, or “what needs to be proved”. An example of this situation is shown in Excerpt 4.5.2.2.1. (2) When solvers use one of the expressions orally (usually “when”), but they immediately write the conjecture using the other expression (usually “if”), as shown in Excerpt 4.5.2.2.2 and in Excerpt 4.5.2.2.3. (3) When one student uses one word (“if” or “when”) and the other in a very similar expression, immediately after the first statement, as shown in Excerpt 4.5.2.2.4.

<u>Excerpt 4.5.2.2.1.</u> This excerpt is taken from a student’s work on Problem 1.

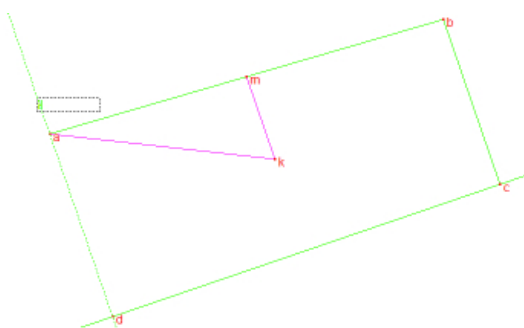
Episode	**Brief Analysis**
[1] Ste: written conjecture: “When K coincides with M, the quadrilateral ABCD becomes a triangle because b and c coincide.”  [Italian: “Quando K coincide con M, il quadrilatero ABCD diventa un triangolo in quanto b e c coincidono.”] [2] Ste: argumentation in which he uses the premise “K coincident with M” to prove “B and C coincide” and so “ABCD becomes a triangle.”	The solver writes down this conjecture ([1]). He immediately delves into an argumentation ([2]). This shows that what Ste refers to after “when” is the premise of his conjecture.

Table 4.5.2.2.1: Analysis of Excerpt 4.5.2.2.1

In the excerpt above we saw an example of “when” being used, logically, as “if”. Sometimes it seems that the use of “when” or “if” can refer to a distinction between the phenomenological domain of Cabri and the theoretical world of geometry, that is, a reinterpretation in geometrical terms of what has been observed in Cabri. Such reinterpretation seems to frequently culminate with the transition from an oral statement to a written statement. Moreover we notice how Ste states his conjecture in a way that seems to be "dynamic." He writes about a quadrilateral *becoming* a triangle, but then has no trouble providing a correct proof of his conjecture. The appearance of dynamic elements in conjectures seems to be a recurring phenomenon when conjectures are developed as the outcome of explorations in dynamic geometry.

Excerpt 4.5.2.2.2. This excerpt is taken from two students' exploration of Problem 1.

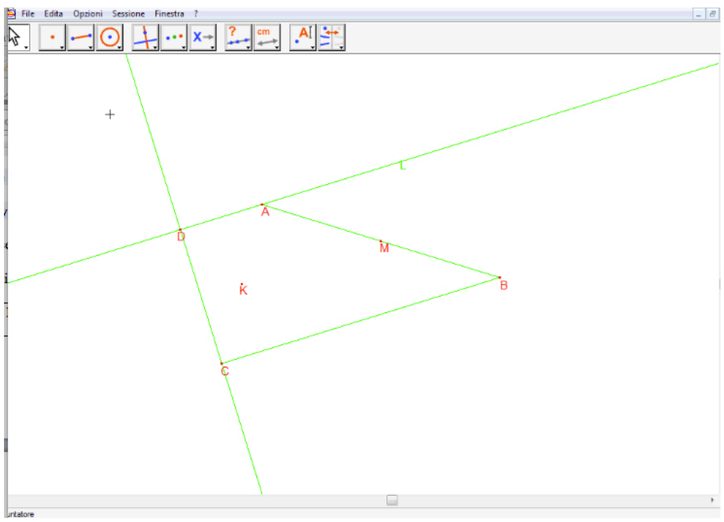


Figure 4.5.2.2.2 A screenshot of the solvers' exploration at the moment of the conjecture.

Episode	**Brief Analysis**
[1] Vale: When DA is equal to CB, that is, when BA is parallel to DC, so also when these here are right angles, it is a rectangle. [Italian: "Quando DA è uguale a CB, cioè quando BA è parallelo a DC, quindi anche quando questi qua son retti è un rettangolo."] [2] Vale: [writing] "If DA=CB then rectangle"	Orally the solvers use the "when" to refer to what is written after "if" as the premise in the written conjecture. The two conjectures are expressing the same CL. In the written conjecture the argumentation chain linked by

[Italian: “se DA=CB allora rettangolo”]	“when” has disappeared.

Table 4.5.2.2.2: Analysis of Excerpt 4.5.2.2.2

Excerpt 4.5.2.2.3. The excerpt below is taken from the student G’s work on Problem 2.

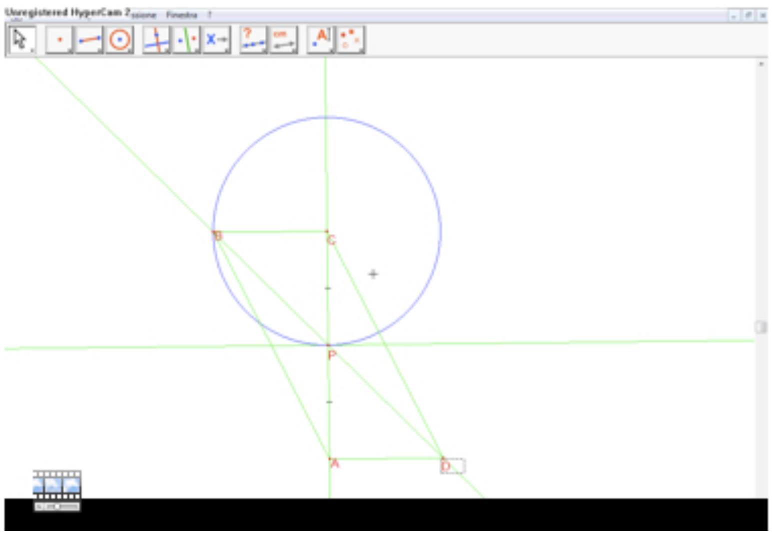


Figure 4.5.2.2.3 A screenshot of the solver’s exploration at the moment of the conjecture

Episode	**Brief Analysis**
[1] G: So when AD is equal to AP... [2] I: Ok [3] G: ..it could be a parallelogram. [4] G: [writing] If...AD is equal to AP [5] I: Ok [6] G: ABCD is a parallelogram.	In lines [1]-[3] G expresses her conjecture orally. This statement is not in the form “if…then…” however the student seems to interpret it as such because when she formulates it in writing immediately after speaking, she writes: “If AD is equal to AP, ABCD is a parallelogram” ([4]-[6]).

Table 4.5.2.2.3: Analysis of Excerpt 4.5.2.2.3

The word “when” (Italian: “quando”) is used in the oral statement, in which G also expresses a degree of insecurity (“it could be”). “When” seems to mark the transition from what is observed on the screen, related to movement, and what can be stated in the static formal world of Euclidean geometry. On the screen G can observe a sequence of instances (that may seem continuous) in which the property “AD equal to AP” may seem to be satisfied. Therefore a reference to time is appropriate, and “when” seems to catch the occurrences of this event. However the word “when” also refers to a CL between events (or occurrences of properties in our case) and this may explain the use of the term immediately followed by the reformulation in formal language in the written conjecture ([4]-[6]). The fluid transition from the oral statement to the written one seems to indicate that the terms “when” and “if” are used by the solver as synonyms, or at the very least, as two ways of referring to the premise of the conjecture.

Excerpt 4.5.2.2.4. This excerpt is taken from two students’ work on Problem 2.

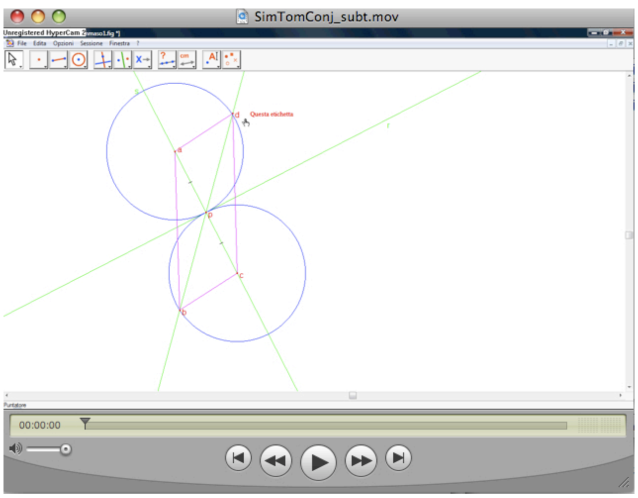


Figure 4.5.2.2.4 A screenshot of the solver’s exploration at the moment of the conjecture.

Episode 1	**Brief Analysis**
[1] Sim: So, ... [2] I: hmmm [3] Sim: eh, when AD [4] Sim: when D belongs to the circle, we have a parallelogram, [5] Sim: because...uh, but now...	In lines [3] and [4] Sim uses the word “when” referring first to AD and then to D belonging to the circle they have drawn as a GDP. In line [4] there is a first formulation of a conjecture: the CL between the IOD (D

[6] Sim: D...because AD, since AP is equal to CP, it means that the radii are the same, and so also AD equals BC. [7] Sim: and since the two circles...are tangent... [8] Sim: eh, they are...how can we say that they are parallel? [9] Tom: Wait, first mark them [he murmurs something]. [10] I: So, what is your conjecture? [11] Sim: So, since we constructed the circle, AD, uh D...ADCB is a parallelogram when D belongs to the circle. [12] Tom: So...[writing]	belongs to the circle) and the III (ABCD parallelogram) is made explicit through the "when". Evidence for such interpretation is provided in lines [6]-[9] when the solvers seem to engage in an argumentation in which they attempt to prove their conjecture. In particular, in [8] Sim is looking for a way to "say that they are parallel", i.e. to prove what is missing in order to "say that there is always a parallelogram" ([25]). Again Sim uses "when" ([11]) to separate the condition "D belongs to the circle" from "ABCD is a parallelogram" ([11]).
Episode 2 [13] Tom: ...if [14] Sim: Because these two [15] Tom: ...[murmurs as he writes] [16] Sim...[murmurs as he thinks and draws the segments] [17] Tom: [writing] ...with center...and radius AP... [18] Sim: Now, so... [19] Tom: [writing]...the quadrilateral...	**Brief Analysis** Tom immediately interprets this as an "if…then…" statement which he writes down ([12]-[28]) as: "If we construct a circle with center in A and radius AP the quadrilateral ABCD is a parallelogram." In the meantime Sim proceeds in his attempt at proving his conjecture, but stops again at the need to prove the two

[20] Sim: these are also radii [marking AD, AP, PC, and BC]. [21] Sim: and so these two are equal [pointing to AD and BC]. [22] I: uhm. [23] Tom: [writing]...ABCD... [24] Sim: so. but now we need to prove that they are parallel. [25] Sim: because this way we can say that there is always a parallelogram.	opposite sides to be parallel ([24]).
[26] I: ok. [27] Sim: and so [28] Tom: Right? [reading what he wrote] If we construct a circle with center in A and radius AP the quadrilateral ABCD is a parallelogram.	With respect to the conjecture, when Tom reads to Sim what he has written, Sim instantly translates his original "when D belongs to the circle" ([4]) into "if D belongs to the circle" ([29]).
[29] Sim: Eh, not always...you have to say "if D belongs to the circle". [30] Tom: [writing]...when [31] Sim: when D belongs to the circle.	Tom adds the new condition to the written conjecture as "when…" ([30]) and Sim repeats his original "when D belongs to the circle " ([31]).

Table 4.5.2.2.4: Analysis of Excerpt 4.5.2.2.4

This almost unconscious switching the terms with great ease seems to indicate interchangeable use of the words "if" and "when", as synonyms to refer to a condition that leads to the conclusion stated in the conjecture.

In the next subsection we give examples of conjectures without traces of the dynamic exploration, in which the "if…then…" form is used.

4.5.2.3 Conjectures without Traces of the Dynamic Exploration. In this subsection we provide some examples of conjectures stated in formal language, and belonging completely to the "static" world of Euclidean geometry. These conjectures clearly show that the transition from "dynamic" to "static" has successfully occurred through a proper interpretation of the Cabri experience in mathematical terms. This subsection contains four examples of conjectures formulated in a "static form", through different techniques: use of the logical "if…then…" form (potentially omitting the "then"); use of the symbol of logic implication; or separation of the premise from the conclusion through labeling.

The first example (4.5.2.3.1) is what two students wrote during their exploration (PS_Fin_GiuAlb_p6_c2) in Problem 2.

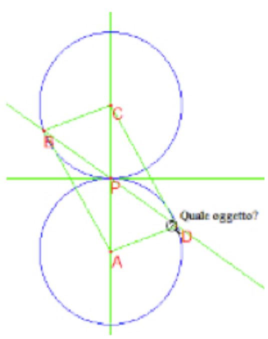


Figure 4.5.2.3.1 A screenshot of the figure at the moment of the expression of the conjecture.

"If D is on the circle with radius PA then the quadrilateral ABCD is a parallelogram." [Italian: "Se D è sulla circonferenza di raggio PA allora il quadrilatero ABCD è un parallelogramma."]

The premise and the conclusion are clearly separated by the "if" and the "then", and the language used does not suggest movement. The only traces of the exploration may be found in the words "D is on the circle with radius PA", in which D plays the main role as the acting-element. From "moving along a circle" it is now conceived as "being on". Possibly the premise of the conjecture could have been expressed in an even more "static" form as "D belongs to the circle".

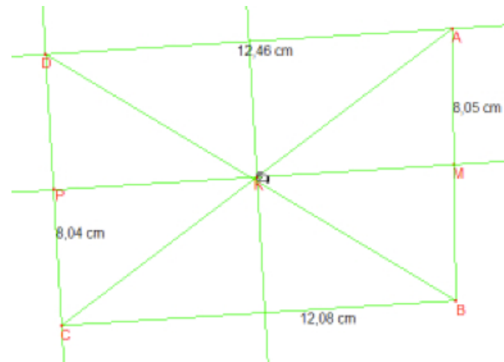


Figure 4.5.2.3.2 A screenshot of the figure at the moment of the expression of the conjecture.

The following example (4.5.2.3.2) is what two students wrote during their exploration (PS_Fin_ValeRic_p3_c5) in Problem 1.

"If K belongs to the perpendicular bisector of AB, ABCD rectangle"

[Italian: "Se K appartiene all'asse di AB, ABCD rettangolo."]

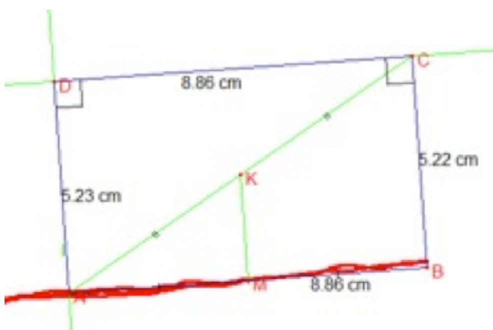


Figure 4.5.2.3.3 A screenshot of the figure at the moment of the expression of the conjecture.

The "then" and the verb in the conclusion of the statement are omitted, but the distinction between the premise and the conclusion is marked clearly by the "if" and the comma after "AB".

The statement (4.5.2.3.3) below is what two students wrote during their exploration in Problem 1, using the symbol of logic implication.

"If A belongs to the line $\perp$ to l through M $\Rightarrow$ ABCD is a rectangle."

[Italian: "Se A appartiene alla retta $\perp$ ad l passante per M $\Rightarrow$ ABCD è rettangolo"]

This written conjecture is formulated in completely static geometric terms, and it even makes use of the symbol of logic implication to link the premise and the conclusion.

In this example (4.5.2.3.4) two students working on Problem 1 shows how a conjecture may be stated by separating explicitly the premise from the conclusion.

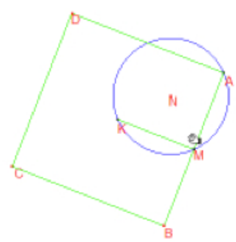


Figure 4.5.2.3.4 A screenshot of the figure at the moment of the expression of the conjecture.

“hyp: M ∈ circle with center N (midpoint of KA) and radius NA. Ths: <ABC=<BCD=<CDA=<DAB”

[Italian: “hp: M ∈ circonferenza con centro N (punto medio di KA) e raggio NA. Ts: <ABC=<BCD=<CDA=<DAB.”]

Here the premise and the conclusion are labeled as such explicitly (“hyp”, “ths”). We can infer that the labeling yields meaning for the students because when proving the conjecture they start by assuming as true what is described in their “hyp”.

4.5.3 The Last Step of the MD-conjecturing Model: Testing the Conjecture

We have noticed that some solvers choose to perform a robust dragging test once their conjecture is formulated. Through this form of dragging, they seem to be checking that a robust construction of the IOD generates a robust III on the Cabri-figure they have explored. Excerpts 4.5.3.1 and 4.5.3.2 provide examples of this. In Excerpt 4.5.3.1 the solver redefines the dragged base point as a point on the object she has constructed as her GDP, and then proceeds to drag the linked base point. In Excerpt 4.5.3.2 the solvers reconstruct the Cabri-figure following the steps of the construction and adding a property to one of the bade points in order to construct the IOD robustly and proceed with the dragging test. Finally Excerpt 4.5.3.3 is the continuation of Excerpt 4.5.2.2.3, and it shows how the command "ask Cabri" can be used to test a conjecture.

Excerpt 4.5.3.1. This excerpt shows how a student makes use of the robust dragging test to test her conjecture, after having written it down. The excerpt is taken from a student's work on Problem 2, and it is the continuation of the exploration from which Excerpt 4.4.1 is taken. Through maintaining dragging with the trace activated, Isa has conceived a GDP and expressed the IOD as D moving along a circle. She has not constructed the GDP or performed a dragging test, and when she writes her conjecture (at the beginning of the excerpt below) she does not seem to be convinced enough to start proving it, but instead she prefers to test it with a robust dragging test.

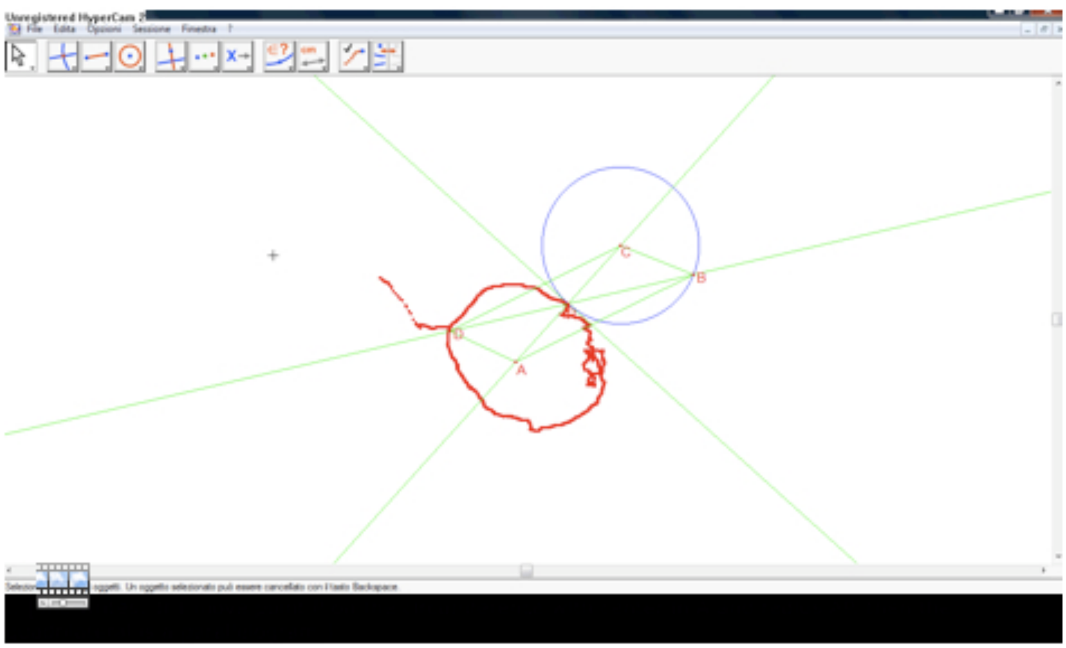


Figure 4.5.3.1 A screenshot of the solver's exploration during the following episode.

Episode	Brief Analysis
[1]...[she writes: "If I move point P on the circle with center in A and radius AP, then	After Isa writes her conjecture which still contains traces of the dynamic exploration

the quadrilateral is a parallelogram."] [2] Isa: eh, for now I'll try to construct it… [3] I: Ok. [4] Isa: So...this, now I need to construct a circle [she constructs a circle centered in A with radius AP]…where is it [the command]? to link D to the circle?	([1]), she proceeds by constructing the IOD robustly: she constructs the object that represents the GDP she has provided ([4]), and she then redefines point D upon it.
[5] I: Under the perpendiculars. [6] Isa: Ok, now let's try to move... [7] [she starts dragging D, now linked to	As she does this, Isa does not disactivate the trace.
the circle] [8] Isa: Yes...here it becomes a single point...and here again…Now we can also turn the trace off.	Isa drags D and notices that in certain points the quadrilateral "collapses", but she seems to conceive these as special cases of the general parallelogram.
[9] I: Now you seem to be pretty convinced. [10] Isa: Yes.	At this point she seems to be looking at both the III and the IOD simultaneously thus conceiving the Cabri-figure as a whole. In fact she even wants to turn off the trace as it is of no use any more.

Table 4.5.3.1: Analysis of Excerpt 4.5.3.1

The fact that Isa does not disactivate the trace when she first constructs the IOD robustly ([4]-[5]) may indicate that she is not completely convinced that her GDP is correct. As she drags she seems to get confirmation that the whole circle is actually a good GDP ([7]-[8]), and thus is ready to disactivate the trace. Visualizing the two invariants simultaneously being verified and knowing how the construction was modified seem to make Isa become convinced of her conjecture, as she confirms in [10].

Excerpt 4.5.3.2. This excerpt is the continuation of Excerpt 4.5.1.1 (numbering is continued), and it shows how the two students test their conjecture ("ABCD is a parallelogram if PA is equal to AD") by reconstructing the Cabri-figure and performing a robust dragging test.

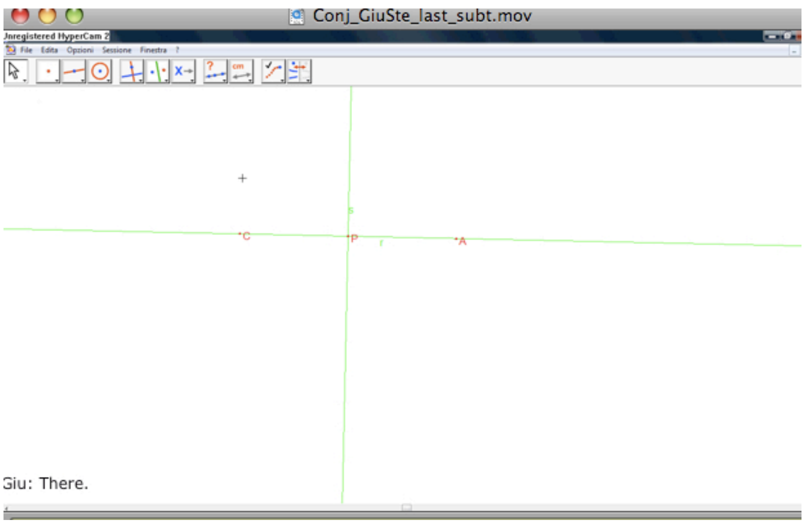


Figure 4.5.3.2 A screenshot of the solvers' exploration during the following episode.

Episode	Brief Analysis
[18] Giu: There [19] Giu: Now we need to make all those nice circles [20] Ste: Yaayy!!! [21] Giu: So this...through there [22] Ste: This one...yay! [23] Giu: So...	To test their conjecture, the students proceed by reconstructing the whole Cabri-figure, adding the premise of their conjecture as a new robust property ([18]-[28]). They use the IOD ("D on the circle with center in A and radius AP") to construct the property "PA equal to AD". After

[24] Giu: Bravo! Wait...that one is B...no, no, no don't do it. [25] Ste: Yes! [26] Giu: No, because...because it depends on that one [pointing to D]!! [27] Ste: Really? Oh yeah! That's right. [28] Giu: Eh!! So draw the circle that goes through this one and through this one [with center in A and radius AP]. [29] [Ste drags point D] [smiles from both the solvers]	constructing point B, the solvers insist on constructing the circle centered in P ([28]), with radius PB, which probably indicates their desire to check the property "PB congruent to BD". This makes sense because this was the bridge property they used as an III to induce "ABCD parallelogram" through maintaining dragging. They seem quite satisfied with the robust dragging test they perform in [29].

Table 4.5.3.2: Analysis of Excerpt 4.5.3.2

As Ste reconstructs the Cabri-figure adding the new condition that they are testing, Giu seems to be guiding the choice of which points to use to construct the new elements of the figure: in particular the circle representing the GDP, on which D will be chosen ([24]-[27]). Although it can lead to the same outcome, the idea of reconstructing the whole figure "adding" a new robust property to the properties that descend from the steps of the construction is a different technique with respect to simply constructing the IOD robustly by constructing the GDP and linking the dragged-base-point to it. As the solvers reconstruct the whole quadrilateral they seem to revisit and summarize steps of the exploration process. Finally, the robust dragging test allows the solvers to simultaneously observe "AD congruent to PA" (or "D on the circle") and "PD equals PB" (or "ABCD parallelogram"), and thus confirm their conjecture.

Excerpt 4.5.3.3. This excerpt is the continuation of Excerpt 4.5.2.2.3 (numbering is continued), and it shows how the solver used the command “ask Cabri” to test her conjecture.

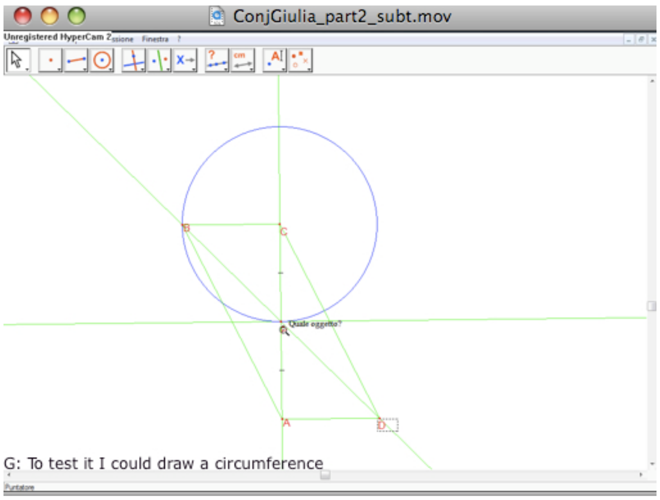


Figure 4.5.3.3 A screenshot of the solver’s exploration during the following episode.

Episode	**Brief Analysis**
[7] G: To test it I could draw a circle [8] I: yes... [9] G: with radius AP [10] I: ok [11] G: So then I could put D on this circle and then see... … [17] G: I wanted "redefine object", yes but first I wanted to ...ok	G has written the conjecture and expresses the desire to test the conjecture explaining what she intends to do.
[18] I: Ah, you wanted to do it over.. [19] G: Then I do "redefine object"...this point...point on an object?	She constructs the circle and links the dragged base point D to it successfully ([19]-[21]).

[20] I: yes... [21] G: On this circle [22] G: Now what should I try, should I ask it if they are parallel? … [27] G: Is this segment is parallel to this? [as she clicks on the objects] [28] I: There, now it should open. [29] G: Ok [murmuring something and she seems satisfied] [30] I: Ok. [31] G: Should I try to move it? I'll try to change the position.	Giu wants to use the command “ask Cabri” to see whether the pairs of opposites sides of ABCD are in fact parallel ([22]). When G uses the command “ask Cabri” she inquires about a property which defines “parallelogram” and is therefore basically the III ([25]-[28]). Reading Cabri’s reply “the two segments are parallel” on the screen seems to convince G of her conjecture more than dragging the redefined point.

Table 4.5.3.3: Analysis of Excerpt 4.5.3.3

It seems that for G it is less important to visualize the two invariants simultaneously than to be sure that according to Cabri her new construction *is* a parallelogram. Only after having read the answer does G spontaneously propose to “move it” and “change the position” ([31]).

Concluding Remarks. At this point we have completed our introduction of the main elements of our model and the relationships between them. Our model describes the perception of invariants, the search for new invariants, the conceived link between them, and how the premise and conclusion of the conjecture fall into place. Below is a visual representation of our model that summarizes the various elements and their mutual relationships.

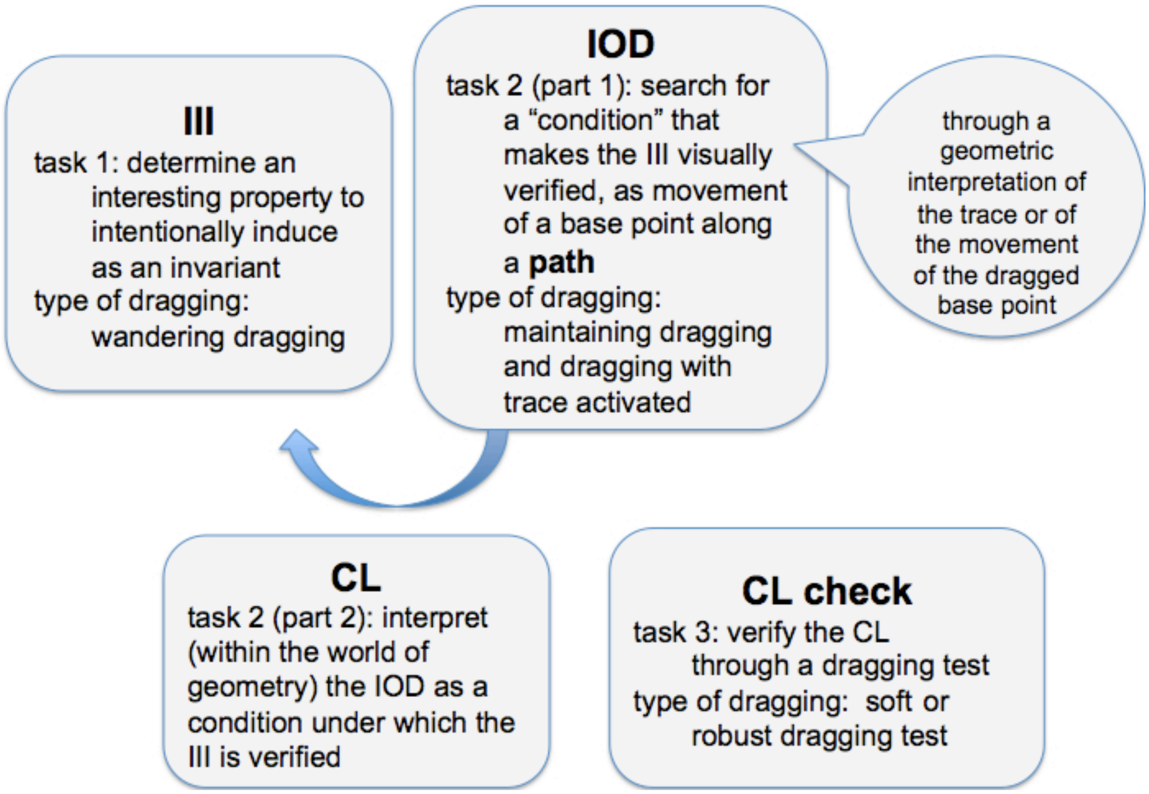


Figure 4.5.1: A representation of the interplay of the elements of the MD-conjecturing Model

Using the model as a tool of analysis led us to some refinements and new notions, many of which related to various types of invariants and how they emerge from the exploration. The central role of these different invariants within our model has led us to a new description and partial generalization of the model itself. This new description is the main focus of Section 4.6.

4.6 Model as Invariant-Type Phases

Through the analyses of transcripts and video recordings of students' work on the activities proposed, we have shown how our initial model seems to appropriately describe processes that may occur during the explorations. Moreover the model was enriched with new elements that were recognized as recurring in many explorations. In particular, with respect to the initial model, we conceived and added new notions, many of which were related to a characterization of invariants that seemed to help describe students' work. The types of invariants we added are point-invariants and construction-invariants (either basic or derived), and additional construction-invariants, that is, invariants that are constructed as a robust invariant after having been observed (or induced) as a soft invariant, or potential property of the Cabri-figure considered, as described in Section 2.1. These new notions and further reflection upon the analyzed transcripts led us to focus on the central role played by invariants throughout the explorations. Therefore we now provide a new description of our model as phases, each characterized by the particular type of invariant investigated. The phases are: (1) the point-invariant and construction-invariant phase; (2) the intentionally-induced-invariant phase; and the (3) additional-construction-invariant phase. Before delving into the descriptions of each phase, we present a second hypothetical exploration of Problem 1, in which we highlight the new elements introduced in the preceding sections of this chapter with particular attention towards different types of invariants.

4.6.1 The Invariant-type Phases

Many students' behaviors during the exploration of Cabri-figures seem to be characterized by the perception of invariants of different types. As we have seen in Section 4.2.1 many solvers start to drag the base points looking for regularities in the behavior of the Cabri-figure (or of subfigures), noticing what we have defined as point-invariants and construction-invariants (section 4.2.1.1 and 4.2.1.2). Solvers may express their first conjectures relating these invariants at this time. Such conjectures do not deal with "conditions under which a certain configuration is obtained", but rather they are "general statements" about the step-by-step-construction, that relate basic and derived construction invariants, and potentially point invariants.

Then solvers may proceed by noticing that a particular property has the potential of being induced upon the Cabri-figure, as described in Section 4.2.1.3 and in the rest of Section 4.2. Therefore a second phase of the exploration may be characterized by the solver's attempt to explain (through a conjecture) how to induce a particular III through dragging. The conjectures that arise during this phase of the exploration are the ones our initial model described in detail. However, we have observed that sometimes the discovery of basic properties (section 4.2.1.3) leads to conjectures (*basic conjectures*, which we introduce in Chapter 5) in which they are expressed in the premise instead of being overcome by a condition found during MD. We will describe this phenomenon in Chapter 5.

Finally, solvers may construct a new property (typically an IOD) robustly in order to continue the exploration within a subset of Cabri-figures of the initial set defined by the step-by-step construction. In this case we can define a new class of invariants, *additional-construction-invariants*.

After having constructed new *additional-construction-invariants*, the exploration may continue, starting from the first phase we described, since the solver is now in front of a new figure. During this phase new construction-invariants and point-invariants may be noticed (a new phase 1), and successively new IIIs may be induced by the subject with the intention of producing new conjectures (a new phase 2). Below is a more detailed description of the three phases.

Phase 1: point-invariant and construction-invariant phase

The solver uses wondering dragging of the various base points of the construction and notices a certain property that seems to always be true (for the dragging of a specific base point or of different base points). Conjectures during this phase may be of the type: “the figure is always a …” or “the figure always has the property …”. The premises of such conjectures are frequently implicit in the final formulations of these conjectures. Such premises are the properties (or a subset of them) assigned to the figure by the steps of the construction.

During this phase it is also possible for the solver to notice two construction (or point) invariants in particular and try to link them. The solver may either

a) link the two invariants through a conditional link (CL) choosing a rule of which they are a case of from his/her bag of mathematical knowledge (known theorems);

b) or he/she links the two invariants through a CL expressed as a conjecture to be proved (the conjecture is not a known theorem).

Phase 2: intentionally-induced-invariant phase

The solver encounters an interesting configuration (frequently through wondering dragging), and decides to investigate “when the initial construction falls into this case” using *maintaining dragging*. Here our cognitive model described in sections 4.1, 4.2, 4.3, 4.4, and 4.5 applies, leading to a conjecture that links and IOD and an III.

The exploration of the particular interesting configuration may continue with the repetition of the phases described above, when dragging a different base point of the construction.

Phase 3: additional-construction-invariant phase

The solver notices (or looks for) a new interesting configuration, which s/he recognize(s) as a subcase of a previously explored case. In order to investigate this new case (for example the case “square” after having analyzed “rectangle” as an III) the solver modifies the initial construction by linking a base point to a curve (the geometrical description of a path) that s/he has discovered implies the more general case. The solver then proceeds through the phases described above, exploring the new construction. More cycles of exploration of this type may be added depending on the possible subcases of a given initial construction. For example, the exploration of a quadrilateral may have at most four cycles[2]: 1) trapezoid; 2) parallelogram; 3) rectangle and rhombus; 4) square.

In the following section we provide an example of what an exploration that takes into account all the elements of the model we have introduced might look like. We will then re-describe the model in terms of tasks and subtasks that the solver can engage in during each phase described above, to relate our new description of the model to our previous task-based one.

4.6.2 A Complete Hypothetical Exploration

The step-by-step open construction problem presented in Problem 1 is the following.

- Draw three points: A, M, K;
- construct point B as the symmetric image of A with respect to M;
- construct point C as the symmetric image of A with respect to K;
- construct the parallel line *l* to BC through A;
- construct point D as the intersection of *l* with the perpendicular to *l* through C.
- Consider the quadrilateral ABCD. Make

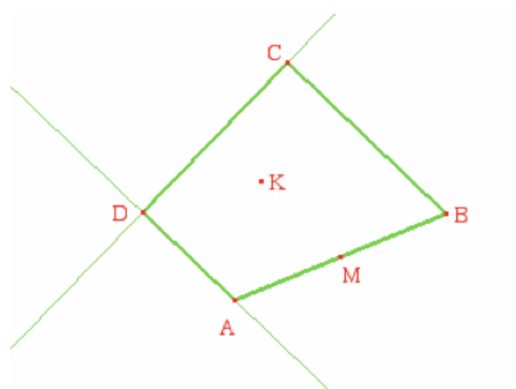


Figure 4.6.2 ABCD as a result of the step-by-step construction.

[2] Given a construction with additional constraints like in our activities there are fewer cycles, because some cases become coincident under the construction constraints.

conjectures on the types of quadrilaterals that it can become, describing all the possible ways it can become a certain quadrilateral. Write your conjectures and then prove them.

We can start by dragging the base point A and noticing that points B, C and D move as a consequence of A's movement. The length of segment BC, however, seems to remain invariant for any movement imposed on A. This can lead to perceiving the length of BC as an A-invariant or as a (derived) construction invariant. At this point we could either go back to the steps of the construction and try to get a better grip on the nature of the length of BC, or we could drag a different base point to see if it still seems to be invariant. Let's assume we try to drag point M. As soon as we start dragging this point, if we are still focused on the length of BC, we will very likely see that the length is not an M-invariant. Therefore the length of CB cannot be a construction-invariant. We might now focus on what seems to be another property of ABCD, that it appears to "always" be a right trapezoid. Therefore this property is likely to be a construction-invariant. The observation may lead to a first conjecture: "ABCD is a right trapezoid", and we could provide an argumentation involving basic and derived construction-invariants (the right angle in D and thus in C, and the parallel bases BC and *l*) as to why this might be the case.

At this point we could start looking for other possible types of quadrilaterals that ABCD might become. We could have noticed during our previous dragging that the configuration "rectangle" seemed to appear sometimes, or we might not have noticed this and we can start dragging a base point, say M, to see if this configuration is possible to obtain visually. It could help us to use a characterizing property of rectangles like "a rectangle is a quadrilateral with four congruent right angles" (basic property), that we can slim down to "the angle ABC is right" thanks to the construction-invariants of our figure.

Having seen a few different possible rectangle-configurations may help us believe in the existence of a path along which dragging the base point M will induce the angle ABC to be right (thus our potential III). Therefore, now we can try to use maintaining dragging to maintain the III and search for a GDP in order to reach an IOD. Activating the trace of M may help us to perceive and describe a GDP, as shown in the figure.

The red mark left by the trace tool together with the haptic perception can lead us to a GDP like "the circle with diameter AK". The IOD, therefore, could be "M moves along the circle with diameter AK". Once we have reached an idea for an IOD we may try to focus our attention both on the III and on the IOD (or quickly alternate or focus from one to the other repeatedly) and try to check their simultaneity. At this point we can try to check our idea by intentionally dragging along the GDP (which we may also decide to construct geometrically) and checking that the III is actually maintained. This is a soft dragging test that allows us to check the existence of a CL between the IOD and the III.

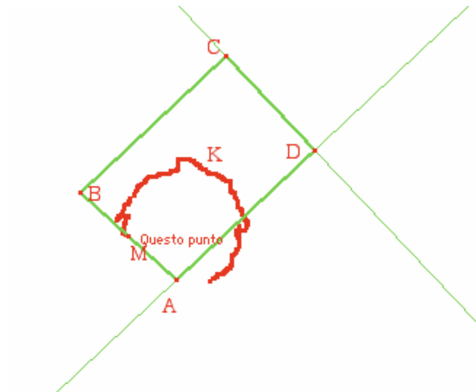


Figure 4.6.3 ABCD as maintaining dragging with the trace activated is performed.

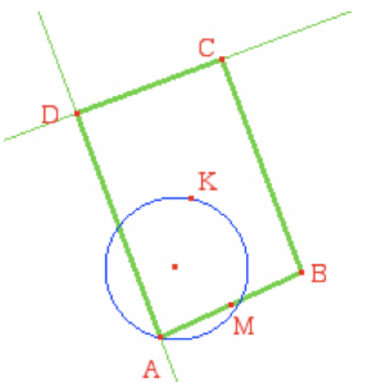


Figure 4.6.4 Soft dragging test after having constructed the GDP.

At this point we may feel convinced enough to formulate a conjecture, but we might also decide to construct the IOD robustly, thus creating an additional-construction-invariant to the Cabri-figure. We can do this by linking M to the circle we constructed. Now any base point we drag should allow us to perceive our original III (angle ABC is right) as a (derived) construction-invariant. Moreover, since the III was a bridge property, a sufficient condition for ABCD to be a rectangle, the property "ABCD rectangle" should now appear to be a (derived) construction-invariant. The verification of these facts occurs during a robust dragging test.

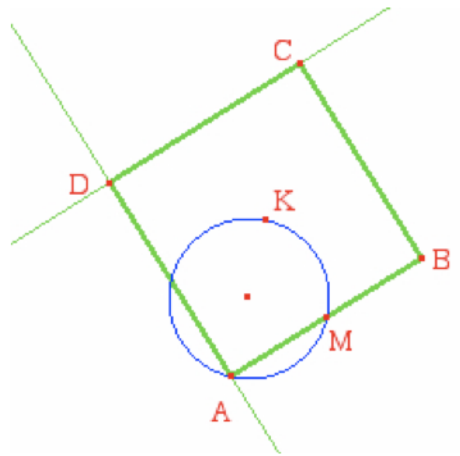
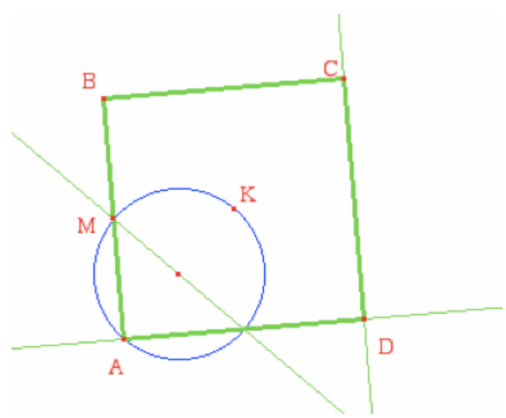


Figure 4.6.5 Wandering dragging on the new Cabri-figure dragging base point M.

A possible conjecture we could formulate is: "If M is on the circle of diameter AK, then ABCD is a rectangle." In this case our figure passes the robust dragging test and our conjecture seems like a "good one" that we can now try to prove.

We can decide to continue our exploration by seeing whether ABCD can become other types of quadrilaterals. Since we have robustly constructed a rectangle at this point, adding a construction-invariant to the initial figure produced by the step-by-step construction, we might decide to use wandering dragging on the new Cabri-figure to try to induce types of quadrilaterals that are particular types of rectangles, for example squares.

Once we visually perceive that a new particular configuration is possible, we can



4.6.6 The property AM congruent to MK is constructed robustly.

proceed as before, trying to induce this property (or a minimum basic property) as an invariant through maintaining dragging. In this case wandering dragging shows us that it is not possible to "maintain" continuously the property "ABCD square", however we can find two choices for M along the circle which seem to induce the desired property. We can try to characterize these positions (a sort of discrete path) and formulate a new conjecture. For example, in both "good positions" the segments KM and AM seem to be the same length. We can check the sufficiency of such property by dragging M (a sort of soft dragging test). Moreover we can construct the property robustly by constructing the perpendicular bisector of KA and redefining M on the intersection of the circle and such perpendicular bisector, as shown below.

We now have a new additional-construction-invariant, which induces a whole new set of (derived) construction-invariants on the new Cabri-figure. Performing a (robust) dragging test on the new Cabri-figure visually confirms the simultaneity of the occurrence of the invariants we are interested in ("ABCD square" and "AM congruent to MK"), thus verifying a CL and leading to the formulation (or confirming it) of a conjecture like: "If M lies on the circle of diameter AK and AM congruent to MK, then ABCD is a square."

Conclusion. We can generalize the steps introduced in the simulated exploration, as we did in section 4.1, adding the new tasks introduced in the simulated exploration above.

