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Higher brackets and Moduli space of vector bundles

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Introduction

Deformation theory is closely related to moduli problem, that is the problem of classifying geometric objects. Let $\mathcal{M}$ be a class of geometric objects, for example, let

$$\mathcal{M} = \{\text{isomorphism classes of complex manifolds}\},$$

the moduli problem for this class is that of describing $\mathcal{M}$, investigating if $\mathcal{M}$ has some kind of algebro-geometric structure, for example if it is a scheme, a quasi-projective or a projective variety and so on. Fixed the class of objects, the space $\mathcal{M}$ is called the moduli space of the classification problem.

The existence of an algebro-geometric structure on the space $\mathcal{M}$ and its local structure are linked to the notions of family and infinitesimal deformations. The concepts of family and deformation are very different for every different class $\mathcal{M}$.
In the example in which $\mathcal{M}$ is the class of complex manifolds, a family is the data of a morphism

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"] \\ S \end{tikzcd}$$

where S is an analytic space, $\mathcal{X}$ is a manifold, π is a flat morphism also required to be smooth or proper. Two of such families $\mathcal{X}$ and $\mathcal{X}'$ over the same analytic space S are isomorphic if there exists an isomorphism $f : \mathcal{X} \to \mathcal{X}'$ which commutes with the maps π and π' to S.
If the space S is connected, the family is called a deformation of $\pi^{-1}(s_0)$, for any $s_0 \in S$.
An infinitesimal deformation is a deformation in which the base space S is the spectrum of a local Artinian $\mathbb{C}$-algebra with residue field isomorphic to $\mathbb{C}$.

The problem of classifying complex manifold is obviously one of the first example classically studied. Globally there are a lot of results about moduli spaces of complex manifolds of a fixed dimension and with given numerical characters. For example about closed subschemes of a projective space with a given Hilbert polynomial, curves of a fixed genus and so on. From a local point of view, Kodaira and Spencer studied deformations of compact complex manifolds, giving rise to the modern deformation theory ([15]).

A fundamental property of the notions of family and deformation is their functoriality. In the above example, if $\pi : \mathcal{X} \to S$ is a family of complex manifolds and $f : T \to S$ is a morphism of analitic spaces, it is induced a family over T by pulling back the family $\mathcal{X}$ via f, it is given by $f^*\mathcal{X} = T \times_S \mathcal{X}$. The functoriality leads naturally to the definition of a contravariant functor

$$\begin{array}{rccc} F: & \mathbf{An} & \to & \mathbf{Set} \\ & S & \mapsto & F(S), \end{array}$$

where $F(S) = \{$isomorphism classes of families of complex manifolds over $S\}$.
The same can be done for every class of objects, once defined the notion of family and of isomorphism of families, leading to the definition of the contravariant functor of the isomorphism classes of families of the objects considered.
This approach to deformation theory, started by Grothendieck about fifty years ago ([11]) and developed by Artin ([2]) and Schlessinger ([33]), allows to formalize deformation theory translating it in functorial language.
Following this point of view it could be important to understand whether the functor F associated to a classification problem, is represented by an analytic space M, i.e. whether there exists an isomorphism of functors:

$$\mathrm{Hom}(-, M) \to F.$$

If such M exists, it is called the fine moduli space for the class $\mathcal{M}$. Schlessinger studied conditions under which such M exists and conditions under which weaker properties, linked to representability, held for the functor F.

The classes $\mathcal{M}$ for which the functor F associated is representable are very rare. In order to understand the role of infinitesimal deformations in classification problem, let's suppose there exists an analitic space M that represents the class $\mathcal{M}$ choosen, i.e. such that the functors $\mathrm{Hom}(-, M)$ and F are isomorphic.
Let X be an object in the class $\mathcal{M}$. The isomorphism classes of infinitesimal deformations of X over the spectrum of a local Artinian $\mathbb{C}$-algebra A are in one to one correspondence with the morphisms of $\operatorname{Spec} A$ to M that send the closed point to X:

$$\{\text{infinitesimal deformations of } X \text{ over } \operatorname{Spec} A\}/\sim \quad \longleftrightarrow \quad \mathrm{Hom}_X(\operatorname{Spec} A, M),$$

then infinitesimal deformations give informations on the infinitesimal structure of the moduli space M in a neighbourhood of X. In particular, if $A = \mathbb{C}[\epsilon] = \mathbb{C}[x]/(x^2)$ is the ring of the dual numbers, the deformations over $\operatorname{Spec} A$ are called first order deformations and there is the following one to one correspondence:

$$\{\text{first order deformations of } X\}/\sim \quad \longleftrightarrow \quad \mathrm{Hom}_X(\operatorname{Spec} \mathbb{C}[\epsilon], M) \quad \longleftrightarrow \quad T_{M,X}.$$

These facts explain some of the links of deformation theory with the moduli theory.

Let now concentrate our attention on deformation theory. Following Grothendieck's approach, to study infinitesimal deformations of a geometric object χ, it is natural to define a deformation functor from the category of local Artinian $\mathbb{C}$-algebras with residue field isomorphic to $\mathbb{C}$ to the category of sets:

$$\begin{array}{rccll} \mathrm{Def}_\chi : & \mathbf{Art}_{\mathbb{C}} & \to & \mathbf{Set} & \\ & A & \mapsto & \mathrm{Def}_\chi(A) = & \{\text{infinitesimal deformations of } \chi \text{ over } \operatorname{Spec} A\}/\sim, \end{array}$$

it is called the functor of deformations of χ. As briefly explained, the study of this functor has a strong geometric meaning, in particular it is interesting to study its tangent space and its obstructions.
In the example of deformations of a complex manifold X, the functor $\mathrm{Def}_X : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$ associates to every local Artinian $\mathbb{C}$-algebra A the set of isomorphism classes of deformations of X over $\operatorname{Spec} A$.

The classical approach, developed by Kodaira and Spencer, tackles the study of the functor of deformations of a geometric object in the following way. In several cases it associates a sheaf of Lie algebras on a topological space to every deformation problem:

$$\text{Deformation problem of the object } \chi \ \rightsquigarrow \text{Sheaf of Lie algebras } \ \mathcal{L},$$

that controls the deformations via the Čech functor $\check{\mathrm{H}}^1(\exp \mathcal{L})$:

$$\check{\mathrm{H}}^1(\exp \mathcal{L}) : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set},$$

$$\check{\mathrm{H}}^1(\exp \mathcal{L})(A) = \check{\mathrm{H}}^1(\exp(\mathcal{L} \otimes \mathfrak{m}_A)),$$

This means that, if the sheaf is appropriately choosen, there is an isomorphism of functors:

$$\check{\mathrm{H}}^1(\exp \mathcal{L}) \cong \mathrm{Def}_\chi .$$

In their studies of deformations of a compact complex manifold X, Kodaira and Spencer identified the holomorphic tangent sheaf $\mathcal{T}_X$ of X to control deformations.

Since thirty years a new approach to deformation theory has been developed. It is based on the principle, due to Deligne, Drinfeld, Kontsevich and Quillen ([16], [17]), for which, in characteristic zero, every deformation problem is governed by a differential graded Lie algebra. A differential graded Lie algebra (DGLA) is simply a graded vector space, with a differential and a Lie bracket, which satisfy some compatibility relations. This approach associates a differential graded Lie algebra to every deformation problem:

$$\text{Deformation problem of the object } \chi \ \rightsquigarrow \text{DGLA } \ L.$$

For every DGLA it is defined a deformation functor associated to it, as the quotient of the Maurer-Cartan functor with the gauge action:

$$\mathrm{Def}_L : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set},$$

$$\mathrm{Def}_L(A) = \left\{ x \in L^1 \otimes \mathfrak{m}_A \mid dx + \frac{1}{2}[x,x] = 0 \right\} / \sim_{gauge} .$$

The philosophy underlying this approach is that, if the DGLA L individuated by the problem of deformations of the object χ is appropriately choosen, the deformation functor associated to L is isomorphic to the functor of the deformations of χ, i.e.

$$\mathrm{Def}_L \cong \mathrm{Def}_\chi .$$

The DGLA associated to the problem of deformations of a complex manifold X is the space $A_X^{0,*}(\mathcal{T}_X)$ of the $(0,*)$-forms on X with values in the tangent sheaf of X.
The importance of this approach is that the study of a deformation functor associated to a DGLA is quite easy and some classical results, like the calculation of the tangent space and of obstructions, are simply consequences of the definitions. On the other hand, given a problem of deformation, in general, it is not an easy task to find a DGLA which governs it.

Although the theory of deformations via DGLA is very useful on its own, in many situations it is unavoidable to recognize that the category of DGLA is too rigid for a good theory. The appropriate way of extending this category is the introduction of L_∞-algebras, which are graded vector spaces V with a sequence of linear maps q_k :

$\bigodot^k V \to V$ which satisfy some compatibility conditions. Every DGLA is a L_∞-algebra. A deformation functor is associated to every L_∞-algebra; it is given by the quotient of the generalized Maurer-Cartan functor with the homotopy relation:

$$\mathrm{Def}_V : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$$
$$\mathrm{Def}_V(A) = \left\{ x \in V^1 \otimes \mathfrak{m}_A \mid \frac{\sum_k q_k(v^{\odot k})}{k!} \right\} / \sim_{homotopy},$$

this functor is a generalization of the deformation functor associated to a DGLA.

All these approaches can be applied in many different cases: in the above example of deformations of a complex manifold and also for deformations of a locally free sheaf, of a pair (manifold, sheaf), of a submanifold in a fixed manifold, of holomorphic maps and so on. In the first part of this work we study different deformation problems linked to the classical problems of deforming a locally free sheaf on a complex manifold and of deforming a pair (complex manifold, locally free sheaf), and we determine the different DGLAs that control these problems.

An infinitesimal deformation the pair $(X,\mathcal{E})$=(complex manifold, locally free sheaf) over a local Artinian $\mathbb{C}$-algebra A is the data of a deformation $\mathcal{X}_A$ of the manifold X over A, i.e. a cartesian diagram of morphisms of schemes

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X}_A \arrow[d, "\pi"] \\ \mathrm{Spec}\,\mathbb{C} \arrow[r] & \mathrm{Spec}\,A, \end{tikzcd}$$

where π is flat; and a locally free sheaf $\mathcal{E}_A$ of $\mathcal{O}_{\mathcal{X}_A}$-modules on $\mathcal{X}_A$, with a morphism $\pi_A : \mathcal{E}_A \to \mathcal{E}$, such that $\pi_A : \mathcal{E}_A \otimes_A \mathbb{C} \to \mathcal{E}$ is an isomorphism. If the deformation $(\mathcal{X}_A, \mathcal{E}_A)$ of the pair $(X,\mathcal{E})$ is such that $\mathcal{X}_A = X \times \mathrm{Spec}\,A$, then it is simply a deformation of the sheaf $\mathcal{E}$ over A.

The classical approach to deformation theory associates to the problem of deformations of $\mathcal{E}$ the sheaf $\mathrm{End}\,\mathcal{E}$ of endomorphism of $\mathcal{E}$ and to the problem of deformations of the pair $(X,\mathcal{E})$ the sheaf $D^1(\mathcal{E})$ of the first order differential operators on sections of $\mathcal{E}$ with scalar principal symbol. While the DGLAs approach associates to these problems the DGLA $A_X^{0,*}(\mathrm{End}\,\mathcal{E})$ of the $(0,*)$-forms on X with values in the sheaf $\mathrm{End}\,\mathcal{E}$ and the DGLA $A_X^{0,*}(D^1(\mathcal{E}))$ of the $(0,*)$-forms on X with values in the sheaf $D^1(\mathcal{E})$, respectively. These algebraic objects govern the deformations in the sense explained above.

In the first part of this work, we consider the pair $(X,\mathcal{E})$=(complex manifold, locally free sheaf) for which a subspace V of global sections of $\mathcal{E}$ is fixed and we study deformations of the pair $(X,\mathcal{E})$ such that the subspace V can be lifted to the deformed pair. As a generalization of this case we study deformations of the pair $(X,\mathcal{E})$ such that some fixed subspaces V^i of the cohomology spaces $H^i(X,\mathcal{E})$ can be lifted to the deformed pair. In both cases we determine the DGLA which controls the deformations (Sections 4.1 and 4.2).

Moreover we study the deformations of a locally free sheaf $\mathcal{E}$ on a complex manifold X which preserve the dimensions of the cohomology spaces. Translating these geometric problem using DGLAs approach, we determine the type of singularity of the strata of the Brill-Noether stratification of the moduli space of stable and flat locally free sheaves on a compact complex Kähler manifold (Section 4.3 and [27]).

Recall that, if X is a compact complex Kähler manifold, the moduli space $\mathcal{M}$ of stable and flat locally free sheaves of $\mathcal{O}_X$-modules on X exists, it is a coarse moduli space and it is a complex analytic space ([31]), or, if X is a projective algebraic variety, it is a quasi-projective variety ([20]). However very little is known about the finer structure of this moduli space except in a few special cases, for example the case in which X is a curve or a projective space. In the Eighties, Nadel ([29]), Goldman and Millson ([9]), in independent works, proved that this moduli space has quadratic algebraic singularities, that is, it is locally defined by finitely many quadratic polynomials.
In the same spirit of this result, we study the Brill-Noether stratification of the moduli space $\mathcal{M}$. The strata of the Brill-Noether stratification are defined in the following way: fixed integers $h_i \in \mathbb{N}$, for all $i = 0 \dots n = \dim X$, the stratum $\mathcal{N}(h_0 \dots h_n) \subset \mathcal{M}$ is the subspace of stable and flat locally free sheaves of $\mathcal{O}_X$-modules on X, with cohomology spaces dimension fixed, given by $\dim H^i = h_i$, for all $i = 0 \dots n$.
Obviously the local structure of one of these strata $\mathcal{N}(h_0 \dots h_n)$ in a neighbourhood of one of its point $\mathcal{E}$ is linked to the infinitesimal deformations of the sheaf $\mathcal{E}$ which preserve the dimensions of cohomology spaces and to the functor $\mathrm{Def}^0_{\mathcal{E}}$ of these deformations. The study of the functor $\mathrm{Def}^0_{\mathcal{E}}$ using DGLA's tecniques allow us to construct a chain of functors with hulls linked by smooth morphisms which starts from $\mathrm{Def}^0_{\mathcal{E}}$ and ends with a functor $\mathcal{F}$ for which it is very simple to prove that it is represented by an analytic space with quadratic algebraic singularities. A priori this is not sufficient to have the same result for the functor $\mathrm{Def}^0_{\mathcal{E}}$. Thanks to a deeper study of the property of having the same type of singularities and in particular of having quadratic algebraic singularities (Sections 1.1 and 1.2), we obtain that these properties are, in some sense, preserved by smooth morphisms (Theorem 1.1.17 and Proposition 1.2.26). Then we prove the following (Theorem 4.3.8):

Main Theorem. *The strata of the Brill-Noether stratification of the moduli space $\mathcal{M}$ have quadratic algebraic singularities.*

The second part of this work is devoted to the study of deformation theory via L_∞-algebras and semicosimplicial DGLAs. All the results of this part are obtained in a joint work with D. Fiorenza and M. Manetti ([7]). To explain our studies and their geometric motivations, we have to introduce some technical tools.

As said before, an L_∞-algebra is a graded vector space V with a sequence of linear maps $q_k : \bigodot^k V \to V$ which satisfy some compatibility conditions and to every L_∞-algebra V is associated a deformation functor Def_V. A semicosimplicial DGLA $\mathfrak{g}^\Delta$ is a diagram:

$$\mathfrak{g}^\Delta : \quad \mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathfrak{g}_2 \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

where $\mathfrak{g}_i$ are DGLAs and the arrows $\partial_{k,i} : \mathfrak{g}_{i-1} \to \mathfrak{g}_i$ are DGLAs morphisms that satisfy some relations. Then the space $\mathfrak{g}^*_* = \bigoplus_{i,j} \mathfrak{g}^j_i$ has a differential graded bicomplex structure, the 'vertical' differentials are the ones of the DGLAs $\mathfrak{g}_i$ and the 'orizzontal' ones are given by $\partial_i = \partial_{0,i} - \partial_{1,i} + \cdots + (-1)^i \partial_{i,i}$. The total complex $(\mathrm{Tot}(\mathfrak{g}^\Delta), \delta)$ has no natural DGLA structure, but, using the homotopical transfer of structure theorem (Theorem 5.1.14), it can be endowed with a canonical L_∞-algebra structure, indicated with $\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)$. To this L_∞-algebra is associated the deformation functor $\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)}$. Our first aim is to describe explicitly this functor.

Before stating the results, let's explain the motivations for this study. The first motivation is linked to some works by M. Manetti ([25]), D. Fiorenza and M. Manetti

([6]) and D. Iacono ([14]), in which they studied some problems of deformations, as deformations of a submanifold in a fixed manifold or deformations of holomorphic maps, using DGLAs and L_∞-algebras tecniques. To tackle these deformation problems, they introduced deformation functors, which are exactly the functor $\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)}$ with semicosimplicial DGLA $\mathfrak{g}^\Delta$ different from zero only in the first two DGLAs. Then the explicit description of the functor $\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)}$ could be seen as a generalization of these previous results.

Another motivation is linked to the attempt of finding an algebraic way to control deformations of a geometric object which is independent from the $\mathcal{C}^\infty$-structure of the object itself.
Let's analyse an example. As said before, the DGLAs approach to deformation theory proves that infinitesimal deformations of a locally free sheaf $\mathcal{E}$ on a complex manifold X, are governed by the DGLA $A_X^{0,*}(\operatorname{End}\mathcal{E})$ of the $(0,*)$-forms on X with values in the sheaf of the endomorphisms of $\mathcal{E}$; obviously this DGLA depends on the $\mathcal{C}^\infty$-structure of X. The same happens to the new examples of infinitesimal deformations studied in Sections 4.1 and 4.2, in fact the DGLAs individuated involve the differential forms on the complex manifold.
Our aim of finding out algebraic objects independent from the $\mathcal{C}^\infty$-structure of X which control these deformations is linked to the study of semicosimplicial DGLAs, because we expect that these algebraic objects are the L_∞-algebras of the total complexes of semicosimplicial DGLAs appropriately choosen.
For example, for deformations of the sheaf $\mathcal{E}$, the semicosimplicial Lie algebra naturally considered is $\operatorname{End}\mathcal{E}(\mathcal{U})$, with $\mathcal{U}$ open covering of X. It is given by:

$$\textstyle\prod_i \operatorname{End}\mathcal{E}(U_i) \rightrightarrows \prod_{i<j} \operatorname{End}\mathcal{E}(U_{ij}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \prod_{i<j<k} \operatorname{End}\mathcal{E}(U_{ijk}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots,$$

where the morphisms $\partial_{j,k} : \prod_{i_0\ldots i_{k-1}} \operatorname{End}\mathcal{E}(U_{i_0\ldots i_{k-1}}) \to \prod_{i_0\ldots i_k} \operatorname{End}\mathcal{E}(U_{i_0\ldots i_k})$ are given by $(\partial_{j,k}(x))_{i_0\ldots i_k} = x_{i_0\ldots \widehat{i_j}\ldots i_k}$, for all $j = 0,\ldots,k$. The total complex $\mathrm{Tot}(\operatorname{End}\mathcal{E}(\mathcal{U}))$ associated to this semicosimplicial Lie algebra is the Čech complex of the sheaf $\operatorname{End}\mathcal{E}$, relative to the covering $\mathcal{U}$, which obviously is independent from the $\mathcal{C}^\infty$-structure of X. We expect that the functor $\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\operatorname{End}\mathcal{E}(\mathcal{U}))}$ is linked to the deformations of $\mathcal{E}$.

Our first result in this direction is the following (Theorem 5.3.6):

Main Theorem (Fiorenza, Manetti, -). *Let $\mathfrak{g}^\Delta$ be a semicosimplicial Lie algebra. Then the two deformation functors $\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)}$ and $H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)$ are isomorphic.*

If $\mathfrak{g}^\Delta$ is a semicosimplicial Lie algebra, the functor $H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta) : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$ is defined, for all $A \in \mathbf{Art}_{\mathbb{C}}$, by:

$$H^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta \otimes \mathfrak{m}_A)) = Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta \otimes \mathfrak{m}_A))/\sim,$$

where $Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta \otimes \mathfrak{m}_A)) = \{x \in \mathfrak{g}_1 \otimes \mathfrak{m}_A \mid e^{\partial_{0,2}(x)} e^{-\partial_{1,2}(x)} e^{\partial_{2,2}(x)} = 1\}$ and two of its elements x, y are equivalent under the relation '$\sim$' if and only if there exists $l \in \mathfrak{g}_0 \otimes \mathfrak{m}_A$ such that $e^{-\partial_{1,1}(l)} e^x e^{\partial_{0,1}(l)} = e^y$.

In the example of the semicosimplicial Lie algebra $\operatorname{End}\mathcal{E}(\mathcal{U})$, this functor has a clear geometric meaning. Infact $Z^1_{\mathrm{sc}}(\exp(\operatorname{End}\mathcal{E}(\mathcal{U}) \otimes \mathfrak{m}_A))$ is the set of the closed 1-Čech cochain of the sheaf $\exp(\operatorname{End}\mathcal{E} \otimes \mathfrak{m}_A)$, while the relation '$\sim$' is the condition to differ by the differential of a 0-cochain. Then the functor $H^1_{\mathrm{sc}}(\exp(\operatorname{End}\mathcal{E}(\mathcal{U}))$ associates, to every $A \in \mathbf{Art}_{\mathbb{C}}$, the first Čech cohomology space for the covering $\mathcal{U}$ with coefficients in the sheaf of groups $\exp(\operatorname{End}\mathcal{E} \otimes \mathfrak{m}_A)$. Since the sheaf $\operatorname{End}\mathcal{E}$ is the one associated by the classical approach to deformations of $\mathcal{E}$, the functor obtained describes these deformations. Then the theorem allows to individuate the algebraic object $\widetilde{\operatorname{Tot}}(\operatorname{End}\mathcal{E}(\mathcal{U}))$ independent from the $\mathcal{C}^\infty$-structure of X to control deformations of $\mathcal{E}$.

Observe that our above result gives an explicit description of the functor $\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)}$ in the case $\mathfrak{g}^\Delta$ is a semicosimplicial Lie algebra, but it could be of some interest to describe it also in the general case $\mathfrak{g}^\Delta$ is a semicosimplicial DGLA.
This not only would be the natural generalization of our result, but would allow to find an algebraic object independent from the $\mathcal{C}^\infty$-structure also for other kind of deformations. For example for the deformations of the triple $(X, \mathcal{E}, V)$=(complex manifold, sheaf, subspace of sections) or of $(X, \mathcal{E}, V^i)$ =(complex manifold, sheaf, subspaces of cohomology), for which the semicosimplicial object naturally associated is a semicosimplicial DGLA and not simply a semicosimplicial Lie algebra.
The study of this general case is a work in progress with D. Iacono.

In the last part of our work, we investigate in a deeper way the functor $\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathcal{C}(\mathcal{U}))}$, where $\mathcal{C}$ is a sheaf of Lie algebras or of DGLAs over a topological space X and $\mathcal{U}$ is an open covering of X, and we define augmented semicosimplicial DGLAs. We prove that the functor $\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathcal{C}(\mathcal{U}))}$ is naturally isomorphic to the deformation functor associated to the DGLA of global sections of an acyclic resolution of $\mathcal{C}$ (Theorem 5.4.8):

Main Theorem (D. Fiorenza, M. Manetti, -)**.** *Let X be a paracompact Hausdorff topological space. Let $\mathcal{C}^\cdot$ be a sheaf of differential graded Lie algebras on X and let $\varphi\colon \mathcal{C}^\cdot \to \mathcal{F}^\cdot$ be an acyclic resolution of $\mathcal{C}^\cdot$. Let $F^\cdot = \mathcal{F}^\cdot(X)$ be the DGLA of global sections of $\mathcal{F}^\cdot$. Then, if $\mathcal{U}$ is an open covering of X which is acyclic with respect to both $\mathcal{C}^\cdot$ and $\mathcal{F}^\cdot$, then there is an isomorphism of functors:*

$$\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathcal{C}^\cdot(\mathcal{U}))} \cong \operatorname{Def}_F\,.$$

The constructions analysed and results obtained in this part have strong geometric meaning, in fact they link the two major different approaches to deformation theory - the classical approach and the DGLAs one - in a concrete and rigorous way. As said before, the classical approach to deformation theory in several cases identifies a sheaf of Lie algebras $\mathcal{L}$ on a topological space X, which controls deformations via the Čech functor $H^1(X, \exp\mathcal{L})$. On the other hand, the theory of deformation via DGLAs is based on the principle that, in characteristic zero, every deformation problem is governed by a differential graded Lie algebra, via the deformation functor associated to it. These two approaches suggest that there should exists a canonical isomorphism between the Čech functor of the sheaf of Lie algebras identified by the problem and the deformation functor associated to the DGLA that governs it.
We obtain this isomorphism as a consequence of our previous results (Corollary 5.4.10):

Main Theorem (D. Fiorenza, M. Manetti, -)**.** *Let X be a paracompact Hausdorff topological space. Let $\mathcal{L}$ be a sheaf of Lie algebras on X and $\varphi\colon \mathcal{L} \to \mathcal{F}$ an acyclic resolution of $\mathcal{L}$. Let F be the DGLA of global sections of $\mathcal{F}$. If open coverings of X which are acyclic for both $\mathcal{L}$ and $\mathcal{F}$ are cofinal in the directed family of open coverings of X, then there is a natural isomorphism of deformation functors*

$$H^1(X; \exp\mathcal{L}) \simeq \operatorname{Def}_F\,.$$

Let's see that this theorem gives the expected isomorphism in the example of deformations of a sheaf $\mathcal{E}$ on a complex manifold X. Consider the sheaf of Lie algeras $\mathcal{L} = \operatorname{End}\mathcal{E}$ and as its acyclic resolution take the Dolbeault's one $\varphi : \operatorname{End}\mathcal{E} \to \mathcal{A}_X^{0,*}(\operatorname{End}\mathcal{E})$, then the DGLA F is $A_X^{0,*}(\operatorname{End}\mathcal{E})$. As we want, the Theorem states that:

$$\check{\mathrm{H}}^1(X; \exp(\operatorname{End}\mathcal{E})) \simeq \operatorname{Def}_{A_X^{0,*}(\operatorname{End}\mathcal{E})}.$$

This thesis is organized as follows.

In *Chapter 1* we collect some abstract tools for the study of formal deformation theory. At first (Section 1.1) we study analytic algebras and germs of analytic spaces, concentrating particular attention to the notion of smoothness. Then (Section 1.2), we recall functor of Artin rings theory: we define tangent space, obstruction theory, the notion of smoothness and we explain the well-known Schlessinger's conditions. In this chapter we define and study equivalence relations between analytic algebras, between germs of analytic spaces and between functors, under which two of these objects are said to have the same type of singularities. We prove that these relations for analitic algebras and for germs of anlaytic spaces are formal, i.e. can be controlled at the level of functors, and that the set of germs with quadratic algebraic singularities is closed under this relation.

In *Chapter 2* we recall some known facts about infinitesimal deformations of geometric objects, like schemes (Section 2.1), locally free sheaves (Section 2.2) and pairs (scheme, sheaf) (Section 2.3). We define geometrically what a deformation of these objects is, we construct the associated functors of deformations, we analyse conditions under which these functors have hulls or they are prorepresentable and we calculate their tangent spaces and their obstructions.

In *Chapter 3* we introduce the basic tools of deformation theory via differential graded Lie algebras. We define DGLAs (Section 3.1), the deformation functor associated to a DGLA (Section 3.2) and the deformation functor associated to a morphism of DGLAs (Section 3.3). For these functors, we calculate the tangent and an obstruction spaces, we discuss the Schlessinger's conditions and we collect important properties that are usefull in the following. All this study is supported by the translation in DGLAs language of the geometric examples studied in the previous chapter and by the analysis of other classical examples.

In *Chapter 4* we analyse, using DGLAs approach, some new examples of deformations of geometric objects. In Section 4.1, we study deformations of the pair (complex manifold, locally free sheaf) with a fixed subspace of global sections of the sheaf that is required to be deformed, determining the DGLA that controls these deformations. In Section 4.2, as a generalization of this case, we study deformations of the pair (complex manifold, locally free sheaf) with fixed subspaces of the cohomology spaces of the sheaf that are required to be deformed, determining the DGLA that controls these deformations. In Section 4.3, we analyse deformations of a locally free sheaf that preserve the dimensions of the cohomology spaces and, using DGLAs tecniques, we prove that the strata of the Brill-Noether stratification of the moduli space of stable and flat locally free sheaves on a compact complex Kähler manifold have quadratic algebraic singularities.

In *Chapter 5* we study deformation theory via L_∞-algebras and semicosimplicial DGLAs. Section 5.1 is an introduction of the basic notions of L_∞-theory: we define what an L_∞-algebra is, we explain how to associate to a L_∞-algebra a deformation functor and we state the fundamental homotopical transfert of structure theorem. In Section 5.2, we define semicosimplicial DGLAs, we construct a canonical L_∞-structure on the total complex of a semicosimplicial DGLA, obtained by homotopical transfer from the Thom-Whitney DGLA, and we define deformation functors associated to these objects. Section 5.3 is dedicated to the case of semicosimplicial Lie algebras: we obtain an explicit descripition of the deformation functor associated to the total complex of a semicosimplicial Lie algebra. In Section 5.4 we introduce and study augmented semicosimplicial DGLAs. Using them, we prove the existence of a natural isomorphism between the functor associated to the total complex of the semicosimplicial DGLA of a sheaf of DGLAs and the deformation functor associated to the DGLA of global sections of an acyclic resolution of the sheaf. This allows us to find a link between the classical approach and the DGLAs approach to deformation theory.

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Contents

Chapter 1

Formal deformation theory

This Chapter deals with the abstract tools for the study of formal deformation theory. In the first section we study analytic algebras and germs of analytic spaces. We introduce the notion of smoothness, recalling the main results related to it, and we define and study an equivalence relation, linked to smoothness, under which two of these objects are said to have the same type of singularities.
In the second section we collect basic facts about the functors of Artin rings: we define tangent space, obstruction theory, the notion of smoothness and we explain the well-known Schlessinger's Theorem.

1.1 Analytic algebras and germs of analytic spaces

We begin fixing the set up and some notations. We always work on a fixed field $\mathbb{K}$ of characteristic zero. We indicate with **Set** the category of sets, with a fixed one point set. We indicate with $\mathbf{Art}_{\mathbb{K}}$ the category of local Artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$, whose arrows are local morphisms. While $\widehat{\mathbf{Art}}_{\mathbb{K}}$ indicates the category of complete Noetherian local $\mathbb{K}$-algebras with residue field $\mathbb{K}$, arrows are local morphisms.

Remark 1.1.1. The categories $\mathbf{Art}_{\mathbb{K}}$ and $\widehat{\mathbf{Art}}_{\mathbb{K}}$ are closed under fibered products. Infact, let $\beta : B \to A$ and $\gamma : C \to A$ be morphisms in $\mathbf{Art}_{\mathbb{K}}$, or in $\widehat{\mathbf{Art}}_{\mathbb{K}}$, then the fibered product

$$B \times_A C = \{(b, c) \in B \times C \mid \beta(b) = \gamma(c)\}$$

is an object in $\mathbf{Art}_{\mathbb{K}}$, or in $\widehat{\mathbf{Art}}_{\mathbb{K}}$, respectively.

Definition 1.1.2. *A* small extension *in* $\mathbf{Art}_{\mathbb{K}}$ *is a short exact sequence:*

$$e: \quad 0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0,$$

where α *is a morphism in* $\mathbf{Art}_{\mathbb{K}}$ *and* e *is such that the kernel* J *of* α *is annihilated by the maximal ideal* $\mathfrak{m}_B$ *of* B*, i.e.* $J \cdot \mathfrak{m}_B = 0$.
A small extension is called principal*, if* $J \cong \mathbb{K}$.

Remark 1.1.3. Let $f : B \to A$ be surjective morphism in $\mathbf{Art}_{\mathbb{K}}$, then it can be expressed as a composition of a finite number of small extensions. Infact, since B is local Artinian, its maximal ideal $\mathfrak{m}_B$ is nilpotent, i.e. there exists $n \in \mathbb{N}$ such that $\mathfrak{m}_B^n \neq 0$ and $\mathfrak{m}_B^m = 0$ for all $m > n$. Let $J = \ker f$ and consider the following exact sequences:

$$0 \to J/\mathfrak{m}_B J \to B/\mathfrak{m}_B J \to B/J \cong A \to 0 \ldots$$

$$0 \to \mathfrak{m}_B^{n-1} J/\mathfrak{m}_B^n J \to B/\mathfrak{m}_B^n J \to B/\mathfrak{m}_B^{n-1} J \to 0,$$
$$0 \to \mathfrak{m}_B^n J \to B \to B/\mathfrak{m}_B^n J \to 0,$$

which are all small extensions and which give the decomposition of f.

The notion of formal smoothness plays an important role in deformation theory. After having introduce it in complete generality for homomorphisms of rings, we concentrate our attention at first on the category of analytic algebras and then on the category of germs of analytic spaces.

Definition 1.1.4. *A homomorphism of rings $\psi : R \to S$ is called* formally smooth *if, for every small extension of local Artinian R-algebras: $0 \to J \to B \to A \to 0$, the induced map $\mathrm{Hom}_R(S, B) \to \mathrm{Hom}_R(S, A)$ is surjective.*

Proposition 1.1.5. *Let $\psi : R \to S$ be a local homomorphism of local Noetherian $\mathbb{K}$-algebras conteining a field isomorphic to their residue field $\mathbb{K}$. Then the following conditions are equivalent:*

- *ψ is formally smooth,*
- *$\hat{S}$ is isomorphic to a formal power series ring over $\hat{R}$,*
- *the homomorphism $\hat{\psi} : \hat{R} \to \hat{S}$, induced by ψ, is formally smooth.*

Proof. See [34], Proposition C.50. □

Now we restrict our study to analytic algebras. We recall that an *analytic algebra* is a $\mathbb{K}$-algebra which can be written in the form $\mathbb{K}\{x_1 \dots x_n\}/I$ and a morphism of analytic algebras is a local homomorphism of $\mathbb{K}$-algebras. Let $\mathbf{An}_{\mathbb{K}}$ be the category of analytic algebras and let $\widehat{\mathbf{An}}_{\mathbb{K}}$ be the category of complete analytic algebras.

Remark 1.1.6. Every analytic algebra is a local Noetherian $\mathbb{K}$-algebras. Every local Artinian $\mathbb{K}$-algebra with residue field $\mathbb{K}$ is an analytic algebra.

We recall two Artin's important results (see [3], Theorem 1.5a and Corollary 1.6):

Theorem 1.1.7. *Let R and S be analytic algebras and let $\hat{R}$ and $\hat{S}$ be their completions. Let $\bar{\psi} : R \to \hat{S}$ be a morphism of analytic algebras, then, for all $n \in \mathbb{N}$, there exists a morphism of analytic algebras $\psi_n : R \to S$, such that the following diagram is commutative:*

$$\begin{tikzcd}
R \arrow[r, "\psi_n"] \arrow[d, equal, "\mathrm{Id}"] & S \arrow[r, "\pi_n"] & S/\mathfrak{m}_S^n \arrow[d, equal, "\mathrm{Id}"] \\
R \arrow[r, "\bar{\psi}"] & \hat{S} \arrow[r, "\pi_n"] & S/\mathfrak{m}_S^n.
\end{tikzcd}$$

Corollary 1.1.8. *With the notation of Theorem 1.1.7, if in addition $\bar{\psi}$ induces an isomorphims $\hat{\bar{\psi}} : \hat{R} \to \hat{S}$, then ψ_n is an isomorphism, provided $n \geq 2$.*

Using these results, we can prove the following

Proposition 1.1.9. *Let R and S be analytic algebras and let $\hat{R}$ and $\hat{S}$ be their completions. Let $\hat{\psi} : \hat{R} \to \hat{S}$ be a smooth morphism, then there exists a smooth morphism $R \to S$.*

Proof. By Thereom 1.1.5, there exists an isomorphism $\hat{\phi} : \hat{R}[[x]] \to \hat{S}$, Corollary 1.1.8 implies that there exists an isomorphism $\phi : R\{x\} \to S$, which is obviously smooth by Theorem 1.1.5. Thus the morphism $\phi \circ i : R \hookrightarrow R\{x\} \to S$ is smooth. □

To complete our study of analytic algebras, we prove the following

Proposition 1.1.10. *Let R and S be analytic algebras, such that*

- $\dim_{\mathbb{K}} \mathfrak{m}_R/\mathfrak{m}_R^2 = \dim_{\mathbb{K}} \mathfrak{m}_S/\mathfrak{m}_S^2$ *and*

- $R\{z_1, \dots, z_N\} \cong S\{z_1, \dots, z_M\}$, *for some N and M,*

then R and S are isomorphic.

Proof. The first hypothesis implies that, in the isomorphism $R\{z_1, \dots, z_N\} \cong S\{z_1, \dots, z_M\}$, $N = M$. Moreover, proving the proposition by induction on N, the first hypothesis makes the inductive step trivial. Thus it is sufficient to prove the proposition for $N = 1$.

Let $R = \mathbb{K}\{x_1, \dots, x_n\}/I$ and $S = \mathbb{K}\{y_1, \dots, y_m\}/J$ be analytic algebras, with $I \subset (x_1, \dots, x_n)^2$ and $J \subset (y_1, \dots, y_m)^2$. Let $\phi : \mathbb{K}\{\underline{x}\}\{z\}/I \to \mathbb{K}\{\underline{y}\}\{z\}/J$ be an isomorphism and let ψ its inverse. Let $\phi(z) = \alpha z + \beta(\underline{y}) + \gamma(\underline{y}, z)$ and let $\psi(z) = az + b(\underline{x}) + c(\underline{x}, z)$, where $\alpha, a \in \mathbb{K}$ are costant, β, γ, b and c are polynomial, γ and c do not contain degree one terms and, with a linear change of variables, we can suppose that ϕ and ψ do not contain constant term.
If at least one between α and a is different from zero, then the thesis follows easely. For example, if $\alpha \neq 0$, the image $\phi(z)$ satisfies the hypothesis of Weierstrass Preparation Theorem and so it can be written as $\phi(z) = (z + h(\underline{y})) \cdot u(\underline{y}, z)$, where u is a unit and $h(\underline{y})$ is a polynomial. Then ϕ is well defined and induces an isomorphism on quotients: $\phi : \mathbb{K}\{\underline{x}\}/I \to \mathbb{K}\{\underline{y}\}\{z\}/J \cdot (z + h(\underline{y})) \cong \mathbb{K}\{\underline{y}\}/J$.
Let's now analyse the case $\alpha = a = 0$. Let $\nu : \mathbb{K}\{\underline{x}\}\{z\}/I \to \mathbb{K}\{\underline{x}\}\{z\}/I$ be a homomorphism defined by $\nu(x_i) = x_i$, for all i, and $\nu(z) = z + b(\underline{x})$. It is obviously an isomorphims and the composition $\phi \circ \nu$ is an isomorphims from $\mathbb{K}\{\underline{x}\}\{z\}/I$ to $\mathbb{K}\{\underline{y}\}\{z\}/J$, such that $\phi \circ \nu(z)$ contains a linear term in z, thus, passing to the quotient, it induces an isomorphism $\mathbb{K}\{\underline{x}\}/I \cong \mathbb{K}\{\underline{y}\}/J$. □

Related to the notion of formal smoothness, there is the following relation between analytic algebras:

$$R \propto S \quad \text{iff} \quad \exists \ R \longrightarrow S \quad \text{formally smooth morphism},$$

let $\sim$ be the equivalence relation between analytic algebras generated by $\propto$. We define another equivalence relation:

$$R \approx S \quad \text{iff} \quad R\{x_1 \dots x_n\} \cong S\{y_1 \dots y_m\} \quad \text{are isomorphic, for some } n \text{ and } m.$$

Remark 1.1.11. The relation $\approx$ is the same as the relation $\sim$. Infact, if $R \sim S$, there exists a chain of formally smooth morphisms $R \to T_1 \leftarrow T_2 \to \dots \to T_n \leftarrow S$, that, by Theorem 1.1.5 and Corollary 1.1.8, gives an isomorphism $R\{\underline{x}\} \cong S\{\underline{y}\}$, then $R \approx S$. Viceversa, if $R \approx S$, there exists an isomorphism $R\{\underline{x}\} \cong S\{\underline{y}\}$ which is formally smooth, thus we have the chain of formally smooth morphisms $R \to R\{\underline{x}\} \to S\{\underline{x}\} \leftarrow S$ and $R \sim S$.

We consider the following relation between complete analytic algebras:

$$\hat{R} \propto \hat{S} \quad \text{iff} \quad \exists\ \hat{R} \to \hat{S} \quad \text{formally smooth morphism,}$$

let $\sim$ be the equivalence relation between analytic algebras generated by $\propto$. We define an other equivalence relation:

$$\hat{R} \equiv \hat{S} \quad \text{iff} \quad \hat{R}[[x_1 \dots x_n]] \cong \hat{S}[[x_1 \dots x_m]] \quad \text{are isomorphic, for some } n \text{ and } m.$$

Remark 1.1.12. As before, the relation $\equiv$ is the same as the relation $\sim$.
Furthermore, the equivalence relation $\sim$ on completions of analytic algebras coincides with the relation $\sim$ between the analytic algebras themselves, because obviously the two relations $\equiv$ and $\approx$ are the same.

The opposite category of the category of analytic algebras $\mathbf{An}^{\mathbf{o}}_{\mathbb{K}}$ is called the category of *germs of analytic spaces*. We indicate it with $\mathbf{Germ}_{\mathbb{K}}$. The geometric meaning of this definition is that a germ A^o can be represented by (X, x, α), where X is a complex space with a distinguished point x and α is a fixed isomorphism of $\mathbb{K}$-algebras $\mathcal{O}_{X,x} \cong A$. Two triples, (X, x, α) and (Y, y, β), are equivalent if there exists an isomorphism from a neighborhood of x in X to a neighborhood of y in Y which sends x in y and which induces an isomorphism $\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,y}$.

Let (X, x) and (Y, y) be germs of analytic spaces, given by the analytic algebras S and R respectively, let $\Psi : (X, x) \to (Y, y)$ be a morphism of germs of analytic spaces and let $\psi : R \to S$ be the corresponding morphism of analytic algebras.

Definition 1.1.13. *The morphism $\Psi : (X, x) \to (Y, y)$ is called* smooth *if the morphism $\psi : R \to S$ is formally smooth.*

We consider the following relation between germs of analytic spaces:

$$(X, x) \propto (Y, y) \quad \text{iff} \quad \exists\ (X, x) \longrightarrow (Y, y) \quad \text{smooth morphism}$$

and we define $\sim$ to be the equivalence relation between germs of analytic spaces generated by the relation $\propto$.

Remark 1.1.14. It is obvious that the relation $\sim$ defined between germs of analytic spaces is the same as the relation $\sim$ defined between their corresponding analytic algebras.

As in [37], we give the following

Definition 1.1.15. *The analytic spaces (X, x) and (Y, y) are said to have the same* type of singularities *if they are equivalent under the relation $\sim$.*

Definition 1.1.16. *Let X be a complex affine scheme, it is said to have* quadratic algebraic singularities *if it is defined by finitely many quadratic homogeneous polynomials. Let $\mathcal{X}$ be an analytic space, it is said to have* quadratic algebraic singularities *if it is locally isomorphic to complex affine schemes with quadratic algebraic singularities.*

For germs of analytic spaces we want to prove the following

Theorem 1.1.17. *Let $(X, 0)$ and $(Y, 0)$ be two germs of analytic spaces and let $\phi : (X, 0) \longrightarrow (Y, 0)$ be a smooth morphism. Then $(X, 0)$ has quadratic algebraic singularities if and only if $(Y, 0)$ has quadratic algebraic singularities.*

We need the following

Lemma 1.1.18. *Let $(X,0)$ and $(Y,0)$ be two germs of analytic spaces and let $\phi : (X,0) \longrightarrow (Y,0)$ be a smooth morphism. Let $X \subset \mathbb{K}^N$ and let $H = \{x \in \mathbb{K}^N \mid h(x) = 0\}$ be an hypersurface of $\mathbb{K}^N$, such that:*

- $dh(0) \neq 0$

- $TH \not\supset T\phi^{-1}(0)$,

then: $\phi|_{X\cap H} : (X \cap H, 0) \longrightarrow (Y,0)$ is a smooth morphism.

Proof. Let $(X,0)$ and $(Y,0)$ be defined by $\mathbb{K}\{x_1,\dots,x_n\}/I$ and $\mathbb{K}\{y_1,\dots,y_m\}/J$ respectively. Since ϕ is smooth, $\mathcal{O}_{X,0}$ is a power series ring over $\mathcal{O}_{Y,0}$, i.e. $\mathcal{O}_{X,0} \cong \mathcal{O}_{Y,0}\{t_1,\dots,t_s\}$, for some s.
Let $X' = X \cap H$ be the intersection, then $\mathcal{O}_{X',0} \cong \mathcal{O}_{X,0}/(h)$. If g corresponds to h by the isomorphism $\mathcal{O}_{X,0} \cong \mathcal{O}_{Y,0}\{t_1,\dots,t_s\}$, then $\mathcal{O}_{X',0} \cong \mathcal{O}_{Y,0}\{t_1,\dots,t_s\}/(g)$.
The hypothesis $dh(0) \neq 0$ becomes $dg(0) \neq 0$, that implies that there exists an indeterminate between y_i and t_i, such that the partial derivative of g with respect to this indeterminate calculated in zero does not vanish. Moreover, the hypothesis $TH \not\supset T\phi^{-1}(0)$ implies that this indeterminate must be one of the t_i, for example $t_{\bar{i}}$.
Thus, using the Implicit Function Theorem, we obtain $\mathcal{O}_{X',0} \cong \mathcal{O}_{Y,0}\{t_1,\dots,t_s\}/(g) \cong \mathcal{O}_{Y,0}\{t_1,\dots,\widehat{t_{\bar{i}}},\dots,t_s\}$ and $\phi|_{X'}$ is a smooth morphism. □

Now we can prove Theorem 1.1.17:

Proof. We start by assuming that $(Y,0)$ has quadratic algebraic singularities, so $(Y,0)$ is defined by the analytic algebra $\mathbb{K}\{y_1,\dots,y_m\}/J$, where J is an ideal generated by quadratic polynomials. Since ϕ is smooth, we have $\mathcal{O}_{X,0} \cong \mathcal{O}_{Y,0}\{t_1,\dots,t_s\} \cong \mathbb{K}\{y_1,\dots,y_m\}\{t_1,\dots,t_s\}/J$, for some s, and X has quadratic algebraic singularities.

Now we prove the other implication. Let $\mathbb{K}\{x_1,\dots,x_n\}/I$ and $\mathbb{K}\{y_1,\dots,y_m\}/J$ be the analytic algebras, that define the germs $(X,0)$ and $(Y,0)$ respectively, where I is an ideal generated by quadratic polynomials. We can assume that ϕ is not an isomorphism, otherwise the theorem is trivial. Since ϕ is smooth, $\mathcal{O}_{X,0} \cong \mathcal{O}_{Y,0}\{t_1,\dots,t_s\}$, for some $s > 0$.
Now we can intersect $X \subset \mathbb{K}^N_x$ with hyperplanes $h_1,\dots,h_s$ of $\mathbb{K}^N_x$, which correspond, by the isomorphism $\mathcal{O}_{X,0} \cong \mathcal{O}_{Y,0}\{t_1,\dots,t_s\}$, to the hyperplanes of equations $t_1 = 0,\dots,t_s = 0$ of $\mathbb{K}^{m+s}_{y,t}$ and we call the intersection X'. Then $(X',0)$ has quadratic algebraic singularities. Moreover, by lemma 1.1.18, ϕ restricted to $(X',0)$ is a smooth morphism and it is bijective because $\mathcal{O}_{X',0} \cong \mathcal{O}_{X,0}/(h_1,\dots,h_s) \cong \mathcal{O}_{Y,0}\{t_1,\dots,t_s\}/(t_1,\dots,t_s) \cong \mathcal{O}_{Y,0}$. Thus $(Y,0)$ has quadratic algebraic singularities. □

Remark 1.1.19. This theorem assures that the set of germs of analytic spaces with quadratic algebraic singularities is closed under the relation $\sim$ and so it is a union of equivalent classes under this relation.

Our aim is to prove that the property that two germs of analytic spaces have the same type of singularities is formal, that is that it can be controlled at the level of functors.

1.2 Functors of Artin rings

In this section we collect some definitions and properties of functors of Artin rings which are used in the following. We mainly follow [5], [21] and [33].

Definition 1.2.1. *A* functor of Artin rings *is a covariant functor* $\mathcal{F} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ *from the category of local Artinian* $\mathbb{K}$*-algebras with residue field* $\mathbb{K}$ *to the category of sets, such that* $\mathcal{F}(\mathbb{K})$ = *fixed one point set.*

Example 1.2.2. We start with some examples of functors of Artin rings that often appear in the following.

- Let $\hat{R} \in \widehat{\mathbf{Art}}_{\mathbb{K}}$ be a complete Noetherian local $\mathbb{K}$-algebra with residue field $\mathbb{K}$ and let $\mathrm{Hom}(\hat{R}, -) : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ be the functor which associates, to every $A \in \mathbf{Art}_{\mathbb{K}}$, the set $\mathrm{Hom}(\hat{R}, A)$ of local homomorphisms of $\mathbb{K}$-algebras.

- Let $\mathcal{X}$ be a geometric object, the functor of infinitesimal deformations of $\mathcal{X}$ is the functor $\mathrm{Def}_{\mathcal{X}} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ which associates to every $A \in \mathbf{Art}_{\mathbb{K}}$ the set of isomorphism classes of deformations of $\mathcal{X}$ over $\mathrm{Spec}\, A$. We analyse precisely some examples of infinitesimal deformations of geometric objects in Chapter 2.

- Let L be a DGLA, there exists a deformation functor $\mathrm{Def}_L : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ associated to L; we define it in Section 3.2. Let $\chi : L \to M$ be a morphism of DGLAs, there exists a deformation functor $\mathrm{Def}_\chi : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ associated to χ; we define it in Section 3.3.

As seen also in our examples, the main interest to functors of Artin rings comes from deformation theory and moduli problems. From this point of view, the following notions of functor with hull and of prorepresentable functor are two of the most important in the study of functors of Artin rings.

Let $\mathcal{F}$ be a functor of Artin rings, a *couple* for $\mathcal{F}$ is a pair (A, ξ), where $A \in \mathbf{Art}_{\mathbb{K}}$ and $\xi \in \mathcal{F}(A)$. A couple (A, ξ) for $\mathcal{F}$ induces an obvious morphism of functors, $\mathrm{Hom}(A, -) \to \mathcal{F}$, which associates, to every $B \in \mathbf{Art}_{\mathbb{K}}$ and $\phi \in \mathrm{Hom}(A, B)$, the element $\phi(\xi) \in \mathcal{F}(B)$. We can extend the functor $\mathcal{F}$ to the category $\widehat{\mathbf{Art}}_{\mathbb{K}}$ of local Noetherian complete $\mathbb{K}$-algebras with residue field $\mathbb{K}$ by the formula $\widehat{\mathcal{F}}(A) = \lim\limits_{\leftarrow} \mathcal{F}(A/\mathfrak{m}^n)$. A *procouple* for $\mathcal{F}$ is a pair (A, ξ), where $A \in \widehat{\mathbf{Art}}_{\mathbb{K}}$ and $\xi \in \widehat{\mathcal{F}}(A)$. It induces an obvious morphism of functors: $\mathrm{Hom}(A, -) \to \mathcal{F}$.

Definition 1.2.3. *Let* $\mathcal{F}$ *be a functor of Artin rings. A procouple* (A, ξ) *for* $\mathcal{F}$ *is called a* prorepresentable hull *of* $\mathcal{F}$*, or just a* hull *of* $\mathcal{F}$*, if the induced morphism* $\mathrm{Hom}(A, -) \to \mathcal{F}$ *is smooth and it is bijective on* $\mathbb{K}[\epsilon]$*, i.e.* $\mathrm{Hom}(A, \mathbb{K}[\epsilon]) \to \mathcal{F}(\mathbb{K}[\epsilon])$ *is bijective.*
A functor $\mathcal{F}$ *is called* prorepresentable *by the procouple* (A, ξ) *if the induced morphism* $\mathrm{Hom}(A, -) \to \mathcal{F}$ *is an isomorphism of functors.*

Remark 1.2.4. Note that, if (A, ξ) prorepresents $\mathcal{F}$, then (A, ξ) is a hull of $\mathcal{F}$. Moreover a couple which prorepresents a functor is unique up to canonical isomorphism, while in general two hulls of a functor are isomorphic in a noncanonical way.

The existence for a functor of Artin rings $\mathcal{F}$ of a procouple which is a hull of it or which prorepresents it is regulated by the well-known following Schlessinger's Theorem.

Let $\mathcal{F} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ be a functor of Artin rings. Let $B \to A$ and $C \to A$ be morphisms in $\mathbf{Art}_{\mathbb{K}}$ and let

$$\eta : \mathcal{F}(B \times_A C) \to \mathcal{F}(B) \times_{\mathcal{F}(A)} \mathcal{F}(C)$$

be the induced morphism. The *Schlessinger conditions* are the following:

(H_1) η is surjective, if $C \to A$ is a small extension,

(H_2) η is bijective, if $C = \mathbb{K}[\epsilon]$ and $A = \mathbb{K}$,

(H_3) $\dim_{\mathbb{K}} t_{\mathcal{F}} < +\infty$,

(H_4) η is bijective, if $B = C \to A$ is a small extension.

Remark 1.2.5. Since every surjective morphism in $\mathbf{Art}_{\mathbb{K}}$ can be expressed as a finite composition of small extensions (see Remark 1.1.3), conditions (H_1) and (H_4) above can be replaced by the following:

$(H_1)'$ η is surjective, if $C \to A$ is surjective,

$(H_4)'$ η is bijective, if $B = C \to A$ is surjective.

Infact, if $C \to A$ is a surjection and there exist small extensions: $0 \to J_0 \to C \to D_0 \to 0$, $0 \to J_1 \to D_0 \to D_1 \to 0$, ... , $0 \to J_{n+1} \to D_n \to A \to 0$, we have successive fibered products:

$$\begin{tikzcd}
C \times_A B \arrow[r] \arrow[d] & D_0 \times_A B \arrow[r] \arrow[d] & D_1 \times_A B \arrow[r] \arrow[d] & \cdots \arrow[r] & D_n \times_A B \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D_0 \arrow[r] & D_1 \arrow[r] & \cdots \arrow[r] & D_n \arrow[r] & A
\end{tikzcd}$$

and successive morphisms of functors:

$$\mathcal{F}(D_n \times_A B) \to \mathcal{F}(D_n) \times_{\mathcal{F}(A)} \mathcal{F}(B) \dots \tag{1.1}$$

$\mathcal{F}(D_0 \times_A B) \to \mathcal{F}(D_0) \times_{\mathcal{F}(D_1)} \mathcal{F}(D_1 \times_A B)$ and $\mathcal{F}(C \times_A B) \to \mathcal{F}(C) \times_{\mathcal{F}(D_0)} \mathcal{F}(D_0 \times_A B)$,

which give a morphism of functors

$$\mathcal{F}(C \times_A B) \to \mathcal{F}(C) \times_{\mathcal{F}(A)} \mathcal{F}(B). \tag{1.2}$$

Thus, if condition (H_1) or (H_4) holds, the morphisms (1.1) are surjective or bijective and the same is true for the morphism (1.2), then the correspondent condition $(H_1)'$ or $(H_4)'$ holds.

Remark 1.2.6. The condition (H_2) is equivalent to require that

$(H_2)'$ η is bijective, if $C \to A$ is a small extension and $A = \mathbb{K}$.

Infact, let $0 \to J \to C \to \mathbb{K} \to 0$ be a small extension, where C is necessary of the form $\mathbb{K}[\epsilon_1 \dots \epsilon_n]$, with $\epsilon_i \cdot \epsilon_j = 0$ for all $i, j = 0, \dots n$; if condition (H_2) holds, prove by induction on n that $(H_2)'$ holds. For $n=1$, condition $(H_2)'$ concides with (H_2). Now suppose that condition $(H_2)'$ holds for $n-1$ and prove it for n. The cartesian diagram:

$$\begin{tikzcd} B \times_{\mathbb{K}} \mathbb{K}[\epsilon_1 \dots \epsilon_{n-1}] \arrow[r] \arrow[d] & \mathbb{K}[\epsilon_1 \dots \epsilon_{n-1}] \arrow[d] \\ B \arrow[r] & \mathbb{K}, \end{tikzcd}$$

by inductive hypothesis, gives the bijective map $\mathcal{F}(B \times_{\mathbb{K}} \mathbb{K}[\epsilon_1 \dots \epsilon_{n-1}]) \to \mathcal{F}(B) \times_{\mathcal{F}(\mathbb{K})} \mathcal{F}(\mathbb{K}[\epsilon_1 \dots \epsilon_{n-1}])$, and the cartesian diagram

$$\begin{tikzcd} B \times_{\mathbb{K}} \mathbb{K}[\epsilon_1 \dots \epsilon_n] \arrow[r] \arrow[d] & \mathbb{K}[\epsilon_n] \arrow[d] \\ B \times_{\mathbb{K}} \mathbb{K}[\epsilon_1 \dots \epsilon_{n-1}] \arrow[r] & \mathbb{K}, \end{tikzcd}$$

by (H_2), gives the bijective map $\mathcal{F}(B \times_{\mathbb{K}} \mathbb{K}[\epsilon_1 \dots \epsilon_n]) \to \mathcal{F}(B \times_{\mathbb{K}} \mathbb{K}[\epsilon_1 \dots \epsilon_{n-1}]) \times_{\mathcal{F}(\mathbb{K})} \mathcal{F}(\mathbb{K}[\epsilon_n])$. Moreover, by condition (H_2) the map $\mathcal{F}(\mathbb{K}[\epsilon_1 \dots \epsilon_n]) \to \mathcal{F}(\mathbb{K}[\epsilon_1 \dots \epsilon_{n-1}]) \times_{\mathcal{F}(\mathbb{K})} \mathcal{F}(\mathbb{K}[\epsilon_n])$ is bijective. Thus also the map

$$\eta : \mathcal{F}(B \times \mathbb{K}[\epsilon_1 \dots \epsilon_n]) \to \mathcal{F}(B) \times_{\mathcal{F}(\mathbb{K})} \mathcal{F}(\mathbb{K}[\epsilon_1 \dots \epsilon_n])$$

is bijective and this proves condition $(H2)'$.

Theorem 1.2.7 (Schlessinger's Theorem)**.** *Let $\mathcal{F} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ be a functor of Artin rings. Let $B \to A$ and $C \to A$ be morphisms in $\mathbf{Art}_{\mathbb{K}}$ and let*

$$\eta : \mathcal{F}(B \times_A C) \to \mathcal{F}(B) \times_{\mathcal{F}(A)} \mathcal{F}(C)$$

be the induced morphism.

- *The functor $\mathcal{F}$ has a hull if and only if it satisfies conditions (H_1), (H_2) and (H_3).*
- *The functor $\mathcal{F}$ is prorepresentable if and only if it satisfies conditions (H_1), (H_2), (H_3) and (H_4).*

Proof. See [33], Theorem 2.11. □

Definition 1.2.8. *Let $\mathcal{F}$ be a functor of Artin rings. $\mathcal{F}$ is a functor with a* good deformation theory *if conditions (H_1) and (H_2) hold. $\mathcal{F}$ is* homogeneous*, if η is bijective, whenever $C \to A$ is surjective.*

Remark 1.2.9. Obviously an homogeneous functor is also a functor with a good deformation theory, but the converse is not true.

The following notions of tangent space and obstruction theory for a functor are linked to the ones defined for a moduli space and for a deformation problem.

Definition 1.2.10. *Let $\mathcal{F}$ be a functor of Artin rings. The* tangent space *to $\mathcal{F}$ is the set $t_{\mathcal{F}} = \mathcal{F}(\mathbb{K}[\epsilon])$, where $\mathbb{K}[\epsilon]$ is the local Artinian $\mathbb{K}$-algebra of dual numbers, i.e. $\mathbb{K}[\epsilon] = \mathbb{K}[x]/(x^2)$.*

Proposition 1.2.11. *Let $\mathcal{F}$ be a functor of Artin rings. The set $t_{\mathcal{F}}$ has a structure of $\mathbb{K}$-vector space. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of functors of Artin rings, then the induced map $\phi : t_{\mathcal{F}} \to t_{\mathcal{G}}$ is a linear map.*

Proof. The scalar multiplication by an element $\alpha \in \mathbb{K}$ on $t_{\mathcal{F}}$ is given by the morphism of sets obtained applying $\mathcal{F}$ to the morphism in $\mathbf{Art}_{\mathbb{K}}$:

$$\begin{array}{rcl} \mathbb{K}[\epsilon] & \longrightarrow & \mathbb{K}[\epsilon] \\ a + \epsilon b & \longrightarrow & a + \epsilon \alpha b. \end{array}$$

The sum on $t_{\mathcal{F}}$ is given by the morphism of sets obtained applying $\mathcal{F}$ to the morphism in $\mathbf{Art}_{\mathbb{K}}$:

$$\begin{array}{rcl} \mathbb{K}[\epsilon] \times_{\mathbb{K}} \mathbb{K}[\epsilon] & \longrightarrow & \mathbb{K}[\epsilon] \\ (a + \epsilon b, a + \epsilon b') & \longrightarrow & (a + \epsilon(b + c)), \end{array}$$

noting that $t_{\mathcal{F}} \times t_{\mathcal{F}} = \mathcal{F}(\mathbb{K}[\epsilon] \times_{\mathbb{K}} \mathbb{K}[\epsilon])$. By this definition of the structure of $\mathbb{K}$-vector space on $t_{\mathcal{F}}$, the last assertion follows. $\square$

Definition 1.2.12. *Let $\mathcal{F}$ be a functor of Artin rings. An* obstruction theory (V, v_e) *for $\mathcal{F}$ is the data of a $\mathbb{K}$-vector space V, called* obstruction space*, and, for every small extension in $\mathbf{Art}_{\mathbb{K}}$:*

$$e : 0 \longrightarrow J \longrightarrow B \longrightarrow A \longrightarrow 0,$$

a map $v_e : \mathcal{F}(A) \to V \otimes_{\mathbb{K}} J$ called obstruction map*. The data (V, v_e) have to satisfy the following conditions:*

- *if $\xi \in \mathcal{F}(A)$ can be lifted to $\mathcal{F}(B)$, then $v_e(\xi) = 0$,*
- *(base change) for every morphism $f : e_1 \to e_2$ of small extensions, i.e. for every commutative diagram*

$$\begin{tikzcd} e_1 : & 0 \arrow[r] & J_1 \arrow[r] \arrow[d, "f_J"] & B_1 \arrow[r] \arrow[d, "f_B"] & A_1 \arrow[r] \arrow[d, "f_A"] & 0 \\ e_2 : & 0 \arrow[r] & J_2 \arrow[r] & B_2 \arrow[r] & A_2 \arrow[r] & 0 \end{tikzcd} \tag{1.3}$$

then $v_{e_2}(f_A(\xi)) = (\mathrm{Id}_V \otimes f_J)(v_{e_1}(\xi))$, for every $\xi \in \mathcal{F}(A_1)$.

An obstruction theory (V, v_e) for $\mathcal{F}$ is called complete *if the converse of the first item above holds, i.e. the lifting of $\xi \in \mathcal{F}(A)$ to $\mathcal{F}(B)$ exists if and only if the obstruction $v_e(\xi)$ vanishes.*

Definition 1.2.13. *Let $\mathcal{F}$ and $\mathcal{G}$ be two functors of Artin rings with obstruction theories (V, v_e) and (W, w_e) respectively and let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of functors. A linear morphism $\phi' : V \to W$ is* compatible *with ϕ, if $w_e \phi = (\phi' \otimes \mathrm{Id}) v_e$, for every small extension e.*

Now we define relative obstruction theory for a morphism of functors of Artin rings $\phi : \mathcal{F} \to \mathcal{G}$. The minimal assumption is that $\mathcal{G}$ is a functor with a good deformation theory, in fact a different definition of relative obstructions can be given also without this hypothesis, but this general notion seems to be of little interest.

Definition 1.2.14. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of functors of Artin rings, where $\mathcal{G}$ is a functor with a good deformation theory. A* relative obstruction theory *(V, v_e) for ϕ is the data of a $\mathbb{K}$-vector space V, called* relative obstruction space*, and, for every small extension in $\mathbf{Art}_{\mathbb{K}}$:*

$$e : 0 \longrightarrow J \longrightarrow B \longrightarrow A \longrightarrow 0,$$

a map $v_e : \mathcal{G}(B) \times_{\mathcal{G}(A)} \mathcal{F}(A) \to V \otimes_{\mathbb{K}} J$ called relative obstruction map*. The data (V, v_e) have to satisfy the following conditions:*

- *if $(\eta, \xi) \in \mathcal{G}(B) \times_{\mathcal{G}(A)} \mathcal{F}(A)$ can be lifted to $\mathcal{F}(B)$, then $v_e(\eta, \xi) = 0$,*
- *(base change) for every morphism $f : e_1 \to e_2$ of small extensions, as in (1.3), then $v_{e_2}((f_B \times f_A)(\eta, \xi)) = (\mathrm{Id}_V \otimes f_J)(v_{e_1}(\eta, \xi))$, for every $(\eta, \xi) \in \mathcal{G}(B_1) \times_{\mathcal{G}(A_1)} \mathcal{F}(A_1)$.*

A relative obstruction theory (V, v_e) for ϕ is called complete *if the converse of the first item above holds, i.e. the lifting of $(\eta, \xi) \in \mathcal{G}(B) \times_{\mathcal{G}(A)} \mathcal{F}(A)$ to $\mathcal{F}(B)$ exists if and only if the obstruction $v_e(\eta, \xi)$ vanishes.*

Definition 1.2.15. *Let*

$$\begin{tikzcd} \mathcal{F}_1 \arrow[r, "\phi_1"] \arrow[d, "\eta"] & \mathcal{G}_1 \arrow[d, "\nu"] \\ \mathcal{F}_2 \arrow[r, "\phi_2"] & \mathcal{G}_2 \end{tikzcd}$$

be a commutative diagram of morphism of functors of Artin rings, where $\mathcal{F}_2$ and $\mathcal{G}_2$ are functors with a good deformation theory. Let (V, v_e) and (W, w_e) be relative obstruction theories for η and ν respectively. A linear morphism $\phi' : V \to W$ is compatible *with ϕ_1 and ϕ_2, if $w_e(\phi_2 \times \phi_1) = (\phi' \otimes \mathrm{Id})v_e$, for every small extension e.*

Remark 1.2.16. In Definition 1.2.14, if we replace the functor $\mathcal{G}$ with the costant functor $\mathcal{K}$, which associates to every local Artinian $\mathbb{K}$-algebras the same fixed one point set, and the morphism ϕ with the obvious one $\mathcal{F} \to \mathcal{K}$, we obtain Definition 1.2.12.

Related to obstruction theory for a functor of Artin rings and to relative obstruction theory for a morphism of functors, there is the notion of smoothness, which also linked to the notion of formal smoothness given in section 1.1.

Definition 1.2.17. *Let $\mathcal{F}$ be a functor of Artin rings, it is* smooth*, if for every surjection $B \to A$ in $\mathbf{Art}_{\mathbb{K}}$ the induced map $\mathcal{F}(B) \to \mathcal{F}(A)$ is surjective.*

Definition 1.2.18. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of functors of Artin rings, it is* smooth *if, for every surjective morphism $B \to A$ in $\mathbf{Art}_{\mathbb{K}}$, the induced map $\mathcal{F}(B) \to \mathcal{G}(B) \times_{\mathcal{G}(A)} \mathcal{F}(A)$ is surjective.*

Remark 1.2.19. It is clear that a smooth morphism of functors $\phi : \mathcal{F} \to \mathcal{G}$ is surjective, i.e. for all $A \in \mathbf{Art}_{\mathbb{K}}$ the morphism $\mathcal{F}(A) \to \mathcal{G}(A)$ is surjective. Infact it is sufficient to apply definition of smoothness to the projection onto the residue field $A \to \mathbb{K}$.

Remark 1.2.20. The definition of smooth functor can be rephrased in terms of smooth morphisms of functors, in the following way. A functor of Artin rings $\mathcal{F}$ is smooth, if the morphism $\phi : \mathcal{F} \to \mathcal{K}$ is smooth, where $\mathcal{K}$ is the costant functor.

Remark 1.2.21. By the above definitions it is clear that a functor of Artin rings $\mathcal{F}$, or a morphism of functors $\phi : \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a funtor with a good deformation theory, is smooth if and only if the trivial obstruction theory, or respectively the trivial relative obstruction theory, is complete.

With the above notions, we can state the following Standard Smoothness Criterion (for the proof, see [21], Proposition 2.17):

Theorem 1.2.22. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of functors of Artin rings and let $\mathcal{G}$ be a functor with a good deformation theory. Let (V, v_e) and (W, w_e) be two obstruction theories for $\mathcal{F}$ and $\mathcal{G}$ respectively. If:*

- *(V, v_e) is a complete obstruction theory,*
- *ϕ is injective between obstructions,*
- *ϕ is surjective between tangent spaces,*

then ϕ is smooth.

For morphisms between Hom functors the following proposition holds (for the proof, see [33], Proposition 2.5):

Proposition 1.2.23. *Let $\hat{\psi} : \hat{R} \to \hat{S}$ be a local homomorphism of local Noetherian complete $\mathbb{K}$-algebras, let $\hat{\phi} : \mathrm{Hom}(\hat{S}, -) \to \mathrm{Hom}(\hat{R}, -)$ be the morphism of functors induced by $\hat{\psi}$. Then $\hat{\phi}$ is smooth if and only if $\hat{S}$ is isomorphic to a formal power series ring over $\hat{R}$.*

Having in mind the equivalent relations $\sim$ defined in section 1.1 between analytic algebras and between gems of analytic spaces, we consider the following relation between functors:

$$\mathcal{F} \propto \mathcal{G} \quad \text{iff} \quad \exists \ \mathcal{F} \longrightarrow \mathcal{G} \quad \text{smooth morphism}$$

and we define $\sim$ to be the equivalence relation generated by $\propto$.

Definition 1.2.24. *The functors $\mathcal{F}$ and $\mathcal{G}$ are said to have the same* type of singularity *if they are equivalent under the relation $\sim$.*

Now we want to link Definitions 1.1.15 and 1.2.24 in the case the functors considered have hulls.

Remark 1.2.25. If two functors with hulls are such that their hulls are equivalent under the relation $\sim$, then the functors themselves are equivalent under the relation $\sim$.
Infact, let $\mathcal{F}$ and $\mathcal{G}$ be two functors with hulls, given by the germs of analytic spaces (X, x) and (Y, y), defined by the analytic algebras S and R respectively. If $(X, x) \sim (Y, y)$, or equivalently, if $R \sim S$, there exists a chain of smooth morphisms of analytic algebras $S \leftarrow T_1 \to T_2 \leftarrow \ldots \to R$ that induces a chain of smooth morphisms of functors $\mathcal{F} \leftarrow \mathrm{Hom}(\hat{S}, -) \to \mathrm{Hom}(\hat{T}_1, -) \leftarrow \mathrm{Hom}(\hat{T}_2, -) \ldots \mathrm{Hom}(\hat{R}, -) \to \mathcal{G}$, by Propositions 1.1.5 and 1.2.23 and by Definition of hull, thus $\mathcal{F} \sim \mathcal{G}$.

For the other implication we need the following

Proposition 1.2.26. *Let $\mathcal{F}$ and $\mathcal{G}$ be two functors with hulls given by the germs of analytic spaces (X, x) and (Y, y) respectively and let $\phi : \mathcal{F} \to \mathcal{G}$ be a smooth morphism. Then there exists a smooth morphism between the two germs (X, x) and (Y, y).*

Proof. Let S and R be the analytic algebras that define the germ $(X, x = 0)$ and $(Y, y = 0)$ respectively. By hypothesis, we have the following diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] & \mathcal{G} \\ \mathrm{Hom}(\hat{S}, -) \arrow[u, "\alpha"] \arrow[r, dashed, "\hat{\phi}"] & \mathrm{Hom}(\hat{R}, -) \arrow[u, "\beta"'] \end{tikzcd}$$

where, by definition of hull, α and β are smooth morphism and they are bijective on tangent spaces. Then, by smoothness, there exists a morphism $\hat{\phi} : \mathrm{Hom}(\hat{S}, -) \to \mathrm{Hom}(\hat{R}, -)$ that makes the diagram commutative. By hypothesis on α, β and ϕ, it is surjective on tangent spaces and it is injective on obstruction spaces. Thus $\hat{\phi}$ is a smooth morphism, by the Standard Smoothness Criterion (Theorem 1.2.22).
The morphism $\hat{\phi}$ determines uniquely an homomorphism $\hat{\psi} : \hat{R} \to \hat{S}$, which is formally smooth, by Propositions 1.1.5 and 1.2.23. Now, by Proposition 1.1.9, there exists a formally smooth morphism $\psi : R \to S$ and so a smooth morphism between the germs (X, x) and (Y, y). $\square$

Remark 1.2.27. If two functors with hulls are equivalent under the relation $\sim$, then their hulls are equivalent under the relation $\sim$.
Infact, if $\mathcal{F} \sim \mathcal{G}$, there exists a chain of smooth morphisms of functors $\mathcal{F} \leftarrow \mathcal{H}_1 \to \mathcal{H}_2 \leftarrow \ldots \to \mathcal{G}$. Then $\mathcal{H}_i$ necessary have hulls, we indicate with T_i the complete analytic algebra that is an hull for $\mathcal{H}_i$. By Proposition 1.2.26, the chain of smooth morphisms of functors gives a chain of smooth morphisms of complete analytic algebras $\hat{S} \to T_1 \leftarrow T_2 \to \ldots \leftarrow \hat{R}$, thus $\hat{S} \equiv \hat{R}$, so, as we have observed, $S \sim R$ and $(X, x) \sim (Y, y)$.

Remark 1.2.28. The Remarks 1.2.25 and 1.2.27 state that the relation $\sim$ between germs of analytic space (or analytic algebras), i.e. the property that two germs of analytic spaces have the same type of singularities, is formal, that is it can be controlled at the level of functors.

Thus is natural to introduce the following definition for functors:

Definition 1.2.29. *Let $\mathcal{F}$ be a functor with hull the germ of analytic space (X, x). It is said to have* quadratic algebraic singularities *if (X, x) has quadratic algebraic singularities.*

Remark 1.2.30. This definition is independent by the choice of the germ of analytic space which is a hull of $\mathcal{F}$, because the isomorphism class of a hull is uniquely determined.

Chapter 2

Geometric deformations

In this Chapter we recall some well-known facts about infinitesimal deformations of geometric objects, like schemes, locally free sheaves and pairs (scheme, sheaf). We define geometrically what a deformation of these objects is and we construct the associated functors of deformations. The study of these functors proceeds analizing conditions under which they have hulls or they are prorepresentable and calculating their tangent spaces and their obstructions.
In all this analysis we mainly follow [2], [33] and [34].

Let's start with some algebraic properties usefull for our purposes.

Lemma 2.0.31. *Let A be a ring, J a nilpotent ideal of A and $u : M \to N$ a homomorphism of A-modules, with N flat over A. If $\bar{u} : M/JM \to N/JN$ is an isomorphism, then u is an isomorphism.*

Proof. See [33], Lemma 3.3. □

Lemma 2.0.32. *Consider the following commutative diagram*

$$\begin{tikzcd}
N \arrow[rr, "p''"] \arrow[dr, "p'"'] & & M'' \arrow[dr, "u''"] & \\
& M' \arrow[rr, "u'"] & & M \\
B \arrow[uu] \arrow[rr] \arrow[dr] & & A'' \arrow[uu] \arrow[dr] & \\
& A' \arrow[uu] \arrow[rr] & & A \arrow[uu]
\end{tikzcd}$$

of rings and modules homomorphisms, where $B = A' \times_A A''$ and $N = M' \times_M M''$, M' is a flat A'-module and M'' is a flat A''-module. Suppose that:

- *$A''/J \xrightarrow{\cong} A$ is an isomorphism, where J is a nilpotent ideal of A'',*
- *u' and u'' induce isomorphisms $M' \otimes_{A'} A \xrightarrow{\cong} M$ and $M'' \otimes_{A''} A \xrightarrow{\cong} M$ respectively.*

Then N is flat over B, p' induces an isomorphism $N \otimes_B A' \xrightarrow{\cong} M'$ and p'' induces an isomorphism $N \otimes_B A'' \xrightarrow{\cong} M''$.

Proof. See [33], Lemma 3.4. □

Corollary 2.0.33. *Using the notation above, let L be a $B = A' \times_A A''$-module that makes the following diagram commutative*

$$\begin{tikzcd} L \arrow[r, "q'"] \arrow[d, "q''"] & M' \arrow[d, "u'"] \\ M'' \arrow[r, "u''"] & M \end{tikzcd}$$

and such that q' induces an isomorphism $q' \otimes_B A' : L \otimes_B A' \xrightarrow{\cong} M'$. Then the morphism $q' \times q'' : L \longrightarrow N = M' \times_M M''$ is an isomorphism.

Proof. Apply Lemma 2.0.31 to the morphism $u = q' \times q''$. □

2.1 Deformations of a scheme

Definition 2.1.1. *Let X be an algebraic scheme over $\mathbb{K}$. An* infinitesimal deformation *of X over $A \in \mathbf{Art}_{\mathbb{K}}$ is a cartesian diagram of morphisms of schemes:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X}_A \arrow[d, "\pi"] \\ \operatorname{Spec} \mathbb{K} \arrow[r] & \operatorname{Spec} A, \end{tikzcd}$$

where π is flat.

Definition 2.1.2. *Let X be an algebraic scheme and let $\mathcal{X}_A$ and $\mathcal{X}'_A$ be two infinitesimal deformations of X over $A \in \mathbf{Art}_{\mathbb{K}}$. An* isomorphism *of these two deformations is an isomorphism $\phi : \mathcal{X}_A \to \mathcal{X}'_A$ that makes the following diagram commutative*

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ \mathcal{X}_A \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{X}'_A \arrow[dl] \\ & \operatorname{Spec} A. & \end{tikzcd} \tag{2.1}$$

Example 2.1.3. For every algebraic scheme X and for every $A \in \mathbf{Art}_{\mathbb{K}}$, there exists at least one deformation of X over A, called the *product deformation*, given by

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \times \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} \mathbb{K} \arrow[r] & \operatorname{Spec} A. \end{tikzcd}$$

A deformation of X over $A \in \mathbf{Art}_{\mathbb{K}}$ is called *trivial*, if it is isomorphic to the product deformation.

Definition 2.1.4. *Let X be an algebraic scheme. An infinitesimal deformation $\mathcal{X}_A$ of X over $A \in \mathbf{Art}_{\mathbb{K}}$ is* locally trivial, *if every point $x \in X$ has a neighborhood $U_x \subset X$,* *such that*

$$\begin{array}{ccc} U_x & \longrightarrow & \mathcal{X}_A|_{U_x} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{K} & \longrightarrow & \operatorname{Spec}A \end{array}$$

is a trivial deformation of U_x.

For an affine scheme the following hold

Proposition 2.1.5. *Let X be an affine algebraic scheme. Every infinitesimal deformation of X is affine.*

Proof. See [2], Remark 1.1. □

Proposition 2.1.6. *Every affine non singular algebraic scheme is rigid, i.e. all its infinitesimal deformations are trivial.*

Proof. See [34], Theorem 1.2.4. □

Remark 2.1.7. Let $\mathcal{X}_A$ and $\mathcal{X}'_A$ be two infinitesimal deformations of an algebraic scheme X over $A \in \mathbf{Art}_{\mathbb{K}}$ and let $\phi : \mathcal{X}_A \to \mathcal{X}'_A$ be a morphism that makes commutative diagram (2.1), then it is an isomorphism.
Infact, applying Proposition 2.1.5 to an affine covering of X, we can reduce to the affine case. Then we have a morphism $\phi : \operatorname{Spec}R \to \operatorname{Spec}R'$ of infinitesimal deformations of $\operatorname{Spec}S$ over A, that defines a morphism of A-modules $\psi : R' \to R$, which induces the identity $R' \otimes A/\mathfrak{m}_A \cong R/\mathfrak{m}_A R \cong S$. Thus applying Lemma 2.0.31, we obtain the result.

Definition 2.1.8. *Let X be an algebraic scheme. The* functor of infinitesimal deformations of X *is the functor of Artin rings:*

$$\operatorname{Def}_X : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

which associates, to every local Artinian $\mathbb{K}$-algebra A, the set of isomorphism classes of infinitesimal deformations of X over A.
The functor of locally trivial infinitesimal deformations of X *is the functor of Artin rings:*

$$\operatorname{Def}'_X : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

which associates, to every local Artinian $\mathbb{K}$-algebra A, the set of isomorphism classes of locally trivial infinitesimal deformations of X over A.

Remark 2.1.9. Propositions 2.1.5 and 2.1.6 state that, if X is a non singular algebraic scheme, all its infinitesimal deformations are locally trivial and then $\operatorname{Def}_X \cong \operatorname{Def}'_X$. Infact, it is sufficient to take an affine covering of X to obtain trivializations.

Theorem 2.1.10. *Let X be an algebraic scheme, then the functors Def_X and Def'_X are functors with a good deformation theory.*

Proof. Let start verifing condition (H_1). Let $B \to A$ and $C \to A$ be morphisms in $\mathbf{Art}_{\mathbb{K}}$, with $C \to A$ surjective, and prove that

$$\eta : \mathrm{Def}_X(B \times_A C) \to \mathrm{Def}_X(B) \times_{\mathrm{Def}_X(A)} \mathrm{Def}_X(C)$$

is surjective. Let $(\mathcal{X}_B, \mathcal{X}_C) \in \mathrm{Def}_X(B) \times_{\mathrm{Def}_X(A)} \mathrm{Def}_X(C)$ and let $\mathcal{X}_A \in \mathrm{Def}_X(A)$ be the deformation, such that $\mathcal{X}_B \times_{\mathrm{Spec}\, B} \mathrm{Spec}\, A \cong \mathcal{X}_A \cong \mathcal{X}_C \times_{\mathrm{Spec}\, C} \mathrm{Spec}\, A$.
Define $\tilde{\mathcal{X}}$ to be the scheme with X as underlying topological space and with $\mathcal{O}_{\tilde{\mathcal{X}}} = \mathcal{O}_{\mathcal{X}_B} \times_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{O}_{\mathcal{X}_C}$ as structure sheaf.
To prove that $\tilde{\mathcal{X}} \in \mathrm{Def}_X(B \times_A C)$ and $\eta(\tilde{\mathcal{X}}) = (\mathcal{X}_B, \mathcal{X}_C)$, taking an affine covering of X and using Proposition 2.1.5, reduce to the affine case. Let $(\mathcal{X}_B, \mathcal{X}_C) = (\mathrm{Spec}\, B', \mathrm{Spec}\, C')$ and $\mathcal{X}_A = \mathrm{Spec}\, A'$. Then there is a commutative diagram

$$\begin{tikzcd}
B' \times_{A'} C' \arrow[rr] \arrow[dr] & & C' \arrow[dr] & \\
& B' \arrow[rr] & & A' \\
B \times_A C \arrow[uu] \arrow[rr] \arrow[dr] & & C \arrow[uu] \arrow[dr] & \\
& B \arrow[uu] \arrow[rr] & & A \arrow[uu]
\end{tikzcd}$$

in which, B' is a flat B-module and C' is a flat C-module, $B' \otimes_B A \cong A'$ and $C' \otimes_C A \cong A'$, since $\mathcal{X}_B$ and $\mathcal{X}_C$ are deformation of X over B and C respectively and their restrictions on $\mathrm{Spec}\, A$ are isomorphic to $\mathcal{X}_A$. Since $C \to A$ is surjective, with kernel J, we have $C/J \cong A$, where $J \subset \mathfrak{m}_C$ is nilpotent. Then, by Lemma 2.0.31, $B' \times_{A'} C'$ is a flat $B \times_A C$-module, $(B' \times_{A'} C') \otimes_{B \times_A C} B \cong B'$ and $(B' \times_{A'} C') \otimes_{B \times_A C} C \cong C'$.
This proves condition (H_1) for Def_X. If the deformations $\mathcal{X}_A$, $\mathcal{X}_B$ and $\mathcal{X}_C$ are locally trivial, the same is true for $\tilde{\mathcal{X}}$ and therefore (H_1) holds also for Def'_X.

Now verify condition (H_2). Let $B \to A$ and $C \to A$ be morphisms in $\mathbf{Art}_{\mathbb{K}}$, with $C \to A$ small extension and $A = \mathbb{K}$, prove that

$$\eta : \mathrm{Def}_X(B \times_A C) \to \mathrm{Def}_X(B) \times_{\mathrm{Def}_X(A)} \mathrm{Def}_X(C)$$

is bijective. Let $(\mathcal{X}_B, \mathcal{X}_C) \in \mathrm{Def}_X(B) \times_{\mathrm{Def}_X(A)} \mathrm{Def}_X(C)$. In this case, any $\tilde{\mathcal{X}}' \in \mathrm{Def}_X(B \times_A C)$ inducing the pair $(\mathcal{X}_B, \mathcal{X}_C)$, is such that fits in the commutative diagram

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] & \\
\mathcal{X}_B \arrow[dr] & & \mathcal{X}_C \arrow[dl] \\
& \tilde{\mathcal{X}}'. &
\end{tikzcd}$$

By the universal property of fibered sum of scheme, there exists a morphism $\tilde{\mathcal{X}} \to \tilde{\mathcal{X}}'$ such that

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] & \\
\tilde{\mathcal{X}} \arrow[rr] \arrow[dr] & & \tilde{\mathcal{X}}' \arrow[dl] \\
& \mathrm{Spec}(B \times_A C), &
\end{tikzcd}$$

where $\tilde{\mathcal{X}}$ is the deformation of X over $B \times_A C$ defined before. Then, by Remark 2.1.7, this morphism is an isomorphism and the fibres of η contains only one element.
This proves condition (H_2) for Def_X. The same arguments hold also for Def'_X. □

Now we recall, without proof, the most important results about tangent spaces and obstructions for the functors Def_X and Def'_X.

Theorem 2.1.11. *There are canonical identifications*

- *For an algebraic scheme X, $t_{\mathrm{Def}'_X} \cong H^1(X, \mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent sheaf.*
- *For an algebraic scheme X, $t_{\mathrm{Def}_X} \cong \mathrm{Ex}_{\mathbb{K}}(X, \mathcal{O}_X)$, where $\mathrm{Ex}_{\mathbb{K}}(X, \mathcal{O}_X)$ is the $\mathbb{K}$-vector space of extensions of X with kernel $\mathcal{O}_X$.*
- *If X is non singular, $\mathrm{Def}_X \cong \mathrm{Def}'_X$ and then $t_{\mathrm{Def}'_X} \cong t_{\mathrm{Def}'_X} \cong H^1(X, \mathcal{T}_X)$.*
- *If $X = \mathrm{Spec}\, R$ is affine, $t_{\mathrm{Def}_X} \cong \mathrm{Ex}_{\mathbb{K}}(R, R) = T^1_R$, where $\mathrm{Ex}_{\mathbb{K}}(R, R) = T^1_R$ is the $\mathbb{K}$-vector space of extensions of R with R, called the first cotangent module.*
- *If X is reduced, then $t_{\mathrm{Def}_X} \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$.*

Proof. See [34], Theorem 2.4.1. □

Theorem 2.1.12. *Let X be an algebraic scheme, then $H^2(X, \mathcal{T}_X)$ is an obstruction space for the functor Def'_X.*
If X is a non singular algebraic scheme, then $\mathrm{Def}'_X \cong \mathrm{Def}_X$ and $H^2(X, \mathcal{T}_X)$ is an obstruction space also for the functor Def_X.
If X is a reduced local complete intersection algebraic scheme, then $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ is an obstruction space for the functor Def_X.

Proof. See [34], Proposition 1.2.12 and Proposition 2.4.8. □

Knowing the tangent space for the functor Def_X, one can prove the following

Corollary 2.1.13. *If X is a projective scheme or an affine scheme with at most isolated singularities, then the functor Def_X has a hull.*

Proof. If $X = \mathrm{Spec}\, R$ is affine with isolated singularities, then T^1_R is a finite dimensional vector space (see [34], Corollary 3.1.2.).
If X is projective, the vector spaces $H^1(X, \mathcal{T}_X)$ and $H^0(X, T^1_X)$ are finite dimensional, and from the exact sequence

$$0 \to H^1(X, \mathcal{T}_X) \to \mathrm{Ex}_{\mathbb{K}}(X, \mathcal{O}_X) \to H^0(X, T^1_X) \to H^2(X, \mathcal{T}_X)$$

we obtain that also $\mathrm{Ex}_{\mathbb{K}}(X, \mathcal{O}_X)$ is finite dimensional.
Thus the Schlessinger's condition (H_3) holds in both cases. □

The stronger property of being prorepresentable is not satisfied in general by Def_X. Analizing the automorphisms of the algebraic scheme X, one can prove the following

Proposition 2.1.14. *If X is a projective scheme such that $H^0(X, \mathcal{T}_X) = 0$, then the functor Def_X is prorepresentable.*

Proof. See [34], Corollary 2.6.4. □

The problem for the functor Def_X to be prorepresentable, to have a hull or to satisfy some weaker similar properties, is linked to the existence of a universal, semiuniversal or versal deformation of X (see [19]). These problems were analysed for example in [4] and [32].

2.2 Deformations of a sheaf

Definition 2.2.1. *Let X be an algebraic scheme and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. An* infinitesimal deformation *of $\mathcal{E}$ over $A \in \mathbf{Art}_{\mathbb{K}}$ is a locally free sheaf of $\mathcal{O}_X \otimes A$-module $\mathcal{E}_A$ on $X \times \operatorname{Spec} A$ with a morphism of sheaves $\pi_A : \mathcal{E}_A \to \mathcal{E}$, such that $\pi_A : \mathcal{E}_A \otimes \mathbb{K} \to \mathcal{E}$ is an isomorphism.*

Definition 2.2.2. *Let X be an algebraic scheme and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. Two infinitesimal deformations of $\mathcal{E}$ over $A \in \mathbf{Art}_{\mathbb{K}}$, $\mathcal{E}_A$ and $\mathcal{E}'_A$, are* isomorphic*, if there exists an isomorphism of sheaves $\phi : \mathcal{E}_A \to \mathcal{E}'_A$ that commutes with the maps $\pi_A : \mathcal{E}_A \to \mathcal{E}$ and $\pi'_A : \mathcal{E}'_A \to \mathcal{E}$, i.e. $\pi'_A \circ \phi = \pi_A$.*

Definition 2.2.3. *Let X be an algebraic scheme and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. The* functor of infinitesimal deformations of $\mathcal{E}$ *is the functor of Artin rings:*

$$\mathrm{Def}_{\mathcal{E}} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

which associates to every local Artinian $\mathbb{K}$-algebra A, the set of isomorphism classes of infinitesimal deformations of $\mathcal{E}$ over A.

Theorem 2.2.4. *Let X be an algebraic scheme and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. Then the functor $\mathrm{Def}_{\mathcal{E}}$ is a functor with a good deformation theory.*

Proof. Start verifing condition (H_1). Let $B \to A$ and $C \to A$ be morphisms in $\mathbf{Art}_{\mathbb{K}}$, with $C \to A$ surjective, and prove that

$$\eta : \mathrm{Def}_{\mathcal{E}}(B \times_A C) \to \mathrm{Def}_{\mathcal{E}}(B) \times_{\mathrm{Def}_{\mathcal{E}}(A)} \mathrm{Def}_{\mathcal{E}}(C)$$

is surjective. Let $(\mathcal{E}_B, \mathcal{E}_C) \in \mathrm{Def}_{\mathcal{E}}(B) \times_{\mathrm{Def}_{\mathcal{E}}(A)} \mathrm{Def}_{\mathcal{E}}(C)$ and let $\mathcal{E}_A \in \mathrm{Def}_{\mathcal{E}}(A)$ be the deformation, such that $\mathcal{E}_B \otimes_B A \cong \mathcal{E}_A \cong \mathcal{E}_C \otimes_C A$.
For simplicity, denote with $\mathcal{X}_R = X \times \operatorname{Spec} R$, for every $R \in \mathbf{Art}_{\mathbb{K}}$. Define $\tilde{\mathcal{E}}$ to be the sheaf $\mathcal{E}_B \times_{\mathcal{E}_A} \mathcal{E}_C$. A priori $\tilde{\mathcal{E}}$ is a sheaf of $\mathcal{O}_{\mathcal{X}_B} \times_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{O}_{\mathcal{X}_C}$-module on $\mathcal{X}_B \times_{\mathcal{X}_A} \mathcal{X}_C$.
Note that $\mathcal{X}_{B \times_A C}$ and $\mathcal{X}_B \times_{\mathcal{X}_A} \mathcal{X}_C$ are both homeomorphic to X as topological space.
Prove that $\mathcal{O}_{\mathcal{X}_B} \times_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{O}_{\mathcal{X}_C} = \mathcal{O}_{\mathcal{X}_{B \times_A C}}$. Let analyse these sheaves on every open set $U \subset X$. By universality of fibered product, there is a map $\phi : \mathcal{O}_{\mathcal{X}_{B \times_A C}}(U) \to \mathcal{O}_{\mathcal{X}_B}(U) \times_{\mathcal{O}_{\mathcal{X}_A}(U)} \mathcal{O}_{\mathcal{X}_C}(U)$; since $C \to A$ is surjective, it induces an isomorphism $\widehat{\phi \otimes_{B \times_A C} C} : \mathcal{O}_{\mathcal{X}_{B \times_A C}}(U) \otimes_{B \times_A C} C \to \mathcal{O}_{\mathcal{X}_C}(U)$ and, by Corollary 2.0.33, we get that ϕ is an isomorphism.
Then $\tilde{\mathcal{E}}$ is a sheaf of $\mathcal{O}_{\mathcal{X}_{B \times_A C}}$-modules on $\mathcal{X}_{B \times_A C}$.
By Lemma 2.0.32, $\tilde{\mathcal{E}}(U) \otimes_{B \times_A C} B \cong \mathcal{E}_B(U)$ and $\tilde{\mathcal{E}}(U) \otimes_{B \times_A C} C \cong \mathcal{E}_C(U)$, for every U. This leads to the surjectivity of the map η.

Let now verify condition (H_2). Let $B \to A$ and $C \to A$ be morphisms in $\mathbf{Art}_{\mathbb{K}}$, with $C \to A$ small extension and $A = \mathbb{K}$, prove that

$$\eta : \mathrm{Def}_{\mathcal{E}}(B \times_A C) \to \mathrm{Def}_{\mathcal{E}}(B) \times_{\mathrm{Def}_{\mathcal{E}}(A)} \mathrm{Def}_{\mathcal{E}}(C)$$

is bijective. Let $(\mathcal{E}_B, \mathcal{E}_C) \in \mathrm{Def}_{\mathcal{E}}(B) \times_{\mathrm{Def}_{\mathcal{E}}(A)} \mathrm{Def}_{\mathcal{E}}(C)$. In this case, any $\hat{\mathcal{E}} \in \mathrm{Def}_{\mathcal{E}}(B \times_A C)$ inducing the pair $(\mathcal{E}_B, \mathcal{E}_C)$, is such that fits in the commutative diagram

$$\begin{tikzcd} & \hat{\mathcal{E}} \arrow[dl] \arrow[dr] & \\ \mathcal{E}_B \arrow[dr] & & \mathcal{E}_C \arrow[dl] \\ & \mathcal{E} & \end{tikzcd} \tag{2.2}$$

and, by Corollary 2.0.33, this implies that $\hat{\mathcal{E}} \cong \tilde{\mathcal{E}}$, where $\tilde{\mathcal{E}}$ is the deformation of $\mathcal{E}$ over $B \times_A C$ defined before. Then the fibres of η contains only one element. □

Theorem 2.2.5. *Let X be an algebraic scheme and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. Let $\mathrm{End}\,\mathcal{E}$ be the sheaf of endomorphisms of $\mathcal{E}$. The tangent space to the functor $\mathrm{Def}_{\mathcal{E}}$ is isomorphic to $H^1(X, \mathrm{End}\,\mathcal{E})$ and $H^2(X, \mathrm{End}\,\mathcal{E})$ is an obstruction space for it.*

Proof. Let $\mathcal{U} = \{U_\alpha\}$ be an affine open covering of X, such that the sheaf $\mathcal{E}$ is defined with respect to it by the system of transition function $\{f_{\alpha\beta}\}$, $f_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{G}\mathrm{l}_X)$, where $\mathcal{G}\mathrm{l}_X$ is the sheaf of matrices of functions in $\mathcal{O}_X$ that are invertible as matrices of costants at every point $x \in X$.
Let $\tilde{\mathcal{E}}$ be a locally free sheaf of $\mathcal{O}_X \otimes \mathbb{K}[\epsilon]$-modules on $X \times \mathrm{Spec}\,\mathbb{K}[\epsilon]$ that gives a deformation of $\mathcal{E}$ over $\mathbb{K}[\epsilon]$. The deformation $\tilde{\mathcal{E}}$ is defined, in the same covering $\mathcal{U}$ of $X \times \mathrm{Spec}\,\mathbb{K}[\epsilon]$, by transition functions

$$\tilde{f}_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{G}\mathrm{l}_{X \times \mathrm{Spec}\,\mathbb{K}[\epsilon]}).$$

Since $\mathcal{G}\mathrm{l}_{X \times \mathrm{Spec}\,\mathbb{K}[\epsilon]} = \mathcal{G}\mathrm{l}_X + \epsilon\,\mathcal{M}_X$, where $\mathcal{M}_X$ is the sheaf of matrices of functions in $\mathcal{O}_X$, and since the $\tilde{f}_{\alpha\beta}$ have restrictions modulo ϵ equal to $f_{\alpha\beta}$, they can be written as

$$\tilde{f}_{\alpha\beta} = f_{\alpha\beta}(1 + \epsilon g_{\alpha\beta}), \quad \text{with} \quad g_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{M}_X). \tag{2.3}$$

Moreover the transition functions $\tilde{f}_{\alpha\beta}$ satisfy conditions $\tilde{f}_{\alpha\beta}\tilde{f}_{\beta\gamma} = \tilde{f}_{\alpha\gamma}$, that, using (2.3), give

$$f_{\alpha\beta}f_{\beta\gamma} = f_{\alpha\gamma} \quad \text{and} \quad g_{\alpha\beta} + g_{\beta\gamma} = g_{\alpha\gamma}.$$

The first relations are the cocicle conditions for the transition functions $\{f_{\alpha\beta}\}$, while the second ones state that the system $\{g_{\alpha\beta}\}$ is a Čech 1-cocicle of the sheaf $\mathcal{M}_X \cong \mathrm{End}\,\mathcal{E}$ and it defines an element of $\check{\mathrm{H}}^1(X, \mathcal{M}_X) \cong \check{\mathrm{H}}^1(X, \mathrm{End}\,\mathcal{E})$.