- Task 1: Search for construction invariants.
    - This can occur through a distinction of point-invariants from construction-invariants.
    - Initial conjectures may be expressed on derived-construction-invariants.
- Task 2: Determine a configuration to be explored by inducing it as a (soft) invariant (III): through wandering dragging the solver can look for interesting configurations and conceive them as potential invariants to be intentionally induced. It may help to
    - search for a basic property (usually a necessary and sufficient condition) that induces the interesting configuration;
    - slim down the basic property to a minimum basic property (sufficient condition).
- Task 3: While maintaining the interesting configuration (or the minimum basic property) using maintaining dragging and maintaining dragging with the trace activated, look for "a condition" that makes the III be visually verified. This can occur through
    - a geometric interpretation of the trace
    - a geometric interpretation of the movement of the dragged base point.

  The "condition" may be considered the movement of the dragged base point along a path which can be described geometrically. The belonging of the dragged base point to a path with a geometric description (GDP) determines the invariant observed during dragging (IOD), and since this invariant arose as a “cause” for the III, a conditional link (CL) between the IOD and III may be also determined.

- Task 4: Verify the CL through the dragging test. This requires the accomplishment of (at least some of) the following subtasks:
  - represent the IOD through a construction of the proposed GDP;
  - perform a soft dragging test by dragging the base point along the constructed path;
  - perform a robust dragging test by providing (and constructing) a GDP that is not dependent upon the dragged base point and redefine the base point on it in order to have a robust invariant, then perform the dragging test.
- Task 5: Construct the additional-construction invariant (the IOD found above) robustly (if not already done in previous step) and continue the exploration investigating configurations that are subcases of the previously induced configuration by repeating tasks 1, 2 and 3.

Table 4.6.1: A more complete task-based description of the MD-conjecturing Model

(Basic or Derived) Construction-Invariant	Geometrical property of a construction that is described explicitly in its steps (basic construction-invariant) or that can be deductively derived from the basic construction invariants (derived construction-invariant). A construction invariant is property that is true for any choice of the base points, and therefore it is a robust property.
Point-Invariant (P-invariant)	Geometrical property that is true for any choice of base point P of the construction, while the others remain fixed. a P-invariant is a robust property under dragging of P.
Basic Property	Geometrical property that characterizes the interesting configuration that the solver

	wants to investigate
Minimum Basic Property	Basic property “slimmed down” (thanks to the properties derived from the steps of the construction) to a sufficient condition to induce the interesting configuration.
Additional-Construction-Invariant	Newly added robust property of the Cabri-figure

Table 4.6.2: New key elements of the MD-conjecturing Model

4.7 Concluding Remarks

Throughout this Chapter we have described our cognitive model for conjecture-generation, and used it as a tool of analysis for different excerpts of students' explorations. During the exposition we have highlighted some critical moments of the process of conjecture-generation described by the model, such as the determination of an III, using maintaining dragging to induce it as an invariant, conceiving a path and an IOD and conditionally linking it to the III, checking the CL, and formulating a conjecture. In Chapter 5 we will describe students' difficulties that arose with respect to these critical moments. Before doing so, we provide a table summarizing the subtasks related to the invariant-type phases of the model and the dragging modalities used during each of them.

Phase of the Model	Subtasks	Dragging Schemes Used
point-invariant and construction-invariant phase	distinction of point-invariants from construction-invariants	wandering dragging
	formulation of initial conjectures	dragging test (robust)
intentionally-induced-invariant phase	determine an III	wandering dragging
	find a (minimum) basic property	no dragging, wandering dragging, dragging test (soft) to test sufficiency of condition
	maintain the III	maintaining dragging
	find a GDP and provide an IOD	maintaining dragging, dragging with trace activated
	verify the CL	dragging test (soft and/or robust version)
additional-construction-invariant phase	construct the IOD from previous phase robustly	redefinition of point on object
	repeat previous phases on new construction	all the dragging above

Table 4.6.3: Model as invariant-type phases with related subtasks.

CONJECTURING IN DYNAMIC GEOMETRY: A MODEL FOR CONJECTURE-GENERATION THROUGH MAINTAINING DRAGGING

(VOL. II: CHAPTERS 5-7)

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CHAPTER V

THE CONJECTURING PROCESS UNDER THE LENS OF THE MD-CONJECTURING MODEL: SOME STUDENTS' DIFFICULTIES

In the previous chapter we saw how different solvers seemed to use maintaining dragging in an efficient and spontaneous way, after the in-class introduction, and we referred to such appropriated use as “expert use”. In this chapter we interpret students' difficulties related to expert use of MD that arose during the activity-based interviews. We base such interpretation on what we have identified as four fundamental components that a solver seems to need to master in order to use the MD as a tool for conjecture-generation. The components seem to be rooted in some of the major differences between conjecturing in a paper-and-pencil environment and in a DGS that we have described in Chapter 2. In particular, we advance the hypothesis that if a solver does not perceive a Cabri-figure dynamically but statically as if s/he were using paper and pencil, s/he will encounter difficulties in differentiating geometrical properties of a figure (even though this may be a Cabri-figure) from invariants of a dynamic-figure. An outcome of this behavior seems to be what we will describe as is a difficulty to overcome “basic conjectures” (Section 5.1).

In addition, the MD-conjecturing Model can be used to highlight difficulties in the perception of invariants, especially of soft invariants. In this regard we advance the hypothesis that the solver needs to be mentally flexible: as a soft invariant is being induced the solver might perceive new induced invariant properties which appear simultaneously, but this can occur only if the solver is able to balance his/her expectations with mental flexibility in order to not lock onto particular ideas that inhibit the formation of new ones. In this chapter we will introduce these and other difficulties that have to do specifically with the process of conjecture-generation described by the MD-conjecturing Model through examples that arose during students' explorations.

The first four sections of this chapter are each dedicated to one of these components: developing transitional basic conjectures (Section 5.1), conceiving a property as an III (Section 5.2), being mentally flexible (Section 5.3), being aware of the status of objects (Section 5.4). Finally in the last section of the chapter we introduce some spontaneous behaviors that solvers exhibited for overcoming difficulties related to maintaining dragging (Section 5.5), and from which we developed prompts to help other students address similar difficulties.

5.1 Developing Transitional Basic Conjectures

Analyzing the data generated from the activity-based interviews, we found that many solvers start their explorations with a preliminary phase, before starting to use maintaining dragging. During this phase the solvers would develop what we call "basic conjectures". Per se basic conjectures are not "inappropriate" with respect to conjecture-generation as described by our model, however if not overcome, they may hinder the use of maintaining dragging during the rest of the exploration. Moreover, for some students basic conjectures seem to dominate the exploration, inhibiting the generation of other conjectures that link an III and an IOD even when these invariants have been found through use of maintaining dragging.

In the introduction to this chapter we highlighted differences in possible processes of conjecture-generation in a paper-and-pencil environment and in a DGS. We advance the hypothesis that solvers' inability to perceive such differences can lead to difficulties in exploiting the potential of the DGS. In particular, if a solver does not perceive a Cabri-figure dynamically but statically as if s/he were using paper and pencil, s/he will encounter difficulties in differentiating geometrical properties of a figure (even though this may be a Cabri-figure) from invariants of a dynamic-figure. For example, let's assume a solver has constructed a Cabri-figure corresponding to the steps of an activity, and s/he starts dragging and stops when s/he thinks the configuration is "interesting" because "it is a parallelogram". At this point the solver freezes the image and treats the figure as if it were in a paper and pencil environment, formulating conjectures about the configuration "parallelogram". Therefore these conjectures will have "the quadrilateral is a parallelogram" as a conclusion and some basic property the solver has thought of as a premise. In this case the solvers seem to perceive a relationship of logical dependency between the basic property and the interesting configuration – treated also as a geometrical property – but they do not seem to conceive these properties as invariants with respect to dragging, that is with respect to the movement of a particular base point or even of any base point at all. It is with respect to such frozen figure and to the properties that solvers assign to them that we observe the emergence of what we called a *basic conjecture*:

> *a particular type of conditional statement in which the conclusion is an "interesting configuration" and the premise is a basic property (or minimum basic property) with respect to the interesting configuration described in the conclusion of the conjecture itself.*

Therefore basic conjectures do not lead to the introduction of new information with respect to the interesting configuration, however basic conjectures can be held by the solver with a strong degree of belief. This seems to be the case because they are based on definite knowledge (definitions or theorems, usually, that the solver knows). Therefore some solvers seem to be satisfied with them and do not feel the need to continue their exploration in a different direction. In subsection 5.1.1 we provide examples of basic conjectures developed in a first phase of different solvers' explorations, and we show how sometimes these are spontaneously overcome whereas other times solvers seem to "fix on" them and do not spontaneously feel the need to continue their exploration. In this case we speak of a "block at a basic conjecture" which needs to be overcome in order to proceed with maintaining dragging. Then, in subsection 5.1.2 we show how "fixing on" basic conjectures may inhibit the accomplishment of other tasks described in the MD-conjecturing Model even when maintaining dragging is performed by the solvers.

5.1.1 Basic Conjectures in the Preliminary Phase

For various solvers, once a basic conjecture is expressed on a static configuration, there does not seem to be a need to go on and further explore the particular interesting configuration. Below are some examples of students reaching basic conjectures during a preliminary phase of the exploration. The first excerpt (5.1.1.1) shows an example of two solvers developing a basic conjecture, but immediately overcoming it and initiating maintaining dragging spontaneously. The second (5.1.1.2) and third (5.1.1.3) excerpts show examples of solvers feeling satisfied with basic conjectures. In these cases the solvers did not initiate maintaining dragging spontaneously, and felt they had completed the activity by providing the basic conjectures they wrote down.

Excerpt 5.1.1.1 (same as 4.2.5). Let us now consider this excerpt, from two students' exploration of Problem 2. We used it in Chapter 4 to exemplifies the identification of a basic property, slimmed down to a minimum basic property, which the solvers use to obtain the configuration they are interested in. We use the excerpt here to show how the solvers have actually developed a basic conjecture, but immediately overcome it, with the intention of performing maintaining dragging. The name of the solver who is performing the dragging is in bold letters.

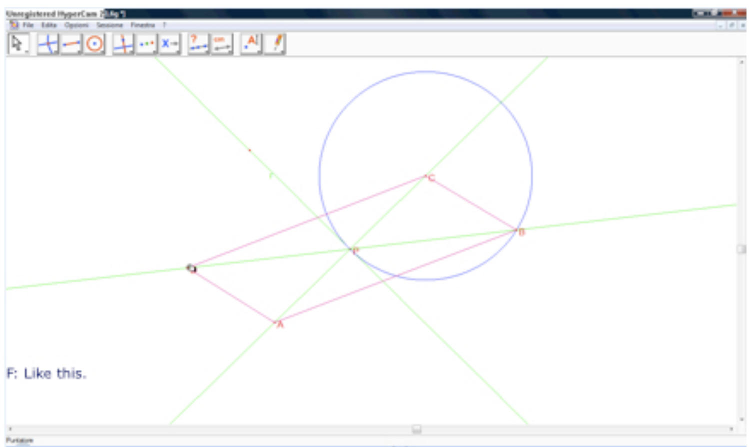


Figure 5.1.1.1: A screenshot of F & G's exploration

Episode	**Brief Analysis**
[1] **F**: wait, it is a…let's try to for example make it become a parallelogram. [2] G: No… yes, go. [3] **F**: Like this. …	F proposes to try to make ABCD a parallelogram ([1]) and seems to be unsure about how to drag the base point D in order to do this.
[8] G: I understand! so, C… we have	G conceives a basic property ([8]), which

to have the diagonals that intersect each other at their midpoints, right? [9] **F**: Right. [10] G: And we know that CA is always divided by P. [11] **F**: exactly, so…	implies a basic conjecture like: "If the diagonals of ABCD intersect at their midpoints, it is a parallelogram". Notice the "have to have" ([8]) implying logical dependency with the property "ABCD parallelogram".
[12] G: therefore it's enough that PB is equal to PD. [13] **F**: exactly. [14] G: you see that if you do, like, "maintaining dragging"… trying to let them more or less be the same [15] **F**: exactly… well, okay.	G proceeds to "slim down" the basic property making it into a minimum basic property, leading to a second implied basic conjecture: "If PB is equal to PB then ABCD is a parallelogram." The solvers do not stop at this basic conjecture, but use its premise as a bridge property to pursue maintaining dragging.

Table 5.1.1.1: Analysis of Excerpt 5.1.1.1

In this case the solvers do not even seem to be interested in writing down the basic conjecture they have developed. Instead they seem to make use of the condition expressed in the premise as a bridge property to help induce the III they have chosen through maintaining dragging. In other words, the solvers do not consider the basic conjecture to be a solution to the initial task, but instead an intermediate step in the description of the configuration they are investigating. As we will discuss in further depth in Chapter 6, overcoming a basic conjecture seems to become spontaneous in expert solvers, and in particular who have developed such scheme as a tool for searching for a "cause" of a given invariance – which will be interpreted geometrically as a "condition under which" the given invariance occurs. For solvers who intend to "search for a cause" of invariance of the III, the premise of a basic conjecture does not provide a satisfactory answer, thus they will spontaneously continue the exploration using maintaining dragging.

We would like to note that these behaviors provide insight into the solvers' interpretation of the task of formulating conjectures. The mathematical meaning of such a request is not obvious or simple to capture, nor had it been explicitly clarified. However it seems like the development of expert use of maintaining dragging comes together with a particular interpretation of the task of formulating conjectures. We will discuss this further in Chapter 7.

Excerpt 5.1.1.2. In this excerpt the solvers start from the interesting configuration of "rectangle" and find a potential minimum basic property through dragging. They then argue why this is enough using a basic property which they then try to make into their minimum basic property and justify their choice through an argumentation that does not involve any dragging at all. They are satisfied with their conjecture and write it down, without wanting to continue the exploration any further. The solvers then explain why they are convinced to have answered the question of the activity.

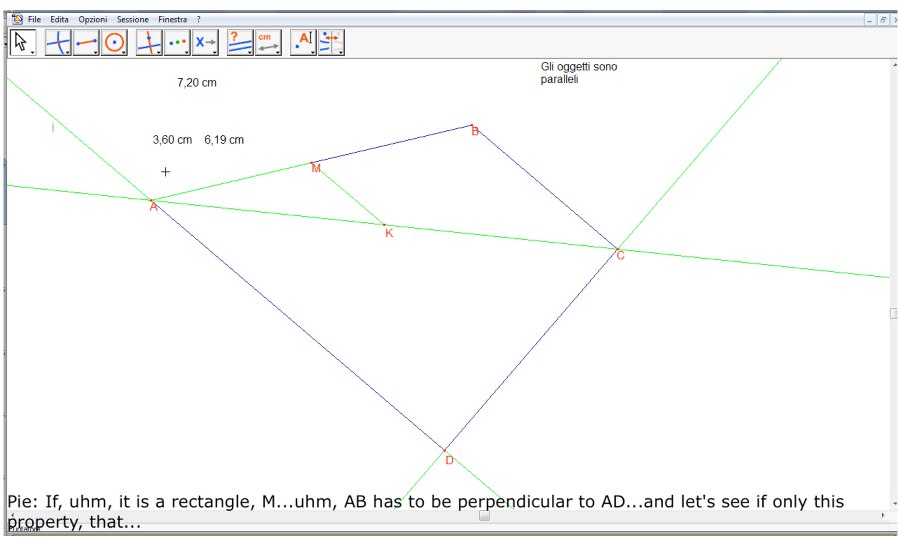
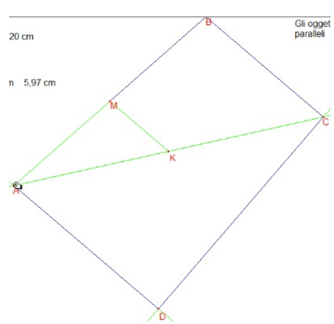
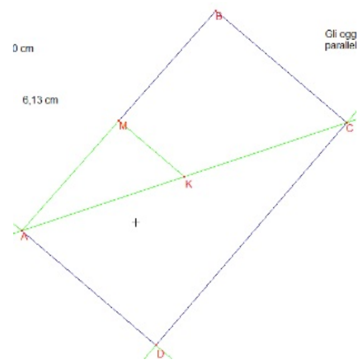
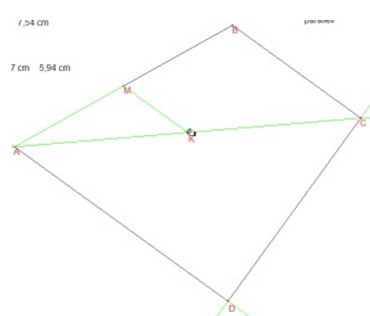


Figure 5.1.1.2: A screenshot of the solvers' exploration.

Episode 1	**Brief Analysis**
[1] Pie: If, uhm, it is a rectangle, M...uhm, AB has to be perpendicular to AD...and let's see if only this property, that... [2] Ale: Yes, [3] Pie: that brings along, let's say, all the others. [4] Pie&Ale: It looks like yes...[they speak together and it is hard to understand] [5] Pie: If AB were perpendicular to l, ...it means that... [6] Ale: So if it's perpendicular to l it means [7] Pie: ..that AD is equal to BC...let's see why [whispering]...because BC [8] Ale: No, AB is equal to CD. Because AD is equal to BC, uh...is by construction, I mean we constructed it parallel. [9] Ale: If we have, uhm, AB perpendicular to l, and CD perpendicular to l, then they are both parallel.  [10] Pie: Well, CD perpendicular to l...	Pie seems to start from the interesting property "rectangle" ([1]) and work backwards through basic properties which he slims down through an argumentation. Pie picks up his minimum basic property ("AB perpendicular to l" ([5])) again and seems to argue why it is sufficient, calling into the picture properties derived from the steps of the construction.
Episode 2	**Brief Analysis**
[11] I: Uhm, so the conjecture, excuse me, what is it? because I do not know what you are starting from. [12] Ale: ...from the rectangle. A rectangle is the figure that	The interviewer asks for the solvers to make their conjecture explicit.

has two sides...the opposite sides parallel...	
[13] I: uh huh... [14] Ale: and all the angles of 90 degrees. [15] I: Ok.	Ale restates a basic property his is starting his reasoning with ([12], [14]).
[16] Ale: So, if we know that by construction we have AD parallel to BC, ...by construction...then we made CD by construction parallel to l... 	
… [21] I: Yes, perpendicular. [22] Pie: Yes, it's right. Yes, because, I mean the segment AD is always parallel to BC. [23] I: Ok.	The solvers go over the argumentation once again leading to the minimum basic property they have obtained ("AB perpendicular to *l*" ([29]).
[24] Ale: Yes. [25] Pie: CD by construction is perpendicular to AD, [26] I: ok... [27] Pie: so therefore we have...this way we have one pair of parallel sides [28] I: Yes... [29] Pie: So if we put that AB is perpendicular to l...and since CD is perpendicular to l... [30] Ale: Then they are...	The solvers perform no dragging in this episode.

[31] Pie: two straight lines that are perpendicular to the same object are parallel themselves...we could say.	
Episode 3 [32] I: So...what's the conjecture? [33] Pie: That if AB is perpendicular to l, then the quadrilateral ABCD is a rectangle. …	**Brief Analysis** Pie states the conjecture, a basic conjecture, which the solvers are satisfied with.
Episode 4 [I has asked whether they feel that they have answered the question proposed in the activity] [34] Ale: Yes, because they are the only figures that have two sides, uh two right angles.. [35] Pie: and two parallel sides. … [38] **Pie**: Therefore we could do ...some other exploration [starting to drag the base point A]. [39] **Pie**: I mean it doesn't...[he starts dragging K]...see it doesn't  [40] **Pie**: without taking those types of figures. [41] Ale: Uh, we had to ... [42] **Pie**: In this case we have always varied...[he goes back to dragging A].	**Brief Analysis** The solvers seem to be uncertain how to continue the exploration, but they seem to be satisfied having looked at the cases of figures with two parallel sides and two right angles. Pie tries to move different base points, but Ale interrupts emphasizing the fact that they have already obtained all possible figures.

[43] Ale: The only figures that we can obtain are those.	
Episode 5	**Brief Analysis**
[44] I: Ok. So let's try to answer the question "trying to describe all the ways in which it is possible to obtain a certain type of quadrilateral." [45] Ale: so... [46] I: You can maybe concentrate on the rectangle?	The interviewer prompts the solvers to think about the initial question and to try to respond thoroughly.
[47] Ale: So, first of all we can say that in order to obtain a quadrilateral, I mean the quadrilateral that we have to obtain has to have to sides, uh two right angles and two parallel sides. [48] Pie: It always has two, I mean the quadrilateral ABCD by construction always has a pair of parallel sides and two consecutive right angles, C and D. [49] Ale: Ok. Therefore the figures that we can obtain are a rectangle, a square, or a right trapezoid...	The solvers give their response and seem to be satisfied with having provided basic conjectures for the different types of quadrilaterals that they thought it was possible to obtain.
[50] I: Ok. [51] Ale: We have said...we made the conjectures on each of these figures.	The solvers perform no dragging in this episode.

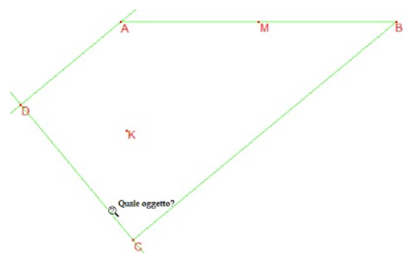
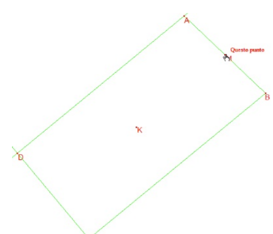
Table 5.1.1.2: Analysis of Excerpt 5.1.1.2

In Episode 1 the solvers develop the premise of their basic conjecture: "AB parallel to *l*" by slimming down a basic property. They finally state their conjecture and write it down when prompted (for the second time) in Episode 3. It seems that they need to convince themselves of the conjecture through oral argumentations (Episodes 1 and 2) and not through dragging. Using oral argumentations seems to be a recurring feature of preliminary phases of explorations in which basic conjectures are developed. Moreover we can notice how in the argumentations related to the slimming down of the basic property and to the basic conjecture there are all the necessary steps for a formal proof of the conjectured-statement. It is possible that the solvers feel satisfied in having produced such a convincing argument (very close to a proof) and thus that they feel confident they have "explained the case of the rectangle."

The solvers' attention to basic properties seems to inhibit their perception of other properties or the relationships between them as invariants with respect to dragging. Instead it seems as if they perceive simultaneity of properties and relationships between them in a particular instant that they want to freeze. Episodes 4 and 5 show how the solvers are not able to overcome their basic conjecture, feeling that they have thoroughly answered the question asked in the Problem. Although Pie starts to drag some base points (A and then K) in Episode 4 to "do some other exploration" ([38]), when Ale interrupts him and then explains why it is enough to do what they had done, Pie seems to become convinced that no more dragging is necessary. So no more conjectures are generated and no maintaining dragging is used.

<u>Excerpt 5.1.1.3</u>. This excerpt provides an example of the formulation of a basic conjecture in the preliminary phase of an exploration, in terms of "finding conditions to add" in order to obtain a particular type of quadrilateral. The solvers seem to be satisfied with their basic conjecture, and are not able to overcome it and start using maintaining dragging. Up to this point the excerpt is similar to the previous one, however, after a destabilizing prompt of the interviewer who asks them to re-read the question in the activity, they check which points can be dragged and formulate a new conjecture, though

still a basic one. This shows the strength of basic conjectures and the difficulty to overcome them.

Episode 1	**Brief Analysis**
[1] Sa: because it's perpendicular to that other one. [2] **Gian**: Yes. [3] **Gian**: So, if we don't add any condition, it's a right trapezoid. We have two right angles, and perpendicularity.  [4] **Gian**: Then if we add the condition that also AB is perpendicular to l, we have a rectangle.  [5] Sa: uh huh [The solvers get involved in formulating a basic conjecture for the "case of the square".] …	The solvers seem to interpret the task in terms of conditions on the base points, to add in order to obtain particular types of quadrilaterals. They choose the condition "AB perpendicular to *l*" and state their basic conjecture ([4]).
Episode 2	**Brief Analysis**
[15] I: So let's work on the rectangle, like before. [16] Gian: Yes. [17] I: Try to answer the question to describe all the ways in which it is possible to obtain a certain type of quadrilateral. [18] **Gian**: uh huh. So, these cannot be moved, so...	I prompts the solvers to go back to answering the question in the activity, concentrating on the case of the rectangle.

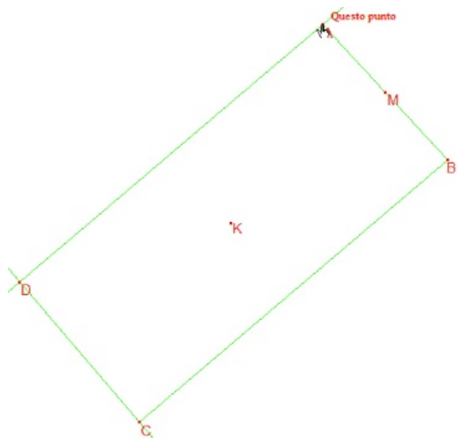
[19] Sa: B can't be moved [20] **Gian**: No, and C neither, so only A, M and K. [21] Sa: like before. [22] **Gian**: So, [23] **Gian**: when...AB is perpendicular to ... [24] Sa: when MK is perpendicular to MA...it is a rectangle. [25] **Gian**: Yes. 	The question leads Gian to trying to drag to check which points move and thus which points are base points. This seems to lead Sa to finding a new condition related to the base points, which she uses as a new premise to the basic conjecture.

Table 5.1.1.3: Analysis of Excerpt 5.1.1.3

In Episode 1 the solvers seem to interpret the task of the activity in terms of “adding a condition” to a quadrilateral in order to obtain a more particular type of quadrilateral. The basic conjecture they formulate is reached through a wandering dragging strategy which only allows the solvers to reach a case of the interesting configuration and visualize and confirm hypotheses on what a sufficient condition might be to obtain the interesting configuration. The conjecture they reach is a basic conjecture because the condition expressed in the premise is a minimum basic property.

In Episode 2 the solvers are prompted to reply to the question in the activity, and although this seems to lead Gian to some dragging, it does not lead the solvers to overcoming their basic conjecture. The attention to the base points seems to only lead Sa to perceiving a new minimum basic property referred to the base points of the Cabri-figure instead of only to vertices or sides of the quadrilateral ABCD. Although the basic conjectures have not been overcome, the new conjecture is a step forward with respect to the search for a condition that depends on the base points.

5.1.2 Persistence of Basic Conjectures in Later Phases of the Exploration

As we described in the previous section (5.1.1) we found that the fixity of basic conjectures may influence the preliminary phase of explorations in which maintaining dragging might otherwise be used. The exploration leads to an interesting configuration which the solver freezes and treats as if it were in a paper and pencil environment, developing basic conjectures strengthened by arguments based on theorems and definitions. At this point solvers feel satisfied and convinced that they have answered the question in the activity.

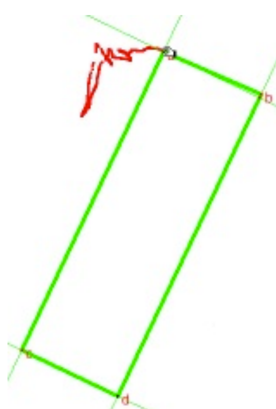
We have also found that in cases in which an exploration apparently is coherent with what we describe in our model – and solvers use maintaining dragging either prompted by the interviewer or on their own – some solvers are not able to perceive an IOD, or, when they are, they might not be able to (or interested in?) reach(ing) a conjecture that links the IOD and the III conditionally, and they resort to a basic conjecture. In particular, in this subsection, we will show how persistence of a basic conjecture can inhibit the discovery of an IOD in a case in which solvers are prompted to use maintaining dragging by the interviewer (Excerpt 5.1.2.1). Moreover, especially when maintaining dragging is prompted by the interviewer, we have witnessed different cases in which even after the emergence of an IOD, the solvers would resort to their basic conjecture instead of linking the III and the IOD at the end of their exploration (see Excerpt 6.1.2). What we found even more interesting were cases in which solvers would spontaneously use maintaining dragging but then be unable to put together the III and the IOD in their final conjecture, ultimately resorting to a previous basic conjecture. We will show an example of this in Excerpt 5.1.2.2.

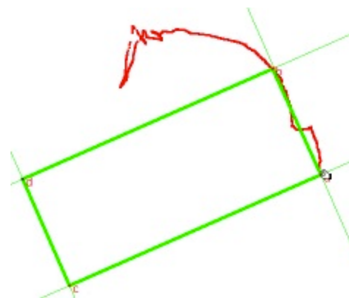
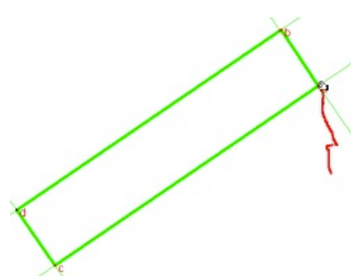
Of course probably difficulties in conceiving the invariants in the terms we describe in our model will have been present before the final phase of the formulation of the conjecture, but as external observers we can only catch such difficulties when they arise and lead to behaviors that are not consistent with what our model might predict. Thus we say that the fixity of basic conjectures may have influence over the final phase of conjecture-formulation, since this is the phase in which such difficulties surface in most cases.

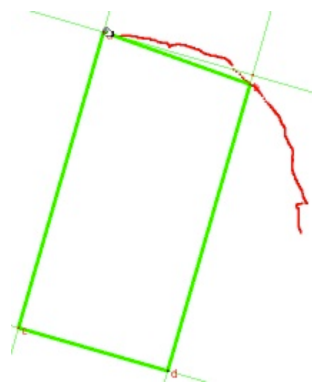
The origin of difficulties which are manifested as resorting to a basic conjecture even after what seems to have been appropriate use – in the eyes of an external observer – of maintaining dragging may be different for different solvers. Definitely making the final transition from the physical experience and the perception of invariants in a dynamic environment to the static world of Euclidean geometry is not a simple matter since it involves conceiving the invariants (properties with respect to movement) once again as static geometrical properties (as traditionally perceived in a paper-and-pencil environment, for example). Moreover there may be difficulties in interpreting the haptic perception in terms of logical dependency of the geometrical properties corresponding to the perceived invariants, that is, in making the transition from simultaneity plus direct or indirect control to logical dependency. However we propose an explanation as to why solvers might not be able to overcome a basic conjecture even after having performed maintaining dragging in a way that seems coherent with our model. Such explanation involves the solvers' interpretation of what is happening during the exploration from a meta-level, as a key to most of the difficulties we have witnessed at this point of the process of conjecture-generation, as we will describe also in Chapter 6. The key element that seems to lead solvers to success or non-success in the formulation of conjectures as described by our model seems to be the solvers' understanding of maintaining dragging as a tool to search for a “condition” or a “cause” of a certain III to be visually maintained. Moreover such “cause” may be *expected* as dragging a point along some *path* to be made explicit during the explorations. It seems like when there is such an intention in the solvers’ actions, the exploration is easily “made sense of” and the pieces of the conjecture seem to naturally fall into place. On the other hand, when there does not seem to be such awareness or the intention of searching for a cause and conception of a path, maintaining dragging may be performed in a technically “correct” manner, but it may not lead to insight in developing a meaningful conjecture that links the IOD and the III logically. Thus many solvers seem to resort to basic conjectures even after having performed maintaining dragging in a way that (in the eyes of an external observer might have) seemed coherent with the model. We will discuss this in further detail in Chapter 6 when we introduce the notion of *instrumented abduction* through which we describe the overarching cognitive process that seems to be associated with solvers’ use of maintaining dragging as an instrument.

Once again, we are dealing with indirect evidence, since we cannot directly access what is going on in solvers’ minds, but only make inferences based on their words and behavior. As described in earlier sections of this chapter and in Section 4.4, it is difficult, within the data we have collected, to obtain evidence of the fact that the solver has perceived a conditional link, as it can only be observed indirectly through behaviors that can be considered “symptoms” of its existence or not in the mind of the solver. The relationship between what can be directly seen, the figure, the solvers’ words, and their thoughts is very delicate and it may only be inferred through interpretation of the observable data. The main evidence we use to infer a difficulties in overcoming basic conjectures once maintaining dragging has been performed is a hesitation or block at the formulation of a conjecture after an investigation. We also consider evidence of these difficulties to be cases in which solvers seem to use maintaining dragging in a way that is apparently coherent with our model, but then formulate conjectures which do not take into consideration the IOD or the III they had seemed to be working with. We hypothesize that there are difficulties at different levels in this final process, and we will analyze some in detail in the excerpts below.

Excerpt 5.1.2.1. This excerpt is taken from two solvers' exploration of Problem 4. The solvers seem to properly perform maintaining dragging with the trace activated, and even recognize a circle from the trace, but they do not link this finding to the property being maintained. They even explicitly state that maintaining dragging is not possible, after having recognized the circle, and explain their experience in terms of a basic conjecture. The name of the solver who is holding the mouse is in bold letters.

Episode 1	**Brief Analysis**
[1] Ila: ...parallel to AB and CA has to always be parallel [perpendicular?] to AB. [2] I: Alright. And so you are saying that there is no way of dragging A maintaining this property?  [3] **Em**: [murmuring]...because wait [4] Ila: Yes, well, but even if we move it in this case...it is as if there were a circle. [5] **Em**: you don't say!! [ironic]	Ila restates the previous conjecture ([1]), in an attempt to explain why maintaining dragging is not possible in this case. The interviewer tries to make the explanation explicit by asking for confirmation of this impossibility of performing maintaining dragging ([2]). The solvers once again attempt to perform maintaining dragging

[6] Ila: [murmurs something] [7] **Ila**: Excuse me, do a circum...give me! [she grabs the mouse] … [15] **Ila**: So, I think there is ...in order for it to be a rectangle [it: "perchè sia"]...well, but ...  [16] **Ila**: or... [17] Em: Or maybe, I think we have to do, put [18] Ila: B there! [19] **Em**: B there...and see when it maintains the property, no? [20] Ila: When it moves...it forms a circle. [21] **Em**: Yes, but try to see where the center is. I think the center... …  [The solvers have some difficulties constructing the circle.]	with the trace activated. This time they seem to be successful, and they even recognize a circle ([4]) in the trace, which they proceed to describe and construct ([7]-[30]). Ila seems to repeat how she sees a circle and a rectangle, but she does not seem able to relate them logically ([15], [29], [31]). Em seems to be attempting to make the connection. In particular she seems to be interested in seeing “when it maintains the property” ([19]), but she does not seem to see “dragging along the circle” to be this “when” or even less a cause for the maintaining of the property. The solvers seem to notice the circle, but not be able to

	conceive movement along such circle as an IOD.
Episode 2 [28] I: What are you looking at while you...? [29] Ila: It's that I think a circle is being formed. I mean... [30] **Em**: Yes. [31] Ila: There is a rectangle and we can move A...  …	**Brief Analysis** The interviewer asks the solvers to explain what they are "seeing". Ila seems to notice the circle ([29]) in correspondence with the "rectangle" and movement of A ([31]) but she does not seem to relate these elements logically.
Episode 3 [37] Ila: and B, too, has to stay on the circle. [38] Ila: In order for it to be a rectangle... [39] **Em**: Eh! [40] Ila: yes. [41] **Em**: but if I move A. Our intent is that we have to start... …	**Brief Analysis** Although an III and an IOD seem to be present the solvers do not seem to be able to make sense of them.
Episode 4 [46] **Em**: No, I think it is not possible to move it, because we start between...from the instant in which,	**Brief Analysis** Finally Em states that "it is not possible to move it" ([46]), even

eh, it is a rectangle. I mean I already say that this is perpendicular. [47] Ila: Right.	though she is unable to provide a satisfactory argument to this claim, and Ila seems to agree.

Table 5.1.2.1: Analysis of Excerpt 5.1.2.1

Probably since the solvers are not completely convinced by their argument, they once again attempt to perform maintaining dragging with the trace activated. Although they proceed to describe and construct this "circle" ([7]-[30]), it does not seem to be related to the properties the solvers are interested in. In other words, they seem to dissociate the circle which they observe as an independent object ([4], [29]) from two lines remaining perpendicular, which seems to the minimum basic property they want to use for their III ("ABCD rectangle"). Ila seems to repeat how she sees a circle and a rectangle, but she does not seem able to relate them logically ([15], [29], [31]). Moreover Ila seems to be relating the circle to other parts of the Cabri-figure: point B ([36]), the rectangle as a whole being "inside the circle" ([33], [35]); however she is not relating it to the movement of A. This further supports our claim that the circle is not conceived as a GDP and furthermore a path does not even seem to be conceived at a generic level as a "cause" for maintaining the III.

We found this excerpt to be quite interesting and surprising since to an external observer, all the elements seemed to be in place for the solvers to conceive an IOD and formulate a conjecture that put the belonging of A to the circle in relationship with ABCD being a rectangle. However since in the end the solvers do not even believe it to be possible to perform maintaining dragging in this case, we are led to interpret the episode as being due to a difficulty in properly conceiving a path. The circle that is recognized does not seem to be linked to movement or to the maintaining of the III, thus it is not a path according to our model.

Excerpt 5.1.2.2. In their exploration, before this excerpt the solvers have found a basic conjecture, written it, and then continued their exploration using maintaining dragging. They reach what seems to be an IOD and they state a conjecture linking their III and IOD. However when they write their conjecture they switch the premise and the conclusion, and they mix it with their previous basic conjecture. The dominance of the basic conjecture over the new conjecture appears also in the solvers' answer to the interviewer's request to repeat the conjecture: the solvers repeat their basic conjecture, not the one obtained by linking the IOD and the III that emerged during maintaining dragging. The excerpt is taken from two solvers' exploration of Problem 3.

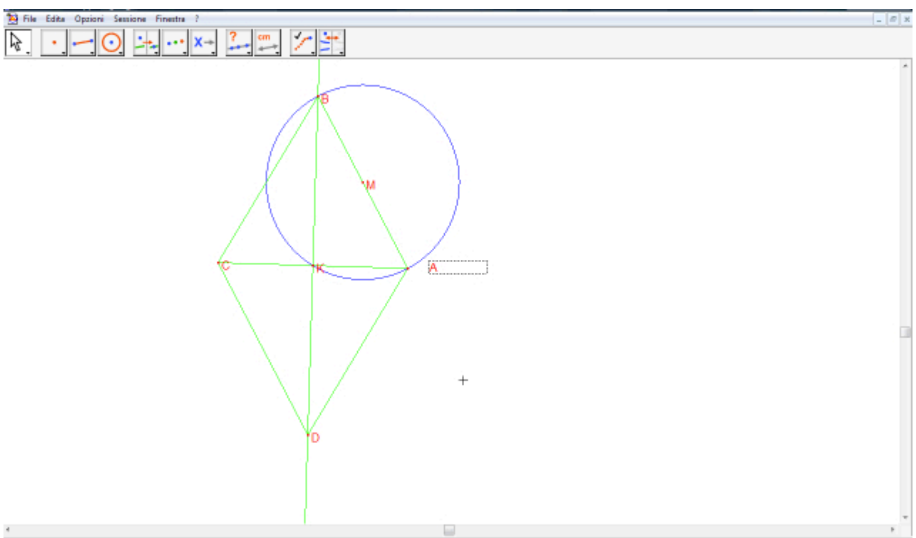
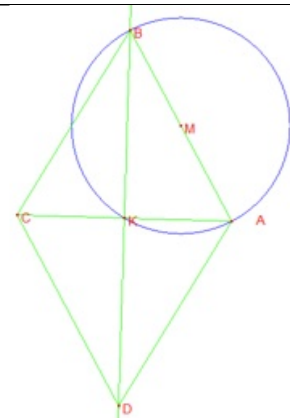


Figure 5.1.2.2 A screenshot of the solvers' exploration.

Episode 1	**Brief Analysis**
[1] Gin: I was thinking [murmurs something]	
[2] Gin: So, therefore, eh yes, for now this part of the circle.	The solvers seem have
[3] Dav: Eh!	conceived a path a provided

[4] Gin: We have to say that to construct...that [5] Gin: Eh, moving A it always remains a rhombus... [6] Gin: if A belongs to a circle with center M and radius MK? [7] Dav: Yes. [8] I: So, write this one... … [13] Gin: A belongs to the circle with center M [He writes: "$AC \perp BD \Rightarrow$ ABCD rhombus $\Rightarrow A \in C_M$"] …	a GDP as the circle with center M and radius MK ([6]). The first conjecture they state is: "ABCD rhombus implies A belongs to the circle with center M and radius MK" ([10]-[15]). There is no dragging in this episode.
Episode 2 [18] Gin: Because this way BKA [19] Dav: Yes. Yes, because... [20] Gin: ..is right... [21] Dav: Exactly, to this way ...yes, necessarily because it is inscribed in a semicircle... [22] Gin: So necessarily also the other three are right... [23] Gin: and it necessarily remains a rectangle.	**Brief Analysis** Argumentation about why the conjecture makes sense. It seems that the solvers are using "$A \in C_M$" as their premise an trying to prove that ABCD is a rhombus.
Episode 3 [24] I: Wait, what are you starting from to make these considerations? [25] Gin: Well, so... that ABCD is a parallelogram.	**Brief Analysis** I asks for a clarification about what the solvers are arguing. The new argumentation provided seems to invert premise and conclusion,

[26] I: Ok.  [27] Gin: In order to ...a parallelogram with perpendicular diagonals is a rhombus. [28] I: Hmm... [29] Gin: Therefore, in order for ABCD to be a rhombus, it has to have AC and BD perpendicular. [30] I: Ok. [31] Dav: So BK... [32] Gin: So BKA is 90 degrees. [33] Gin: eh, here it happens, here BKA is 90 degrees, in this picture, because [34] Gin: It is an angle inscribed in a circle, that insists on a diameter, which is AB. [35] I: Ok. [36] Dav: Yes.	again, showing instability in the status of the two properties the solvers try to link in their conjecture. There is no dragging in this episode.
Episode 4 [37] I: and this proves what conjecture? [38] Gin: That... [39] I: can you repeat the statement? [40] Gin: Well, we said that if AC is perpendicular to BD, ABCD is a parall, is a rhombus. [41] Dav: Yes.	**Brief Analysis** This leads to the interviewer's question about what the conjecture they want to prove in ([37]). After a slight hesitation Gin gives the original basic

	conjecture as the answer.

Table 5.1.2.2: Analysis of Excerpt 5.1.2.2

Although the solvers seem to have reached a new conjecture through the use of maintaining dragging, this conjecture seems to be destabilized by the original basic conjecture the solvers have formulated. The instability of the new conjecture can also be seen thanks to the following elements of the episode. First we notice that the direction of the logical implication in the first conjecture is reversed with respect to what we describe in our model. It is not incorrect mathematically, and moreover it is provable, however it seems to denote instability in the perception of causality (if in fact there is any). They first seem to use “A belongs to a circle with center M and radius MK” as the premise of the conjecture ([6]), however then Dav and Gin state the conjecture together using this property as the conclusion. This may happen because there is no dragging going on during this excerpt. Therefore the haptic sensation of dependent and independent objects and properties is completely absent (it could have been present only in the sensory memory of the solver who had performed the maintaining dragging) and cannot guide the transition to a logical interpretation of the relationship between the perceived invariants.

Moreover the fact that the figure is left static seems to foster the “flattening” of all properties onto a same level, as in the paper-and-pencil environment. Through the argumentation the solvers use the theorem that any angle inscribed in a semicircle is a right angle ([18]-[22]), and then the focus on what property to use in order to prove that ABCD is a rhombus ([29]) seems to lead the solvers to no longer take into account any experience of movement. Their argumentation also shows instability in the status of the two properties the solvers try to link in their conjecture, because once again premise and conclusion seem to be reversed.

When the interviewer asks what the conjecture they want to prove is ([37]), after a slight hesitation Gin restates the original basic conjecture. This shows interference and moreover dominance of the basic conjecture over the new conjecture.

Concluding Remarks. In this section we have introduced basic conjectures and discussed how they can interfere with other tasks described in our MD-conjecturing Model. In the following section we introduce a second necessary ingredient that solvers need to use in order to be able to formulate conjectures according to the MD-conjecturing Model. In particular we will describe difficulties in conceiving a property of a dynamic figure as an III. If such difficulties are present they can inhibit the perception of an III and the possibility of continuing the exploration using maintaining dragging.

5.2 Conceiving a Property as an III

In Section 5.1 we described basic conjectures and how some solvers would feel satisfied with such conjectures, instead of using them to transition to conjectures developed according to our model, or return to them even after "discovering" properties that could have been used to formulate a conjecture according to the process described by our model. In this section we will analyze solvers' behaviors that are not consistent with Task 1 of our model (Section 4.1): "Determine a configuration to be explored by inducing it as a (soft) invariant intentionally induced invariant (III)". We describe these behaviors as difficulties in conceiving a property as an III. We attempt to provide a fine analysis of such difficulties by separating the different factors that need to be considered when accomplishing Task 1 of our model. In the paragraph below we highlight each of these factors and then use them in the analyses of excerpts from solvers' work during the interviews.

In Chapter 4 we define the III as: "a property (or configuration) that the solver finds interesting and chooses to try to maintain during dragging" (Section 4.2). The idea of "maintaining during dragging" condenses the awareness that the III is a property that may become an invariant thanks to some induced continuous movement of a specific base point. We can separate out four factors that seem to be condensed in such awareness, and that seem to cause the difficulties encountered by solvers at this point of the exploration. These factors are described below:

1) The III is a potential invariant of the dynamic-figure, that is, it does not vary with respect to some movement, as described in Section 2.1.1, and such movement is produced by dragging a base point in a particular way. Conceiving a property as invariant with respect to the movement of a base point occurs through haptic perception, a "feeling" that the solver can experience and that is generated by visual and manual feedback from the Cabri-figure. Therefore, as illustrated in section 5.1, an III is fundamentally *different from a "static" property* that can be perceived in the paper and pencil environment.

2) The possible movement through which the III may be maintained as a property is intimately related to the *base point* chosen for the dragging. In particular, different choices of the base point to drag will imply different movements necessary to maintain the selected property. Moreover, for any choice of the base point to drag, some points will remain fixed while others will move, depending on their status with respect to the construction that generated the Cabri-figure. Difficulties in conceiving a property as being induced by the movement of a specific base point seem to occur in cases in which solvers lack control over of the status of the various points of the Cabri-figure. We will discuss this issue further in section 5.4. Difficulties in conceiving this aspect of the III may also arise from a particular configuration of the dynamic-

figure, in which the trajectory of the movement of the dragged-base-point is difficult to distinguish from an element of the figure (for example, if the trajectory is a line that seems to “go through” a side of the dynamic-figure). In this case the solver might perceive the variation of the element of higher dimension (the side in our example) instead of the variation of the dragged-base-point alone (Duval,1995, 1998).

3) The movement of the dragged-base-point is perceived as *continuous*, and therefore it guarantees the maintaining of the III “always” during the time lapse in which the dragging is performed. When trying to determine whether a certain property is maintainable, the solver may proceed by making “small perturbations” in order to get a feeling for how to carry out the movement, if in fact it is possible, and by searching for “good positions”, as described in the analysis of Excerpt 4.2.1. In this case, difficulties may arise if the solver does recognize occurrences of the desired property in any “close position” and thus interprets the “good position” as being isolated and guaranteeing a form of “stable equilibrium” to the Cabri-figure. Even in cases in which the solver does recognize a number of discrete “good positions” that give this kind of perception of “equilibrium”, s/he may not be inclined to think that it is possible to “connect” these positions continuously while maintaining the interesting property, which in this case would become the III.

4) The III is a soft invariant (Section 2.1.2 and Section 4.2), so maintaining the III is *“controlled” or “caused”* by dragging within the DGS, and in particular by the *specific movement* induced by the solver on the base point s/he is dragging. Solvers who seem to be aware of this and who want to focus both on the III and on the movement of the dragged-base point can encounter difficulties in coordinating haptic perception and multiple visual perceptions. In fact some solvers seem to be unable to proceed using maintaining dragging if they have not previously envisioned some “way” of

carrying out the dragging. These solvers, who need to conceptualize the movement to induce on the dragged-base point in order to carry out maintaining dragging, may encounter difficulties in conceiving a property as an III, as they may tend not to separate the property to induce from the idea of how to move the base point through which it can be induced. We have observed that this sort of difficulty arises frequently during solvers' attempts to use maintaining dragging. However it seems to be a particular consequence of difficulties arising from a more general factor that comes into play in various problem-solving activities, that of being flexible/ having a free mind. In this case the solver seems to fix his/her attention on specific properties (usually basic properties) of the configuration and tries to link the idea of how to move the base point to such properties even though they might not be directly related. We discuss other consequences of difficulties related to being flexible/having a free mind in section 5.3.

Each of these aspects of an III seems to potentially be a source of difficulty for solvers attempting to identify an III and perform maintaining dragging. In the excerpts below we will show how difficulties emerge during this phase of the exploration, and how they can be interpreted with respect to the aspects we separated and described above. Below is a brief overview of the Excerpts we present in this section.

Excerpt 5.2.1: The solver performs maintaining dragging with the trace activated in a way that seems successful to the interviewer, but he quickly formulates a conjecture that has nothing to do with the trace. The solver seems to not be conceiving the property to induce through dragging with respect to movement (aspect 1) and he seems to not relate the movement or the induced property to the base point being dragged (aspect 2).

Excerpt 5.2.2: The two solvers do not seem to conceive the property to induce as an III with respect to movement (aspect 1). Moreover, when the solvers try to perform maintaining dragging in response to the interviewer's prompting, they seem to recognize the regularity in the movement in terms of a basic property (aspect 4).

Excerpt 5.2.3: The solvers oscillate between acknowledging the possibility of using MD or not, unsure whether there are only a few discrete "good positions" (aspect 3) or whether the induced property can be maintained through a continuous movement. Almost "by chance" (and through symmetry of the figure) the solvers notice the first "good positions", and then rapidly more and more, which leads them to treat the induced property as an III and perform maintaining dragging.

Excerpt 5.2.4: The solvers initially conceive only one good position, as a sort of stable equilibrium (aspect 3), but then they find more good spots for the point they are dragging. Unlike the solvers in Excerpt 5.2.3, these solvers are not able to proceed using maintaining dragging, and they resort to their original basic conjecture, probably due to a lack of flexibility (aspect 4): the solvers seem to not let go of the property they have initially conceived and to not separate it from a potential movement of the dragged base point.

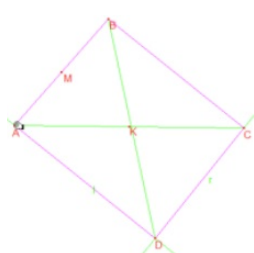
Excerpt 5.2.5: The solvers seem to conceive a property with respect to movement, but they do not "let go" of basic properties (aspect 4) which seem to dominate their perception and inhibit the proper conception of an III. The solvers limit their description of how to maintain the property "ABCD rectangle" to an "up and down" movement that they do not clearly define with respect to the dragged-base-point (aspect 2), and they seem to be satisfied with their original basic conjecture.

Excerpt 5.2.6: The perception of basic properties seems to inhibit the conception of an III (aspect 4). Unlike the previous example in Excerpt 5.2.5 in which the regularity in the movement seemed coherent with the basic property (they both involved a "line"), in this case the trace produced during maintaining dragging seems to create a conflict with what the solver has in mind, and this seems to generate confusion.

<u>**Excerpt 5.2.1**</u>

This excerpt is taken from a student's work on Problem 1; it shows an example in which the solver performs maintaining dragging with the trace activated in a way that seems successful to the interviewer, but he quickly formulates a conjecture that has nothing to do with the trace, as if it was of no importance at all. From this excerpt the solver seems to not be conceiving the property to induce through dragging with respect to movement (aspect 1) and he seems to not relate the movement or the induced property to the base point being dragged (aspect 2).

In the previous part of this exploration, Sim has fixed points M and K with nails in order to concentrate on dragging A. He has become interested in the property "BD passes through K", a property that he seemed to want to use as a minimum basic property.

Episode	**Brief Analysis**
[1] I: You just have to move A...you already have M and K fixed, right? [2] Sim: Yes. [3] I: Ok, so now you move A trying to maintain BD passing through K. [4] Sim: Yes. [5] I: Ok. Let's try to see if we are able to say something about it. 	The interviewer proposes to use the property "BD passes through K" as an III ([3] and [4]).

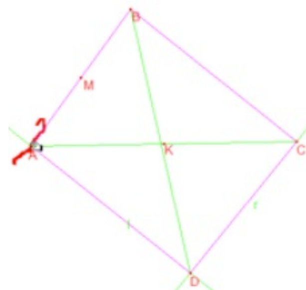
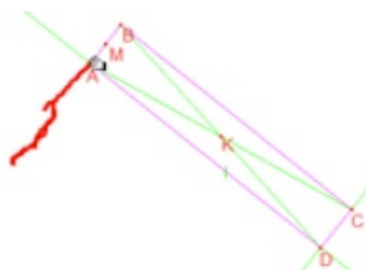
[6] I: Uhm [observing the dragging] [7] I: If you want, you can help yourself... [8] Sim: It looks to me like it is always .... [9] I: ...with the trace tool, eh? …  [15] Sim: now...[murmuring] [16] I: Ok....yes. [17] Sim: no...[murmuring] [18] I: there [whispering] [19] I: Ok...	The interviewer is quite insistent in trying to prompt Sim to use maintaining dragging and activate the trace ([9], [11]), and this seems to lead Sim to performing maintaining dragging in a proper way ([13]-[22]).
[20] I: Try to go the other way...to the other side, so you know that this mark is good...  [21] I: continue... [22] I: uh huh [23] Sim: I wanted to consider that if K is the intersection of the diagonals, it is always a rectangle. [24] I: You think that it is always a rectangle [25] Sim: Yes. [26] I: Yes. [27] Sim: because...	However what Sim seems to "see" as an outcome of his dragging are the properties "K is the intersection of the diagonals" and "it is always a rectangle" ([23]), which he links logically in his conjecture. That is, the III basically becomes his premise and the conclusion is the original interesting case "ABCD rectangle".

Table 5.2.1: Analysis of Excerpt 5.2.1

Although the trace appears in a neat manner on the screen and a lot of attention seems to be devoted to performing the dragging correctly, Sim does not seem to pay attention to it at all, but instead he seems to "use" the dragging to strengthen a conjecture on statically-conceived properties (aspect 1). It seems unclear what Sim is trying to maintain during the dragging even though the interviewer had suggested trying to maintain "BD passing through K" ([3]). From what he states in his conjecture, he seems to transition from "BD passing through K" to "K is the intersection of the diagonals". In any case the new premise of the conjecture does not involve A, the base point being dragged, nor the trace conceived as any representation of the *path*, which does not seem to be conceived at all.

Moreover, the fact that Sim had fixed with nails the other base points could have helped him relate an object appearing from the movement of the base point to the base point being dragged (the only free one), and conceive an III (aspect 2). However this did not occur even though the maintaining dragging was carried out precisely, and everything seemed to be in place for the solver to proceed according to the model and conceive an IOD as "A belonging to a line".

We may provide different interpretations and give different hypotheses as to why this might be the case. Here we prefer to insist on the lack of conception of an III, according to all the aspects described in the introduction of the section. The lack of such conception seems to be clearly visible, and it may explain the solver's inability to perceive properties related to movement of particular points and to make sense of what is happening in his DGS experience in the terms described by our model.

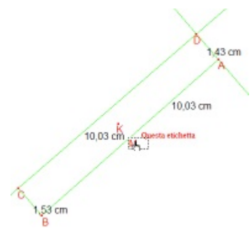
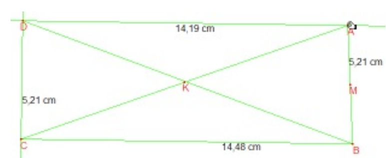
Finally this excerpt shows that it is possible to "provoke" behaviors that are coherent with the ones described in our model, but this does not mean that awareness of "what maintaining dragging can be used for" has been achieved by the solver. In particular, the solver does not seem to use maintaining dragging to search for a cause of the induction of a certain invariant (2). We will describe this in detail in Chapter 6 and Chapter 7. Here we argue that this excerpt provides evidence that "performing" maintaining dragging does not mean being aware of what it can show. That is, a solver can use maintaining dragging as a tool only if s/he has developed a *mental scheme* associated with it that allows the various elements to be identified and geometrically interpreted according to our model.

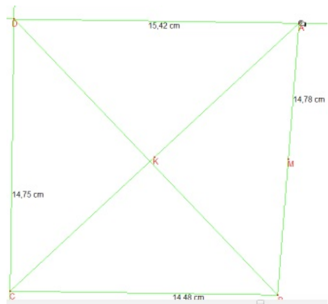
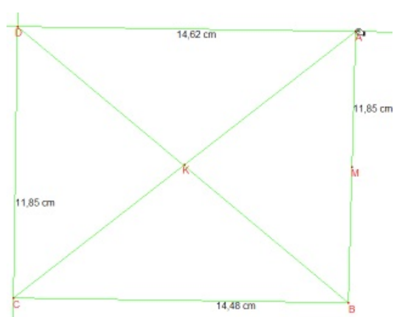
<u>Excerpt 5.2.2</u>

This excerpt shows how two solvers do not seem to conceive the property to induce as an III with respect to movement (aspect 1). Moreover, when the solvers try to perform maintaining dragging in response to the interviewer's prompting, they seem to recognize the regularity in the movement in terms of a basic property (aspect 4). The excerpt is taken from the solvers' exploration of Problem 1. In this excerpt and all of the following ones the bold refers to the solver who is using the mouse.

Before the beginning of this excerpt the solvers had formulated two basic conjectures. The oral conjecture was: "If AD is perpendicular to CD, then ABCD is a rectangle." The written conjecture was: "If DA=CB then rectangle."

Episode 1	**Brief Analysis**
[1] Vale: …rectangle... [2] I: For example...maybe let's try to think about other ways in which we can obtain a rectangle... [3] I: Uhm [4] I: So Ric seems to be dragging M...with the	The solvers are interested in the configuration "rectangle" so the interviewer proposes to look for other ways of obtaining a rectangle ([2]). Ric, who was dragging, states

idea of maintaining rectangle?...or not? [5] **Ric**: Well, no, I don't know. [6] I: You were dragging freely?  [7] **Ric**: I was studying the figure... [8] I: Ok. [9] **Ric**: Ok, yes it is possible... [10] I: So you were doing wandering dragging? [11] Vale: Maybe ...if...adding the diagonals DB and CA. Try adding DB and ...  [12] **Ric**: [murmuring as he draws] DB and CA. [13] Vale: and putting like rectangle. [14] **Ric**: With M? [15] Vale: I don't know [It: "boh"] [16] **Ric**: Whatever [he starts dragging A]	that he was only "studying the figure" ([7]) through wandering dragging ([5], [7]), while Vale suggests drawing the diagonals ([11]) and using them to look for a new property ([13], [16]). Ric seems to share this perception, as can be inferred from his words: "Well, no. I don't know. I was studying the figure." which he states even though the interviewer was insisting on prompting the use of maintaining dragging ([5], [7]). Vale's suggestion leads us to infer that she is not relating the property "ABCD rectangle" to movement in any way. Ric switches from dragging M to dragging A ([14], [16]), and seems unsure about any difference this choice would make.
Episode 2 … [24] I: Uhm, is it only possible to choose A like that	**Brief Analysis** When the interviewer asks whether

to have a rectangle? [25] Vale: I don't think so. [26] **Ric**: Well like this too I can say it is a rectangle [as he drags A in different "good positions"]  … [29] **Ric**: Yes.	there might be other positions for A in order to have a rectangle, the solvers seem to agree that there are other positions.
Episode 3 [30] Vale: Well more or less I think ... [31] **Ric**: I think all the positions in which AB is perpendicular to CB. [32] I: ...in which AB... is perpendicular... [33] **Ric**: and as she said DA is congruent to CB. [34] I: Ok. Wait, so try to tell me the conjecture again. It seems similar to what you had said before: AB...ah, no, you had said...  [35] **Ric**: So, [36] I: ...you said DC... [37] **Ric**: If...no I had said before if AD is	**Brief Analysis** These positions do not seem to be conceived with respect to a trajectory, but more “statically” with respect to the basic properties “AB perpendicular to CB” ([31]) and “DA congruent to CB” ([33]). For Ric the exploration seems to have only strengthened his original basic conjecture. Here he seems to conceive a new premise, that is “AB is perpendicular to CB” ([31]), but he recognizes the equivalence of the premises (“It is the same

perpendicular to CD, then ABCD is ... [38] I: Ok. [39] **Ric**: ...a rectangle [40] I: Ok, on the other hand now you said: If AB... [41] **Ric**: Well, no, I said the same thing... [42] I:...is parallel	thing" [41]).

Table 5.2.2: Analysis of Excerpt 5.2.2

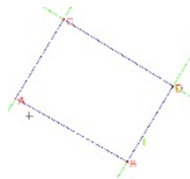
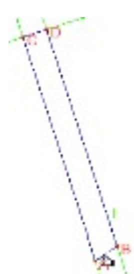
Vale's behavior characterized by looking at a static configuration and "guessing" at some additional property to use as a premise in the conjecture ([11]) seems to be typical of a paper and pencil environment. In fact Vale's suggestion leads us to infer that she is not relating the property "ABCD rectangle" to movement in any way (aspect 1). Instead she seems to perceive it as an interesting configuration with nothing more to it than if it had been in a paper and pencil environment. The property "ABCD rectangle" never becomes an III, because throughout the excerpt it never seems to be perceived with respect to movement (aspect 1). This can also be seen in the ease with which Ric switches from dragging M to dragging A ([14], [16]), unsure about any difference this choice would make (aspect 2).

When the interviewer asks whether there might be other positions for A in order to have a rectangle, the solvers seem to agree that there are other positions. However these positions do not seem to be conceived with respect to a trajectory. Instead they seem to be conceived "statically" with respect to the basic properties "AB perpendicular to CB" ([31]) and "DA congruent to CB" ([33]), that might have easily been perceived this way in a paper and pencil environment. The solvers' perception seems to be dominated by basic properties, and the little dragging that Ric does perform seems only to strengthen his original basic conjecture. In fact in his new conjecture the only difference is in the premise, that is “AB is perpendicular to CB” ([31]), but he recognizes the equivalence of the hypotheses (“It is the same thing” [41]).

Moreover, the solvers resist using maintaining dragging throughout the interviewer’s prompting and they resort to techniques that are typical of the paper and pencil environment, using dragging at most to confirm their statically-developed insights, leading to a more robust belief in the original basic conjecture. In Chapter 6 we will discuss how this behavior may hinder the development of the notion of *path*. In particular, the solvers might be seeing the vertical movement of the base point A as the invariance of “perpendicularity of segment AB to BC” instead of as the movement of A along a line. The fact that the figure-specific path in this case is a line on which a whole segment (AB) rests when the III is maintained may be leading the solvers to continue “seeing” the basic properties of ABCD that led to the original basic conjecture (aspect 4), instead of to overcome them and conceive a *path* with respect to point A. In the next excerpts we will show examples in which such interpretation seems to be the most convincing.

Excerpt 5.2.3

In this excerpt the solvers oscillate between acknowledging the possibility of using MD or not, unsure whether there are only a few discrete “good positions” (aspect 3) or whether the induced property can be maintained through a continuous movement. Almost “by chance” (and through symmetry of the figure) the solvers notice the first “good positions”, and then rapidly more and more, which leads them to treat the induced property as an III and perform maintaining dragging. The excerpt is taken from two students’ exploration of Problem 4.

Episode	Brief Analysis
[1] G: and when... do like maintaining dragging when it is a rectangle. [2] **F**: Never... I mean one point and that's it.  [3] G: really? If you move... moving A... let's write moving...[G starts to write]  [4] **F**: Moving A... [5] G: Moving A... there is only one point... but are you sure, even going over there? Can't you go over there?  [6] G: There... Already two... [7] **F**: two... [8] G: eh, no. [9] **F**: No, here...no it does funny things. [10] G: wait,... no that is the one from before. [11] **F**: Exactly. This is the one from before... [12] G: two...  [13] **F**: two... I mean, one... [14] G: one...two...three, four...twenty thousand! [15] **F**: yes, there are really many of them	Initially F thinks that the property “ABCD rectangle” cannot be maintained through dragging, as it is verified in “one point and that’s it” ([2]). G seems uncertain and proposes to check “over there” ([5]). His idea seems to be guided by a sort of perception of symmetry, which in fact leads to the discovery of another “good point” ([6]). This strengthens his belief that there are other good points and F’s difficulty in dragging is soon overcome: when he goes back to start at the original good position, he discovers another “good position” along the way ([13]) and immediately after a whole set of good points ([14]). This seems to encourage the solvers who now propose to perform maintaining dragging with the trace

[laughing]... let's do trace... we made a mistake. There are really too many.	activated.

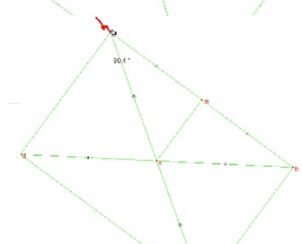
Table 5.2.3: Analysis of Excerpt 5.2.3

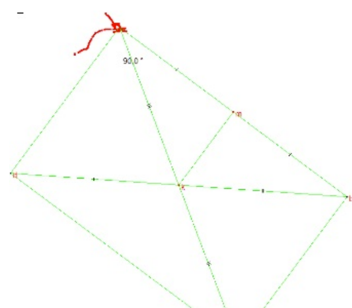
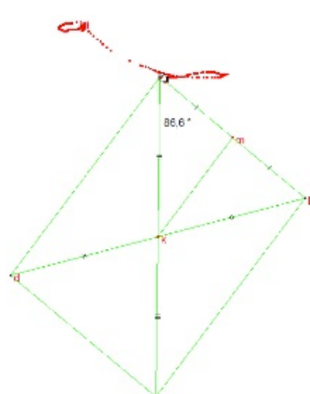
In this excerpt the solvers find discrete "good configurations" that guarantee the visual verification of the property "ABCD rectangle" that they are interested in. As they discover more and more "good positions", for the solvers, the property "ABCD rectangle" seems to transit from the status of "potential III" to proper III. Evidence of this proper conception can be seen in the solvers' desire to activate the trace and *make the path explicit*. As we will discuss further in Chapter 6, what seems to be guiding the solvers' experience is the "expectation" of being able to induce the property "ABCD rectangle" through dragging along a *path*. This allows them to overcome the initial perception of their being only discrete good positions, and expect to describe a regularity in the movement of the dragged-base-point by observing the trace mark left during maintaining dragging.

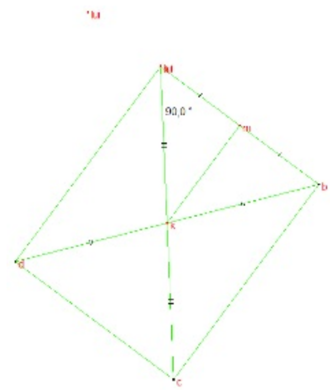
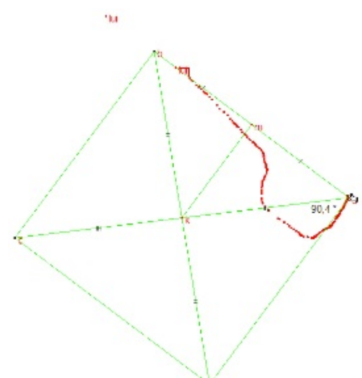
In the following excerpt the solvers use a similar technique to explore the Cabri-figure and in particular the possibility to maintain a certain property. However the solvers will not be able to overcome the block. We think the difference in the behavior resides in the expectations developed by the different solvers with respect to maintaining dragging. We will describe this theory in further detail in Chapter 7, through the notion of *maintaining dragging scheme*.

Excerpt 5.2.4

This excerpt from Em and Ila's work on Problem 1, shows an example of solvers who attempt to perform maintaining dragging. They start by looking for "good positions", that is choices of the dragged base point that seem to induce the desired property (the potential III). Initially they conceive only one good position, as a sort of stable equilibrium (aspect 3), but then they find more good spots for the point they are dragging. This technique seems to give them a hint about some regularity in a possible continuous movement of the dragged base point, however, unlike the solvers in Excerpt 5.2.3, these solvers are not able to proceed using maintaining dragging, and they resort to their original basic conjecture. We interpret this second difficulty as related to flexibility (aspect 4): the solvers seem to not let go of the property they have initially conceived and to not separate it from a potential movement of the dragged base point.

Episode 1	**Brief Analysis**
[1] **Ila**: So, …	
[5] I: Because you are telling me that it is possible, but you are not showing it to me. [6] Emi: Uhm. [7] I: and so maybe it is not possible. [8] **Ila**: I do not think it is possible, because you see that...in any case if I move point A farther away, it is never equal to 90! [9] **Ila**: There will never be a point equal to 90. There is only that point there. 	Ila states that she does not think it is possible to perform maintaining dragging with point A and the property “ABCD rectangle” ([8]). As an argument she uses her perception that she does not think there will ever be another “point equal to 90” ([9]).
[10] Emi: Can I try? [11] **Emi:** No, there is no point. 	Emi, too, seems to believe that it is not possible to perform maintaining dragging, as she tries to drag A.

Episode 2	**Brief Analysis**
[12] Ila: Yes, there is. See, look. [13] Ila: You see? [she murmurs something as she takes the mouse back] [14] **Ila**: Excuse me but now let's do something. [15] Emi: Uhm. [16] **Ila**: Pointer. [17] **Ila**: This has to be 90 degrees. [18] **Ila**: Eh...90 [19] **Ila**: Uhm...now like this. I make a point  … [27] **Ila**: and I'll call it [she writes "lui" (English: "him") on the point she draws] [28] I: ...go back and get it...ok.	However Ila does not seem to be completely convinced, and is ready to change her mind, spotting another “good position”. She comes up with a strategy ([14]) for looking for other “good points”. She proceeds by placing a free point called “him” ([27]) on the “good position” for A ([17]-[28]) which she recognizes by the measure of the angle she has marked on the Cabri-figure ([17], [18]).
[29] **Ila**: I go get A again [30] I: Ok [31] **Ila**: B is 90... [32] I: and look for another one. [33] **Ila**: But see that...no, wait. 	Ila, on her own, seems unable to find other good positions in the vicinity of the point she has marked “lui” ([29]-[33]), and seems to think there is only one good position ([34], [35]).
[34] **Ila**: 90! You always go back THERE.	The interviewer tries to perturb

[35] **Ila**: always there. [36] I: Move a lot. Let's see if there is something else. There now try to look for... [37] **Ila**: There seems to be one here too...theeeere! [38] **Ila**: No. [39] Val: There! [40] **Ila**: There [she labels the point "lui" again] 	her belief of there being a single good position by asking her to “move a lot” ([36]). She still seems quite uncertain, but maybe seeing the angle measure become very close to 90 in a place “so far” from her original good point leads her to believe that there is another good point, which she finds and marks “lui” again ([40]).
Episode 3 [41] **Ila**: Another 90...there! It's along there. [42] Emi: Eh, yes. [43] **Ila**: See? [44] Emi: Uhm. [45] Emi: Ok. [46] **Ila**: Therefore, [47] **Ila**:...they have to ...I mean the points ...eh they have to... [48] **Ila**: But then we are ... [49] Emi: They have to be on… [50] **Ila**: It's always the same thing as before! [51] Emi: Right! 	**Brief Analysis** At this point Ila seems confident enough to look for another good point and she seems to recognize a path when she exclaims “along there” ([41]). The difficulty might have been overcome, and Emi seems to conceive something the points need to be on, as she starts murmuring in line 49.

[52] **Ila**: It has to always be, uh, CD has to always be parallel to AB	However Ila interrupts her and states that it is "the same thing as before" ([50]) and imposes her original basic conjecture.

Table 5.2.4: Analysis of Excerpt 5.2.4

The technique used by Ila to decide whether maintaining dragging is possible seems similar to the one used by F and G in Excerpt 5.2.1, and although it was not a strategy presented in class during the introductory lessons, it seems similar to the spontaneous scheme described as "line dragging" by Arzarello et al. (2002). However in the previous excerpt, the solvers were able to overcome the initial uncertainty, and propose to use maintaining dragging with the trace activated. Here the solvers do seem to recognize a regularity ("It's along there!" [41]), which suggests a seed of conceiving the III as "caused" by dragging (aspect 4), but such regularity does not seem to be conceived with respect to the movement of the base point A (aspect 2). Instead it seems to be a sort of generalization of a statically-conceived set of good positions, which cannot be considered in relation to a movement and, therefore, to the invariance of any property with respect to such movement.

The solvers do not seem to conceive dragging along the discovered "good positions" as the "cause" of the invariance (aspect 4) of the induced property ("ABCD rectangle"), and they resort back to a basic conjecture to explain the figure's behavior. This shows how strongly basic conjectures can be rooted and how they can guide other perceptions during an exploration. In the end Ila's original basic conjecture appears to be strengthened by this episode, and not overcome. We hypothesize that this can occur when solvers have not developed an adequate way of thinking with respect to the use of maintaining dragging (we will discuss this in further detail in Chapter 6), so, in particular they are not using it to search for a cause of the maintaining of an invariant conceived dynamically. Moreover, Ila's perception seems to be dominated by basic properties of the class of quadrilaterals she is interested in, and she does not seem to be able to free her mind and overcome this view of what she is experiencing (aspect 4).

Since Ila seems to be unable to conceive "how to move her dragged-base-point" (aspect 4), she seems to proceed by "trial and error". Moreover, she seems to perceive the initial "good position" that she has named "lui" as a sort of point of stable equilibrium for the dynamic-figure (aspect 3). Ila moves point A very slightly and seems to be using "small perturbations" to explore whether a property can be imposed at some level of generality on the Cabri-figure, and she keeps returning to what she thinks is her initial good position, in which the angle she has marked is 90 (according to the software). Only after being prompted in line 36 (I:"Move a lot. Let's see if there is something else. There now try to look for...") does Ila start looking for another good position "far away" from her marked point. Again she behaves as if this were another point of stable equilibrium for the dynamic-figure. Even after identifying a third good position she does not seem to conceive a "good movement" that might connect them. Instead she recognizes the basic property she had used in the first conjecture. Therefore we assume Ila has not properly conceived a *path*, nor an III according to our model.

Excerpt 5.2.5

This excerpt is taken from Val and Ric's exploration of Problem 1. The solvers seem to conceive a property with respect to movement, but they do not "let go" of basic properties (aspect 4) which seem to dominate their perception and inhibit the proper conception of an III and of a potential *path* as an object to drag along *in order to* induce the III. Instead the solvers limit their description of how to maintain the property "ABCD rectangle” to an “up and down” movement. The solvers seem to be satisfied with their original basic conjecture.

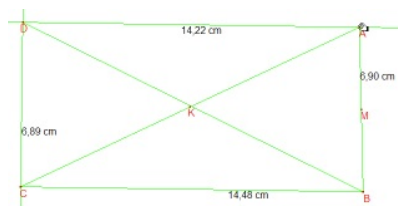
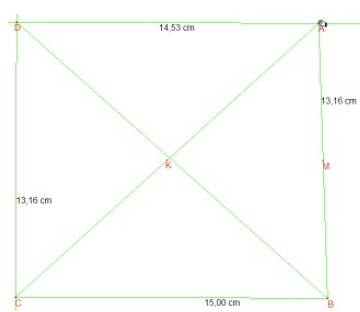
Episode	Brief Analysis
[1] I: and it was called "maintaining dragging", and so you now are interested in the property rectangle	The solvers describe the dragging as “moving up and down” ([2]).
[2] Val: Yes, but in the end, like moving A up and down... 	
[3] I: Alright, so you already saw that moving A up and down...what is this "up and down"?	However they do not seem to perceive the movement not as a movement of A along some object. Instead Val seems to recognize it as “AB has to remain parallel to DC” ([6]).
[4] Val: Yes, alright, uh...I mean that in any case, right, AB	
[5] **Ric**: You have to move...	
[6] Val: AB has to remain parallel to DC, or anyway ABC has to be right...yes.	
[7] I: uhm.	In her final remark ([8]) Val seems to try to describe how this dragging occurs, by “making, let’s say, segment AB longer”.
[8] Val: Always...and so you can do...making, let's say, segment AB longer. 	

Table 5.2.5: Analysis of Excerpt 5.2.5

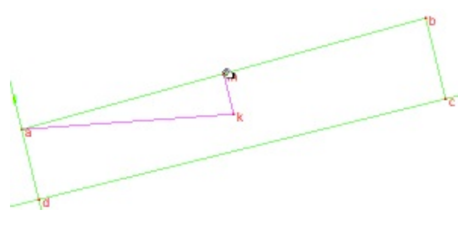
The solvers do not seem able to conceive the property “ABCD rectangle” as an III as we describe in our model, because they do not seem to be able to conceive it with respect a movement of A alone (aspect 2). In fact Val seems to see the “up and down” movement as the “making segment AB longer” instead of A moving along a *path*. We advance the following hypothesis. Val may be unable to conceive the movement of A as independent from that of AB because the trajectory of its movement (a sort of line parallel to CD) guides her attention (aspect 4) towards the basic property she "sees" and uses as a premise in her basic conjecture: "AB parallel to DC". The fact that she does not seem able to conceive the regularity in the movement of A as the movement along an object which is independent from segment AB – a *path* – seems to inhibit her conception of an III and the process of conjecture-generation through maintaining dragging in general, as we will describe in Chapter 6. The fact that Val is able to recognize her basic property in the movement of the base point A probably strengthens her basic conjecture and definitely it does not seem to create confusion or perplex her in any way.

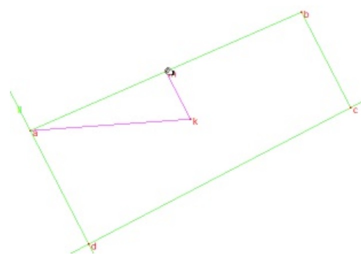
In the following excerpt we recognize a similar phenomenon: the solver's inability to properly conceive an III as an invariant with respect to some movement of the dragged-base-point which is independent from any basic properties (aspect 4). However in the following example the movement of the dragged-base point does not seem to help the solver recognize basic properties, instead it seems to create a conflict with what the he has in mind, and to create confusion and uncertainty.

Excerpt 5.2.6

This excerpt is taken from a student's exploration of Problem 1 and it is an example of how the perception of basic properties (aspect 4) seems to inhibit the conception of an III as an invariant with respect to some movement of the dragged-base-point. Unlike the previous example in Excerpt 5.2.5 in which the regularity in the movement seemed coherent with the basic property (they both involved a "line"), in this case the trace produced during maintaining dragging seems to create a conflict with what the solver has in mind, and this seems to generate confusion.

Before this excerpt, in this exploration, Ste had dragged the base point A, and used maintaining dragging to reach a conjecture, which he wrote as: “Maintaining A on the line through M and perpendicular to MK, the quadrilateral ABCD is a rectangle.”

Episode 1	**Brief Analysis**
[1] Ste: M [2] I: Ok. [3] Ste: Also M has to... … [12] Ste: So...[he starts to drag M] [13] I: Maybe to stay on the screen we could move A closer to K... [14] Ste: Uhm, yes. [15] I: Because then it's smaller, the triangle.  [16] I: Ok. [17] Ste: Theoretically, uh...I always have the rectangle [18] I: uh huh... [19] Ste: Uh, yes, if M, uh...if the line through A and M [20] I: uh huh...	Now, once Ste has erased the line and repositioned his figure ([1]-[16]) he tries to use MD dragging M and maintaining the property “ABCD rectangle”. He proposes a first conjecture ([17]-[19]): “I always have a rectangle …if the line through A and M…” Even though it is no longer drawn on the page, the line from the previous conjecture seems to still play a main role in Ste’s perception of properties of the figure. Ste does not seem to see M as moving along a path, but instead he seems to see a “property” that should be satisfied by the rectangle he is trying to maintain during dragging,

	conceived as a generic rectangle.
Episode 2 [21] Ste: and therefore, eh, yes, it's the same thing as before. [22] I: The same thing as before only [23] Ste: M has to stay on the line  [24] I: Wait now you are moving M. [25] Ste: Yes. [26] I: Right? So there is not the line from before any more, because the line from before was defined by M and K. But now M is moving. [27] Ste: Uh huh...	**Brief Analysis** Ste realizes the conjecture is the same as before ([21]). Ste does not seem to be able to conceive the movement of M independently from the basic property he has in mind which has to do with the perpendicular line to MK through M.
Episode 3 [28] I: Ok. So maybe try to move very freely with M, ok, and try to see if you are able to maintain this rectangle. [29] I: Ok, now when you move M it leaves the red mark. [30] Ste: So, maintaining rectangle,	**Brief Analysis** Ste is using the trace and dragging the base point M in a way that the interviewer perceives as successful maintaining dragging. Ste seems to be linking the maintained property to movement ("Moving M" [34]). However, Ste does not seem to be able to

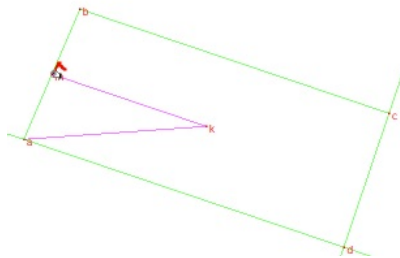
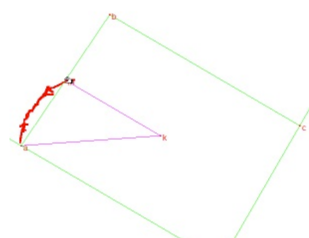
[31] I: Go. [32] Ste: Eh, only in that point there...  [33] I: Uhm. [34] Ste: moving M. [35] I: Try, try. You are doing it. [36] Ste: Ah! I understand!	perceive regularity in the movement, interpret the trace as the path becoming explicit, or even conceive a path, probably because the basic property he has in mind is creating a conflict with the trace mark that is appearing on the screen.
 [37] Ste: I mean, ...no. [38] I: What are you seeing?… [39] Ste: Uh, ...no, that...  [40] I: Keep going, maybe go back along there and see if you did well and keep going on the other side to see...if you can still do it.	There is a moment in which the trace seems to change status (Ah! I understand!” [36]), but the transition does not seem to occur ([39]) and Ste ends up not does not continuing the investigation in this direction.

Table 5.2.6: Analysis of Excerpt 5.2.6

Ste seems to be having a conflict between the basic property he has in mind and the trace mark left on the screen by the dragged-base-point during maintaining dragging. We interpret this excerpt as representative of an improper conception of the III, since Ste does not seem to be able to conceive the movement of M independently from the basic property (aspect 4), as the movement along a *path* (Episode 2). While in other cases the same lack in conception of the III during maintaining dragging would lead to strengthening of a basic conjecture (for example in Excerpt 5.2.5), in this case it leads to a conflict because the trace mark does not resemble any of the basic properties Ste seems to be considering while looking at the Cabri-figure.

Concluding Remarks on the Section

In the analyses of the excerpts above we started to introduce the issue of conceiving a *path* as a source of various difficulties in performing maintaining dragging and proceeding coherently with respect to what we describe in our model. In particular, in the last excerpt we presented (Excerpt 5.2.6) there seems to be no reference to any kind of path: neither at a "general" level, as something (not better described) to drag the base point along in order to maintain the desired property; nor at a "figure-specific" level, as a particular geometrical curve described in relation to specific points of the figure. In line [36] the solver exclaimed: "Ah! I understand!", but then goes back to his original conjecture without interpreting the trace at all. We believe that if the solver had properly conceived an III relating the movement of the dragged-base-point to some regularity – dragging along a generic trajectory which the trace could have been made figure-specific – he probably would have anticipated a path and "seen" the trace mark as an arc of circumference (GDP) along which the dragged-base-point was moving. This would have allowed him to overcome the conflict with the basic property involving the perpendicular line to MK through M, and probably conceive an IOD as M belonging to the figure-specific curve described through the GDP.

In Chapter 6 we will explain how we consider the conception of a generic *path* to be at the base of expert use of maintaining dragging. In fact the generic notion of path withholds the possibility of maintaining a property as an III through dragging along a trajectory – a figure-specific path – and dragging along such trajectory may be interpreted as a regularity in the movement of the dragged-base-point, a new invariant, the IOD. Thus the notion of path connects the two invariants and leads to an interpretation of the IOD as a cause of the maintaining of the III.

5.3 Being Mentally Flexible

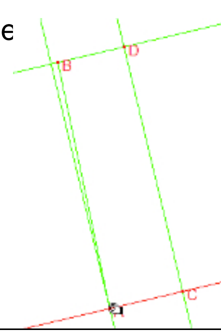
In the previous sections we have analyzed two factors that seem to be necessary for the elaboration of a conjecture according to our model; first the necessity to overcome a basic conjecture, and second that of conceiving a property as an invariant to intentionally induce. We have identified a third necessary component which we will describe in this section: being mentally flexible, that is being able to "let go" of the various properties that one might have in mind, in order to perceive "new" properties during the exploration. This ability could be described as a particular case of a more general problem-solving technique introduced by Polya as a "change in perspective" (1988). A change in perspective can help the solver overcome a perceptual block that might have occurred because s/he is only seeing what s/he expects to see or because s/he is locking on an idea that came to mind previously and is ignoring further ideas. This is not to say that a solver should not have expectations. On the contrary, success depends on a dynamic tension between the solver's expectations and his/her being mentally flexible. Mason describes this key problem-solving ability as being able to perform a shift in attention, alternatively "seeking for relationships and perceiving or applying properties" (Johnston-Wilder & Mason, 2005, p. 251).

In terms of figural concepts (Fischbein, 1993; Mariotti, 1995, p. 112), we could say that what needs to be "let go" are particular aspects of the conceptual component evoked in the solver by the Cabri-figure. We consider having in mind a necessary component because it seems that in cases in which the solver is not able to “let go” of a property that is pre-conceived with respect to the exploration s/he seems unable to perceive new properties that could make continuing the exploration easier or possible at all. For example, in Excerpts 5.2.6 and 5.2.7 we saw how pre-conceived basic properties can inhibit the conception of an III and/or the performance of maintaining dragging. As described in the analyses of these excerpts, when the solver seems to be concentrated on a basic property and s/he attempts to perform maintaining dragging, the movement of the Cabri-figure seems to either strengthen the solver’s perception of the pre-conceived basic property (as in Excerpt 5.2.6) or create a conflict with it (as in Excerpt 5.2.7). In the first case the strengthening of the basic property frequently leads to a basic conjecture which the solver tends to be satisfied with, therefore preventing the search for new conjectures involving the particular type of quadrilateral. In the second case the conflict unfolds into an inability to perform maintaining dragging until the solver is able to be mentally flexible and free his/her mind from the property guiding his/her expectations.

In this section we will show two examples (Excerpt 5.3.1 and Excerpt 5.3.2) of how the inability to be mentally flexible and free their mind from a pre-conceived property inhibits the performance of maintaining dragging, or the perception of an IOD while maintaining dragging is attempted. In particular, in Excerpt 5.3.2 the solvers are not able to strike a balance between expectations and being mentally flexible. Their pre-conceived properties inhibit the development of appropriate expectations with respect to maintaining dragging. On the other hand, in Excerpt 5.3.3 the solvers elaborate proper expectations with respect to maintaining dragging, but a strong pre-conceived idea for the GDP does not allow them to properly interpret the trace mark. In this case the resistance to letting go of a previous idea leads to a conflict between the solvers’ expectations and the trace mark that appears on the screen. However the conflict does not hinder the solvers' correct expectations with respect to the possibility of performing maintaining dragging. This is a sign of expert behavior, as we will describe later in Chapter 6.

<u>Excerpt 5.3.1</u>

In this excerpt the solver seems to be unable to perform maintaining dragging, because of a conflict created between the movement of the dragged-base-point and the basic property he has in mind and from which he cannot free his mind. The excerpt is taken from a solvers' exploration of Problem 4.

Episode	**Brief Analysis**
[1] Gin: I was thinking...I mean, moving A...we can't, we can't solve it. [2] Gin: It should remain...B... [3] I: You think that moving A it does not remain a rectangle? [4] Gin: I mean, yes... [5] I: Try to explain to me why [6] Gin: I mean yes, but B would have to anyway be on that perpendicular line.	Gin seems to be trying to maintain the property "ABCD rectangle" while dragging the base point A, in order to "solve it" (line [1]). The interviewer inquires about this in lines [3] and [5], which leads to Gin's to explain why he thinks the property cannot be maintained dragging A.
[7] Gin: Because...uh, since this line rotates...with center C, I mean the rotates with center C, basically... [8] I: Uh huh...  	Gin appears to be confused about the behavior of the figure when dragging A. He first keeps on moving A left and right, sort of maintaining AC at a constant inclination, as if that

[9] Gin: Uh, B on the other hand does not move. I mean, it always stays in the same position. [10] Gin: Therefore B, uh, I mean in order for this figure to be a rectangle, B has to anyway [Italian: "comunque"] be on the perpendicular line.	were the only movement possible. Moreover, Gin seems to concentrate on B, which appears to be fixed ([9]) and on the perpendicular line ([10]) while he thinks that the rest of the figure “rotates with center C” ([7]).

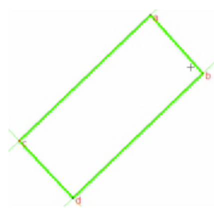
Table 5.3.1: Analysis of Excerpt 5.3.1

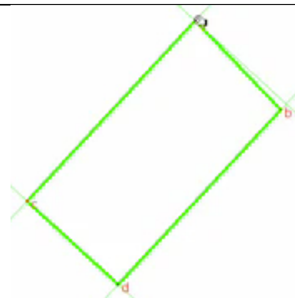
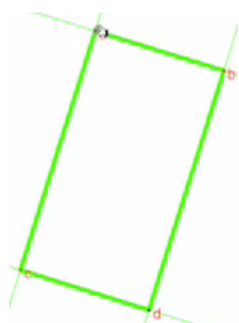
From what has happened during the exploration, before the beginning of this excerpt, we infer that with “solve it” he is probably referring to the problem of maintaining the property “ABCD rectangle” while dragging. Gin seems to be considering a minimum basic property during dragging, that is “B would have to anyway be on that perpendicular line” ([6]), which seems to inhibit his dragging. He does not seem to be able to be mentally flexible and free his mind from the property. Moreover this property combined with the observation that B “does not move” ([9]) during dragging seems to generate confusion, as can be seen when Gin is not able to explain both the “rotation” he perceives and the basic property “B on the perpendicular line” at the same time. We might infer from Gin’s attempts to perform maintaining dragging that (at least for some time) he also thinks that the only way of maintaining a “general rectangle” is dragging A so that the line through AC maintains a constant inclination (see his dragging in lines [1]-[5]). Such idea together with the inability to “let go” of the property “B belonging to the perpendicular line” (which seemed possible only when A was in a particular position) seems to lead Gin to the conclusion that maintaining dragging is not possible. However Gin does not explicitly state whether maintaining dragging is or is not possible. Instead he prefers to state the property he is convinced of ([10]). This property may have such a strong appeal to Gin because it seems to come from the conceptual part of the figural concept he has developed (a rectangle has four right angles, in particular <ABD must be right, so B must be on the perpendicular line he has constructed), and so *it must* be correct and important.