Conversely, let $\{g_{\alpha\beta}\} \in \check{Z}^1(X, \mathcal{M}_X) \cong \check{Z}^1(X, \mathrm{End}\,\mathcal{E})$, the transition functions in (2.3) define a deformation $\tilde{\mathcal{E}}$ of $\mathcal{E}$ over $\mathbb{K}[\epsilon]$.

If we modify the system $\{g_{\alpha\beta}\}$ by the coboundary of an element $\{a_\alpha\} \in \Gamma(U_\alpha, \mathcal{M}_X)$, we have

$$h_{\alpha\beta} = g_{\alpha\beta} + a_\beta - a_\alpha \in \check{Z}^1(X, \mathcal{M}_X),$$

the correspondent transition functions are

$$\hat{f}_{\alpha\beta} = f_{\alpha\beta}(1 + \epsilon h_{\alpha\beta}) = f_{\alpha\beta}(1 + \epsilon(g_{\alpha\beta} + a_\beta - a_\alpha)) =$$

$$= f_{\alpha\beta}(1 + \epsilon g_{\alpha\beta})(1 + \epsilon a_\beta)(1 - \epsilon a_\alpha) = \tilde{f}_{\alpha\beta}(1 + \epsilon a_\beta)(1 + \epsilon a_\alpha)^{-1},$$

that define a sheaf $\hat{\mathcal{E}}$ which is obviously a deformation of the sheaf $\mathcal{E}$ over $\mathbb{K}[\epsilon]$ and which is isomorphic to $\tilde{\mathcal{E}}$, because their transition functions differ by multiplication by an invertible term. Thus there is an isomorphism $t_{\mathrm{Def}_{\mathcal{E}}} \cong H^1(X, \mathcal{M}_X) \cong H^1(X, \operatorname{End}\mathcal{E})$.

Now analyse obstructions for $\mathrm{Def}_{\mathcal{E}}$. Given a principal extension in $\mathbf{Art}_{\mathbb{K}}$

$$0 \longrightarrow t\mathbb{K} \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0,$$

we have the exact sequence

$$0 \longrightarrow t\,\mathcal{M}_X \xrightarrow{\exp} \mathcal{G}\mathrm{l}_{X\times \operatorname{Spec} B} = \mathcal{G}\mathrm{l}_X + \mathfrak{m}_B\,\mathcal{M}_X \xrightarrow{\mathrm{Id}+\alpha} \mathcal{G}\mathrm{l}_{X\times \operatorname{Spec} A} = \mathcal{G}\mathrm{l}_X + \mathfrak{m}_A\,\mathcal{M}_X \longrightarrow 0,$$

that induces a long exact sequence in cohomology

$$H^1(X \times \operatorname{Spec} B, \mathcal{G}\mathrm{l}_{X\times \operatorname{Spec} B}) \longrightarrow H^1(X \times \operatorname{Spec} A, \mathcal{G}\mathrm{l}_{X\times \operatorname{Spec} A}) \xrightarrow{\delta} H^2(X, \mathcal{M}_X).$$

A deformation of the sheaf $\mathcal{E}$ over A defines an element $\mathcal{E}_A \in H^1(X\times \operatorname{Spec} A, \mathcal{G}\mathrm{l}_{X\times \operatorname{Spec} A})$, it can be lifted to a deformation of $\mathcal{E}$ over B if and only if $\delta(\mathcal{E}_A) = 0$. Then $H^2(X, \mathcal{M}_X) \cong H^2(X, \operatorname{End}\mathcal{E})$ is an ostruction space for $\mathrm{Def}_{\mathcal{E}}$. $\square$

The following theorem state conditions for the existence of a hull for $\mathrm{Def}_{\mathcal{E}}$ and for prorepresentability.

Theorem 2.2.6. *Let X be an algebraic scheme and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. If $\dim_{\mathbb{K}} H^1(X, \operatorname{End}\mathcal{E}) < \infty$, the functor $\mathrm{Def}_{\mathcal{E}}$ has a hull. If moreover $H^0(X, \operatorname{End}\mathcal{E}) = \mathbb{K}$ then the functor $\mathrm{Def}_{\mathcal{E}}$ is prorepresentable.*

Proof. By Theorem 2.2.4, the functor $\mathrm{Def}_{\mathcal{E}}$ verifies conditions (H_1) and (H_2). By Theorem 2.2.5, the first hypothesis corrsponds to condition (H_3) and, by Schlessinger's Theorem, $\mathrm{Def}_{\mathcal{E}}$ has a hull. We prove the last assertion using differential graded Lie algebras tecniques (see Example 3.2.15). $\square$

Remark 2.2.7. Let X be an algebraic scheme. We recall that the Picard group $\mathrm{Pic}(X)$ is the group of the isomorphism classes of invertible sheaves of $\mathcal{O}_X$-modules on X. Classically the *Picard functor* is defined:

$$\begin{array}{rccl} \mathrm{Pic}: & \mathbf{Art}_{\mathbb{K}} & \longrightarrow & \mathbf{Group} \\ & A & \longrightarrow & \mathrm{Pic}(X \times \operatorname{Spec} A), \end{array}$$

note that the functoriality is given by the pullback of sheaves. Consider now an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules X with isomorphism class $[\mathcal{L}] \in \mathrm{Pic}(X)$. The classical *local Picard functor* associated to $[\mathcal{L}]$ is given by:

$$\begin{array}{rccl} \mathrm{Pic}_{[\mathcal{L}]}: & \mathbf{Art}_{\mathbb{K}} & \longrightarrow & \mathbf{Set} \\ & A & \longrightarrow & \mathrm{Pic}_{[\mathcal{L}]}(X \times \operatorname{Spec} A), \end{array}$$

where $\mathrm{Pic}_{[\mathcal{L}]}(X \times \operatorname{Spec} A) = \{[\mathcal{F}] \in \mathrm{Pic}(X \times \operatorname{Spec} A) \mid [\mathcal{F} \otimes_A \mathbb{K}] = [\mathcal{L}]\}$. Then $\mathrm{Pic}_{[\mathcal{L}]}$ coincides with $\mathrm{Def}_{\mathcal{L}}$.

Applying Theorem 2.2.5 and Theorem 2.2.6 to an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules on an algebraic scheme X, we obtain the following result

Theorem 2.2.8. *Let X be an algebraic scheme and let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$-modules on X. If the following conditions are satisfied:*

- $H^0(X, \mathcal{O}_X) = \mathbb{K}$,
- $\dim_{\mathbb{K}} H^1(X, \mathcal{O}_X) < \infty$,

then the functor $\mathrm{Def}_{\mathcal{L}}$ *is prorepresentable. Its tangent space is* $t_{\mathrm{Def}_{\mathcal{L}}} = H^1(X, \mathcal{O}_X)$ *and* $H^2(X, \mathcal{O}_X)$ *is an obstruction space for it.*

Proof. For a direct proof, see [33], Proposition 3.2, or [34], Theorem 3.3.1. □

In more generality, if $\mathcal{E}$ is an analytic coherent sheaf on a complex space X with compact support, it was proved that there exists a versal deformation for $\mathcal{E}$ (see [36]).

2.3 Deformations of the pair (scheme, sheaf)

Definition 2.3.1. *Let X be an algebraic scheme and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. An* infinitesimal deformation *of the pair $(X, \mathcal{E})$=(scheme, sheaf) over $A \in Art_{\mathbb{K}}$ is the data of:*

- *a deformation $\mathcal{X}_A$ of the scheme X over A, i.e. a cartesian diagram of morphisms of schemes*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X}_A \arrow[d, "\pi"] \\ \operatorname{Spec} \mathbb{K} \arrow[r] & \operatorname{Spec} A, \end{tikzcd}$$

  where π is flat;

- *a locally free sheaf $\mathcal{E}_A$ of $\mathcal{O}_{\mathcal{X}_A}$-modules on $\mathcal{X}_A$, with a morphism $\pi_A : \mathcal{E}_A \to \mathcal{E}$, such that $\pi_A : \mathcal{E}_A \otimes_A \mathbb{K} \to \mathcal{E}$ is an isomorphism.*

Definition 2.3.2. *Let X be an algebraic scheme and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. Two infinitesimal deformations, $(\mathcal{X}_A, \mathcal{E}_A)$ and $(\mathcal{X}'_A, \mathcal{E}'_A)$, of the pair $(X, \mathcal{E})$ over $A \in \mathbf{Art}_{\mathbb{K}}$ are* isomorphic, *if*

- *$\exists\ \phi : \mathcal{X}_A \to \mathcal{X}'_A$ isomorphism of deformations of the scheme X, i.e. ϕ is an isomorphism that makes the following diagram commutative:*

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ \mathcal{X}_A \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{X}'_A \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$

- *$\exists\ \psi : \mathcal{E}_A \to \mathcal{E}'_A$ isomorphism of sheaves of $\mathcal{O}_{\mathcal{X}_A}$-modules, where the structure of sheaf of $\mathcal{O}_{\mathcal{X}_A}$-module on $\mathcal{E}'_A$ is the one induce by ϕ, such that $\pi_A = \pi'_A \circ \psi$.*

Remark 2.3.3. Observe that, if the deformation $(\mathcal{X}_A, \mathcal{E}_A)$ of the pair $(X, \mathcal{E})$ over $A \in \mathbf{Art}_{\mathbb{K}}$ is such that $\mathcal{X}_A$ is trivial, then it defines a deformation of the sheaf $\mathcal{E}$ over A.

Definition 2.3.4. *Let X be an algebraic scheme and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. The* functor of infinitesimal deformations of the pair $(X,\mathcal{E})$ *is the functor of Artin rings:*

$$\mathrm{Def}_{(X,\mathcal{E})} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

which associates to every local Artinian $\mathbb{K}$-algebra A, the set of isomorphism classes of infinitesimal deformations of the pair $(X,\mathcal{E})$ over A.

Theorem 2.3.5. *Let X be an algebraic scheme and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. The functor $\mathrm{Def}_{(X,\mathcal{E})}$ is a functor with a good deformation theory.*

Proof. Let start verifying condition (H_1). Let $B \to A$ and $C \to A$ be morphisms in $\mathbf{Art}_{\mathbb{K}}$, with $C \to A$ surjective, and prove that

$$\eta : \mathrm{Def}_{(X,\mathcal{E})}(B \times_A C) \to \mathrm{Def}_{(X,\mathcal{E})}(B) \times_{\mathrm{Def}_{(X,\mathcal{E})}(A)} \mathrm{Def}_{(X,\mathcal{E})}(C)$$

is surjective. Let

$$((\mathcal{X}_B,\mathcal{E}_B),(\mathcal{X}_C,\mathcal{E}_C)) \in \mathrm{Def}_{(X,\mathcal{E})}(B) \times_{\mathrm{Def}_{(X,\mathcal{E})}(A)} \mathrm{Def}_{(X,\mathcal{E})}(C)$$

and let $(\mathcal{X}_A,\mathcal{E}_A) \in \mathrm{Def}_{(X,\mathcal{E})}(A)$ be the deformation, such that

$$\mathcal{X}_B \times_{\mathrm{Spec}\, B} \mathrm{Spec}\, A \cong \mathcal{X}_A \cong \mathcal{X}_C \times_{\mathrm{Spec}\, C} \mathrm{Spec}\, A \quad \text{and}$$

$$\mathcal{E}_B \otimes_B A \cong \mathcal{E}_A \cong \mathcal{E}_C \otimes_C A.$$

As in proof of Theorem 2.1.10, define $\tilde{\mathcal{X}}$ to be the scheme with X as underlying topological space and with $\mathcal{O}_{\tilde{\mathcal{X}}} = \mathcal{O}_{\mathcal{X}_B} \times_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{O}_{\mathcal{X}_C}$ as structure sheaf. As before, it is a deformation of X over $B \times_A C$ and it induces $(\mathcal{X}_B,\mathcal{X}_C) \in \mathrm{Def}_X(B) \times_{\mathrm{Def}_X(A)} \mathrm{Def}_X(C)$.
As in proof of Theorem 2.2.4, define $\tilde{\mathcal{E}}$ to be the sheaf $\mathcal{E}_B \times_{\mathcal{E}_A} \mathcal{E}_C$. It is a sheaf of $\mathcal{O}_{\mathcal{X}_B} \times_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{O}_{\mathcal{X}_C}$-modules on the topological space $\mathcal{X}_B \times_{\mathcal{X}_A} \mathcal{X}_C \cong X$ and it induces $(\mathcal{E}_B,\mathcal{E}_C) \in \mathrm{Def}_{\mathcal{E}}(B) \times_{\mathrm{Def}_{\mathcal{E}}(A)} \mathrm{Def}_{\mathcal{E}}(C)$.
This proves the surjectivity of the map η.

Now verify condition (H_2). Let $B \to A$ and $C \to A$ be morphisms in $\mathbf{Art}_{\mathbb{K}}$, with $C \to A$ small extension and $A = \mathbb{K}$, prove that

$$\eta : \mathrm{Def}_{(X,\mathcal{E})}(B \times_A C) \to \mathrm{Def}_{(X,\mathcal{E})}(B) \times_{\mathrm{Def}_{(X,\mathcal{E})}(A)} \mathrm{Def}_{(X,\mathcal{E})}(C)$$

is bijective. Let

$$((\mathcal{X}_B,\mathcal{E}_B),(\mathcal{X}_C,\mathcal{E}_C)) \in \mathrm{Def}_{(X,\mathcal{E})}(B) \times_{\mathrm{Def}_{(X,\mathcal{E})}(A)} \mathrm{Def}_{(X,\mathcal{E})}(C).$$

By Theorem 2.1.10, the deformation $\tilde{\mathcal{X}}$, defined before, is the unique element in $\mathrm{Def}_X(B \times_A C)$ that induces $(\mathcal{X}_B,\mathcal{X}_C) \in \mathrm{Def}_X(B) \times_{\mathrm{Def}_X(A)} \mathrm{Def}_X(C)$.
Moreover, as in Theorem 2.2.4, the sheaf $\tilde{\mathcal{E}}$, defined before, fits into the diagram (2.2), and so it is the unique element that induces $(\mathcal{E}_B,\mathcal{E}_C) \in \mathrm{Def}_{\mathcal{E}}(B) \times_{\mathrm{Def}_{\mathcal{E}}(A)} \mathrm{Def}_{\mathcal{E}}(C)$.
This proves the bijectivity of the map η. □

Now let X be a non singular projective algebraic scheme of dimension n and let $\mathcal{T}_X$ be its tangent sheaf. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X and let $\mathrm{End}\,\mathcal{E}$ be the sheaf of endomorphisms of $\mathcal{E}$.

Let $D^1(\mathcal{E})$ be the locally free sheaf of *first order differential operators with scalar principal symbol on the sections of the sheaf* $\mathcal{E}$.
Recall that, if $\mathcal{U} = \{U_\alpha\}$ is an open covering of X and $\{x_{k,\alpha}\}_k$ are local coordinates of X on U_α, a first order differential operator $P : \Gamma(X, \mathcal{E}) \to \Gamma(X, \mathcal{E})$ can be written locally on U_α as

$$P|_{U_\alpha} = (g_{ij})_\alpha + \sum_{k=1}^{n} (h_{ij})_{k,\alpha} \frac{\partial}{\partial x_{k,\alpha}} = g_\alpha + d_\alpha,$$

where $g_\alpha = (g_{ij})_\alpha \in \mathcal{M}_X(U_\alpha)$ is a matrix of functions in $\mathcal{O}_X(U_\alpha)$ and $d_\alpha = \sum_{k=1}^{n} (h_{ij})_{k,\alpha} \frac{\partial}{\partial x_{k,\alpha}}$ is a matrix with coefficients in $\mathcal{T}_X(U_\alpha)$.
The *principal symbol* of the first order differential operator P is the section σ of the sheaf $\operatorname{End}\mathcal{E} \otimes \mathcal{T}_X$ given, in the open set U_α, by:

$$\sigma(x) = \sum_{k=1}^{n} (h_{ij})_{k,\alpha}(x) \frac{\partial}{\partial x_{k,\alpha}}, \qquad \text{for } x \in U_\alpha.$$

The principal symbol is called *scalar* if the endomorphisms $(h_{ij})_{k,\alpha}$ of $\mathcal{E}$ are scalar, i.e. $(h_{ij})_{k,\alpha} = h_{k,\alpha} \cdot I$, where $h_{k,\alpha} \in \mathcal{O}_X(U_\alpha)$ and I is the identity matrix. If P has scalar principal symbol, d_α can be consider as an element in $\mathcal{T}_X(U_\alpha)$.
The symbol defines an exact sequence of sheaves:

$$0 \longrightarrow \operatorname{End}\mathcal{E} \longrightarrow D^1(\mathcal{E}) \xrightarrow{\sigma} \mathcal{T}_X \longrightarrow 0,$$

called *Atiyah extension*.

As before, let $\mathcal{U} = \{U_\alpha\}$ be an open covering of X and let $P \in D^1(\mathcal{E})$, consider the local expressions of P on the open sets U_α and U_β of $\mathcal{U}$:

$$P|_{U_\alpha} = g_\alpha + d_\alpha \quad \text{and} \quad P|_{U_\beta} = g_\beta + d_\beta,$$

where $g_\alpha \in \mathcal{M}_X(U_\alpha)$ and $d_\alpha \in \mathcal{T}_X(U_\alpha)$, $g_\beta \in \mathcal{M}_X(U_\beta)$ and $d_\beta \in \mathcal{T}_X(U_\beta)$. Let explicitate the glueing rules.
Let $\{e_{i,\alpha}\}_\alpha$ be local frames of $\mathcal{E}$ on the open sets U_α and $\{f_{\alpha\beta}\}_{\alpha\beta}$ be a system of transition functions of the sheaf $\mathcal{E}$ with respect to the open covering $\mathcal{U}$. Let $s \in \Gamma(X, \mathcal{E})$ be a section of $\mathcal{E}$, written locally on U_α as $s|_{U_\alpha} = \sum_i h_{i,\alpha} e_{i,\alpha}$ and on U_β as $s|_{U_\beta} = \sum_i h_{i,\beta} e_{i,\beta}$.
Using matrices notation, on the intersection $U_\alpha \cap U_\beta$ we have

$$P|_{U_\beta}(h_\beta e_\beta) = g_\beta h_\beta e_\beta + d_\beta(h_\beta) e_\beta \quad \text{and}$$

$$P|_{U_\alpha}(h_\beta e_\beta) = P|_{U_\alpha}(h_\beta f_{\alpha\beta} e_\alpha) = g_\alpha h_\beta f_{\alpha\beta} f_{\alpha\beta}^{-1} e_\beta + d_\alpha(h_\beta) f_{\alpha\beta} f_{\alpha\beta}^{-1} e_\beta + h_\beta d_\alpha(f_{\alpha\beta}) f_{\alpha\beta}^{-1} e_\beta,$$

then balancing the above expressions, we have

$$g_\beta = g_\alpha + \frac{d_\alpha f_{\alpha\beta}}{f_{\alpha\beta}} \quad \text{and} \quad d_\beta = d_\alpha.$$

Now we are ready to define the Čech differential on the Čech complex $\mathcal{C}^*(\mathcal{U}, D^1(\mathcal{E}))$. By the above calculation:

$$\begin{array}{rlcl} \delta : & \mathcal{C}^0(\mathcal{U}, D^1(\mathcal{E})) & \longrightarrow & \mathcal{C}^0(\mathcal{U}, D^1(\mathcal{E})) \\ & \{(g_\alpha, d_\alpha)\}_\alpha & \longrightarrow & \{(g_\beta - g_\alpha - \frac{d_\alpha f_{\alpha\beta}}{f_{\alpha\beta}}, d_\beta - d_\alpha)\}_{\alpha\beta}. \end{array}$$

By similiar calculations on the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$:

$$\begin{array}{rlcl} \delta : & \mathcal{C}^1(\mathcal{U}, D^1(\mathcal{E})) & \longrightarrow & \mathcal{C}^2(\mathcal{U}, D^1(\mathcal{E})) \\ & \{(g_{\alpha\beta}, d_{\alpha\beta})\}_{\alpha\beta} & \longrightarrow & \{(g_{\beta\gamma} - g_{\alpha\gamma} + g_{\alpha\beta} + \frac{d_{\alpha\beta} f_{\beta\gamma}}{f_{\beta\gamma}}, d_{\beta\gamma} - d_{\alpha\gamma} + d_{\alpha\beta})\}_{\alpha\beta\gamma}. \end{array}$$

and on the intersection $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$:

$$\begin{array}{rlcl} \delta : & \mathcal{C}^2(\mathcal{U}, D^1(\mathcal{E})) & \longrightarrow & \mathcal{C}^3(\mathcal{U}, D^1(\mathcal{E})) \\ & \{(g_{\alpha\beta\gamma}, d_{\alpha\beta\gamma})\}_{\alpha\beta\gamma} & \longrightarrow & \{(g_{\beta\gamma\delta} - g_{\alpha\gamma\delta} + g_{\alpha\beta\delta} - g_{\alpha\beta\gamma} + \frac{d_{\alpha\beta\gamma} f_{\gamma\delta}}{f_{\gamma\delta}}, \\ & & & d_{\beta\gamma\delta} - d_{\alpha\gamma\delta} + d_{\alpha\beta\delta} - d_{\alpha\beta\gamma})\}_{\alpha\beta\gamma\delta}. \end{array}$$

Theorem 2.3.6. *Let X be a non singular projective algebraic variety and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. The tangent space to the functor* $\mathrm{Def}_{(X,\mathcal{E})}$ *is isomorphic to $H^1(X, D^1(\mathcal{E}))$ and $H^2(X, D^1(\mathcal{E}))$ is an obstruction space for it.*

Proof. Let $\mathcal{U} = \{U_\alpha\}$ be an affine open covering of X and let $\{f_{\alpha\beta}\}$, with $f_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{G}\mathrm{l}_X)$, be a system of transition functions for the sheaf $\mathcal{E}$ with respect to this covering.
Let $(\tilde{\mathcal{X}}, \tilde{\mathcal{E}})$ be a deformation of $(X, \mathcal{E})$ over $\operatorname{Spec} \mathbb{K}[\epsilon]$. Since $\mathcal{U}$ is an affine covering, the deformation $\tilde{\mathcal{X}}$, restricted to every open set of it, is trivial. The composition of the trivializations over the intersections of open sets gives automorphisms

$$\theta_{\alpha\beta} : U_{\alpha\beta} \times \operatorname{Spec} \mathbb{K}[\epsilon] \xrightarrow{\theta_\beta} \tilde{\mathcal{X}}|_{U_{\alpha\beta}} \xrightarrow{\theta_\alpha^{-1}} U_{\alpha\beta} \times \operatorname{Spec} \mathbb{K}[\epsilon].$$

As in Theorem 2.1.11, the element in $H^1(X, \mathcal{T}_X)$ which corresponds to the deformation $\tilde{\mathcal{X}}$ is the class of the cocicle $\{d_{\alpha\beta}\} \in \check{Z}^1(\mathcal{U}, \mathcal{T}_X)$ associated to the automorphisms $\theta_{\alpha\beta}$ by

$$\theta_{\alpha\beta} = 1 + \epsilon d_{\alpha\beta}. \tag{2.4}$$

Let the sheaf $\tilde{\mathcal{E}}$ be defined in the same open covering $\mathcal{U}$ by transition functions $F_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{G}\mathrm{l}_{\tilde{\mathcal{X}}})$. Since the deformation $\tilde{\mathcal{X}}$ is trivial on every $U_{\alpha\beta}$, $\Gamma(U_{\alpha\beta}, \mathcal{G}\mathrm{l}_{\tilde{\mathcal{X}}}) \cong \Gamma(U_{\alpha\beta}, \mathcal{G}\mathrm{l}_{X \times \operatorname{Spec} \mathbb{K}[\epsilon]}) \cong \Gamma(U_{\alpha\beta}, \mathcal{G}\mathrm{l}_X) + \epsilon \Gamma(U_{\alpha\beta}, \mathcal{M}_X)$. The transition functions $F_{\alpha\beta}$ reduces to the $f_{\alpha\beta}$ modulo ϵ, then they can be written as

$$F_{\alpha\beta} = f_{\alpha\beta}(1 + \epsilon g_{\alpha\beta}). \tag{2.5}$$

Using the automorphisms $\theta_{\alpha\beta}$, the cocicle conditions satisfied by the transition functions are

$$F_{\alpha\beta} \theta_{\alpha\beta}(F_{\beta\gamma}) = F_{\alpha\gamma},$$

that, using the above equality (2.4) and (2.5), become

$$f_{\alpha\beta}(1 + \epsilon g_{\alpha\beta})[f_{\beta\gamma}(1 + \epsilon g_{\beta\gamma}) + \epsilon d_{\alpha\beta} f_{\beta\gamma}(1 + \epsilon g_{\beta\gamma})] = f_{\alpha\gamma}(1 + \epsilon g_{\alpha\gamma}),$$

that give

$$f_{\alpha\beta} f_{\beta\gamma} = f_{\alpha\gamma} \quad \text{and} \quad g_{\alpha\beta} + g_{\beta\gamma} - g_{\alpha\gamma} + \frac{d_{\alpha\beta}(f_{\beta\gamma})}{f_{\beta\gamma}} = 0.$$

Then the system $\{(g_{\alpha\beta}, d_{\alpha\beta})\}$ is a Čech 1-cocicle of the sheaf $D^1(\mathcal{E})$ and it defines an element in $\check{\mathrm{H}}^1(X, D^1(\mathcal{E}))$.

Conversely, let $\{(g_{\alpha\beta}, d_{\alpha\beta})\} \in \check{Z}^1(\mathcal{U}, D^1(\mathcal{E}))$, the automorphisms $\theta_{\alpha\beta}$ in (2.4) define a deformation $\tilde{\mathcal{X}}$ of X over $\mathbb{K}[\epsilon]$ and the transition functions in (2.5) define a locally free sheaf $\tilde{\mathcal{E}}$ of $\mathcal{O}_{\tilde{\mathcal{X}}}$-modules on $\tilde{\mathcal{X}}$, such that the pair $(\tilde{\mathcal{X}}, \tilde{\mathcal{E}})$ is a deformation of $(X, \mathcal{E})$ over $\mathbb{K}[\epsilon]$. It can be verified that this deformation do not depend on the cicle that represents the class in $\check{\mathrm{H}}(X, D^1(\mathcal{E}))$. Thus there is an isomorphism $t_{\mathrm{Def}_{(X,\mathcal{E})}} \cong H^1(X, D^1(\mathcal{E}))$.

Now analyse the obstruction of $\mathrm{Def}_{(X,\mathcal{E})}$. Let

$$0 \to t\mathbb{K} \to B \to A \to 0$$

be a principal extension in $\mathbf{Art}_{\mathbb{K}}$ and let $(\mathcal{X}_A, \mathcal{E}_A)$ be a deformation of the pair $(X, \mathcal{E})$ over A. Let $\mathcal{U} = \{U_\alpha\}$ be the affine open covering of X considered before. The deformation $\mathcal{X}_A$ is trivial when restricted to the affine open sets of $\mathcal{U}$, let

$$\theta_{\alpha\beta} : U_{\alpha\beta} \times \operatorname{Spec} A \xrightarrow{\theta_\beta} \mathcal{X}_A|_{U_{\alpha\beta}} \xrightarrow{\theta_\alpha^{-1}} U_{\alpha\beta} \times \operatorname{Spec} A$$

be the automorphisms given by the composition of its trivializations. Let the sheaf $\mathcal{E}_A$ be defined in the same open covering $\mathcal{U}$ by transition functions $F_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{G}\mathrm{l}_{\mathcal{X}_A}) \cong \Gamma(U_{\alpha\beta}, \mathcal{G}\mathrm{l}_{X \times \operatorname{Spec} A})$, such that

$$F_{\alpha\beta}\theta_{\alpha\beta}(F_{\beta\gamma}) = F_{\alpha\gamma}.$$

To construct a deformation of the pair $(X, \mathcal{E})$ over B which is a lifting of the deformation $(\mathcal{X}_A, \mathcal{E}_A)$, consider the system

$$\{(\chi_{\alpha\beta}, G_{\alpha\beta})\}. \tag{2.6}$$

The elements $\chi_{\alpha\beta}$ are automorphisms of $U_{\alpha\beta} \times \operatorname{Spec} B$, which restrict to $\theta_{\alpha\beta}$ on $U_{\alpha\beta} \times \operatorname{Spec} A$, thus they satisfy

$$\chi_{\alpha\beta}\chi_{\beta\gamma}\chi_{\alpha\gamma}^{-1} = \mathrm{Id} + t d_{\alpha\beta\gamma},$$

where $\{d_{\alpha\beta\gamma}\} \in \check{Z}^2(X, \mathcal{T}_X)$ is the Čech 2-cocicle which represents the obstruction to lift $\mathcal{X}_A$ to a deformation of X over B, as in Theorem 2.1.12.
The $G_{\alpha\beta}$ are elements in $\Gamma(U_{\alpha\beta}, \mathcal{G}\mathrm{l}_{X \times \operatorname{Spec} B})$, which restrict to $F_{\alpha\beta}$ on $U_{\alpha\beta} \times \operatorname{Spec} A$, thus they satisfy

$$G_{\alpha\beta}\chi_{\alpha\beta}(G_{\beta\gamma})G_{\alpha\gamma}^{-1} = \mathrm{Id} + t g_{\alpha\beta\gamma},$$

where $g_{\alpha\beta\gamma} \in \Gamma(U_{\alpha\beta\gamma}, \mathcal{M}_X)$.
Therefore we have

$$[\chi_{\alpha\beta}(G_{\beta\gamma}\chi_{\beta\gamma}(G_{\gamma\delta})G_{\beta\delta}^{-1})][G_{\alpha\gamma}\chi_{\alpha\gamma}(G_{\gamma\delta})G_{\alpha\delta}^{-1}]^{-1}[G_{\alpha\beta}\chi_{\alpha\beta}(G_{\beta\delta})G_{\alpha\delta}^{-1}][G_{\alpha\beta}\chi_{\alpha\beta}(G_{\beta\gamma})G_{\alpha\gamma}^{-1}]^{-1} =$$

$$= \mathrm{Id} + t(g_{\beta\gamma\delta} - g_{\alpha\gamma\delta} + g_{\alpha\beta\delta} - g_{\alpha\beta\gamma}),$$

the left side can be written also as

$$\chi_{\alpha\beta}\chi_{\beta\gamma}(G_{\gamma\delta})[\chi_{\alpha\gamma}(G_{\gamma\delta})]^{-1} = \mathrm{Id} + t d_{\alpha\beta\gamma} = \mathrm{Id} + t\frac{d_{\alpha\beta\gamma}(f_{\gamma\delta})}{f_{\gamma\delta}},$$

then

$$g_{\beta\gamma\delta} - g_{\alpha\gamma\delta} + g_{\alpha\beta\delta} - g_{\alpha\beta\gamma} = \frac{d_{\alpha\beta\gamma}(f_{\gamma\delta})}{f_{\gamma\delta}}.$$

This shows that the system $\{(g_{\alpha\beta\gamma}, d_{\alpha\beta\gamma})\}$ is a Čech 2-cocicle of the sheaf $D^1(\mathcal{E})$ and it defines a class in $\check{\mathrm{H}}^2(X, D^1(\mathcal{E}))$.
It can be proved that this cohomology class does not dependent on the choice of the system (2.6). Moreover it is clear that the deformation $(\mathcal{X}_A, \mathcal{E}_A)$ can be lifted to a deformation over B if and only if the system $\{(g_{\alpha\beta\gamma}, d_{\alpha\beta\gamma})\}$ is zero, for a choice of a pair $\{(\chi_{\alpha\beta}, G_{\alpha\beta})\}$. Thus $H^2(X, D^1(\mathcal{E}))$ is an obstruction space for $\mathrm{Def}_{(X,\mathcal{E})}$. □

Theorem 2.3.7. *Let X be a non singular projective algebraic variety and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. If $\dim_{\mathbb{K}} H^1(X, D^1(\mathcal{E})) < \infty$, the functor $\mathrm{Def}_{(X,\mathcal{E})}$ has a hull. If moreover $H^0(X, D^1(\mathcal{E})) = \mathbb{K}$, the functor $\mathrm{Def}_{(X,\mathcal{E})}$ is prorepresentable.*

Proof. By Theorem 2.3.5, the functor $\mathrm{Def}_{(X,\mathcal{E})}$ verifies conditions (H_1) and (H_2). By Theorem 2.3.6, the first hypothesis corrsponds to condition (H_3) and, by Schlessinger's Theorem, $\mathrm{Def}_{(X,\mathcal{E})}$ has a hull. We prove the last assertion using differential graded Lie algebras tecniques (see Example 3.2.16).

□

The problem of finding conditions under which the functor $\mathrm{Def}_{(X,\mathcal{E})}$ is prorepresentable, has a hull or satisfies some weaker properties linked to these, was tackled also without the use of Schlessinger's conditions, by the direct construction of deformations of $(X, \mathcal{E})$. We recall that Siu and Trautmann (see [36]), using results about the existence of versal deformations for a complex space X and for a coherent analytic sheaf $\mathcal{E}$, find out explicitly a versal deformation for the pair $(X, \mathcal{E})$ in the case X is a compact analytic space and $\mathcal{E}$ is a coherent analytic sheaf.

Chapter 3

Deformation theory via differential graded Lie algebras

In this Chapter we introduce the basic tools for the study of deformations via differential graded Lie algebras. The philosophy underlying this approach is that, in characteristic 0, every deformation problem individuates a differential graded Lie algebra that controls the problem via a deformation functor canonically associated to the DGLA:

$$\text{Deformation problem} \rightsquigarrow \text{DGLA} \rightsquigarrow \text{Deformation functor.}$$

For a good choice of the DGLA, this last deformation functor is isomorphic to the one associated to the problem in the classical way. This approach allows to preserve a lot of information on the deformation problem which is lost with the classical method, moreover some well-known results of the classical theory are easy consequences of the definitions and of the formal constructions.

To understand this method, in Sections 3.1 and 3.2 we give definitions and recall some basic facts about DGLAs and deformation functors associated to them.
The techniques of the study of deformations via DGLAs encrease with the introduction of the deformation functor associated to a morphism of DGLAs, that we analyse in Section 3.3. The motivation of the introduction of this tool is the study of particular deformations, called semitrivialized deformations, whose most important example is the case of embedded deformations of a submanifold.

3.1 Differential graded Lie algebras

Definition 3.1.1. *A* differential graded Lie algebra*, DGLA, is the data* $(L, d, [\ ,\ ])$*, where* $L = \bigoplus_{i\in\mathbb{Z}} L_i$ *is a* $\mathbb{Z}$*-graded vector space over a field* $\mathbb{K}$*,* $d : L_i \to L_{i+1}$ *is a linear map, such that* $d \circ d = 0$*, and* $[\ ,\ ] : L_i \times L_j \to L_{i+j}$ *is a bilinear map, such that:*

1. $[\ ,\ ]$ *is graded skewsymmetric, i.e.* $[a, b] = -(-1)^{\deg(a)\deg(b)}[b, a]$*, for every* a, b *homogeneous;*

2. *Every triple* a, b, c *of homogeneous elements verifies the graded Jacoby identity, i.e.* $[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a)\deg(b)}[b, [a, c]]$*;*

3. $[\ ,\ ]$ *and* d *verify the graded Leibniz's rule, i.e.* $d[a, b] = [da, b] + (-1)^{\deg(a)}[a, db]$*, for every* a, b *homogeneous.*

Definition 3.1.2. *Let $(L, d_L, [\ ,\ ]_L)$ and $(M, d_M, [\ ,\ ]_M)$ be two DGLAs, a* morphism $\phi : L \to M$ of DGLAs *is a linear morphism which preserves graduation and commutes with the brackets and the differentials.*

We indicate with **DGLA** the category of differential graded Lie algebras, whose objects are DGLAs and whose arrows are morphisms of DGLAs.

Remark 3.1.3. Since a DGLA L is a differential graded vector space, it make sense to calculate its cohomology spaces $H^*(L)$. The Leibniz's rule implies that the bracket of a DGLA L induces a structure of graded Lie algebra on its cohomology $H^*(L)$.
Moreover a morphism of DGLAs $\phi : L \to M$ induces a morphism of graded Lie algebras between the cohomology spaces $\overline{\phi} : H^*(L) \to H^*(M)$. The morphism ϕ is called a *quasi-isomorphism* of DGLAs, if the induced $\overline{\phi}$ is an isomorphism, and two differential graded Lie algebras are called *quasi-isomorphic*, if there exists a quasi-isomorphism between them.

Example 3.1.4. Let $(V = \bigoplus_{i\in\mathbb{Z}} V^i, d)$ be a differential $\mathbb{Z}$-graded $\mathbb{K}$-vector space. Consider the $\mathbb{Z}$-graded $\mathbb{K}$-vector space

$$\mathrm{Hom}(V, V) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^i(V, V)$$

where $\mathrm{Hom}^i(V, V) = \{f : V \to V \ \text{ linear} \mid f(V^n) \subset f(V^{n+i}) \ \text{ for every } n\}$. The bracket

$$[f, g] = f \circ g - (-1)^{\deg f \deg g} g \circ f$$

and the differential

$$df = [d, f] = d \circ f - (-1)^{\deg f} f \circ d$$

make $\mathrm{Hom}(V, V)$ a differential graded Lie algebra.

Example 3.1.5. Given a differential graded Lie algebra L and a commutative $\mathbb{K}$-algebra $\mathfrak{m}$, there exists a natural structure of DGLA in the tensor product $L\otimes\mathfrak{m}$ with differential and bracket given by:

$$d(x \otimes r) = dx \otimes r \quad \text{and} \quad [x \otimes r, y \otimes s] = [x, y] \otimes rs.$$

If $\mathfrak{m}$ is nilpotent (for example, if $\mathfrak{m}$ is the maximal ideal of a local Artinian $\mathbb{K}$-algebra), then the DGLA $L \otimes \mathfrak{m}$ is nilpotent.

Example 3.1.6. Let X be a compact connected complex manifold, let $\mathcal{T}_X$ be the holomorphic tangent bundle of X, let $\mathcal{A}_X^{p,q}$ be the sheaf of differentiable (p, q)-forms of X and let $\mathcal{A}_X^{p,q}(\mathcal{T}_X) = \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{T}_X$ be the sheaf of (p, q)-forms of X with values on the tangent bundle. The *Kodaira-Spencer* algebra is

$$\mathrm{KS}_X = \bigoplus_{i\in\mathbb{N}} \Gamma(X, \mathcal{A}_X^{0,i}(\mathcal{T}_X)) = \bigoplus_{i\in\mathbb{N}} A_X^{0,i}(\mathcal{T}_X)$$

the graded vector space of the global sections of the sheaf of the $(0, i)$-forms of X with values on the tangent bundle $\mathcal{T}_X$. A DGLA structure can be defined on KS_X. The differential on KS_X is the Dolbeault differential and the bracket is defined in local coordinates extending the standard bracket on $\mathcal{A}_X^{0,0}(\mathcal{T}_X)$ bilinearly with respect to the sheaf of the antiholomorphic differential forms. In local coordinates $z_1, \dots, z_n$, the explicit expressions for the differential and the bracket are:

$$\bar{\partial}(f d\bar{z}_I \frac{\partial}{\partial z_i}) = \bar{\partial} f \wedge d\bar{z}_I \frac{\partial}{\partial z_i}$$

$$[f d\bar{z}_I \frac{\partial}{\partial z_i}, g d\bar{z}_J \frac{\partial}{\partial z_j}] = (f \frac{\partial g}{\partial z_i} \frac{\partial}{\partial z_j} - g \frac{\partial f}{\partial z_j} \frac{\partial}{\partial z_i}) d\bar{z}_I \wedge d\bar{z}_J,$$

for every $f, g \in \mathcal{A}_X^{0,0}$.

Example 3.1.7. Let X be a compact connected complex manifold and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. Let $\mathcal{A}_X^{p,q}(\operatorname{End}\mathcal{E})$ be the sheaf of differentiable (p,q)-forms of X with values on the sheaf of the endomorphisms of $\mathcal{E}$ and let $A_X^{p,q}(\operatorname{End}\mathcal{E})$ be the space of its global sections. A DGLA structure on the graded vector space:

$$A_X^{0,*}(\operatorname{End}\mathcal{E}) = \bigoplus_{i \in \mathbb{N}} A_X^{0,i}(\operatorname{End}\mathcal{E})$$

is defined using as differential the Dolbeault differential on forms and as bracket the wedge product on forms and the composition of endomorphism. In local coordinates $z_1, \dots, z_n$, the explicit expressions for the differential and the bracket are:

$$\bar{\partial}(f d\bar{z}_I \otimes \phi) = \bar{\partial} f \wedge d\bar{z}_I \otimes \phi$$

$$[f d\bar{z}_I \otimes \phi, g d\bar{z}_J \otimes \psi] = f g d\bar{z}_I \wedge d\bar{z}_J \otimes (\phi \circ \psi - (-1)^{|I| \cdot |J|} \psi \circ \phi),$$

for all $f, g \in \mathcal{A}_X^{0,0}$ and $\phi, \psi \in \operatorname{End}\mathcal{E}$.

Example 3.1.8. Let X be a compact connected complex manifold and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. Let $D^1(\mathcal{E})$ be the space of the first order differential operator on $\mathcal{E}$, with scalar principal symbol. Let $\mathcal{A}_X^{p,q}(D^1(\mathcal{E}))$ be sheaf of differentiable (p,q)-forms of X with values on the sheaf of the differential operators on $\mathcal{E}$ with scalar symbol and let $A_X^{p,q}(D^1(\mathcal{E}))$ be the space of its global sections. A DGLA structure on the graded vector space:

$$A_X^{0,*}(D^1(\mathcal{E})) = \bigoplus_{i \in \mathbb{N}} A_X^{0,i}(D^1(\mathcal{E}))$$

is defined using as differential the Dolbeault differential on forms and as bracket the wedge product on forms and the composition of differential operators. In local coordinates $z_1, \dots, z_n$, the explicit expressions for the differential and the bracket are:

$$\bar{\partial}(f d\bar{z}_I \otimes P) = \bar{\partial} f \wedge d\bar{z}_I \otimes P$$

$$[f d\bar{z}_I \otimes P, g d\bar{z}_J \otimes Q] = f g d\bar{z}_I \wedge d\bar{z}_J \otimes (P \circ Q - (-1)^{|I| \cdot |J|} Q \circ P),$$

for all $f, g \in \mathcal{A}_X^{0,0}$ and $P, Q \in \operatorname{End}\mathcal{E}$.

3.2 Deformation functor associated to a DGLA

In this section we explain how to associate to differential graded Lie algebra some functors of Artin rings, following mainly [21].

Definition 3.2.1. *Let L be a differential graded Lie algebra, the* Maurer-Cartan functor *associated to L is the functor:*

$$\mathrm{MC}_L : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

defined, for all $(A, \mathfrak{m}_A) \in \mathbf{Art}_{\mathbb{K}}$, by:

$$\mathrm{MC}_L(A) = \left\{ x \in L^1 \otimes \mathfrak{m}_A \mid dx + \frac{1}{2}[x,x] = 0 \right\}.$$

The equation $dx + \frac{1}{2}[x,x] = 0$ is called the Maurer-Cartan equation.

Definition 3.2.2. *Two elements $x, y \in L^1 \otimes \mathfrak{m}_A$ are said to be* gauge equivalent *if there exists $a \in L^0 \otimes \mathfrak{m}_A$ such that*

$$y = e^a * x = x + \sum_{n=0}^{+\infty} \frac{([a,-])^n}{(n+1)!}([a,x] - da).$$

Remark 3.2.3. The above gauge relation is defined by the action $*$ of the exponential group $\exp(L^0 \otimes \mathfrak{m}_A)$ on the set $L^1 \otimes \mathfrak{m}_A$. It is an action because:

$$e^a * (e^b * x) = e^{a \bullet b} * x, \quad \text{for all } a, b \in L^0 \otimes \mathfrak{m}_A \text{ and } \ x \in L^1 \otimes \mathfrak{m}_A,$$

where $\bullet$ is the Baker-Camper-Hausdorff product (see [10]).

Remark 3.2.4. The solutions of the Maurer-Cartan equation are preserved under the gauge action.
It is useful to consider the following construction. Given a DGLA $(L, [\ ,\ ], d)$ we can construct a new DGLA $(\hat{L}, [\ ,\ ]', \hat{d})$, by setting:

$$\hat{L}^i = L^i \ \text{ for all } \ i \neq 1 \ \text{ and } \quad \hat{L}^1 = L^1 \oplus \mathbb{C}d$$

with differential and bracket given by

$$\hat{d}(x + vd) = dx \ \text{ and } \quad [x + vd, y + wd]' = [x,y] + vdy - (-1)^{\deg x} wdx$$

The natural inclusion $L \subset \hat{L}$ is a morphism of DGLA; while the map $\phi : L \hookrightarrow \hat{L}$ given by $\phi(x) = x + d$ is only a linear morphism and, for every $x \in L$, we have:

$$dx + \frac{1}{2}[x,x] = \frac{1}{2}[\phi(x), \phi(x)]'.$$

We observe that:

$$e^{[a,\ ]'}\phi(x) = e^{[a,\ ]'}(x + d) = \sum_{n=0}^{\infty} \frac{([a,-]')^n}{n!}(x+d) = x + d + \sum_{n=0}^{\infty} \frac{([a,-]')^n}{(n+1)!}([a, x+d]) =$$

$$= x + \sum_{n=0}^{+\infty} \frac{[a,-]^n}{(n+1)!}([a,x] - da) + d = \phi(e^a * x).$$

Let's calculate:

$$d(e^a * x) = d(e^{[a,\ ]'}\phi(x) - d) = [d, e^{[a,\ ]'}\phi(x) - d]' = [d, e^{[a,\ ]'}\phi(x)]'$$

and

$$\begin{aligned}&[e^a * x, e^a * x] = [e^{[a,\ ]'}\phi(x) - d, e^{[a,\ ]'}\phi(x) - d]' = \\ &= [e^{[a,\ ]'}\phi(x), e^{[a,\ ]'}\phi(x)]' - 2[d, e^{[a,\ ]'}\phi(x)]' = e^{[a,\ ]'}[\phi(x), \phi(x)]' - 2[d, e^{[a,\ ]'}\phi(x)]'.\end{aligned}$$

Now let $x \in L^1 \otimes \mathfrak{m}_A$ be a solution of the Maurer-Cartan equation and let $a \in L^0 \otimes \mathfrak{m}_A$, by the above calculations, we have:

$$d(e^a * x) + \frac{1}{2}[e^a * x, e^a * x] = [d, e^{[a,\ ]'}\phi(x)]' + \frac{1}{2}\left(e^{[a,\ ]'}[\phi(x), \phi(x)]' - 2[d, e^{[a,\ ]'}\phi(x)]'\right) = 0,$$

as we said.

Definition 3.2.5. *Let L be a differential graded Lie algebra, the* deformation functor *associated to L is the functor:*

$$\mathrm{Def}_L : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

defined, for all $(A, \mathfrak{m}_A) \in \mathbf{Art}_{\mathbb{K}}$, by:

$$\mathrm{Def}_L(A) = \frac{\mathrm{MC}_L(A)}{\sim_{gauge}}.$$

We are now ready to compute tangent space and obstructions for the above functors. The tangent space for MC_L is the space

$$t_{\mathrm{MC}_L} = \mathrm{MC}_L(\mathbb{K}[\epsilon]) = \{x \in L^1 \otimes \mathbb{K}\epsilon \mid dx + \frac{1}{2}[x, x] = dx = 0\} = Z^1(L) \otimes \mathbb{K}\epsilon.$$

The functor MC_L has a natural obstruction theory $(H^2(L), v_e)$. Let

$$e : \quad 0 \to J \to B \to A \to 0$$

be a small extension in $\mathbf{Art}_{\mathbb{K}}$ and let $x \in L^1 \otimes \mathfrak{m}_A$ be a solution of the Maurer-Cartan equation; we define an obstruction $v_e(x) \in H^2(L \otimes J) = H^2(L) \otimes J$ in the following way. Take a lifting $\tilde{x} \in L^1 \otimes \mathfrak{m}_B$ of x and consider

$$h = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] \in L^2 \otimes B.$$

First observe that $h \in L^2 \otimes J$, because x satisfies the Maurer-Cartan equation, moreover

$$dh = d^2\tilde{x} + [d\tilde{x}, \tilde{x}] = [h, \tilde{x}] - \frac{1}{2}[[\tilde{x}, \tilde{x}], \tilde{x}] = 0$$

because $J \cdot \mathfrak{m}_B = 0$ and because of the Jacoby identity. Define $v_e(x) = [h] \in H^2(L) \otimes J$. The first thing to prove is that $v_e(x)$ is independent from the choice of the lifting $\tilde{x}$. Let $y \in L^1 \otimes \mathfrak{m}_B$ be an other lifting of x, it is of the form $y = \tilde{x} + z$, with $z \in L^1 \otimes J$, then

$$k = dy + \frac{1}{2}[y, y] = d\tilde{x} + dz + \frac{1}{2}[\tilde{x}, \tilde{x}] + [\tilde{x}, z] = h + dz,$$

because $J \cdot \mathfrak{m}_B = 0$, and thus h and k have the same cohomology class. Moreover it is obvious that the class $v_e(x)$ is zero if and only if there exists a lifiting of x in $L^1 \otimes \mathfrak{m}_B$ which satisfies the Maurer-Cartan equation, then $(H^2(L), v_e)$ is a complete obstruction theory for MC_L.