Finally, we remark that the solvers in this excerpt, which comes from the first exploration they engage in, do not seem to be expert solvers, yet. We consider the fact that they do not seem to be expecting a *path* evidence for such interpretation, as we will discuss in Chapter 6. Moreover, the solvers' resistance to letting go of their previously-conceived property hinders the development of such expectation, and therefore the possibility of using the maintaining dragging scheme. In this episode, the solvers do not seem to have perceived any regular movement of the base point being dragged as "a cause" for their III to be visually verified, and instead of expecting a *path*, they seem to accept some basic property (B on the perpendicular line) as the "cause" of the III, which becomes a condition and the premise in their conjecture.

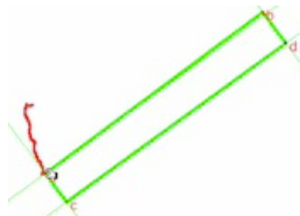
Excerpt 5.3.2

This excerpt is taken from two students' exploration of Problem 4. The solvers have formulated a first written conjecture on how ABCD can be a rectangle: "ABCD rectangle (when AB is perpendicular to CA and AB ≠ AC)." This Excerpt shows how the solvers' inability to be mentally flexible and free their mind from pre-conceived properties inhibits the carrying out of maintaining dragging. In particular, the properties the solvers seem to be thinking of involve parallel and perpendicular lines, while an appropriate GDP for the dragged-base-point would be a circle. This contributes to making the conflict that emerges as the solvers try to interpret the trace mark particularly evident. The student who is holding the mouse is marked in bold in the transcript below.

Episode 1	**Brief Analysis**
[1] Em: Basically a line [murmuring] [2] **Ila**: Yes [3] **Ila**: Basically the parallel line to CD. [4] I: What were... [5] **Ila**: Uhm, it can move along ... [6] I:...you looking at while you were moving it? [7] **Ila**: Because basically I was looking at the fact that this segment here... [8] I: uhm... [9] **Ila**: Has to always be parallel to this [pointing to AB and CD]. [10] I: Ok.  [11] **Ila**: So that the angles are always 90 and also if I do...I activate trace, for example, I will get the parallel line to CD. [12] **Ila**: It will always be a rectangle when I move A and B, so on the parallel that I can construct.	Em and Ila provide a first GDP, with respect to the movement of A, as a “parallel line to CD” ([1], [3]). Ila describes the property she has in mind, which seems to be guiding her perception. She even anticipates what the trace mark will look like and proposes to construct the object representing her GDP.
Episode 2	**Brief Analysis**
… [15] **Ila**: Parallel line through this point ... [16] **Ila**: ...through this...There, now if I move A... [17] **Ila**: Ah no, but I need to fix B too. [18] **Ila**: This	Ila realizes that the figure does not behave as she expected. She tries to

[19] Em: [murmurs something] [20] **Ila**: Wait! Right! [21] Em: Move it. [22] **Ila**: Wait, no no. [23] **Ila**: No, it's enough to do... [24] **Ila**: No... [25] **Ila**: [murmuring] Theoretically I need "parallel" [26] **Ila**: parallel [27] **Ila**: through this point... [28] **Ila**: No! What the heck! 	take B into consideration and she seems to want to “fix” it ([17]) in order to maintain the parallelism she was expecting.
Episode 3 [29] Em: Why are you...I don't understand. [30] **Ila**: No, no. I made a mistake. [31] **Ila**: Because. [32] **Ila**: I also need to fix this point ... … [38] **Ila**: but this point too has to be fixed on the parallel line. So.. … [42] **Ila**: It's the same thing as before! I mean...A has to always belong to that famous line that we put in the hypothesis. 	**Brief Analysis** Both solvers seem confused, and Ila returns to her idea of wanting to have B “fixed to the parallel line” ([38]). She finally goes back to considering either the condition expressed in the first conjecture AB perpendicular to CA or AB parallel to CD.
Episode 4 [Ila tries to perform MD again]	**Brief Analysis** The solvers seem unsure

… [55] Em: Eh, every time that you...the more you make it longer, the more you [56] Em&**Ila**: [together] take it down. [57] **Ila**: and the more I go up...no...the more [58] Em: the more you shorten the more you raise. [59] **Ila**: It's as if it followed...the line...but [60] Em: Raise a little. [61] Em: Lower. [62] **Ila**: see that if I...I mean [63] I: What are you looking at?  [64] **Ila**: I don't know. I am looking at the fact that it is as if...I am trying to follow this line here, that is the parallel to CD, [65] I: Uhm... [66] **Ila**: However, even if I follow it [showing the movement], B goes farther and farther away. [Ila decides to activate the trace] … [79] **Ila**: So, trace...this point here. [80] **Ila**: It's as if it is only in that point there. [81] **Ila**: Wait, right. [82] **Ila**: Yes!! Because if I move A,	whether it is possible or not to perform maintaining dragging. Em tries to guide Ila in her attempt to perform maintaining dragging. Ila describes the property she is using to guide her attempt at performing maintaining dragging. As she tries to do this she realizes once again that the figure, in particular B is not behaving as expected. She decides to activate trace on the dragged-base-point. Once again

[83] I: Yes.. [84] **Ila**: B...no, it doesn't stay still...but if I move A  … [87] **Ila**: No, it's there. [88] **Ila**: See that... [89] Em: Try to maintain... [90] **Ila**: ...only in that point, I think. [91] Em: Go down! Go further down. [92] Em: Lower...ok...keep going down. [93] **Ila**: down...[murmuring]...there	Em tries to help her guiding the movement orally.
Episode 5 [94] **Ila**: It has to follow...it has to be...see that it is... [95] Em: [murmurs something] [96] **Ila**:...basically the parallel. [97] Em: No... [98] **Ila**: Look: if I follow... [99] **Ila**: this parallel line here... [100] **Ila**: See? Look. … [105] **Ila**: Point A... [106] Em: Bring it up. [107] **Ila:** There! See that ...no. [108] Em: No, I don't think...it's a rectangle [109] **Ila**: It's not a rectangle, you're right.	**Brief Analysis** Again Ila seems to only be able to interpret the movement of A only in terms of "following a parallel line" ([96]-[99]).

[110] **Ila**: No, but there is something that ... [111] **Ila**: Because ... [112] **Ila**: It's as if A had to stay fixed there. [113] I: Uhm. [114] **Ila**: It has to... [115] I: But you were moving it... [116] **Ila**: Eh. [117] **Ila**: I mean, yes, but I'm saying in order for it to remain a rectangle.  [118] I: Uhm... [119] **Ila**: It's as if it had to stay fixed there. [120] I: "There" where? [121] **Ila**: In, uhm, between the intersection...between the line that... [122] **Ila**: Between A..., basically between this line here [pointing to AC], and this one here [pointing to AB].	Ila seems to be confused again, and decides that A, too, needs to be "fixed" ([112], [119]) between the intersection of AB and AC, as she indicates to the interviewer ([121]-[122]).

Table 5.3.2: Analysis of Excerpt 5.3.2

Initially the solvers seem to be interested in moving point A and maintaining the property "ABCD rectangle" (their III). Then the Ila seems to shift her attention to a property she recognized in the Cabri-figure she is exploring ([7], [9]): "the segment here…has to always be parallel to this." We can interpret this as part of the conceptual component of the figural concept Ila has built from the Cabri-figure. In other words, Ila seems to be interpreting the Cabri-figure as a rectangle, a figural concept, with the parallelism between two sides as a property of the conceptual component. She seems to show a desire to relate this property to the trace, and in line 11 she even predicts (incorrectly) what the trace would represent if she activated it and dragged A trying to maintain her III.

Ila's prediction leads her to draw the parallel line to CD through A (lines [14] and [15]) and to try to move A along it. As doing so she expresses the need to "fix B too" ([17]), which indicates the beginning of a conflict arising between the predicted and the actual behavior of the Cabri-figure. She repeats her intention in lines 32 and 38 while she is trying to redraw her line and explain her thinking (unsuccessfully) to Em. Ila's argumentation is built around her pre-conceived property which she does not appear to want to abandon. Even though Ila does not seem to be able to successfully drag A along the line she has conceived, she states again that "A has to always belong to that famous line" ([42]).

The conflict becomes more evident when, trying to perform maintaining dragging again and getting help from her partner (we will discuss this collaborative behavior in section 5.5), Ila keeps on looking at "this line here, that is the parallel to CD" ([64]) and at the behavior of B as well ([50], [66], [84], [122]). Initially Ila seems to be successful at performing maintaining dragging, however shifting her attention to the movement of the dragged-base-point A, she is not able to overcome her original idea of moving along "the parallel" ([96]). Ila's pre-conceived property, AB parallel to CD, leading to her idea of having to move A along a parallel line, seems to inhibit the carrying out of maintaining dragging, even after Em tries to guide her in an attempt that in the eyes of the interviewer seems successful ([49]-[67]). Furthermore, Ila seems to reach the conclusion, and be pretty convinced, that it is not possible to perform maintaining dragging with this base point and this III ([80], [82], [90], [112]). The conflict is now evident and Ila seems to be confused, but still unable to let go of her pre-conceived property. She only seems to be able to let go of the particular parallel line she was considering to substitute it with another (possibly to BD this time) in order to try to resolve the conflict when Em prompts her to continue dragging, since she seems not to agree with Ila ([97]) and wants to "do all the trace" some other way ([102]). All Ila seems to be able to perceive is that dragging along her imaginary parallel line does not induce the III ([107]-[109]), which strengthens the conflict. In the end, Ila states that maintaining dragging is not possible, since A need to "stay fixed" ([112], [119]) in an intersection ([121], [122]). Again her reasoning seems to revolve around the pre-conceived property which she cannot let go of.

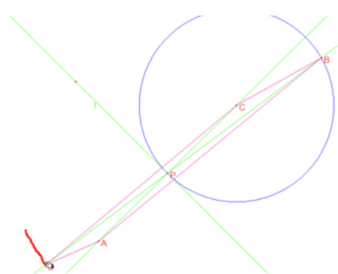
Ila's attachment to her pre-conceived property leads her to uncertainty and difficulties in performing maintaining dragging. Moreover the strength of her belief may be augmented by the roots of the pre-conceived property. Again Ila seems to be interpreting the Cabri-figure as a rectangle, a figural concept, of which the pre-conceived property is part of the conceptual component. The strength of Ila's pre-conceived property appears again clearly later in the exploration, which unfolds in the following way (see Excerpt 6.2.2 in Ch6). The solvers, prompted by the interviewer, are eventually able to perform maintaining dragging in a way that seems consistent with our model, but are unable to make sense of the "circle" that appears on the screen when the trace is activated (Excerpt 6.2.2). Although all the elements were in place for the solvers to let go of their incorrect GDP (the "parallel line" [99], Excerpt 5.3.2) and provide a new one ("the circle" [8], Excerpt 6.2.2), they do not do this. Instead they ask themselves "why" a couple times and, unable to reach an explanation, settle on a basic conjecture in which the premise is "AB parallel to DC, that is when AB is perpendicular to CA".

Overall, we seem to have been able to describe the solvers' difficulties in this excerpt in terms of solvers' reluctance to freeing their mind. We did so by showing that the pre-conceived property inhibited the performance of maintaining dragging and led to a conflict that the solvers were not able to resolve. The conflict originated from the coexisting idea of parallel and perpendicular lines and the observation and haptic perception of the movement of the base point A (circular) during attempts to perform maintaining dragging. The inability to be mentally flexible and free the mind from the pre-conceived property made it impossible for the solvers to reach a harmonic interpretation of their experience. In particular it seems like any interpretation of the trace mark was linked to the pre-conceived property instead of potentially leading to a new detached geometrical object along which to drag.

Finally, as in Excerpt 5.3.1, the solvers in this excerpt do not seem to be expert solvers. We will discuss this aspect in further detail when we discuss issues related to the appropriation of the maintaining dragging scheme, in Chapter 6. Although there seems to be expectation of a *path*, the prediction of its geometric description is dominated by the strong conceptual components of the figures the solvers seem to be dealing with. Moreover, the properties of the conceptual component are conceived statically and the solvers' resistance to letting go of these properties inhibits their perception of a regular movement of the base point being dragged as "a cause" for their III to be visually verified. As in Excerpt 5.3.1, the solvers seem to accept a basic property as the "cause" of the III, which becomes a condition and the premise in their conjecture.

Excerpt 5.3.3

This Excerpt features two solvers, who we consider "experts" with respect to maintaining dragging, but that encounter difficulties performing maintaining dragging because they resist letting go of a previously-conceived idea. This leads to a conflict between the solvers' expectations and the trace mark that appears on the screen.

Episode	**Brief Analysis**
[1] G: you see that if you do, like, maintaining dragging ["trascinamento di mantenimento"]… trying to keep them more or less the same [2] **F**: exactly [murmuring]… well, okay. …	G has identified a minimum basic property (PD=PB) to use as an III.
[3] G: Ok, uh, then what had we done? parallelogram. [4] **F**: For the parallelogram, uh, let's try to use "trace" to see if we can see something. [5] G: go, let's try [speaking together with him]…uh, "trace" is over there. There, no there, there! [6] **F**: Trace, we have to do D, well for now let's do a parallelogram like this, okay, so of this point… with respect to what?	F decides to activate trace on the base point D and maintain "ABCD parallelogram" as an III.
[7] G: With respect to what? [not understanding] only that point. [8] **F**: Only this point. Okay so I'll take it and go. [9] G: and now what are we doing? Oh yes, for the parallelogram?  [10] **F**: yes [as he drags D with the trace activated] yes, we are trying to see when it remains a parallelogram. [11] G: yes, okay the usual circle comes out.	The solvers seem to give a first GDP as a circle, however the circle they have in mind does not seem to "fit" what appears on the screen as F performs maintaining dragging. This conflict

[12] **F**: wait, because here…oh dear! [“accidenti”] where is it going? … [13] **F**: So, maybe it’s not necessarily the case that D is on a circle so that [“in modo che”] ABCD is the parallelogram.  	leads F to question the hypothesis of D being on a circle in order to maintain the III.

Table 5.3.3: Analysis of Excerpt 5.3.3

We consider the solvers in this excerpt to be “experts” since they have successfully used maintaining dragging and generated conjectures in a way that was coherent with our maintaining dragging scheme in previous explorations. In this exploration as well the solvers seem to expect the maintaining of the III to be “caused” by the movement of the chosen base point along a *path* ([4]), which can indicate appropriation of the scheme as we will describe in more detail in Chapter 6. However, as soon as the idea of “circle” comes to mind – and moreover of a particular circle ([11]) – the solvers seem unable to free their mind from such conception and are unable to make sense of the trace mark, even doubting that the GDP is a circle at all ([13]). The circle they seem to have in mind seems to be the circle centered in P and with radius 2PC. This idea seems to inhibit the conception of other circles and even the performance of maintaining dragging; the exploration continues with an argumentation about why their initial idea does not work, during a second attempt at performing maintaining dragging, and eventually with G deciding to “think about it” without the trace or any dragging (we will show this in Chapter 6). In other words the idea of movement along a certain circle the solvers cannot free their mind from is strong: it inhibits the perception of other invariants and even the performance of maintaining dragging since it creates a conflict with what appears on the screen. In the end the conflict is overcome when one of the solvers lets go of the idea and thinks about the situation in a different way (Excerpt 6.2.3).

5.4 Being Aware of the Status of Objects

As described in Chapter 2, a Cabri-figure is constructed from a set of basic objects which the user initially places on the screen as s/he pleases. New objects are then constructed from this basic set according to specific geometrical properties. Such properties (and all derived properties) are maintained by the Cabri-figure during dragging, that is when the user acts upon the figure. The user can act upon the Cabri-figure by dragging any of the basic objects through which it is defined. In the step-by-step constructions that lead to the Cabri-figures in our activities, the basic objects through which figures can be acted upon are mainly points.

Awareness of the different status of objects of a Cabri-figure – that is of the basic elements, those that can be acted upon directly, as opposed to the elements that are dependent from these, and that cannot be directly acted upon – can guide the solver when s/he is deciding how to proceed in the exploration. However gaining such *general awareness* is not trivial and many solvers seem to exhibit a lack of it. For example, recall Excerpt 5.3.2 from the previous section. Ila constructs the parallel line to CD through A (lines [26] and [27]) and then tries to drag A along it, as if the line were independent from A. Her hypothesis is that A moves along such line, as she repeatedly expresses (lines: [42], [66], [98], [99]). Her lack of general awareness over the status of the different objects leads to perplexity ([28]) and a state of confusion when she tries to move A along the line and realizing that the line moves with A ([112]-[117]). Moreover, her lack of awareness of the status of different objects seems to even lead to an erroneous perception of objects that move or stay still during the dragging she is performing. Ila explicitly states that B does not stay still when she tries to move A ([82]-[84]) even though it actually does, as shown on the screen and justified by the fact that B is not constructed as dependent from A in any way. However Ila does not seem to be able to realize this, and instead concludes that the whole figure must basically "stay still" as it is good "only in that point" ([90]).

As described in the example above, a general level of awareness of the different status of objects of a Cabri-figure is fundamental for dynamic explorations, whether they include the use of maintaining dragging or not. This type of awareness is necessary for the solver to be able to act upon the dynamic-figure, either dragging its base points or constructing new robust properties to add to the ones inherited from the steps of the construction. In particular, being aware of the different status of the geometrical objects that the Cabri-figure is made of – which is necessary for having control over the Cabri-figure – fundamental for generating conjectures according to our model.

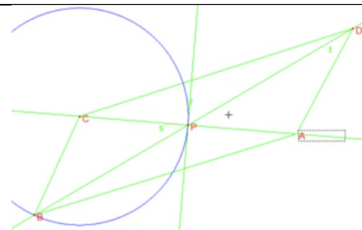
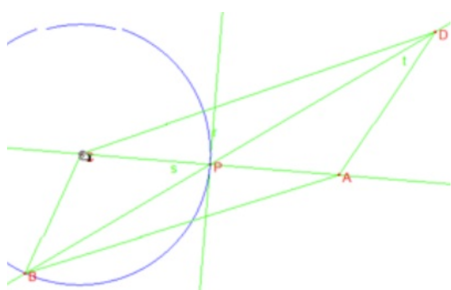
Although general awareness is necessary for exploring and making sense of the dynamic-figure, it is not sufficient. There seems to also be a *figure-specific level of awareness* that allows solvers to control the Cabri-figure. When general awareness is present, even in cases in which initially the solvers do not seem to have control at a figure-specific level, solvers seem to be able to reason about the various elements of the dynamic-figure and quickly gain control over their different status. On the other hand, when general awareness seems to be lacking, solvers do not seem to be able to proceed in the exploration. In the excerpts we present in this section we will provide an analysis that takes into consideration the general level and the figure-specific level of awareness, to show the roles played by each of them and how they are woven into processes of conjecture-generation.

First we present two excerpts that show evidence of solvers' awareness of the different status of objects of a dynamic-figure. In particular Excerpts 5.4.1 and 5.4.2 show respectively how awareness of the dependence of certain objects from other basic ones allows solvers to decide how to proceed in an exploration, and how a discussion over which points are base points allows the solvers to overcome a block at a basic conjecture. We then proceed by analyzing three excerpts shed light onto other consequences that the lack of awareness of the different status of objects of a Cabri-figure either at a general level and/or at a figure-specific level can have on the explorations. In particular, for solvers who have general awareness, a lack of figure-specific control may just make the dragging test manually harder but not hinder the process of conjecture generation (Excerpt 5.4.3), while solvers who do not seem to have awareness at a general level (Excerpts 5.4.4 and 5.4.5), may experience blocks in the process and difficulties in developing the maintaining dragging scheme, the utilization scheme associated to maintaining dragging and the task of conjecture-generation, as described by our cognitive model.

Excerpt 5.4.1

This excerpt shows how general awareness of the different status of objects of the Cabri-figure allows the solvers to decide how to proceed in the exploration. The solvers are exploring Problem 2. As usual, the name of the solver who is holding the mouse is marked in bold letters.

[1] Giu: So D is independent and it stays on its own... [2] **Ste**: ...however...yes	

[3] Giu: Yes...exactly.  [4] **Ste**: A depends...[they speak together] … [8] Giu: A is dependent from C because it is at the same distance to is remains like that. [9] **Ste**: It's an axial symmetry, so I can't do anything about it. [10] Giu: Good for you. [11] Giu: uh, B is dependent both from C and from D, right? [12] **Ste**: But can I move it?...no [13] Giu: Of course not!! [they laugh] [14] **Ste**: right, actually... [15] Giu: Because if you move C... [16] **Ste**: if I move C... [17] Giu: IF YOU MOVE C... [18] **Ste**: I am moving C! what's wrong? …  [They tease each other and Giu takes the mouse] [21] **Giu**: Ok. I am not responsible for whatever it is that I am doing...yes...if you move C, B also moves...if you move D. [22] Ste: It's the same. [23] **Giu**: it moves...so B is dependent from D and from C...	The solvers refer to the steps of the construction in which C is defined as the symmetric image of C with respect to P. They predict that A is not a base point and try to drag it, which confirms their reasoning. Referring to how B was constructed they also conclude that B is “dependent” and therefore not draggable. They seem unsure as to whether B is dependent from C as well or not, so they test it by dragging. They see that C does influence B, and

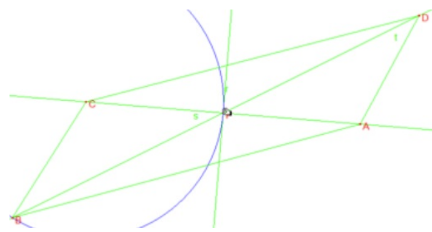
[24] Ste: from C and from D. [25] **Giu**: from C and D exactly. [Ste takes the mouse back] [26] Giu: Therefore we need to find the way to...so the possible conditions are C and D, because only moving C and D we can have something that changes. [27] **Ste**: right. [28] Giu: Otherwise... [29] **Ste**: right, actually here I can't do...oh no, I can... [30] Giu: eh, I can chaaange [he drags P] yes  [31] **Ste**: Yes, because that point too can move this line here, so... [32] Giu: But it is like changing D.	decide that the points that influence the behavior (in order to "have something that changes") of the Cabri-figure are C and D. They finally notice that P is also a base point, but decide not to use it for the exploration, since they see dragging it as "like changing D".

Table 5.4.1: Analysis of Excerpt 5.4.1

The excerpt shows how the solvers are aware of the hierarchy of objects of the Cabri-figure at a general level. Among the objects that the Cabri-figure is made of, they seem to pay particular attention to points, deciding which ones depend on others, thus gaining figure-specific awareness. While they figure out which points are base points they mix theoretical properties ([8], [9]) derived from the steps of the construction with empirical arguments based on trying to move the points with the pointer ([13], [18], [21]). The solvers' reasoning and exploring general awareness of the hierarchy of the various points of the Cabri-figure allow them to quickly gain figure-specific control, identifying the base points. This is a prerequisite for deciding which base point to choose in order to perform maintaining dragging. They choose to use D as their selected base point and the property “ABCD parallelogram” as their III, and proceed according to our model.

<u>Excerpt 5.4.2</u>

This excerpt shows how awareness of the different status of objects and of the role of the base points in determining the behavior of the Cabri-figure allows the solvers to overcome a block at a basic conjecture they had formulated earlier in the exploration. The solvers had been working on Activity 3 and had written the following basic conjecture: “If AB is perpendicular to *l*, then ABCD is a rectangle.” Not knowing how to continue the exploration, Pie thinks of dragging the base points and tries to explain his idea. The solvers then continue the exploration trying to perform maintaining dragging with the different base points.

[1] Ale: [murmurs something about angles.]	
[2] **Pie**: Let's say this: If it is a rectangle, we can say that AB has to be, I mean the only case in which, uh, the quadrilateral is a rectangle, is when AB is perpendicular to line *l*.	Pie repeats the basic conjecture they had reached earlier in the exploration.
[3] **Pie**: and that seems to make sense to us.	
[4] Ale: Yes.	
[5] **Pie**: Now we would need to see if moving the base points we can obtain more...I mean in other ways, changing the base points, we can obtain AB...perpendicular to *l*.	He then argues that the condition “AB perpendicular to *l*”, necessary and sufficient for ABCD to be a rectangle, may
[6] Ale: [murmuring] only in this case...	
[7] **Pie**: perpendicular to l. Which is what I was saying before, that maybe it could be that moving K or M...that is [dragging M].	

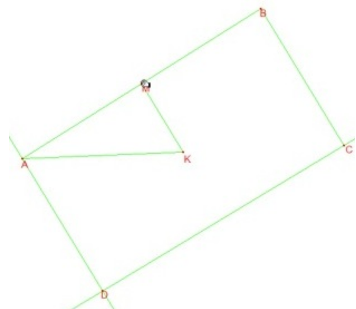
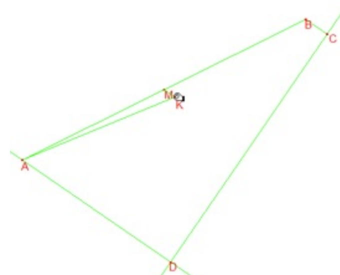
[8] I: Ok, so let's work in this direction. [9] Ale: Yes, but in any case, uh, even if we move M or K...AB has to in any case be always perpendicular to I in order to have a rectangle.  [10] **Pie**: Yes, and that we said is OK. Only in that case, in the sense that if and only if AB is, uh, perpendicular to *l*, we have a rectangle. [11] **Pie**: But what I'm saying is that maybe having three base points... [12] Ale: Yes. [13] **Pie**: ...that we can move, it could be that moving, in particular one of those, [as he drags K] uh points, we can obtain that AB is a...that AB is, uh, perpendicular. 	be obtained by "changing [dragging] the base points". Although Ale is not convinced by this argument, Pie – maybe encouraged by the interviewer's comment – tries to explain again how it might be possible to obtain the desired condition by "moving, in particular one of those… points", and simultaneously he tries dragging them.

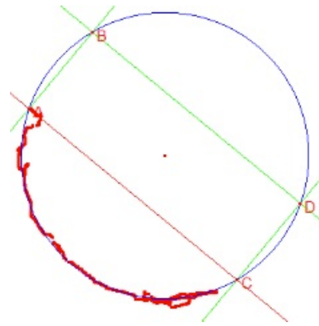
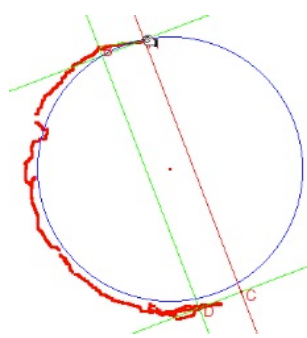
Table 5.4.2: Analysis of Excerpt 5.4.2

In this excerpt Pie seems to have awareness of the different status of the points A, M, and K with respect to the other points of the Cabri-figure both at a generic level and at a figure-specific level. This is evident in his argumentation about why there may be other ways to explore the case of "ABCD rectangle". In particular he argues that although the condition they had expressed in the first conjecture (AB perpendicular to *l*) is necessary and sufficient for ABCD to be a rectangle ([2], [10]) there may be other ways to “obtain it” by dragging the base points ([5], [11], [13]). This awareness allows Pie to conceive the condition of the first conjecture as a bridge property, which can be used as a temporary III (see Ch 4), in order to find new conditions under which ABCD might be a rectangle. In other words, the awareness at both levels seems to allow Pie to overcome a global static apprehension of the figure and of the conceptual relations between its elements – which had been used for generating basic conjectures – and choose to proceed inducing the property “ABCD rectangle” through movement of the base points. This way the solvers overcome their original (basic) conjecture and proceed in the exploration using maintaining dragging.

Excerpt 5.4.3

The excerpt shows consequences that the lack of figure-specific control over the different status of objects of a Cabri-figure can have on the explorations. This excerpt is from an exploration of Problem 4 in which the solvers are experts and have performed maintaining dragging using the base-point A, activated the trace, and reached a GDP, which they describe at the beginning of the excerpt. Instead of constructing an A-invariant object that represents their GDP, they construct an object that is dependent on the dragged-base-point. In doing this they do not seem to be controlling the different status of points. Although in this case the decision does not hinder the process of conjecture generation, it makes the (soft) dragging test manually more difficult to perform and a robust dragging test impossible to perform.

[1] **Gin**: So...circle again. [2] I: Hmm. [3] **Gin**: Yes. [4] **Gin**: so...	Gin describes the GDP as a circle. The solvers successively refine the

[5] Dav: [murmurs something] [6] **Gin**: Yes...it is [7] Dav: ...it is the midpoint of C and B  [8] **Gin**: It is the midpoint of... [9] Dav: It is the intersection of the diagonals [10] **Gin**: diagonals [11] Dav: of the diagonals. [12] Dav: and since it is a rectangle, it is also the...the...uh the center of the circumscribed circle. [13] **Gin**: whatever. [14] Dav: Eh, they are all on the circle. [15] **Gin**: yes. [16] **Gin**: hmm. [17] I: Now, are you sure of this? [18] **Gin**: eh, yes.... [19] I: Because you have traced only ...	GDP trying to decide where the center of the circle might lie. They then proceed by constructing the circle that represents their GDP as the circle with center the midpoint of BC and passing through A. The solvers seem to be describing aspects of the new Cabri-figure on the screen.
 [20] **Gin**: ...pretty much [21] I: a little piece. Hmm. [22] **Gin**: there. [23] **Gin**: Well, we could try to continue.	The solvers seem convinced by their GDP and are able to predict what the rest of the trace mark should look

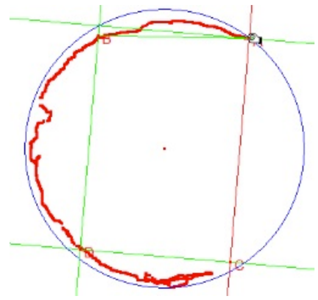
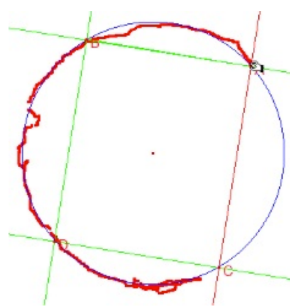
[24] Dav: exactly. [25] **Gin**: So now let's ...  [26] **Gin**: more or less along there [27] **Gin**: nooo [as a little circle appears when he clicks another point on the screen because he had not finished using the command "circle"] [28] **Gin**: Good here... [29] Dav: No... [30] **Gin**: Yes, alright, it looks like it is good [Italian: "sembra di sì"]  [31] **Gin**: Yes, good. It could be. [32] Dav: Yes, it looks like it is good. [33] **Gin**: yes. [34] Dav: Careful you are going out...	like. Although the circle they have constructed is not A-invariant the solvers seem convinced that it correctly describes their observations as they perform a soft dragging test.

Table 5.4.3: Analysis of Excerpt 5.4.3

The solvers have performed maintaining dragging and activated the trace on the base point that they are dragging. They seem to notice a circle appearing ([1]). They proceed to give further details of their GDP, describing the center of the circle as the midpoint of BC ([7]), or the intersection of the diagonals ([9]-[11]). Notice how these descriptions do not take into account the status of different objects with respect to the construction: defining a circle by its diameter defined by base points that are not being dragged (B, C) is fundamentally different – in terms of behavior of the resulting construction – than defining the center of the circle as the intersection of the diagonals (thus necessarily dependent from the dragged base point and other dependent objects).

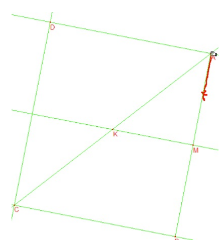
However they construct the object that represents their GDP as the circle with center the midpoint of BC, through A, the base point being dragged. This is a GDP that is not an A-invariant, and it creates difficulties in performing any type of dragging test, as shown in the excerpt. The solvers need to drag point A trying to keep the constructed circle still and the III to be visually verified, and at the same time check their IOD. Gin, with some difficulty, does seem to be able to perform the dragging and both students seem to have conceived the IOD as “A belongs to the circle”.

The solvers do not seem to be aware of the difficulties that their GDP is creating in performing the dragging, and they seem to be able to overcome such difficulties by cooperating: Dav seems to check that all vertices of the rectangle are on the circle (a bridge property he perceives as an III instead of “ABCD rectangle”), while Gin seems to be trying to keep the circle still and drag A “along it”. It seems like this collaboration is fundamental given all that the solvers need to keep under control (for more on “collaboration” see Section 5.5). Such difficulty would not have arisen if the GDP had been an A-invariant object.

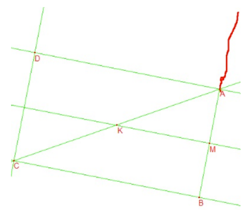
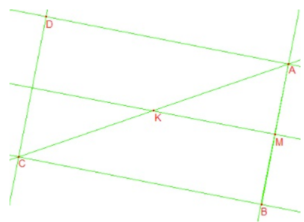
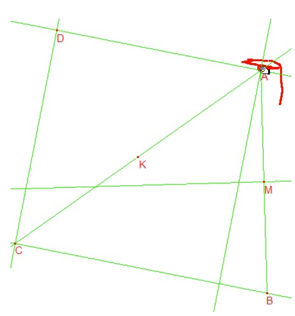
This example shows how the figure-specific control over the different status of objects comes into play at different phases of our model. In Excerpt 5.4.2 we have seen how it contributes to the initial phase of a dynamic exploration, and now we have seen how it can affect the behavior of the object constructed to represent the GDP, making such behavior non-ideal for the dragging test (especially a robust dragging test). The solvers in this excerpt did not seem to have difficulty formulating their conjecture even if they could only perform a soft dragging test. This seems to be the case because they were aware, at a general level, of the hierarchy of constructed elements that determined the dependence of certain objects from others. However, for solvers who do not seem to have this general awareness this situation might have been puzzling. In the next two excerpts we will show cases in which the solvers do not seem to have awareness at a generic level, and how these affects their explorations.

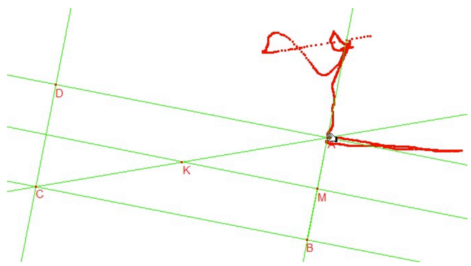
Excerpt 5.4.4

This excerpt shows how a student provides a GDP for the base point she is dragging. However she does not seem to take into account the different status of the objects she considers in her description, and does not define an object that is independent from the base point she is dragging. The solver does not seem to have awareness at a generic level, which seems to make it difficult for her to providing a GDP. This interpretation of ours is supported by the fact that she overcomes the difficulties only through an intervention of the interviewer aimed at fostering awareness of the base points. The excerpt is taken from a student's exploration of Problem 1[1].

Episode 1	**Brief Analysis**
[1] I: you are moving A...with the intention of?	
[2] Giu: Of leaving it...	Giu is using a bridge property to maintain ABCD a rectangle while dragging A.
[3] Giu: of making coincide the line AC and ... 	
[4] I: uh huh	
[5] Giu: I mean, uh, to make the perpendicular bisector of AB go through K.	
[6] I: Ok.	
[7] I: and what are you seeing?	

[1] Since this exploration was part of the pilot study, the interviewer was "more active" than in the interviews of the final study. However the solver's lack of control over the status of objects seems quite apparent, so we chose this excerpt even given this limitation.

[8] Giu: Eh, that it moves along the perpendicular line. [9] I: Uhm [10] Giu: I mean.  [11] I: Ok, so let's try to describe this perpendicular line. [12] Giu: Eh...it's the the the … [16] Giu: Ehm, the perpendicular line to AD, that is to the line / that we constructed [17] I: uh huh...  [18] Giu: through A. [19] I: Ok, let's try. [20] Giu: there.	She provides a first GDP as “the perpendicular line”. When prompted she tries to refine her GDP and describes it as the perpendicular line to AD through A. She proceeds to construct the object that represents her GDP.
Episode 2	**Brief Analysis**
[21] Giu: Eh, B has to be... [22] Giu: I would have to fix B. [23] I: Hmm...B is a constructed object... [24] Giu: Then I have to do the line through... [25] Giu: Hmm...so...[she starts dragging A again.] [26] Giu: So I want A... [27] Giu:..here. [28] Giu: There...so... 	She seems to realize the line she constructed does not have all the properties she had hypothesized, in particular it does not contain B. She tries to find another way of describing “the place” to

[29] I: Or also in another place, you may want it, right? There are a number of places... [30] Giu: Yes. [31] I: Ok. Also all that [as Giu drags A]..you may want. [32] Giu: All that. [33] I: So let's try to describe this. [34] I: There no [as Giu goes "off track"] [35] Giu: No. [36] I: Ok. [37] I: So try to explain  to Cabri and to me what is the "here yes and here no". [38] Giu: [smiling] [39] I: Let's define a bit better the "here yes". [40] Giu: [sighing] eh, it would be the perpendicular...only I have already drawn it! but it doesn't stay...	put A in order to obtain the property she wants. She describes "the place" visually in terms of good and bad zones on the plane: "here yes" and "here no". However she seems to keep on conceiving the "here yes" zone as the perpendicular line she has already constructed.
Episode 3 [41] I: Eh, but it moves. Why does it move? [42] Giu: Because... [43] I: Because who are you moving? [44] Giu: Because I move A and so the parallel, uh, the perpendicular moves too.	**Brief Analysis** The interviewer prompts her to think about which points make other objects move. Giu does not seem to be

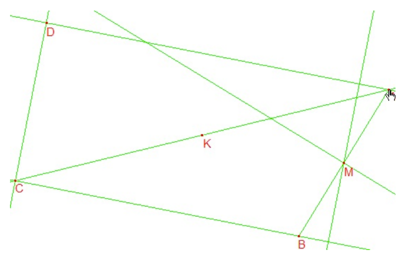
[45] I: Yes, because you defined it through A. [46] Giu: Ok. [47] I: So let's maybe try to define it not using A. [48] Giu: [together]...not using A... [49] Giu:...but instead...through B. [50] Giu: ...through M? [51] I: Ok, M does not move, for example, [52] Giu: uhm. [53] I:...so it might be a good one. [54] Giu: was it this one? [56] I: That one, yes, we don't want it any more.  [57] Giu: Hmm,...I have to do the perpendicular line [as she constructs the line] to this one [she selects BC] through this [she selects M].	aware of the hierarchy of the objects of the Cabri-figure: she proposes to use B to define the line, as if she were just guessing randomly, and finally she decides to use M – maybe because of the response of the interviewer. Giu finally constructs an object, representing her GDP, that is A-invariant.

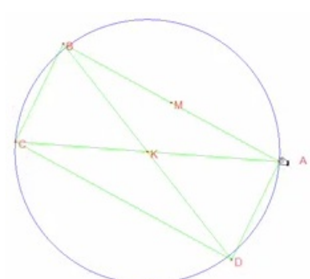
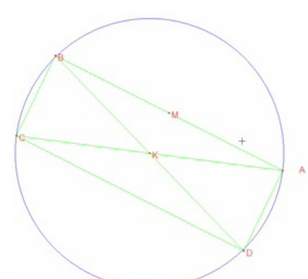
Table 5.4.4: Analysis of Excerpt 5.4.4

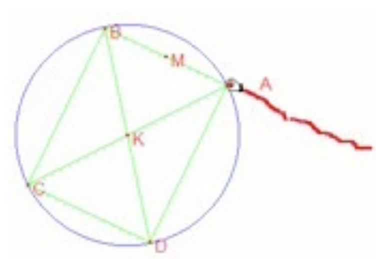
The III that Giu seems to be inducing is “the perpendicular bisector of AB goes through K” ([5]), a bridge property that she seems to be using to study the case of the rectangle (previous episodes of this exploration). She notices that A “moves along the perpendicular line” ([11]), but she seems to have trouble describing it geometrically in a more precise manner. Giu does not seem be aware (neither at a generic level nor at a figure-specific level) of the hierarchy of the various elements of the Cabri-figure she is investigating. She constructs a line to represent her GDP which is not an A-invariant: she describes it as being “perpendicular to AD” ([16]) and “through A” ([18]), so actually it is doubly-dependent on the dragged base point. The object she constructs therefore moves when she tries to drag A along it. This causes difficulties in Giu's exploration, since she does not seem to be able to distinguish (at a generic level) between a line defined through a point versus a point belonging to a line. In dynamic geometry these two situations are fundamentally different: while in the second case the point can only be dragged along the line (it would be linked to it and therefore dependent from it) and the line would not move, in the first case dragging the point makes the whole line move and only dragging in a particular way (which Giu was trying to do) can the solver obtain the perception of a point moving on a fixed line. Eventually, only after intense prompting aimed at fostering awareness of the different status of elements of the dynamic-figure, she reaches a GDP which is A-invariant and she formulates the following conjecture: "If A lies on the perpendicular line to KM through M, then it's a rectangle."

We consider the difficulties portrayed in this excerpt to depend on Giu's lack of generic awareness of the different status of objects of which the Cabri-figure is made. It seems as if she were conceiving the figure "statically", or still very much in paper and pencil mode, which leads to construction of an object that she imagines dragging A along, and that moves when A is dragged. In other words Giu seems to conceive the Cabri-figure as a whole, with properties that are analogous to paper-and-pencil properties, which are not related to movement, and thus not invariants according to our definition. In a static paper-and-pencil environment the situations "point on a line" or "line through a point" are represented in the same way, and therefore in a certain sense "equivalent", however in a DGS they are clearly different situations that the solver needs to become aware of. We advance the hypothesis that this might be causing her lack of generic awareness of the hierarchy of objects of the Cabri-figure, which of course implies not having control over the status of different objects in a Cabri-figure generated through the construction steps.

Excerpt 5.4.5

In this excerpt the solvers recognize “a line” in the trace mark left when they perform maintaining dragging using the base-point A. However lack of generic awareness of the hierarchy of elements of the dynamic-figure seems to block their construction of the IOD, since they are not able to provide a GDP which is A-invariant. The excerpt is taken from two students’ exploration of Problem 1.

Episode	**Brief Analysis**
[1] Gin: What did you want to do? [2] **Dav**: Eh, we should be able to move A... [3] Gin: [murmurs something] [4] Gin: you need to move it... [5] **Dav**: eh...hmmm... [6] **Dav**: no, it's not a...[murmurs something] [7] **Dav**: Yes, it is a rectangle...before it goes out. 	Dav seems to consider A a base point, but he wants to check by trying to drag it. He tries to perform maintaining dragging with “ABCD rectangle” as his III.
[8] Gin: I see...in the meantime...what movement it makes. [9] **Dav**: Yes, it could be ... [10] **Dav**: Only I think moving [11] Gin: Yes. [12] I: Eh, Gin, what movement is it 	Gin concentrates on the movement of the dragged-base-point, but does not seem to be able to describe any regularity at first. Then

making?	he gives a first GDP as
[13] Gin: Eh, I don't know, I don't	a line, which Dav
[14] **Dav**: Let's try to trace.	interprets as potentially
[15] Gin: It looks like a line, but I'm not sure.	a “very big circle”.
[16] Gin: I mean, I don't understand well.	
[17] **Dav**: Let's try now...or a very big circle.	
[18] Gin: I was...	
[19] **Dav**: No, I'm inside	Dav continues to
[murmuring] 	perform maintaining
[20] Gin: Ok, now you are.	dragging with the trace
[21] Gin: Hmm..out	activated.
[22] **Dav**: yucky!	
[23] **Dav**: Let's try on the other side.	
[24] Gin: Yes, it could be a line.	
[25] **Dav**: Yes, like a line.	
[26] Gin: It is a line ...uh...a line through	They refine their GDP
[27] **Dav**: It looks like a line on AB.	as a “line on AB” or “the
[28] Gin: Yes.	extension of AB”.
[29] **Dav**: Yes, it looks like a line 	
[30] **Dav**: The extension of AB.	However the solvers do
[31] **Dav**: why does it disappear?	not seem satisfied with
[referring to the trace mark disappearing]	this GDP. In particular
[32] Gin: Yes, but the extension of AB is a particular case.	Gin doesn’t think their
[33] Gin: it could be "any" extension of AB.	definition is cyclic ([37],

[34] **Dav**: when...eh, yes. [35] **Dav**: eh, when ...AB has to lie on... [36] **Dav**: when it is on the circle. [37] Gin: AB has to stay on the extension of AB seems to be a bit... [38] Gin:...forced [Italian: forzato]. [39] [they murmur something] [40] **Dav**: Otherwise what? What else can we say? AB...	[38]). Instead of refining the GDP Dav takes into consideration a new condition "when it is on the circle", and seems to be confused about how to give a different description of the proposed line.

Table 5.4.5: Analysis of Excerpt 5.4.5

The solvers seem to be able to perform maintaining dragging, and they perceive a regularity in the movement of the dragged-base-point (it moves along a "line" [15] or a "very big circle" [17]). The solver also seems to be using a bridge property as an III during maintaining dragging: keeping the rectangle inscribed in a circle they have drawn ([4]-[7]). The solvers spontaneously activate trace ([14]) to see what movement the base point is following ([8]), and they seem to recognize "a line" ([15]) or a piece of "a very big circle" ([17]). They then proceed to provide a more detailed GDP as a "line through AB" ([25]-[27]), and then as "the extension of AB" ([30]), which is dependent upon the base point being dragged. This seems to create difficulties for the solvers, and in particular Gin seems to be unsatisfied with this description ([37]), but is unable to provide an alternative one. We attribute this behavior to lack of generic awareness of the different status of objects of the Cabri-figure they are exploring.

What seems to confuse the solvers is the GDP being dependent on the elements through which they have defined it. Gin claims such a description is "forced [Italian: forzato]" ([38]), and the solvers start looking for different conditions, like all the vertices of the quadrilateral being on the circle ([36]) and are not able to describe an IOD that has to do with the regularity of the movement of the dragged-base-point. In this sense we consider the solvers to lack awareness of the status of objects at a generic and figure-specific level. Only through an intervention of the interviewer aimed at fostering such awareness will the solvers be able to formulate a conjecture coherently with the maintaining dragging scheme model. We will describe this in Chapter 6.

5.5 Some Spontaneous Behaviors for Overcoming Difficulties Related to Maintaining Dragging

In the previous sections of this chapter we introduced four components that seem to be necessary for expert use of maintaining dragging for conjecture-generation as described by our model. We argued that each component seems to be necessary in the process of conjecture-generation, and we did this by showing examples in which a lack of a specific component hinders or inhibits expert use of MD. In particular this involved analyzing cases in which the solvers' behavior was (completely or in part) not coherent with our model. In this section we describe two spontaneous behaviors that solvers exhibited in overcoming difficulties related to maintaining dragging. We consider such behaviors particularly interesting because they recurred during different solvers' explorations, in other words they were somewhat *general*. Moreover, after witnessing different spontaneous occurrences of the behaviors, we developed prompts to foster the behaviors in other solvers experiencing similar difficulties.

The first behavior, that we show in Excerpts 5.5.1 and 5.2.2, has to do with performing maintaining dragging. Let us quickly return to what seems to happen when this way of dragging is used. Once a property has been conceived as a potential III, in order to carry out maintaining dragging successfully, the expert solver seems to concentrate on the property to maintain and trust that the dragging strategy will allow him/her to “see” something. In order for the solver to observe the “something” arising from the movement of the dragged-base-point and/or from the trace mark s/he must simultaneously exercise haptic control – and therefore deal with the manual aspects this task – over the dragged-base-point, checking that the III is maintained at every instant, and concentrate on the movement of the dragged-base-point as a whole – and therefore deal with theoretical aspects of the task. In various cases we have observed solvers unable to drag because they seemed to feel the need to know “how to move” the chosen base-point, as if trying to control both the induction of the III and the perception of an unknown “way of moving” was too much to manage simultaneously. Some solvers spontaneously developed the following strategy: a separation of tasks involved in performing maintaining dragging. The solver holding the mouse would concentrate on maintaining the III, ignoring potential regularities in the movement of the dragged-base-point, while the solver watching would concentrate on perceiving regularities in the movement of the dragged-base-point. In this manner one solver would conquer the manual difficulties of inducing the III while the other solver could identify a GDP.

The second behavior we observed, and that we show in Excerpt 5.5.3, has to do with the construction of an object that represents a particular GDP identified by a solver. This behavior consists in constructing the object by “approximate” points that the solver marks on the screen “by eye [Italian: a occhio]”. For example, a solver might describe a GDP as “a circle” and then approximately mark various points on the screen that s/he thinks the circle goes through and construct a circle that seems to go through the marked points. This behavior arose particularly in cases in which the solvers would search for a GDP by generalizing from a discrete set of points, as, for example, described in Excerpt 5.2.2.

In Excerpts 5.5.4 and 5.5.5 of this section we show how we used the prompts we developed after analyzing the spontaneous behaviors, to help other solvers overcome similar difficulties.

<u>Excerpt 5.5.1</u>

This excerpt shows how two solvers spontaneously separate tasks involved in performing maintaining dragging.

Episode	**Brief Analysis**
[1] **Ste**: There. Ok. It'll be difficult. [He starts dragging A with the trace activated.] [2] **Ste**: umh [murmuring] [3] I: So Ste, what are you looking at to try to maintain it?	Ste is trying to maintain "ABCD rectangle" as an III by dragging the base point A.
[3] **Ste**: Uhm, now I am basically looking at B to do something decent, but… [4] I: Are you looking to make sure that the line goes through B? [5] **Ste**: Yes, exactly, otherwise it comes out too sloppy…	Ste explains that he is trying to maintain the perpendicular line to BD through A going through B, a bridge property for the III.
[6] I: and you, Giu, what are you looking at? [7] Giu: That is seems to be a circle… 	Giu, on the other hand, is concentrating on the trace mark.

Table 5.5.1: Analysis of Excerpt 5.5.1

Before the beginning of this episode the solvers had found a bridge property ("the perpendicular line to BD through A passing through B") for inducing the property "ABCD rectangle", the III they had chosen. In this excerpt the two solvers seem to be carrying out very different tasks: Ste is dragging, exercising haptic control over the figure, but concentrating on a very local property (the bridge property "the perpendicular line to BD through A passing through B") in order to do so; while Giu can concentrate on the figure as a whole and perceive the regularity in the movement of the dragged-base point, highlighted by the trace mark.

Excerpt 5.5.2

This excerpt is another example of how solvers spontaneously separate tasks involved in performing maintaining dragging in order to overcome difficulties related to this way of dragging.

Episode	**Brief Analysis**
[1] Giu: Try to maintain these things here. [2] **Ste**: It'll be hard. [3] Giu: You try! [4] **Ste**: eh, what am I doing? [5] Giu: There, more or less...yes, yes, yes, not too much, there. [6] I: In the meantime you, Giu, tell me what you are looking at. [7] Giu: Come on, come on... [8] Giu: Uhm...it seems to be a curve. Unless it's him who is	Ste is trying to simultaneously maintain the concurrence of the two circles and of the line through PD, a bridge property for the III "ABCD parallelogram". Giu is guiding Ste orally, helping him adjust the manual movements, and

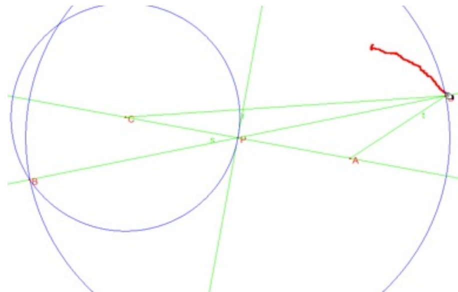
not able to do anything... [9] **Ste**: It's really hard! It moves!! [laughing] [10] Giu: I know. [11] Giu: I can only imagine...but I think that is it also, uh, that it is a circle...with center in A.  [12] Giu: and maybe with radius P. [13] Giu:...exactly... [14] **Ste**: What do you mean with center in A and radius P?! [15] Giu: AP! [16] **Ste**: Ah! No, eh, I didn't... [17] Giu: Radius a point is impossible! But... [18] **Ste**: No, I think the radius is AD necessarily, in any case, you should have AP equal to AD. [19] Giu: Maybe I also understand why...	simultaneously concentrating on the regularity in the movement of the dragged-base-point A. Once Ste has overcome some manual difficulties, he can shift his attention to the trace mark and to properties of the circle that Ste has proposed as a GDP.

Table 5.5.2: Analysis of Excerpt 5.5.2

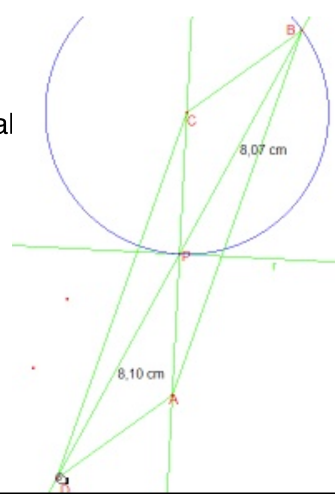
Controlling the simultaneous concurrence of the two circles and the line through PD during dragging is a manually-difficult task that seems to require all of Ste’s (the solver who is dragging) attention. Spontaneously Giu offers oral guidance during the dragging task, as he can concentrate on the figure as a whole, not having to exercise manual control over the figure. Moreover Giu takes on the task of interpreting the regularity in the movement of the dragged-base-point using the hint of the trace mark left on the screen.

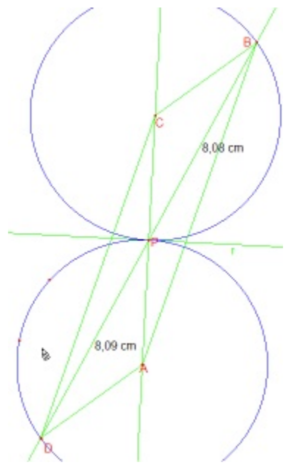
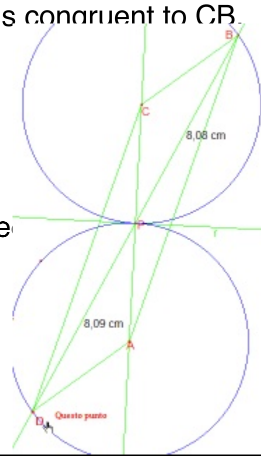
Through this collaboration the two solvers are able to overcome difficulties involved in performing maintaining dragging and perceiving an IOD.

Through first two excerpts we showed how some solvers spontaneously developed the strategy of separating some tasks involved in performing maintaining dragging. The solver holding the mouse would concentrate on maintaining the III, ignoring potential regularities in the movement of the dragged-base-point, while the solver watching would concentrate on perceiving regularities in the movement of the dragged-base-point. In this manner one solver can conquer the manual difficulties of inducing the III while the other solver can identify a GDP.

<u>Excerpt 5.5.3</u>

This excerpt shows a particular way of providing a GDP: the solver marks “good positions” on the screen and then connects them with a curve that he thinks represents the GDP. The constructed GDP therefore is an object that does not depend on the dragged base point.

Episode	**Brief Analysis**
[1] I: Ah, so you clicked the...	
[2] An: Eh, yes.	
[3] I: Ok.	
[4] An: Eh, I got the line instead of tal	
[5] I: Ah.	Giu searchers for
[6] I: Go on "undo".	positions of D in which
[7] An: Point D.	the measures of PB and
	PD seem to be the

[8] An: Where are you...there [murmuring] [9] [he murmurs as he reads the measures of the segments he has marked] [10] An: In the end...if I do a nice drawing...	same (up to two decimals).
[11] An: Yes, more or less we have it. [12] I: Hmm. [13] An: I'll do...a circle [as he constructs it] [14] An: Yes, that ...I was saying	He now perceives the points as belonging to a circle, which he proceeds to construct.
[15] An: If it is a parallelogram, the side, well, obviously the diagonals have to be congruent, they have to cut each other at their midpoints.  [16] An: But even more importantly the opposite sides have to be congruent. [17] I: Yes. [18] An: In this case, if we already know that AC are, uhm, well, AC is divided in half, is always divided in half [19] An: and we put as condition that AD is congruent to CB, so that D [20] I: uh huh [21] An:..lies on the circle that, uh, centere P [22] I: uh huh 	An perceives a property of the parallelogram he had not considered previously: opposite sides being congruent. This strengthens his belief of having found an appropriate condition for obtaining a parallelogram, that is D belonging to a circle centered in A and through P.

[23] An: We know that AD is congruent to CB..always. [24] An: And this puts us closer to the, uh [25] An: to the parallelogram...now let's see...moving D a bit...	An then proceeds to perform a dragging test.

Table 5.5.3: Analysis of Excerpt 5.5.3

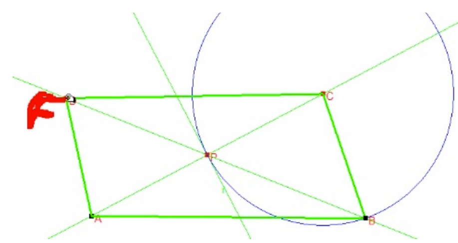
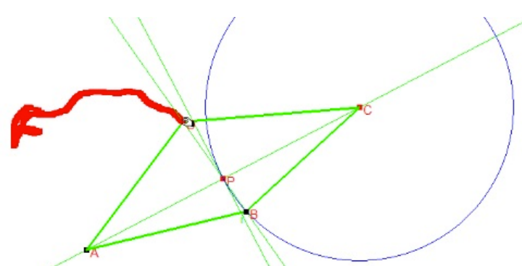
This second behavior seems to potentially help when maintaining dragging is difficult to carry out, or when a solver has trouble perceiving regularity in the movement of the dragged-base-point during maintaining dragging. Although An describes the circle he constructed as the circle centered in A and through P, he constructs it as the circle centered in P and through one of the points he had marked as a good position for D. A consequence of such choice is that the dragging test can only be performed "approximately". However this does not seem to bother the solver.

An "approximate" construction of a GDP can help in such cases, by providing visual feedback to check an initial idea. The constructed object is "approximate" in the sense that it depends on points that were placed "by eye". However it can provide good support for transitioning towards a new GDP that depends on the base points of the dynamic-figure that are not being dragged.

After we had observed solvers spontaneously use these first two behaviors to overcome difficulties they encountered when using the maintaining dragging scheme, we decided to develop two types of interventions aimed at inducing such behaviors in other solvers. We show how through these interventions solvers were able to overcome difficulties in the following two excerpts (5.5.4 and 5.5.5).

<u>Excerpt 5.5.4</u>

This Excerpt shows how prompting aimed at inducing a separation of tasks can be used to help solvers succeed in performing maintaining dragging and reaching an IOD. The solvers are exploring Problem 2.

Episode	**Brief Analysis**
[1] V: No [2] I: Eh, you Val maybe tell her a little more up, down, right, left...  [3] V: Go down...no, no, no up...up...up, up. [4] V: Go up moving a little [3] **M**: But it's not any more [4] V: go up moving a bit to the right... it's still a parallelogram. [5] V: Up, no, no, go like in diagonal a little....there. [6] V: There, perfect, a bit further down...like that. [7] **M**: Maybe... [8] V: a circle [together] [9] **M**: ...a circle  [10] V: With center A and radius AP? [11] **M**: Let's try to do it. [12] **M**: One second... [13] **M**: So, circle...with center in A	The interviewer prompts the solvers to help each other by asking the solver who is not dragging to orally guide the solver dragging. The solver who is guiding (Val) is able to consider the figure as a whole, while the solver dragging seems to concentrate locally on the point she is dragging. The solvers seem to simultaneously

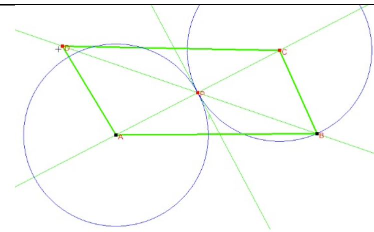
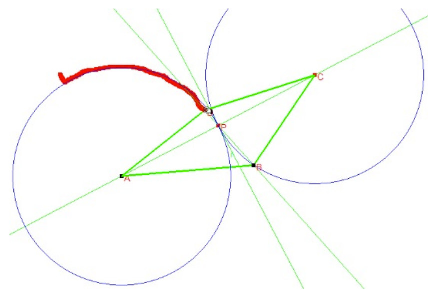
[14] **M**: center A, are you sure? [15] V: uh huh... [16] **M**: and radius AP. [17] **M**: Trace on D...Let's start from here and let me go...hey, tell me if it remains, ok? [18] V: Yes. [19] V: Yes, yes yes yes...yes, it remains.	 	recognize a circle. Val is able to provide a more precise GDP as a circle centered in A and with radius AP. M constructs the GDP and performs a dragging test.

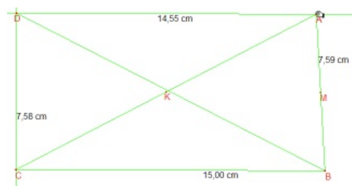
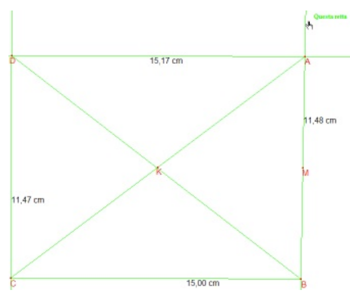
Table 5.5.4: Analysis of Excerpt 5.5.4

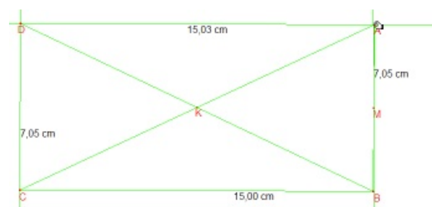
The interviewer's prompt seems to induce the appropriate behavior, the same that other solvers had spontaneously exhibited. One solver looked at the figure globally and was able to guide the other solver, who was dragging and concentrated on local properties of the figure. The separation of tasks led to success in conceiving a GDP and IOD. The solvers in fact perform a dragging test and formulate the conjecture: "If D belongs to the circle with radius AP, then ABCD is a parallelogram."

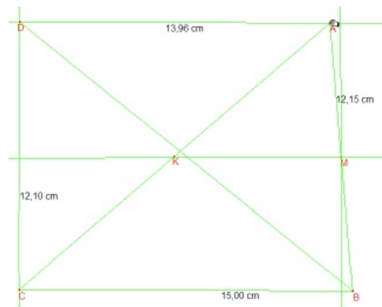
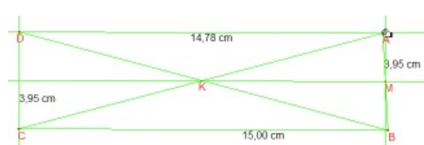
<u>Excerpt 5.5.5</u>

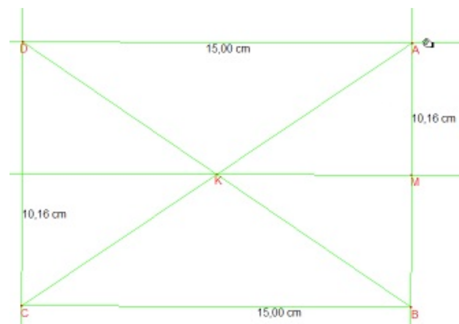
This Excerpt shows how an intervention that suggests the construction of an object that stays fixed during dragging of a certain base point seems to help the solvers overcome difficulties in providing a GDP and reaching an IOD for their conjecture.

Episode 1	**Brief Analysis**
[1] I: There, so you saw that moving A "up and down"...what is this "up and down"? [2] **Val**: Yes, well, uh, I mean that in any case...	The interviewer prompts the solvers to describe their observations by

[3] Ric: it has to move... [4] **Val**: AB has to remain parallel to DC, and yes, well, ABC has to be right... [5] I: uhm. [6] **Val**: ...always, and so you can do...extending segment AB.  [7] I: ...extending AB [thinking to herself]. So you say "drag A on the extension of AB"? [8] **Val**: Yes. [9] **Val**: Ah, ok [as she constructs a line through M and A]  [10] **Val**: Eh, but this line here varies when we vary...	using the same words they had used to describe the movement of the dragged-base-point, “up and down”. Val immediately provides basic properties as explanations, then she suggests “extending AB”. She tries to construct the object representing her GDP, but realizes that it moves when dragging A.
Episode 2 [11] I: So what do we need? An object that does not vary. [12] Ric: Yes. [13] **Val**: Eh no, because if you move A... [14] Ric: Then let's do... [15] **Val**: ...it is not a rectangle any more, I mean they are not right any more, the angles that we move. [16] **Val**: I mean... [17] Ric: I know but I wanted to add a point that does not move.	**Brief Analysis** Ric suggests drawing points that stay fixed, but is not able to carry out his suggestion. The interviewer then suggests drawing an object the “approximately” represents the GDP to

[18] **Val**: like... [19] I: ...the line that you drew...maybe for now you could draw it approximately [Italian: "a occhio"] and then we can see, and ...we'll keep it still and then let's see if we can redefine it in a better way. Ok [as Val constructs it]. [20] I: You think that it's more or less this line, right? [21] **Val**: More or less. [22] I: along which you to drag A. Ok.  [23] **Val**: More or less it looks like a rectangle. [24] I: More or less it seems like a rectangle. [25] **Val**: Yes.	then redefine it some other way. Val proceeds with the construction of a line through M and roughly in the "up and down" direction. The solvers seem satisfied.
Episode 3 [26] I: So how could we describe this line? [27] I: Who are the base points? [28] **Val**: A, M, K [29] I: uh huh [30] **Val**: Well, yes, it goes through M. [31] I: Ok. [32] **Val**: because M is the midpoint of AB. [33] I: Great. [34] **Val**: and it should theoretically be parallel to DC, but they derive from... [35] I: But DC..uhm.	**Brief Analysis** The interviewer prompts the solvers to focus on the base points of the figure to redefine the GDP. Val reaches a refined GDP. Although she defines the new GDP as a line through M and

[36] **Val**: Maybe ...it's perpendicular to KM. [37] I: Eh, so try. [38] **Val**: I should do...[while she constructs it]...perpendicular to the line I had constructed through M, it should go through K. I don't know if it does... [39] I: Ok. Yes, well in any case you did it approximately [Italian: "a occhio"]. [40] **Val**: Yes, well ok.  	perpendicular to KM she leaves the approximate line and constructs the perpendicular line to it through M, hoping it will go through K. There seems to be a lack of control over the status of objects, which leads to the next intervention of the interviewer.
Episode 4 [41] I: Ok, so try to maybe to the opposite. Construct KM and then do the perpendicular and see if it was exactly that one. [42] **Val**: So [constructing the line through KM] [43] **Val**: Through K...and M [44] I: Ok. [45] **Val**: Perpendicular to this through M. [46] I: Ok. [47] **Val**: Eh, it looks like it could do.  [48] I: It looks like it could... [49] **Val**: So A can be redefined on the line. [50] I: Ok and now you are testing the conjecture.	**Brief Analysis** The solvers seem satisfied and they redefine the dragged-base-point on the line representing their GDP, and test their conjecture with a robust dragging test. The conjecture they

<table>
<tr>
<td>
[51] I: Right? How was this conjecture? It was: if...
[52] Val: If, uhm A moves on a line through M and perpendicular to ...to the segment KM,
[53] I: Ok.
[54] Val: the figure is a rectangle, it remains a rectangle.
[55] Ric: Yes.


[56] I: Ok.
[57] Val: and also from the measures it looks like it because in any case, yes, well, the sides…
</td>
<td>formulate uses the IOD they have reached as premise and the interesting configuration (ABCD rectangle) as its conclusion, as described by our model.</td>
</tr>
</table>

Table 5.5.5: Analysis of Excerpt 5.5.5

In Episode 1 the solvers are facing difficulties providing an appropriate GDP, so the interviewer's first prompt is aimed at guiding the solvers to a new one that does not vary as the dragged-base-point moves ([11]) even if it might be constructed "approximately" for the time being ([19]). In her second prompt in Episode 2 ([19]), the interviewer remarks on how the description of the object the solvers are tying to deal with can be refined successively. The solvers seem to be satisfied with the "approximate" GDP that they construct in response to the interviewer's prompt. Then, in Episode 3 the interviewer tries to guide the solvers to re-describing the constructed line in terms of the base point of the dynamic-figure. This leads to the construction of a new object, a perpendicular line to the one constructed "approximately", that according to the solvers highlights another property that the figure – interpreted as a rectangle – should exhibit: this newly constructed line should go through K (and it does "by eye").

The solvers now are aware of conceptual properties that link the object representing the GDP to the base points of the dynamic-figure. Therefore in Episode 4 they respond to the last prompt by constructing a new object that represents the GDP appropriately. Moreover they are able to redefine the dragged-base-point upon it and perform a robust dragging test.

The prompting sequence used by the interviewer in these episodes is representative of the type of intervention that would be carried out during the interviews when solvers faced difficulties providing an appropriate GDP. In section 6.3 of Chapter 6 we will describe the interviewer's prompts in greater depth.

CHAPTER VI

A SECOND LEVEL OF FINDINGS: THE MAINTAINING DRAGGING SCHEME

Throughout the previous chapters we have mentioned "expert use" of maintaining dragging (MD), intending cases in which the exploration of the Cabri-figure that emerged from the steps of the construction proceeded according to our model. A key element, necessary for expert use of MD, seems to be conceiving MD as a tool that can help answer the question "what might cause the maintaining of the property I am interested in?" by leading to the answer "dragging a particular base point along a (generic) path that I will try to make explicit". A second key element that seems to be tightly connected to expert use of MD and with the response to the question above is the notion of *path*. In Chapter 4 we introduced the notion of path which we had conceived in our first description of the MD-conjecturing Model, and now we will present a further elaboration of such notion. In particular, we will distinguish two components of the notion of path: a "generic path" and a "figure-specific path".

In this chapter we will also describe how the becoming conscious of how an invariant may be induced by dragging a specific base-point along a generic path seems to belong to a meta-cognitive level with respect to the dynamic exploration being carried out. This meta-cognitive level seems to influence the interpretation of the phenomena that occur on the screen, and to control the whole development of the exploration process. Constructing this meta-level knowledge seems to allow some students to transition to using MD during an exploration, and exhibit expert behavior. In other words, having focused on searching for a cause for a certain type of quadrilateral to be maintained may guide the interpretation of the task in terms of developing conjectures in which the condition in the premise may be reached through MD. The description of expert behavior requires an extension of the analysis, from the cognitive level to the meta-cognitive level. While the figure-specific component of the notion of path resides at the cognitive level, the generic component of such notion, conceived as a *cause*, resides at the meta-cognitive level.

These considerations have led us to a new interpretation of "where" abduction may lie within our model. In section 6.2 we therefore introduce a new notion, that of *instrumented abduction*, describing the type of abduction that may be seen in explorations leading to conjecture-generation that feature expert use of MD. Moreover we describe instrumented abduction as a particular type of instrumented argument, which we also introduce in this section. Finally, in Section 6.3, we provide a glimpse into recurring aspects of a process of development of expert use of MD. We accomplish this by describing a possible sequence of prompts that was used by the interviewer to foster solvers' awareness about the use of MD for producing a conjecture, and that seemed to lead solvers to progress in a process of development of expert use of MD.

6.1 Elaborating the Notion of Path

In Chapter 4 we introduced the notion of *path* (Section 4.3) that we had elaborated during the data analysis we used to confirm and refine our model. New considerations about our idea of path, when focusing on expert use of the MD, have led us to further elaboration of the notion, that we will present in this section. We will do this through an example offered by an excerpt that we have previously introduced, and that we include again in this section for ease of the reader. In particular we will illustrate the generic and figure-specific components of the notion of path. For simplicity we will refer to these components as *generic path* and *figure-specific path*.

This distinction between “generic” and “figure-specific” has revealed to be very effective in the analysis of solvers’ explorations. On one hand the idea of generic path seems to play a key role in the development of expert use of MD. While on the other hand, the notion of figure-specific path seems to be very helpful when analyzing the emergence of elements related to the path in a *specific* exploration. Moreover, we note and recall that this distinction has been quite useful in the description of other factors involved in the exploration process leading to the conjecture, that we described in Chapter 5. For instance the distinction between generic and figure-specific awareness of the status of objects of the Cabri-figure allowed us to analyze and explain different difficulties that solvers encountered during their explorations (Section 5.4).

Let us recall the first two episodes of Excerpt 4.3.2 that we introduced and analyzed in Chapter 4.

Episode 1	**Brief Analysis**
(0:41) **F**: exactly. [he drags D a bit, in a way that looks like he is trying to maintain the property parallelogram] (0:48) G: you see that if you do, like, maintaining dragging … trying to keep them [the diagonals] more or less the same… (0:57) **F**: exactly [murmuring]… well, okay.	F and G decide to use maintaining dragging to investigate “when ABCD is a parallelogram” (intent repeated in (2:41) and (3:05)). In a previous episode they have noticed that the property “ABCD parallelogram” can be substituted with the sufficient property “diagonals of ABCD congruent”, a bridge property (0:48).

…	
Episode 2 (2:41) **F:** For the parallelogram, uh, let's try to use "trace" to see if we can see something. G: go, let's try [speaking together with him]…uh, "trace" is over there.	**Brief Analysis** F proposes to activate the trace in order to "see something" (2:41).

Table 6.1.1: Analysis of Excerpt 6.1.1

In Episode 2 F says: "For the parallelogram, uh, let's try to use "trace" to see if we can see something" (2:41). There seems to be "something" he is referring to that has to *happen for the parallelogram to happen*. At this point it is not important "what" the "object" along which dragging will occur is, but that such an *object exists* in the mind of the solver. Moreover, the analyses of this and of other transcripts showed that for solvers to apply MD they need to have conceived "something" to look for. We describe this behavior as "anticipation of a *generic path"*, and it seems to occur through an objectification of the movement induced on the dragged-base-point. This seems to be a key aspect of expert use of MD.

On one hand this "something" at this point of the exploration is not associated to a particular geometric shape (or curve), in this sense it is a *generic* idea. Therefore it is what we have identified as *generic path*. On the other hand, this generic idea may be developed during the solvers' experience with MD, and it may lead the solvers to determine a *figure- specific path.* The anticipation of a figure-specific path can be seen in the excerpt above when the solvers state their intention to use the trace tool to help them "see" something (2:41).

6.1.1 Generic Path and Figure-Specific Path

In the previous paragraph we introduced the distinction between "generic" and "figure-specific" to describe different components of the notion of path. We now provide definitions for these two components. We will refer to *generic path* as the condensation of a complex relation of elements:

> *a movement of a base point that is recognized to be regular and causing an interesting property to be maintained, the possibility of describing such movement through a property of points belonging to a potential trajectory, that is positioning the dragged-base-point on any point of such trajectory the III is visually verified.*

This characterizing property leads from a dynamic conception to a static one, allowing the movement to be objectified into a static whole. We define the *figure-specific path* to be

> *the particular set of points that satisfy the characterizing property described above, and that is related to the particular Cabri-figure being considered.*

The "figure-specific path" is what may be recognized as a particular geometrical curve and described geometrically in the GDP.

In the example above, the idea of generic path seems to be present since the solvers talk about "something" to look for, while the figure-specific path is what becomes explicit through the trace mark. The figure-specific path can be described as a geometrical object with the property that when the dragged base point D is on "it", in this particular exploration "the parallelogram happens" or the property "ABCD parallelogram" is (visually) verified. As the exploration continues, the solvers interpret the trace mark as the figure-specific path, providing various GDPs until they reach a satisfactory one as the circle with center in A and radius AP. We would like to stress how the trace mark is *not* the path, but a new image that may provide hints as to what an appropriate geometric description of the figure-specific path may be, that is what form the generic path might take on in this specific case.

In Chapter 5 when analyzing solvers' difficulties in conceiving a property as an III (Section 5.2) we frequently referred to the issue of "conceiving a path" and how this seemed to influence expert use of MD. We may now look at such difficulties as being related to conceiving a generic path. In particular, Excerpt 5.2.4 showed an example of behavior that was consistent with various steps of our model, but in which the solver was not able to make sense of his findings. Excerpt 6.2.2 will show another example of a similar behavior, which we will interpret in terms of difficulties in conceiving a generic path. In the following paragraph we discuss conceiving a generic path and contend that it is a key notion which is necessary for developing expert use of MD, and a notion that, like the necessary component "generic control over the status of objects" resides at a meta-level with respect to the specific exploration. This means it has to be developed a priori with respect to the exploration in order for the solver to exhibit expert behavior in performing MD for generating-conjectures according to our model. On the other hand, conceiving a figure-specific path resides at the level of the specific exploration, and occurs during the exploration, much like the development of figure-specific awareness of the status of different objects of the Cabri-figure.

6.1.2 Conceiving a Generic Path

We find it useful to view the process of becoming experts in using MD for conjecture-generation and making sense of an exploration in terms of the developing the notion of generic path as a response to conceiving a new task to solve. This new task consists in searching for a cause for the III to be maintained (see Section 4.3). We discuss this idea of searching for a cause and what cognitive processes it leads to more in depth in Section 6.2, while here we focus on the necessity of conceiving a generic path in order to look for and make sense of the soft invariants that emerge during the exploration. Conceiving the idea of generic path is necessary because it "incorporates" both the III, since dragging along such object makes the III visually verified, and the potentiality of an IOD, since a regularity may emerge as the movement of the dragged-base-point along a trajectory that may be described geometrically.

Moreover, the generic path has aspects that belong to the phenomenology of the DGS and aspects that belong to the world of Euclidean geometry, so it provides a bridge that can guide the interpretation of the experience within the phenomenology of the DGS in geometrical terms. Specifically, the generic path has a "dynamic" nature in that it can be conceived as a trajectory along which a point can be dragged, and dragging along such trajectory "makes something happen". That is, the generic path is part of an "action" that leads to phenomena that occur in particular ways and times: we described how the roles of simultaneity and a feeling of "control" are fundamental in making sense of the exploration within the phenomenology of the DGS. Therefore within the phenomenology of the DGS the generic path withholds the seed of a causal link between the invariants perceived during the exploration. However the generic path can also be conceived as a continuous set of points that a certain point of the Cabri-figure may "belong to". Such "belonging to" is a geometrical property that may be perceived as "the condition" for a second property to be verified, since this is exactly the defining property of the points of such continuous set. In other words, the generic path can be considered, within the phenomenology of a DGS, as a trajectory with respect to movement, a movement that coordinates the dragged-base-point with the III, causing the III to be visually verified. In geometry, this trajectory which becomes figure-specific, may be seen as a geometrical object that a point can belong to, a mathematical locus (in the cases in which the belonging of the point implies an “if and only if” with the geometrical property expressed by the III, otherwise it is a subset of the locus), a condition for a second property to be verified.

As discussed above, in order to develop the idea of generic path, it seems to be necessary to have developed awareness of the fact that an answer to the “search for a cause” (Section 4.3 and 6.2) for the III to be maintained within the phenomenology of the DGS may be found as a regularity in the movement of a base point that can be induced by the solver, and that can “make” the III be visually verified. Such awareness is dependent upon another form of awareness, that of the dependencies of objects of a Cabri-figure from one another, and of how these dependencies influence the behavior of the dynamic-figure. Awareness of these dependencies, or of the different status of objects of the Cabri-figure, as we called it in Chapter 5, must be previously developed by the solver at a generic level. Therefore we now have introduced another reason why the component “awareness of the different status of objects of the Cabri-figure”, introduced in Chapter 5, is necessary for carrying out an exploration using MD as an expert. Moreover, as for generic and figure-specific awareness of the status of objects of the Cabri-figure, the figure-specific component of path can be developed “on the spot” and fostered easily through prompting if the solver has developed the idea of the generic component. On the other hand, the idea of generic path needs to be developed a priori with respect to the specific exploration, it is more difficult to foster, and it may take a longer time to develop.

6.2 Argumentative Processes in the Model: Where is Abduction Situated? The Notion of Instrumented Abduction and of Instrumented Argument

In the previous section we introduced the idea of a meta-level of sense-making, necessary for the development of expert use of MD. In particular we discussed the necessity of developing the idea of *generic path* in order to make use of MD in a manner that seems to be coherent with our model. As mentioned in the introduction to this chapter, another idea that seems to be necessary for the development of expert use of MD, leading to making sense of what emerges during an exploration, seems to be conceiving MD as a tool that may help answer the question "what might cause the property I am interested in to be maintained". This question paired with the developed notion of generic path allows the solver to search for a cause of the maintaining of the III as dragging the considered base point along a path which will have a figure-specific description in each particular exploration, depending on the construction, the property chosen to maintain, and the base point chosen to drag.

We have introduced the idea of developing the subtask of "searching for a cause" in Chapter 4 and used it to analyze transcripts in Chapter 4 and Chapter 5. Here we highlight how setting this subtask resides at a meta-level with respect to the particular exploration being carried out, and it seems to be a key intuition leading to becoming an expert with respect to MD. We may now describe expert solvers as solvers who have developed the necessary meta-level knowledge related to the use of MD, specifically the notion of generic path and the idea of using MD to "search for a cause". With this in mind, we re-analyzed the solvers' explorations and conceived the following table describing how the explorations could now be classified. The two components we considered are the presence of the "meta-level knowledge", or in other words, whether the solvers were experts or not, and how MD was used. We described the use of MD as “use that leads to an III and an IOD”, and “no (or incomplete use of MD”. The first use is present in cases in which MD allowed the solvers to behave coherently with our MD-conjecturing Model up to (at least) what, to an external observer, might have seemed an IOD. While we classified as cases with “no (or incomplete) use of MD” cases in which the solvers did not use MD or tried to use it but could not reach an IOD and abandoned this type of dragging.

	MD that leads to an III and an IOD	**no (or incomplete) use of MD**
meta-level not present	*non-expert use of MD* Excerpt 6.2.2 in this section	*no use of MD or use inhibited by difficulties* Various cases of non-appropriation described in Chapter 5
meta-level present	*expert-use of MD* Excerpt 6.2.1 in this section (which is also Excerpt 4.3.1 of Chapter 4)	*interiorized MD* Excerpt 6.2.3 in this section (which is the continuation of Excerpts 4.2.5 and 4.3.2)

Table 6.2.1 Different uses of MD together (or not) with the presence of the meta-level.

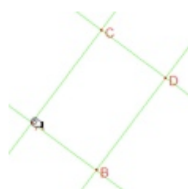
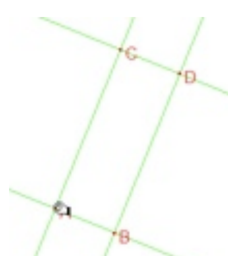
In Chapter 5 we have already discussed cases in which the meta-level was not present and MD was not carried out thoroughly by the solvers. In Chapter 4 we have shown expert use of MD and so we will re-analyze one of these excerpts here (Excerpt 6.2.1) to show how the presence of meta-level knowledge leads to the formulation of conjectures as described by our model. We will use such excerpt to describe how sense is made of the elements that emerge during the exploration, and how once solvers become experts, use of MD can become “automatic”. We compare such behavior to that described in an excerpt in which two solvers were not able to make sense of their findings even though these were coherent with the MD- conjecturing Model (Excerpt 6.2.2). We argue that in this second case the meta-level understanding is not present, in other words appropriation is not complete and a conjecture that puts together the findings cannot be formulated. Moreover, we use such examples to advance our hypothesis that when considering expert use of MD the abduction that previous research has focused on (Arzarello et al., 2002) is "incorporated" into the meta-level knowledge and in the *utilization scheme* developed by the solver with respect to MD, the maintaining dragging scheme (MDS), and it no longer occurs at the level of the exploration.

Finally, we take our considerations one step further and describe how a different form of "expert use" of MD may occur even when MD is not actually used. That is, we have evidence (in Excerpt 6.2.3) that some solvers *interiorize* the MD-artifact to the point that it becomes a psychological tool (Vygotsky, 1978, p. 52 ff.) and no longer needs to be supported by the physical enactment of MD. In this case abduction does seem to occur at the level of the exploration, allowing the conception of a second invariant property which plays the role of the IOD described in the maintaining dragging scheme (MDS).

<u>6.2.1 Expert and Non-expert Use of MD</u>

Let us first analyze a case of expert solvers using MD to reach what, to an external observer, seem to be an III and an IOD, consistently with our MD-conjecturing Model. Using MD the perception of a second invariant, the IOD, can occur in a rather automatic way. As a matter of fact, when MD is possible, the IOD may "automatically" become "the regular movement of the dragged-base-point along the curve" recognized through the trace mark, and this can be interpreted geometrically as the property "dragged-base-point belongs to the curve (described through a GDP)". In Excerpt 4.3.1, which we include below for ease of the reader, we saw how two experts, Giu and Ste, reached a conjecture through MD coherently with our model. They seem to behave in an "automatic" way, that is, the solvers proceed smoothly through the perception of the III and the IOD and immediately interpret them appropriately, as conclusion and premise respectively, in the final conjecture.

Excerpt 6.2.1 (also Excerpt 4.3.1). This Excerpt is taken from Giu and Ste's exploration of Problem 4. The solvers followed the steps that led to the construction of ABCD, as shown in Figure 6.2.1, and soon noticed that it could become a rectangle. Ste was holding the mouse (as shown by his name being in bold letters in the excerpt below), and followed Giu's suggestion to use MD to "see what happens" when trying to maintain the property "ABCD rectangle" while dragging the base point A. In such situation the selected property "ABCD rectangle".

Episode 1	**Brief Analysis**
[1] **Ste**: I have to make it so that the... [2] Giu: B stays [3] **Ste**: that...uh, B remains on the intersection. [4] Giu: Exactly. [5] **Ste**: which is...I mean I have to drag this, right? [6] I: Maintaining the property rectangle...  	The solvers resort to the bridge property (see section 4.2.1.3) "B on the intersection" ([3]) to make the task of maintaining dragging easier. The solvers have chosen "ABCD is a rectangle" as an III.
Episode 2	**Brief Analysis**
[12] **Ste**: Identical...ta-ta-ta-ta...ta-ta-ta [13] I: Giu, what are you seeing? [14] Giu: Uhm, I don't know...I thought it was making a pretty  	While Ste is concentrated on maintaining the III ([12]-[14]), Giu seems to be looking for a GDP, and recognizes a continuous curve ("pretty precise curve" [14])

precise curve...but it's hard to ...to understand. We could try to do "trace" [15] **Ste**: trace! [16] Giu: This way at least we can see if...	instead of discrete positions. He then wants to better understand ([14]) and "see" ([16]), so he proposes the use of the trace tool ([14]).
Episode 3 [17] **Ste**: Where is it? [18] Giu: Uh, if you ask me... [19] **Ste**: Trace! [they giggle as they search for it in the menus] [20] **Ste**: Trace of A... …	**Brief Analysis** After the trace is activated ([17]-[20]) Ste starts maintaining dragging again.
Episode 4 [28] I: So Ste, what are you looking at to maintain it? [29] **Ste**: Uhm, now I am basically looking at B to do something decent, but...  [30] I: Are you looking to make sure that the line goes through B? [31] **Ste**: Yes, exactly. Otherwise it comes out too sloppy... [32] I: and you, Giu what are you looking at? [33] Giu: That it seems to be a circle...	**Brief Analysis** Ste is using the property "the line goes through B" as his III ([29], [30]). Both students show the intention of uncovering a path by referring to "it" ([31], [33], [34]). Giu, in particular concentrates on describing the path geometrically

[34] **Ste**: I'm not sure if it is a cir... [35] Giu: It's an arc of a circle, I think the curvature suggests that.	and he seems to recognize in the trace a circle ([33]) or an arc of a circle ([35]).
Episode 5 [36] **Ste**: Yes, but.. [37] Giu: But passing through B [38] **Ste**: Ah yes, B [39] Giu: B because it can also become a line [40] **Ste**: Yes, it could be B. [41] **Ste**: I would dare to say with center in C?...no, it seems more, no. [42] **Ste**: It seemed like  [43] Giu: No, the center is more or less over there...in any case inside [44] **Ste**: Hmm [45] Giu: Ok, do half and then more or less you understand it, where it goes through. [46] **Ste**: But C is staying there, so it could be that BC is...is [47] Giu: right! because considering BC a diameter of a circle, [48] **Ste**: Well yes, actually it passes through C also because if then I make it collapse, uh,	**Brief Analysis** The solvers' attention seems to shift to the mark left on the screen by the trace. Now that a first GDP is given, the solvers try to ameliorate the description by adding properties: "(a circle) passing through B" ([37], [38], [39], [40]), "with center in C" ([41]), with BC as a diameter ([46], [47]). As Ste continues to drag, Giu checks and confirms the suggested properties and tries to justify them providing argumentations based on visual observations, recognition of geometrical properties, and the knowledge of particular theorems ([49], [55]). Ste seems to have some difficulty

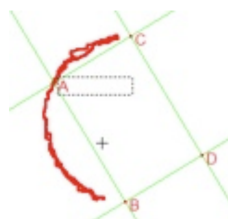
… [59] **Ste**: Well... [60] **Ste**: I wouldn't call this...aaaa...there [61] **Ste**: No, but it jumps, when it's closer it's easier.	dragging as he drags A closer to C, but is able to overcome the manual difficulty.
Episode 6 [62] **Ste**: It surely can look like a circle. [63] Giu: Well, in theory...you can see it goes through B and C. [64] I: Ok, are you sure of this? [65] Giu and Ste: Yes.  … [They construct the circle and drag A along it, and then they write the conjecture: “ABCD is a rectangle when A is on the circle with diameter BC.”]	**Brief Analysis** Ste continues to drag and both solvers seem to be checking the proposed GDP, confirming it ([62], [63]) with considerable confidence ([65]).

Table 6.2.2: Analysis of Excerpt 6.2.1

Giu seems to be looking for something, which he describes for the time being as a “pretty precise curve” ([14]). This intention seems to indicate that Giu has conceived a generic path. Moreover he is trying to “understand” ([14]) what the figure-specific path might be, that is he is searching for a geometric description of the path (GDP). To do this he suggests activating the trace tool. Giu then identifies a regularity in the movement of the dragged-base-point, “a pretty precise curve” ([14]), then “a circle” ([33], [34]) “considering BC a diameter” ([46], [47]). This seems to all occur in a smooth, “automatic” way. The solvers have used MD before and exhibit expert behavior which in this case guarantees a transition to the conjecture with no apparent difficulties involved.

Reaching expert behavior is not trivial, as shown by the fact that many solvers we interviewed did not seem able to make sense of their discoveries even when they appeared to be using MD in a way that seemed to lead to the perception of an III and an IOD. In particular, even when invariants are perceived, it seems that their simultaneous perception does not guarantee the interpretation of such phenomenon in causal terms. Moreover, putting the geometrical properties which correspond to the III and the IOD in a conditional relationship with each other within the world of Euclidean geometry is not always straightforward. The excerpt below shows a case in which two non-expert solvers have used MD maintaining the property “ABCD rectangle” as their III dragging A, they have provided a GDP and perceived the invariant “A on the circle” as an IOD. However they do not seem to make sense of what they have discovered.

Excerpt 6.2.2. In this excerpt the solvers carefully carry out maintaining dragging with the trace activated and reach a GDP, that they seem to use in constructing an IOD and in performing what seems to be a dragging test. However they do not consider the IOD in the conjecture that they formulate; instead they go back to a basic conjecture they had used previously. One solver even explicitly refers to what she sees now as “like we said before”, and seems to completely ignore the circle that has appeared and that was constructed.