The tangent space for Def_L is the space

$$t_{\mathrm{Def}_L} = \mathrm{Def}_L(\mathbb{K}[\epsilon]) = \frac{\mathrm{MC}_L(\mathbb{K}[\epsilon])}{\sim_{gauge}} = \frac{Z^1(L) \otimes \mathbb{K}\epsilon}{B^1(L) \otimes \mathbb{K}\epsilon} = H^1(L) \otimes \mathbb{K}\epsilon,$$

because the gauge action of $L^0 \otimes \mathbb{K}\epsilon$ on $L^1 \otimes \mathbb{K}[\epsilon]$ is given by:

$$e^a * x = x + \sum_{n=0}^{+\infty} \frac{([a,-])^n}{(n+1)!}([a,x] - da) = x - da.$$

Lemma 3.2.6. *The projection* $\pi : \mathrm{MC}_L \to \mathrm{Def}_L$ *is a smooth morphism of functors.*

Proof. Let $\alpha : B \to A$ be a surjection in $\mathbf{Art}_{\mathbb{K}}$ and prove that the induced map $\mathrm{MC}_L(B) \to \mathrm{MC}_L(A) \times_{\mathrm{Def}_L(A)} \mathrm{Def}_L(B)$ is surjective. Let $(a', b) \in \mathrm{MC}_L(A) \times_{\mathrm{Def}_L(A)} \mathrm{Def}_L(B)$ and $\tilde{b} \in \mathrm{MC}_L(B)$ a lifting of b, then $\alpha(\tilde{b})$ and a' have the same image in $\mathrm{Def}_L(A)$, i.e. there exists $t \in L^0 \otimes \mathfrak{m}_A$ such that $a' = e^t * \alpha(\tilde{b})$. Let $s \in L^0 \otimes \mathfrak{m}_B$ be a lifting of t and define $b' = e^s * \tilde{b} \in \mathrm{MC}_L(B)$, then $\alpha(b') = e^t * \alpha(\tilde{b}) = a'$ and $\pi(b') = \pi(\tilde{b}) = b$. □

The functor Def_L has a natural obstruction theory $(H^2(L), w_e)$, defined as follow. Let

$$e: \quad 0 \to J \to B \to A \to 0$$

be a small extension in $\mathbf{Art}_{\mathbb{K}}$, let $x \in \mathrm{Def}_L(A)$ and let $x' \in \mathrm{MC}_L(A)$ be a lifting of x, we define an obstruction $w_e(x) = v_e(x') \in H^2(L) \otimes J$. Since the morphism π is smooth, $(H^2(L), w_e)$ is a complete obstruction theory for Def_L.

Lemma 3.2.7. *The functor* MC_L *is homogeneous.*

Proof. For every $\beta : B \to A$ and $\gamma : C \to A$ morphisms in $\mathbf{Art}_{\mathbb{K}}$, we have $\mathrm{MC}_L(B \times_A C) \cong \mathrm{MC}_L(B) \times_{\mathrm{MC}_L(A)} \mathrm{MC}_L(C)$. □

Lemma 3.2.8. *The functor* Def_L *is a functor with a good deformation theory. Thus, if* $H^1(L)$ *has finite dimension,* Def_L *has a hull, but in general it is not prorepresentable.*

Proof. We have to verify that Schlessinger's conditions $(H1)$ and $(H2)$ of are satisfied.

Let $\beta : B \to A$ and $\gamma : C \to A$ be morphisms in $\mathbf{Art}_{\mathbb{K}}$, with γ surjective, we have to prove that the induced map $\eta : \mathrm{Def}_L(B \times_A C) \to \mathrm{Def}_L(B) \times_{\mathrm{Def}_L(A)} \mathrm{Def}_L(C)$ is surjective. Let $(b, c) \in \mathrm{Def}_L(B) \times_{\mathrm{Def}_L(A)} \mathrm{Def}_L(C)$, let $b' \in \mathrm{MC}_L(B)$ and $c' \in \mathrm{MC}_L(C)$ liftings of b and c respectively. Since $\beta(b'), \gamma(c') \in \mathrm{MC}_L(A)$ have the same image in $\mathrm{Def}_L(A)$, modifying b' and c' via gauge action, we can suppose that $\beta(b') = \gamma(c') \in \mathrm{MC}_L(A)$. Since MC_L is a homogeneous functor, for $(b', c') \in \mathrm{MC}_L(B) \times_{\mathrm{MC}_L(A)} \mathrm{MC}_L(C)$, there exists $x \in \mathrm{MC}_L(B \times_A C)$ lifting of (b', c'). Then $\eta(\pi(x)) = (b, c)$.

Let now $A = \mathbb{K}$, then $\mathrm{Def}_L(B \times_A C) = \mathrm{Def}_L(B \times C) = \mathrm{Def}_L(B) \times_{\mathrm{Def}_L(A)} \mathrm{Def}_L(C) = \mathrm{Def}_L(B) \times \mathrm{Def}_L(C)$.

The last assertion of the Theorem follows from Theorem 1.2.7. □

Now we prove the following property that we use in the following

Lemma 3.2.9. *Let L be a DGLA such that $H^0(L) \cong \mathbb{K}$, then* Def_L *is homogeneous.*

Proof. Consider the vector space decomposition $L^1 = dL^0 \oplus N^1$ and define a subDGLA of L as follows

$$N^i = 0 \quad \text{for } i < 0, \qquad N^0 = \mathbb{K}, \qquad N^1 \quad \text{as above}, \qquad N^i = L^i \quad \text{for } i \geq 2,$$

with the same differential and bracket as in L. The inclusion $N \hookrightarrow L$ gives isomorphisms $H^i(N) \to H^i(L)$, for $i = 0, 1, 2$, thus $\operatorname{Def}_N \cong \operatorname{Def}_L$. Moreover, since the gauge action via elements in $\mathbb{K}$ is trivial, $\operatorname{Def}_N = \operatorname{MC}_N$ is homogeneous. □

The utility of define the deformation functor associated to a DGLA relies on the following result, sometimes called *basic theorem of deformation theory* (see [21], Theorem 3.1).

Proposition 3.2.10. *Let $f : L \to M$ be a morphism of DGLAs and let $\bar{f} : \operatorname{Def}_L \to \operatorname{Def}_M$ be the induced morphism of functors. If*

- *$f : H^0(L) \to H^0(M)$ is surjective,*
- *$f : H^1(L) \to H^1(M)$ is bijective,*
- *$f : H^2(L) \to H^2(M)$ is injective,*

then $\bar{f}$ is an isomorphism between the deformation functors Def_L *and* Def_M*.*

Corollary 3.2.11. *Let L and M be quasi-isomorphic DGLAs, then the deformation functors* Def_L *and* Def_M *are isomorphic.*

As we have already explained, the motivation for the introduction of differential graded Lie algebras in deformation theory, and for the consequent definition of the deformation functor associated to a DGLA, is the principle for which in characteristic 0 every deformation problem is governed by a differential graded Lie algebra. To be more precise:

Definition 3.2.12. *If the functor of deformations of a geometric object $\mathcal{X}$ is isomorphic to the deformation functor associated to a DGLA L, then we say that L* governs the deformations of *$\mathcal{X}$.*

There are some well-known examples of deformations of geometric objects in which this principle is applied. We start with a trivial example.

Example 3.2.13. Let $(V = \bigoplus_i V^i, d)$ be a differential graded vector space, or, that is the same, a complex of vector spaces. To define a deformation of it, let start with the simpler case of a vector space.
An *infinitesimal deformation* of a $\mathbb{K}$-vector space V over $A \in \mathbf{Art}_{\mathbb{K}}$ is the data of a flat A-module V_A, such that the projection onto the residue field induces an isomorphism $V_A \otimes_A \mathbb{K} \cong V$. It is easy to see that every infinitesimal deformation of a vector space V over A is trivial, i.e. it is isomorphic to $V \otimes A$.
Thus to deform a complex of vector spaces it is sufficient to deform the differential. An *infinitesimal deformation* of a complex of vector spaces $(V = \bigoplus_i V^i, d)$ over $A \in \mathbf{Art}_{\mathbb{K}}$ is a complex of A-modules of the form $(V \otimes A = \bigoplus_i V^i \otimes A, d_A)$, such that the projection onto the residue field induces an isomorphism between $(V \otimes A, d_A)$ and (V, d).

Let $\mathrm{Hom}(V,V)$ be the DGLA of the homomorphism of the complex (V,d), defined in Example 3.1.4.
It is easy to proved (see [24], pages 3-4) that the deformation functor $\mathrm{Def}_{\mathrm{Hom}(V,V)}$ is isomorphic to the functor of deformations of the complex (V,d), $\mathrm{Def}_{(V,d)}$. The isomorphism is given, for all $A \in \mathbf{Art}_{\mathbb{K}}$, by

$$\begin{array}{rcl} \mathrm{Def}_{\mathrm{Hom}(V,V)}(A) & \longrightarrow & \mathrm{Def}_{(V,d)}(A) \\ f & \longrightarrow & (V \otimes A, d+f). \end{array}$$

Now we consider some geometric examples.

Example 3.2.14. Let X be a compact and connected complex manifold and let KS_X be the Kodaira-Spencer DGLA, defined in Example 3.1.6. It can be proved (see [24]) that the deformation functor $\mathrm{Def}_{\mathrm{KS}_X}$ is isomorphic to the functor of deformations of the manifold X, Def_X. The isomorphism is given, for all $A \in \mathbf{Art}_{\mathbb{C}}$, by

$$\begin{array}{rcl} \mathrm{Def}_{\mathrm{KS}_X}(A) & \longrightarrow & \mathrm{Def}_X(A) \\ x & \longrightarrow & \ker(\bar{\partial} + \mathfrak{l}_x), \end{array}$$

The symbol $\mathfrak{l}_x$ indicates the holomorphic Lie derivative

$$\mathfrak{l} : \mathcal{A}_X^{0,*}(\mathcal{T}_X) \to \mathcal{D}er^*(\mathcal{A}_X^{0,*}, \mathcal{A}_X^{0,*})$$

which associates, to every $x \in \mathcal{A}_X^{0,*}(\mathcal{T}_X)$, the derivation $\mathfrak{l}_x$, given by

$$\mathfrak{l}_x(\omega) = \partial(x \lrcorner \omega) + (-1)^{\deg x}(x \lrcorner \partial\omega),$$

for all $\omega \in \mathcal{A}_X^{0,*}$, where $\lrcorner$ is the contraction of the differential forms with vector fields.
Knowing that the DGLA KS_X governs the deformations of X, from calculation of Section 3.2, we recover the isomorphism $t_{\mathrm{Def}_X} \cong H^1(X, \mathcal{T}_X)$ and the fact that $H^2(X, \mathcal{T}_X)$ is an obstruction space for Def_X, stated in Theorems 2.1.11 and 2.1.12.

Example 3.2.15. Let X be a compact and connected complex manifold and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. Let $A_X^{0,*}(\mathrm{End}\,\mathcal{E})$ be the DGLA of the differentiable $(0,*)$-forms of X with values in the sheaf of the endomorphisms of the sheaf $\mathcal{E}$, defined in Example 3.1.7. It can be proved (see [8], Theorem 1.1.1) that the deformation functor $\mathrm{Def}_{A_X^{(0,*)}(\mathrm{End}\,\mathcal{E})}$ is isomorphic to the functor of deformations of the sheaf $\mathcal{E}$, $\mathrm{Def}_{\mathcal{E}}$. The isomorphism is given, for all $A \in \mathbf{Art}_{\mathbb{C}}$, by

$$\begin{array}{rcl} \mathrm{Def}_{A_X^{(0,*)}(\mathrm{End}\,\mathcal{E})}(A) & \longrightarrow & \mathrm{Def}_{\mathcal{E}}(A) \\ x & \longrightarrow & \ker(\bar{\partial} + x). \end{array}$$

Knowing that the DGLA $\mathrm{End}\,\mathcal{E}$ governs the deformations of $\mathcal{E}$, from calculation of Section 3.2, we recover the isomorphism $t_{\mathrm{Def}_{\mathcal{E}}} \cong H^1(X, \mathrm{End}\,\mathcal{E})$ and the fact that $H^2(X, \mathrm{End}\,\mathcal{E})$ is an obstruction space for $\mathrm{Def}_{\mathcal{E}}$, proved in Theorem 2.2.5.

Moreover, if $H^0(X, \mathrm{End}\,\mathcal{E}) = \mathbb{C}$, the functor $\mathrm{Def}_{\mathcal{E}}$ is homogeneous, for Lemma 3.2.9, and this conclude the proof of Theorem 2.2.6.

Example 3.2.16. Let X be a compact and connected complex manifold and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. Let $A_X^{0,*}(D^1(\mathcal{E}))$ be the DGLA of the differentiable $(0,*)$-forms of X with values in the sheaf of the first order differential operator on $\mathcal{E}$ with scalar symbol, defined in Example 3.1.8. It can be proved that the deformation functor $\mathrm{Def}_{A_X^{0,*}(D^1(\mathcal{E}))}$ is isomorphic to the functor of deformations of the pair $(X,\mathcal{E})$, $\mathrm{Def}_{(X,\mathcal{E})}$. The isomorphism is given, for all $A \in \mathbf{Art}_{\mathbb{C}}$, by

$$\begin{array}{rcl} \mathrm{Def}_{A_X^{(0,*)}(D^1(\mathcal{E}))}(A) & \longrightarrow & \mathrm{Def}_{(X,\mathcal{E})}(A) \\ x & \longrightarrow & (\ker(\bar{\partial} + \mathfrak{l}_{\sigma(x)}), \ker(\bar{\partial} + x)). \end{array}$$

Knowing that the DGLA $D^1(\mathcal{E})$ governs the deformations of $(X,\mathcal{E})$, from calculation of Section 3.2, we recover the isomorphism $t_{\mathrm{Def}_{(X,\mathcal{E})}} \cong H^1(X, D^1(\mathcal{E}))$ and the fact that $H^2(X, D^1(\mathcal{E}))$ is an obstruction space for $\mathrm{Def}_{\mathcal{E}}$, proved in Theorem 2.3.6.

Moreover, if $H^0(X, D^1(\mathcal{E})) = \mathbb{C}$, the functor $\mathrm{Def}_{(X,\mathcal{E})}$ is homogeneous, for Lemma 3.2.9, and this conclude the proof of Theorem 2.3.7

3.3 Deformation functor associated to a morphism of DGLAs

In this section we explain how to associate to a morphism of DGLAs some functors of Artin rings, following mainly [25].

Definition 3.3.1. *Let L and M be two DGLAs and let $\chi : L \to M$ be a morphism of DGLAs. The* Maurer-Cartan functor *associated to χ is the functor:*

$$\mathrm{MC}_\chi : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

defined, for all $(A, \mathfrak{m}_A) \in \mathbf{Art}_{\mathbb{K}}$, by:

$$\mathrm{MC}_\chi(A) = \left\{ (x, e^a) \in (L^1 \otimes \mathfrak{m}_A) \times \exp(M^0 \otimes \mathfrak{m}_A) \mid dx + \frac{1}{2}[x,x] = 0, e^a * \chi(x) = 0 \right\}.$$

Definition 3.3.2. *Two elements (x, e^a) and (y, e^b) in $(L^1 \otimes \mathfrak{m}_A) \times \exp(M^0 \otimes \mathfrak{m}_A)$ are said to be* gauge equivalent *if there exists $(e^l, e^{dm}) \in \exp(L^0 \otimes \mathfrak{m}_A) \times \exp(dM^{-1} \otimes \mathfrak{m}_A)$, such that:*

$$(y, e^b) = (e^l, e^{dm}) * (x, e^a) = (e^l * x, e^{dm} e^a e^{-\chi(l)}).$$

Remark 3.3.3. The above gauge relation is defined by the action $*$ of the exponential group $G = \exp(L^0 \otimes \mathfrak{m}_A) \times \exp(dM^{-1} \otimes \mathfrak{m}_A)$ on the set $S = (L^1 \otimes \mathfrak{m}_A) \times \exp(M^0 \otimes \mathfrak{m}_A)$. It is an action because, for all (e^l, e^{dm}), $(e^\lambda, e^{d\mu}) \in G$ and $(x, e^a) \in S$:

$$(e^l, e^{dm}) * ((e^\lambda, e^{d\mu}) * (x, e^a)) = (e^l, e^{dm}) * (e^\lambda * x, e^{d\mu} e^a e^{-\chi(\lambda)}) = (e^{l \bullet \lambda} * x, e^{d(m \bullet \mu)} e^a e^{-\chi(l \bullet \lambda)}),$$

where $\bullet$ is the Baker-Camper-Hausdorff product (see [10]).

Remark 3.3.4. The gauge action is well defined on $\mathrm{MC}_\chi(A)$.
Let $(x, e^a) \in \mathrm{MC}_\chi(A)$, let $(e^l, e^{dm}) \in \exp(L^0 \otimes \mathfrak{m}_A) \times \exp(dM^{-1} \otimes \mathfrak{m}_A)$ and let

$$(e^l, e^{dm}) * (x, e^a) = (e^l * x, e^{dm} e^a e^{-\chi(l)}).$$

As proved in Remark 3.2.4, $e^l * x$ satisfied the Maurer-Cartan equation, moreover

$$e^{dm} e^a e^{-\chi(l)} * \chi(e^l * x) = e^{dm} e^a e^{-\chi(l)} * e^{\chi(l)} * \chi(x) = e^{dm} e^a \chi(x) = e^{dm} * 0 = 0,$$

as we want.

Definition 3.3.5. *Let L and M be two DGLAs and let $\chi : L \to M$ be a morphism of DGLAs. The* deformation functor *associated to χ is the functor:*

$$\mathrm{Def}_\chi : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

defined, for all $(A, \mathfrak{m}_A) \in \mathbf{Art}_{\mathbb{K}}$, by:

$$\mathrm{Def}_\chi(A) = \frac{\mathrm{MC}_\chi(A)}{\sim_{gauge}}.$$

Let $\chi : L \to M$ be a morphism of DGLAs, the *suspension of the mapping cone* of χ is defined to be the differential graded vector space (C_χ, δ), where $C_\chi^i = L^i \oplus M^{i-1}$ and the differential is given by:

$$\delta(l, m) = (d_L l, \chi(l) - d_M m).$$

The projection $C_\chi \to L$ is a morphism of differential graded vector spaces. In general does not exist any bracket on this cone, making it a DGLA and the projection a morphism of DGLAs.
Moreover, associated to the morphism of DGLAs χ, there exists a long exact sequence:

$$\ldots \to H^i(C_\chi) \to H^i(L) \to H^i(M) \to H^{i+1}(C_\chi) \to \ldots.$$

We also observe that every commutative diagram of differential graded Lie algebras

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "\chi"] & H \arrow[d, "\eta"] \\ M \arrow[r, "f'"] & I \end{tikzcd} \tag{3.1}$$

induces a morphism between the cones $C_\chi \to C_\eta$ and a morphism of functors $\mathrm{Def}_\chi \to \mathrm{Def}_\eta$, for which the following *Inverse Function Theorem* (see [25], Theorem 2.1) holds:

Theorem 3.3.6. *If the diagram (3.1) induces a quasi isomorphism between the cones $C_\eta \to C_\chi$, then the induced morphism of functor $\mathrm{Def}_\chi \to \mathrm{Def}_\eta$ is an isomorphim.*

We are now ready to compute tangent space and the obstructions for the functors MC_χ and Def_χ.

The tangent space for MC_χ is the space $t_{\mathrm{MC}_\chi} = \mathrm{MC}_\chi(\mathbb{K}[\epsilon])$ given by

$$\{(x, e^a) \in (L^1 \otimes \mathbb{K}\epsilon) \times \exp(M^0 \otimes \mathbb{K}\epsilon) \mid dx = 0, e^a * \chi(x) = \chi(x) - da = 0\}$$

$$\cong \{(x, a) \in L^1 \times M^0 \mid dx = 0, \chi(x) - da = 0\} = \ker(\delta : C_\chi^1 \to C_\chi^2).$$

The functor MC_χ has a natural obstruction theory $(H^2(C_\chi), v_e)$. Let

$$e : \quad 0 \to J \to B \to A \to 0$$

be a small extension in $\mathbf{Art}_{\mathbb{K}}$ and let $(x, e^a) \in \mathrm{MC}_\chi(A)$; we define an ostruction $v_e(x, e^a) \in H^2(C_\chi) \otimes J$ in the following way. Take a lifting $(y, e^b) \in (L^1 \otimes \mathfrak{m}_B) \times \exp(M^0 \otimes \mathfrak{m}_B)$ and consider

$$h = dy + \frac{1}{2}[y, y] \in L^2 \otimes \mathfrak{m}_B, \quad r = e^b * \chi(y) \in M^1 \otimes \mathfrak{m}_B.$$

First observe that $h \in L^2 \otimes J$ and $r \in M^1 \otimes J$, because $(x, e^a) \in \mathrm{MC}_\chi(A)$. Moreover $\delta(h, r) = (dh, \chi(h) - dr) = 0$, in fact

$$dh = d^2y + [dy, y] = [h, y] - \frac{1}{2}[[y, y], y] = 0,$$

by $J \cdot \mathfrak{m}_B = 0$ and by Jacoby identity, and

$$\begin{aligned} \chi(h) &= \chi(y + \frac{1}{2}[y, y]) = \chi(y) + \frac{1}{2}[\chi(y), \chi(y)] = \\ &= e^{-b} * r + \frac{1}{2}[e^{-b} * r, e^{-b} * r] = d(r + e^{-b} * 0) + \frac{1}{2}[r + e^{-b} * 0, r + e^{-b} * 0] = \\ &= dr + d(e^{-b} * 0) + \frac{1}{2}[e^{-b} * 0, e^{-b} * 0] = dr, \end{aligned}$$

since $e^{-b} * 0$ satifies the Maurer-Cartan equation. Define $v_e(x, e^a) = [(h, r)] \in H^2(C_\chi) \otimes J$. The first thing to prove is that $v_e(x, e^a)$ is independent from the choice of the lifting of (x, e^a). Let $(z, e^c) \in (L^1 \otimes \mathfrak{m}_B) \times \exp(M^0 \otimes \mathfrak{m}_B)$ be an other lifting of (x, e^a), it is of the form $(z = y + w, e^c = e^{b+\gamma})$, with $w \in L^1 \otimes J$ and $\gamma \in M^0 \otimes J$, then

$$k = dz + \frac{1}{2}[z, z] = h + dw$$

$$\begin{aligned} s &= e^c * \chi(z) = e^{b+\gamma} * \chi(y + w) = \\ &= \chi(y) + \chi(w) + \sum_{n=0}^{+\infty} \frac{[b + \gamma, -]^n}{(n+1)!}([b + \gamma, \chi(y + w)] - d(b + \gamma)) = \\ &= \chi(y) + \chi(w) + \sum_{n=0}^{+\infty} \frac{[b + \gamma, -]^n}{(n+1)!}([b, \chi(y)] - db - d\gamma) = \\ &= \chi(y) + \chi(w) + [b, \chi(y)] - db - d\gamma + \sum_{n=1}^{+\infty} \frac{[b, -]^n}{(n+1)!}([b, \chi(y)] - d(b)) = \\ &= e^b * \chi(y) + \chi(y) - d\gamma \end{aligned}$$

where it is used repeatedly the fact that $J \cdot \mathfrak{m}_B = 0$. Thus $(h, r) - (k, s) = (dw, \chi(w) - d\gamma)$ and (h, r) and (k, s) have the same class in $H^2(C_\chi) \otimes J$.
Moreover it is obvious that the class $v_e(x, e^a)$ is zero if and only if there exists a lfting of (x, e^a) in $\mathrm{MC}_\chi(B)$, then $(H^2(C_\chi), v_e)$ is a complete obstruction theory for MC_χ.

The tangent space for Def_χ is the space

$$t_{\mathrm{Def}_\chi} = \mathrm{Def}_\chi(\mathbb{K}[\epsilon]) = \frac{\mathrm{MC}_\chi(\mathbb{K}[\epsilon])}{\sim_{gauge}} = \frac{Z^1(C_\chi) \otimes \mathbb{K}\epsilon}{B^1(C_\chi) \otimes \mathbb{K}\epsilon} = H^1(C_\chi) \otimes \mathbb{K}\epsilon,$$

because the gauge action of $(e^l, e^{dm}) \in \exp(L^0 \otimes \mathbb{K}\epsilon) \times \exp(dM^{-1} \otimes \mathbb{K}\epsilon)$ on $(x, e^a) \in (L^1 \otimes \mathbb{K}[\epsilon]) \times \exp(M^0 \otimes \mathbb{K}[\epsilon])$ is given by:

$$(e^l, e^{dm}) * (x, e^a) = (e^l * x, e^{dm} e^a e^{-\chi(l)}) = (x - dl, e^a e^{-(\chi(l) - dm)}).$$

Lemma 3.3.7. *The projection $\pi : \mathrm{MC}_\chi \to \mathrm{Def}_\chi$ is a smooth morphism of functors.*

Proof. Let $\alpha : B \to A$ be a surjection in $\mathbf{Art}_{\mathbb{K}}$ and prove that the induced map $\mathrm{MC}_\chi(B) \to \mathrm{MC}_\chi(A) \times_{\mathrm{Def}_\chi(A)} \mathrm{Def}_\chi(B)$ is surjective. Let $((x', e^{a'}), (y, e^b)) \in \mathrm{MC}_\chi(A) \times_{\mathrm{Def}_\chi(A)} \mathrm{Def}_\chi(B)$ and $(\tilde{y}, e^{\tilde{b}}) \in \mathrm{MC}_\chi(B)$ a lifting of (y, e^b). Then $\alpha(\tilde{y}, e^{\tilde{b}})$ and $(x', e^{a'})$ have the same image in $\mathrm{Def}_\chi(A)$, i.e. there exists $(e^t, e^{dm}) \in \exp(L^0 \otimes \mathfrak{m}_A) \times \exp(dM^{-1} \otimes \mathfrak{m}_A)$ such that $(x', e^{a'}) = (e^t, e^{dm}) * \alpha(\tilde{y}, e^{\tilde{b}})$. Let $s \in L^0 \otimes \mathfrak{m}_B$ be a lifting of t and $n \in M^{-1} \otimes \mathfrak{m}_B$ a lifting of m, define $(y', e^{b'}) = (e^s, e^{dn}) * (\tilde{y}, e^{\tilde{b}}) \in \mathrm{MC}_\chi(B)$, then $\alpha(y', e^{b'}) = (e^t, e^{dm}) * \alpha(\tilde{y}, e^{\tilde{b}}) = (x', e^{a'})$ and $\pi(y', e^{b'}) = \pi(\tilde{y}, e^{\tilde{b}}) = (y, e^b)$. □

The functor Def_χ has a natural obstruction theory $(H^2(C_\chi), w_e)$, defined as follow. Let

$$e: \quad 0 \to J \to B \to A \to 0$$

be a small extension in $\mathbf{Art}_{\mathbb{K}}$, let $(x, e^a) \in \mathrm{Def}_\chi(A)$ and let $(x', e^{a'}) \in \mathrm{MC}_\chi(A)$ be a lifting of (x, e^a), we define an obstruction $w_e(x, e^a) = v_e(x', e^{a'}) \in H^2(C_\chi) \otimes J$. Since the morphism π is smooth, $(H^2(C_\chi), w_e)$ is a complete obstruction theory for Def_χ.

Lemma 3.3.8. *The functor* MC_χ *is homogeneous.*

Proof. For every $\beta : B \to A$ and $\gamma : C \to A$ morphisms in $\mathbf{Art}_{\mathbb{K}}$, we have $\mathrm{MC}_\chi(B \times_A C) \cong \mathrm{MC}_\chi(B) \times_{\mathrm{MC}_\chi(A)} \mathrm{MC}_\chi(C)$. □

Lemma 3.3.9. *The functor* Def_χ *is a functor with a good deformation theory. Thus, if* $H^1(C_\chi)$ *has finite dimension,* Def_χ *has a hull, but in general it is not prorepresentable.*

Proof. We have to verify that Schlessinger's conditions $(H1)$ and $(H2)$ of are satisfied.

Let $\beta : B \to A$ and $\gamma : C \to A$ be morphisms in $\mathbf{Art}_{\mathbb{K}}$, with γ surjective, we have to prove that the induced map $\eta : \mathrm{Def}_\chi(B \times_A C) \to \mathrm{Def}_\chi(B) \times_{\mathrm{Def}_\chi(A)} \mathrm{Def}_\chi(C)$ is surjective. Let $((y, e^b), (z, e^c)) \in \mathrm{Def}_\chi(B) \times_{\mathrm{Def}_\chi(A)} \mathrm{Def}_\chi(C)$, let $(y', e^{b'}) \in \mathrm{MC}_\chi(B)$ and $(z', e^{c'}) \in \mathrm{MC}_\chi(C)$ liftings of (y, e^b) and (z, e^c) respectively. Since $\beta(y', e^{b'}), \gamma(z' e^{c'}) \in \mathrm{MC}_\chi(A)$ have the same image in $\mathrm{Def}_\chi(A)$, modifying $(y', e^{b'})$ and $(z', e^{c'})$ via gauge action, we can suppose that $\beta(y', e^{b'}) = \gamma(z', e^{c'}) \in \mathrm{MC}_\chi(A)$. Since MC_χ is a homogeneous functor, for $((y', e^{b'}), (z', e^{c'})) \in \mathrm{MC}_\chi(B) \times_{\mathrm{MC}_\chi(A)} \mathrm{MC}_\chi(C)$, there exists $(x, e^a) \in \mathrm{MC}_\chi(B \times_A C)$, such that $\eta(x, e^a) = ((y', e^{b'}), (z', e^{c'}))$. Then $\eta(\pi(x, e^a)) = ((y, e^b), (z, e^c))$.

Let now $A = \mathbb{K}$, then $\mathrm{Def}_\chi(B \times_A C) = \mathrm{Def}_\chi(B \times C) = \mathrm{Def}_\chi(B) \times_{\mathrm{Def}_\chi(A)} \mathrm{Def}_\chi(C) = \mathrm{Def}_\chi(B) \times \mathrm{Def}_\chi(C)$.

The last assertion of the Theorem follows from Theorem 1.2.7. □

As we have already explained, the motivation for the introduction of the deformation functor associated to a morphism of DGLAs is the study of semitrivialized deformations. Here we recall the most classical example of embedded deformations of a submanifold and the example of deformations of a complex that preserve the i-th cohomology space.

Example 3.3.10. Let X be a compact and connected complex manifold and let Z be an analytic subvariety of X defined by the sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$.
An *embedded deformation* of Z in X over $A \in \mathbf{Art}_{\mathbb{C}}$ is the data of a sheaf of ideals $\mathcal{I}_A \subset \mathcal{O}_X \otimes_{\mathbb{C}} A$, flat over A and such that $\mathcal{I}_A \otimes_A \mathbb{C} = \mathcal{I}$. We denote with $\mathrm{Hilb}_X^Z : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$ the *Hilbert functor* which is the functor of these deformations.
Let $\mathrm{KS}_X = \bigoplus_{i \in \mathbb{N}} A_X^{0,i}(\mathcal{T}_X)$ be the Kodaira-Spencer DGLA of X and let $\bigoplus_{i \in \mathbb{N}} A_X^{0,i}(\mathcal{T}_X)(-\log Z)$ be its subalgebra defined by:

$$A_X^{0,i}(\mathcal{T}_X)(-\log Z) = \{x \in A_X^{0,i}(\mathcal{T}_X) \mid x(\ker i^*) \subset \ker i^*\},$$

where $i : Z \to X$ is the inclusion of closed smooth complex submanifold and $i^* : A_X^{0,i} \to A_Z^{0,i}$ is the morphism given by the restriction of forms on Z. Let $\chi : A_X^{0,i}(\mathcal{T}_X)(-\log Z) \to A_X^{0,i}(\mathcal{T}_X)$ be the natural inclusion of DGLAs and let Def_χ be the deformation functor associated to it.
It can be proved that Def_χ and Hilb_X^Z are isomorphic (see [25], Theorem 5.2) and the isomorphism is given, for all $A \in \mathbf{Art}_{\mathbb{C}}$, by:

$$\begin{array}{rcl} \mathrm{Def}_\chi(A) & \longrightarrow & \mathrm{Hilb}_X^Z(A) \\ (x, e^a) & \longrightarrow & (\mathcal{O}_X \otimes A) \cap e^a(\ker i^* \otimes A). \end{array}$$

Example 3.3.11. Let

$$(V = \bigoplus_{j \in \mathbb{Z}} V^j, d) : \quad \dots \xrightarrow{d} V^{j-1} \xrightarrow{d} V^j \xrightarrow{d} V^{j+1} \xrightarrow{d} \dots$$

be a complex of vector spaces and let $\mathrm{Hom}(V, V)$ be the DGLA of the homomorphism of the complex, that governs deformations of it (see Example 3.2.13).
Let analyse deformations of this complex that *preserves the i-th cohomology space*. To be precise, if $(V \otimes A, d + x)$ is a deformation of (V, d) over $A \in \mathbf{Art}_{\mathbb{K}}$, we say that it preserves the i-th cohomology space, if $H^i(V \otimes A, d + x) \cong H^i(V, d) \otimes A$. We indicate with $\mathrm{Def}^i_{(V,d)}$ the functor of these deformations.
Let

$$(T^i V, d) : \quad V^{i-1} \xrightarrow{d} V^i \xrightarrow{d} V^{i+1}$$

be the truncated complex, let $\mathrm{Hom}(T^i V, T^i V)$ be the DGLA of the homomorphism of it and let $T^i : \mathrm{Hom}(T^i V, T^i V) \to \mathrm{Hom}(V, V)$ be the natural morphism of DGLAs. Let Def_{T^i} be the deformation functors associated to T^i and let $\mathrm{Def}_{T^i} \to \mathrm{Def}_{\mathrm{Hom}(V,V)}$ the forgetful morphism of functors.
It can be proved (see [25], Proposition 3.5) that the functor $\mathrm{Def}^i_{(V,d)}$ is isomorphic to the image of the morphism $\mathrm{Def}_{T^i} \to \mathrm{Def}_{\mathrm{Hom}(V,V)}$.
Moreover observe that a deformation of the complex (V, d) preserve all the cohomology spaces if and only if it is trivial.

Chapter 4

New examples of deformations

In this Chapter we analyse some other examples of deformations of geometric objects: firstly the deformations of the pair (manifold, sheaf) with a fixed subspace of global sections of the sheaf that is required to be deformed, secondly, as a generalization of this case, the deformations of the pair (manifold, sheaf) with fixed subspaces of the cohomology spaces of the sheaf that are required to be deformed. In Sections 4.1 and 4.2, we define these deformations precisely and determine the differential graded Lie algebras that govern these problems.
In Section 4.3 we concentrate our interests on deformations of a locally free sheaf that preserve the dimensions of the cohomology spaces and we define the stratification of the coarse moduli space of stable and flat locally free sheaves on a compact complex Kähler manifold that corresponds to these deformations. Using the DGLAs approach to this problem, we prove that the strata of this stratification have quadratic algebraic singularities.

4.1 Deformations of (manifold, sheaf, subspace of sections)

Let X be a compact and connected complex manifold, let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X and let $V \subset H^0(X, \mathcal{E})$ be a subspace of its global sections.
This section is devoted to study infinitesimal deformations of the triple $(X, \mathcal{E}, V)$=(manifold, sheaf, subspace of sections) and to define a DGLA which governs these deformations.
Let's define these deformations geometrically.

Definition 4.1.1. *An* infinitesimal deformation of the triple $(X, \mathcal{E}, V)$=(manifold, sheaf, subspace of sections) *over $A \in \mathbf{Art}_{\mathbb{C}}$ is the data of:*

- *a deformation $\mathcal{X}_A$ of the manifold X over A, i.e. a cartesian diagram of complex spaces*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X}_A \arrow[d, "\pi"] \\ \operatorname{Spec}\mathbb{C} \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

  where π is flat;

- *a locally free sheaf $\mathcal{E}_A$ of $\mathcal{O}_{\mathcal{X}_A}$-modules on $\mathcal{X}_A$ with a morphism $\pi_A : \mathcal{E}_A \to \mathcal{E}$ such that $\pi_A : \mathcal{E}_A \otimes_A \mathbb{C} \to \mathcal{E}$ is an isomorphism;*

- *a map* $i_A : V \otimes A \to H^0(\mathcal{X}_A, \mathcal{E}_A)$*, such that the following diagram is commutative:*

$$\begin{tikzcd} V \otimes A \arrow[r, "i_A"] \arrow[d, "\pi"] & H^0(\mathcal{X}_A, \mathcal{E}_A) \arrow[d, "\pi_A"] \\ V \arrow[r, "i"] & H^0(X, \mathcal{E}). \end{tikzcd} \tag{4.1}$$

Definition 4.1.2. *Two infinitesimal deformations* $(\mathcal{X}_A, \mathcal{E}_A, i_A)$ *and* $(\mathcal{X}'_A, \mathcal{E}'_A, i'_A)$ *of the triple* $(X, \mathcal{E}, V)$ *over* $A \in \mathbf{Art}_{\mathbb{C}}$ *are* isomorphic *if:*

- $\exists\ \phi : \mathcal{X}_A \to \mathcal{X}'_A$ *isomorphism of deformations of the manifold* X *over* A*, i.e.* ϕ *is an isomorphism that makes the following diagram commutative:*

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ \mathcal{X}_A \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{X}'_A \arrow[dl] \\ & \operatorname{Spec} A; & \end{tikzcd}$$

- $\exists\ \psi : \mathcal{E}_A \to \mathcal{E}'_A$ *isomorphism of sheaves of* $\mathcal{O}_{\mathcal{X}_A}$*-modules, where the structure of sheaf of* $\mathcal{O}_{\mathcal{X}_A}$*-module on* $\mathcal{E}'_A$ *is the one induced by* ϕ*, that commutes with the maps* $\pi_A : \mathcal{E}_A \to \mathcal{E}$ *and* $\pi'_A : \mathcal{E}'_A \to \mathcal{E}$*, i.e.* $\pi'_A \circ \phi = \pi_A$*;*

- $\exists\ \zeta : V \otimes A \to V \otimes A$ *isomorphism, such that the following diagram is commutative:*

$$\begin{tikzcd} V \otimes A \arrow[r, "i_A"] \arrow[d, "\zeta"] & \mathcal{E}_A \arrow[d, "\psi"] \\ V \otimes A \arrow[r, "i'_A"] & \mathcal{E}'_A, \end{tikzcd} \tag{4.2}$$

this last condition is equivalent to the condition that $\psi : i_A(V \otimes A) \to i'_A(V \otimes A)$ *is an isomorphism.*

Definition 4.1.3. *The* functor of infinitesimal deformations of the triple $(X, \mathcal{E}, V)$ *is the functor of Artin rings:*

$$\mathrm{Def}_{(X,\mathcal{E},V)} : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$$

which, to every $A \in \mathbf{Art}_{\mathbb{C}}$*, associates the set of the isomorphism classes of deformations of the triple* $(X, \mathcal{E}, V)$ *over* A*.*

Let $\mathcal{A}_X^{0,*}(\mathcal{E})$ be the complex of sheaves:

$$0 \longrightarrow \mathcal{A}_X^{0,0}(\mathcal{E}) \xrightarrow{\bar\partial} \mathcal{A}_X^{0,1}(\mathcal{E}) \xrightarrow{\bar\partial} \mathcal{A}_X^{0,2}(\mathcal{E}) \longrightarrow \dots,$$

where $\bar\partial$ is the Dolbeault differential, let Q be the complex of sheaves:

$$0 \longrightarrow V \xrightarrow{i} \mathcal{A}_X^{0,0}(\mathcal{E}) \xrightarrow{\bar\partial} \mathcal{A}_X^{0,1}(\mathcal{E}) \xrightarrow{\bar\partial} \mathcal{A}_X^{0,2}(\mathcal{E}) \longrightarrow \dots,$$

where i is the natural inclusion, and let $\mathrm{Hom}^*(Q,Q)$ be the DGLA of the homomorphisms of this complex, in which the differential and the bracket are the ones defined in Example 3.1.4. We define the DGLA L as

$$L^* = \{x \in \mathrm{Hom}^*(Q,Q) \mid x|_{\mathcal{A}_X^{0,i}(\mathcal{E})} \in A_X^{0,*}(D^1(\mathcal{E}))\},$$

in which the structure of DGLA is the one inherited by $\mathrm{Hom}^*(Q,Q)$. We indicate an element of L as $x = (x_{-1}, x_i)$, where $x_{-1} = x|_V$ and $x_i = x|_{\mathcal{A}_X^{0,i}(\mathcal{E})}$ for $i \geq 0$.
Let write explicitly the action of the elements in $A_X^{0,*}(D^1(\mathcal{E}))$ as homomorphisms of the complex $\mathcal{A}_X^{0,*}(\mathcal{E})$. Let $x \in A_X^{0,i}(D^1(\mathcal{E}))$, written locally in an open set $U \subset X$ as $x = \omega \otimes P$, and let $z \in \mathcal{A}_X^{0,j}(\mathcal{E})(U)$, written as $z = \eta \otimes s$, then

$$x(z) = (\omega \otimes P)(\eta \otimes s) = \omega \wedge \eta \otimes P(s) + \omega \wedge \sigma(P)(\eta) \otimes s.$$

To explicitate the differential of the DGLA L, let $x = (x_{-1}, x_i) \in L$

- if $x \in L^0$, then $dx = (i \circ x_{-1} - x_0 \circ i, \bar{\partial} \circ x_i - x_{i+1} \circ \bar{\partial})$,
- if $x \in L^j$ with $j > 0$, then $dx = (\bar{\partial} \circ x_{-1} - x_0 \circ i, \bar{\partial} \circ x_i - (-1)^j x_{i+1} \circ \bar{\partial})$.

We note that for the restriction of an element $x \in L$ to the complex $\mathcal{A}_X^{0,*}(\mathcal{E})$, two differentials are defined:

- the differential of x as restriction of an element of L: $\bar{\partial} \circ x_i - (-1)^j x_{i+1} \circ \bar{\partial}$,
- the differential of x as element in $A_X^{0,*}(D^1(\mathcal{E}))$: $dx|_{\mathcal{A}_X^{0,i}(\mathcal{E})}$;

these two differentials coincide. Infact, making a local computation on an open set $U \subset X$, if $x|_{\mathcal{A}_X^{0,*}(\mathcal{E})(U)} = \omega \otimes P \in A_X^{0,i}(D^1(\mathcal{E}))|_U$ and $z = \eta \otimes s \in \mathcal{A}_X^{0,j}(\mathcal{E})(U)$, we have:

$$dx(z) = (\bar{\partial}\omega \otimes P)(\eta \otimes s) = \bar{\partial}\omega \wedge \eta \otimes P(s) + \bar{\partial}\omega \wedge \sigma(P)(\eta) \otimes s$$

and on the other hand we have:

$$\bar{\partial} \circ x(z) - (-1)^i x \circ \bar{\partial}(z) = \bar{\partial}(\omega \wedge \eta \otimes P(s) + \omega \wedge \sigma(P)(\eta) \otimes s) - (-1)^i (\omega \otimes P)(\bar{\partial}\eta \otimes s) =$$

$$= (\bar{\partial}\omega) \wedge \eta \otimes P(s) + (-1)^i \omega \wedge (\bar{\partial}\eta) \otimes P(s) + (\bar{\partial}\omega) \wedge \sigma(P)(\eta) \otimes s + (-1)^i \omega \wedge \bar{\partial}\sigma(P)(\eta) \otimes s +$$

$$-(-1)^i \omega \wedge \bar{\partial}\eta \otimes P(s) - (-1)^i \omega \wedge \sigma(P)(\bar{\partial}\eta) \otimes s = \bar{\partial}\omega \wedge \eta \otimes P(s) + \bar{\partial}\omega \wedge \sigma(P)(\eta) \otimes s.$$

The bracket of the DGLA L is simply defined by:

$$[x,y] = x \circ y - (-1)^{\deg x \deg y} y \circ x, \quad \text{for all} \quad x, y \in L$$

and obviously the bracket of the restrictions of two elements $x, y \in L$ to the complex $\mathcal{A}_X^{0,*}(\mathcal{E})$ is the same as their bracket as elements in the DGLA $A_X^{0,*}(D^1(\mathcal{E}))$.

Definition 4.1.4. *Let*

$$\mathrm{Def}_L : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$$

be the deformation functor associated to the DGLA L.

Our aim is to prove that the functors Def_L and $\mathrm{Def}_{(X,\mathcal{E},V)}$ are isomorphic. We start by defining a morphism of functors:

$$\Phi_L : \mathrm{MC}_L \longrightarrow \mathrm{Def}_{(X,\mathcal{E},V)};$$

for all $A \in \mathbf{Art}_{\mathbb{C}}$ and for all elements $x = (x_{-1}, x_i) \in \mathrm{MC}_L(A)$, $\Phi_L(x)$ is defined to be the isomorphism class of the deformation of the triple $(X, \mathcal{E}, V)$ over A given by $(\mathcal{X}_A, \mathcal{E}_A, i_A)$, where:

- $\mathcal{X}_A = \ker(\bar{\partial} + \mathfrak{l}_{\sigma(x_i)} : \mathcal{A}_X^{0,0} \otimes A \longrightarrow \mathcal{A}_X^{0,1} \otimes A)$, where $\mathfrak{l}$ is the Lie derivative,
- $\mathcal{E}_A = \ker(\bar{\partial} + x_0 : \mathcal{A}_X^{0,0}(\mathcal{E}) \otimes A \longrightarrow \mathcal{A}_X^{0,1}(\mathcal{E}) \otimes A)$,
- $i_A = i + x_{-1} : V \otimes A \longrightarrow H^0(\mathcal{X}_A, \mathcal{E}_A)$.

Lemma 4.1.5. *The above morphism $\Phi_L : \mathrm{MC}_L \longrightarrow \mathrm{Def}_{(X,\mathcal{E},V)}$ is well defined.*

Proof. Let $x = (x_{-1}, x_i) \in \mathrm{MC}_L(A)$. First of all, the restriction to the $\mathcal{A}_X^{0,*}(\mathcal{E})$ complex of the Maurer-Cartan equation for x gives the Maurer-Cartan equation for $x_i \in A_X^{0,1}(D^1(\mathcal{E}))$ and, taking the principal symbol, it gives the Maurer-Cartan equation for $\sigma(x_i) \in KS_X^1 = A_X^{0,1}(\mathcal{T}_X)$. Thus $\sigma(x_i) \in \mathrm{MC}_{KS_X}(A)$ and $\mathcal{X}_A = \ker(\bar{\partial} + \mathfrak{l}_{\sigma(x_i)})$ is a deformation of the manifold X over A (see Example 3.2.14).

Secondly, $\mathcal{E}_A = \ker(\bar{\partial} + x_0)$ is a locally free sheaf of $\mathcal{O}_{\mathcal{X}_A}$-modules and the morphism $\pi_A : \mathcal{E}_A \to \mathcal{E}$ is the obvious projection to the residue field.
Let $y \in \mathcal{E}_A$ and $g \in \mathcal{O}_{\mathcal{X}_A}$, then $g \cdot y \in \mathcal{E}_A$, in fact $(\bar{\partial} + x_0)(g \cdot y) = \bar{\partial}(g)y + g\bar{\partial}(y) + \mathfrak{l}_{\sigma(x_i)}(g)y + gx_0(y) = (\bar{\partial} + \mathfrak{l}_{\sigma(x_i)})g + (\bar{\partial} + x_0)y = 0$. This verifies that $\mathcal{E}_A$ is a sheaf of $\mathcal{O}_{\mathcal{X}_A}$-modules.
To verify that it is locally free do the following. Since $x_i \in \mathrm{MC}_{A_X^{0,*}(D^1(\mathcal{E}))}(A)$, it is locally gauge equivalent to zero, i.e. there exist an open covering $\mathcal{U} = \{U_\alpha\}_\alpha$ of X and elements $a_\alpha \in A_X^{0,0}(D^1(\mathcal{E}))|_{U_\alpha} \otimes \mathfrak{m}_A$ such that $e^{a_\alpha} * x_i|_{U_\alpha} = 0$. As we will explain in (4.5), from these equations we obtain isomorphisms $e^{a_\alpha} : \ker(\bar{\partial} + x_0)|_{U_\alpha} \to \ker \bar{\partial}|_{U_\alpha}$. Moreover $\sigma(x_i)$ satisfies $e^{\sigma(a_\alpha)} * \sigma(x_i)|_{U_\alpha} = 0$, so we have isomorphisms $e^{\sigma(a_\alpha)} : \ker(\bar{\partial} + \mathfrak{l}_{\sigma(x_i)})|_{U_\alpha} \to \ker \bar{\partial}|_{U_\alpha}$. Thus we obtain isomorphisms $e^{-\sigma(a_\alpha)} \circ e^{a_\alpha} : \ker(\bar{\partial} + x_0)|_{U_\alpha} \to \ker(\bar{\partial} + \mathfrak{l}_{\sigma(x_i)})|_{U_\alpha}$ that make $\mathcal{E}_A$ a locally free sheaf.
Moreover the map $\pi_A : \mathcal{A}_X^{0,*}(\mathcal{E}) \otimes A \to \mathcal{A}_X^{0,*}(\mathcal{E})$ induced by the projection onto the residue field commutes with the differentials $\bar{\partial} + x_i$ and $\bar{\partial}$, so it is well defined from $\mathcal{E}_A$ to $\mathcal{E}$ and induces an isomorphism $\mathcal{E}_A \otimes_A \mathbb{C} \cong \mathcal{E}$.

Lastly, the morphisms i_A and π_A make the diagram (4.1) commutative, because $i_A = i + x_{-1}$, with $x_{-1} : V \otimes A \to A_X^{0,0}(\mathcal{E}) \otimes \mathfrak{m}_A$; and $i_A(V \otimes A) = (i + x_{-1})(V \otimes A)$ is a subset of $H^0(X, \mathcal{E}_A) = \ker(\bar{\partial} + x_0)$, because on $V \otimes A$ we have $(\bar{\partial} + x_0) \circ (i + x_{-1}) = \bar{\partial} \circ i + \bar{\partial} \circ x_{-1} + x_0 \circ i + x_0 \circ x_{-1} = 0 + (\bar{\partial} \circ x_{-1} + x_0 \circ i) + x_0 \circ x_{-1} = (dx)_{-1} + \frac{1}{2}[x, x]_{-1} = 0$, since $V \subset \ker \bar{\partial}$ and x satisfies the Maurer-Cartan equation. □

Lemma 4.1.6. *The above morphism Φ_L induces a morphism between the deformation functors $\Phi_L : \mathrm{Def}_L \to \mathrm{Def}_{(X,\mathcal{E},V)}$.*

Proof. Let $x = (x_{-1}, x_i)$ and $y = (y_{-1}, y_i)$ be gauge equivalent elements in $\mathrm{MC}_L(A)$ and let $(\mathcal{X}_A, \mathcal{E}_A, i_A)$ and $(\mathcal{X}'_A, \mathcal{E}'_A, i'_A)$ be their images via Φ_L. By definition of gauge relation, it exists an element $a = (a_{-1}, a_i) \in L^0 \otimes \mathfrak{m}_A$, such that:

$$y = e^a * x = x + \sum_{n=0}^{+\infty} \frac{([a, -])^n}{(n+1)!}([a, x] - da). \tag{4.3}$$

This gauge relation restricted to the $\mathcal{A}_X^{0,*}(\mathcal{E})$ complex becomes

$$y_i = x_i + \sum_{n=0}^{+\infty} \frac{([a_i,-])^n}{(n+1)!}([a_i,x_i] - da_i) = e^{a_i} * x_i \tag{4.4}$$

taking the principal symbol, it becomes:

$$\begin{aligned}\sigma(y_i) &= \sigma(e^{a_i} * x_i) = \sigma(x_i) + \sum_{n=0}^{+\infty} \sigma\left[\frac{([a_i,-])^n}{(n+1)!}([a_i,x_i] - da_i)\right] = \\ &= \sigma(x_i) + \sum_{n=0}^{+\infty} \frac{([\sigma(a_i),-])^n}{(n+1)!}([\sigma(a_i),\sigma(x_i)] - d\sigma(a_i)) = e^{\sigma(a_i)} * \sigma(x_i),\end{aligned}$$

then $\sigma(x_i)$ and $\sigma(y_i)$ are gauge equivalent via $\sigma(a_i)$ in the DGLA KS_X and so the deformations $\mathcal{X}_A = \ker(\bar{\partial} + \mathfrak{l}_{\sigma(x_i)})$ and $\mathcal{X}'_A = \ker(\bar{\partial} + \mathfrak{l}_{\sigma(y_i)})$ are isomorphic via $e^{\sigma(a_i)}$ (see Example 3.2.14).

Making calculation on the gauge relation (4.4), obtain:

$$\begin{aligned}y_i &= e^{a_i} * x_i = x_i + \sum_{n=0}^{+\infty} \frac{([a_i,-])^n}{(n+1)!}([a_i,x_i] - da_i) = \\ &= x_i + \sum_{n=0}^{+\infty} \frac{([a_i,-])^n}{(n+1)!}([a_i,x_i] + [a_i,\bar{\partial}]) = x_i + \sum_{n=1}^{+\infty} \frac{([a_i,-])^n}{n!}(\bar{\partial} + x_i) = \\ &= \sum_{n=0}^{+\infty} \frac{([a_i,-])^n}{n!}(\bar{\partial} + x_i) - \bar{\partial} = e^{[a_i,-]}(\bar{\partial} + x_i) - \bar{\partial},\end{aligned} \tag{4.5}$$

where it is used that, since the DGLA structure on $A_X^{0,*}(D^1(\mathcal{E}))$ is the same as the one on $\mathrm{Hom}^*(Q,Q)$, $da_i = -[a_i,\bar{\partial}]$. Therefore $\bar{\partial} + y_i = e^{[a_i,-]}(\bar{\partial} + x_i) = e^{a_i} \circ (\bar{\partial} + x_i) \circ e^{-a_i}$, i.e. e^{a_i} commutes with the derivations, thus e^{a_i} is an isomorphism between the two shaves $\mathcal{E}_A = \ker(\bar{\partial} + x_0)$ e $\mathcal{E}'_A = \ker(\bar{\partial} + y_0)$.
Moreover e^{a_i} is an isomorphism of sheaves of $\mathcal{O}_{\mathcal{X}_A}$-modules, where the structure of sheaf of $\mathcal{O}_{\mathcal{X}_A}$-modules on $\mathcal{E}'_A$ is the one induced by the isomorphism $e^{\sigma(a_i)}$. Need to verify that, for all $\eta \otimes s \in \ker(\bar{\partial} + x_0)$ and for all $g \in \ker(\bar{\partial} + l_{\sigma(x_i)})$, $e^{a_i}(g \cdot (\eta \otimes s)) = e^{\sigma(a_i)}(g) \cdot e^{a_i}(\eta \otimes s)$.
Let $a_i = h \otimes P \in A_X^{0,0}(D^1(\mathcal{E})) \otimes \mathfrak{m}_A$, prove by induction on $n \in \mathbb{N}$ that:

$$(h \otimes P)^n(g \cdot (\eta \otimes s)) = \sum_{k=0}^{n} \binom{n}{k} (h \otimes \sigma(P))^k(g) \cdot (h \otimes P)^{n-k}(\eta \otimes s). \tag{4.6}$$

For $n = 0, 1$ it is clear. Suppose it is true for $n-1$ and prove it for n:

$$\begin{aligned}
& (h\otimes P)^n(g\cdot(\eta\otimes s)) = (h\otimes P)\left((h\otimes P)^{n-1}(g\cdot\eta\otimes s)\right) = \\
= \;& (h\otimes P)\sum_{k=0}^{n-1}\begin{pmatrix} n-1 \\ k \end{pmatrix}(h\otimes\sigma(P))^k(g)\cdot(h\otimes P)^{n-k-1}(\eta\otimes s) = \\
= \;& \sum_{k=0}^{n-1}\begin{pmatrix} n-1 \\ k \end{pmatrix}\Big((h\otimes\sigma(P))^{k+1}(g)\cdot(h\otimes P)^{n-k-1}(\eta\otimes s) + \\
+ \;& (h\otimes\sigma(P))^k(g)\cdot(h\otimes P)^{n-k}(\eta\otimes s)\Big) = \\
= \;& \sum_{j=1}^{n}\begin{pmatrix} n-1 \\ j-1 \end{pmatrix}(h\otimes\sigma(P))^j(g)\cdot(h\otimes P)^{n-j}(\eta\otimes s) + \\
+ \;& \sum_{k=0}^{n-1}\begin{pmatrix} n-1 \\ k \end{pmatrix}(h\otimes\sigma(P))^k(g)\cdot(h\otimes P)^{n-k}(\eta\otimes s) = \\
= \;& \sum_{i=1}^{n-1}\begin{pmatrix} n \\ i \end{pmatrix}(h\otimes\sigma(P))^i(g)\cdot(h\otimes P)^{n-i}(\eta\otimes s) + (h\otimes\sigma(P))^n(g) + (h\otimes P)^n(\eta\otimes s) \\
= \;& \sum_{i=0}^{n}\begin{pmatrix} n \\ i \end{pmatrix}(h\otimes\sigma(P))^i(g)\cdot(h\otimes P)^{n-i}(\eta\otimes s).
\end{aligned}$$

This calculation prove formula (4.6). Now, e^{a_i} is a morphisms of sheaves of $\mathcal{O}_{\mathcal{X}_A}$-modules, in fact:

$$\begin{aligned}
e^{a_i}(g\cdot(\eta\otimes s)) &= \sum_{n=0}^{+\infty}\frac{a_i^n}{n!}(g\cdot(\eta\otimes s)) = \sum_{n=0}^{+\infty}\frac{(h\otimes P)^n}{n!}(g\cdot(\eta\otimes s)) = \\
&= \sum_{n=0}^{+\infty}\sum_{k=0}^{n}\frac{1}{n!}\begin{pmatrix} n \\ k \end{pmatrix}(h\otimes\sigma(P))^k(g)\cdot(h\otimes P)^{n-k}(\eta\otimes s) = \\
&= \sum_{n=0}^{+\infty}\sum_{i+j=n}\frac{1}{i!j!}(h\otimes\sigma(P))^i(g)\cdot(h\otimes P)^j(\eta\otimes s) = \\
&= \sum_{n=0}^{+\infty}\sum_{i+j=n}\frac{\sigma(a_i)^i}{i!}(g)\cdot\frac{a_i^j}{j!}(\eta\otimes s) = e^{\sigma(a_i)}(g)\cdot e^{a_i}(\eta\otimes s).
\end{aligned}$$

Lastly, consider the gauge relation (4.3) when the elements in the DGLA L are restricted to V:

$$\begin{aligned}
y_{-1} &= e^z * x_{-1} = x_{-1} + \sum_{n=0}^{+\infty}\frac{([z,-])^n}{(n+1)!}([z,x]_{-1} - (dz)_{-1}) = \qquad (4.7)\\
&= x_{-1} + \sum_{n=0}^{+\infty}\frac{([z,-])^n}{(n+1)!}([z,x]_{-1} + [z,i]_{-1}) = x_{-1} + \sum_{n=1}^{+\infty}\frac{([z,-])^n}{n!}(i + x_{-1}) = \\
&= \sum_{n=0}^{+\infty}\frac{([z,-])^n}{n!}(i + x_{-1}) - i = e^{[z,-]}(i + x_{-1}) - i\ ,
\end{aligned}$$

where the equality $(dz)_{-1} = i \circ z_{-1} - z_0 \circ i = -[z,i]_{-1}$ is used, which is true for the restriction of the elements of L to homomorphisms from V. Therefore:

$$i + y_{-1} = e^{[z,-]}(i + x_{-1}) = e^{z_0} \circ (i + x_{-1}) \circ e^{-z_{-1}},$$

thus e^{z_i} is an isomorphism, it commutes with $i + x_{-1}$ and $i + y_{-1}$, i.e. it makes the diagram (4.2) commutative and it is obvious that it commutes with the projections in the residue field because $z \in L^0 \otimes \mathfrak{m}_A$.

Therefore, if x and y are gauge equivalent elements in $\mathrm{MC}_L(A)$, the associated deformations $\Phi_L(x)$ and $\Phi_L(y)$ are isomorphic thus Φ_L is well defined on deformation functors. $\square$

Remark 4.1.7. The above calculations can be used also in a simpler case.
Let X be a compact and connected complex manifold, let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X and let $V \subset H^0(X, \mathcal{E})$ be a subspace of its global sections.
Consider *infinitesimal deformations of the pair* $(\mathcal{E}, V)$=(sheaf, subspace of sections) over $A \in \mathbf{Art}_{\mathbb{C}}$. They are defined as in Definition 4.1.1, with trivial deformation of the manifold X, i.e. $\mathcal{X}_A = X \times \operatorname{Spec} A$. Two of such deformations are isomorphic if they satisfy conditions of Definition 4.1.2, where the isomorphism between the trivial deformations of the manifold X is the identity. Let

$$\mathrm{Def}_{(\mathcal{E},V)} : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$$

be the functor of deformations of the pair $(\mathcal{E}, V)$.

Let M be the DGLA $M^* = \{x \in \mathrm{Hom}^*(Q,Q) \mid x|_{\mathcal{A}_X^{0,i}(\mathcal{E})} \in A_X^{0,*}(\mathrm{End}(\mathcal{E}))\}$, in which the structure of DGLA is the one inherited by $\mathrm{Hom}^*(Q,Q)$, it is a subDGLA of L. Let

$$\mathrm{Def}_M : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$$

be the deformation functor associated to M.

Let

$$\Phi_M : \mathrm{Def}_M \to \mathrm{Def}_{(\mathcal{E},V)}$$

be the morphism which associates, to every $A \in \mathbf{Art}_{\mathbb{C}}$ and to every element $x = (x_{-1} = x|_V, x_i = x|_{\mathcal{A}_X^{0,i}(\mathcal{E})}) \in \mathrm{MC}_M(A)$, the isomorphism class of the deformation of the pair $(\mathcal{E}, V)$ over A given by $(\mathcal{E}_A, i_A)$, where:

- $\mathcal{E}_A = \ker(\bar\partial + x_0 : \mathcal{A}_X^{0,0}(\mathcal{E}) \otimes A \longrightarrow \mathcal{A}_X^{0,1}(\mathcal{E}) \otimes A)$,
- $i_A = i + x_{-1} : V \otimes A \longrightarrow H^0(X, \mathcal{E}_A)$.

The above calculations assure that Φ_M is a well defined morphism of deformation functors.

Theorem 4.1.8. *The morphism* $\Phi_L : \mathrm{Def}_L \to \mathrm{Def}_{(X,\mathcal{E},V)}$ *is injective.*

Proof. Let $x = (x_{-1}, x_i)$ and $y = (y_{-1}, y_i)$ in $\mathrm{MC}_L(A)$ such that the two deformations $\Phi_L(x) = (\mathcal{X}_A, \mathcal{E}_A, i_A)$ and $\Phi_L(y) = (\mathcal{X}'_A, \mathcal{E}'_A, i'_A)$, defined as above, are isomorphic. Then there exist isomorphisms ϕ, ψ and ζ as in Definition 4.1.2.
Consider the following diagram

$$\begin{tikzcd} \mathrm{Def}_L \arrow[r, "\Phi_L"] \arrow[d, "p"] & \mathrm{Def}_{(X,\mathcal{E},V)} \arrow[d, "\pi"] \\ \mathrm{Def}_{A_X^{0,*}(D^1(\mathcal{E}))} \arrow[r, "\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}"] & \mathrm{Def}_{(X,\mathcal{E})}, \end{tikzcd} \tag{4.8}$$

where the morphism p is given, for all $x = (x_{-1}, x_i) \in L^1 \otimes \mathfrak{m}_A$, by $p(x) = x_i$, the morphism π is given by $\pi(\mathcal{X}_A, \mathcal{E}_A, i_A) = (\mathcal{X}_A, \mathcal{E}_A)$, for all deformations, and the morphism $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}$ is defined in Example 3.2.16. These maps are well defined on deformation functors and make the diagram commutative.
The deformations $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}(x_i) = (\mathcal{X}_A, \mathcal{E}_A)$ and $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}(y_i) = (\mathcal{X}'_A, \mathcal{E}'_A)$ of the pair $(X, \mathcal{E})$ over A are isomorphic. Since $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}$ is an isomorphism (see Example 3.2.16), the elements x_i and y_i are gauge equivalent in the DGLA $A_X^{0,*}(D^1(\mathcal{E}))$, i.e. there exists $a \in A_X^{0,0}(D^1(\mathcal{E})) \otimes \mathfrak{m}_A$ such that $e^a * x_i = y_i$. Then the isomorphism $\psi : \mathcal{E}_A \to \mathcal{E}'_A$ can be lifted to an isomorphism of complexes $e^a : (\mathcal{A}_X^{0,*}(\mathcal{E}) \otimes A, \bar{\partial} + x_i) \to (\mathcal{A}_X^{0,*}(\mathcal{E}) \otimes A, \bar{\partial} + y_i)$. Moreover ζ can be written as e^b, with $b \in \mathrm{Hom}(V, V) \otimes \mathfrak{m}_A$, because it is the identity on the residue field $A/\mathfrak{m}_A \cong \mathbb{C}$.
Therefore it exists an element $c = (b, a) \in L^0 \otimes \mathfrak{m}_A$, such that e^c is an isomorphism which makes commutative diagram (4.2). As verified in (4.7), this commutativity property is equivalent to the relation $y = e^c * x$.
Then $x = y \in \mathrm{Def}_L(A)$ and the morphism Φ_L is injective. □

Corollary 4.1.9. *The morphism $\Phi_M : \mathrm{Def}_M \to \mathrm{Def}_{(\mathcal{E},V)}$ is injective.*

Proof. The proof is the same as the previous one, where now the deformations $\mathcal{X}_A$ and $\mathcal{X}'_A$ are trivial. □

Theorem 4.1.10. *The morphism $\Phi_L : \mathrm{Def}_L \to \mathrm{Def}_{(X,\mathcal{E},V)}$ is smooth.*

Proof. The morphism Φ_L is smooth if and only if, given a small extension in $\mathbf{Art}_{\mathbb{C}}$, $0 \to J \to B \xrightarrow{\alpha} A \to 0$, an element $x \in \mathrm{Def}_L(A)$ and its image $\Phi_L(x) = (\mathcal{X}_A, \mathcal{E}_A, i_A) \in \mathrm{Def}_{(X,\mathcal{E},V)}(A)$, $(\mathcal{X}_A, \mathcal{E}_A, i_A)$ has a lifting $(\mathcal{X}_B, \mathcal{E}_B, i_B) \in \mathrm{Def}_{(X,\mathcal{E},V)}(B)$ if and only if x has a lifting $\tilde{x} \in \mathrm{Def}_L(B)$.
It is clear that if $x \in \mathrm{Def}_L(A)$ has a lifting $\tilde{x} \in \mathrm{Def}_L(B)$, then $\Phi_L(\tilde{x})$ lifts $\Phi_L(x)$. Thus it is sufficient to prove the other implication.
Consider the diagram

$$\begin{tikzcd}
\mathrm{Def}_L \arrow[r, "\Phi_L"] \arrow[d, "p"] & \mathrm{Def}_{(X,\mathcal{E},V)} \arrow[d, "\pi"] \\
\mathrm{Def}_{A_X^{0,*}(D^1(\mathcal{E}))} \arrow[r, "\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}"] & \mathrm{Def}_{(X,\mathcal{E})},
\end{tikzcd}$$

defined as in (4.8), let $x = (x_{-1}, x_i) \in \mathrm{MC}_L(A)$, $\Phi_L(x) = (\mathcal{X}_A, \mathcal{E}_A, i_A) \in \mathrm{Def}_{(X,\mathcal{E},V)}(A)$, $p(x) = x_i \in \mathrm{MC}_{A_X^{0,*}(D^1(\mathcal{E}))}(A)$ and $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}(p(x)) = \pi(\Phi_L(x)) = (\mathcal{X}_A, \mathcal{E}_A) \in \mathrm{Def}_{(X,\mathcal{E})}(A)$.
Suppose that it exists a lifting $(\mathcal{X}_B, \mathcal{E}_B, i_B) \in \mathrm{Def}_{(X,\mathcal{E},V)}(B)$ of $(\mathcal{X}_A, \mathcal{E}_A, i_A)$, thus $(\mathcal{X}_B, \mathcal{E}_B) \in \mathrm{Def}_{(X,\mathcal{E})}(B)$ is a lifting of $(\mathcal{X}_A, \mathcal{E}_A)$. Since $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}$ is an isomorphism (see Example 3.2.16), it is smooth and it exists a lifting $\tilde{x}_i \in \mathrm{MC}_{A_X^{0,*}(D^1(\mathcal{E}))}(B)$ of x_i.
To obtain a lifting of x in $\mathrm{MC}_L(B)$, consider the homomorphism $z = i_B - i : V \otimes B \to \mathcal{A}_X^{0,0}(\mathcal{E}) \otimes B$ and define $\tilde{x} = (z, \tilde{x}_i) \in \mathrm{Hom}(Q, Q) \otimes \mathfrak{m}_B$. The image of z via α is given by

$$\alpha(z)(v \otimes b) = \alpha(i_B - i)(v \otimes b) = \alpha i_B(v \otimes b) - \alpha i(v \otimes b) = i_A(v \otimes \alpha(b)) - i(v) \otimes \alpha(b) =$$

$$= (i + x_{-1})(v \otimes \alpha(b)) - i(v) \otimes \alpha(b) = x_{-1}(v \otimes \alpha(b)) \quad \text{for} \quad v \otimes b \in V \otimes B,$$

then z is a lifting of x_{-1}.
Moreover $\tilde{x}$ satisfies the Maurer-Cartan equation in the DGLA $L \otimes \mathfrak{m}_B$. Infact, since $\tilde{x}_i \in$

$\mathrm{MC}_{A_X^{0,*}(D^1(\mathcal{E}))}(B)$, it is sufficient to verify the Maurer-Cartan equation for $\tilde{x}$ restricted to $V\otimes B$, for all $v\in V\otimes B$ the following holds:

$$d\tilde{x}+\frac{1}{2}[\tilde{x},\tilde{x}](v)=\bar{\partial}z(v)+\tilde{x}_i\circ i(v)+\tilde{x}_i\circ z(v)=(\bar{\partial}+\tilde{x}_i)\circ(i+z)(v)=(\bar{\partial}+\tilde{x}_i)\circ i_B(v)=0,$$

since $i_B:V\otimes B\to H^0(\mathcal{X}_B,\mathcal{E}_B)$. Then $\tilde{x}\in\mathrm{MC}_L(B)$ is a lifting of x. □

This Theorem can be proven also using relative obstruction theory.
It could be of some interests the analisis of it. Let $0\to J\to B\xrightarrow{\alpha}A\to 0$ be a small extension in $\mathbf{Art}_{\mathbb{C}}$ and consider the commutative diagram (4.8).

Analyse obstructions for the functor Def_L and relative obstructions for the morphism p. Given an element $x\in\mathrm{MC}_L(A)$, if $\tilde{x}$ is a generic lifting of x, i.e. $\tilde{x}\in L^1\otimes\mathfrak{m}_B$ such that $\alpha(\tilde{x})=x$, then the obstruction of x is $H(x)=d\tilde{x}+\frac{1}{2}[\tilde{x},\tilde{x}]\in H^2(L)\otimes J$. Given an element $x=(x_{-1},x_i)\in\mathrm{MC}_L(A)$, now suppose that a lifting $\tilde{x}_i\in\mathrm{MC}_{A_X^{0,*}(D^1(\mathcal{E}))}(B)$ of x_i exists, then take a generic lifting h of x_{-1} and consider the lifting $\tilde{x}=(h,\tilde{x}_i)\in L^1\otimes\mathfrak{m}_B$ of x. In this case the relative obstruction of $(\tilde{x}_i,x)$ is $H(x)=d\tilde{x}+\frac{1}{2}[\tilde{x},\tilde{x}]=(\bar{\partial}+\tilde{x}_i)\circ(i+h)\in\mathrm{Hom}(V,\mathcal{A}_X^{0,1}(\mathcal{E}))\otimes J$, and, to eliminate the dipendence on the choice of the lifting, consider its class in $\mathrm{Hom}(V,H^1_{\bar{\partial}}(\mathcal{E}))\otimes J$.

Now analyze relative obstructions for the morphism π. Given an element $(\mathcal{X}_A,\mathcal{E}_A,i_A)\in\mathrm{Def}_{(X,\mathcal{E},V)}(A)$, suppose that a lifting $(\mathcal{X}_B,\mathcal{E}_B)\in\mathrm{Def}_{(X,\mathcal{E})}(B)$ of $(\mathcal{X}_A,\mathcal{E}_A)$ exists. Then there is the short exact sequence of sheaves: $0\to\mathcal{E}\to\mathcal{E}_B\to\mathcal{E}_A\to 0$ and the long exact sequence of cohomology: $\ldots\to H^0(\mathcal{E}_B)\to H^0(\mathcal{E}_A)\xrightarrow{\delta}H^1(\mathcal{E})\to\ldots$, thus a section $s\in H^0(\mathcal{E}_A)$ is lifted by a section $\tilde{s}\in H^0(\mathcal{E}_B)$ if and only if $\delta(s)=0$. Define the relative obstruction of $((\mathcal{X}_B,\mathcal{E}_B),(\mathcal{X}_A,\mathcal{E}_A,i_A))$ as $\delta(i_A(-))\in\mathrm{Hom}(V,\check{\mathrm{H}}^1(\mathcal{E}))\otimes J$, which associates to a section $v\in V$ the element $\delta(i_A(v))\in\check{\mathrm{H}}^1(\mathcal{E})\otimes J$.

Lemma 4.1.11. *The map* $\mathrm{Hom}(V,H^1_{\bar{\partial}}(\mathcal{E}))\to\mathrm{Hom}(V,\check{\mathrm{H}}^1(\mathcal{E}))$ *given by the isomorphism between Dolbeault and Čech cohomology is a linear isomorphism between the relative obstruction theories for* $p:\mathrm{Def}_L\to\mathrm{Def}_{A_X^{0,*}(D^1(\mathcal{E}))}$ *and* $\pi:\mathrm{Def}_{(X,\mathcal{E},V)}\to\mathrm{Def}_{(X,\mathcal{E})}$ *compatible with the morphisms* Φ_L *and* $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}$.

Proof. Let

$$(\tilde{x}_i,x)\in\mathrm{Def}_{A_X^{0,*}(D^1(\mathcal{E}))}(B)\times_{\mathrm{Def}_{A_X^{0,*}(D^1(\mathcal{E}))}(A)}\mathrm{Def}_L(A)\qquad\text{and}$$

$$(\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}(\tilde{x}_i),\Phi_L(x))=((\mathcal{X}_B,\mathcal{E}_B),(\mathcal{X}_A,\mathcal{E}_A,i_A))\in\mathrm{Def}_{(X,\mathcal{E})}(B)\times_{\mathrm{Def}_{(X,\mathcal{E})}(A)}\mathrm{Def}_{(X,\mathcal{E},V)}(A),$$

as in the previous analysis.

To find the element which corresponds to $H(x)$ in the Čech cohomology, proceed in the following way. Since $H(x)\in\mathrm{Hom}(V,H^1_{\bar{\partial}}(\mathcal{E}))\otimes J$, for all $v\in V$, $H(x)(v)$ is closed and so locally exact. Thus there exist an open covering $\mathcal{U}=\{U_a\}$ of X and elements $\tau_a(v)\in\mathcal{A}_X^{0,0}(\mathcal{E})|_{U_a}\otimes J$, such that $\bar{\partial}\tau_a(v)=H(x)(v)|_{U_a}$. Define $\sigma_{ab}(v)=\tau_a(v)-\tau_b(v)$, then $\bar{\partial}\sigma_{ab}(v)=\bar{\partial}\tau_a(v)-\bar{\partial}\tau_b(v)=H(x)(v)|_{U_a}-H(x)(v)|_{U_b}=0$ and $\{\delta\sigma(v)\}_{abc}=\sigma_{bc}(v)-\sigma_{ac}(v)+\sigma_{ab}(v)=\tau_b(v)-\tau_c(v)-\tau_a(v)+\tau_c(v)+\tau_a(v)-\tau_b(v)=0$, thus $\{\sigma_{ab}(v)\}_{ab}$ is a 1-Čech cocicle and we can consider its class $[\{\sigma_{ab}(v)\}_{ab}]\in\check{\mathrm{H}}^1(\mathcal{E})\otimes J$. The element which corresponds to $H(x)$ in Čech coomology is the morphism which associates to all $v\in V$ the class $[\{\sigma_{ab}(v)\}_{ab}]\in\check{\mathrm{H}}^1(\mathcal{E})\otimes J$.

Now calculate the relative obstruction of $((X_B, \mathcal{E}_B), (\mathcal{X}_A, \mathcal{E}_A, i_A))$. It is given by $\delta(i_A(-)) \in \operatorname{Hom}(V, \check{\mathrm{H}}^1(\mathcal{E})) \otimes J$, i.e. for all $v \in V$ by $\delta(i_A(v)) \in \check{\mathrm{H}}^1(\mathcal{E}) \otimes J$, and so it is usefull to analyze the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & C^0(\mathcal{E}) \arrow[r] \arrow[d] & C^0(\mathcal{E}_B) \arrow[r] \arrow[d] & C^0(\mathcal{E}_A) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^1(\mathcal{E}) \arrow[r] & C^1(\mathcal{E}_B) \arrow[r] & C^1(\mathcal{E}_A) \arrow[r] & 0 \end{tikzcd}$$

Let $v \in V$ and $i_A(v) = (i + x_{-1})(v) \in C^0(\mathcal{E}_A)$, consider $(i+h)(v)|_{U_a} - \tau_a(v)$, where h and τ_a are the above ones. Then $\alpha((i+h)(v)|_{U_a} - \tau_a(v)) = (i + x_{-1})(v)|_{U_a}$, because h is a lifting of x_{-1} and $\tau_a(v) \in \mathcal{A}_X^{0,0}(\mathcal{E})|_{U_a} \otimes J$; and $(\bar{\partial} + \tilde{x}_i) \circ ((i+h)(v)|_{U_a} - \tau_a(v)) = (\bar{\partial} + \tilde{x}_i) \circ (i+h)(v)|_{U_a} - \bar{\partial}\tau_a(v) - \tilde{x}_i \tau_a(v) = H(x)(v)|_{U_a} - \bar{\partial}\tau_a(v) = 0$, for the definitions of $H(x)(v)$ and $\tau_a(v)$ and because $J \cdot \mathfrak{m}_B = 0$. Thus $\{(i+h)(v)|_{U_a} - \tau_a(v)\}_a$ is a lifting of $i_A(v)$ in $C^0(\mathcal{E}_B)$.

Calculating the Čech differential of this element, obtain $\{\delta((i+h)(v)|_U - \tau(v))\}_{ab} = (\operatorname{Id} + h)(v)|_{U_b} - \tau_b(v) - (\operatorname{Id} + h)(v)|_{U_a} + \tau_a(v) = \sigma_{ab}(v)$. The cohomology class $[\{\sigma_{ab}(v)\}_{ab}] \in \check{\mathrm{H}}^1(\mathcal{E})$ is the obstruction $\delta(i_A(v))$. □

Now we can prove Theorem 4.1.10, using relative obstructions.

Proof. Let $0 \to J \to B \xrightarrow{\alpha} A \to 0$ be a small extension in $\mathbf{Art}_{\mathbb{C}}$, let $x \in \operatorname{Def}_L(A)$ and its image $\Phi_L(x) = (\mathcal{X}_A, \mathcal{E}_A, i_A) \in \operatorname{Def}_{(X,\mathcal{E},V)}(A)$, it is sufficient to prove that, if $(\mathcal{X}_A, \mathcal{E}_A, i_A)$ has a lifting $(\mathcal{X}_B, \mathcal{E}_B, i_B) \in \operatorname{Def}_{(X,\mathcal{E},V)}(B)$, then x has a lifting $\tilde{x} \in \operatorname{Def}_L(B)$.

Having in mind the diagram (4.8), let $x = (x_{-1}, x_i) \in \operatorname{Def}_L(A)$, $\Phi_L(x) = (\mathcal{X}_A, \mathcal{E}_A, i_A) \in \operatorname{Def}_{(X,\mathcal{E},V)}(A)$, $p(x) = x_i \in \operatorname{Def}_{A_X^{0,*}(D^1(\mathcal{E}))}(A)$ and $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}(p(x)) = \pi(\Phi_L(x)) = (\mathcal{X}_A, \mathcal{E}_A) \in \operatorname{Def}_{(X,\mathcal{E})}(A)$.

Suppose that it exists a lifting $(\mathcal{X}_B, \mathcal{E}_B, i_B) \in \operatorname{Def}_{(X,\mathcal{E},V)}(B)$ of $(\mathcal{X}_A, \mathcal{E}_A, i_A)$, thus $(\mathcal{X}_B, \mathcal{E}_B) \in \operatorname{Def}_{(X,\mathcal{E})}(B)$ is a lifting of $(\mathcal{X}_A, \mathcal{E}_A)$. For smoothness of $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}$, it exists a lifting $\tilde{x}_i \in \operatorname{Def}_{A_X^{0,*}(D^1(\mathcal{E}))}(B)$ of x_i and the relative obstruction of $(\tilde{x}_i, x)$ is $H(x) \in \operatorname{Hom}(V, H^1_{\bar{\partial}}(\mathcal{E})) \otimes J$. Its image $(\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}(\tilde{x}_i), \Phi_L(x)) = ((\mathcal{X}_B, \mathcal{E}_B), (\mathcal{X}_A, \mathcal{E}_A, i_A))$ has a lifting $(\mathcal{X}_B, \mathcal{E}_B, i_B) \in \operatorname{Def}_{(X,\mathcal{E},V)}(B)$, so its relative obstruction is $\delta(i_A(-)) = 0 \in \operatorname{Hom}(V, \check{\mathrm{H}}^1(\mathcal{E})) \otimes J$. Since the relative obstructions are isomorphic, also $H(x)$ must be zero, moreover, since the obstruction of a deformation functor associated to a DGLA is complete, a lifting $\tilde{x} \in \operatorname{Def}_L(B)$ of x must exist. □

Corollary 4.1.12. *The morphism $\Phi_M : \operatorname{Def}_M \to \operatorname{Def}_{(\mathcal{E},V)}$ is smooth.*

Proof. Arguments are the same as in the previous case. Instead of diagram (4.8), consider the following:

$$\begin{tikzcd} \operatorname{Def}_M \arrow[r, "\Phi_M"] \arrow[d, "b"] & \operatorname{Def}_{(\mathcal{E},V)} \arrow[d, "\pi"] \\ \operatorname{Def}_{A_X^{0,*}(\operatorname{End}\mathcal{E})} \arrow[r, "\Phi_{A_X^{0,*}(\operatorname{End}\mathcal{E})}"] & \operatorname{Def}_{\mathcal{E}} . \end{tikzcd}$$

Let $x = (x_{-1}, x_i) \in \operatorname{MC}_M(A)$, $\Phi_M(x) = (\mathcal{E}_A, i_A) \in \operatorname{Def}_{(\mathcal{E},V)}(A)$, $b(x) = x_i \in \operatorname{MC}_{A_X^{0,*}(\operatorname{End}\mathcal{E})}(A)$ and $\Phi_{A_X^{0,*}(\operatorname{End}\mathcal{E})}(b(x)) = \pi(\Phi_M(x)) = \mathcal{E}_A \in \operatorname{Def}_{\mathcal{E}}(A)$.

Suppose that it exists a lifting $(\mathcal{E}_B, i_B) \in \operatorname{Def}_{(\mathcal{E},V)}(B)$ of $(\mathcal{E}_A, i_A)$, thus $\mathcal{E}_B \in \operatorname{Def}_{\mathcal{E}}(B)$ is a

lifting of $\mathcal{E}_A$ and, for smoothness of $\Phi_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}$, it exists a lifting $\tilde{x}_i \in \mathrm{MC}_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}(B)$ of x_i.
Consider the homomorphism $z = i_B - i$ and define $\tilde{x} = (z, \tilde{x}_i) \in \mathrm{Hom}(Q, Q) \otimes B$. As before $\tilde{x}$ satisfies the Maurer-Cartan equation in the DGLA $M \otimes \mathfrak{m}_B$ and $\alpha(\tilde{x}) = x$.

Also relative obstruction theory can be done in this case. As before analyze relative obstructions for $b : \mathrm{Def}_M \to \mathrm{Def}_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}$ and $\pi : \mathrm{Def}_{(\mathcal{E},V)} \to \mathrm{Def}_{\mathcal{E}}$.
The relative obstruction of an element $x = (x_{-1}, x_i) \in \mathrm{MC}_M(A)$, such that x_i has a lifting $\tilde{x}_i \in MC_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}(B)$, is $H(x) \in \mathrm{Hom}(V, H^1_{\bar\partial}(\mathcal{E})) \otimes J$. The relative obstruction of an element $(\mathcal{E}_A, i_A) \in \mathrm{Def}_{(\mathcal{E},V)}(A)$, such that $\mathcal{E}_A$ has a lifting $\mathcal{E}_B \in \mathrm{Def}_{\mathcal{E}}(B)$, is $\delta(i_A(-)) \in \mathrm{Hom}(V, \check{\mathrm{H}}^1(\mathcal{E})) \otimes J$.
Lemma 4.1.11 is still true for the morphisms Φ_M and $\Phi_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}$. Since $\Phi_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}$ is an isomorphims (see Example 3.2.15), it is smooth and the assertion is proved. □

Putting together Theorem 4.1.8 and Theorem 4.1.10 and remembering that smoothness implies surjectivity, the following holds

Theorem 4.1.13 (Main Theorem)**.** *The morphism $\Phi_L : \mathrm{Def}_L \to \mathrm{Def}_{(X,\mathcal{E},V)}$ is an isomorphism of deformation functors, thus the DGLA L governs the deformations of the triple $(X, \mathcal{E}, V)$.*

Putting together Corollary 4.1.9 and Corollary 4.1.12, the following holds

Corollary 4.1.14. *The morphism $\Phi_M : \mathrm{Def}_M \to \mathrm{Def}_{(\mathcal{E},V)}$ is an isomorphism of deformation functors, thus the DGLA M governs the deformations of the pair $(\mathcal{E}, V)$.*

4.2 Deformations of (manifold, sheaf, subspaces of cohomology)

Let X be a compact and connected complex manifold of dimension n, let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-module on X and, for all $i = 1 \ldots n$, let $\varphi^i : V^i \to H^i(X, \mathcal{E})$ be linear morphisms from fixed finite dimensional vector spaces V^i to the cohomology spaces of X with coefficients in the sheaf $\mathcal{E}$.
In the following we indicate all these data, the manifold X, the sheaf $\mathcal{E}$ and the subspaces of the cohomology of X with coefficients in $\mathcal{E}$ seattled by the morphisms φ^i, simply with $(X, \mathcal{E}, V^i)$ forgetting the morphisms φ^i, if there is no ambiguity.
This section is devoted to study infinitesimal deformations of the data $(X, \mathcal{E}, V^i)$ and to define a DGLA which governs these deformations.

Definition 4.2.1. *An* infinitesimal deformation *of $(X, \mathcal{E}, V^i)$, i.e. an infinitesimal deformation of the manifold X and of the sheaf $\mathcal{E}$ with the subspaces of the cohomology seattled by the morphisms $\varphi^i : V^i \to H^i(X, \mathcal{E})$, over $A \in \mathbf{Art}_{\mathbb{C}}$ is the data of:*

- *a deformation $\mathcal{X}_A$ of the manifold X over A;*

- *a locally free sheaf $\mathcal{E}_A$ of $\mathcal{O}_{\mathcal{X}_A}$-modules on $\mathcal{X}_A$, with a morphism of sheaves $\pi_A : \mathcal{E}_A \to \mathcal{E}$ such that $\pi_A : \mathcal{E}_A \otimes \mathbb{C} \to \mathcal{E}$ is an isomorphism;*

- *for all $i = 1 \dots n$, a morphism $\varphi^i_A : V^i \otimes A \to H^i(\mathcal{X}_A, \mathcal{E}_A)$, such that the following diagram is commutative:*

$$\begin{tikzcd} V^i \otimes A \arrow[r, "\varphi^i_A"] \arrow[d, "\pi"] & H^i(\mathcal{X}_A, \mathcal{E}_A) \arrow[d, "\pi_A"] \\ V^i \arrow[r, "\varphi^i"] & H^i(X, \mathcal{E}). \end{tikzcd} \tag{4.9}$$

Definition 4.2.2. *Two infinitesimal deformations $(\mathcal{X}_A, \mathcal{E}_A, \varphi^i_A)$ and $(\mathcal{X}'_A, \mathcal{E}'_A, \varphi'^i_A)$ of $(X, \mathcal{E}, V^i)$ over $A \in \mathbf{Art}_{\mathbb{C}}$ are called* isomorphic *if:*

- *$\exists\ \phi : \mathcal{X}_A \to \mathcal{X}'_A$ isomorphism of deformations of the manifold X over A;*
- *$\exists\ \psi : \mathcal{E}_A \to \mathcal{E}'_A$ isomorphism of sheaves of $\mathcal{O}_{\mathcal{X}_A}$-modules, where the structure of sheaf of $\mathcal{O}_{\mathcal{X}_A}$-module on $\mathcal{E}'_A$ is the one induced by ϕ, such that $\pi'_A \circ \psi = \pi_A$;*
- *for all $i = 1 \dots n$, $\exists\ \zeta^i : V^i \otimes A \to V^i \otimes A$ isomorphism which makes the following diagram commutative:*

$$\begin{tikzcd} V^i \otimes A \arrow[r, "\varphi^i_A"] \arrow[d, "\zeta^i"] & H^i(\mathcal{X}_A, \mathcal{E}_A) \arrow[d, "\psi"] \\ V^i \otimes A \arrow[r, "\varphi'^i_A"] & H^i(\mathcal{X}'_A, \mathcal{E}'_A). \end{tikzcd} \tag{4.10}$$

Definition 4.2.3. *The* functor of infinitesimal deformations of $(X, \mathcal{E}, V^i)$ *is the functor of Artin rings:*

$$\mathrm{Def}_{(X,\mathcal{E},V^i)} : \mathbf{Art}_{\mathbb{C}} \longrightarrow \mathbf{Set}$$

which, to every $A \in \mathbf{Art}_{\mathbb{C}}$, associates the set of the isomorphism classes of infinitesimal deformations of $(X, \mathcal{E}, V^i)$ over A.

Let $\mathcal{C}$ be the complex of sheaves:

$$\mathcal{C}: \qquad 0 \longrightarrow V^0 \xrightarrow{\varphi^0} \mathcal{A}^{0,0}_X(\mathcal{E}) \oplus V^1 \xrightarrow{(\bar\partial, \varphi^1)} \mathcal{A}^{0,1}_X(\mathcal{E}) \oplus V^2 \longrightarrow \cdots \tag{4.11}$$

$$\cdots \longrightarrow \mathcal{A}^{0,i-1}_X(\mathcal{E}) \oplus V^i \xrightarrow{(\bar\partial, \varphi^i)} \mathcal{A}^{0,i}_X(\mathcal{E}) \oplus V^{i+1} \longrightarrow \cdots,$$

where $\bar\partial$ is the Dolbeault differential and φ^i are liftings of the above morphisms $\varphi^i : V^i \to H^i(X, \mathcal{E})$ to $\mathcal{A}^{0,i}_X(\mathcal{E})$. Let $\mathrm{Hom}(\mathcal{C}, \mathcal{C})$ be the DGLA of the homomorphisms of this complex, in which the differential and the bracket are defined as in Example 3.1.4. We define the DGLA F as

$$F^0 = \{x \in \mathrm{Hom}^0(\mathcal{C}, \mathcal{C}) \mid x|_{\mathcal{A}^{0,*}_X(\mathcal{E})} \in A^{0,0}_X(D^1(\mathcal{E})) \ \text{ and } \ x|_{V^*} : V^* \to V^*\},$$

$$F^i = \{x \in \mathrm{Hom}^i(\mathcal{C}, \mathcal{C}) \mid x|_{\mathcal{A}^{0,*}_X(\mathcal{E})} \in A^{0,i}_X(D^1(\mathcal{E})) \ \text{ and } \ x|_{V^*} : V^* \to \mathcal{A}_X(\mathcal{E})^{*+i-1}\} \ \text{ if } i \geq 1,$$

in which the structure of DGLA is the one inherited by $\mathrm{Hom}^*(\mathcal{C}, \mathcal{C})$. We indicate an element of F as $x = (x_{-1}, x_i)$, where $x_{-1} = x|_{V^0}$ and $x_i = x|_{\mathcal{A}^{0,i}_X(\mathcal{E}) \oplus V^{i+1}}$, for $i \geq 0$.
Remembering definitions of Section 4.1, note that, if $\varphi^0 : V^0 \to H^0(X, \mathcal{E})$ is the inclusion of a subspace of global sections of $\mathcal{E}$, the restriction of an element $x \in F$ to the subcomplex $Q \subset \mathcal{C}$ is an element of the DGLA $L : x|_Q = (x_{-1}, x_i|_{\mathcal{A}^{0,i}_X(\mathcal{E})}) \in L$.

Definition 4.2.4. *Let*

$$\mathrm{Def}_F : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$$

be the deformation functor associated to the DGLA F.

Remark 4.2.5. The complex $\mathcal{C}$ is defined using some liftings of the morphisms φ^i, indicated above again with φ^i. We prove that, if $\phi^i, \psi^i : V^i \to Z^i(X,\mathcal{E})$ are two different liftings of the morphism $\varphi^i : V^i \to H^i(X,\mathcal{E})$, the complexes $\mathcal{C}_\phi$ and $\mathcal{C}_\psi$, defined as in (4.11) with the liftings ϕ^i and ψ^i respectively, are isomorphic and the DGLAs F_ϕ and F_ψ, defined as above using the complexes $\mathcal{C}_\phi$ and $\mathcal{C}_\psi$ respectively, are isomorphic.
Let $\Lambda : \mathcal{C}_\phi \to \mathcal{C}_\psi$ be the morphism given by

$$\Lambda|_{V^0} = \mathrm{Id} \quad \text{and} \quad \Lambda(x,v) = (x + z_v, v), \quad \text{for all} \;\; (x,v) \in \mathcal{A}_X^{0,i}(\mathcal{E}) \oplus V^{i+1} \; i \geq 0,$$

where z_v is an element in $\mathcal{A}_X^{0,i}(\mathcal{E})$, such that $\bar{\partial}(z_v) = (\phi^{i+1} - \psi^{i+1})(v)$, that exists because ϕ^{i+1} and ψ^{i+1} are liftings of the same morphism φ^{i+1}. It is easy to verify that Λ commutes with the differentials of the complexes. Moreover it has an inverse Θ defined by

$$\Theta|_{V^0} = \mathrm{Id} \quad \text{and} \quad \Theta(x,v) = (x - z_v, v) \;\; \text{for all} \;\; (x,v) \in \mathcal{A}_X^{0,i}(\mathcal{E}) \oplus V^{i+1}, \;\; i \geq 0.$$

Now let $\bar{\Lambda} : \mathrm{Hom}(\mathcal{C}_\phi, \mathcal{C}_\phi) \to \mathrm{Hom}(\mathcal{C}_\psi, \mathcal{C}_\psi)$ be the morphism of DGLAs given by $\bar{\Lambda}(f) = \Lambda \circ f \circ \Lambda^{-1}$. It is easy to verify that $\bar{\Lambda}$ is well defined as morphism from F_ϕ to F_ψ. Moreover $\bar{\Lambda}$ has an inverse $\bar{\Theta} = \Theta \circ f \circ \Theta^{-1}$, which is well defined from F_ψ to F_ϕ, so it is an isomorphism between the two DGLAs F_ϕ and F_ψ.
This assures that the deformation functor Def_F do not depend, unless isomorphism, on the choice of the liftings of the morphisms φ^i with which we construct the complex $\mathcal{C}$ and the DGLA F, in what follows we indicate these liftings again with φ^i.

Our aim is to prove that the functors Def_F and $\mathrm{Def}_{(X,\mathcal{E},V^i)}$ are isomorphic. We start by defining a morphism of functors:

$$\Phi_F : \mathrm{MC}_F \longrightarrow \mathrm{Def}_{(X,\mathcal{E},V^i)};$$

for all $A \in \mathbf{Art}_{\mathbb{C}}$ and for all elements $x = (x_{-1}, x_i) \in \mathrm{MC}_F(A)$, $\Phi_F(x)$ is defined to be the isomorphism class of the deformation of $(X,\mathcal{E},V^i)$ over A given by $(\mathcal{X}_A, \mathcal{E}_A, \varphi^i_A)$, where:

- $\mathcal{X}_A = \ker(\bar{\partial} + \mathfrak{l}_{\sigma(x_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})})} : \mathcal{A}_X^{0,0} \otimes A \longrightarrow \mathcal{A}_X^{0,1} \otimes A)$, where $\mathfrak{l}$ is the Lie derivative,
- $\mathcal{E}_A = \ker(\bar{\partial} + x_0|_{\mathcal{A}_X^{0,0}(\mathcal{E})} : \mathcal{A}_X^{0,0}(\mathcal{E}) \otimes A \longrightarrow \mathcal{A}_X^{0,1}(\mathcal{E}) \otimes A)$,
- $\varphi^i_A = \varphi^i + x_{i-1}|_{V^i} : V^i \otimes A \longrightarrow H^i(\mathcal{X}_A, \mathcal{E}_A)$.