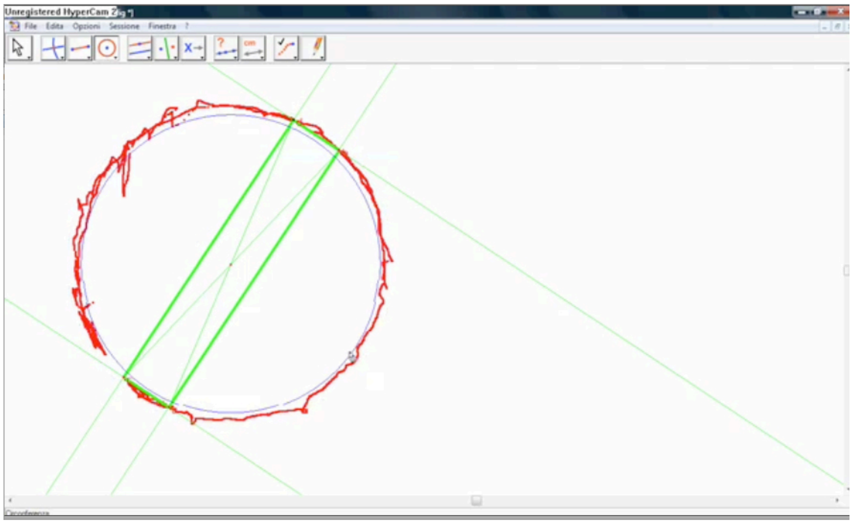


Figure 6.2.2 A Screenshot of the solvers' exploration

Episode 1	**Brief Analysis**
[1] Ila: passing through...through...oh my goodness! [2] **Val**: no. [3] Ila: Yes, but make it go through, eh, it isn't... [4] Ila: I mean you have to ... [5] **Val**: Do "control Z" [6] Ila: Nooo! [7] **Val**: But ok, it doesn't matter! [8] Ila: Circle... [9] Ila: Good! We have seen that it follows. [10] **Val**: Yes, this is the trace...in brief.	The solvers have constructed a circle that is not A-invariant and seem to be trying to compare it to the trace mark that they have obtained through MD.
Episode 2	**Brief Analysis**
[11] **Ila**: But...wait, because there there are points. [12] Val: what points do you have to make? [13] **Ila**: Well,...oh dear! No.	The solvers do not seem to be convinced of what they have

[14] Val: Wait. [15] **Ila**: I am tracing now... [16] Val: Yes. [17] Val: Move A on the circle. [18] **Ila**: Eh! [19] **Ila**: You look to check that it stays... [20] Val: There, it remains, it remains a parallelogram. [21] Val: Yes, I mean a parallelo...it remains a rectangle. [22] **Ila**: a rectangle. [23] Val: Yes, more or less. [24] **Ila**: Yes, ok. [25] **Ila**: But... [26] Val: Ok....why? [27] **Ila**: Because... [28] Val: Why?	found. Val proposes to try to drag A "on the circle" even thought the circle is not A-invariant. They seem to notice that the properties "A on circle" and "ABCD rectangle" occur simultaneously. However the solvers do not seem to be able to make sense of this.
Episode 3 [29] Val: So...I know that, uh, so [30] **Ila**: But B has to always be in that point there. [31] Val: Where? [32] Val: So I think...this remains a rectangle [33] Val: ...when AB is perpendicular to DC, ok but in this case it would also be BA is equ, perpendicular to CA. [34] **Ila**: Basically, uh, it's like we said before [35] Val: and...	**Brief Analysis** Ila tries to make sense of the behavior of the figure, but she does not seem to be able to. Val then suggests the same property as they had used in a previous basic conjecture and Ila

[36] **Ila**: No that basically uh DB has to always be parallel to CA, and, uh the segments AB, also AB, AB, no we had the points... [37] **Ila**: Wait...this was fixed...these two were, right! I mean that CA and DB have to always be parall, uh perpendicular to, uh... [38] **Ila**: to the line, uh, parallel to DC.	seems to agree. The solvers end up "explaining" the exploration through a basic conjecture containing this property they had previously used.

Table 6.2.3: Analysis of Excerpt 6.2.2

The solvers seem to have conceived a figure-specific path, and they even manage to provide a GDP which is independent from the base point being dragged ([8]-[11]). Val suggests to move A on the circle ([17]) and she notices that in this case it "remains a rectangle" ([20]-[21]). The solvers seem to agree and we would think they have successfully performed a soft dragging test, having proceeded according to our model. However when they start asking themselves "why" ([26], [28]) they seem to exhibit confusion. Ila starts talking about point B ([30]) and they start discussing properties of the figure as a whole, looking at sides of the quadrilateral, and recognizing only "what we said before" ([34]), that is a basic conjecture involving DB being parallel to CA ([36]). The solvers do not seem able to make sense of what they have discovered in terms of what we describe in our model.

Although they seemed to have used an III and conceived an IOD during the exploration, there does not seem to be understanding at the meta-level which allows the interpretation of the IOD as a "cause" for the maintaining of the III. In other words, they do not seem to have conceived a generic path. This can also be inferred from the solvers' insistence on trying to conceive "why" ([26], [28]). Even though this question might have arisen out of surprise as to "why" a circle (the figure-specific path), it seems that it also refers to the meta-level of "why dragging along a path" would guarantee the maintaining of an invariant, an important aspect of the generic path. In any case the solvers do not seem to be aware of the meta-level relationship between the arising invariants.

The solvers do not seem to be able to establish a connection between the static property they were using to characterize the rectangle (AB parallel to CD) and the idea of dragging the base point they were considering along a path. In particular they do not seem to have developed the idea of *generic path*, so they are unable to interpret the property "A on the circle" as a *cause* for the property "ABCD rectangle" to occur.

6.2.2 The Notion of Instrumented Abduction

Unlike Ila and Val, expert solvers seem to withhold the key for "making sense" of their findings, which seems to be conceiving the IOD as a *cause* of the III within the phenomenology of the DGS, and then interpreting such cause as a geometrical *condition* for the III to be verified. In other words, the solvers establish a causal relationship between the two invariants generating – as Magnani says – an explanatory hypothesis (Magnani, 2001) for the observed phenomenon. Moreover, as soon as they decide to use MD to explore the construction, experts seem to *search for a cause* of the III in terms of a regular movement of the dragged-base-point. This idea is key; it seems to lie at a meta-level with respect to each specific investigation the solvers engage in, and possessing it, together with the notion of *path*, seems to lead to expert behavior with respect to MD, culminating in the formulation of the conjecture. If we consider MD to be an instrument with respect to the task of conjecture-generation, we can consider the utilization scheme associated to it, which we will call the *maintaining dragging scheme*

(MDS). The MDS is described by our model, and we will from now on refer to “expert behavior” as exploitation or use of the MDS.

We mentioned above that the process of conjecture-generation as described by our model seems to become “automatic” for expert solvers. Moreover automatic use of the MDS seems to condense and hide the abductive process that occurs during the process of conjecture-generation in a specific exploration: the solver proceeds through steps that lead smoothly to the discovery of invariants and to the generation of a conjecture, with no apparent abductive ruptures in the process. Thus our research seems to show that,

> *for the expert, the abduction that previous research described as occurring within the dynamic exploration occurs at a meta-level and is concealed within the MD-instrument.*

We introduce the new notion of *instrumented abduction* to refer to the inference the solver makes when exploiting the MDS to formulate a conjecture.

<u>6.2.3 Interiorization of MD</u>

We now take our reflections on the MDS one step further. We have found evidence that experts may use the MDS as a “way of thinking” freeing it from the physical dragging-support. In the following excerpts we will show how the MDS guided the process of conjecture-generation of two experts, F and G, even though they were not able to reach an IOD through MD.

<u>Excerpt 6.2.3</u>

The solvers were working on Problem 2. Excerpts 4.2.5 and 4.3.2 are taken from the solvers’ exploration that originated from this Problem. We provide summaries of what was is contained in such excepts and we pick up from the end of Excerpt 4.3.2. Then we provide excerpts from how the exploration ended. We refer to the sequence of Excerpt 4.2.5, Excerpt 4.3.2 and the additional excerpt as Excerpt 6.2.3. In Excerpt 4.2.5 the solvers identify a basic property, slim it down to a minimum basic property, which they use to obtain the configuration they are interested in. Excerpt 4.3.2 shows the solvers' belief in the existence of a path and traces of an implicit idea for the GDP. However the conceived GDP doesn't seem to correspond to what they observe during the maintaining dragging. The solvers want to therefore make the path explicit through activation of the trace, and they use the trace to reject an incorrect GDP. The lines of the transcript are marked by their times relative to the beginning of the excerpt in order to show the development over time of this part of the investigation. In particular we chose to cut parts of the exploration in which the solvers were not investigating "the case of the parallelogram", as they refer to it. The bold refers to the solver who is holding the mouse.

Episode 1	**Brief Analysis**
[1] (3:05) G: and now what are we doing? Oh yes, for the parallelogram? [2] (3:07) **F**: yes, yes, we are trying to see when it remains a parallelogram. [3] (3:18) G: yes, okay the usual circle comes out. [4] (3:23) **F**: wait, because here…oh dear! where is it going? [5] (3:35) I: What are you looking at as you drag? [6] (3:38) **F**: I am looking at when ABCD is a	G reminds himself what their intention was and seems to be concentrating on the movement of the dragged-base-point, while F, who is dragging, concentrates on maintaining the property "ABCD parallelogram" (3:07). G (too?) quickly proposes a GDP (3:18). It is not clear what "usual" refers to: maybe to a previous investigation. However what F sees does not seem to be the circle he had

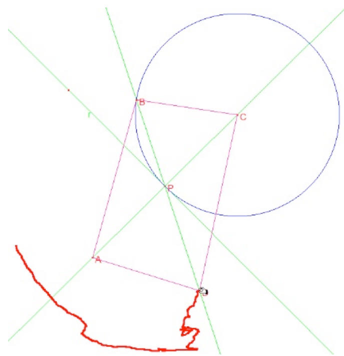
parallelogram. You try [handing the mouse to G] … [16] **F**: So, maybe it's not necessarily the case that D is on a circle so that ABCD is the parallelogram.  … [40] **F**: Because you see, if we then do a kind of circle starting from here, like this, it's good it's good it's good it's good, and then here… see, if I go more or less along a circumference that seemed good, instead it's no good…so when is it any good?	in mind (maybe the circle centered in P with radius AC) and he appears unhappy and confused when he does not understand "where it is going" (3:23). After repeating his intention of investigating "when ABCD is a parallelogram" (3:38) F hands the mouse to G, asking him to try. F and G seem to have conceived a GDP ([3]) that does not coincide with the trace mark they see on the screen as F performs MD ([4]). This leads the solvers to reject the original GDP ([16]) and search for a new condition ("when" [40]).

Table 6.2.4: Analysis of Excerpt 6.2.3

The solvers are not able to reach a condition for ABCD to be a rectangle using MD because of manual difficulties they encounter as the exploration continues. This leads G, who is not holding the mouse, to conceive a condition without external support from the MD-instrument as shown in the following excerpt.

Episode 2	**Brief Analysis**
[43] G: eh, since this is a chord, it's a chord right? We have to, it means that this has to be an equal chord of another circle, in my opinion with center in A. because I	The solvers' search for a condition as the belonging of D to a curve defined through

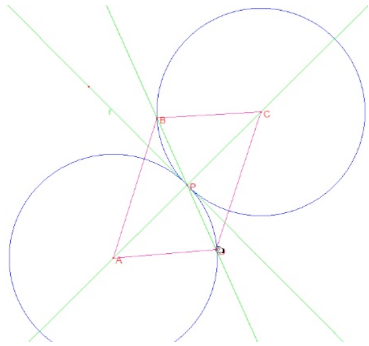
think if you do, like, a circle with center [44] **F**: A, you say… [45] G: symmetric with respect to this one, you have to make it with center A.  [46] **F**: uh huh [47] G: Do it! [48] **F**: with center A and radius AP? [49] G: with center A and radius AP. I, I think… [50] **F**: let's move D. more or less… [51] G: it looks right doesn't it? [52] **F**: yes. [53] G: Maybe we found it!	other base points of the construction is now complete, as they construct the circle with center in A and radius AP and proceed to link D to it in order to check the CL. The solvers seem quite satisfied and formulate their conjecture immediately after the dragging test, proceeding in accordance to the MDS model.

Table 6.2.5: Analysis of Excerpt 6.2.3

Although the "search for a cause" through use of MD with the trace activated failed, the solvers are able to overcome the technical difficulties and reach a conjecture by conceiving a new GDP without help from the MD-instrument. In other words the solvers seem to have interiorized the instrument of MD to the extent that it has become a psychological tool which no longer needs external support. Moreover the abductive process supported by MD in the case of an instrumented abduction now occurs *internally* and is supported by the theory of Euclidean geometry (BP and PD are conceived as chords of symmetric circles). Taking a Vygotskian perspective (Vygotsky, 1978, p. 52

ff.), we can say that the MD has been internalized and the actual use of the MD-artifact has been transformed, becoming internally oriented, into a psychological tool.

Concluding Remarks. Summarizing, we have now seen examples from each of the four situations described by our table. In particular, the model of the MDS seems appropriate for describing the processes of conjecture-generation when MD is used by experts, providing evidence to a correlation between the introduction of the dragging schemes, and MD in particular, and a specific new (with respect to those in literature) cognitive process described by the model. We have referred to such process as a form of *instrumented abduction*, a new notion that we hope can be generalized to other contexts in which abduction is supported by other instruments. Furthermore, we seem to have captured the key ideas which may lead to developing and using the MDS, and we described how these key ideas reside at a meta-level with respect to each specific exploration in which MD is exploited by experts. Finally we described how for expert solvers the MDS might be transformed into a way of thinking that can take place when MD is not used at all. In this sense it may lead to the construction of fruitful "mathematical habits of mind" (Cuoco, 2008) that may be exploited in various mathematical explorations leading to the generation of conjectures. We will discuss this further in Chapter 7.

6.2.4 The Notion of Instrumented Argument

Stepping back for a moment we may consider abductive arguments to be particular types of arguments within the argumentation that a solver can make during the conjecturing phase of his/her exploration. We have developed the notion of instrumented abduction to describe a particular type of abductive process of which the reasoning described in the MDS seems to contain an example. At this point it seems reasonable to extend the potential of being “instrumented” to other types of arguments, which naturally leads to the more general notion of *instrumented argument*. In this section we would like to introduce some examples of what we might call “instrumented arguments”. However at this point we will not define the notion in general, since we believe further discussion and richness of examples – potentially in which different instruments are used – seems to be necessary. For now we discuss characteristics of the notion limited to examples we noticed in some of the episodes we analyzed in this study.

Instrumented arguments seem to be used when the solvers need to convince themselves or each other that of a certain idea. For example, in Episode 1 of Excerpt 6.2.3 F continues his argumentation leading to the rejection of the previous GDP ([40]). In such argumentation he uses arguments with visual and haptic characteristics: “kind of circle starting from here [as he drags point D showing G what he means]”, “you see”, “in a certain sense it goes…down along a slope [mimicking the movement with his hand]”.

Another example can be found in Excerpt 4.3.3, which showed how checking a CL can lead to the generalization of a preconceived path. The solvers provide a GDP that they do not seem sure of. In particular F does not seem to be convinced that ABCD remains a parallelogram when D is dragged along the whole hypothesized circle. He therefore performs a soft dragging test which definitively convinces him and G of the GDP. Frequently we have observed that students use the words “try it” with respect to an idea (or possibly yet unexpressed conjecture) when they intend to perform a robust dragging test. From the transcripts we have so far analyzed within our study, this seems to be an even more convincing argument for solvers.

The analysis of our protocols highlighted a particular form of argument used by the solvers and strictly related to the exploration within the DGS. We called it *instrumented argument*, and see it as

> *an argument – thus part of an argumentation supporting a logical step – in which the warrants are supported by an instrument, in this case dragging.*

Its goal is to convince oneself or someone else of a specific claim, thus changing its epistemic status. In other cases the instrument could be other features of the DGS, the DGS itself seen as an instrument, or other types of instruments. Instrumented arguments in DGSs seem to be frequently used in conjunction with different versions of the dragging test, as in the episodes analyzed above.

We have also observed other examples of instrumented arguments in solvers' explorations. One example can be seen in the transition from a soft to a robust construction before a final (robust) dragging test is performed. Redefining the dragged base point to a constructed object that represents the GDP the IOD becomes robust, and the solver may subsequently refer to this property, in the argument, as being "true". This may also occur if the solver reconstructs the Cabri-figure in order to add a property, with respect to the ones that already originate from the steps, to its base points. These are acts that may correspond to geometrical ideas, but that first of all acquire meaning (and not necessarily a geometrical meaning) within a DGS. The (implicit) claim to defend is that a CL holds between the IOD and the III, and the instrumented argument consists in showing that when the IOD becomes "true", the III in the new construction also becomes a construction-invariant (at a visual and physical perceptual level), thus robust, and therefore "true". The warrants for such claim rely heavily on the software.

Another example can be found in arguments in favor of a certain GDP. Before constructing the geometrical object that hypothetically represents the path, the solver may try to argue that his/her idea is right, speaking about the movement of the dragged base point in physical terms and showing what s/he means by physically enacting the dragging movement on the screen. Thus, dragging and the feedback provided by the software are used as warrants supporting the solver's ideas about the GDP. This type of instrumented argument is also used to reject a given GDP. This can be seen in Episode 1 ([4]) of Excerpt 6.2.3 when the visual feedback seems to provide F with confirmation that what he had thought of as the GDP was "no good". Frequently the instrumented arguments used to reject a GDP (in the most convincing way) make use of the *dragging test* after the GDP has been constructed. In this case the solver argues that while dragging the base point along (or even having linked it to) the hypothetical object that represents the GDP the III is not maintained.

Moreover, when a solver has found a good candidate for basic property, with respect to a certain type of geometrical figure, to use as a bridge property (section 4.2.1.3) to continue the exploration, s/he may provide an argumentation in defense of such candidate. The implicit claim is: "if the (minimum) basic property is true, then the interesting type of geometrical figure is obtained." In the argument s/he may drag a base point to visually obtain a configuration (or various configurations) that seems to exhibit the candidate property and *show* him/herself and/or another person that in these cases the Cabri-figure seems to also become the geometrical figure s/he was initially interested in. In a way we can consider this argument as a kind of soft dragging test: imposing the hypothesized property, the solver checks that the original property that s/he was interested in is also visually verified. Furthermore, the solver may use what apparently looks like maintaining dragging in his/her argument when it is possible to continuously drag a base point in a way that the basic property is visually maintained. However, the focus of the instrumented argument in this case is the fact that the interesting type of geometrical figure is maintained *while* the basic property is maintained. Thus we say that the instrumented argument makes use of a sort of soft dragging test that gives rise to simultaneity, a warrant that is supported by the DGS.

Below is a flow chart that shows typical occurrences of instrumented arguments during explorations in which MD is used.

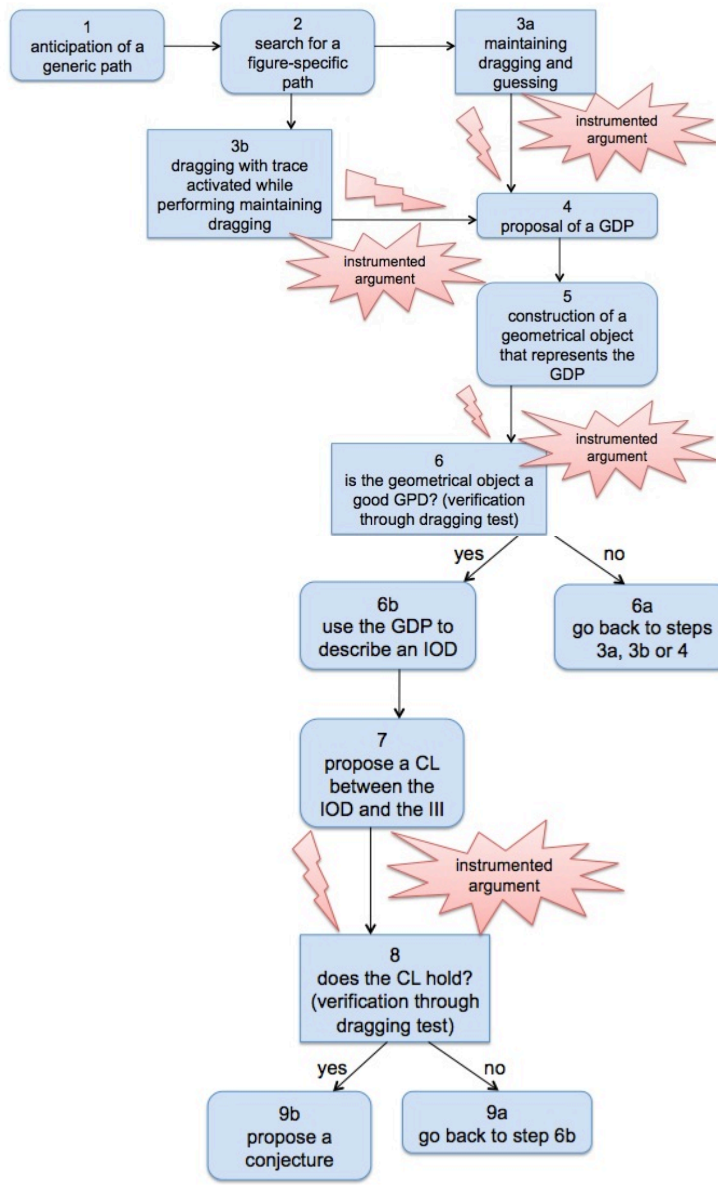


Figure 6.2.4: Typical occurrences of instrumented arguments during an expert use of MD.

We conclude with a last example of what we consider to be an instrumented argument of a slightly different nature than the ones described above.

<u>Excerpt 6.2.4.</u> In this Excerpt two solvers use an instrumented argument in an indirect argumentation to decide whether a certain property of a triangle should be included in the premise of a conjecture they develop. The excerpt is taken from the solvers' exploration of Problem 1 during the pilot study. This is why the interviewer's prompts are more frequent than in excerpts from the final study.

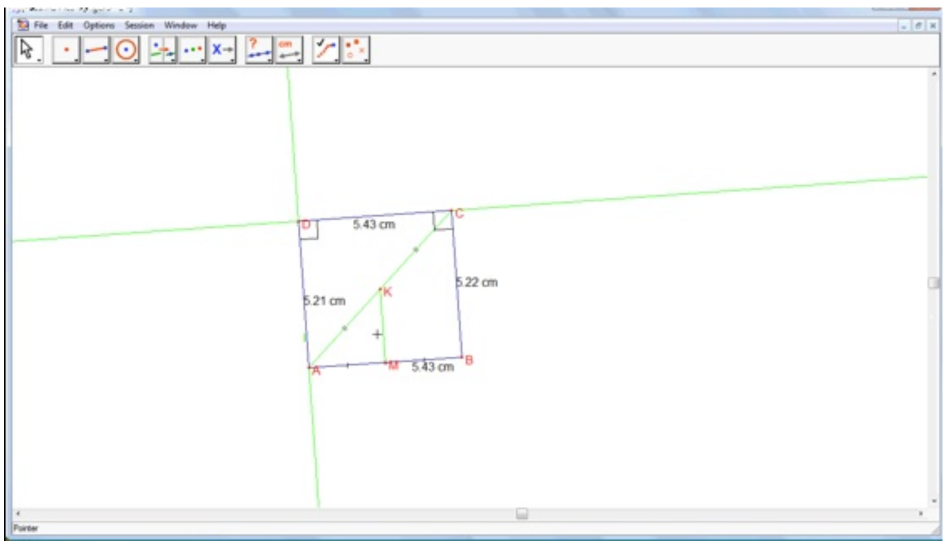


Figure 6.2.5 A screenshot of the solvers' exploration

Episode 1	**Brief Analysis**
[1] G: Eh, wait. I was thinking...should we try with the square?	The solvers identify a
[2] **F**: Eh, right! Let's try to obtain a square, moving A.	potential III.
[3] I: Ok, moving A.	
[4] **F**: Like this.	G notices a property
[5] I: Ok.	that emerges
[6] G: I think it looks like when AM is equal to MK.	simultaneously with
[7] **F**: Ah, you mean when AMK is an isosceles right triangle!	the property "ABCD

[8] I: Uhm. So how is this conjecture? [9] I: If AMK... [10] **F**: Is a right isosceles triangle, then ABCD is a square. [11] I: Ok. Write. [They write the conjecture: "If AMK is a right isosceles triangle, ..." then F interrupts G's writing]	square" and F adds a second property. The solvers state their conjecture, but when writing it seem unsure about the premise.
Episode 2	**Brief Analysis**
[12] **F**: No, we don't know it! [13] I: It depends on what you want. [14] I:...to put in the premise. [15] **F**: No, we have to say it, because I think if this is not right [pointing to the angle AMK]... [16] I: Well, try to move and see. [17] **F**: Wait, let's see. [18] G: in the meantime... [19] **F**: We have to move M, yes, so I vary AMK. [20] I: You were moving A before. [21] **F**: Yes. [22] I: But ok. Because you are moving M to try to get rid of the right angle? [23] **F**: Exactly, I was verifying that if I get rid of, eh see, if I get rid of ... [24] G: Eh, but you have to put AM and KM equal. [25] **F**: KM and AM equal...	The solvers start investigating whether triangle AMK should also be "right" or not in the premise of their conjecture. F decides to seek an answer by varying angle AMK. F tries to make angle AMK not right, but maintain AM equal to KM in order to have an

[26] G: Yes, because we have already written that AMK is an isosceles triangle, we know... [27] **F**: Eh no, wait, let's see.	isosceles triangle.
Episode 3	**Brief Analysis**
[28] I: So to maintain an isosceles triangle how should you move M? [29] G: Eh, along, along the perpendicular bisector of AK. [30] I: Ok, so try to move M like that. [31] **F**: Like this. No, what do I have to do? [32] I: He wanted to maintain only the property "isosceles triangle"... [33] G: You have to move, ...that is what we were discussing, right? [34] **F**: Yes. [35] G: Eh, so more or less like this...eh, yes, see? Here it is more or less isosceles. [36] **F**: Ahhhh... [37] G: Here it is more or less isosceles... [38] **F**: Yes, but do you see a square? [39] G: Exactly, it is not a square, so we need to write that ... [40] **F**: It has to also be right. [41] G: Yes. [42] **F**: Eh, you see?! [43] G: I was also writing it!!	In response to the interviewer's prompt, G suggests dragging M along the perpendicular bisector of AK. F seems to be interested in trying to maintain "ABCD square" but performs the dragging along the perpendicular bisector and notices that his property is not maintained. This leads him to conclude, through an indirect instrumented argumentation, that

[They laugh] [44] **F**: Ok, write: "and is also right" [45] **F**:...then ABCD is...a square.	“AMK right” must also be in the premise of their conjecture.

Table 6.2.6: Analysis of Excerpt 6.2.4

The argumentation is indirect, because F is trying to convince himself and G that if AMK is not right, ABCD is not a square, as he starts to state in line 15: “…because I think if this is not right [pointing to the angle AMK]...” That is, that the condition “AMK is right” is necessary for ABCD to be a square. Furthermore F wants to convince himself that “AMK isosceles” alone is not a sufficient condition for ABCD to be a square. G as well seems to engage in trying to convince himself that such condition alone is not necessary and proposes to maintain the condition “AMK isosceles” by dragging the base point M along the perpendicular bisector of AK. To propose this, G has implicitly used the conjecture (or theoretical knowledge) that “if M belongs to the perpendicular bisector of AK, AMK is an isosceles triangle (with base AK)”, together with the idea that maintaining dragging along the figure-specific path “perpendicular bisector of AK” will assure the invariance of the property “AMK isosceles” but not necessarily the property “AMK right”. This way the effect of the condition “AMK isosceles but not right” can be seen upon the quadrilateral ABCD. The “dragging argument” seems to be decisive in convincing the solvers that both conditions need to be included in the premise of the conjecture. Since the argumentation relies on the use of the instrument (in particular on a form of maintaining dragging, in this case) we claim it is another significant example of instrumented argument.

6.3 Overcoming Difficulties: Induction of Maintaining Dragging Leading to Development of the MDS

In Chapter 4 we introduced our cognitive model of the MDS and gave various examples of solvers behaving according to such model. In Chapter 5 we then proceeded to describe difficulties that various solvers encountered, due to the lack of certain fundamental components that we identified. Some of these difficulties would inhibit the expert use of MD. Moreover, in the first two sections of this chapter we discussed how acquiring the notion of "path" and reaching the idea of "searching for a cause" for the maintaining of the III, seen as a phenomenon within the world of the DGS, are key aspects of the MDS that lie at a meta-level with respect to the figure-specific elements described in our model that emerge during an exploration when MD is used. However we have not yet described *how* solvers might develop the MDS. In this section we would like to describe a basic sequence of prompts that the interviewer would use to "guide" the development of expert behavior in cases in which the solvers did not exhibit it spontaneously during their explorations. The prompting sequence emerged a posteriori from the analysis of our interventions and of solvers' responses. In particular we noticed the recurring use of a sequence of prompts that would foster similar patterns of responses. Moreover, in many cases, once the solvers had worked through a sequence (or two) of prompts, they would proceed in the following explorations using MD by themselves, and showing expert behavior.

We stress that the prompts were not aimed at leading solvers to behave according to the MD-conjecturing Model, but to foster awareness of aspects of the exploration that might lead them to overcoming the impasse. In other words, the prompts were conceived to act at the meta-cognitive level, to foster development of the MDS.

<u>6.3.1 The Prompting Sequence</u>

<u>Step 1</u>: A first "new" task

When solvers would not feel the need to overcome a basic conjecture, the interviewer would ask them to consider the particular type of quadrilateral they used in their conjecture and try to construct one that passed the dragging test and that respected all the steps of the initial construction in the Problem. The idea behind this intervention was to lead the solvers to become aware of the different status of objects of the construction and look for "constructable properties" to add to the steps of the construction that would induce the desired type of quadrilateral robustly. With "constructable properties" we intend properties that are compatible with the steps of the construction and that can be added to the steps of the construction without altering them. Solving this task should not only lead to awareness of the different status of the objects of the Cabri-figure, but it should also plant the seed of the idea of needing to "search for a cause" for the particular type of quadrilateral to "happen". Moreover, the property in the premise of most solvers' initial conjecture would be a (minimum) basic property that was not immediately constructable, so this new task would lead to the search for a new property that could potentially induce the initial property in the premise of the original conjecture. In this sense the task would lead the solvers to make their (minimum) basic property into a bridge property (defined in section 4.2.1.3).

Most solvers would respond by thinking about the construction and the status of the different objects of which the Cabri-figure was made. Some would think of a new property through an abduction, using known theorems; others would find difficulties and not be able to quickly find a way to solve the task; a few thought of using maintaining dragging and proceeded according to the MDS model from here.

<u>Step 2</u>: Prompting MD

Once the first "new" task had been given, if solvers had not started using MD on their own, the interviewer would prompt them to use it, in different ways. If the solvers had been able to reconstruct the type of quadrilateral they were considering so that it passed the dragging test, the interviewer would ask them whether it was possible to obtain that type of quadrilateral robustly "in other ways", and, shortly after, she would explicitly propose using MD. If the solvers were having difficulties with the first task, the interviewer would ask if they remembered "maintaining dragging" used in class during the introductory lessons, and suggest trying it.

Step 3: Overcoming difficulties with MD

Performing MD leads to various conceptual and manual difficulties, and frequently solvers who had not decided to use it spontaneously would experience various difficulties. In some cases after a first try solvers seemed to decide maintaining dragging was not possible for that given base point and III to maintain, so the interviewer would explicitly ask them whether they thought it was possible or not, and if it was not she would ask for an explanation. When solvers seemed to believe maintaining dragging was possible, but still not be able to perform it, the interviewer would ask one solver to concentrate on the property to maintain, and the other to "help" their partner by telling him/her in which direction to go. Sometimes the interviewer would also guide the solvers to use a property they had thought of as a bridge property for the MD. Usually once solvers were able to perform maintaining dragging they would perceive some regularity in the movement of the dragged base point and try to describe it, or help themselves "see" by activating the trace. When solvers didn't seem able to "see" and did not think of using the trace tool, the interviewer would suggest to activate it.

Step 4: Reaching a new conjecture

In different cases "seeing" the movement of the dragged base point and/or "recognizing" the trace mark as some known geometrical object was enough for the solvers to spontaneously formulate a new conjecture describing their (guided) exploration. However in some cases it did not seem to be. The interviewer at this point would explicitly ask for a conjecture. In some cases this would lead to a statement with dynamic elements which then the solvers would translate into an "if …then" statement in more "static" terms. However in other cases it would lead to uncertainty and to a return to the original conjecture or to one containing a new property that had been found and used as a bridge property for MD. If the solvers' conjecture at this point still did not include the IOD perceived during MD, the interviewer would try to get the solvers to focus on such IOD again and construct it robustly. The interviewer would either ask for the solvers to construct the object they "discovered" using MD and try to solve the reconstruction task (in Step 1), or she would ask the solvers to formulate a "constructable conjecture" from what they had found in their exploration, one that would lead to a quadrilateral passing the dragging test.

Once the solvers had successfully responded to this sequence (or a to a subsequence of this sequence) of prompts they tended to then use MD spontaneously in later explorations, and with a scheme that was coherent with the MDS.

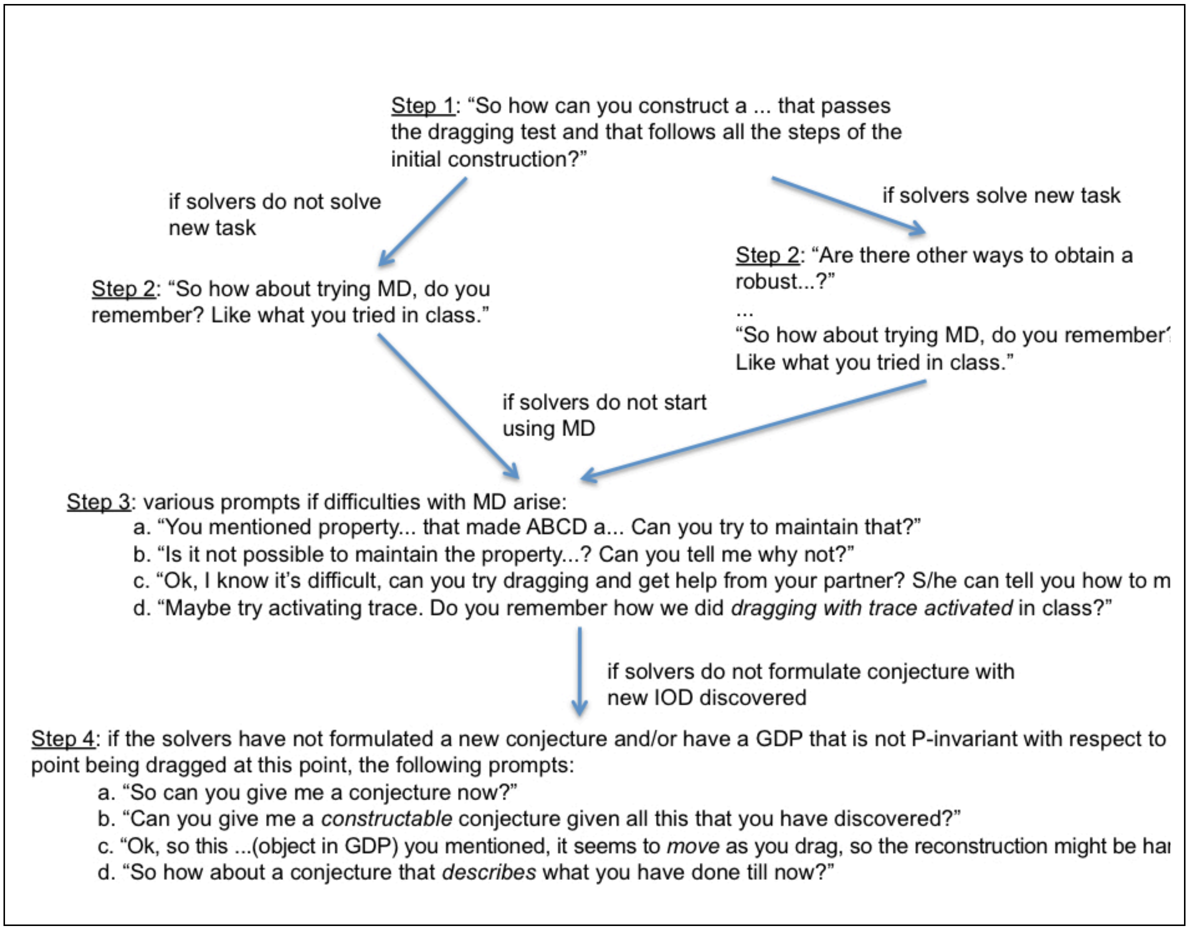


Figure 6.3.1: Sequence of prompts to guide development of expert behavior with respect to MD.

We would now like to give an example of how this sequence of prompts played out. The solvers' responses to the interviewer's prompts in the example we propose below were similar to various others' responses. We provide summaries of the various episodes from the sequence, and brief excerpts of particularly significant moments. The episodes are taken from two solvers' exploration of problem 1, and they lead to the solvers' 6th conjecture on this Problem.

Episode 1 (t17:03-t19:57). The first episode starts after the solvers have constructed a robust rectangle by linking the base point B to the perpendicular line to AC through A, having successfully solved the task posed by the interviewer, as in Step 1 of the prompting sequence. This excerpt begins with the interviewer explicitly prompting the use of MD, starting from the initial construction. The solvers seem to still have in mind a basic property as they start dragging and then activate trace. They also seem to be uncertain which base point to drag. They do not seem to have conceived the idea of generic path yet, even though they are able to maintain ABCD a rectangle by dragging B. There is a discussion about whether the quadrilateral is or is not a rectangle.

Episode 1	**Brief Analysis**
…	
[3] I: Can you try to do maintaining dragging? You aren't too used to it	Prompting according
[4] Dav: Yes, ok.	to Step 2.
[5] I: ...so I'll push you a bit to do it. So given the initial construction,	
[6] I: So B anywhere...it's enough to just unlink B.	The solvers follow
[7] **Dav**: So this away...what did we have to do? [rereading the	the steps in the
steps of the construction] B anywhere...	activity and
[8] I: Yes.	construct ABCD.
[they murmur as they remake the construction]	
[9] **Dav**: Eh, B....parallel...[as he constructs]	
…	
[14] **Dav**: and then segment AB. Ok now we have to try to drag.	They start dragging
[15] Gin: Yes...it has to be along, uh the perpendicular to AC	the base point A. Gin
through A.	seems to have in
[16] **Dav**: Ok.	mind a basic

[17] Gin: Take it, do "trace". [18] Gin: Mark an angle of 90. [19] **Dav**: where? "trace" [20] Gin: Trace of B. [21] **Dav**: and moving...A, [22] Gin: A and B. [23] Gin: but we have to...do, wait, do the perpendicular through A. [23] Gin: a line perpendicular through A. [24] **Dav**: Ah! Ok, now I understand. [25] Gin: Yes, good. [26] Gin: Yes, but now we are not sure it is a rectangle...we have to mark the angle or else we do not know it is a rectangle... [27] **Dav**: Ok...Yes, well, ok that's true. [28] Gin: I mean put like DBA, put the angle so it's 90 and we know that it is a rectangle. [29] **Dav**: I put DBA 90? eh, it's what we did before. [30] Gin: Yes, no, put the measure of the angle. [31] **Dav**: Yes, that is equivalent to putting B on this line, since here in any case it would be 90, and here 90. [32] Gin: but here you can also move B like this [showing a horizontal movement with his hand.] I mean B, in this case you can also move it like this. [33] **Dav**: Yes.	property. The solvers activate trace on B and drag the base point B. Dav switches to dragging B after Gin mentions both points. Dragging along the line described in the basic property turns out to work. Gin seems to associate ABCD's "being a rectangle" to it having angle DBA being right. However Dav suggests that that is equivalent to having B on the line he was dragging along. Gin seems to insist

[34] Gin: But if you put this angle here...we know ...when, uh the quadrilateral is a rectangle. [35] **Dav**: Yes. [36] Gin: Otherwise this way we do not know that it is a rectangle, we only hypothesize it. [37] Gin:...moving like this. [38] **Dav**: Yes, but we can prove that if B remains on ...on the...line there, on the parallel to, perpendicular on A, it is a rectangle. [39] Dav: We proved it before. [40] Gin: Yes. Oh, yes, that's right. [41] **Dav**: If B is on that line, we already know it is a rectangle, in theory.	on marking the angle, as if that would give the quadrilateral the status of rectangle. Dav, instead argues that they can prove that if B is on the line, ABCD is a rectangle.

Table 6.3.1.1: Analysis of Episode 1

In Episode 1 the solvers respond to the interviewer's prompt by trying to first briefly drag the base point A, and then they switch to dragging B as they decide to activate the trace. The solvers' behavior seems to show that they have trouble freeing their minds from the minimum basic property (angle DBA right) they had reached earlier in the exploration. Dav seems to be uncertain about how to drag A, so as soon as his partner mentions B, while he is activating the trace, he switches to dragging B. Before he starts dragging, Gin predicts that it will be enough to move B along the perpendicular line through A to AC, which they had used to solve the reconstruction task. This suggests that Gin has not yet conceived key elements of the concept of generic path: its independence with respect to basic properties of the type of quadrilateral being considered, and its dynamic nature, as a trajectory. Moreover, Gin worries about not “knowing” that the quadrilateral is a rectangle unless a certain angle is marked and its measure reads “90 degrees”. This difficulty might arise from the property “ABCD rectangle” not being constructed robustly, unlike in the previous part of the exploration. Another hypothesis is that he might be frustrated because he does not think of mentally deriving the fact that ABCD is a rectangle “given” that B is on the perpendicular line. He seems to be reassured when Dav explains how they had already proved that “if B is on that line, we already know it is a rectangle, in theory” ([41]).

Episode 2 (t19:57- t22:07). The interviewer prompts the solvers to activate the trace on the base point A, and seeing that the solvers are having trouble dragging, she asks questions from Step 3 of the prompting-sequence. The solvers get confused when they redefine B obtaining again a robust rectangle. This does not allow performance of maintaining dragging, since the III is no longer a soft property. The solvers realize the redefinition of B was not useful and proceed to unlink it spontaneously.

Episode 2	**Brief Analysis**
[1] I: Ok, let's go back to what you were doing...you wanted to activate trace on something else...you were dragging A, but I didn't understand ...could you repeat...	Prompt d in Step 3.
[2] **Dav**: No, I was thinking about what to do, I mean... [3] I: hmm. [4] **Dav**: Thinking about it, I mean moving A...we can't solve it...it should stay...	Dav expresses his difficulties in dragging A using maintaining dragging.
[5] I: You think that dragging A it does not remain a rectangle? [6] **Dav**: I mean...	The interviewer uses prompt b from Step 3.

[7] I: Ok, then try to explain why. [8] **Dav**: I mean yes, but B would have to in any case stay on the perpendicular, because since the line, this line rotates...with center in C [as he drags A], I mean all the figure rotates with center in C, basically, [9] I: uh huh... [10] **Dav**: Eh, instead B does not vary. I mean it always remains in the same position.	Dav seems to be mixing the preconceived property with a GDP for A, and seems unable to relate the behavior of the figure that he perceives to conceiving a *path* for A.
[11] **Dav**: Therefore B, uh, I mean, in order for this figure to be a rectangle, B has to in any case be on the perpendicular. I: Ok.	The interviewer uses prompt b from Step 3 again.
[12] **Dav**: Therefore, uh,... [13] I: So it is not possible to move A... [14] **Dav**: So moving A, I mean yes... [15] **Dav**: But we would have to move it like along...a circle? maybe...	Dav seems to conceive a possible new GDP leading him to believe that maintaining dragging is possible.
[16] Gin: but...no, I don't think so. Try. [17] **Dav**: Maybe so... [18] Gin: Link B to ...to the perpendicular... [19] **Dav**: Uh...where is it? "redefinition"? … [24] **Dav**: Point on this line. [murmuring as he goes back to dragging]	However Gin seems to still be confused by soft and robust properties of the Cabri-figure and proposes to redefine B on the perpendicular line, again.

[25] Gin: You have to do trace...the trace of A [the trace is now active on both B and A]...you don't need the trace of B. [26] **Dav**: Yes, but now, I mean, now it always remains a rectangle, however you move A! [27] Gin: Ah, that's true, right. [28] **Dav**: So it's not good. [they murmur as Dav unlinks B]	After they construct a robust rectangle again they realize this was not helpful and it prevents MD from working.

Table 6.3.1.2: Analysis of Episode 2

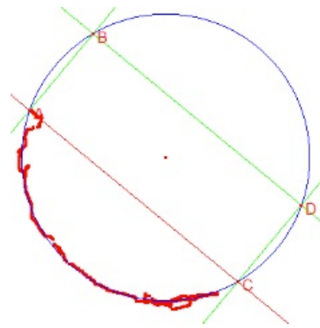
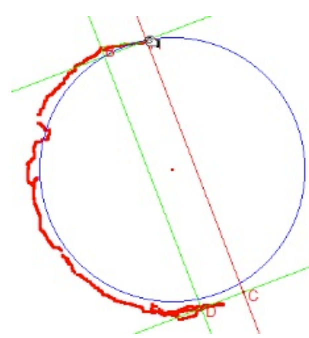
In this episode the interviewer's prompts seem to lead to a destabilization of the solvers' belief that performing maintaining dragging using the base point A was not possible. Dav seems to perceive a regularity in the movement of the base point he is dragging and provides a GDP as “a circle” ([15]). While Dav seems to have developed a proper conception of generic path at this point, Gin does not seem to have developed one yet since he again proposes to construct a robust rectangle by linking B to the same perpendicular line as in Episode 1. Moreover this shows that Gin has not yet managed to free his mind from the preconceived property. However this time both solvers seem to realize that this was not a useful move. Overcoming the belief that maintaining dragging was not possible seems to be what led to the behavior we will see in Episode 3, which was not prompted by the interviewer in any further way.