Lemma 4.2.6. *The above morphism $\Phi_F : \mathrm{MC}_F \longrightarrow \mathrm{Def}_{(X,\mathcal{E},V^i)}$ is well defined.*

Proof. Let $x = (x_{-1}, x_i) \in \mathrm{MC}_F(A)$ and let $z = x|_Q = (x_{-1}, x_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})}) \in L$ be its restriction to the complex Q. Note that z satisfies the Maurer-Cartan equation in the DGLA L, in fact

$$0 = \left(dx + \frac{1}{2}[x,x] \right) |_Q = dz + \frac{1}{2}[z,z].$$

Thus Lemma 4.1.5 assures that the above $\mathcal{X}_A$ is a deformation of X over A and the above $\mathcal{E}_A$ is a locally free shaef of $\mathcal{O}_{\mathcal{X}_A}$-modules on $\mathcal{X}_A$, such that the obvious map $\pi_A : \mathcal{E}_A \to \mathcal{E}$, given by the projection onto the residue field, gives an isomorphism $\pi_A : \mathcal{E}_A \otimes \mathbb{C} \to \mathcal{E}$. Now analyse the above maps φ^i_A. Their images are contained in $Z^i(X, \mathcal{E}_A)$, in fact, for all $v \in V^i$, we have:

$$(\bar{\partial} + x_i|_{\mathcal{A}^{(0,i)}_X(\mathcal{E})}) \circ (\varphi^i + x_{i-1})(v) = \bar{\partial}\varphi^i(v) + \bar{\partial}x_{i-1}(v) + x_i\varphi_i(v) + x_i x_{i-1}(v),$$

on the other hand, the Maurer-Cartan equation satisfied by x, gives:

$$\begin{aligned} 0 = dx + \frac{1}{2}[x, x](v) &= x_i \circ (\bar{\partial}, \varphi_i)(v) + (\bar{\partial}, \varphi_{i+1}) \circ x_{i-1}(v) + x_i \circ x_{i-1}(v) = \\ &= x_i(\varphi_i(v)) + \bar{\partial}(x_{i-1}(v)) + x_i \circ x_{i-1}(v), \end{aligned}$$

thus $(\bar{\partial} + x|_{\mathcal{A}^{(0,i)}_X(\mathcal{E})}) \circ (\varphi^i + x_{i-1}) = 0$ and so the morphisms $\varphi^i : V^i \otimes A \to H^i(X, \mathcal{E}_A)$ are well defined. Lastly, the maps π_A and φ^i_A make the diagram (4.9) commutative, because $\varphi^i_A = \varphi^i + x_{i-1}|_{V^i}$, with $x_{i-1}|_{V^i} : V^i \otimes A \to \mathcal{A}^i_X(\mathcal{E}) \otimes \mathfrak{m}_A$. □

Lemma 4.2.7. *The above morphism Φ_F induces a morphism between the deformation functors $\Phi_F : \mathrm{Def}_F \to \mathrm{Def}_{(X,\mathcal{E},V^i)}$.*

Proof. Let $x = (x_{-1}, x_i)$ and $y = (y_{-1}, y_i)$ be gauge equivalent elements in $\mathrm{MC}_F(A)$ and let $(\mathcal{X}_A, \mathcal{E}_A, \varphi^i_A)$ and $(\mathcal{X}'_A, \mathcal{E}'_A, \varphi'^i_A)$ be their images via Φ_F. By definition of gauge relation, it exists an element $a = (a_{-1}, a_i) \in F^0 \otimes \mathfrak{m}_A$, such that:

$$y = e^a * x = x + \sum_{n=0}^{+\infty} \frac{([a, -])^n}{(n+1)!}([a, x] - da). \tag{4.12}$$

This gauge relation restricted to the complex Q becomes

$$y|_Q = (e^a * x)\,|_Q = e^{a|_Q} * x|_Q,$$

that is the gauge relation for the element $x|_Q$ and $x|_Q$ in $\mathrm{MC}_F(A)$. Thus, for Lemma 4.1.6, the deformations $\mathcal{X}_A$ and $\mathcal{X}_A$ of X over A are isomorphic and there exists an isomorphism of sheaves of $\mathcal{O}_{\mathcal{X}_A}$-modules between $\mathcal{E}_A$ and $\mathcal{E}'_A$.

Now analyse the maps φ^i_A and φ'^i_A. The restriction of the gauge relation (4.12) to the subcomplex $V^* \subset \mathcal{C}$, gives:

$$\begin{aligned} y(v) &= e^a * x(v) = x_i(v) + \sum_{n=0}^{\infty} \frac{([a, -])^n}{(n+1)!}([a, x](v) - da(v)) = \qquad (4.13) \\ &= x_i(v) + \sum_{n=0}^{\infty} \frac{([a, -])^n}{(n+1)!} \big([a, x](v) - ((\bar{\partial}, \varphi^{i+1})a_i(v) - a_{i+1}(\bar{\partial}, \varphi_{i+1})(v))\big) = \\ &= x_i(v) + \sum_{n=0}^{\infty} \frac{([a, -])^n}{(n+1)!} \big([a, x](v) - (\varphi^{i+1}a_i(v) - a_{i+1}\varphi_{i+1}(v))\big) = \\ &= x_i(v) + \sum_{n=0}^{\infty} \frac{([a, -])^n}{(n+1)!} ([a, x](v) + [a, \varphi](v)) = x_i(v) + \sum_{n=1}^{\infty} \frac{([a, -])^n}{n!} \big(x_i(v) + \varphi^{i+1}(v)\big) \\ &= \sum_{n=0}^{\infty} \frac{([a, -])^n}{n!} \big(x_i(v) + \varphi^{i+1}(v)\big) - \varphi^{i+1}(v) = e^{[a,-]}(x_i + \varphi^{i+1})(v) - \varphi^{i+1}(v), \end{aligned}$$

for all $v \in V^{i+1}$. Thus $\varphi^{i+1} + y_i = e^{[a,-]}(\varphi^{i+1} + x_i)$, that is $\varphi_A'^{i+1} = e^{a_{i+1}} \circ \varphi_A^{i+1} \circ e^{-a_i}$. Then e^{a_i} are isomorphisms of $V^{i+1} \otimes A$ in its self, which make commutative the diagram (4.10).

Therefore, if x and y are gauge equivalent elements in $\mathrm{MC}_F(A)$, the associated deformations $\Phi_F(x)$ and $\Phi_F(y)$ are isomorphic and Φ_F is well defined on deformation functors. $\square$

Remark 4.2.8. The above calculations can be used also in a simpler case.
Let X be a compact and connected complex manifold of dimension n, let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-module on X and, for all $i = 1 \dots n$, let $\varphi^i : V^i \to H^i(X, \mathcal{E})$ be linear morphisms from fixed finite dimensional vector spaces V^i to the cohomology spaces of X with coefficients in the sheaf $\mathcal{E}$.

Consider *infinitesimal deformations of* $(\mathcal{E}, V^i)$, i.e. infinitesimal deformations of the sheaf $\mathcal{E}$ with the subspaces of cohomology seattled by the morphisms φ^i, over $A \in \mathbf{Art}_{\mathbb{C}}$. They are defined as in Definition 4.2.1, with trivial deformation of the manifold X, i.e. $\mathcal{X}_A = X \times \operatorname{Spec} A$. Two of such deformations are isomorphic if they satisfy conditions of Definition 4.2.2, where the isomorphism between the trivial deformations of the manifold X is the identity. Let

$$\mathrm{Def}_{(\mathcal{E},V^i)} : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$$

be the functor of deformations of $(\mathcal{E}, V^i)$.

Let N be the DGLA defined as

$$N^0 = \{x \in \mathrm{Hom}^0(\mathcal{C},\mathcal{C}) \mid x|_{\mathcal{A}_X^{0,*}(\mathcal{E})} \in A_X^{0,0}(\operatorname{End}\mathcal{E}) \quad \text{and} \quad x|_{V^*} : V^* \to V^*\},$$

$$N^i = \{x \in \mathrm{Hom}^i(\mathcal{C},\mathcal{C}) \mid x|_{\mathcal{A}_X^{0,*}(\mathcal{E})} \in A_X^{0,i}(\operatorname{End}\mathcal{E}) \quad \text{and} \quad x|_{V^*} : V^* \to \mathcal{A}_X(\mathcal{E})^{*+i-1}\} \quad \text{if} \quad i \geq 1,$$

in which the structure of DGLA is the one inherited by $\mathrm{Hom}^*(\mathcal{C},\mathcal{C})$. Let

$$\mathrm{Def}_N : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$$

be the deformation functor associated to N.

Let

$$\Phi_N : \mathrm{Def}_N \to \mathrm{Def}_{(\mathcal{E},V^i)}$$

be the morphism which associates, to every $A \in \mathbf{Art}_{\mathbb{C}}$ and to every element $x = (x_{-1} = x|_{V^0}, x_i = x|_{\mathcal{A}_X^{0,i} \oplus V^{i+1}}) \in \mathrm{MC}_N(A)$, the isomorphism class of the deformation of the data $(\mathcal{E}, V^i)$ over A given by $(\mathcal{E}_A, \varphi_A^i)$, where:

- $\mathcal{E}_A = \ker(\bar{\partial} + x_0|_{\mathcal{A}_X^{0,0}(\mathcal{E})} : \mathcal{A}_X^{0,0}(\mathcal{E}) \otimes A \longrightarrow \mathcal{A}_X^{0,1}(\mathcal{E}) \otimes A)$,
- $\varphi_A^i = \varphi^i + x_{i-1}|_{V^i} : V^i \otimes A \longrightarrow H^i(\mathcal{X}_A, \mathcal{E}_A)$.

The above calculations assure that Φ_N is a well defined morphism of deformation functors.

Theorem 4.2.9. *The morphism* $\Phi_F : \mathrm{Def}_F \to \mathrm{Def}_{(X,\mathcal{E},V^i)}$ *is injective.*

Proof. Let $x = (x_{-1}, x_i)$ and $y = (y_{-1}, y_i)$ in $\mathrm{MC}_F(A)$ such that the two deformations $\Phi_F(x) = (\mathcal{X}_A, \mathcal{E}_A, \varphi_A^i)$ and $\Phi_F(y) = (\mathcal{X}_A', \mathcal{E}_A', \varphi_A'^i)$, defined as above, are isomorphic.

Then there exist isomorphisms ϕ, ψ and ζ^i as in Definition 4.2.2.
Consider the following diagram

$$\begin{tikzcd} \mathrm{Def}_F \arrow[r, "\Phi_F"] \arrow[d, "r"] & \mathrm{Def}_{(X,\mathcal{E},V^i)} \arrow[d, "\pi"] \\ \mathrm{Def}_{A_X^{0,*}(D^1(\mathcal{E}))} \arrow[r, "\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}"] & \mathrm{Def}_{(X,\mathcal{E})}, \end{tikzcd} \tag{4.14}$$

where the morphism r is given, for all $x = (x_{-1}, x_i) \in F^1 \otimes \mathfrak{m}_A$, by $r(x) = x_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})}$, π is given by $\pi(\mathcal{X}_A, \mathcal{E}_A, \varphi_A^i) = (\mathcal{X}_A, \mathcal{E}_A)$, for all deformations, and the morphism $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}$ is defined in Example 3.2.16. These maps are well defined on deformation functors and make the above diagram commutative.
The deformations $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}(x_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})}) = (\mathcal{X}_A, \mathcal{E}_A)$ and $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}(y_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})}) = (\mathcal{X}'_A, \mathcal{E}'_A)$ of the pair $(X, \mathcal{E})$ over A are isomorphic. Since $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}$ is an isomorphism (see Example 3.2.16), the elements $x_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})}$ and $y_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})}$ are gauge equivalent in the DGLA $A_X^{0,*}(D^1(\mathcal{E}))$, i.e. there exists $a \in A_X^{0,0}(D^1(\mathcal{E})) \otimes \mathfrak{m}_A$ such that $e^a * x_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})} = y_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})}$.
Then the isomorphism $\psi : \mathcal{E}_A \to \mathcal{E}'_A$ can be lifted to an isomorphism of complexes $e^a : (\mathcal{A}_X^{0,*}(\mathcal{E}) \otimes A, \bar{\partial} + x_i) \to (\mathcal{A}_X^{0,*}(\mathcal{E}) \otimes A, \bar{\partial} + y_i)$.
Moreover all the isomorphisms ζ^i can be written as e^{b^i}, with $b^i \in \mathrm{Hom}(V^i, V^i) \otimes \mathfrak{m}_A$, because they are the identity on the residue field $A/\mathfrak{m}_A \cong \mathbb{C}$.
Let c the element in $F^0 \otimes \mathfrak{m}_A$ given by $c|_{\mathcal{A}_X^{0,*}(\mathcal{E})} = a$ and $c|_{V^i} = b^i$, then e^c is an isomorphism which makes commutative the diagram (4.10). As verified in (4.13), the fact that e^{b^i} and e^a make commutative the diagram (4.10), is equivalent to the fact that $y|_{V^i} = e^c * x|_{V^i}$.
Then $x = y \in \mathrm{Def}_F(A)$ and the morphism Φ_F is injective. □

Corollary 4.2.10. *The morphism $\Phi_N : \mathrm{Def}_N \to \mathrm{Def}_{(\mathcal{E},V^i)}$ is injective.*

Proof. The proof is the same as the previous one, where now the deformations $\mathcal{X}_A$ and $\mathcal{X}'_A$ are trivial. □

Theorem 4.2.11. *The morphism $\Phi_F : \mathrm{Def}_F \to \mathrm{Def}_{(X,\mathcal{E},V^i)}$ is smooth.*

Proof. The morphism Φ_F is smooth if and only if, given a small extension in $\mathbf{Art}_{\mathbb{C}}$, $0 \to J \to B \xrightarrow{\alpha} A \to 0$, an element $x \in \mathrm{Def}_F(A)$ and its image $\Phi_F(x) = (\mathcal{X}_A, \mathcal{E}_A, \varphi_A^i) \in \mathrm{Def}_{(X,\mathcal{E},V^i)}(A)$, $(\mathcal{X}_A, \mathcal{E}_A, \varphi_A^i)$ has a lifting $(\mathcal{X}_B, \mathcal{E}_B, \varphi_B^i) \in \mathrm{Def}_{(X,\mathcal{E},V^i)}(B)$ if and only if x has a lifting $\tilde{x} \in \mathrm{Def}_L(B)$.
It is clear that if $x \in \mathrm{Def}_F(A)$ has a lifting $\tilde{x} \in \mathrm{Def}_F(B)$, then $\Phi_F(\tilde{x})$ lifts $\Phi_F(x)$. Thus it is sufficient to prove the other implication.
Consider the diagram

$$\begin{tikzcd} \mathrm{Def}_F \arrow[r, "\Phi_F"] \arrow[d, "r"] & \mathrm{Def}_{(X,\mathcal{E},V^i)} \arrow[d, "\pi"] \\ \mathrm{Def}_{A_X^{0,*}(D^1(\mathcal{E}))} \arrow[r, "\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}"] & \mathrm{Def}_{(X,\mathcal{E})}, \end{tikzcd}$$

defined as in (4.14), let $x = (x_{-1}, x_i) \in \mathrm{MC}_F(A)$, $\Phi_F(x) = (\mathcal{X}_A, \mathcal{E}_A, \varphi_A^i) \in \mathrm{Def}_{(X,\mathcal{E},V^i)}(A)$, $r(x) = x_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})} \in \mathrm{MC}_{A_X^{0,*}(D^1(\mathcal{E}))}(A)$ and $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}(r(x)) = \pi(\Phi_F(x)) = (\mathcal{X}_A, \mathcal{E}_A) \in$

$\mathrm{Def}_{(X,\mathcal{E})}(A)$.
Suppose that it exists a lifting $(\mathcal{X}_B, \mathcal{E}_B, \varphi^i_B) \in \mathrm{Def}_{(X,\mathcal{E},V^i)}(B)$ of $(\mathcal{X}_A, \mathcal{E}_A, \varphi^i_A)$, thus $(\mathcal{X}_B, \mathcal{E}_B) \in \mathrm{Def}_{(X,\mathcal{E})}(B)$ is a lifting of $(\mathcal{X}_A, \mathcal{E}_A)$. Since $\Phi_{A^{0,*}_X(D^1(\mathcal{E}))}$ is an isomorphism (see Example 3.2.16), it is smooth and it exists a lifting $y \in \mathrm{MC}_{A^{0,*}_X(D^1(\mathcal{E}))}(B)$ of $x_i|_{\mathcal{A}^{0,i}_X(\mathcal{E})}$.
To obtain a lifting $\tilde{x} \in \mathrm{MC}_F(B)$ of x, consider homomorphisms $z_{i-1} = \varphi^i_B - \varphi^i$: $V^i \otimes B \to \mathcal{A}^{0,i}_X(\mathcal{E}) \otimes B$ and define $\tilde{x} \in \mathrm{Hom}(\mathcal{C},\mathcal{C}) \otimes \mathfrak{m}_B$ to be given by $\tilde{x}|_{\mathcal{A}^{0,*}_X(\mathcal{E})} = y$ and $\tilde{x}|_{V^i} = z_{i-1}$.
For all i, the image of z_{i-1} via α is given by

$$\alpha(z_{i-1})(v\otimes b) = \alpha(\varphi^i_B - \varphi^i)(v\otimes b) = \alpha\varphi^i_B(v\otimes b) - \alpha\varphi^i(v\otimes b) = \varphi^i_A(v\otimes\alpha(b)) - \varphi^i(v)\otimes\alpha(b) =$$

$$= (\varphi^i + x_{i-1}|_{V^i})(v\otimes\alpha(b)) - \varphi^i(v)\otimes\alpha(b) = x_{i-1}|_{V^i}(v\otimes\alpha(b)) \quad \text{for} \quad v\otimes b \in V\otimes B,$$

then z_{i-1} is a lifting of $x_{i-1}|_{V^i}$.
Moreover $\tilde{x}$ satisfies the Maurer-Cartan equation in the DGLA $F\otimes\mathfrak{m}_B$. Infact, since $y \in \mathrm{MC}_{A^{0,*}_X(D^1(\mathcal{E}))}(B)$, it is sufficient to verify the Maurer-Cartan equation for $\tilde{x}$ restricted to $V^i \otimes B$, for all $v \in V^i \otimes B$ the following holds:

$$d\tilde{x} + \frac{1}{2}[\tilde{x},\tilde{x}](v) = \bar{\partial} z_{i-1}(v) + y\circ\varphi^i(v) + y\circ z_{i-1}(v) = (\bar{\partial}+y)\circ(\varphi^i + z_{i-1})(v) = (\bar{\partial}+y)\circ\varphi^i_B(v) = 0,$$

since $\varphi^i_B : V^i \otimes B \to H^i(X, \mathcal{E}_B)$. Then $\tilde{x}$ is a lifting of x. □

It could be of some interests the analisis of relative obstruction theory.
Let $0 \to J \to B \xrightarrow{\alpha} A \to 0$ be a small extension in $\mathbf{Art}_{\mathbb{C}}$ and consider the commutative diagram (4.14).

Analyze obstructions for the functor Def_F and relative obstructions for the morphism r. Given an element $x \in \mathrm{MC}_F(A)$, if $\tilde{x}$ is a generic lifting of x, i.e. $\tilde{x} \in F^1 \otimes \mathfrak{m}_B$ such that $\alpha(\tilde{x}) = x$, then the obstruction of x is $H(x) = d\tilde{x} + \frac{1}{2}[\tilde{x},\tilde{x}] \in H^2(F) \otimes J$. Given an element $x = (x_{-1}, x_i) \in \mathrm{MC}_F(A)$, now suppose that a lifting $y \in \mathrm{MC}_{A^{0,*}_X(D^1(\mathcal{E}))}(B)$ of $x_i|_{\mathcal{A}^{0,i}_X(\mathcal{E})}$ exists, then take generic liftings $z_{i-1} : V^i \otimes B \to \mathcal{A}^{0,i}_X(\mathcal{E}) \otimes B$ of $x_{i-1}|_{V^i}$ and consider the lifting $\tilde{x} \in F^1 \otimes \mathfrak{m}_B$ of x, given by $\tilde{x}|_{\mathcal{A}^{0,*}_X(\mathcal{E})} = y$ and $\tilde{x}|_{V^i} = z_{i-1}$. In this case the relative obstruction of (y, x) is $H(x) = d\tilde{x} + \frac{1}{2}[\tilde{x},\tilde{x}] \in \mathrm{Hom}^2(\mathcal{C},\mathcal{C}) \otimes J$, it is non zero only on the sets V^i and it given by $H(x)|_{V^i} = (\bar{\partial} + y) \circ (\varphi^i + z_{i-1}) \in \mathrm{Hom}(V^i, \mathcal{A}^{0,i+1}_X(\mathcal{E})) \otimes J$, for all i. Then the relative obstruction of (y, x) is $H(x) \in \bigoplus_i \mathrm{Hom}(V^i, \mathcal{A}^{0,i+1}_X(\mathcal{E}))$ and, to eliminate the dipendence on the choice of the lifting, consider its class in $\bigoplus_i \mathrm{Hom}(V^i, H^{i+1}_{\bar{\partial}}(\mathcal{E})) \otimes J$.

Now analyze relative obstructions for the morphism π. Given an element $(\mathcal{X}_A, \mathcal{E}_A, \varphi^i_A) \in \mathrm{Def}_{(X,\mathcal{E},V^i)}(A)$, suppose that a lifting $(\mathcal{X}_B, \mathcal{E}_B) \in \mathrm{Def}_{(X,\mathcal{E})}(B)$ of $(\mathcal{X}_A, \mathcal{E}_A)$ exists. Then there is the short exact sequence of sheaves: $0 \to \mathcal{E} \to \mathcal{E}_B \to \mathcal{E}_A \to 0$ and the long exact sequence of cohomology:

$$\ldots \to H^0(\mathcal{E}_B) \to H^0(\mathcal{E}_A) \xrightarrow{\delta_0} H^1(\mathcal{E}) \to H^1(\mathcal{E}_B) \to H^1(\mathcal{E}_A) \xrightarrow{\delta_1} H^2(\mathcal{E}) \to \ldots$$

$$\ldots \to H^i(\mathcal{E}_B) \to H^i(\mathcal{E}_A) \xrightarrow{\delta_i} H^{i+1}(\mathcal{E}) \to \ldots,$$

thus an element $s \in H^i(\mathcal{E}_A)$ is lifted by an element $\tilde{s} \in H^i(\mathcal{E}_B)$ if and only if $\delta_i(s) = 0$. The relative obstruction of $((\mathcal{X}_B, \mathcal{E}_B), (\mathcal{X}_A, \mathcal{E}_A, \varphi^i_A))$ is defined to be $(\delta_i(\varphi^i_A(-)))_i \in \bigoplus_i \mathrm{Hom}(V^i, \check{\mathrm{H}}^{i+1}(\mathcal{E})) \otimes J$.

As in Section 4.1 , one can prove the following lemma, which links these two relative obstruction theories. Using it one can reprove smoothness of the morphism Φ_F.

Lemma 4.2.12. *The maps* $\mathrm{Hom}(V^i, H^{i+1}_{\bar\partial}(\mathcal{E})) \to \mathrm{Hom}(V^i, \check{\mathrm{H}}^{i+1}(\mathcal{E}))$ *given by the isomorphism between Dolbeault and Čech cohomology induce a linear isomorphism between the relative obstruction theories for* $r : \mathrm{Def}_F \to \mathrm{Def}_{A_X^{0,*}(D^1(\mathcal{E}))}$ *and* $\pi : \mathrm{Def}_{(X,\mathcal{E},V^i)} \to \mathrm{Def}_{(X,\mathcal{E})}$ *compatible with the morphisms* Φ_F *and* $\Phi_{A_X^{0,*}(D^1(\mathcal{E}))}$.

Corollary 4.2.13. *The morphism* $\Phi_N : \mathrm{Def}_N \to \mathrm{Def}_{(\mathcal{E},V^i)}$ *is smooth.*

Proof. Arguments are the same as in the previous case. Instead of diagram (4.14), consider the following:

$$\begin{tikzcd}
\mathrm{Def}_N \arrow[r, "\Phi_N"] \arrow[d, "s"] & \mathrm{Def}_{(\mathcal{E},V^i)} \arrow[d, "\pi"] \\
\mathrm{Def}_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})} \arrow[r, "\Phi_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}"] & \mathrm{Def}_{\mathcal{E}} .
\end{tikzcd}$$

Let $x = (x_{-1}, x_i) \in \mathrm{MC}_N(A)$, $\Phi_N(x) = (\mathcal{E}_A, \varphi^i_A) \in \mathrm{Def}_{(\mathcal{E},V^i)}(A)$, $s(x) = x_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})} \in \mathrm{MC}_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}(A)$ and $\Phi_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}(s(x)) = \pi(\Phi_N(x)) = \mathcal{E}_A \in \mathrm{Def}_{\mathcal{E}}(A)$.
Suppose that it exists a lifting $(\mathcal{E}_B, \varphi^i_B) \in \mathrm{Def}_{(\mathcal{E},V^i)}(B)$ of $(\mathcal{E}_A, \varphi^i_A)$, thus $\mathcal{E}_B \in \mathrm{Def}_{\mathcal{E}}(B)$ is a lifting of $\mathcal{E}_A$ and, for smoothness of $\Phi_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}$, it exists a lifting $y \in \mathrm{MC}_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}(B)$ of $x_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})}$.
Consider homomorphisms $z_{i-1} = \varphi^i_B - \varphi^i$ and define $\tilde{x} \in \mathrm{Hom}(\mathcal{C}, \mathcal{C}) \otimes B$ to be given by $\tilde{x}|_{\mathcal{A}_X^{0,*}(\mathcal{E})} = y$ and $\tilde{x}|_{V^i} = z_{i-1}$. As before $\tilde{x}$ satisfies the Maurer-Cartan equation in the DGLA $N \otimes B$ and $\alpha(\tilde{x}) = x$. □

Also relative obstruction theory can be done in this case. As before analyze relative obstructions for $s : \mathrm{Def}_N \to \mathrm{Def}_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}$ and $\pi : \mathrm{Def}_{(\mathcal{E},V^i)} \to \mathrm{Def}_{\mathcal{E}}$.
The relative obstruction of an element $x = (x_{-1}, x_i) \in \mathrm{MC}_N(A)$, such that $x_i|_{\mathcal{A}_X^{0,i}(\mathcal{E})}$ has a lifting $y \in MC_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}(B)$, is $H(x) \in \bigoplus_i \mathrm{Hom}(V^i, H^{i+1}_{\bar\partial}(\mathcal{E})) \otimes J$. The relative obstruction of an element $(\mathcal{E}_A, \varphi^i_A) \in \mathrm{Def}_{(\mathcal{E},V)}(A)$, such that $\mathcal{E}_A$ has a lifting $\mathcal{E}_B \in \mathrm{Def}_{\mathcal{E}}(B)$, is given by $(\delta_i(\varphi^i_A(-)))_i \in \bigoplus_i \mathrm{Hom}(V^i, \check{\mathrm{H}}^{i+1}(\mathcal{E})) \otimes J$.
Lemma 4.2.12 is still true for the morphisms Φ_N and $\Phi_{A_X^{0,*}(\mathrm{End}\,\mathcal{E})}$ and it allows to prove smoothness of Φ_N using relative obstructions.

Putting together Theorem 4.2.9 and Theorem 4.2.11 and remembering that smoothness implies surjectivity, the following holds

Theorem 4.2.14 (Main Theorem)**.** *The morphism* $\Phi_F : \mathrm{Def}_F \to \mathrm{Def}_{(X,\mathcal{E},V^i)}$ *is an isomorphism of deformation functors, thus the DGLA* F *governs the deformations of* $(X, \mathcal{E}, V^i)$.

Putting together Corollary 4.2.10 and Corollary 4.2.13, the following holds

Corollary 4.2.15. *The morphism* $\Phi_N : \mathrm{Def}_N \to \mathrm{Def}_{(\mathcal{E},V^i)}$ *is an isomorphism of deformation functors, thus the DGLA* N *governs the deformations of* $(\mathcal{E}, V^i)$.

4.3 Local structure of the Brill-Noether stratification of the moduli space of vector bundles

Let X be a compact and connected complex Kähler manifold of dimension n. In this section we consider complex vector bundles on X, or equivalently locally free sheaves of $\mathcal{O}_X$-modules on X, which are stable and flat. We recall the following

Definition 4.3.1. *Let E be a complex vector bundle on X, a* flat structure *in E is given by an open covering $\{U_i\}_i$ of X with a local frame $\{s_i\}_i$ of E, such that the corresponding transition functions g_{ij} are all constant matrices in* $\mathrm{GL}(\operatorname{rank} E, \mathbb{C})$*. A vector bundle with a flat structure is said to be* flat*.*
The same definition can be given in terms of connection. A connection D in the complex a vector bundle E is said to be flat *if its curvature R vanishes. A vector bundle with a flat connection is said to be* flat*.*

Definition 4.3.2. *Let E be a locally free sheaf of $\mathcal{O}_X$-modules on X. The* slope *of E is*

$$\mu(E) = \frac{\deg E}{\operatorname{rank} E} \ .$$

The sheaf E is called semistable *if for every non-zero coherent subsheaf E' of E, we have $\mu(E') \leq \mu(E)$. The sheaf E is called* stable *if it is semistable and has no non-zero subsheaf of slope equal to $\mu(E)$ other than itself.*

It is known that, if X is a compact and connected complex Kähler manifold, a coarse moduli space of locally free sheaves of $\mathcal{O}_X$-modules on a X, which are stable and flat, can be constructed. Let $\mathcal{M}$ be this moduli space. It has a natural structure of Hausdorff complex analytic space (see [31]) or, if X is a projective algebraic variety, it is a quasi-projective variety (see [20]). Moreover Nadel, Goldman and Millson prove the following result in order to determine the type of singularities of this moduli space (see Definitions 1.1.16 and 1.1.15):

Theorem 4.3.3. *The moduli space $\mathcal{M}$ has quadratic algebraic singularities.*

For an analytic space, to have quadratic algebraic singularities it is a local property. The local analysis of the moduli space $\mathcal{M}$ at a point $\mathcal{E}$, which is a stable and flat locally free sheaf of $\mathcal{O}_X$-modules on X, corresponds to the local study of the base space of a Kuranishi family of deformations of the sheaf $\mathcal{E}$ or, equivalently, it corresponds to the study of a germ of analytic space which prorepresents the functor of infinitesimal deformations of the sheaf $\mathcal{E}$.
In his article [29], Nadel constructs explicitly the Kuranishi family of deformations of a flat and stable locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ on X and he proves that the base space of this family has quadratic algebraic singularities. Whereas the proof given by Goldman and Millson in [9] is based on the study of a germ of analytic space which prorepresents the functor of infinitesimal deformations of a sheaf $\mathcal{E}$. They find out this analytic germ and prove that it has quadratic algebraic singularities.

Now we are interested in the local study of the strata of the Brill-Noether stratification of the moduli space $\mathcal{M}$ and in particular, in the same spirit as Theorem 4.3.3, in the determination of their type of singularities.
Let define this stratification. The subspaces of the moduli space $\mathcal{M}$ involved in this stratification are defined globally as sets in the as follows:

Definition 4.3.4. *Let $h_i \in \mathbb{N}$ be fixed integers, for all $i = 0 \dots n$, we define:*

$$\mathcal{N}(h_0 \dots h_n) = \{\mathcal{E}' \in \mathcal{M} \mid \dim H^i(X, \mathcal{E}') = h_i\}.$$

It is obvious that, for a generic choice of the integers $h_i \in \mathbb{N}$, the subspace $\mathcal{N}(h_0 \dots h_n)$ is empty, from now on we fix our attention on non empty ones.

Let $\mathcal{E} \in \mathcal{M}$ be one fixed stable and flat locally free sheaf of $\mathcal{O}_X$-modules on X and let $h_i = \dim H^i(X, \mathcal{E})$. Let $\mathcal{U} \to M \times X$ be the universal Kuranishi family of deformations of $\mathcal{E}$, parametrized by the germ of analytic space M, which is isomorphic to a neighbourhood of $\mathcal{E}$ in $\mathcal{M}$ (see [29] for the construction). Let $\nu : M \times X \to M$ be projection, thus, for all $\mathcal{E}' \in M$, we have $\nu^{-1}(\mathcal{E}') \cong X$ and $\mathcal{U}|_{\nu^{-1}(\mathcal{E}')} = \mathcal{U}|_{\mathcal{E}'} \cong \mathcal{E}'$.

Now let's define the germ of the strata $\mathcal{N}(h_0 \dots h_n)$ at its point $\mathcal{E}$. Since, for all $i = 0 \dots n$, the function $\mathcal{E}' \in \mathcal{M} \to \dim H^i(X, \mathcal{E}') \in \mathbb{N}$ is upper semicontinuos, for Semicontinuity Theorem (see [12], Theorem 12.8, ch.III), the set $U_i = \{\mathcal{E}' \in \mathcal{M} \mid \dim H^i(X, \mathcal{E}') \leq h_i\}$ and the intersection $U = \bigcap_{i=0 \dots n} U_i = \{\mathcal{E}' \in \mathcal{M} \mid \dim H^i(X, \mathcal{E}') \leq h_i,\ \text{for } i = 0 \dots n\}$ are open subsets of $\mathcal{M}$.

For all $i = 0 \dots n$, let $N_i(\mathcal{E}) = V(F_{h_i - 1}(R^i \nu_* \mathcal{U})) = \{\mathcal{E}' \in M \mid \dim R^i \nu_* \mathcal{U} \otimes_{\mathcal{O}_M} k(\mathcal{E}') > h_i - 1\}$ be the closed subschemes of M defined by the sheaf of ideals $F_{h_i-1}(R^i \nu_* \mathcal{U})$, which is the sheaf of $(h_i - 1)$-th Fitting ideals of the sheaf of $\mathcal{O}_M$-modules $R^i \nu_* \mathcal{U}$. Let $N(\mathcal{E}) = \bigcap_{i=0 \dots n} N_i(\mathcal{E})$ be the closed subscheme of M given by the intersection of the previous ones.

Definition 4.3.5. *The germ of the strata $\mathcal{N}(h_0 \dots h_n)$ at its point $\mathcal{E}$ is given by:*

$$U \cap N(\mathcal{E}) = \{\mathcal{E}' \in \mathcal{M} \mid \dim H^i(X, \mathcal{E}') \leq h_i, \quad \forall i = 0 \dots n\} \cap \bigcap_{i=0 \dots n} V(F_{h_i - 1}(R^i \nu_* \mathcal{U})).$$

Remark 4.3.6. We observe that the support of the germ of the strata $\mathcal{N}(h_0 \dots h_n)$ at $\mathcal{E}$, defined in Definition 4.3.5, coincide with a neighbourhood of $\mathcal{E}$ in the set given in Definition 4.3.4. Infact, for the Theorem of Cohomology and Base Change (see [12], Theorem 12.11, ch.III), we have $R^n \nu_* \mathcal{U} \otimes_{\mathcal{O}_M} k(\mathcal{E}') \cong H^n(X, \mathcal{E}')$, then the condition which defines $N_n(\mathcal{E})$ becomes $\dim H^n(X, \mathcal{E}') \geq h_n$ and the ones which define the intersection $U \cap N_n(\mathcal{E})$ become $\dim H^n(X, \mathcal{E}') = h_n$ and $\dim H^i(X, \mathcal{E}') \leq h_i$, for all $i = 0 \dots n - 1$. Applying iteratively the Theorem of Cohomology and Base Change, we obtain $U \cap N(\mathcal{E}) = \{\mathcal{E}' \in M \mid \dim H^i(X, \mathcal{E}') = h_i,\ \text{for } i = 0 \dots n\}$ as we want.

Now we prove the following:

Proposition 4.3.7. *The germ of the strata $\mathcal{N}(h_0 \dots h_n)$ at $\mathcal{E}$ is the base space of a Kuranishi family of deformations of $\mathcal{E}$ which preserve the dimensions of cohomology spaces.*

Proof. Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_{T \times X}$-module on $T \times X$ which is a deformation of the sheaf $\mathcal{E}$ over the analytic space T that preserve the dimensions of cohomology spaces. If the morphism $g : T \to M$ such that $(g \times \mathrm{Id}_X)^* \mathcal{U} \cong \mathcal{F}$, whose existence is assured by the universality of $\mathcal{U}$, can be factorized as in the following diagram:

$$\begin{tikzcd}
\mathcal{F} \cong (g \times \mathrm{Id}_X)^* \mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\
T \times X \arrow[r, "g \times \mathrm{Id}_X"] \arrow[dr, dashed, "h \times \mathrm{Id}_X"'] & M \times X \\
& (\mathcal{N}(h_0 \dots h_n) \cap M) \times X, \arrow[u, "i \times \mathrm{Id}_X"']
\end{tikzcd}$$

then $\mathcal{F} \cong (g \times \mathrm{Id}_X)^*\mathcal{U} \cong (h \times \mathrm{Id}_X)^*(i \times \mathrm{Id}_X)^*\mathcal{U} \cong (h \times \mathrm{Id}_X)^*\mathcal{U}|_{(\mathcal{N}(h_0 \ldots h_n) \cap M) \times X}$ and the restriction of $\mathcal{U}$ to $(\mathcal{N}(h_0 \ldots h_n) \cap M) \times X$ satisfies the universal property.

Let's analyse the pullback via the map g of the sheaf of ideals $F_k(R^i \nu_* \mathcal{U})$, for $i = 0 \ldots n$, which defines locally $\mathcal{N}(h_0 \ldots h_n) \cap M$. Since Fitting ideals commute with base change, for all i and k we have $g^*(F_k(R^i \nu_* \mathcal{U})) = F_k(g^* R^i \nu_* \mathcal{U})$. Let's consider the diagram:

$$\begin{tikzcd}
\mathcal{F} \cong (g \times \mathrm{Id}_X)^*\mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\
T \times X \arrow[r, "g \times \mathrm{Id}_X"] \arrow[d, "\mu"] & \mathrm{Im}\, g \times X \subset M \times X \arrow[d, "\nu"] \\
T \arrow[r, "g"] & \mathrm{Im}\, g \subset M;
\end{tikzcd}$$

using the Theorem of Cohomology and Base Change, since for all $i = 0 \ldots n$ the functions $\mathcal{E}' \in \mathrm{Im}\, g \subset M \to h^i(X, \mathcal{E}') \in \mathbb{N}$ are costant, we have $F_k(g^* R^i \nu_* \mathcal{U}) \cong F_k(R^i \mu_* (g \times \mathrm{Id}_X)^* \mathcal{U}) \cong F_k(R^i \mu_* \mathcal{F})$, and since $\mathcal{F}$ is a deformation which preserves the dimensions of cohomology spaces, the sheaves $R^i \mu_* \mathcal{F}$ are locally free and so the Fitting ideals $F_k(R^i \mu_* \mathcal{F})$ are equal to zero. Then $g^* F_k(R^i \nu_* \mathcal{U})$ is equal to zero, as we want. $\square$

Our aim is to determine the local structure of these strata $\mathcal{N}(h_0 \ldots h_n)$, obtaining the following

Theorem 4.3.8 (Main Theorem)**.** *The Brill-Noether strata $\mathcal{N}(h_0 \ldots h_n)$ have quadratic algebraic singularities.*

To prove this theorem we study deformations of a stable and flat locally free sheaf of $\mathcal{O}_X$-modules on X that preserve the dimensions of its cohomology spaces and we define the deformation functor associated to this problem.

Let $\mathcal{E}'$ be a deformation of the sheaf $\mathcal{E}$ on the manifold X over the analytic space S with a fixed point s_0, i.e. $\mathcal{E}'$ is a locally free sheaf of $(\mathcal{O}_{X \times S})$-modules on $X \times S$ with a morphism $\mathcal{E}' \to \mathcal{E}$ inducing an isomorphism between $\mathcal{E}'|_{X \times s_0}$ and $\mathcal{E}$.
Let $\pi : X \times S \to S$ be the projection, then, for all $s \in S$, $\mathcal{E}'|_{\pi^{-1}(s)} = \mathcal{E}'|_{X \times s} = \mathcal{E}'_s$ is a locally free sheaf on X and so it makes sense to calculate the cohomology spaces of these sheaves, $H^i(\mathcal{E}'_s)$.
By the Theorem of Cohomology and Base Change, the condition that, for all $i \in \mathbb{N}$, $\dim H^i(\mathcal{E}'_s)$ is costant when s varies in S, is equivalent to the condition that, for all $i \in \mathbb{N}$, the direct image $R^i \pi_* \mathcal{E}'$ is a locally free sheaf on S, and in this case we have that the fibre $R^i \pi_* \mathcal{E}' \otimes k(s)$ is isomorphic to $H^i(\mathcal{E}'_s)$.

Now let $\mathcal{E}_A$ be a deformation of $\mathcal{E}$ over $A \in \mathbf{Art}_{\mathbb{C}}$, i.e. $\mathcal{E}_A$ is a locally free sheaf of $(\mathcal{O}_X \otimes A)$-modules on $X \times \mathrm{Spec}\, A$ with a morphism $\mathcal{E}_A \to \mathcal{E}$ inducing an isomorphism $\mathcal{E}_A \otimes_A \mathbb{C} \cong \mathcal{E}$.
In the case of infinitesimal deformation, we can replace the condition that the dimensions of the cohomology spaces are costant along the fibres of the projection $\pi : X \times \mathrm{Spec}\, A \to \mathrm{Spec}\, A$, with the condition that the direct images $R^i \pi_* \mathcal{E}_A$ are locally free sheaves, and in this case we have isomorphisms $R^i \pi_* \mathcal{E}_A \otimes \mathbb{C} \cong H^i(\mathcal{E})$. We observe also that $H^i(\mathcal{E}_A) \cong R^i \pi_* \mathcal{E}_A(\mathrm{Spec}\, A)$.

The functor associated to this kind of deformations is defined by:

Definition 4.3.9. *Let* $\mathrm{Def}^0_{\mathcal{E}} : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$ *be the covariant functor defined, for all* $A \in \mathbf{Art}_{\mathbb{C}}$*, by:*

$$\mathrm{Def}^0_{\mathcal{E}}(A) = \left\{ \mathcal{E}_A \;\middle|\; \begin{array}{l} \mathcal{E}_A \textit{ is a deformation of the sheaf } \mathcal{E} \textit{ over } A \\ R^i\pi_*\mathcal{E}_A \textit{ is a locally free sheaf on } \operatorname{Spec} A \textit{ for all } i \in \mathbb{N} \end{array} \right\} / \sim .$$

If $\mathcal{E}_A$ is an infinitesimal deformation of $\mathcal{E}$ over A, it belongs to $\mathrm{Def}^0_{\mathcal{E}}(A)$, as defined in Definition 4.3.9, if and only if it is such that $H^i(\mathcal{E}_A)$ are free A-modules, that is the same as flat A-modules since A is local Artinian, and $H^i(\mathcal{E}_A) \otimes_A \mathbb{C} \cong H^i(\mathcal{E})$. Thus, remembering the definition of a deformation of a vector space and that all these deformations are trivial (see Example 3.2.13), we have the two following equivalent definitions for the functor $\mathrm{Def}^0_{\mathcal{E}}$:

Definition 4.3.10. *The functor* $\mathrm{Def}^0_{\mathcal{E}}$ *is defined, for all* $A \in \mathbf{Art}_{\mathbb{C}}$*, by:*

$$\mathrm{Def}^0_{\mathcal{E}}(A) = \left\{ \mathcal{E}_A \;\middle|\; \begin{array}{l} \mathcal{E}_A \textit{ is a deformation of the sheaf } \mathcal{E} \textit{ over } A \\ H^i(\mathcal{E}_A) \textit{ is a deformation of } H^i(\mathcal{E}) \textit{ over } A \textit{ for all } i \in \mathbb{N} \end{array} \right\} / \sim .$$

or equivalently by:

$$\mathrm{Def}^0_{\mathcal{E}}(A) = \left\{ \mathcal{E}_A \;\middle|\; \begin{array}{l} \mathcal{E}_A \textit{ is a deformation of the sheaf } \mathcal{E} \textit{ over } A \\ H^i(\mathcal{E}_A) \textit{ is isomorphic to } H^i(\mathcal{E}) \otimes A \textit{ for all } i \in \mathbb{N} \end{array} \right\} / \sim . \tag{4.15}$$

Now we link this functor of deformations with the theory of deformations via DGLAs. We recall that the DGLA $A^{0,*}_X(\operatorname{End}\mathcal{E})$ governs the deformation of the sheaf $\mathcal{E}$ (see Example 3.2.15) and that the DGLA $\operatorname{Hom}^*(A^{0,*}_X(\mathcal{E}), A^{0,*}_X(\mathcal{E}))$ governs the deformation of the complex $(A^{0,*}_X(\mathcal{E}), \bar{\partial})$ (see Example 3.2.13). Let

$$\chi : A^{0,*}_X(\operatorname{End}\mathcal{E}) \to \operatorname{Hom}^*(A^{0,*}_X(\mathcal{E}), A^{0,*}_X(\mathcal{E}))$$

be the natural inclusion of DGLAs, let analyse the deformation functor Def_χ associated to χ. Let $(x, e^a) \in \mathrm{MC}_\chi(A)$, for $A \in Art_{\mathbb{C}}$. Since $x \in A^{0,1}_X(\operatorname{End}\mathcal{E}) \otimes \mathfrak{m}_A$ satisfies the Maurer-Cartan equation, it gives a deformation $\mathcal{E}_A = \ker(\bar{\partial} + x)$ of $\mathcal{E}$ over A. While $e^a \in \exp(\operatorname{Hom}^0(A^{0,*}_X(\mathcal{E}), A^{0,*}_X(\mathcal{E})) \otimes \mathfrak{m}_A)$ gives a gauge equivalence between $\chi(x) = x$ and zero in the DGLA $\operatorname{Hom}^*(A^{0,*}_X(\mathcal{E}), A^{0,*}_X(\mathcal{E}))$. Thus e^a is an isomorphism between the two correspondent deformations of the complex $(A^{0,*}_X(\mathcal{E}), \bar{\partial})$ or equivalently e^a is an isomorphism between the cohomology spaces $H^i(\mathcal{E}_A)$ and $H^i(\mathcal{E}) \otimes A$, for all $i \in \mathbb{N}$. Thus:

$$\mathrm{Def}_\chi(A) = \left\{ (\mathcal{E}_A, f^i_A) \;\middle|\; \begin{array}{ll} \mathcal{E}_A & \text{is a deformation of the sheaf } \mathcal{E} \text{ over } A \\ f^i_A & \text{is the isomorphism } f^i_A : H^i(\mathcal{E}_A) \to H^i(\mathcal{E}) \otimes A \text{ for all } i \in \mathbb{N} \end{array} \right\} .$$

Now let Φ be the morphism of functors given, for all $A \in \mathbf{Art}_{\mathbb{C}}$, by:

$$\begin{array}{rccc} \Phi : & \mathrm{Def}_\chi(A) & \longrightarrow & \mathrm{Def}_{A^{0,*}_X(\operatorname{End}\mathcal{E})}(A) \\ & (x, e^a) & \longrightarrow & x \end{array}$$

With the above geometric interpretations of the functors Def_χ and $\mathrm{Def}_{A^{0,*}_X(\operatorname{End}\mathcal{E})}(A)$, the morphism Φ is the one which associates to every pair $(\mathcal{E}_A, f^i_A) \in \mathrm{Def}_\chi(A)$ the element $\mathcal{E}_A \in \mathrm{Def}_{A^{0,*}_X(\operatorname{End}\mathcal{E})}(A)$. Thus we have the following characterization of $\mathrm{Def}^0_{\mathcal{E}}$ using DGLAs point of view (see [25], Lemma 4.1).

Proposition 4.3.11. *The functor* $\mathrm{Def}^0_{\mathcal{E}}$ *is isomorphic to the image of the morphism* $\Phi : \mathrm{Def}_\chi \to \mathrm{Def}_{A^{0,*}_X(\mathrm{End}\,\mathcal{E})}$.

Remark 4.3.12. We note that, by Proposition 4.3.7, the functor $\mathrm{Def}^0_{\mathcal{E}}$ is prorepresented by $\mathcal{N}(h_0 \dots h_n)$.

Now we are ready to prove the Main Theorem of this section.

Proof. The local study of the strata $\mathcal{N}(h_0 \dots h_n)$ at one of its point $\mathcal{E}$, is the same as the study of a germ of analytic space which prorepresents the functor $\mathrm{Def}^0_{\mathcal{E}}$. Our proof is divided into four steps in which we find out a chain of functors, linked each other by smooth morphisms, from the functor $\mathrm{Def}^0_{\mathcal{E}}$ to a deformation functor for which it is known that the germ of analytic space that prorepresents it has quadratic algebraic singularities.

First Step. We prove that the morphism $\Phi : \mathrm{Def}_\chi \to \mathrm{Def}^0_{\mathcal{E}}$ is smooth. Then, given a principal extension in $\mathbf{Art}_{\mathbb{C}}$, $0 \to J \to B \xrightarrow{\alpha} A \to 0$, and an element $(\mathcal{E}_A, f^i_A) \in \mathrm{Def}_\chi(A)$, we have to prove that, if its image $\mathcal{E}_A \in \mathrm{Def}^0_{\mathcal{E}}(A)$ has a lifting $\mathcal{E}_B \in \mathrm{Def}^0_{\mathcal{E}}(B)$, it has a lifting in $\mathrm{Def}_\chi(B)$.

Since $\mathcal{E}_A \in \mathrm{Def}^0_{\mathcal{E}}(A)$ and $\mathcal{E}_B \in \mathrm{Def}^0_{\mathcal{E}}(B)$, their cohomology spaces are deformations of $H^i(\mathcal{E})$ over A and B respectively and so $H^i(\mathcal{E}_A) \cong H^i(\mathcal{E}) \otimes A$ and $H^i(\mathcal{E}_B) \cong H^i(\mathcal{E}) \otimes B$. Thus $H^i(\mathcal{E}_B)$ is a lifting of $H^i(\mathcal{E}_A)$ and it is a polynomial algebra over B. It follows that in the diagram

$$\begin{tikzcd}
H^i(\mathcal{E}_B) \arrow[r] \arrow[drr, dashed, "f^i_B"] & H^i(\mathcal{E}_A) \arrow[r, "f^i_A"] & H^i(\mathcal{E}) \otimes A \\
B \arrow[u] \arrow[rr] & & H^i(\mathcal{E}) \otimes B \arrow[u, "\beta=\mathrm{Id}\otimes\alpha"']
\end{tikzcd}$$

there exists a homomorphism $f^i_B : H^i(\mathcal{E}_B) \to H^i(\mathcal{E}) \otimes B$, which lifts f^i_A. Then also the following diagram commutes:

$$\begin{tikzcd}
& H^i(\mathcal{E}) & \\
H^i(\mathcal{E}_B) \arrow[ur] \arrow[r, "f^i_B"] & H^i(\mathcal{E}) \otimes B \arrow[u] & \\
& B \arrow[ul] \arrow[u] &
\end{tikzcd}$$

and so f^i_B is an isomorphism, for all $i \in \mathbb{N}$.

Second Step. Since X is a Kähler manifold and $\mathcal{E}$ is a hermitian sheaf, the operators $\bar\partial^*_{\mathcal{E}}$, adjoint of $\bar\partial_{\mathcal{E}}$, and the Laplacian $\overline{\Box}_{\mathcal{E}} = \bar\partial_{\mathcal{E}}\bar\partial^*_{\mathcal{E}} + \bar\partial^*_{\mathcal{E}}\bar\partial_{\mathcal{E}}$ can be defined between forms on X with values in the sheaf $\mathcal{E}$. Let $\mathcal{H}^{0,*}_X(\mathcal{E}) = \ker \overline{\Box}_{\mathcal{E}}$ be the complex of $(0,*)$-harmonic forms on X with values in $\mathcal{E}$ and let $\mathrm{Hom}^*(\mathcal{H}^{0,*}_X(\mathcal{E}), \mathcal{H}^{0,*}_X(\mathcal{E}))$ be the formal DGLA of the homomorphisms of this complex.

Also for the sheaf $\mathrm{End}\,\mathcal{E}$ the operator $\bar\partial^*_{\mathrm{End}\,\mathcal{E}}$, adjoint of $\bar\partial_{\mathrm{End}\,\mathcal{E}}$, and the Laplacian $\overline{\Box}_{\mathrm{End}\,\mathcal{E}} = \bar\partial_{\mathrm{End}\,\mathcal{E}}\bar\partial^*_{\mathrm{End}\,\mathcal{E}} + \bar\partial^*_{\mathrm{End}\,\mathcal{E}}\bar\partial_{\mathrm{End}\,\mathcal{E}}$ can be defined. Let $\mathcal{H}^{0,*}_X(\mathrm{End}\,\mathcal{E}) = \ker \overline{\Box}_{\mathrm{End}\,\mathcal{E}}$ be the complex of the $(0,*)$-harmonic forms on X with values in $\mathrm{End}\,\mathcal{E}$.

Siu proved (see [35]) that, for a flat holomorphic vector bundle $\mathcal{L}$ on a Kähler manifold X, the two Laplacian operators $\overline{\Box}_{\mathcal{L}}$ and $\Box_{\mathcal{L}}$ coincide. Then a $(0,*)$-form on X with values in $\mathcal{L}$ is harmonic if and only if it anhilates ∂, which is well defined because $\mathcal{L}$ is flat.

Since $\operatorname{End}\mathcal{E}$ is flat, these facts imply that the complex $\mathcal{H}^{0,*}_X(\operatorname{End}\mathcal{E})$ is a DGLA with bracket given by the wedge product on forms and the composition of endomorphisms.

Moreover, we can define a morphism $\Omega : \mathcal{H}^{0,*}_X(\operatorname{End}\mathcal{E}) \to \operatorname{Hom}^*(\mathcal{H}^{0,*}_X(\mathcal{E}), \mathcal{H}^{0,*}_X(\mathcal{E}))$. Every element $x \in \mathcal{A}^{0,*}_X(\operatorname{End}\mathcal{E})$ gives naturally an homomorphism $\Omega(x)$ from $A^{0,*}_X(\mathcal{E})$ in itself, defined locally to be the wedge product between forms and the action of the endomorphism on the elements of $\mathcal{E}$. If we defined it on an open covering of X on which both the sheaves $\mathcal{E}$ and $\operatorname{End}\mathcal{E}$ have costant transition functions, when $x \in \mathcal{H}^{0,*}_X(\operatorname{End}\mathcal{E})$ and $\Omega(x)$ is restricted to the harmonic forms $\mathcal{H}^{0,*}_X(\mathcal{E})$, it gives as a result an harmonic form. Let $\Omega : \mathcal{H}^{0,*}_X(\operatorname{End}\mathcal{E}) \to \operatorname{Hom}^*(\mathcal{H}^{0,*}_X(\mathcal{E}), \mathcal{H}^{0,*}_X(\mathcal{E}))$ be the DGLAs morphism just defined and let $\operatorname{Def}_\Omega$ be the deformation functor associated to it.

We want to prove that the two functors $\operatorname{Def}_\Omega$ and Def_χ are isomorphic. Then we consider the following commutative diagram:

$$\begin{tikzcd}
A^{0,*}_X(\operatorname{End}\mathcal{E}) \arrow[d, "\chi"] & \mathcal{H}^{0,*}_X(\operatorname{End}\mathcal{E}) \arrow[l, "\alpha"'] \arrow[r, "\gamma"] \arrow[d, "\eta"] & \mathcal{H}^{0,*}_X(\operatorname{End}\mathcal{E}) \arrow[d, "\Omega"] \\
\operatorname{Hom}^*(A^{0,*}_X(\mathcal{E}), A^{0,*}_X(\mathcal{E})) & M^* \arrow[l, "\beta"'] \arrow[r, "\delta"] & \operatorname{Hom}^*(\mathcal{H}^{0,*}_X(\mathcal{E}), \mathcal{H}^{0,*}_X(\mathcal{E}))
\end{tikzcd}$$

where $M^* = \left\{ \varphi \in \operatorname{Hom}^*(A^{0,*}_X(\mathcal{E}), A^{0,*}_X(\mathcal{E})) \mid \varphi(\mathcal{H}^{0,*}_X(\mathcal{E})) \subseteq \mathcal{H}^{0,*}_X(\mathcal{E}) \right\}$.

The morphism β is a quasi-isomorphism, in fact it is injective and $\operatorname{coker}\beta = \operatorname{Hom}^*(A^{0,*}_X(\mathcal{E}), A^{0,*}_X(\mathcal{E}))/M$ $\cong \operatorname{Hom}^*(A^{0,*}_X(\mathcal{E}), A^{0,*}_X(\mathcal{E})/\mathcal{H}^{0,*}_X(\mathcal{E}))$ is an acyclic complex. Then α and β induce a quasi-isomorphism between the cones $C_\eta \to C_\chi$ and so, by the Inverse Function Theorem (Theorem 3.3.6), an isomorphism between the functors $\operatorname{Def}_\eta \to \operatorname{Def}_\chi$.

Also the morphism δ is a quasi-isomorphism, in fact it is surjective and its kernel is $\ker\delta = \left\{ \varphi \in \operatorname{Hom}^*(A^{0,*}_X(\mathcal{E}), A^{0,*}_X(\mathcal{E})) \mid \varphi(\mathcal{H}^{0,*}_X(\mathcal{E})) = 0 \right\}$ that isomorphic to the acyclic complex $\operatorname{Hom}^*(A^{0,*}_X(\mathcal{E})/\mathcal{H}^{0,*}_X(\mathcal{E}), A^{0,*}_X(\mathcal{E}))$. Then γ and δ induce a quasi-isomorphism between the cones $C_\eta \to C_\Omega$ and so an isomorphism between the functors $\operatorname{Def}_\eta \to \operatorname{Def}_\Omega$.

Third Step. Let $\tilde{\mathcal{H}}_X(\operatorname{End}\mathcal{E})$ be the DGLA equal to zero in zero degree and equal to $\mathcal{H}^{0,*}_X(\operatorname{End}\mathcal{E})$ in positive degrees, with zero differential and bracket given by wedge product on forms and composition of endomorphisms.
Let $\tilde{\Omega} : \tilde{\mathcal{H}}^{0,*}_X(\operatorname{End}\mathcal{E}) \to \operatorname{Hom}^*(\mathcal{H}^{0,*}_X(\mathcal{E}), \mathcal{H}^{0,*}_X(\mathcal{E}))$ be the DGLAs morphism defined as in the previous step and let $\operatorname{Def}_{\tilde{\Omega}}$ be the deformation functor associated to it.
The inclusion $\tilde{\mathcal{H}}_X(\operatorname{End}\mathcal{E}) \hookrightarrow \mathcal{H}_X(\operatorname{End}\mathcal{E})$ and the identity on $\operatorname{Hom}(\mathcal{H}^{0,*}_X(\mathcal{E}), \mathcal{H}^{0,*}_X(\mathcal{E}))$ induce a morphism of functors $\Psi : \operatorname{Def}_{\tilde{\Omega}} \to \operatorname{Def}_\Omega$.
We note that the morphism induced by Ψ between the cohomology spaces of cones of Ω and $\tilde{\Omega}$ respectively is bijective in degree greater equal than 2 and it is surjective in degree 1. Thus, using the Standard Smootheness Criterion (Theorem 1.2.22), we conclude that Ψ is smooth.

Fourth Step. Let's write explicitly $\operatorname{Def}_{\tilde{\Omega}}$. The functor $\operatorname{MC}_{\tilde{\Omega}}$, for all $A \in \mathbf{Art}_{\mathbb{C}}$, is given by:

$$\mathrm{MC}_{\tilde{\Omega}}(A)=\left\{(x,e^a)\in(L^1\otimes\mathfrak{m}_A)\times\exp(M^0\otimes\mathfrak{m}_A)\mid dx+\frac{1}{2}[x,x]=0,e^a*\tilde{\Omega}(x)=0\right\}$$

where $L^*=\tilde{\mathcal{H}}^{0,*}_X(\operatorname{End}\mathcal{E})$ and $M^*=\operatorname{Hom}^*(\mathcal{H}^{0,*}_X(\mathcal{E}),\mathcal{H}^{0,*}_X(\mathcal{E}))$. Since the differential in the DGLA $\tilde{\mathcal{H}}^{0,*}_X(\operatorname{End}\mathcal{E})$ is zero and since the equation $e^a*\tilde{\Omega}(x)=0$ can be written as $\tilde{\Omega}(x)=e^{-a}*0=0$, we obtain, for all $A\in\mathbf{Art}_{\mathbb{C}}$:

$$\mathrm{MC}_{\tilde{\Omega}}(A)=\left\{x\in\ker\tilde{\Omega}\otimes\mathfrak{m}_A\mid[x,x]=0\right\}\times\exp(\operatorname{Hom}^0(\mathcal{H}^{0,*}_X(\mathcal{E}),\mathcal{H}^{0,*}_X(\mathcal{E}))\otimes\mathfrak{m}_A).$$

Moreover $\exp(\tilde{\mathcal{H}}^{0,0}_X(\operatorname{End}\mathcal{E})\otimes\mathfrak{m}_A)\times\exp(d\operatorname{Hom}^{-1}(\mathcal{H}^{0,*}_X(\mathcal{E}),\mathcal{H}^{0,*}_X(\mathcal{E}))\otimes\mathfrak{m}_A)$ is equal to zero, thus there isn't gauge action. Thus, for all $A\in\mathbf{Art}_{\mathbb{C}}$, we have:

$$\mathrm{Def}_{\tilde{\Omega}}(A)=\left\{x\in\ker\tilde{\Omega}\otimes\mathfrak{m}_A\mid[x,x]=0\right\}\times\exp(\operatorname{Hom}^0(\mathcal{H}^{0,*}_X(\mathcal{E}),\mathcal{H}^{0,*}_X(\mathcal{E}))\otimes\mathfrak{m}_A).$$

Since $\tilde{\Omega}$ is a DGLAs morphism, $\ker\tilde{\Omega}$ is a DGLA and it is defined the deformation functor $\mathrm{Def}_{\ker\tilde{\Omega}}$ associated to it. Now, for all $A\in\mathbf{Art}_{\mathbb{C}}$, we obtain:

$$\mathrm{Def}_{\tilde{\Omega}}(A)=\mathrm{Def}_{\ker\tilde{\Omega}}(A)\times\exp(\operatorname{Hom}^0(\mathcal{H}^{0,*}_X(\mathcal{E}),\mathcal{H}^{0,*}_X(\mathcal{E}))\otimes\mathfrak{m}_A).$$

The DGLA $\ker\tilde{\Omega}$ has zero differential, so the functor $\mathrm{Def}_{\ker\tilde{\Omega}}$ is prorepresented by the germ in zero of the quadratic cone (see [10], Theorem 5.3):

$$X=\{x\in\ker^1\tilde{\Omega}\mid[x,x]=0\},$$

that has quadratic algebraic singularities. Then also the functor $\mathrm{Def}_{\tilde{\Omega}}$ is prorepresented by a germ of analytic space with quadratic algebraic singularities.

Conclusion. Since now we have constructed smooth morphisms between the functor $\mathrm{Def}^0_{\mathcal{E}}$ and the functor $\mathrm{Def}_{\tilde{\Omega}}$:

$$\mathrm{Def}^0_{\mathcal{E}}\xleftarrow{\text{smooth}}\mathrm{Def}_{\chi}\xleftrightarrow[\text{isomorphism}]{\cong}\mathrm{Def}_{\Omega}\xleftarrow{\text{smooth}}\mathrm{Def}_{\tilde{\Omega}}.$$

By Proposition 1.2.26, there exists a smooth morphism between the germs of analytic spaces which are hulls of the two functors $\mathrm{Def}^0_{\mathcal{E}}$ and $\mathrm{Def}_{\tilde{\Omega}}$. Moreover, by Theorem 1.1.17, since the germ which is a hull of $\mathrm{Def}_{\tilde{\Omega}}$ has quadratic algebraic singularities, the same is true for the hull of $\mathrm{Def}^0_{\mathcal{E}}$. $\square$

Chapter 5

L_∞-algebras and semicosimplicial dglas in deformation theory

This Chapter is devoted to the study of deformation theory via L_∞-algebras. These objects were introduced in deformation theory as an extension of the category of DGLAs, that in many situation is not sufficient to understand the deformation problems completely. The approach consists to associate an L_∞-algebra to a deformation problem and a deformation functor canonically to an L_∞-algebra:

$$\text{Deformation problem} \rightsquigarrow L_\infty\text{-algebra} \rightsquigarrow \text{Deformation functor}.$$

If the L_∞-algebra is appropriately choosen, the deformation functor obtained is isomorphic to the one associated to the problem in the classical way.

In this Chapter, at first we introduce the basic notions of L_∞-theory, Section 5.1, then we concentrate our attention to the L_∞ constructions linked to semicosimplicial differential graded Lie algebras, proving some results in this direction, Sections 5.2 and 5.3. These constructions have strong geometric motivations, in fact they allow to state a concrete and rigorous link between the classical approach to deformation theory and the theory of deformation via DGLAs, as explained in Section 5.4.

5.1 L_∞-algebras

In this section we collect some basic preliminaries and definitions of L_∞-algebras theory. We state the homotopical transfer of structure Theorem and explain how to associate a deformation functor to an L_∞-algebra.

Definition 5.1.1. *A* graded coalgebra *is a graded vector space $C = \bigoplus_{i\in\mathbb{Z}} C_i$ with a morphism of graded vector spaces $\Delta : C \to C \otimes C$ called* coproduct*.*
The graded coalgebra (C, Δ) is coassociative if $(\Delta \otimes \mathrm{Id}) \otimes \Delta = (\mathrm{Id} \otimes \Delta) \otimes \Delta$ and it is cocommutative if $\Delta = T\Delta$, where T is given by $T(v \otimes w) = (-1)^{\deg v \deg w} w \otimes v$.
Let (C, Δ_C) and (D, Δ_D) be two graded coalgebras, a linear morphism $f : C \to D$ is a morphism *of coalgebras if $(f \otimes f)\Delta_C = \Delta_D f$.*

Definition 5.1.2. *Let (C, Δ) be a graded coalgebra. A* coderivation *of degree n of it is a linear map of degree n, $d \in \mathrm{Hom}^n(C, C)$, that satisfies the coLeibnitz rule*

$$\Delta d = (d \otimes \mathrm{Id} + \mathrm{Id} \otimes d)\Delta.$$

More generally, let $\theta : (C, \Delta_C) \to (D, \Delta_D)$ *be a morphism of graded coalgebras, a* coderivation *of degree* n *with respect to* θ *is a linear map of degree* n, $d \in \mathrm{Hom}^n(C, D)$, *such that*

$$\Delta_D d = (d \otimes \theta + \theta \otimes d)\Delta_C.$$

A coderivation d *is called a* codifferential *if* $d^2 = d \circ d = 0$.

Example 5.1.3. Let V be a $\mathbb{Z}$-graded vector space over $\mathbb{K}$.
The *tensor coalgebra* generated by V is defined to be the graded vector space

$$T(V) = \bigoplus_{n=0}^{+\infty} \bigotimes^{n} V$$

endowed with the associative coproduct

$$a(v_1 \otimes \ldots \otimes v_n) = \sum_{k=1}^{n-1} (v_1 \otimes \ldots \otimes v_k) \otimes (v_{k+1} \otimes \ldots \otimes v_n).$$

The *reduced tensor coalgebra* generated by V is the sub-coalgebra of $T(V)$ given by $\overline{T(V)} = \bigoplus_{n=1}^{+\infty} \bigotimes^{n} V$.
Let I be the homogeneous ideal of $T(V)$ generated by $\langle v \otimes w - (-1)^{\deg v \deg w} w \otimes v ; \forall \ \ v, w \in V \rangle$.
The *symmetric coalgebra* generated by V is defined as the quotient

$$S(V) = \bigoplus_{n=0}^{+\infty} \bigodot^{n} V, \quad \text{with} \quad \bigodot^{n} V = \frac{\bigotimes^n V}{I \cap \bigotimes^n V},$$

endowed with the associative coproduct

$$\Delta(v_1 \odot \ldots \odot v_n) = \sum_{k=1}^{n-1} \sum_{\sigma \in S_n^k} \epsilon(\sigma) \ (v_{\sigma(1)} \odot \ldots \odot v_{\sigma(k)}) \otimes (v_{\sigma(k+1)} \odot \ldots \odot v_{\sigma(n)}),$$

where $\sigma \in S_n^k$ is a permutation of n elements, such that $\sigma(i) < \sigma(i+1)$ for all $i \neq k$ and $\epsilon(\sigma) = \pm 1$ is the sign determined by the relation in $\bigodot^n V$: $v_1 \odot \ldots \odot v_n = \epsilon(\sigma) \ v_{\sigma(1)} \odot \ldots \odot v_{\sigma(n)}$. The *reduced symmetric coalgebra* generated by V is the sub-coalgebra of $S(V)$ given by $\overline{S(V)} = \bigoplus_{n=1}^{+\infty} \bigodot^{n} V$. Let $\pi : \overline{T(V)} \to \overline{S(V)}$ be the projection.

Proposition 5.1.4. *Let* V *be a graded vector space and let* (C, Δ) *be a locally nilpotent cocommutative graded coalgebra. The composition with the projection* $p : \overline{S(V)} \to V$ *defines a bijective map:*

$$\mathrm{Hom}(C, \overline{S(V)}) \xrightarrow{p\circ} \mathrm{Hom}(C, V)$$

with inverse given by

$$f \mapsto M(f) = \sum_{n=1}^{+\infty} \frac{(f \odot \ldots \odot f) \circ \pi}{n!} \Delta^{n-1}. \tag{5.1}$$

Proof. See [26], Proposition 8.4. $\square$

Proposition 5.1.5. *Let V be a graded vector space, let (C, Δ) be a locally nilpotent cocommutative graded coalgebra and let $\theta : C \to \overline{S(V)}$ be a morphism of coalgebras. The composition with the projection $p : \overline{S(V)} \to V$ defines a bijective map:*

$$\mathrm{Coder}^n(C, \overline{S(V)}, \theta) \xrightarrow{p\circ} \mathrm{Hom}^n(C, V)$$

with inverse given by

$$q \mapsto Q = \pi \sum_{n=1}^{+\infty} \frac{1}{n!}(q \otimes \theta^{\otimes n}) \circ \Delta^n. \tag{5.2}$$

Proof. See [26] Corollary 8.7. □

Remark 5.1.6. In the following definitions, we apply the above Propositions with $C = \overline{S(V)}$ and $\theta = \mathrm{Id}$. In particular formula (5.2) gives for all $q = \sum_k q_k \in \mathrm{Hom}^n(\overline{S(V)}, V)$ and for all $v_1 \odot \ldots \odot v_n \in \overline{S(V)}$

$$Q(v_1 \odot \ldots \odot v_n) = \sum_{k=1}^{n} \sum_{\sigma \in S_n^k} \varepsilon(\sigma) q_k(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \ldots \odot v_{\sigma(n)}. \tag{5.3}$$

Definition 5.1.7. *Let V be a graded vector space, an L_∞-*structure *on V is a sequence of linear maps of degree 1*

$$q_k \colon \bigodot^k V[1] \to V[1], \qquad \text{for} \quad k \geq 1,$$

such that the coderivation Q induced on the reduced symmetric coalgebra $\overline{S(V[1])}$ by the homomorphism $q = \sum_k q_k$, as in formula (5.3), is a codifferential.
*An L_∞-*algebra *is indicated with $(V, q_1, q_2, q_3, \ldots)$ and the morphisms q_i are called the* brackets *of the L_∞-algebra.*

Remark 5.1.8. The condition $Q \circ Q$ implies that $q_1 \circ q_1 = 0$, then a L_∞-algebra is a differential complex.

Remark 5.1.9. A DGLA L has a natural structure of L_∞-algebra. Infact L is a graded vector space and it can be verified that the brackets:

$$q_1(x) = -dx, \qquad q_2(x \odot y) = (-1)^{\deg x}[x, y] \qquad \text{and} \qquad q_k = 0 \ \text{ for all } \ k \geq 3$$

satisfy condition $Q \circ Q = 0$.

Definition 5.1.10. *Let (V, q_i) and $(W, \hat{q}_i)$ be two L_∞-algebras, a* morphism $f_\infty : (V, q_i) \to (W, \hat{q}_1)$ of L_∞-algebras *is a sequence of linear maps of degree* 0

$$f_k \colon \bigodot^k V[1] \to W[1], \qquad \text{for} \quad k \geq 1,$$

such that the morphism of coalgebras induced on the reduced symmetric coalgebras by $f = \sum_k f_k$, as in formula (5.1), commutes with the codifferentials induced by the two L_∞ structures of V and W.

Remark 5.1.11. If f_∞ is an L_∞ morphism between (V, q_i) and $(W, \hat{q}_i)$, then its linear part $f_1 \colon V[1] \to W[1]$ satisfies the equation $f_1 \circ q_1 = \hat{q}_1 \circ f_1$, i.e. f_1 is a morphism of differential complexes $(V[1], q_1) \to (W[1], \hat{q}_1)$.
Thus it makes sense to call a L_∞-morphism f_∞ a *quasi-isomorphism of L_∞-algebras*, if its linear part f_1 is a quasi-isomorphism of differential complexes.

Remark 5.1.12. An L_∞-morphism f_∞ is called *linear* if $f_n = 0$ for every $n \geq 2$.
A linear map $f_1 \colon V[1] \to W[1]$ is a linear L_∞-morphism if and only if $\hat{q}_n(f_1(v_1) \odot \cdots \odot f_1(v_n)) = f_1(q_n(v_1 \odot \cdots \odot v_n))$, for all $n \geq 1$.

Remark 5.1.13. Let L and M be two DGLAs and let $f : L \to M$ be a DGLAs morphism. Then f_∞, given by $f_1 = f$ and $f_k = 0$ for all $k \geq 2$, is a L_∞-morphism between the L_∞ algebras L and M defined from the DGLAs structures of L and M.
So the category of differential graded Lie algebras is a (non full) subcategory of the category of L_∞-algebras.

Let $\mathbf{L}_\infty$ be the category of L_∞-algebras, whose objects are L_∞-algebras and whose arrows are morphisms of L_∞-algebras.

A major result in the theory of L_∞-algebras is the following *homotopical transfer of structure* theorem.

Theorem 5.1.14. *Let $(V, q_1, q_2, q_3, \dots)$ be an L_∞-algebra and (C, δ) be a differential complex. If there exist two morphisms of differential complexes*

$$\imath \colon (C[1], \delta_{[1]}) \to (V[1], q_1) \qquad \text{and} \qquad \pi \colon (V[1], q_1) \to (C[1], \delta_{[1]})$$

such that the composition $\imath\pi$ is homotopic to the identity, then there exist an L_∞-algebra structure $(C, \langle\,\rangle_1, \langle\,\rangle_2, \dots)$ on C extending its differential complex structure and an L_∞-morphism $\imath_\infty$ extending $\imath$.

Remark 5.1.15. Note that, if $\imath$ is a quasi-isomorphism of complexes, by Remark 5.1.11, its extension $\imath_\infty$ is a quasi-isomorphism of L_∞-algebras.

It has been remarked by Kontsevich and Soibelman (see [18]) that the L_∞-morphism $\imath_\infty$ and the brackets $\langle\,\rangle_n$ can be explicitly written as summations over rooted trees.

Let $h \in \mathrm{Hom}^{-1}(V[1], V[1])$ be an homotopy between $\imath\pi$ and $\mathrm{Id}_{V[1]}$, i.e., $q_1 h + h q_1 = \imath\pi - \mathrm{Id}_{V[1]}$. Denote by $\mathcal{T}_{h,n}$ the groupoid whose objects are rooted trees with internal vertices of valence at least two and exactly n tail edges. Trees in $\mathcal{T}_{h,n}$ are decorated as follows: each tail edge of a tree in $\mathcal{T}_{h,n}$ is decorated by the operator $\imath$, each internal edge is decorated by the operator h and also the root edge is decorated by the operator h; every internal vertex v carries the operation q_r, where r is the number of edges having v as endpoint. Isomorphisms between objects in $\mathcal{T}_{h,n}$ are isomorphisms of the underlying trees. Denote by $T_{h,n}$ the set of isomorphism classes of objects of $\mathcal{T}_{h,n}$.

Similarly, let $\mathcal{T}_{\pi,n}$ be the groupoid whose objects are rooted trees with the same decoration as $\mathcal{T}_{h,n}$ except for the root edge, which is decorated by the operator π instead of h. The set of isomorphism classes of objects of $\mathcal{T}_{\pi,n}$ is denoted $T_{\pi,n}$.

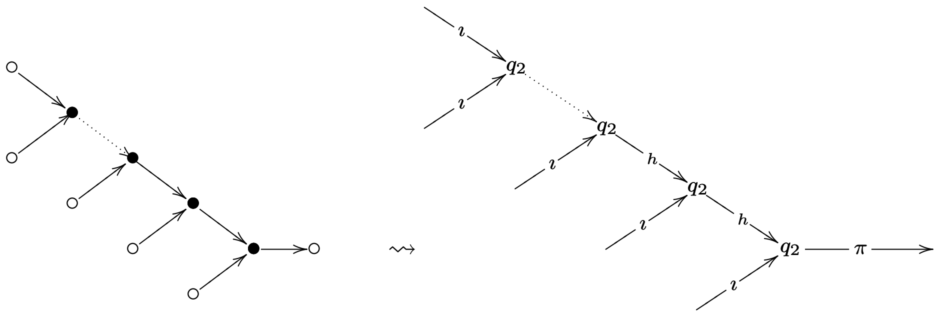
Via the usual operadic rules, each decorated tree $\Gamma \in \mathcal{T}_{h,n}$ gives a linear map

$$Z_\Gamma(\imath, \pi, h, q_i) \colon C[1]^{\odot n} \to V[1],$$

similarly, each decorated tree $\Gamma \in \mathcal{T}_{\pi,n}$ gives a degree one linear map

$$Z_\Gamma(\imath, \pi, h, q_i) \colon C[1]^{\odot n} \to C[1].$$

We recall that the operadic rules associate to the following tree $\Gamma \in \mathcal{T}_{\pi,n}$



the operator

$$
\begin{aligned}
&Z_\Gamma(\imath,\pi,h,q_i)(x_1\odot\cdots\odot x_n)=\\
&\quad=\frac{1}{2}\sum_{\sigma\in S_n}\varepsilon(\sigma)\pi q_2(\imath(x_{\sigma(1)})\odot hq_2(\imath(x_{\sigma(2)})\odot\cdots\odot hq_2(\imath(x_{\sigma(n-1)})\odot\imath(x_{\sigma(n)}))\cdots)).
\end{aligned}
$$

Having introduced these notations, we can write Kontsevich-Soibelman's formulas as follows.

Proposition 5.1.16. *In the above set-up the brackets $\langle\,\rangle_n$, and the L_∞ morphism $\imath_\infty$ can be expressed as sums over decorated rooted trees via the formulas*

$$
\imath_n=\sum_{\Gamma\in T_{h,n}}\frac{Z_\Gamma(\imath,\pi,h,q_i)}{|\operatorname{Aut}\Gamma|},\qquad \langle\,\rangle_n=\sum_{\Gamma\in T_{\pi,n}}\frac{Z_\Gamma(\imath,\pi,h,q_i)}{|\operatorname{Aut}\Gamma|},\qquad \textit{for}\quad n\geq 2.
$$

Two functors are canonically associated to an L_∞-algebra, to define them let specify the L_∞-structures on graded vector spaces that are involved in the definitions.

Example 5.1.17. Given an L_∞-algebra (V,q_i) and a commutative $\mathbb{K}$-algebra $\mathfrak{m}$, there exists a natural L_∞-structure on the tensor product $V\otimes\mathfrak{m}$ with brackets given by:

$$
q_n((x_1\otimes r_1)\odot\ldots\odot(x_n\otimes r_n))=q_n(x_1\odot\ldots\odot x_n)\otimes r_1\cdot\ldots\cdot r_n.
$$

If $\mathfrak{m}$ is nilpotent (for example, if $\mathfrak{m}$ is the maximal ideal of a local Artinian $\mathbb{K}$-algebra), then the L_∞-algebra $V\otimes\mathfrak{m}$ is nilpotent.

Example 5.1.18. Let $\mathbb{K}[t,dt]$ be the DGLA of polynomial differential forms over the affine line, the differential and the bracket are given by:

$$
d(p(t)+r(t)dt)=p'(t)dt,\quad [p(t)+r(t)dt,m(t)+n(t)dt]=p(t)m(t)+p(t)n(t)dt+m(t)r(t)dt.
$$

Let (V,q_i) be an L_∞-algebra. Then there is an L_∞-structure on $V[t,dt]=V\otimes\mathbb{K}[t,dt]$, which brackets are given by:

$$
q_n(v_1p_1(t)\odot\ldots\odot v_np_n(t))=q_n(v_1\odot\ldots\odot v_n)\cdot p_1(t)\ldots p_n(t)\qquad\text{and}
$$

$$
q_n(v_1p_1(t)\odot\ldots\odot v_{n-1}p_{n-1}(t)\odot w_nr_n(t)dt)=q_n(v_1\odot\ldots\odot v_{n-1}\odot w_n)\cdot p_1(t)\ldots p_{n-1}(t)r_n(t)dt.
$$

We are now ready to define two functors associated to a L_∞-algebra.

Definition 5.1.19. *Let (V, q_i) be an L_∞-algebra, the* Maurer-Cartan functor *associated to it is the functor*

$$\mathrm{MC}_V : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

defined, for all $(A, \mathfrak{m}_A) \in \mathbf{Art}_{\mathbb{K}}$, by:

$$\mathrm{MC}_V(A) = \left\{ x \in V[1]^0 \otimes \mathfrak{m}_A \;\middle|\; \sum_{n\geq 1} \frac{q_n(x^{\odot n})}{n!} = 0 \right\}.$$

The sum $\displaystyle\sum_{n\geq 1} \frac{q_n(\gamma^{\odot n})}{n!}$ has a finite number of nonzero terms, because $\mathfrak{m}_A$ is nilpotent. This equation is called the generalized Maurer-Cartan equation*.*

Definition 5.1.20. *Two elements $x, y \in \mathrm{MC}_V(A)$ are said to be* homotopy equivalent*, if there exists $z(t, dt) \in \mathrm{MC}_{V[t,dt]}(A)$, such that $z(0) = x$ and $z(1) = y$.*

Remark 5.1.21. Obviously the set of the Maurer-Cartan solutions $\mathrm{MC}_V(A)$ in closed under homotopy relation, because the homotopy $z(t, dt)$ is an element in $\mathrm{MC}_{V[t,dt]}(A)$, then $z(k) \in \mathrm{MC}_V(A)$ for every $k \in \mathbb{K}$.
Moreover the homotopy relation is an equivalence relation, a proof of this fact can be found in [23], Chapter 9.

Definition 5.1.22. *Let (V, q_i) be an L_∞-algebra, the* deformation functor *associated to it is the functor*

$$\mathrm{Def}_V : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

defined, for all $(A, \mathfrak{m}_A) \in \mathbf{Art}_{\mathbb{K}}$, by:

$$\mathrm{Def}_V(A) = \frac{\mathrm{MC}_V(A)}{\sim_{\text{homotopy}}}.$$

Remark 5.1.23. Let L be a DGLA, as observed in Remark 5.1.9, it has a L_∞-structure. Note that the generalized Maurer-Cartan equation for the L_∞-algebra L is exactly the Maurer-Cartan equation for the DGLA L, since $q_n = 0$ for $n \geq 3$. Moreover, for a DGLA the homotopy equivalence coincide with the gauge equivalence, this fact is proved in [6], Corollary 7.6. Then the deformation functors associated to L as DGLA and as L_∞-algebra coincide.

The correct setting to study the functors MC_V and Def_V associated to an L_∞-algebra V is that of extended deformation functors (see [22] and [23]). An extended functor F is a set valued functor defined on a category containing $\mathbf{Art}_{\mathbb{K}}$ as a full subcategory, for example on the category of $\mathbb{Z}$-graded associative and commutative nilpotent finite dimensional $\mathbb{K}$-algebras. Such a functor is called a *predeformation* or a *deformation* functor, if it satisfies some extended Schlessinger's conditions (see Definition 2.1 in [22]). Moreover this theory associates to every predeformation functor F a deformation functor F^+, defined as the quotient of F with an equivalent relation (see Theorem 2.8 in [22]). In the case of the functors associate to an L_∞-algebra V, the following results hold:

Lemma 5.1.24. *The functor MC_V is a predeformation functor.*

Proof. See Proposition 5.9 in [23]. □

Proposition 5.1.25. *The functor* Def_V *is a deformation functor.*

Proof. It is sufficient to see that the deformation functor MC_V^+ associated to the predeformation functor MC_V is exacly the functor Def_V, this helds because the equivalent relation $\sim$ that defines MC_V^+ is exactly the homotopy relation. □

For a predeformation functor F, the *tangent spaces* T^nF are defined (see Definition V.63 in [23]); in the case F is a deformation functor, the tangent spaces are given by $T^nF = F(\mathbb{K}\epsilon)$, where ϵ is an indeterminate of degree $-n+1$, such that $\epsilon^2 = 0$. The spaces T^1F and T^2F are also is called the *tangent space* and an *obstruction space* for F. In the case of the functors associated to an L_∞-algebra V, the following result helds:

Proposition 5.1.26. *The tangent spaces for the functors* MC_V *and* Def_V *are* $T^i\,\mathrm{MC}_V = T^i\,\mathrm{Def}_V = H^i(V, q_1)$.

Proof. Let's compute directly the tangent space $T^1(\mathrm{Def}_V) = \mathrm{Def}_V(\mathbb{K}[\epsilon])$. We have:

$$\mathrm{MC}_V(\mathbb{K}[\epsilon]) = \left\{ x \in V[1]^0 \otimes \mathbb{K}\epsilon \;\middle|\; \sum_{n\geq 1} \frac{q_n(x^{\odot n})}{n!} = q_1(x) = 0 \right\} = Z^1(V) \otimes \mathbb{K}\epsilon.$$

If $x, y \in \mathrm{MC}_V(\mathbb{K}[\epsilon])$ are hotomopy equivalent, there exists $z(t,dt) \in \mathrm{MC}_{V[t,dt]}(\mathbb{K}[\epsilon])$, such that $z(0) = x$ and $z(1) = y$; since $z(t,dt) = \mu(t) + \eta(t)dt \in \mathrm{MC}_{V[t,dt]}(\mathbb{K}[\epsilon])$, it satisfies

$$\mu'(t) = q_1(\eta(t)) \quad \text{and} \quad q_1(\mu(t)) = 0;$$

the element $w = \displaystyle\int_0^1 \eta(t)dt \in V^0$ is such that $x - y = q_1(w)$. Viceversa, if for $x, y \in \mathrm{MC}_V(\mathbb{K}[\epsilon])$ there exists $w \in V^0$ such that $x - y = q_1(w)$, define $z(t,dt) = (x + q_1(w)t) + wdt$; it is obviously the homotopy we are looking for. Then

$$T^1(\mathrm{Def}_V) = \mathrm{Def}_V(\mathbb{K}[\epsilon]) = \frac{\mathrm{MC}_V(\mathbb{K}[\epsilon])}{\sim_{\text{homotopy}}} = \frac{Z^1(V)\otimes\mathbb{K}\epsilon}{B^1(V)\otimes\mathbb{K}\epsilon} = H^1(V)\otimes\mathbb{K}\epsilon.$$

For the other tangent spaces see Proposition IX.14 and Theorem IX.19 in [23]. □

As in the DGLAs case, the deformation functor associated to an L_∞-algebra is defined up to quasi-isomorphism, i.e. quasi-isomorphic L_∞-algebras have isomorphic deformation functors. Infact:

Proposition 5.1.27. *Let* $f : (V, q_i) \to (W, \hat{q}_j)$ *be a morphism of* L_∞*-algebras and let* $\hat{f} : \mathrm{Def}_V \to \mathrm{Def}_W$ *the induced morphism of functors. If* f *is a quasi-isomorphism, then* $\hat{f}$ *is an isomorphism.*

Proof. See [23], Corollary IX.22. □

5.2 Semicosimplicial dglas

In this section we define semicosimplicial DGLAs and construct a canonical L_∞-structure on the total complex of a semicosimplicial DGLA, obtained by homotopical transfer from the Thom-Whitney DGLA. Then we define deformation functors associated to these objects.

Definition 5.2.1. *A* semicosimplicial differential graded Lie algebra *is a covariant functor $\mathfrak{g}^\Delta : \mathbf{\Delta}_{\mathrm{mon}} \to \mathbf{DGLA}$, from the category $\mathbf{\Delta}_{\mathrm{mon}}$, whose objects are finite ordinal sets and whose arrows are order-preserving injective maps between them, to the category of DGLAs.*
Equivalently, a semicosimplicial DGLA $\mathfrak{g}^\Delta$ *is a diagram*

$$\mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathfrak{g}_2 \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

where each $\mathfrak{g}_i$ is a DGLA, and for each $i > 0$ there are $i + 1$ morphisms of DGLAs

$$\partial_{k,i} \colon \mathfrak{g}_{i-1} \to \mathfrak{g}_i, \qquad \textit{for} \quad k = 0, \dots, i,$$

such that $\partial_{k+1,i+1}\partial_{l,i} = \partial_{l,i+1}\partial_{k,i}$, for any $k \geq l$.

Remark 5.2.2. For future use, we write explicitly the relations $\partial_{k+1,i+1}\partial_{l,i} = \partial_{l,i+1}\partial_{k,i}$, for $k \geq l$, for the maps $\partial_{0,1}, \partial_{1,1} \colon \mathfrak{g}_0 \to \mathfrak{g}_1$ and $\partial_{0,2}, \partial_{1,2}, \partial_{2,2} \colon \mathfrak{g}_1 \to \mathfrak{g}_2$. These morphisms satisfy

$$\partial_{1,2}\partial_{0,1} = \partial_{0,2}\partial_{0,1}, \quad \partial_{2,2}\partial_{0,1} = \partial_{0,2}\partial_{1,1}, \quad \partial_{2,2}\partial_{1,1} = \partial_{1,2}\partial_{1,1}.$$

The maps

$$\partial_i = \partial_{0,i} - \partial_{1,i} + \cdots + (-1)^i \partial_{i,i}$$

endow the vector space $\bigoplus_i \mathfrak{g}_i$ with the structure of a differential complex. Moreover, being a DGLA, each $\mathfrak{g}_i$ is in particular a differential complex and, since the maps $\partial_{k,i}$ are morphisms of DGLAs, the space

$$\mathfrak{g}_*^* = \bigoplus_{i,j} \mathfrak{g}_i^j$$

has a natural bicomplex structure. The associated total complex $(\mathrm{Tot}(\mathfrak{g}^\Delta), \delta)$ has no natural DGLA structure, but it can be endowed with a canonical structure on L_∞-algebra.

Example 5.2.3. Let X be a topological space and let $\mathcal{U} = \{U_i\}$ be an open covering of X. Let $\mathcal{L}$ be a sheaf of Lie algebras on X and let $\mathcal{L}(\mathcal{U})$ be the associated Čech semicosimplicial Lie algebra:

$$\mathcal{L}(\mathcal{U}) : \qquad \textstyle\prod_i \mathcal{L}(U_i) \rightrightarrows \prod_{i<j} \mathcal{L}(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i<j<k} \mathcal{L}(U_{ijk}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots,$$

where the morphisms $\partial_{k,j} : \prod_{i_0 \dots i_{j-1}} \mathcal{L}(U_{i_0 \dots i_{j-1}}) \to \prod_{i_0 \dots i_j} \mathcal{L}(U_{i_0 \dots i_j})$ are given by $(\partial_{k,j}(x))_{i_0 \dots i_j} = x_{i_0 \dots \widehat{i_k} \dots i_j}$, for all $k = 0, \dots, j$. The total complex $\mathrm{Tot}(\mathcal{L}(\mathcal{U}))$ associated to this semicosimplicial Lie algebra is given by:

$$0 \longrightarrow \prod_i \mathcal{L}(U_i) \longrightarrow \prod_{i<j} \mathcal{L}(U_{ij}) \longrightarrow \prod_{i<j<k} \mathcal{L}(U_{ijk}) \longrightarrow \dots,$$

with the Čech differential $(\partial_j(x))_{i_0 \dots i_j} = \partial_{0,j}(x)_{i_0 \dots i_j} - \partial_{1,j}(x)_{i_0 \dots i_j} + \dots + (-1)^j \partial_{j,j}(x)_{i_0 \dots i_j}$ $= x_{i_1 \dots i_j} - x_{i_0 i_2 \dots i_j} + \dots + (-1)^j x_{i_0 \dots i_{j-1}} = (\check{\delta}(x))_{i_0 \dots i_j}$. Thus $\mathrm{Tot}(\mathcal{L}(\mathcal{U}))$ is the Čech complex of the sheaf $\mathcal{L}$.

In the same way, taking a sheaf of differential graded Lie algebras $\mathcal{C}^\cdot$, one can define the associated Čech semicosimplicial DGLA $\mathcal{C}^\cdot(\mathcal{U})$ and the total complex $\mathrm{Tot}(\mathcal{C}^\cdot(\mathcal{U}))$.

Example 5.2.4. Following the above example, let X be a complex manifold and let $\mathcal{U} = \{U_i\}$ be an open covering of X.
Taking $\mathcal{L}$ to be the tangent sheaf $\mathcal{T}_X$ of X, the associated Čech semicosimplicial Lie algebra $\mathcal{T}_X(\mathcal{U})$ has as total complex $\mathrm{Tot}(\mathcal{T}_X(\mathcal{U}))$ the Čech complex of the sheaf $\mathcal{T}_X$.
Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X, taking $\mathcal{L}$ to be $\mathrm{End}\,\mathcal{E}$ the sheaf of the endomorphisms of $\mathcal{E}$ or $D^1(\mathcal{E})$ the sheaf of the first order differential operators on $\mathcal{E}$, the associated Čech semicosimplicial Lie algebras have as total complexes $\mathrm{Tot}(\mathrm{End}\,\mathcal{E}(\mathcal{U}))$ and $\mathrm{Tot}(D^1(\mathcal{E})(\mathcal{U}))$, given by the Čech complexes of the sheaves $\mathrm{End}\,\mathcal{E}$ and $D^1(\mathcal{E})$ respectively.

To a semicosimplicial DGLA $\mathfrak{g}^\Delta$ can be associated the Thom-Whitney DGLA, let's define it.
For every $n \geq 0$, we denote by Ω_n the differential graded commutative algebra of polynomial differential forms on the standard n-simplex Δ^n:

$$\Omega_n = \frac{\mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]}{(\sum_{i=0}^n t_i - 1, \sum_{i=0}^n dt_i)}.$$

For every semicosimplicial DGLA $\mathfrak{g}^\Delta$, the space

$$\bigoplus_n \Omega_n \otimes \mathfrak{g}_n$$

is a DGLA. Its degree k component is $\bigoplus_{p+q=k} \bigoplus_n \Omega_n^p \otimes \mathfrak{g}_n^q$, the differential is given by $d((\omega_n \otimes \gamma_n)_n) = (d\omega_n \otimes \gamma_n + \omega_n \otimes d_{\mathfrak{g}_n}\gamma_n)_n$ and the bracket is given by $[(\omega_n \otimes \gamma_n)_n, (\eta_n \otimes \lambda_n)_n] = (\omega_n \wedge \eta_n \otimes [\gamma_n, \lambda_n]_{\mathfrak{g}_n})_n$.

Definition 5.2.5. *The* Thom-Whitney DGLA *associated to the semicosimplicial DGLA* $\mathfrak{g}^\Delta$ *is*

$$\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta) = \{(x_n)_n \in \bigoplus_n \Omega_n \otimes \mathfrak{g}_n \mid \delta^{k,n} x_n = \partial_{k,n} x_{n-1} \quad \forall\, 0 \leq k \leq n\},$$

where, for $k = 0, \ldots, n$, $\delta^{k,n} \colon \Omega_n \to \Omega_{n-1}$ *are the face maps and* $\partial_{k,n} \colon \mathfrak{g}_{n-1} \to \mathfrak{g}_n$ *are the maps of the semicosimplicial DGLA* $\mathfrak{g}^\Delta$. *Its differential and bracket are defined as above. We indicate by* $C^{p,q}_{TW}(\mathfrak{g}^\Delta) = \mathrm{Tot}_{TW}(\mathfrak{g}^\Delta) \cap (\bigoplus_n \Omega_n^p \otimes \mathfrak{g}_n^q)$.

Remark 5.2.6. For future use, we write explicitly the equations $\delta^{k,n} x_n = \partial_{k,n} x_{n-1}$ that define the Thom-Whitney DGLA, for $n = 1, 2$ and $0 \leq k \leq n$.
Under the identifications $\Omega_1 \simeq \mathbb{K}[t, dt]$ via $(t_0, t_1) \leftrightarrow (t, 1-t)$, and $\Omega_2 \simeq \mathbb{K}[s_0, s_1, ds_0, ds_1]$ via $(t_0, t_1, t_2) \leftrightarrow (s_0, s_1, 1 - s_0 - s_1)$, the face maps read

$$\delta^{0,1} \colon \eta_1(t, dt) \mapsto \eta_1(0), \qquad \delta^{1,1} \colon \eta_1(t, dt) \mapsto \eta_1(1),$$
$$\delta^{0,2} \colon \eta_2(s_0, s_1; ds_0, ds_1) \mapsto \eta_2(0, t, dt), \qquad \delta^{1,2} \colon \eta_2(s_0, s_1, ds_0, ds_1) \mapsto \eta_2(t, 0, dt),$$
$$\delta^{2,2} \colon \eta_2(s_0, s_1, ds_0, ds_1) \mapsto \eta_2(t, 1-t, dt).$$

The Thom-Whitney DGLA $\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$ is then the subalgebra of $\mathfrak{g}_0 \oplus (\mathfrak{g}_1 \otimes \Omega_1) \oplus (\mathfrak{g}_2 \otimes \Omega_2) \oplus \ldots$ consisting of elements $(l, m(t, dt), n(s_0, s_1, ds_0, ds_1), \ldots)$ satisfying the face conditions

$$m(0) = \partial_{0,1}(l), \qquad m(1) = \partial_{1,1}(l),$$

$$n(0, t, dt) = \partial_{0,2}(m(t, dt)), \quad n(t, 0, dt) = \partial_{1,2}(m(t, dt)), \quad n(t, 1-t, dt) = \partial_{2,2}(m(t, dt)) \quad \ldots$$

For the total complex $\operatorname{Tot}(\mathfrak{g}^\Delta)$ and the Thom-Whitney DGLA $\operatorname{Tot}_{TW}(\mathfrak{g}^\Delta)$ the following holds:

Theorem 5.2.7. *There exist $E\colon \operatorname{Tot}(\mathfrak{g}^\Delta) \to \operatorname{Tot}_{TW}(\mathfrak{g}^\Delta)$ and $I\colon \operatorname{Tot}_{TW}(\mathfrak{g}^\Delta) \to \operatorname{Tot}(\mathfrak{g}^\Delta)$ morphisms of complexes and there exists $h : \operatorname{Tot}_{TW}(\mathfrak{g}^\Delta) \to \operatorname{Tot}_{TW}(\mathfrak{g}^\Delta)[-1]$ homotopy, such that $IE = \operatorname{Id}_{\operatorname{Tot}(\mathfrak{g}^\Delta)}$ and $EI - \operatorname{Id}_{\operatorname{Tot}_{TW}(\mathfrak{g}^\Delta)} = [h, d_{TW}]$.*

This theorem allow us to apply the homotopical transfer of structure theorem, Theorem 5.1.14. Then, by homotopical transfer from the Thom-Whitney DGLA $\operatorname{Tot}_{TW}(\mathfrak{g}^\Delta)$, the total complex $\operatorname{Tot}(\mathfrak{g}^\Delta)$ can be endowed with a canonical L_∞-structure that extends its differential complex structure and there exixts a L_∞-morphism E_∞ that extends E. We indicate with $\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)$ this L_∞-algebra.

Remark 5.2.8. Since E is a quasi-isomorphism of complexes, the morphism E_∞ gives a quasi-isomorphism between the L_∞-algebras $\operatorname{Tot}_{TW}(\mathfrak{g}^\Delta)$ and $\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)$.