<u>Episode 3 (t22:17-t26:15).</u> The solvers try to perform maintaining dragging with the trace activated on A, again. This time they seem to anticipate a path, and show a proper conception of such idea. However they have some difficulties providing a GDP. They finally reach a GDP that is not A-invariant but that seems to satisfy them.

Episode 3a	**Brief Analysis**
[1] **Dav**: Let's try to put B like it was before...like... …[the solvers unlink B] [4] Gin: Yes, so now we can move it [5] **Dav**: yes. [6] **Dav**: So [he starts dragging A] now [7] Gin: go back...yes. Ok, now put trace of A. [8] **Dav**: Yes, now we'll do the trace of A and moving A we can see how it comes out... [9] **Dav**: So [as he starts dragging]...I'll take it from here. [10] Gin: Yes. [11] **Dav**: No, no better if you do it [handing the mouse to Gin] [12] **Gin**: Yes, ok, but it's not like I am better...so wait a second let's put it straight. [13] I: If now you could tell me what each of you is looking at... [14] **Gin**: Eh, I am trying to move A maintaining B on the perpendicular... [15] I: Ok.	The solvers unlink B to obtain the initial construction and try maintaining dragging once again with trace on A. Dav shows that he is anticipating a figure-specific path. Gin seems to concentrate on a bridge property (B on the perpendicular line) as he is dragging A.

Table 6.3.1.3: Analysis of Episode 3a

Episode 3b (Excerpt 5.4.3)	**Brief Analysis**
[1] **Gin**: So...circle again. [2] I: Hmm. [3] **Gin**: Yes.	Gin describes the GDP as a circle. The solvers

[4] **Gin**: so... [5] Dav: [murmurs something] [6] **Gin**: Yes...it is [7] Dav: ...it is the midpoint of C and B [8] **Gin**: It is the midpoint of... [9] Dav: It is the intersection of the diagonals  [10] **Gin**: diagonals [11] Dav: of the diagonals. [12] Dav: and since it is a rectangle, it is also the...the...uh the center of the circumscribed circle. [13] **Gin**: whatever. [14] Dav: Eh, they are all on the circle. [15] **Gin**: yes. [16] **Gin**: hmm. [17] I: Now, are you sure of this? [18] **Gin**: eh, yes.... [19] I: Because you have traced only ...  [20] **Gin**: ...pretty much [21] I: a little piece. Hmm. [22] **Gin**: there. [23] **Gin**: Well, we could try to continue.	successively refine the GDP trying to decide where the center of the circle might lie. They then proceed by constructing the circle that represents their GDP as the circle with center the midpoint of BC and passing through A. The solvers seem to be describing aspects of the new Cabri-figure on the screen. The solvers seem convinced by their GDP and are able to predict what the rest of the

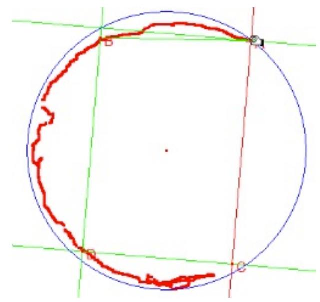
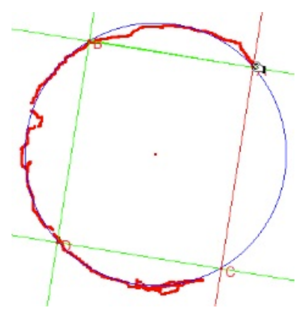
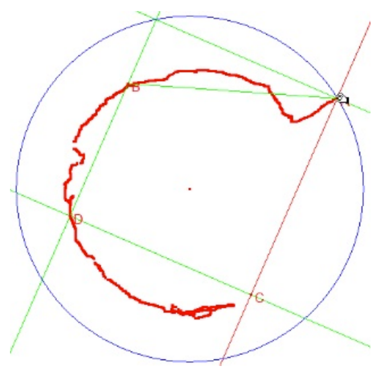
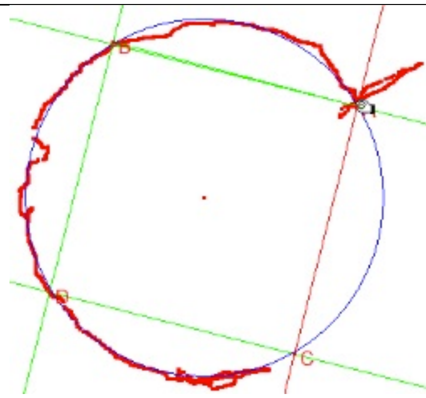
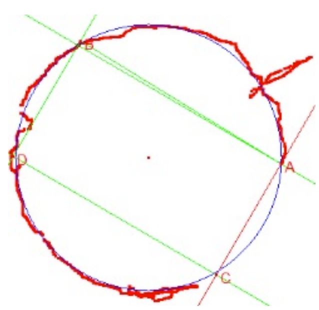
[24] Dav: exactly. [25] **Gin**: So now let's ...  [26] **Gin**: more or less along there [27] **Gin**: nooo [as a little circle appears when he clicks another point on the screen because he had not finished using the command "circle"] [28] **Gin**: Good here... [29] Dav: No... [30] **Gin**: Yes, alright, it looks like it is good [Italian: "sembra di sì"]  [31] **Gin**: Yes, good. It could be. [32] Dav: Yes, it looks like it is good. [33] **Gin**: yes. [34] Dav: Careful you are going out...	trace mark should look like. Although the circle they have constructed is not A-invariant the solvers seem convinced that it correctly describes their observations as they perform a soft dragging test.

Table 6.3.1.4: Analysis of Episode 3b

Now the solvers seem to have properly conceived a path: they have anticipated it ([8]) and seem to be aware that dragging along "something" that can be identified through the trace mark and the movement of the dragged-base-point "causes" the maintaining of the III (in this case the bridge property that they have already proved to be sufficient to obtain a rectangle). The solvers seem to be "convinced" ([31]-[33]) of their findings, as they perform a soft dragging test, but have not yet stated a conjecture.

Episode 4 (t26:15-t27:58). During this episode I asks the solvers to focus on the circle they constructed in the previous episode, questioning its “movement”. The intervention is aimed at overcoming the non-A-invariant GDP so that a robust construction of the added property might be possible. Although the solvers propose an alternative GDP which is A-invariant, they do not spontaneously construct it.

Episode 4	**Brief Analysis**
[1] I: Why are you talking about "one" circle? I mean, I see that it moves...	This is prompt c from Step 4 of the prompting sequence.
[2] Dav: Eh, because...	
[3] **Gin**: Yes, right because moving A theoretically the circle changes... 	
[4] Dav: Yes, but...if it gets bigger it is not any more...wait, move it...	
[5] Dav: Move it up. See, it does not stay any more...it is the circle through A and B and C	The solvers propose alternative GDPs.
[6] Dav: I mean they are together	
[7] **Gin**: Yes, through A and B.	
[8] Dav: Through A, B, and C exactly. I mean a circle through A, B, and C because if I assume a circle this one has to be...	

[9] **Gin**: Yes [as he continues dragging] [10] **Gin**: Yes, right it is a circle through A, B and C [11] Dav: Or else we could say it with...eh, center the midpoint of BC...and radius 	Dav proposes a new GDP which is A-invariant and the solvers seem to agree.
[12] **Gin**: [murmurs something] [13] Dav: Eh...eh, no. Yes, because you can't say that the radius is neither one nor the other, unless you say that they are the same. [14] **Gin**: Yes, exactly. [15] **Gin**: I mean it has to be... [16] I: So can you repeat what your idea is? [17] Dav: That this circle, so... [18] Gin: That the quadrilateral is a rectangle...if A [19] Dav: Yes... [20] **Gin**: rotates around...the circle... 	The interviewer asks the solvers to re-explain their idea, prompting for a conjecture (Step 4).
[21] Dav: it is on the circle with center… [22] **Gin**: with center [23] together: the midpoint of... [24] **Gin**: between B and C, where, uh	The solvers are now able to verbally describe their new GDP.

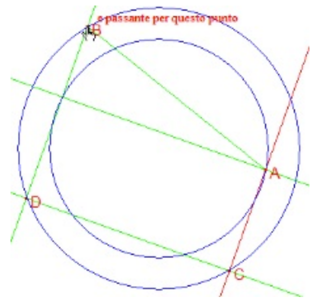
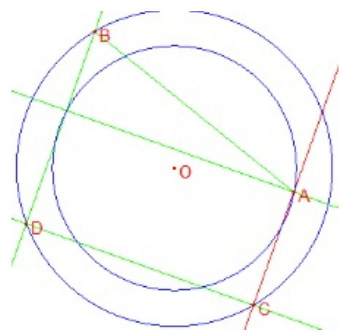
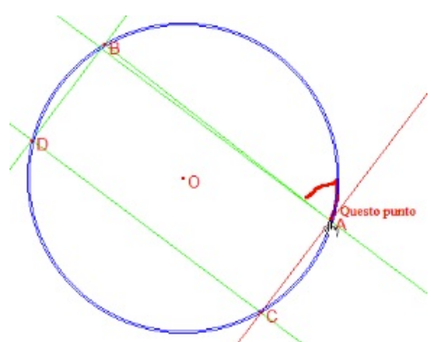
[25] Dav: and radius... [26] **Gin**: and radius...I mean [27] Dav: uh [28] **Gin**: OC equal to OA equal to OB. [29] Dav: and radius OB, because if you say that it is there...	

Table 6.3.1.5: Analysis of Episode 4

The interviewer's prompting leads to a new GDP which is A-invariant. Moreover, by the end of this episode, the solvers are able to verbally formulate what seems to be a conjecture linking the III ("the quadrilateral is a rectangle" [18]) with the IOD ("A rotates around the circle" [20]). We infer that the solvers have now conceived the idea that dragging along a trajectory can induce a configuration to become in invariant property of a dynamic-figure. This is a key aspect of the notion of generic path. However they do not spontaneously write down the conjecture or try to reconstruct the IOD robustly to perform a robust dragging test. Therefore the interviewer prompts such behavior in the following Episode.

<u>Episode 5 (t27:58-t29:39).</u> The interviewer asks for a dragging test for the idea the solvers had expressed in the previous episode ([18]-[28]). This leads to robust construction of the proposed IOD and to further conviction of the appropriateness of the conjecture, which the solvers now write down.

Episode 5	**Brief Analysis**
[1] **Dav**: Because if you say that it is there...yes, and radius OB. [2] Gin: radius OB. [3] I: Ok so try to do the dragging test of this that you have just told me.	 The interviewer asks

[4] **Dav**: We have to drag...uh [as Gin rereads the text of the activity]...yes. [5] **Dav**: Ok, we can, uh... [6] Gin: Eh, construct...wait go down, erase the line... [7] **Dav**: Yes.  [8] **Dav**: We can construct the circle with radius...OB [9] together: yes. [10] **Dav**: Let's call this O. [11] **Dav**: and then...[as he drags]...try to maintain it on this new circle. [12] Gin: but I think you need to link it, wait link it.  [13] **Dav**: ah, we forgot...right. [14] **Dav**: we need to link it...where is it? here. [15] Gin: A. [16] **Dav**: "point on object" [17] Gin: ..."object" [18] **Dav**: and then [19] Gin: circle [20] **Dav**: erase one.  [21] Gin: "hide/show" right. [22] **Dav**: "point on object"...A "point on object" [23] Gin: and now do circle.	the solvers to verify their idea with a dragging test (Step 4). The solvers construct the IOD robustly by constructing a circle that is A-invariant and redefining A upon it.

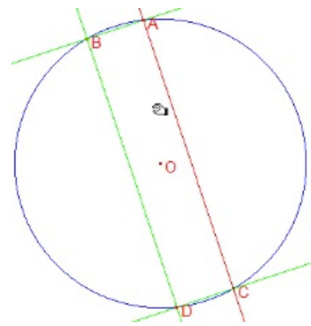
[24] **Dav**: circle, ok. [25] **Dav**: [as he drags the newly linked point] Now we can get rid of the trace. [26] Gin: Yes, ok always a rectangle. [27] **Dav**: Yes. It should be a rectangle, yes. 	The solvers perform a robust dragging test and seem to be convinced of their idea.

Table 6.3.1.6: Analysis of Episode 5

The solvers respond positively to the prompt, and are able to construct a robust IOD. When the perform the dragging test, they seem to be satisfied and almost relieved to see that the figure's behavior corresponds to their expectation that after this reconstruction the quadrilateral should in fact be a rectangle. The robust dragging test seems to be convincing for the solvers, who now write down their conjecture: "ABCD is a rectangle when $A \in C_O$, with O midpoint of BC and radius OB."

Although through the prompting sequence the solvers proceed coherently with our model and reach a conjecture linking the III with the IOD, they do not exhibit "automatic" behavior at this point. They seem to still be developing expertise with respect to MD during the exploration they engage in after this one, hesitating on providing a GDP which is invariant with respect to the dragged-base-point. However after such hesitation the solvers seem to exhibit expert behavior in their final explorations.

6.4 Conclusion

In this chapter we elaborated the notion of path that we had introduced with our model in Chapter 4, emphasizing its centrality in the development of expert use of MD. In particular we described how the generic path resides at a meta-cognitive level with respect to the dynamic exploration being carried out. This meta-cognitive level seems to influence the interpretation of the phenomena that occur on the screen, and to control the whole development of the exploration process. Moreover, constructing this meta-level knowledge seems to allow some students to transition to using MD during an exploration, and exhibit expert behavior. The meta-cognitive level seems to also conceal the abduction that previous studies have identified during dynamic exploration that involve the use of maintaining dragging (previously known as dummy locus dragging). We therefore introduced a new notion, that of *instrumented abduction*, describing this type of abduction, and others that may be supported by an instrument. Finally, in Section 6.3, we identified recurring aspects of a process of development of expert use of MD by describing a possible sequence of prompts that was used by the interviewer to foster solvers' awareness about the use of MD for producing a conjecture, and that seemed to lead solvers to progress in a process of development of expert use of MD.

CHAPTER VII

CONCLUSIONS, IMPLICATIONS AND DIRECTIONS FOR FUTURE RESEARCH

In this concluding chapter we will explicitly explain how the MD-conjecturing Model led to significant findings with respect to the research questions we had set out to investigate. Concisely, the model provides an adequate description of the process of conjecture-generation when maintaining dragging (MD) is used by the solver; it also provided a lens through which it was possible to analyze solvers' explorations and gain further insight into cognitive aspects of this particular process of conjecture-generation. In particular, it shed light onto the relationship between an abductive process and use of the dragging tool, specifically MD.

As mentioned in the description of the methodology, our findings have no statistical ambitions because of the limited number of cases analyzed. However, the fine grain qualitative analysis that was carried out for every case provided a richness in detail and depth which would not have otherwise been possible. Furthermore, many commonalities emerged during the analyses, outlining a common process of conjecture-generation through MD, thus giving sense to a definition of expert use of MD. All this leads us to think that, in a search for more general results, quantitative research can be fruitfully grounded upon our findings.

In this chapter, after answering our research questions, we will contextualize our findings within the field of mathematics education. The contextualization of our research within the broader perspective of the field as a whole will serve to describe implications of this study and directions for further research.

7.1 Answers to The Research Questions

The research questions we proposed to investigate were:

1. What relationship do the forms of reasoning used by solvers during the conjecturing stage of an open problem in a DGS, have with the ways in which solvers use the dragging tool?
2. When a solver engages in the activities proposed in this study within a DGS there seems to be a common *process* used to generate conjectures through use of *maintaining dragging*.
    a. Does our model describe this process adequately?
    b. How does the model describe the dragging scheme and how can we refine the description?
    c. What insight into the process of conjecture-generation can be gained when using our model as a tool of analysis for solvers' explorations?
    d. What is the role of the *path* within this model? Moreover is the path, as a part of the model, a useful tool of analysis?
    e. How does the model highlight abductive processes involved in conjecture generation?
3. In cases where students do not use maintaining dragging (MD), is it possible to outline how they might develop effective use of MD?

In the following paragraphs we will provide answers to each of the questions with respect to our findings described in Chapter 4, Chapter 5, and Chapter 6.

7.1.1 Answer to Question 1

The MD-conjecturing Model unravels the delicate point of transition marked by an abduction and use of *dummy locus dragging* (Arzarello et al., 2002). As such, our model provides a tool of analysis that allows us to “zoom into” this transition point and look at different concurring features that contribute to its complexity. In particular, with our model we were able to analyze in further detail the relationship between *maintaining dragging* and particular forms of reasoning, including *abduction*. The model proposes a classification of robust invariants that provides a window through which solvers’ reasoning can be viewed and analyzed. In particular our notions of *basic* and *derived construction-invariant* and of *point-invariant* have revealed to be insightful tools of analysis. They allow us to highlight the solvers’ ability to use theoretical knowledge to interpret invariants, and, more importantly, the cognitive process through which solvers can link these simultaneously-observed properties together in a conditional relationship. Wandering dragging is used to perceive these robust invariants, which can then be used in what we have defined as basic conjectures, during a preliminary phase of explorations. For example, these notions allow us to interpret exclamations such as “always a trapezoid” ([1], Excerpt 4.2.2) and put them in relation with the subsequent conjectures generated by solvers.

As the exploration proceeds and the solver searches for interesting configurations, we can recognize a form of guided dragging (Arzarello et al., 2002) or use of a drag-to-fit strategy (Lopez-Real & Leung, 2006), which seems to be a manifestation of the solver’s use of his/her conceptual knowledge to induce a particular configuration on the dynamic-figure by acting on its base points. Our model introduces the notions of *basic property* and *minimum basic property* to describe a particular use of theoretical knowledge to reach a desired configuration. These notions are also useful for interpreting solvers' behavior as they are trying to maintain a desired property, through maintaining dragging. For example, when G exclaims: "I understand! so, C… we have to have the diagonals that intersect each other at their midpoints, right?" ([8], Excerpt 4.2.5) he has conceived a minimum basic property which he uses to make the task of maintaining dragging easier.

Moreover, by identifying two types of soft invariants, *intentionally induced invariants* and *invariants observed during dragging*, the model allows us to put the (potentially) subsequent use of *maintaining dragging* in relation with the idea of "searching for a cause" and, in general, with an abductive cognitive process. We will analyze this relationship in depth in our answer to Question 2. Here we highlight an aspect of this cognitive process, related to use of *maintaining dragging with the trace activated* as a means to reach a *GDP*. Recall, for example, episodes like that described in Episode 4 of Excerpt 4.3.1, when, activation of the trace on the base point being dragged leads to Giu's observation: "It's an arc of a circle, I think the curvature suggests that…." ([35] Episode 4, Excerpt 4.3.1).

The terminology we introduce for *soft invariants* helps describe reasoning that occurs in correspondence with the use of the *soft dragging test*. If the solver is exploring the figure dynamically and has perceived two soft invariants, potentially an *III* and an *IOD*, that seem to occur simultaneously, s/he might drag a base point to induce one property directly and the other one indirectly and check that they are visually verified simultaneously. Our model sheds light onto how causality between the invariants in the DGS may be interpreted as *conditionality* between geometrical properties in Euclidean Geometry and to how a *CL* may be established, leading to the formulation of a conjecture. For example, we analyzed how the use of the word "when" can mark the conception of a CL between soft invariants. We can recall exclamations like: "Now there is this problem of the parallelogram in which we can't exactly find when it is" ([6:36], Excerpt 4.4.2), or: "I find that the quadrilateral is a parallelogram, except when, uh, D comes to lie on the line CA" ([17], Excerpt 4.4.1).

A similar form of reasoning seems be used in correspondence to the *robust dragging test*. This may be performed by the solver after a redefinition of the dragged-base-point on the geometrical object s/he constructed to represent the figure-specific path. The solver this way can test his/her conjecture in a robust and "general" way. As a matter of fact, now the solver can only perceive simultaneity of the two invariants, which, if the conjecture is provable, have now become *robust invariants*, and can be conceived as new construction invariants (see also the description of the model in phases, Section 4.6 and 7.2.2).

7.1.2 Answers to Questions 2a, 2b and 2c

The data analysis appears to confirm that there *is* a common process of conjecture-generation when maintaining dragging (MD) is used, and this process is well-described by the MD-conjecturing Model. Moreover, the model provided a lens through which we could analyze students' difficulties, which led to the identification of four components that seem to be necessary for expert use of MD. We used these four components to describe the solvers' difficulties in Chapter 5. The analysis of solvers' difficulties allowed us to gain further insight into cognitive aspects of conjecture-generation that we had set out to study, leading to the identification of a figure-specific level and a generic level of the MD-conjecturing Model. These were described in Chapter 6. In the following paragraphs of this section we will highlight significant aspects of these findings.

This initial model presented in Chapter 2 was found to be appropriate, but not sufficient to describe various aspects of the process we were investigating. Therefore this initial model was refined and elaborated into the MD-conjecturing Model which was introduced in Chapter 4. We found it useful to present the MD-conjecturing Model as a sequence of tasks and sub-tasks that a solver can decide to carry out during his/her dynamic exploration. The tasks we identified and described are the following.

- Task 1: Determine a configuration to be explored by inducing it as a (soft) invariant intentionally induced invariant (III);
- Task 2: Look for a condition that makes the intentionally induced invariant (III) be visually verified through maintaining dragging;
- Task 3: Verify the conditional link (CL) through the dragging test.

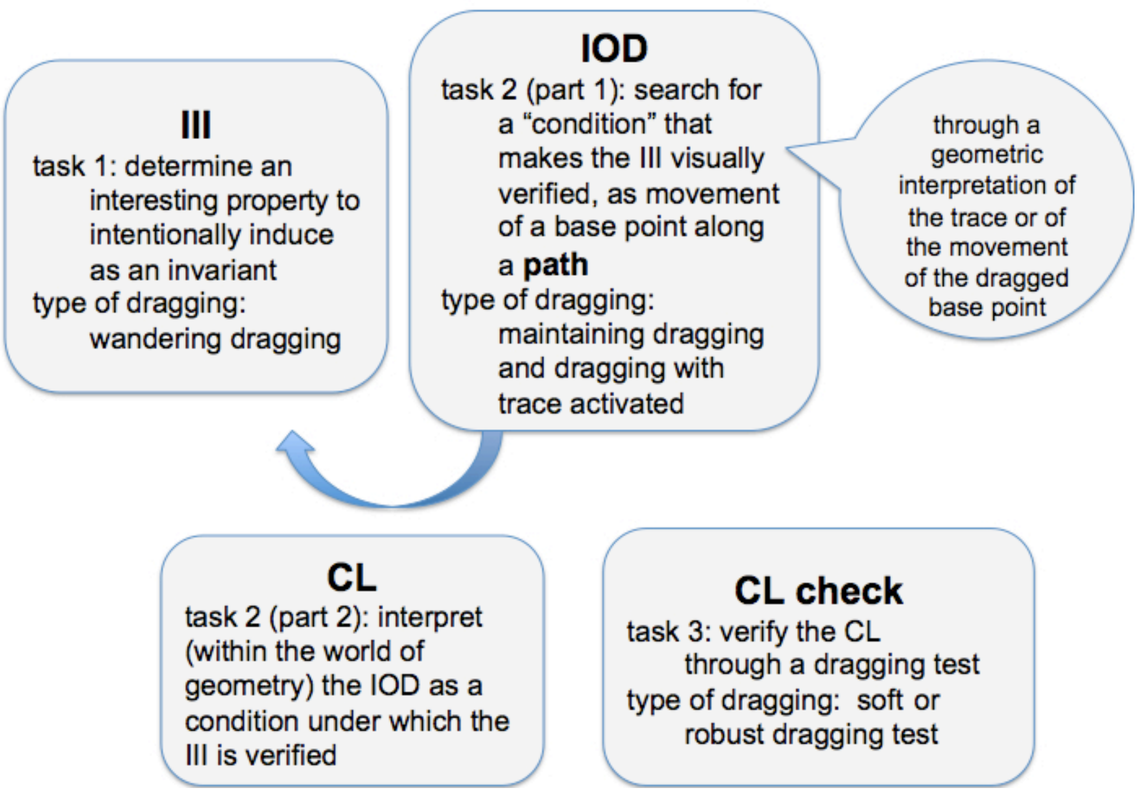


Figure 7.1.1: Interplay of the main elements of the MD-conjecturing Model.

Throughout Chapter 4 we highlighted the additions that the data analysis led to, and broadened our description of the process of conjecture-generation. Although use of MD is still central in this new description of the process of conjecture-generation, we added the description of a phase that appeared in many explorations, in which solvers seemed to explore robust invariants.

Moreover, with respect to the initial model, we noticed how most of the additions to our initial model were related to a *characterization of invariants* that seemed to help describe students' work. The types of invariants we added are point-invariants and construction-invariants (either basic or derived), and additional construction-invariants, that is, invariants that are constructed as a robust invariant after having been observed (or induced) as a soft invariant, or potential property of the Cabri-figure considered. We therefore proposed an alternative description of the process of conjecture-generation characterized by the particular type of invariant investigated: (1) the point-invariant and construction-invariant phase; (2) the intentionally-induced-invariant phase; and the (3) additional-construction-invariant phase. The phases describe how an exploration may be carried out over time, through a process that could repeat cyclically. This second way of describing the model seems to complement the first description, and the combination of the two descriptions revealed to be useful in analyzing solvers' explorations. Below is a table that represents the description of the MD-conjecturing Model as invariant-type phases.

Phase of Model	Subtasks	Dragging Schemes Used
point-invariant and construction-invariant phase	distinction of point-invariants from construction-invariants	wandering dragging
	formulation of initial conjectures	dragging test (robust)
intentionally-induced-invariant phase	determine an III	wandering dragging
	find a (minimum) basic property	no dragging, wandering dragging, dragging test (soft) to test sufficiency of condition
	maintain the III	maintaining dragging
	find a GDP and provide an IOD	maintaining dragging, dragging with trace activated
	verify the CL	dragging test (soft and/or robust version)
additional-construction-invariant phase	construct the IOD from previous phase robustly	redefinition of point on object
	repeat previous phases on new construction	all the dragging above

Table 7.1.2: The MD-conjecturing Model as invariant-type phases with related subtasks.

As mentioned above, the MD-conjecturing Model also allowed us to gain further insight into cognitive aspects of conjecture-generation we had set out to study. If we focus specifically on the solver's use of MD, the analysis we carried out through the lens of the MD-conjecturing model allowed us to describe what we called *expert use of MD* for conjecture-generation. Moving to a meta-cognitive level, it is possible to describe key aspects that seem to determine such expert use. In particular, in Chapters 4 and in

Chapter 6, we have introduced the idea of developing the subtask of "searching for a cause". We highlighted how expert use of MD seems to be characterized by an open and flexible attitude during use of MD. In other words, the expert does not expect anything specific, but simply is open to the possibility of perceiving a regularity that might be transformed into a geometrical condition for verification of the interesting property induced. Conceiving MD as a tool that may help answer the question "what might cause the property I am interested in to be maintained" seems to be necessary for the development of expert use of MD, leading to making sense of what emerges during an exploration. We believe that this question paired with the developed notion of generic path (Section 6.1) supports the solver in searching for a cause of the maintaining of the III as dragging the considered base point along a path which will have a figure-specific description in each particular exploration, depending on the construction, the property chosen to maintain, and the base point chosen to drag.

These considerations allowed us to describe expert solvers as solvers who have developed the necessary meta-level knowledge related to the use of MD, specifically the notion of generic path and the idea of using MD to "search for a cause". Combining our description of the meta-cognitive level with the elements of the model that illustrate the use of MD during the dynamic exploration, leads to what we have defined the *maintaining dragging scheme* (MDS). Taking an instrumental perspective, we can characterize expert use of the MD through the description of the utilization scheme that solvers seem to build in correspondence to MD with respect to the task of conjecture-generation in open problems in a DGS. The utilization scheme is the combination of the two components we described: the cognitive component at the level of the exploration, and the meta-cognitive component that we introduced to describe expert behavior.

7.1.3 Answer to Question 2d

The role of the path is fundamental within the MD-conjecturing model. Through its two components, the figure-specific path and the generic path, it bridges the two levels of the MD-conjecturing Model. Moreover, the notion of path was found to be a useful tool of analysis, giving an indication of what phase of the model the solver seemed to be proceeding through, and providing insight into difficulties when solvers did not use MD effectively. In particular, the (re-elaborated) notion of path, and especially its generic component described in Chapter 6, allowed us frequently to identify the crucial point of many of the difficulties. This is the case, because the notion of generic path "incorporates" fundamental aspects of the intentionally induced invariant (III) – since dragging along "the path" makes the III visually verified – and the potentiality of an invariant observed during dragging (IOD) – since a regularity may emerge as the movement of the dragged-base-point along a trajectory that may be described geometrically.

Furthermore, the generic path expresses a link between the phenomenology of the DGS and the world of Euclidean Geometry. Conceiving a generic path guides the interpretation of the experience within the phenomenology of the DGS in geometrical terms. We described how this seems to be the case because within the phenomenology of the DGS the generic path withholds both the seed of a causal link between the invariants perceived during the exploration and of the conditional link (CL). In particular the generic path can be considered, within the phenomenology of a DGS, as a trajectory with respect to movement, a movement that coordinates the dragged-base-point with the III, causing the III to be visually verified. In Geometry, this trajectory which becomes figure-specific, may be seen as a geometrical object that a point can belong to, a mathematical locus (or a subset of it), a condition for a second property to be verified.

Difficulties can arise in cases in which solvers identify a figure-specific path, but not being able to conceive a generic path, they are not able to relate what they experience within the phenomenology of the DGS to a geometrical statement expressing a conjecture.

7.1.4 Answer to Question 2e

Through the MD-conjecturing Model we were able to successfully "zoom into" the delicate transition point that Arzarello et al. (1998) describe as marked by abduction. There seems to be a correspondence between abduction and use of MD, situating the abduction at a meta-level with respect to the exploration. We express this idea through the notion of *instrumented abduction* (Section 6.2). When conjectures are generated coherently with the MD-conjecturing Model, use of MD seems to become "automatic" for expert solvers who exploit the corresponding utilization scheme (MDS). Moreover automatic use of the MDS seems to condense and hide the abductive process that occurs during the process of conjecture-generation in a specific exploration: the solver proceeds through steps that lead smoothly to the discovery of invariants and to the generation of a conjecture, with no apparent abductive ruptures in the process. In other words, our research seems to show that,

> *for the expert, the abduction that previous research described as occurring within the dynamic exploration occurs at a meta-level and is concealed within the MD-instrument.*

Instrumented abduction is the main type of abduction that we seemed to find occurring in correspondence with MD, and that characterizes the maintaining dragging scheme.

However, our data seemed to also suggest that if MD is also *internalized* by solvers, thus becoming a psychological tool (Vygostky, 1981, p.162), it may be freed from the physical artifact of dragging within the DGS. When MD is developed into a psychological tool, it seems to become a way of thinking that can be used to solve a different problem: no longer that of maintaining a property through dragging, but that of searching for a cause. We described this case at the end of Section 6.2: although the "search for a cause" through use of MD with the trace activated failed, the solvers were able to overcome the technical difficulties and reach a conjecture by conceiving a new GDP without help from the actual use of the MD. In other words the solvers seem to have interiorized the use of the MD to the extent that it has become a psychological tool which no longer needs external support. This is also very interesting with respect to the abduction involved, because our data suggested that when MD is used as a psychological tool, the abduction seems to occur *internally* and is supported by the theory of Euclidean Geometry. This abduction is not an instrumented abduction, but an abduction that resides at the level of the dynamic exploration, and that leads to the emergence of geometrical properties of the GDP which in the case of an instrumented abduction do not emerge.

7.1.5 Answer to Question 3

Many solvers did not exhibit expert behavior during their explorations, especially during their first explorations. In general, during the study we did not observe the possible process of development of expert use of MD, nor did we attempt to describe a process of instrumental genesis (Rabardel, 2002). However some of the solvers did reach an expert or nearly expert behavior by the end of their interviews. The evolution of expert behavior did not seem to be completely spontaneous. In fact we developed a number of prompts to use in situations in which solvers seemed to have encountered some sort of impasse, or would not be able to proceed. These prompts were not aimed at leading solvers to behave according to the MD-conjecturing Model, but to foster awareness of aspects of the exploration that might lead them to overcoming the impasse. In other words, the prompts were conceived to act at the meta-cognitive level, to foster development of the MDS.

Somewhat unexpectedly, as we analyzed our interventions and solvers' responses during the interviews, a prompting sequence emerged. In particular we noticed the recurring use of a sequence of prompts that would foster similar patterns of responses. In Section 6.3 we described the basic sequence of prompts that emerged from the analysis of the interventions and the solvers' responses. From this sequence it is possible to identify a series of four steps that seem to outline how solvers might develop effective use of MD.

We stress that this sequence of prompts is not the only one that may foster the development of expert use of MD, nor can we state that it is the most effective one. Its significance resides in the fact that it emerged from the analyses as a recurrent sequence from an otherwise orderless set of prompts we had prepared for the interviews. The order in which the prompts were used and the consistency of solvers' responses led us to the considerations above. However the relatively small number of cases analyzed in this study does not allow us to make significant claims on the "generality" of the process, which may be studied in future research.

7.2 Contextualization of Our Findings

In this section we situate our findings within the field of mathematics education. In particular we discuss how our results can be considered with respect to Arzarello et al.'s analysis of dragging in Cabri, to Leung's variational analysis of dragging, and to Boero's processes of generation of conditionality.

7.2.1 Our Findings with Respect to Arzarello et al.'s Analysis of Dragging in Cabri

Our research has its roots within the research developed by Arzarello, Olivero, Paola and Robutti (Arzarello et al., 1998a, 1998b, 2002) that provided a cognitive analysis of dragging practices in Cabri environments. Our study advances this line of research by explicitly describing in detail certain possible steps of the cognitive processes that may occur when students engage in particular dragging practices among the ones described by Arzarello et al.'s research. More precisely, our model illustrates a process of conjecture-generation that can occur when maintaining dragging is used by the solver. Maintaining dragging is essentially Arzarello et al.'s *dummy locus dragging* (Arzarello et al., 2002), with the essential difference that it is a way of dragging "given" to solvers instead of observed and classified a posteriori. While Arzarello et al.'s research led to a detailed description of dragging practices during the solution of open problems in Cabri, our primary goal was to further investigate specific cognitive processes that seemed to occur during the phase of conjecture generation in the solution of open problems when the use of the specific MD modality is promoted. In this sense our research aimed at unraveling what Arzarello et al. had described as the delicate transition from ascending to descending control, guided by abduction, and occurring in correspondence to use of dummy locus dragging. Through our model we intended to "zoom into" this delicate transition point and analyze, in a fine manner, the processes involved. Consistently with this goal we took a different approach to studying the use of dragging: we chose to introduce particular dragging modalities, and in particular the maintaining dragging modality, to the participants.

This approach to our investigation allowed us to develop and test our model, which provided insight into processes that Arzarello et al.'s research had hinted at, and in particular it led us to recognize where" abduction seems to lie within this process of conjecture-generation. As we described in Chapter 6, when maintaining dragging is used by expert solvers in an "automatic" way, no abduction seems to occur at the level of the dynamic exploration. Instead it is supported by the instrument of *maintaining dragging*, and concealed within the instrument, in particular at a meta-cognitive level within what we described as the maintaining dragging *scheme*. These considerations led us to define the notion of *instrumented abduction*, a main finding of our research. This way our findings are consistent with previous studies carried out by Arzarello et al., but deepen them with respect to the use of MD and to the presence of abductive processes that become indwelling of the meta-cognitive component of the MD scheme.

7.2.2 Our Findings with Respect to Leung's Variational Lens

Similarly to how we developed our model to gain insight into specific processes in DGS explorations, Leung has developed a different lens that provides a tool of analysis from a cognitive perspective. The lens of variation (Leung, 2008) is introduced to help capture and explain cognitive components of experiences involving dragging, as described in Chapter 1. Moreover, he used such lens to introduce a discernment framework that can mediate geometrical knowledge (Leung, 2008, p.152-153). This opens the delicate issue of the relationship between the phenomenological domain of a DGS and the world of Euclidean Geometry (EG), introduced in previous research (for example Lopez-Real & Leung, 2006; Strässer, 2001).

The perspective introduced by Leung presents an interesting and complementary perspective in respect to our own. Thus in a recent and ongoing research collaboration with Leung, we developed the complementarities between our MD-conjecturing Model and the lens of variation, constructing a combined-lens that sees elements of the MDS and of the lens of variation fused together. We used the combined-lens to analyze students' work to try to gain insight into aspects of the dynamic relationship between the Cabri-world and the world of EG that arise when MD is used. The combined-lens, through the elements that constitute it and their relationships, seems to in fact provide deeper insight into cognitive processes involved in conjecture-generation when MD is used. Moreover, analyzing solvers' explorations through the combined-lens seems to lead to a new perspective on the transition from sense-making within the DGS to mathematical interpretation within EG, a transition that is needed to reach the formulation of a geometrical conjecture (a conjecture in geometrical terms).

The Combined-lens - We now introduce our combined-lens for describing conjecture-generation when MD is used, and summarize it in the table below (Table 7.2.2.1), which spells out the complexity involved. The table combines the main elements that have arisen from our study with those previously developed by Leung (Leung, 2008), describing them within the phenomenology of a DGS together with the cognitive components involved in their perception (column 1) and illustrating their interpretation within EG (column 2). The system of relationships is presented in the table through placing corresponding elements in parallel. Accordingly, we describe the combined lens following the organization of the table row-by-row, which corresponds to separating different key elements of the process of conjecture-generation, expressing some of the possible cognitive components involved and how each element of the process develops across the two worlds.

Phenomenology of a DGS	EG Interpretation
Level-1-Invariants (Perceived Invariants): robust invariants and induced soft invariants. One of these may be chosen as an III. The IOD will later emerge as another of these invariants during the process. Cognitive components: functions of contrast and separation	Geometrical interpretation of level-1 invariants as geometrical properties
Level-2-Invariants (Perceived Invariant Relations between Invariants): perception of an invariant relation between the III and the IOD. Cognitive components: coordination between different functions of variation and synchronic simultaneity	Geometrical interpretation of level-2-invariants as geometrical relations between geometrical properties
Locus of Validity (LoV): a figure-specific *path*. It can be of type I (traced path), type II (soft path), or type III (robust/generalized path) Cognitive components: functions of separation, diachronic to synchronic simultaneity, generalization, fusion	Geometrical interpretation and description of the LoV (GDLoV) as a geometrical object
Critical Link 1 (CrL1): transition from the first to the second level of invariants Critical Link 2 (CrL2): interpretation of CrL1 as the answer to the "search for a cause" Cognitive components: simultaneity together with the sensation of direct and indirect control over the III and the IOD	Interpretation of the CrL2 as a Conditional Link (CL): a relationship of logical dependency between geometrical properties

Table 7.2.2.1: Elements of our combined-lens with respect to the phenomenology of a DGS and their interpretation within the world of EG.

Row 1: A Level-1-invariant is a property of a dynamic-figure that remains invariant while other properties change under different dragging modalities. Level-1-invariants may be interpreted within the domain of EG as geometrical properties of figures. While some invariants are properties that the dynamic-figure maintains for *any* movement of a specific base-point (or all base-points) being dragged, other invariants are properties that may be "induced" to be invariant by *particular* movements of the dragged-base-point. Using Healy's terminology (Healy, 2000), the first type are *robust* invariants, while induced invariants are *soft* invariants. The solver may choose to use MD, i.e. to drag intentionally trying to induce a property as a (soft) invariant, that is to obtain an Intentionally Induced Invariant (III). Other soft invariants may then be perceived. We refer to these other invariants as Invariants Observed during Dragging (IODs), as in our MDS model. Different functions of variation (Leung, 2008) seem to explain how the perception of the different types of invariants may occur. For example, when determining and maintaining an III the solver mainly uses the function of contrast to identify a certain property which the figure can have “sometimes” but not “always”.

Row 2: A more complex type of invariant that can be perceived during a dynamic exploration is an invariant relation between level-1-invariants, we refer to invariants of this type as level-2-invariants. These are perceived within the DGS through awareness of *synchronic simultaneity* between two or more level-1-invariants. In the domain of EG these correspond to relations of logical dependency between geometrical properties.

Row 3: As the solver performs MD, s/he can determine a *locus of validity, LoV* (Leung & Lopez-Real, 2002), that is, a sketch of a trajectory along which to drag the base point in order to maintain the III. Coming up with a LoV can be quite difficult, and using the trace tool activated on the dragged-base-point may help. This can lead to a geometrical description of the LoV (GDLoV). A new invariant may be perceived: the “belonging of the dragged-base-point to the LoV”, an IOD.

Row 4: Within the phenomenological domain of a DGS the transition from perceiving two level-1-invariants to perceiving an invariant relation between them is delicate. We can describe this transition as follows. A first Critical Link (CrL1) is established as awareness of a level-2-invariant between an III and an IOD, such awareness is fostered by synchronic simultaneity of the two level-1-invariants. Moreover, a sense of direct/indirect control over each invariant may guide the conception of a *second critical link* (CrL2) between the invariants. A CrL2 is established when the solver can interpret the IOD as "causing" the III to occur within the phenomenological domain of the DGS. In the realm of Euclidean Geometry, critical links can be interpreted as a *conditional link (CL) between the geometrical properties corresponding to the IOD and the III*. Such a CL may be expressed in the conjecture. Soft or robust dragging tests may be used by the solver to test the hypothesized critical link and CL, using the functions of contrast, synchronic simultaneity, fusion, and generalization.

7.2.3 Our Findings with Respect to Boero's PGCs

Through our model we have described how conditionality seems to arise through the geometrical interpretation of causality determined by a combination of the perception of simultaneity plus direct or indirect control over the invariants observed when MD is used (Section 4.4). The complexity of the process can also be seen from a different perspective: during the process described by our model it is possible to identify several of the *processes of generation of conditionality* (PGC) introduced by Boero, Garuti and Lemut (Boero et al., 1999). In this section we propose a combined analysis to explore the consistency of our model with the PGCs described in the literature. During the complex process of conjecture-generation described by our model we have identified different possible PGCs . We believe that describing complementarities with the PGCs present in literature not only serves to contextualize our research, but it also serves as a basis for future research on the semiotic potential of the dragging tool with respect to the TEG and mathematics in general. Once we have described how our MD-conjecturing Model seems to feature a combination of PGCs, in Section 7.4, we will illustrate the mathematical meanings that can emerge from dynamic explorations that involve MD in generating conjectures, and that could be featured in future research on semiotic potential of dragging.

<u>PGC1 in the MD-conjecturing Model</u> - When solvers are exploring a particular configuration, focusing on a specific property and asking themselves "when" it might occur, they frequently seem to "freeze" the image and suddenly conceive a condition for the particular configuration to occur. This seems to occur mostly during the preliminary phases of an exploration, when basic conjectures are formulated, or bridge properties for MD are conceived. This behavior may be interpreted as an occurrence of a PGC1, that is

> a time section in a *dynamic exploration* of the problem situation: during the exploration one identifies a configuration inside which B happens, then the analysis of that configuration suggests the condition A, hence "*if A then B*". (Boero et al., 1999, p.140).

Consider the following example of such behavior.

Excerpt 7.2.3.1 - The two solvers are working on Problem 1, and they identify an interesting configuration: "ABCD rectangle". They seem to analyze the configuration leaving it static, as if frozen, and they provide a condition for this configuration to occur.

Episode	**Brief Analysis**
[1] F: a rectangle … [2] G: A rectangle. [2] F: More or less [he moves M so that ABCD looks like a rectangle]. [3] G: eh, look at the measures…when it comes out to be a rectangle. [4] F: eh…I don't know, well, about like this… [5] I: ok. [6] F: rectangle when…	The solvers identify an interesting configuration for the figure being a rectangle.

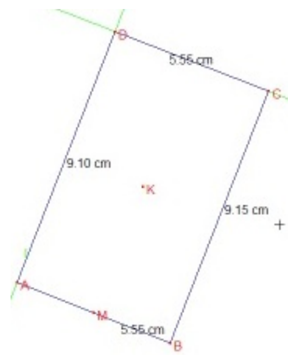
[7] G: when…eh, wait…when the perpendicular, I think, when the perpendicular to AB through M is also through K… [8] F: exactly [together] [9] G: …it's a hypothesis. [10] F: Wait, when the perpendicular…it's a conjecture [he gets ready to write it] [11] G: …through …The perpendicular to AB through M is also through K. [12] F: Ok. [13] G: Try to draw it… 	They identify a condition that they consider sufficient for the interesting configuration to be verified. They leave the image frozen on the screen.

Table 7.2.3.1: Analysis of Excerpt 7.2.3.1

Once they have placed the base-points in a way that makes ABCD look like a rectangle, the solvers do not perform any type of dragging. Instead they seem to freeze the configuration and identify a condition A inside which they think B occurs. The phenomenon B in this case is "ABCD rectangle" ([1], [2]) and the condition A is "the perpendicular to AB through M is also through K" ([7], [11]). The relationship between A and B is expressed by the solvers through the word "when" ([17], [10]). This process of generation of conditionality has also been eloquently described as follows:

> the conditionality of the statement can be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process that will be subsequently detached from it and then "crystallize" from a logic point of view ("if…, then…"). (Boero et al., 1996).

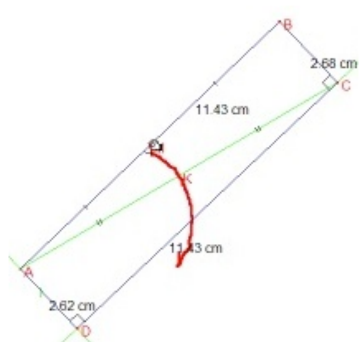
The word "when", used by the solvers to express the conditional relationship between A and B, seems to mark the "crystallization" described by Boero.

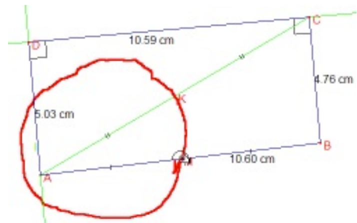
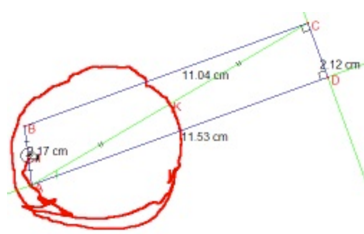
PGC2 in the MD-conjecturing Model - When solvers are determining the figure-specific path by searching for a GDP they frequently use MD (with or without the trace activated) and continuously check “when” the desired regularity, B, is maintained, in a continuous manner. They seem to do this by generating the condition, A dynamically through continuous trials and errors during which they check that “when the dragged-base-point is not on the hypothesized figure-specific path” (“not A”) the regularity B fails to happen. This behavior seems to be well described by a PCG2, that is:

> noticing a regularity B in a given situation then identifying, by exploration performed through a *transformation* of the situation, a condition A, present in the original situation, such that B may fail to happen if A is not satisfied. (Boero et al., 1999, p.141).

Consider the following example of such behavior.

Excerpt 7.2.3.2 - The excerpt is taken from the same exploration as in Excerpt 7.2.3.1. Here the solvers are refining their GDP and they seem to be using a process of generation of conditionality of the second type.

Episode	**Brief Analysis**
[22] **F**: Ah, it looks like a curve!	GDP1: a curve
[23] G: Again a nice circle?	refinement of the GDP1: “a nice circle”
	
[24] **F**: Like this…	Here there is a change in the dragging mode.
[25] **F**: It’s definitely not a straight line.	
[26] I: hmm…	
[27] **F**: So it’s a curve…	

[28] **F**: Let's go the whole way around and see...what happens...  [29] I: Wait, not you are going around without maintaining the property, I think. [30] **F**: Well, more or less…no?...like this? [31] G: eh, here, here…here I don't think it is a rectangle… [32] **F**: No, no...you're right you're right. [33] **F**: So more or less we were starting from here …	Instead of looking at the III, F seems to concentrate on the "circle" and he finishes to "go the whole way around". This is a version of the *soft dragging test,* at least for G who seems to also keep on checking the III.
[34] I: eh… [35] **F**: It looks like it goes through A…  [36] G: …and through K. [37] **F**: Where? [38] G: It looks like a circle…with diameter AK.	Now they refine the GDP1 adding the property "passing through A" and then "through K". Therefore we now have a GDP2: a circle through A and K; and then a GDP 3: a circle with diameter AK.

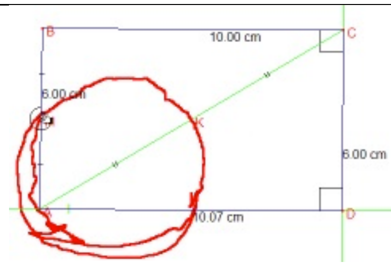
[39] **F**: Yes, that's what it looks like,  exactly. It looks like a circle with diameter AK.	

Table 7.2.3.2: Analysis of Excerpt 7.2.3.2

During this episode the solvers are performing MD, moving the dynamic-figure and proposing successively more refined GDPs. Condition A in this case is "M moves (?) on a circle" and the regularity B is "ABCD rectangle". During the refinement of the GDP, once F has dragged "the whole way around" but without paying attention to the III, the solvers seem to be noticing and describing a regularity A, through the refined GDPs, such that "B may fail to happen if A is not satisfied". In fact the final GDP seems to arise dynamically, from a series continuous trials-and-errors, as an object such that if the dragged-base-point is not on it the regularity B is not verified.

<u>PGC3 in the MD-conjecturing Model</u> - When determining a GDP the solvers start searching for a regularity from the movement (and the trace mark if the trace activated). Solvers seem to be associating some perceived regularity to other regularities previously discovered in other experiences. Moreover, reasoning through "selection and generalization" (Boero et al., 1999) seems to be used by solvers who select a subset of positions from the movement (or points from the trace if activated) that have in common some property (for example that of being equidistant from an imagined point in the case of a circle) and from which a "general rule" can be inferred. We think this process could

also be described as a “continuous” case of Boero, Garuti, and Lemut’s description of PGC 3, that is

> a ‘synthesis and generalization’ process starting with an *exploration* process of a meaningful sample of conveniently *generated* examples (Boero et al., p. 141).

Consider the following example of such behavior.

Excerpt 7.2.3.3 - This excerpt was presented in Chapter 4 (Excerpt 4.3.1), and here we repropose an episode from it to illustrate how PGC of the third type seem to take place when the movement of the dragged-base-point and the trace mark are used to reach a GDP and an IOD.

Episode	**Brief Analysis**
[28] I: So Ste, what are you looking at to maintain it? [29] **Ste**: Uhm, now I am basically looking at B to do something decent, but... [30] I: Are you looking to make sure that the line goes through B? [31] **Ste**: Yes, exactly. Otherwise it comes out too sloppy...  [32] I: and you, Giu what are you looking at? [33] Giu: That it seems to be a circle... [34] **Ste**: I'm not sure if it is a circ... [35] Giu: It's an arc of a circle, I think the curvature suggests that.	Ste is using the property “the line goes through B” as his III ([29], [30]). Both students show the intention of uncovering a path by referring to “it” ([31], [33], [34]). Giu, in particular concentrates on describing the path geometrically and he seems to recognize in the trace a circle ([33]) or an arc of a circle ([35]).

Table 7.2.3.3: Analysis of Excerpt 7.2.3.3

As the solvers look at the trace mark left by the dragged-base-point while they perform MD, they conceive an idea about what a GDP might be. They are able to do this through an exploration with MD in which they conveniently generate a significant sample of examples. From these examples they generalize the perceived regularity, from a movement along an arc of a circle to a whole circle ([33]-[35]).

The complexity of the process of conjecture-generation described by our MD-conjecturing Model becomes evident once again, in a new way, if we emphasize the presence of various PGCs within it, as we have tried to do. Not only does a combination of PGCs seem to be present during the process, but there is also a new element with respect to the initial description of the PGCs: continuity that is induced by the specific kind of motions that occurs in a DGS. While the examples provided for each of the described PGCs in literature have mostly been of a "discrete" nature, the presence of dragging, and MD in particular, attributes a new "continuous" nature to the processes. Although dynamicity seems to provide support for this particular process of conjecture-generation, making it more "natural", it may turn into an obstacle as far as the aim to formulate conjectures within the "static" TEG, where it becomes necessary to "eliminate" time. We will discuss this issue briefly in Section 7.3.2.

7.3 Implications of the Study and Directions for Future Research

In Chapter 1 we introduced the importance within the field of mathematics education of ameliorating the teaching and learning of Geometry, and how the use of open problems can be a means to achieve this goal. Particular issues within this line of research arise when studying the didactic potential of open problems in dynamic geometry. Our results specifically address questions in this field that involve dragging and its possible role within the teaching and learning of Geometry. Our results shed light onto possible answers and avenues of research that could lead to more complete answers to some questions that were posed in different moments by researchers in this field. In particular our MD-conjecturing Model describes a process for generating conjectures in a way that can become "mechanical" as we have described in Chapter 6. Reasoning about the use of MD and fostering awareness of the process of conjecture-generation achieved with its support can be used by the teacher to trigger a process of semiotic mediation centered on the use of dragging with respect to mathematical meanings like "premise", "conclusion", "implication", and "conjecture". In Section 7.3.1 we will interpret our findings within the frame of semiotic mediation and highlight their didactic potential with respect to the construction of these specific mathematical meanings. Specifically, we will describe how our model seems to support the design of activities that could be used in the classroom to exploit the use of MD to mediate these particular mathematical meanings. This didactical implication is important since these activities can be used to have the students engage in discussions with classmates and the teacher that can foster their development of these mathematical notions useful in the overarching context of proof. This is emphasized, for example, in *Principles and Standards for School Mathematics* (NCTM 2000) that states: "Reasoning and proof are not special activities reserved for special times or special topics in the curriculum, but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied." (p. 342). Moreover such activities give students the opportunity to use their prior knowledge as they enhance their learning, while engaging in a physical experience within a DGS, to actively build new mathematical knowledge.

However, with respect to the issue of teaching and learning proof, our findings suggest different hypotheses to be refined and investigated in future research. We will frame our description of these implications considering the theory of reference with respect to which a proof of a particular statement may be constructed. Such a conception has been introduced by Mariotti (2000) through the following characterization of “theorem”:

> ...any mathematical theorem is characterized by a statement and a proof and that the relationship between statement and proof makes sense within a particular theoretical context, i.e. a system of shared principles and inference rules. Historic-epistemological analysis highlights important aspects of this complex link and shows how it has evolved over the centuries. The fact that the reference theory often remains implicit leads one to forget or at least to underevaluate its role in the construction of the meaning of proof. For this reason it seems useful to refer to a 'mathematical theorem' as a system consisting of a statement, a proof and a reference theory (Mariotti, 2000, p.29).

Pedemonte has proposed a similar characterization of “conjecture” (2007), as a triplet consisting of a statement, a system of conceptions (Balacheff, 2000; Balacheff & Margolinas, 2005), and an argumentation. Considering the symmetry between the two definitions we will analyze the potential cognitive gap that emerges between an argumentation developed within a DGS and a proof, if the theory of reference is the Theory of Euclidean Geometry (TEG). Sections 7.3.2 and 7.3.3 are devoted to different aspects of this gap: first a description of elements that may make the transition from the phenomenology of a DGS to the TEG problematic, and then an interpretation of the cognitive gap within the perspective of cognitive unity (Boero, Garuti & Mariotti, 1996). Framing the gap between argumentation and proof, when the solvers’ system of conceptions is related to the phenomenology of a DGS and the theory of reference is the TEG, will serve to outline our hypotheses on how the gap may be (partially) bridged if the MDS is used as a psychological tool, freed from the support of the instrument.

7.3.1 Semiotic Potential of Our Findings with Respect to the Elaboration of a Statement

We have described how our cognitive model sheds light onto a process leading to the formulation of a statement that makes a conditional link (CL) between two invariants explicit (Section 4.5). Within our model we have referred to this statement as a *conjecture*. Maintaining this perspective, we can frame our findings within the theory of semiotic mediation (TMS) and describe the didactic potential withheld by the conjecturing process described by our model with respect to important mathematical notions such as premise, conclusion, implication, conjecture, and theorem. We will first briefly introduce aspects of the TMS that we will use to frame our findings, and then we will describe the specific semiotic potential of dragging highlighted by our findings, and our hypotheses on how this semiotic potential might be exploited. These hypotheses can be used in future long term teaching experiments that investigate the semiotic potential of dragging and of MD specifically.

<u>Brief Introduction to the Theory of Semiotic Mediation (TMS)</u> - Semiotic mediation in the field of mathematics education is a form of mediation between students and mathematical knowledge that occurs through signs. Researchers have recently adapted the idea of semiotic mediation, introduced by Vygotsky (1987), to the context of school mathematics (Mariotti, 2001, 2002; Bartolini Bussi, Mariotti & Ferri, 2005; Falcade, Laborde, & Mariotti, 2007; Mariotti & Maracci, 2009; Bartolini Bussi & Mariotti, 2008). We stress what is intended with *semiotic mediation* as opposed to *mediation tout-court*. The latter is the mediation that occurs when a tool acts as a prothesis, in that it only serves for helping the user accomplish a task. For example, a fishing rod mediates (tout-court) the task of fishing. Instead, the former occurs when a tool is used not only to accomplish a task, but also to put the user in contact with another "theory/world." For example Cabri not only can be used to help solve a problem, but it also puts the user in touch with the

world/theory of Euclidean Geometry, and it can be used purposefully with this intent by the teacher. Of course the two kinds of mediation are interrelated; in particular, acting by means of a tool may constitute the basis of the subsequent functioning of the same tool in the process of semiotic mediation, triggered by the teacher.

Bartolini Bussi and Mariotti developed the ideas of *tool of semiotic mediation* and of *semiotic potential of an artifact*:

> ...any artifact will be referred to as a tool of semiotic mediation as long as it is (or it is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention (Bartolini Bussi & Mariotti, 2008).

When an artifact is used to mediate meanings, we can speak of its *semiotic potential* (Bartolini Bussi & Mariotti, 2008):

> on the one hand, personal meanings are related to the use of the artifact, in particular in relation to the aim of accomplishing the task; on the other hand, mathematical meanings may be related to the artifact and its use. This double semiotic relationship will be named *the semiotic potential of an artifact.*" (p. 754).

The analysis of the semiotic potential of an artifact can focus on the possible interaction between students and the artifact during appropriately designed activities, the artifact and the mathematical meanings evoked during these activities, and on how the teacher can guide the development of mathematical meanings from the personal meanings by interacting with the students and using the artifact. Computers, in general, and a DGS, in particular, can be considered tools of semiotic mediation (Mariotti, 2006; Bartolini Bussi & Mariotti, 2008).

If a goal of education is to have students engage in sense-making and argumentation with respect to specific mathematical content (for example, NCTM, 2000), teachers need to have a variety of activities available to propose and integrate into the Geometry curriculum. This section presents issues to be taken into consideration in designing activities that can be used in the Geometry classroom within the perspective of semiotic mediation. When designing and using activities of this sort it is fundamental not to forget the complexity involved in mathematical sense-making process, leading potentially to a variety of difficulties. These may be analyzed through the lens of our model which hopefully will provide useful insight into both understanding and helping students overcome their difficulties. Further research involving long term teaching experiments in this area is necessary to test our hypotheses and to better describe how the semiotic potential of dragging, and maintaining dragging in particular, may be exploited. In the next section, considering "dragging" as an artifact, we use our findings to highlight the *semiotic potential of dragging* with respect to particular mathematical meanings.

The Semiotic Potential of Dragging from Our Findings - Our model focuses on a particular process of conjecture generation that sees the emergence of a premise and a conclusion from different invariants perceived during a dynamic exploration. In this section we will analyze this process of emergence of the premise and conclusion of a conditional statement, and discuss how these findings contribute to the analysis of the semiotic potential of MD with respect to particular mathematical meanings such as "premise", "conclusion", "implication", "conjecture", and "theorem". Moreover we will describe how our distinction of different types of invariants highlights how the potential of MD could be exploited elaborating on the different types of invariants.

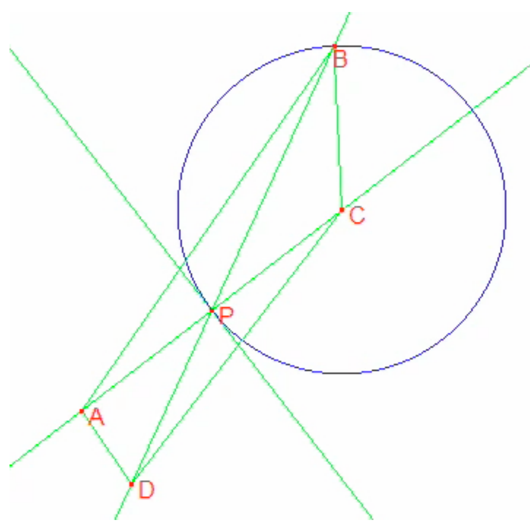


Fig 7.3.1.1 ABCD as a result of the step-by-step construction.

Let us consider an activity like Problem 2 (Section 3.3.3). A step-by-step construction is given and the solver is asked to make a conjecture about the possible configurations that can occur. If we consider activities like this, or in general, activities that contain a series of steps followed by a question like: "what can you say about the figure?, or what can you say about…when…?, or under what conditions can the figure become a…?", it is possible to clearly/explicitly distinguish the invariants destined to originate the conclusion and the premise of the conjecture that is the outcome of the exploration as it can be carried out by the student. In particular the invariant (the III) that is destined to become the conclusion of the conjecture has the following characteristics that make it clearly recognizable:

1) it is a first soft invariant that may be induced,

2) it is induced indirectly and it is a configuration that can be acted-upon by moving different base points,

3) once a second soft invariant is perceived (the IOD) with respect to the dragged-base-point, the two invariants appear simultaneously but the control over the III is indirect.

On the other hand the invariant destined to originate the premise has the following characteristics:

1) it is a soft invariant perceived while a first one (the III) is being induced,

2) it is searched for in response to the question "what might cause the III to be maintained?",

3) it is related to a specific base point and therefore can be induced directly by dragging this base point,

4) it is perceived simultaneously with the III but differs in the type of control that the solver exercises over it.

The characterization of the invariants can be used by the teacher during collective discussions and in so doing exploiting the semiotic potential of maintaining dragging with the aim at developing the mathematical meanings of premise and conclusion of a conditional statement.

Another component of the MD-conjecturing Model that has an important counterpart in the development of the idea of conjecture is what we have described as a "bridge property" (Section 4.2.1.3), that is a property that implies the property corresponding to a previously conceived III, and that therefore can be used during MD in substitution of the original III. The emergence of *bridge properties*, may give the opportunity of introducing the idea of implication. As a matter of fact, the relationship that links the selected property (III) and these new properties has a counterpart in the theory in a logic relationship that may become the aim of the didactic intervention.

<u>The Role of the Task in the Analysis of the Semiotic Potential</u> - Although the analysis of soft invariants in step-by-step construction problems seems to have a strong semiotic potential with respect to the development of mathematical meanings such as premise and conclusion of a conditional statement, analyzing different types of robust invariants in step-by-step construction problems also withholds semiotic potential. Various activities can be constructed around step-by-step constructions in order to foster the development of these mathematical meanings from the analysis of robust invariants. In particular, we will show that the type of problems we developed for the interviews, can be used within a context of semiotic mediation to help students construct the meanings of

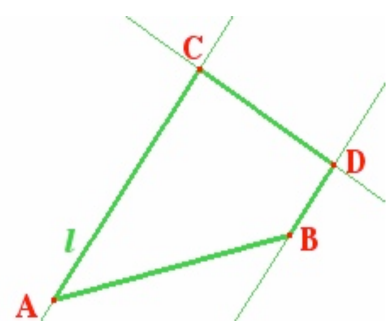


Figure 7.3.1.2: ABCD as a result of the step-by-step construction.

"implication" and more in general of "conjecture" and "theorem". Let us consider the step-by-step construction in our Problem 4 (Section 3.3.3):

- Draw three points: A, B, C.
- Construct the parallel line *l* to AC through B,
- and the perpendicular line to *l* through C.
- Construct D as the intersection of these two lines.
- Consider the quadrilateral ABCD.

Students can be asked to list all the information about ABCD that they know given the steps of the construction. Within such a list different robust invariants will emerge, and basic construction invariants and derived construction invariants (Section 4.2.1.1) may both be present. For example, a student may produce the following list of properties of ABCD:

- AC parallel to BD,
- angle ACD right,
- angle CDB right,
- ABCD right trapezoid.

The first two properties in the list are basic construction invariants, while the second two are derived construction invariants, since "angle CDB right" is not explicitly contained in the steps of the construction, but it can be derived through logical implication from the first two properties. Reflection upon differences between these two types of construction invariants can help the construction of the meaning of "implication" within a theory. In this sense it could be a step towards the construction of the meaning of "theorem" conceived as a triplet (statement, theory, proof) in Mariotti's terms (2000).

Once students have reflected upon the construction invariants, it can be made explicit how the geometrical properties that correspond to these invariants will always be part of the premise (although maybe implicitly) of any conjecture developed on ABCD. In order to foster awareness of these properties and of their meanings, the teacher can ask students to list them explicitly for a number of conjectures, before allowing that these properties be used implicitly.

We showed how the notions of basic and derived construction invariants can be used to distinguish between properties of a figure that emerges from a step-by-step construction, leading to the development of the meaning of “implication”. The notion of point-invariants (Section 4.2.1.2) may also be useful to distinguish between robust invariants that correspond to derived-construction invariants as opposed to invariants that are robust only for the dragging of particular base points, and that therefore do not correspond to general properties of the geometrical figure represented by the product of the step-by-step construction.

Asking students to compare and discuss their solutions to activities like the ones described, designed to foster the emergence of meanings of particular mathematical notions, can be useful within a process of semiotic mediation towards notions like premise and conclusion of a conditional statement. Moreover, as described, students can gain awareness of logical dependencies between geometrical properties by constructing and perceiving the corresponding invariants. In particular students can be guided to reason about what they perceive, on how a dynamic-construction can be used to show relationships between properties, and, more generally, about what a logical implication might me, abstracting from the situated context (Noss & Hoyles, 1996). During the discussion various issues may arise, such as how the perception of simultaneity plus direct or indirect control over an invariant property can be interpreted statically as logical dependence of one property from another. A discussion centered around the relationship between steps of a construction and geometrical properties explicitly stated in the conjecture may serve to develop further understanding of the notions of premise and conclusion of a conditional statement, and of logical implication. Moreover, engaging in activities similar to the ones described students will have the opportunity of engaging in explorations that require flexibility in recalling and using different definitions and representations of the figures involved.

7.3.2 The MD-conjecturing Model with Respect to a Theory

In the previous section and throughout our study we used the word "conjecture" to refer to particular kinds of statements originating from an open problem and still requiring a proof. Now we will consider these statements with respect to the solver's system of conceptions (Balacheff, 2000; Balacheff & Margolinas, 2005) and to the argumentation they are generated through, according to Pedemonte's definition (2007). This conception of conjecture is symmetric with respect to Mariotti's definition of theorem as the triplet consisting of a statement, a theory of reference, and a proof (Mariotti, 2000). Thus this conception of conjecture introduces a correspondence that may be used to describe the relationship between the exploration phase, when the conjecture statement is produced and the proof phase when such statement is proved, or is to be proved.

Let us consider the case in focus when the production of the conjecture is accomplished through the use of MD and the proof is expected in the TEG. The cognitive gap that may arise is potentially quite wide if the argumentation is constructed within the solver's system of conceptions in the phenomenological domain of a DGS and the theory of reference is the Theory of Euclidean Geometry (TEG). Although there might be the possibility of bridging such cognitive gap by choosing to introduce a different theory of reference that might be constructed upon "axioms of a DGS", we choose the

TEG as the theory of reference. Therefore, we must consider the complex issue of transitioning from a dynamic conception in which dynamism (and therefore time) is present, to a *generalized* and *static* domain, that of the TEG, ordered by logical implications and in which time is no longer present. In the following sub-section we will discuss the complexity of this task, through a few considerations on the elimination of dynamism in order to interpret the findings geometrically and generate a conjecture with a statement that is provable within the TEG.

Here we would like to highlight an interesting feature of our findings. Mechanical use of MD can be a powerful tool for generating conjectures: expert use of MD seems to lead smoothly to sense-making of the findings of a dynamic exploration in terms of a conjecture that could be proved within the TEG. However few elements of the argumentation leading to the conjectures are transferrable to the TEG. In fact frequently only the invariants corresponding to the premise and the conclusion of the final conditional statement are interpreted within the TEG. This contributes to widening a discontinuity between argumentation and proof. The phenomenon can be interpreted within the perspective of cognitive unity as we will do in Section 7.3.3.

<u>Transitioning from the Phenomenology of a DGS to the Theory of Euclidean Geometry: the Elimination of Dynamism</u> - We have described how personal meanings concern the idea of dependent movement as it emerges from students' activities in a DGS, characterized by dynamism; while mathematical meanings concern the ideas of logical dependence between premise and conclusion of a conjecture in the context of the TEG. The dragging tool is the means connecting dynamism to logical statements, in a process through which the solver gains theoretical control, moving from personal meanings to mathematical meanings of his/her observations. Goldenberg and Cuoco

(1998) provide an insightful example of how invariants are such with respect to the dragging and therefore to a dynamic perception.

> We hypothesize that when an endpoint of a stretchy segment is moved, and the segment is the only object present, the user perceives the movement as a *translation* of the point. That is, dragging A to A' may feel psychologically like a translation. The display may also tend to be seen more as a mapping of A (in its various positions) to C (the midpoint of AB), than as a mapping of A and C to A' and C' respectively. But other situations may lead to very different perceptions. For example consider the same construction with a perpendicular to AB at B. A comparable movement of A now appears to rotate the system; the sense that A is being translated is now considerably diminished…What do students make out of this we don't yet fully know (p. 352).

A major difficulty is that it is hard to "translate" these dynamic observations into logical propositions. The literature indicates that dynamic thinking seems to be useful for generating conjectures (for example, Hadamard, 1949; Polya, 1962; Schoenfeld, 1985; Thurston, 1995; Simon, 1996; Boero et al., 1996, 1999) long before the advent of dynamic geometry. However little is known on how the elimination of the dynamic components of processes of conjecture-generation may occur. In the following paragraphs we will describe aspects of the complexity of this translation when the dynamism is situated within the domain of a DGS.

Conjectures generated within a DGS can be based on a crucial element, which has a dynamic nature, but the dynamic nature of this element can conflict with the static nature of the theorems available in the TEG (Mariotti, 2000). The literature is filled with cases in which subjects are not able to find compatibility between geometric static knowledge and the perceptions of "movement" generated by the software. This can be explained as follows. When the figural part is dynamic and the conceptual part is static, there is a conflict. For example, it can be very difficult to conceptually control the phenomenon of a point moving on a circle through the definition of *locus of points* in Geometry. When using dragging, and in particular movement along a path, it is possible to end up in similar situations because of the simultaneously dynamic and static nature of the path, as described in Section 6.1.

Another aspect of the translation from dynamism within the DGS to staticity within the TEG has to do with the perception of generality of a figure accomplished through the "condensation of dynamism". As described by Mariotti (2010), the dynamism of a Cabri-figure is perceived as change in contrast to what remains simultaneously invariant: the interaction between what changes and what does not is at the basis of the perception of *movement of the image*. The invariants, that remain unchanged constitute the identity of the figure on the screen, that is they allow recognition of the image on the screen as a unitary object "in movement" and perhaps as a particular "geometric figure", for example a trapezoid or a parallelogram. The dialogue between invariants and variation is at the basis of the process of conceptualization: it allows us to recognize very different objects as belonging to a same class of geometrical objects, or to recognize a person's face after many years. So in a DGS variation represents generality of a concept. For example, a Cabri-figure represents a "general square" because of its potential variation during dragging, a variation that maintains the theoretical properties of a square as invariants (Mariotti, 2010).

Dominating generality in dynamic terms is not trivial, because it requires "condensing" the dynamism. When does a solver say that a certain figure (or part of a figure) "is the same" object, or "is always" something? Let us think a bit more about it using an example. Assume that a certain Cabri-figure is constructed so that it is a robust parallelogram. What does perceiving a Cabri-figure as a generic parallelogram mean? First of all, the perception lies within the mind of the perceiver, in our case the solver, so the Cabri-figure will be compared to the solver's figural concept of parallelogram (Fischbein, 1993; Mariotti, 1995). As the solver moves the Cabri-figure, s/he may recognize various instances that correspond to his/her figural concept of parallelogram, and no instances that do not correspond to such image. In this case the solver may mentally “condense” the instances and recognize the Cabri-figure as a *generic* parallelogram. However, depending on how it has been constructed, the Cabri-figure may only represent a subset of all possible parallelograms. In this case the solver will probably recognize it as a “parallelogram”, but is it still a generic parallelogram?

Difficulties may emerge as the solver compares his/her conception of a figure with the dynamic-figure on the screen. For example, the solver may be thinking of a specific subset of parallelograms, say all homothetic parallelograms with respect to a particular one, and s/he may be identifying “parallelogram” with this conception. In this case, if the Cabri-figure is dragged into a configuration that does not belong to the set of homothetic parallelograms, the solver may not perceive it as a parallelogram any longer. There may be further subtleties in the process of recognizing different screen images as instances of something more general. Moreover, when a property is not constructed robustly within a Cabri-figure, complications in the process of perceiving generality seem to increase.

Concluding Remarks - In conclusion, our model describes how the process of conjecture-generation through expert use of MD makes use of dynamism within the phenomenological domain of the DGS. This, on one hand seems to facilitate the process of conjecture-generation, but, on the other, it makes it necessary to eliminate the dynamic component if we choose to work towards a theorem that has the Theory of Euclidean Geometry (TEG) as the theory of reference (Mariotti, 2000). Thus there is a potential cognitive gap between an argumentation within the phenomenology of a DGS, based on a system of conceptions that is dynamic, and a proof within the TEG. The next section is dedicated to a further analysis of this gap, and to some new hypotheses we advance with respect to conjectures generated when the MDS is used as a psychological tool, freed from the external support of the instrument.

7.3.3 Links to Proof

In this section, according to Mariotti's definition of theorem, we will consider the conditional statement of a conjecture as a potential statement of a theorem. Within this perspective we will discuss implications and hypotheses that arise from our findings with respect to proof. First we will consider different types of conjectures that arose from the dynamic explorations our solvers engaged in, characterizing them through the process by which they were generated. Then we will advance hypotheses on how the process of generation of each conjecture may foster (or not) its proof within the TEG. We will frame these considerations within the construct of cognitive unity (Boero, Garuti & Mariotti, 1996; Pedemonte, 2007b).

We described how expert use of MD leads to "automaticity" in the process of conjecture-generation in which it is used. On the other hand this automaticity seemed not to be present in the case of *internalization* of MD (Section 6.2.3). If we consider conjectures generated in these two ways, the differences do not reside in the statement of the conjecture: expert use of MD seems to lead to statements in which the premise and the conclusion are "distant". In other words, conjectures generated through expert use of MD seem to exhibit a "gap" between the premise and the conclusion, and no bridging geometrical properties emerge from the exploration leading to the statement of the conjecture. On the other hand, it seems that internalization of MD leads to conjectures accompanied by geometrical arguments bridging the premise and the conclusion.

The relatively small amount of data analyzed in our study does not allow us to make general statements about the observation we illustrated above. Moreover our study was not focused on investigating the internalization of MD and its transformation into a psychological tool. These are secondary findings that we briefly introduced in Chapter 6. However they can be considered as potentially interesting directions for future research. At this point we focus on the two types of conjectures, those with a "gap" that emerge through expert use of MD and those that emerge as a product of an internalization of MD, and we advance our hypotheses on their respective relationships with proof.

Although proof was not taken into consideration in this study, in some cases solvers would proceed to give an oral proof of some of their conjectures. This happened after F and G reached their strong conjecture described in Episode 2 of Excerpt 6.2.3. We present this episode below for ease of the reader, highlight the geometric properties that emerge through an abduction, and then describe the oral proof provided by the solvers.

We remind the reader that F and G in this exploration, before this episode, have attempted to use MD having chosen "ABCD parallelogram" as their III and "PB=PD" as a bridge property.

Episode 2 of Excerpt 6.2.3	**Brief Analysis**
[43] G: eh, since this is a chord, it's a chord right? We have to, it means that this has to be an equal chord of another circle, in my opinion with center in A. because I think if you do, like, a circle with center	G uses the theory to interpret what he is seeing. G seems to focus on DP and PB and interpret them as chords of symmetric circles. As if the movement of these chords (not

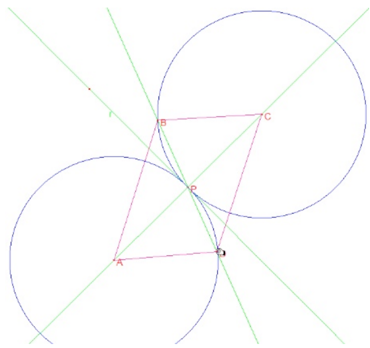
[44] **F**: A, you say… [45] G: symmetric with respect to this one, you have to make it with center A. [46] **F**: uh huh [47] G: Do it! [48] **F**: with center A and radius AP? [49] G: with center A and radius AP. I, I think…  [50] **F**: let's move D. more or less… [51] G: it looks right doesn't it? [52] **F**: yes.	of D) led him to the second circle. The abduction (in Pierce's terms) seems to proceed as follows: • fact: DP=PB (and their behavior during maintaining dragging) • rule: given symmetric circles with PB and PD symmetric chords, then PB=PD (and they would behave like this) • abductive hypothesis: there exists a symmetric circle with center in A and radius AP.

Table 7.3.3.1: Analysis of Episode 2 of Excerpt 6.2.3

In the brief analysis we presented next to the excerpt we highlighted how the abduction (described in Pierce's terms) makes use of elements of the TEG, in particular geometrical properties that link the circle on which D is assumed to move to the III ("ABCD parallelogram"). Once the solvers have tested the conjecture "D belongs to the circle centered in A with radius AP implies ABCD parallelogram", they engage in an oral proof. The proof they develop proceeds as follows:

- the circles are symmetric so AD is congruent to AP which is congruent to PD and to therefore to BC;
- the isosceles triangles APD and PBC are congruent because they have congruent angles, since the angle DPA is opposite at its vertex to CPB;
- therefore PD is congruent to PB,

- so ABCD has diagonals that intersect at their midpoints and therefore it is a parallelogram.

A key idea (Raman, 2003) in the proof is the interpretation of PD and PB as chords of symmetric circles, which emerged in the conjecturing phase of the investigation. The use of properties of symmetric circles is fundamental both to the development of the conjecture and of the proof. We advance the hypothesis that when MD is internalized and used as a psychological tool, reasoning used in the conjecturing phase (and abduction in particular) leads to the emergence of geometrical properties that logically relate the premise to the conclusion of the conjecture and that can be re-used in the proving phase.

When such a way of thinking is developed the abductive reasoning has the advantage of involving geometrical concepts, like in the case of F and G. Our hypothesis is that the geometrical concepts that emerge in this case can become “bridging elements” with respect to the proving phase, since they can be re-elaborated into the deductive steps of a proof. On the other hand, expert use of the MD seems to lead to conjectures in which no geometrical elements arise to “bridge the gap” between the premise and the conclusion. In other words, although expert use of MD seems to offer the possibility of generating “powerful” conjectures that solvers might have trouble reaching without support of the dragging-support (since the IOD which becomes the premise may be cognitively “quite distant” from the conclusion), generating conjectures “automatically” through the MDS supported by the dragging-support, may hinder the proving phase in which these “bridging elements” are essential.

In terms of cognitive unity (Boero, Garuti & Mariotti, 1996), it seems like strong conjectures generated through mechanical use of MD will lead to cognitive rupture. This seems to be the case because the process of conjecture-generation, or the argumentation phase, is supported by the DGS. In particular we have described particular types of arguments that are used by solvers during the conjecturing phase of an open problem activity and that are supported by the DGS conceived as an instrument. In Section 6.2 we introduced the notion of instrumented abduction as a particular type of instrumented argument. Further research is necessary to generalize and elaborate these notions, however what we stress here is that the warrants of such arguments are supported by an instrument, in our case dragging or the DGS more in general. As a consequence the arguments make use of many elements that do not directly correspond to geometrical properties and that therefore cannot be re-used in a proof residing within the TEG. This leads to a potential strong rupture between the conjecturing phase and the proving phase that may be manifested through solvers' potential difficulties with proof of a statement generated through mechanical use of MD.

On the other hand, we hypothesize that if expert solvers interiorize MD transforming it into a psychological tool, or a fruitful "mathematical habit of mind" (Cuoco, 2008) that may be exploited in various mathematical explorations leading to the generation of conjectures, there might be a greater cognitive unity between the conjecturing phase and the proving phase. In other words, our hypothesis is that when the MDS is used as a psychological tool, the conjecturing phase is characterized by the emergence of arguments that the solver can set in chain in a deductive way when constructing a proof (Boero et al., 1996). We think this may occur if, as in the case of F and G, abduction in which the rules are taken from the domain of TEG is used during the conjecturing phase. An abduction of this sort seems to expose key ideas to use in the proof, and geometrical properties that bridge the gap between the premise and the conclusion. At this point a logical re-ordering of these properties might be sufficient for the construction of the proof. Again, we do not have enough data to support the claims we are making in this section, but our data suggests that these may be important issues to study in order to gain insight into how a DGS can be used (or not) in the context of proof.

7.3.4 Directions for Future Research

We would like to conclude this Chapter by introducing some general questions that arise from our study, and by outlining two possible directions for future research that might be carried on from our study. First, given our findings, a discussion should be opened about whether, as a mathematics education community, we are interested in fostering a process of conjecture-generation as described by our model, and therefore whether specific dragging modalities, and maintaining dragging in particular, should be taught as part of the mathematics curriculum. If we decide to add *the dragging schemes* to curricula we must consider issues related to fostering an instrumental genesis of MD. In particular, how to develop students' construction of both components of the MDS, but also related to fostering the internalization of MD that might induce use of abduction leading to the emergence of bridging elements in sight of proof. Moreover, we would need to consider students' difficulties in developing the maintaining dragging scheme; how long a potential teaching sequence should be; which dragging modalities (and schemes?) should be taught and how; what (if any) elements of our model should be made explicit during the teaching sequence. Moreover, might it be possible, through particular teaching strategies, to avoid some of the potential cognitive difficulties that the dragging schemes seem to induce? If not, what strategies might be developed to overcome such difficulties? Furthermore, it would be beneficial to investigate whether there are particular types of students who benefit more (or less) from being introduced to the dragging modalities (and schemes). On the other hand, if we choose not to introduce specific dragging modalities at the classroom level, would it be beneficial (and in what ways) for teachers to be aware of possible utilization schemes like the one for maintaining dragging described by our model, since dynamic geometry is already being used in many classrooms?

As for the two lines of research we outline, one aims at developing research from our findings within the theory semiotic mediation, to investigate how the semiotic potential of dragging, and maintaining dragging in particular, might be exploited; the second investigates our hypotheses with respect to proof and cognitive unity that we introduced in Section 7.3.3.

Studies on the Semiotic Potential of Dragging - Studies on the semiotic potential of the artifact *dragging* in a DGS based on the development of precise hypotheses from our study, with respect to tasks that involve conjecture-generation. The hypotheses would emerge from our reflection on our findings with respect to semiotic mediation, as presented in Section 7.3.1, involving the relationship between the use of dragging and in particular *maintaining dragging* and the mathematical meaning of conjecture and the related notions of premise, conclusion, conditionality, and implication. An appropriate methodology could be a long term teaching experiment to allow a first validation of the hypotheses arising for our study.

In particular a long term teaching experiment could allow to observe the hypothesized unfolding of the semiotic potential of the MD and the evolution of personal meanings into the mathematical meanings through the semiotic processes triggered and orchestrated by the teacher in classroom discussions (Mariotti & Maracci, 2010).
Our notions of *instrumented abduction* and *instrumented argument* could be further elaborated in light of the analysis of the effectiveness of the didactical intervention aimed

at developing mathematical meanings, from the use of maintaining dragging relative to the notion of conjecture.

Studies on Proof in a DGS - A second line of research could investigate the hypotheses we advanced in Section 7.3.3 with respect cognitive unity. In particular it could be insightful to study the process of generation of conjectures in solvers who have interiorized MD and who are using it as a psychological tool. This way it would be possible to test out hypothesis on the presence, within this process, of abduction that uses rules from the TEG, like in the case of F and G, and of potentially other forms of reasoning that lead to geometrical properties that can bridge the gap between the premise and the conclusion of the produced conjecture. If this were to be the case, the study should then compare the conjecturing phases in which the two types of conjectures are developed with the subsequent proving phase. This analysis could be used to test our hypothesis on the emergence of geometrical properties during the conjecturing phase, in the case of conjectures developed through the use of the MDS as a psychological tool that can be used as key ideas in a proof of the statement of the conjecture. Confirmation of this hypothesis would be a significant result for designing activities in dynamic geometry that foster cognitive unity.

Of course we acknowledge the difficulty of implementing such a study, since finding subjects who have interiorized MD would not be a trivial task. However, some possible subjects of this kind might be identified during the teaching experiment carried our during the first study we outlined. This could be a viable possibility since during the teaching experiment the dragging modalities we introduced in this study would be introduced again and in a more thorough way with respect to the task of conjecture-generation.

Finally, if the first line of research we outlined were to give insight into how to foster the development of the MDS as a psychological tool, and the second line of research confirmed our hypothesis on cognitive unity, we would be able to develop activities in a DGS that involve a process of conjecture-generation with strong links to a subsequent proving phase. Such activities would be particularly beneficial in the teaching and learning of proof.

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APPENDIX A

STUDENT CONSENT FORM

Al Dirigente Scolastico del Liceo Scientifico_________________

Al fine di attuare lo studio per la tesi di dottorato, la sottoscritta Anna Baccaglini-Frank, dottoranda alla University of New Hampshire (USA) sotto la direzione della Prof.ssa Maria Alessandra Mariotti, chiede di poter svolgere due lezioni durante ore di Matematica della classe ____ ed alcune osservazioni/interviste ad alunni della stessa classe in orario pomeridiano, sotto la guida della Prof.ssa _________, nei periodi ottobre-novembre e febbraio-marzo dell'anno scolastico in corso.

Presentazione

L'obiettivo della tesi, "Sviluppo di Congetture e Dimostrazioni in Geometria Dinamica," è di confermare ipotesi di ricerca su processi cognitivi che avvengono nelle fasi di congettura e di dimostrazione in problemi aperti proposti con lo strumento della geometria dinamica. In particolare, il software che verrà utilizzato è Cabri, un software didattico usato correntemente dall'insegnante della classe, Prof.ssa __________.

Le attività proposte saranno costruite appositamente per la classe in cui verranno attuate e saranno complementari al regolare percorso didattico della classe. Inoltre le attività saranno svolte sotto la sorveglianza e con la collaborazione della Prof.ssa __________.

Sono previsti due cicli (uno a ottobre–novembre ed uno a febbraio-marzo) composti dai seguenti interventi: lezione introduttiva sugli schemi di trascinamento in Cabri (in orario di lezione mattutina), e serie di osservazioni di coppie di (o di singoli) studenti che lavorano alle attività proposte (in orario pomeridiano). I dati raccolti saranno analizzati dalla dott.ssa Anna Baccaglini-Frank e dalla Prof.ssa Maria Alessandra Mariotti con la partecipazione attiva dell'insegnante della classe.

Cordialmente,

La Dottoranda
dott.ssa Anna Baccaglini-Frank

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APPENDIX B

IRB APPROVAL FORMS

University *of* New Hampshire

Research Conduct and Compliance Services, Office of Sponsored Research
Service Building, 51 College Road, Durham, NH 03824-3585
Fax: 603-862-3564

02-Apr-2008

Baccaglini-Frank, Anna
Math & Statistics, Kingsbury Hall
14 McDaniel Drive, Box 2102
Durham, NH 03824

IRB #: 4250
Study: Conjecturing and Proving in a Dynamic Geometry Environment
Approval Date: 31-Mar-2008

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved the protocol for your study as Expedited as described in Title 45, Code of Federal Regulations (CFR), Part 46, Subsection 110.

Approval is granted to conduct your study as described in your protocol for one year from the approval date above. At the end of the approval period, you will be asked to submit a report with regard to the involvement of human subjects in this study. If your study is still active, you may request an extension of IRB approval.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the attached document, *Responsibilities of Directors of Research Studies Involving Human Subjects.* (This document is also available at http://www.unh.edu/osr/compliance/irb.html.) Please read this document carefully before commencing your work involving human subjects.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie F. Simpson

Julie F. Simpson
Manager

cc: File
Graham, Karen

University *of* New Hampshire

Research Integrity Services, Office of Sponsored Research
Service Building, 51 College Road, Durham, NH 03824-3585
Fax: 603-862-3564

11-Mar-2010

Baccaglini-Frank, Anna
Math & Statistics, Kingsbury Hall
14 McDaniel Drive, Box 2102
Durham, NH 03824

IRB #: 4250
Study: Conjecturing and Proving in a Dynamic Geometry Environment
Review Level: Expedited
Approval Expiration Date: 31-Mar-2011

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved your request for time extension for this study. Approval for this study expires on the date indicated above. At the end of the approval period you will be asked to submit a report with regard to the involvement of human subjects. If your study is still active, you may apply for extension of IRB approval through this office.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the document, *Responsibilities of Directors of Research Studies Involving Human Subjects.* This document is available at http://www.unh.edu/osr/compliance/irb.html or from me.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie F. Simpson

Julie F. Simpson
Manager

cc: File
Graham, Karen

University *of* New Hampshire

Research Integrity Services, Office of Sponsored Research
Service Building, 51 College Road, Durham, NH 03824-3585
Fax: 603-862-3564

05-Mar-2009

Baccaglini-Frank, Anna
Math & Statistics, Kingsbury Hall
14 McDaniel Drive, Box 2102
Durham, NH 03824

IRB #: 4250
Study: Conjecturing and Proving in a Dynamic Geometry Environment
Review Level: Expedited
Approval Expiration Date: 31-Mar-2010

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved your request for time extension for this study. Approval for this study expires on the date indicated above. At the end of the approval period you will be asked to submit a report with regard to the involvement of human subjects. If your study is still active, you may apply for extension of IRB approval through this office.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the document, *Responsibilities of Directors of Research Studies Involving Human Subjects.* This document is available at http://www.unh.edu/osr/compliance/irb.html or from me.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie Simpson

Julie F. Simpson
Manager

cc: File
Graham, Karen