Moreover, Dupont has described explicitly the morphisms I, E and h (see [30]). The morphism of complexes $I : \operatorname{Tot}_{TW}(\mathfrak{g}^\Delta) \to \operatorname{Tot}(\mathfrak{g}^\Delta)$ is simply given by the integration map

$$\int_{\Delta^n} \otimes \operatorname{Id}\colon \Omega_n \otimes \mathfrak{g}_n \to \mathfrak{g}_n.$$

The morphism of complexes $E : \operatorname{Tot}(\mathfrak{g}^\Delta) \to \operatorname{Tot}_{TW}(\mathfrak{g}^\Delta)$ is defined in the following way. If $\gamma \in \mathfrak{g}_i^j$, the element $E(\gamma) = (E(\gamma)_n) \in C_{TW}^{i,j}(\mathfrak{g}^\Delta)$ is given by:

$$\begin{cases} E(\gamma)_n = 0 & \text{if} \quad n < i, \\ E(\gamma)_n = i! \displaystyle\sum_{I \in I(i,n)} \omega_I \otimes \partial^{\bar{I}}\gamma & \text{if} \quad n \geq i, \end{cases}$$

where $I(i,n)$ is the set of all multiindices $I = (a_0, a_1, \dots, a_i) \in \mathbb{Z}^{i+1}$, such that $0 \leq a_0 < a_1 < \dots a_i \leq n$ and, if $I \in I(i,n)$, $\bar{I}$ is the complementary multiindex. If $I = (a_0, a_1, \dots, a_i) \in I(i,n)$, we indicate with ω_I the differential form:

$$\omega_I = \sum_{s=0}^{i} (-1)^s t_{a_s} dt_{a_0} \wedge dt_{a_1} \wedge \dots \wedge \widehat{dt_{a_s}} \wedge \dots \wedge dt_{a_i} \in \Omega_n^i.$$

If $\bar{I} = (b_1, b_2, \dots, b_{n-i})$ is the complementary multiindex of I, and $\gamma \in \mathfrak{g}_i^j$, we indicate with $\partial^{\bar{I}}\gamma$ the element

$$\partial^{\bar{I}}\gamma = \partial_{b_{n-i},n} \circ \dots \circ \partial_{b_2,i+2} \circ \partial_{b_1,i+1}\gamma \in \mathfrak{g}_n^j.$$

The homotopy $h : \operatorname{Tot}_{TW}(\mathfrak{g}^\Delta) \to \operatorname{Tot}_{TW}(\mathfrak{g}^\Delta)[-1]$ is defined in the following way. If $x = (x_n) = (\eta_n \otimes \gamma_n) \in C_{TW}^{i,j}(\mathfrak{g}^\Delta)$, the image $h(x) = (h(x)_n) \in C_{TW}^{i-1,j}(\mathfrak{g}^\Delta)$ is given by:

$$h(x)_n = \sum_{0 \leq r < i} \sum_{I \in I(r,n)} r!\, \omega_I \wedge h_I(\eta_n) \otimes \gamma_n,$$

where $I(r,n)$ and ω_I are as before, and, if $I = (a_0, \dots, a_r) \in I(r,n)$, then the map h_I is given by the composition $h_I = h_{a_r} \circ \dots \circ h_{a_0}$, where the maps $h_a = \pi \circ \psi_a^*\colon \Omega_n^* \to \Omega_n^{*-1}$ are the compositions of the integration over the first factor

$$\pi : \Omega^*([0,1] \times \Delta_n) \to \Omega^*(\Delta_n) = \Omega_n^*$$

$$\eta(u, t_a, du, dt_a) \mapsto \int_{u \in [0,1]} \eta(u, t_a, du, dt_a)$$

and the pull-back by the dilation maps:

$$\psi_a : [0,1] \times \Delta_n \to \Delta_n$$
$$(u, t_0, \dots t_n) \mapsto ((1-u)t_0, \dots, (1-u)t_a + u, \dots (1-u)t_n).$$

More explicitly, the map $h_a : \Omega_n^* \to \Omega_n^{*-1}$ is given by:

$$h_a(\eta(t_0, \dots, t_n, dt_0, \dots, dt_n)) = \int_{u\in[0,1]} \eta((1-u)t_0, \dots, (1-u)t_a + u, \dots (1-u)t_n, du, dt_a).$$

Using formulas of Proposition 5.1.16 and the above descriptions of the morphisms I, E and h, it is possible to explicitate the L_∞-structure defined on $\mathrm{Tot}(\mathfrak{g}^\Delta)$ by homotopical transfert writing its brackets.

These constructions lead to the definition of two deformation functors associated to a semicosimplicial DGLA. Let

$$\mathfrak{g}^\Delta : \ \mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \mathfrak{g}_2 \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \cdots$$

be a semicosimplicial DGLA. Let $\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$ be its Thom-Whitney DGLA, then there is a deformation functor $\mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)}$ associated to it as DGLA and this functor coincide with the functor associated to it as L_∞-algebra (see Remark 5.1.23).
Let $\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)$ be the total complex of $\mathfrak{g}^\Delta$ endowed with the L_∞-structure constructed above, then there is a deformation functor $\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)}$ associated to it as L_∞-algebra.
Since the L_∞-algebras $\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$ and $\widetilde{\mathrm{Tot}(\mathfrak{g}^\Delta)}$ are quasi-isomorphic (see Remark 5.2.8), the associated deformation functors are canonically isomorphic $\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)} \simeq \mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)}$.

5.3 The semicosimplicial Lie algebras case

In this section we restrict our study to semicosimplicial Lie algebras and we find an explicit and more significant way to rewrite functors $\mathrm{MC}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)}$ and $\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)}$. All the results of this Section are obtained in a joint work with D. Fiorneza and M. Manetti (see [7]).

Let

$$\mathfrak{g}^\Delta : \ \mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \mathfrak{g}_2 \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \cdots ,$$

be a semicosimplicial Lie algebra, where $\mathfrak{g}_i$ are Lie algebras. As before we define the total complex $\mathrm{Tot}(\mathfrak{g}^\Delta)$, the Thom-Whitney DGLA $\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$, from which we transfert the L_∞-structure to $\mathrm{Tot}(\mathfrak{g}^\Delta)$ and the functors $\mathrm{MC}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)}$, $\mathrm{MC}_{\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)}$ and $\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)}$, $\mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)}$.

We introduce two other functors associated to a semicosimplicial Lie algebra:

Definition 5.3.1. *Let $\mathfrak{g}^\Delta$ be a semicosimplicial Lie algebra, the functor*

$$Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta) : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

is defined, for all $(A,\mathfrak{m}_A)\in\mathbf{Art}_{\mathbb{K}}$, by:

$$Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta\otimes\mathfrak{m}_A))=\{x\in\mathfrak{g}_1\otimes\mathfrak{m}_A\mid e^{\partial_{0,2}(x)}e^{-\partial_{1,2}(x)}e^{\partial_{2,2}(x)}=1\}.$$

This functor can be rewritten in terms of the Baker-Camper-Hausdorff product as:

$$Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta\otimes\mathfrak{m}_A))=\{x\in\mathfrak{g}_1\otimes\mathfrak{m}_A\mid \partial_{2,2}(x)\bullet\partial_{0,2}(x)=\partial_{1,2}(x)\},\qquad (A,\mathfrak{m}_A)\in\mathbf{Art}_{\mathbb{K}}.$$

Definition 5.3.2. *Let $x,y\in Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta\otimes\mathfrak{m}_A))$, they are said to be* equivalent *under the relation '$\sim$' iff*

$$\exists\ l\in\mathfrak{g}_0\otimes\mathfrak{m}_A,\quad e^{-\partial_{1,1}(l)}e^{x}e^{\partial_{0,1}(l)}=e^{y}.$$

This equivalent relation can be rewritten in terms of the Baker-Camper-Hausdorff product as:

$$x\sim y\quad\textit{iff}\quad\exists\ l\in\mathfrak{g}_0\otimes\mathfrak{m}_A,\quad(-\partial_{1,1}(l))\bullet x\bullet\partial_{0,1}(l)=y.$$

Remark 5.3.3. Note that the set $Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta\otimes\mathfrak{m}_A))$ is closed under the relation '$\sim$', i.e., if $x\in Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta\otimes\mathfrak{m}_A))$, $y\in\mathfrak{g}_1\otimes\mathfrak{m}_A$ and $x\sim y$ via $l\in\mathfrak{g}_0\otimes\mathfrak{m}_A$, then $y\in Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta\otimes\mathfrak{m}_A))$. Infact

$$e^{\partial_{0,2}(y)}e^{-\partial_{1,2}(y)}e^{\partial_{2,2}(y)}=$$
$$=e^{-\partial_{0,2}\partial_{1,1}(l)}e^{\partial_{0,2}(x)}e^{\partial_{0,2}\partial_{0,1}(l)}e^{-\partial_{1,2}\partial_{0,1}(l)}e^{-\partial_{1,2}(x)}e^{\partial_{1,2}\partial_{1,1}(l)}e^{-\partial_{2,2}\partial_{1,1}(l)}e^{\partial_{2,2}(x)}e^{\partial_{2,2}\partial_{0,1}(l)}=$$
$$=e^{-\partial_{0,2}\partial_{1,1}(l)}\cdot 1\cdot e^{\partial_{2,2}\partial_{0,1}(l)}=1,$$

where we used Remark 5.2.2 and the equation that define $Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta\otimes\mathfrak{m}_A))$.

Note that the relation $\sim$ is an equivalent relation, in fact if

$$x\sim y\quad\text{i.e.}\quad e^{-\partial_{1,1}(l)}e^{x}e^{\partial_{0,1}(l)}=e^{y}\quad\text{and}\quad y\sim z\quad\text{i.e.}\quad e^{-\partial_{1,1}(m)}e^{y}e^{\partial_{0,1}(m)}=e^{z},$$

then

$$e^{z}=e^{-\partial_{1,1}(m)}e^{y}e^{\partial_{0,1}(m)}=e^{-\partial_{1,1}(l\bullet m)}e^{x}e^{\partial_{0,1}(l\bullet m)},$$

as we want.

Definition 5.3.4. *Let $\mathfrak{g}^\Delta$ be a semicosimplicial Lie algebra, the functor*

$$H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta):\mathbf{Art}_{\mathbb{K}}\to\mathbf{Set}$$

is defined, for all $(A,\mathfrak{m}_A)\in\mathbf{Art}_{\mathbb{K}}$, by:

$$H^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta\otimes\mathfrak{m}_A))=Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta\otimes\mathfrak{m}_A))/\sim.$$

Thus to a semicosimplicial Lie algebra $\mathfrak{g}^\Delta$ are associated now the two functors $Z^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)$ and $H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)$ defined above. The geometric meaning of them is evident in the following example.

Example 5.3.5. Let X be a topological space and let $\mathcal{U}=\{U_i\}$ be an open covering of X. Let $\mathcal{L}$ be a sheaf of Lie algebras on X and let $\mathcal{L}(\mathcal{U})$ be the associated Čech semicosimplicial Lie algebra (see Example 5.2.3). The functor $H^1_{\mathrm{sc}}(\exp\mathcal{L}(\mathcal{U}))$ is given, for all $(A,\mathfrak{m}_A)\in\mathbf{Art}_{\mathbb{K}}$, by:

$$A\to H^1(\mathcal{U},\exp(\mathcal{L}\otimes\mathfrak{m}_A));$$

it associates, to every $(A,\mathfrak{m}_A)\in\mathbf{Art}_{\mathbb{K}}$, the first Čech cohomology space for the covering $\mathcal{U}$ with coefficients in the sheaf of groups $\exp(\mathcal{L}\otimes\mathfrak{m}_A)$. Taking the direct limit over the open coverings of X, we obtain the functor $H^1(X;\exp\mathcal{L})$, given, for all $(A,\mathfrak{m}_A)\in\mathbf{Art}_{\mathbb{K}}$, by:

$$A \to \varinjlim_{\mathcal{U}} H^1_{\rm sc}(\exp(\mathcal{L}(\mathcal{U})\otimes\mathfrak{m}_A)) = \varinjlim_{\mathcal{U}} H^1(\mathcal{U},\exp(\mathcal{L}\otimes\mathfrak{m}_A)) = H^1(X;\exp(\mathcal{L}\otimes\mathfrak{m}_A));$$

it associates, to every $(A,\mathfrak{m}_A)\in\mathbf{Art}_{\mathbb{K}}$, the first Čech cohomology space of X with coefficients in the sheaf of groups $\exp(\mathcal{L}\otimes\mathfrak{m}_A)$.

Our aim is to prove the following:

Theorem 5.3.6 (Main Theorem). *Let $\mathfrak{g}^\Delta$ be a semicosimplicial Lie algebra. Then the two deformation functors $\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)}$ and $H^1_{\rm sc}(\exp\mathfrak{g}^\Delta)$ are isomorphic.*

First Step. At first we note that the up to isomorphism the functors $\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)}$ and $\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)}$ are defined by a truncation of the semicosimplicial Lie algebra $\mathfrak{g}^\Delta$.

Let $m_1\in\mathbb{N}$ and $m_2\in\mathbb{N}\cup\{\infty\}$ with $m_1\le m_2$, we denote by $\mathfrak{g}^{\Delta_{[m_1,m_2]}}$ the *truncated between levels m_1 and m_2* semicosimplicial Lie algebra defined by $(\mathfrak{g}^{\Delta_{[m_1,m_2]}})_n=\mathfrak{g}_n$ for $m_1\le n\le m_2$ and $(\mathfrak{g}^{\Delta_{[m_1,m_2]}})_n=0$ otherwise, with the obvious maps $\partial_{k,i}^{[m_1,m_2]}=\partial_{k,i}$ for $m_1<i\le m_2$ and $\partial_{k,i}^{[m_1,m_2]}=0$ otherwise. For any positive integers m_1,m_2,r_1,r_2, such that $r_i\le m_i$, the map $\operatorname{Id}_{[m_1,r_2]}\colon\,\mathfrak{g}^{\Delta_{[m_1,m_2]}}\to\mathfrak{g}^{\Delta_{[r_1,r_2]}}$ given by

$$\operatorname{Id}_{[m_1,r_2]}\Big|_{(\mathfrak{g}^{\Delta_{[m_1,m_2]}})_n}=\begin{cases}\operatorname{Id}_{\mathfrak{g}_n} & \text{if } m_1\le n\le r_2\\ 0 & \text{otherwise.}\end{cases}$$

is a morphism of semicosimplicial Lie algebras, and so it induces a linear morphism of L_∞-algebras $\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[m_1,m_2]}})\to\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[r_1,r_2]}})$.

Proposition 5.3.7. *Let $\mathfrak{g}^\Delta$ be a semicosimplicial Lie algebra. Then the morphism $\operatorname{Id}_{[m_1,r_2]}$ induces natural isomorphisms*

$$\begin{aligned}\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[m_1,m_2]}})}&\to\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[r_1,r_2]}})}, &&\text{for } m_1\le 1 \text{ and } 2\le r_2\le m_2;\\ \operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[0,m_2]}})}&\to\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[0,r_2]}})}, &&\text{for } 2\le r_2\le m_2.\end{aligned}$$

Proof. We have $\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[m_1,m_2]}})^i=\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[r_1,r_2]}})^i=\mathfrak{g}_i$, for $m_1\le i\le r_2$. In particular the linear L_∞ morphism $\operatorname{Id}_{[m_1,r_2]}:\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[m_1,m_2]}})\to\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[r_1,r_2]}})$ is the identity on degree one and degree two elements. Thus $\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[m_1,m_2]}})}=\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[r_1,r_2]}})}$.

The same argument says that $\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[0,m_2]}})[\xi,d\xi]}=\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[0,r_2]}})[\xi,d\xi]}$, thus the homotopy relation between elements in $\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[0,m_2]}})}$ is the same as the homotopy relation between elements in $\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[0,r_2]}})}$. $\square$

Remark 5.3.8. In what follows we use the above proposition only for $m_1=0$ and $r_2=2$, obtaining the isomorphisms

$$\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)}\to\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[0,2]}})}\quad\text{and}\quad\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)}\to\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[0,2]}})}.$$

Second Step. To prove the Main Theorem we pass through the Thom-Whitney DGLA, remembering that it is quasi-isomorphic to the L_∞-algebra $\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)$ (see Remark 5.2.8). For this we need an explicit description of the solutions of the Maurer-Cartan equation in the DGLA $\operatorname{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})$.

We recall the following result (see [6], Proposition 7.2):

Lemma 5.3.9. *Let $(L, d, [\ ,\ ])$ be a differential graded Lie algebra such that:*

1. *$L = M \oplus C \oplus D$ as graded vector spaces.*
2. *M is a differential graded subalgebra of L.*
3. *$d\colon C \to D[1]$ is an isomorphism of graded vector spaces.*

Then, for every $A \in \mathbf{Art}$ there exists a bijection

$$\alpha\colon\ \mathrm{MC}_M(A) \times (C^0 \otimes \mathfrak{m}_A) \xrightarrow{\sim} \mathrm{MC}_L(A), \qquad (x, c) \mapsto e^c * x.$$

Corollary 5.3.10. *Let $\mathfrak{g}$ be a DGLA concentrated in nonnegative degrees, and ξ_0 a point in the n-simplex Δ_n. Then every Maurer-Cartan element in $\mathfrak{g}\otimes\Omega_n$ is of the form $e^p * x$, for a unique polynomial $p(\xi) \in \mathfrak{g}^0 \otimes \Omega_n^0$ with $p(\xi_0) = 0$ and a unique Maurer-Cartan element x for $\mathfrak{g}$. In particular, if $\mathfrak{g}$ is a Lie algebra, then every Maurer-Cartan element in $\mathfrak{g} \otimes \Omega_n$ is of the form $e^p * 0$, for a unique polynomial $p(\xi) \in \mathfrak{g}^0 \otimes \Omega_n^0$ with $p(\xi_0) = 0$.*

Proof. The evaluation at ξ_0 is a quasi-isomorphism of differential complexes $\mathrm{ev}_{\xi_0}\colon\ \mathfrak{g}\otimes\Omega^n \to \mathfrak{g}$. Let $H = \ker(\mathrm{ev}_{\xi_0})$, and let $H = C \oplus \ker d\big|_H$ be a graded vector spaces decomposition of H; since H is an acyclic complex, $dC = \ker d\big|_H$. Then we have a decomposition $\mathfrak{g}\otimes\Omega^n = \mathfrak{g} \oplus C \oplus dC$ as in Lemma 5.3.9. Moreover $C^0 = H^0 = \{p \in \mathfrak{g}^0[s_1, \dots, s_n] \mid p(\xi_0) = 0\}$. □

Corollary 5.3.11. *Let $\mathfrak{g}^\Delta$ be a semicosimplicial Lie algebra, and let*

$$\mathfrak{g}^{\Delta_{[0,2]}}:\qquad \mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\ \mathfrak{g}_2 \ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\ 0$$

be its truncation at level three. Then the Maurer-Cartan functor associated to the Thom-Whitney DGLA of the semicosimplicial Lie algebra $\mathfrak{g}^{\Delta_{[0,2]}}$ is given by

$$\mathrm{MC}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}(A) = \left\{(0, e^{p(t)} * 0, e^{q(s_0,s_1)} * 0)\right\}$$

with uniquely determined polynomials $p \in \mathfrak{g}_1[t]$ and $q \in \mathfrak{g}_2[s_0, s_1]$ such that

$$p(0) = 0, \qquad q(0,t) = \partial_{0,2}p(t), \qquad q(t,0) = \partial_{1,2}p(t), \qquad q(t,1-t) = \partial_{2,2}p(t) \bullet \partial_{0,2}p(1).$$

Proof. Since $\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})$ is a sub-DGLA of $\mathfrak{g}_0 \oplus \mathfrak{g}_1[t] \oplus \mathfrak{g}_2[s_0, s_1]$, by Lemma 5.3.10, the solutions of the Maurer-Cartan equation on $\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})$ have the form $(0, e^{p(t)} * 0, e^{q(s_0,s_1)} * 0))$ with uniquely determined polynomials $p \in \mathfrak{g}_1[t]$ and $q \in \mathfrak{g}_2[s_0, s_1]$. The statement then follows by the face conditions and uniqueness. □

Third Step. Now we construct two morphisms between the two functors $\mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}$ and $H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)$ and we prove that are inverse to each other.

Proposition 5.3.12. *Let $\mathfrak{g}^\Delta$ be a semicosimplicial Lie algebra. The map*

$$\Phi : \mathrm{MC}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}(A) \to \mathfrak{g}_1 \otimes \mathfrak{m}_A,$$

given by $(0, e^{p(t)} * 0, e^{q(s_0,s_1)} * 0) \mapsto p(1)$ induces a morphism of functors $\mathrm{MC}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})} \to Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)$.

Proof. The polynomial $q(s_0, s_1) \bullet (-q(s_0, 0)) \bullet (-q(0, s_1))$ vanishes on the lines $s_0 = 0$ and $s_1 = 0$, so it is divisible by $s_0 s_1$. Let

$$\rho(s_0, s_1) = \frac{q(s_0, s_1) \bullet (-q(s_0, 0)) \bullet (-q(0, s_1))}{s_0 s_1}.$$

Then $e^{q(t,1-t)} = e^{t(1-t)\,\rho(t,1-t)} e^{q(0,1-t)} e^{q(t,0)}$, that is $e^{\partial_{2,2}p(t) \bullet \partial_{0,2}p(1)} = e^{t(1-t)\,\rho(t,1-t)} \cdot e^{\partial_{0,2}p(1-t)} e^{\partial_{1,2}p(t)}$. Evaluating at $t = 1$ we find

$$e^{\partial_{0,2}p(1)} e^{-\partial_{1,2}p(1)} e^{\partial_{2,2}p(1)} = 1.$$

□

Proposition 5.3.13. *The map* $\Phi\colon \mathrm{MC}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})} \to Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^{\Delta})$ *induces a morphism of functors* $\mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})} \to H^1_{\mathrm{sc}}(\exp \mathfrak{g}^{\Delta})$

Proof. We have to show that, if two elements $x = (0, e^{p_0(t)} * 0, e^{q_0(s_0,s_1)} * 0)$ and $y = (0, e^{p_1(t)} * 0_1, e^{q_1(s_0,s_1)} * 0)$ in $\mathrm{MC}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta})}(A)$ are homotopy equivalent, then there exists $\lambda \in \mathfrak{g}_0 \otimes \mathfrak{m}_A$ such that $\Phi(y) = -\partial_{1,1}(\lambda) \bullet \Phi(x) \bullet \partial_{0,1}(\lambda)$. Let $z(\xi, d\xi)$ be a homotopy between x an y. It is a Maurer-Cartan element in $\mathfrak{g}_0[\xi] \oplus \mathfrak{g}_1[t;\xi] \oplus \mathfrak{g}_2[s_0, s_1; \xi]$, with $z(0) = x$, then

$$z(\xi, d\xi) = (e^{H_0(\xi)} * 0, e^{H_1(t;\xi)} * e^{p_0(t)} * 0, e^{H_2(s_0,s_1;\xi)} * e^{q_0(s_0,s_1)} * 0),$$

with $H_0(0) = H_1(t; 0) = H_2(s_0, s_1; 0) = 0$. The face conditions for $z(\xi, d\xi)$ and uniqueness give us

$$H_1(0; \xi) = \partial_{0,1}(H_0(\xi)) \quad \text{and} \quad H_1(1; \xi) = \partial_{1,1}(H_0(\xi)).$$

Moreover $z(1) = y$ so, by uniqueness, we get

$$p_1(t) = H_1(t; 1) \bullet p_0(t) \bullet (-H_1(0; 1)),$$

and that, evaluating at $t = 1$ and using the face conditions, give

$$p_1(1) = \partial_{1,1}(H_0(1)) \bullet p_0(1) \bullet (-\partial_{0,1}(H_0(1)).$$

Hence the thesis, with $\lambda = -H_0(1)$. □

Lemma 5.3.14. *For* $x \in Z^1_{sc}(\exp(\mathfrak{g}^{\Delta} \otimes \mathfrak{m}_A))$, *let*

$$\theta(x, y, s) = \frac{(s(y \bullet x)) \bullet (-sx) \bullet (-sy)}{s(1-s)},$$

and

$$R(x; s_0, s_1) = (s_0 s_1 \theta(-\partial_{0,2}(x), \partial_{1,2}(x), s_0)) \bullet (s_0 \partial_{1,2}(x)) \bullet (s_1 \partial_{0,2}(x)).$$

Then we have

$$R(x; 0, t) = t\partial_{0,2}(x); \qquad R(x; t, 0) = t\partial_{1,2}(x); \qquad R(x; t, 1-t) = t(\partial_{2,2}(x)) \bullet \partial_{0,2}(x).$$

Proof. The only nontrivial identity is the last one, which uses the fact that x is an element of $Z^1_{sc}(\exp(\mathfrak{g}^{\Delta} \otimes \mathfrak{m}_A))$. □

Proposition 5.3.15. *The map* $\Psi : Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta \otimes \mathfrak{m}_A)) \to \mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}}) \otimes \mathfrak{m}_A$ *given by*

$$x \mapsto (0, e^{tx} * 0, e^{R(x;s_0,s_1)} * 0),$$

induces a morphism between the functors $Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta) \to \mathrm{MC}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}$.

Proof. The element $\Psi(x)$ clearly satisfies the Maurer-Cartan equation and the face conditions. □

Proposition 5.3.16. *The map* $\Psi\colon Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta) \to \mathrm{MC}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}$ *induces a morphism of functors* $H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta) \to \mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}$.

Proof. We have to show that if two elements x and y in $Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta \otimes \mathfrak{m}_A))$ are equivalent, i.e., there exists $\lambda \in \mathfrak{g}_0 \otimes \mathfrak{m}_A$ such that $y = (-\partial_{1,1}(\lambda)) \bullet x \bullet \partial_{0,1}(\lambda)$, then $\Psi(x)$ and $\Psi(y)$ are homotopy equivalent. Let $x(\xi) = (-\partial_{1,1}(\xi\lambda)) \bullet x \bullet \partial_{0,1}(\xi\lambda)$; then, $x(\xi) \sim x$, and so $x(\xi) \in Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta \otimes \mathfrak{m}_A))$ for any ξ. We define the element $z(\xi, d\xi)$ in $\mathfrak{g}_0[\xi, d\xi] \oplus \mathfrak{g}_1[t, dt; \xi, d\xi] \oplus \mathfrak{g}_2[s_0, s_1, ds_0, ds_1; \xi, d\xi]$ by

$$z(\xi, d\xi) = (e^{-\xi\lambda} * 0, e^{t\, x(\xi)\bullet(-\partial_{0,1}(\xi\lambda))} * 0, e^{R(x(\xi);s_0,s_1)\bullet(-\partial_{0,2}\partial_{0,1}(\xi\lambda))} * 0).$$

The element $z(\xi, d\xi)$ obviously satisfies the Maurer-Cartan equation in $\mathrm{MC}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}[\xi, d\xi]$; moreover, it satisfies the face conditions as a a consequence of the identities $\partial_{0,2}\partial_{0,1} = \partial_{1,2}\partial_{0,1}$ and $\partial_{0,2}\partial_{1,1} = \partial_{2,2}\partial_{0,1}$. Moreover the element $z(\xi, d\xi)$ gives a homotopy between $\Psi(x)$ and $\Psi(y)$, since $x(0) = x$ and $x(1) = y$. □

We now show that the natural transformations Φ and Ψ induce isomorphisms at the level of deformation functors. First, we need the following simple lemma.

Lemma 5.3.17. *If* $(0, e^{p(t)}*0, e^{q_0(s_0,s_1)}*0)$ *and* $(0, e^{p(t)}*0, e^{q_1(s_0,s_1)}*0)$ *are Maurer-Cartan elements in* $\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})$*, then* $(0, e^{p(t)} * 0, e^{q_0(s_0,s_1)} * 0) \sim (0, e^{p(t)} * 0, e^{q_1(s_0,s_1)} * 0)$.

Proof. The element $(0, e^{p(t)} * 0, e^{\xi q_1(s_0,s_1)+(1-\xi)q_0(s_0,s_1)} * 0)$ is a Maurer-Cartan element satisfying the face conditions and providing the desired homotopy. □

Proposition 5.3.18. *The two morphisms* $\Phi\colon \mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})} \to H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)$ *and* $\Psi\colon H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta) \to \mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}$*, are inverse to each other.*

Proof. The composition $\Phi \circ \Psi$ is clearly the identity, even at the Maurer-Cartan level. Now we prove that the composition

$$\mathrm{MC}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})} \xrightarrow{\Phi} Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta) \xrightarrow{\Psi} \mathrm{MC}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}$$

is homotopic to the identity, i.e. that

$$(0, e^{tp(1)} * 0, e^{R(p(1);s_0,s_1)} * 0) \sim (0, e^{p(t)} * 0, e^{q(s_0,s_1)} * 0)...$$

Let $p(t;\xi)$ be the convex combination $p(t;\xi) = \xi tp(1) + (1-\xi)p(t)$. Since $p(0;\xi) = 0$ and $p(1;\xi) = p(1) \in Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^\Delta \otimes \mathfrak{m}_A))$, the polynomial

$$\partial_{2,2}p(t;\xi) \bullet \partial_{0,2}p(1) \bullet (-\partial_{0,2}p(1-t;\xi)) \bullet (-\partial_{1,2}p(t;\xi))$$

vanishes at $t=0$ and at $t=1$. Let $\sigma(t;\xi)$ be the polynomial

$$\sigma(t;\xi)=\frac{\partial_{2,2}p(t;\xi)\bullet\partial_{0,2}p(1)\bullet(-\partial_{0,2}p(1-t;\xi))\bullet(-\partial_{1,2}p(t;\xi))}{1-t},$$

and let

$$S(s_0,s_1;\xi)=s_1\sigma(s_0;\xi)\bullet\partial_{1,2}p(s_0;\xi)\bullet\partial_{0,2}p(s_1;\xi).$$

Then we have $S(0,t;\xi)=\partial_{0,2}p(t;\xi)$, $S(t,0;\xi)=\partial_{1,2}p(t;\xi)$, and $S(t,1-t;\xi)=\partial_{2,2}p(t;\xi)\bullet\partial_{0,2}p(1)$. Let $q_0(s_0,s_1)=S(s_0,s_1;0)$ and $q_1(s_0,s_1)=S(s_0,s_1;1)$. Then the element

$$z(\xi,d\xi)=(0,e^{p(t;\xi)}*0,e^{S(s_0,s_1;\xi)}*0).$$

is a Maurer-Cartan element in $\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})[\xi,d\xi]$, and provides a homotopy

$$(0,e^{p(t)}*0,e^{q_0(s_0,s_1)}*0)\sim(0,e^{tp(1)}*0,e^{q_1(s_0,s_1)}*0).$$

Since, by Lemma 5.3.17 we have homotopies

$$(0,e^{p(t)}*0,e^{q(s_0,s_1)}*0)\sim(0,e^{p(t)}*0,e^{q_0(s_0,s_1)}*0)$$

and

$$(0,e^{tp(1)}*0,e^{q_1(s_0,s_1)}*0)\sim(0,e^{tp(1)}*0,e^{R(p(1);s_0,s_1)}*0),$$

the two Maurer-Cartan elements $(0,e^{p(t)}*0,e^{q(s_0,s_1)}*0)$ and $(0,e^{tp(1)}*0,e^{R(p(1);s_0,s_1)}*0)$ are homotopy equivalent. □

Now we are ready to prove the Main Theorem 5.3.6:

Proof. By Propositions 5.3.7 and 5.3.18 and by Remark 5.2.8, we have the following composition of isomorphisms:

$$\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^{\Delta})}=\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^{\Delta_{[0,2]}})}\xrightarrow{E_\infty}\mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}\xrightarrow{\Phi}H^1_{\mathrm{sc}}(\exp\mathfrak{g}^{\Delta}).$$

□

Remark 5.3.19. The Main Theorem 5.3.6 also assures that $H^1_{\mathrm{sc}}(\exp\mathfrak{g}^{\Delta})$ is a deformation functor.

The Main Theorem 5.3.6 gives an explicit and significant expression for the functor $\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^{\Delta})}$ associated to a semicosimplicial Lie algebras $\mathfrak{g}^{\Delta}$, moreover it allows to obtain an explicit description for the functor $\mathrm{MC}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^{\Delta})}$:

Corollary 5.3.20. *For any local artinian $\mathbb{C}$-algebra A, we have*

$$\mathrm{MC}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^{\Delta})}(\mathrm{A})=\mathrm{Z}^1_{\mathrm{sc}}(\exp(\mathfrak{g}^{\Delta}\otimes\mathfrak{m}_{\mathrm{A}})),$$

as subsets of $\mathfrak{g}_1\otimes\mathfrak{m}_A$.

Proof. By definition of Z^1_{sc} and of H^1_{sc} we have $Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^{\Delta}\otimes\mathfrak{m}_A))=Z^1_{\mathrm{sc}}(\exp(\mathfrak{g}^{\Delta_{[1,2]}}\otimes\mathfrak{m}_A))=H^1_{\mathrm{sc}}(\exp(\mathfrak{g}^{\Delta_{[1,2]}}\otimes\mathfrak{m}_A))$ and by propositon 5.3.7 we have $\mathrm{MC}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^{\Delta_{[1,2]}})}(A)=\mathrm{MC}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^{\Delta})}(A)$ By Theorem 5.3.6, the composition $\Phi\circ E_\infty:\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^{\Delta_{[1,2]}})}(A)\to H^1_{\mathrm{sc}}(\exp(\mathfrak{g}^{\Delta_{[1,2]}}\otimes\mathfrak{m}_A))$ is an isomorphism.

Now we show that $\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[1,2]}})}(A) = \operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[1,2]}})}(A)$, i.e., that the homotopy equivalence on $\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[1,2]}})}(A)$ is trivial. Indeed, since $(\operatorname{Tot}(\mathfrak{g}^{\Delta_{[1,2]}})[\xi, d\xi])^1 = \mathfrak{g}_1[\xi]$ and $(\operatorname{Tot}(\mathfrak{g}^{\Delta_{[1,2]}})[\xi, d\xi])^2 = \mathfrak{g}_1[\xi]d\xi \oplus \mathfrak{g}_2[\xi]$, the Maurer-Cartan equation for an element $x(\xi) \in \mathfrak{g}_1[\xi] \otimes \mathfrak{m}_A$ splits into the equations:

$$\begin{cases} \dfrac{\partial x(\xi)}{\partial \xi} = 0 \\ -d_{\mathfrak{g}_1} x(\xi) + \displaystyle\sum_{n \geq 2} \frac{q_n((x(\xi))^{\odot n})}{n!} = 0, \end{cases}$$

which tell us that $x(\xi) \equiv x(0) \in \operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[1,2]}})}$. Moreover, by Proposition 5.3.7, we have $\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[1,2]}})} = \operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[0,2]}})} = \operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta})}$.

To conclude, we prove that the composition $\Phi \circ E_\infty$ is the identity on elements in $\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[1,2]}})}$. Let $x \in \operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^{\Delta_{[1,2]}})}$, by definition, Φ reads only the $(\Omega_1 \otimes \mathfrak{g}_1)$-component of $E_\infty(x)$. We have $E_1(x) = E(x) = (t_0\, dt_1 - t_1\, dt_0)x$, which, under the isomorphism $\Omega_1 \simeq \mathbb{C}[t, dt]$ reads $E_1(x) = -x\, dt$. If $n \geq 2$, the formulas for $E_n(x^{\odot n})$ involve the operation $q_2(E \odot E)$ on $\mathfrak{g}_1 \odot \mathfrak{g}_1$. Since we have:

$$E : \mathfrak{g}_1 \to \Omega_1^1 \otimes \mathfrak{g}_1 \oplus \Omega_2^1 \otimes \mathfrak{g}_2, \qquad q_2 : (\Omega_1^1 \otimes \mathfrak{g}_1 \oplus \Omega_2^1 \otimes \mathfrak{g}_2)^{\odot 2} \to \Omega_2^2 \otimes \mathfrak{g}_2,$$

the element $E_n(x^n)$ has no $(\Omega_1 \otimes \mathfrak{g}_1)$-component, for $n \geq 2$. Then $\Phi E_\infty(x) = \Phi E_1(x) = \Phi(-xdt) = \Phi(e^{tx} * 0) = x$. $\square$

The geometric meaning of the Main Theorem 5.3.6 is more clear if we reanalyse Example 5.3.5.

Let X be a topological space and let $\mathcal{L}$ be a sheaf of Lie algebras on X. For every open covering $\mathcal{U}$ of X, the Čech semicosimplicial Lie algebra $\mathcal{L}(\mathcal{U})$ and the functor $\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathcal{L}(\mathcal{U}))}$ are defined.
Let $\mathcal{V} \geq \mathcal{U}$ be a refinement of open coverings of X, and let τ be a refinement function, i.e., the choice, for every open set $V \in \mathcal{V}$ of an open set $U \in \mathcal{U}$ with $V \subseteq U$. The refinement function τ induces a natural morphism of semicosimplicial Lie algebras $\mathcal{L}(\mathcal{U}) \to \mathcal{L}(\mathcal{V})$ and so a natural morphism of deformation functors $\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathcal{L}(\mathcal{U}))} \to \operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathcal{L}(\mathcal{V}))}$. Note that a priori these morphisms depends on the refinement map choosen, so it does not make sense to take the direct limit over the refinement of the functors $\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathcal{L}(\mathcal{U}))}$.

On the other hand in Example 5.3.5, we observed that the functor $H^1_{\mathrm{sc}}(\exp \mathcal{L}(\mathcal{U}))$ is given, for all $(A, \mathfrak{m}_A) \in \mathbf{Art}_{\mathbb{K}}$, by the first Čech cohomology space for the covering $\mathcal{U}$ with coefficients in the sheaf of groups $\exp(\mathcal{L} \otimes \mathfrak{m}_A)$ and that it makes sense to take the direct limit over the refinement of the functors $H^1_{\mathrm{sc}}(\exp \mathcal{L}(\mathcal{U}))$, obtaining the functor $H^1(X; \exp \mathcal{L})$, that associates to every $(A, \mathfrak{m}_A) \in \mathbf{Art}_{\mathbb{K}}$, the first Čech cohomology space of X with coefficients in the sheaf of groups $\exp(\mathcal{L} \otimes \mathfrak{m}_A)$.

Now, using the isomorphism $\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathcal{L}(\mathcal{U}))} \cong H^1_{\mathrm{sc}}(\exp(\mathcal{L}(\mathcal{U})))$, we obtain the following Theorem.

Theorem 5.3.21. *Let X be a paracompact Hausdorff topological space, and let $\mathcal{L}$ be a sheaf of Lie algebras on X. The direct limit*

$$\operatorname{Def}_{[\mathcal{L}]} = \lim_{\overrightarrow{\mathcal{U}}} \operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathcal{L}(\mathcal{U}))}$$

is well defined and there is a natural isomorphism of functors

$$H^1(X;\exp\mathcal{L})\simeq \operatorname{Def}_{[\mathcal{L}]}\,.$$

Moreover, if acyclic open coverings for $\mathcal{L}$ are cofinal in the directed family of all open coverings of X, there are isomorphisms

$$H^1(X;\exp\mathcal{L})\simeq H^1(\mathcal{U};\exp\mathcal{L})\qquad \textit{and}\qquad \operatorname{Def}_{[\mathcal{L}]}\simeq \operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{U}))},$$

for every $\mathcal{L}$-acyclic open covering $\mathcal{U}$ of X.

Proof. Let $\mathcal{V}\geq\mathcal{U}$ be a refinement of open coverings of X, and let τ be a refinement function, that induces a natural morphism of semicosimplicial Lie algebras $\mathcal{L}(\mathcal{U})\to\mathcal{L}(\mathcal{V})$. There is a commutative diagram of morphism of functors:

$$\begin{tikzcd}
\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{U}))} \arrow[r, "\sim"] \arrow[d] & H^1_{\mathrm{sc}}(\exp\mathcal{L}(\mathcal{U})) \arrow[r, equal] \arrow[d] & H^1(\mathcal{U},\exp\mathcal{L}) \arrow[d] \\
\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{V}))} \arrow[r, "\sim"] & H^1_{\mathrm{sc}}(\exp\mathcal{L}(\mathcal{V})) \arrow[r, equal] & H^1(\mathcal{V},\exp\mathcal{L})
\end{tikzcd}$$

in which, by Theorem 5.3.6, the horizontal arrows are isomorphisms and the rightmost vertical arrow is independent of the refinement function τ, see, e.g., [13]. Hence, also the leftmost morphism $\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{U}))}\to\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{V}))}$ is independent of τ. Then, the direct limit

$$\operatorname{Def}_{[\mathcal{L}]}=\varinjlim_{\mathcal{U}}\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{U}))}$$

is well defined and we have a natural isomorphism $\operatorname{Def}_{[\mathcal{L}]}\simeq H^1(X;\exp\mathcal{L})$.
Assume now that acyclic open coverings for $\mathcal{L}$ are cofinal in the family of all open coverings of X, this implies that we can use acyclic open coverings to make direct limits.
For any refinement $\mathcal{V}\geq\mathcal{U}$ of acyclic open coverings, by Leray's Theorem, the cohomology spaces $H^i(\mathcal{U},\mathcal{L})$ and $H^i(\mathcal{V},\mathcal{L})$ are isomorphic to $H^i(X,\mathcal{L})$, then the L_∞-morphism $\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{U}))\to\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{V}))$ is a quasi-isomorphism and the morphism $\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{U}))}\to\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{V}))}$ is an isomorphism of functors. Therefore, by Theorem 5.3.6, we have a commutative diagram of isomorphisms

$$\begin{tikzcd}
\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{U}))} \arrow[r, "\sim"] \arrow[d, "\wr"] & H^1_{\mathrm{sc}}(\exp\mathcal{L}(\mathcal{U})) \arrow[r, equal] \arrow[d, "\wr"] & H^1(\mathcal{U},\exp\mathcal{L}) \arrow[d, "\wr"] \\
\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{V}))} \arrow[r, "\sim"] & H^1_{\mathrm{sc}}(\exp\mathcal{L}(\mathcal{V})) \arrow[r, equal] & H^1(\mathcal{V},\exp\mathcal{L})
\end{tikzcd}$$

Taking the direct limit over $\mathcal{L}$-acyclic coverings, we obtain that $H^1(X;\exp\mathcal{L})\simeq H^1(\mathcal{U};\exp\mathcal{L})$ and $\operatorname{Def}_{[\mathcal{L}]}\simeq\operatorname{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{U}))}$, for any $\mathcal{U}$ $\mathcal{L}$-acyclic open covering of X. □

Remark 5.3.22. Note that, if $\mathcal{U}$ is an open covering of X acyclic with respect to $\mathcal{L}$, in general it is not true that it is also acyclic with respect to $\exp\mathcal{L}$. Then the isomorphism $H^1(\mathcal{U},\exp\mathcal{L})\cong H^1(X,\exp\mathcal{L})$ is a consequece of our construction.

5.4 Augmentations and deformations

In this section we introduce augmented semicosimplicial DGLAs and we use them to describe an isomorphism between the functor $\mathrm{Def}_{[\mathcal{L}]}$, associated to a sheaf of Lie algebras $\mathcal{L}$ as in the previous sections, and the functor Def_F, associated to the DGLA of global sections of an acyclic resolution $\mathcal{F}$ of $\mathcal{L}$. These constructions have a strong geometric motivation in deformation theory, as we discuss at the end of this section. These results are obtained in a joint work with D. Fiorenza and M. Manetti (see [7]).

Definition 5.4.1. *Let $\mathbf{\Delta}^+_{\mathrm{mon}}$ be the category obtained by adding the empty set $\emptyset$ to the category $\mathbf{\Delta}_{\mathrm{mon}}$ of finite ordinal sets with order-preserving injective maps.*
An augmented semicosimplicial differential graded Lie algebra *is a covariant functor $\mathfrak{g}^{\Delta^+}: \mathbf{\Delta}^+_{\mathrm{mon}} \to \mathbf{DGLA}$, from the category $\mathbf{\Delta}^+_{\mathrm{mon}}$ to the category of DGLAs.*
Equivalently, an augmented semicosimplicial DGLA $\mathfrak{g}^{\Delta^+}$ *is a diagram*

$$\mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \substack{\longrightarrow\\[-1em] \longrightarrow \\[-1em] \longrightarrow} \mathfrak{g}_2 \substack{\longrightarrow\\[-1em] \longrightarrow \\[-1em] \longrightarrow\\[-1em] \longrightarrow} \cdots$$

where the truncated diagram $\mathfrak{g}^{\Delta}$

$$\mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \substack{\longrightarrow\\[-1em] \longrightarrow \\[-1em] \longrightarrow} \mathfrak{g}_2 \substack{\longrightarrow\\[-1em] \longrightarrow \\[-1em] \longrightarrow\\[-1em] \longrightarrow} \cdots$$

is a semicosimplicial DGLA and

$$\partial_{0,0} \colon \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_0$$

is a DGLA morphism, such that $\partial_{0,1}\partial_{0,0} = \partial_{1,1}\partial_{0,0}$.

Remark 5.4.2. Given an augmented semicosimplicial DGLA $\mathfrak{g}^{\Delta^+}$, the composition of $\partial_{0,0}$ with the natural inclusion $i : \mathfrak{g}_0 \to \mathrm{Tot}(\mathfrak{g}^{\Delta})$ is a morphism of complexes $\mathfrak{g}_{-1} \to \mathrm{Tot}(\mathfrak{g}^{\Delta})$. Infact, for all $x \in \mathfrak{g}_{-1}$, we have $\delta(i \circ \partial_{0,0}(x)) = \partial_1\partial_{0,0}(x) + d_{\mathfrak{g}_0}(\partial_{0,0}(x)) = \partial_{0,1}\partial_{0,0}(x) - \partial_{1,1}\partial_{0,0}(x) + d_{\mathfrak{g}_0}(\partial_{0,0}(x)) = d_{\mathfrak{g}_0}(\partial_{0,0}(x)) = i \circ \partial_{0,0}(d_{\mathfrak{g}_{-1}}(x))$.

We recall that the DGLA $\mathfrak{g}_{-1}$ and the complex $\mathrm{Tot}(\mathfrak{g}^{\Delta})$ have natural structures of L_∞-algebras. We consider the morphisms

$$\alpha_n : \underbrace{\mathfrak{g}_{-1}[1] \odot \ldots \odot \mathfrak{g}_{-1}[1]}_{n} \to \mathrm{Tot}(\mathfrak{g}^{\Delta})[1], \quad \text{where } \alpha_1 = \partial_{0,0} \text{ and } \alpha_n = 0 \text{ for all } n \geq 2.$$

We want to prove the following

Theorem 5.4.3. *The natural map*

$$\alpha\colon\ \mathfrak{g}_{-1} \to \widetilde{\mathrm{Tot}}(\mathfrak{g}^{\Delta})$$

defined by the α_n above is a linear L_∞-morphism.

Proof. Because of definition of α and of Remark 5.1.12, it suffices to prove the following condition:

$$\hat{q}_n(\alpha(x_1) \odot \ldots \odot \alpha(x_n)) = \alpha(q_{n,\mathfrak{g}_{-1}}(x_1 \odot \ldots \odot x_n)) \qquad \forall x_i \in \mathfrak{g}_{-1} \quad \text{and} \quad \forall n \in \mathbb{N}, \tag{5.4}$$

where $\hat{q}_n$ are the brackets on $\widetilde{\mathrm{Tot}}(\mathfrak{g}^{\Delta})$ and $q_{n,\mathfrak{g}_{-1}}$ are the brackets on $\mathfrak{g}_{-1}$.

For $n=1$, condition (5.4) becomes $\hat{q}_1(\alpha(x)) = \alpha(q_{1,\mathfrak{g}_{-1}}(x))$, i.e. $-\delta(\partial_{0,0}(x)) = \partial_{0,0}(-d_{\mathfrak{g}_{-1}}(x))$, that is verified, because $\partial_{0,0}$ is a morphism of complexes, as noticed in Remark 5.4.2.

For $n=2$, condition (5.4) becomes $\hat{q}_2(\alpha(x)\odot\alpha(y)) = \alpha(q_{2,\mathfrak{g}_{-1}}(x\odot y))$, i.e. $\hat{q}_2(\partial_{0,0}(x)\odot \partial_{0,0}(y)) = \partial_{0,0}((-1)^{\deg x}[x,y]_{\mathfrak{g}_{-1}})$ and, since $\partial_{0,0}$ is a DGLAs morphism, it is $\hat{q}_2(\partial_{0,0}(x)\odot \partial_{0,0}(y)) = (-1)^{\deg x}[\partial_{0,0}(x),\partial_{0,0}(y)]_{\mathfrak{g}_0}$.
To verify this equality holds, we have to write explicitly the bracket $\hat{q}_2$ defined on $\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)$ by the formula $\hat{q}_2(x,y) = Iq_2(E(x)\odot E(y))$. If $x\in\mathfrak{g}_k^h$ and $y\in\mathfrak{g}_p^q$, then $E(x)\in C_{TW}^{k,h}$ and $E(y)\in C_{TW}^{p,q}$ are given by:

$$E(x)_n = \begin{cases} 0 & n<k, \\ k! \displaystyle\sum_{I\in I(k,n)} \omega_I \otimes \partial^{\bar{I}} x & n\geq k, \end{cases} \qquad E(y)_n = \begin{cases} 0 & n<p, \\ p! \displaystyle\sum_{I\in I(p,n)} \omega_I \otimes \partial^{\bar{I}} x & n\geq p, \end{cases}$$

then

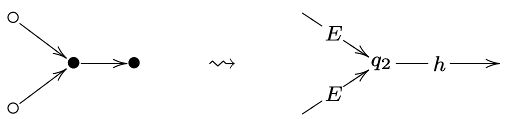
$$(q_2(E(x)\odot E(y))_n = \begin{cases} 0 & n<\max(p,k), \\ (-1)^{k+h} k!p! \displaystyle\sum_{\substack{I\in I(k,n)\\ J\in I(p,n)}} \omega_I\wedge\omega_J\otimes[\partial^{\bar{I}}x,\partial^{\bar{J}}y] & n\geq \max(p,k), \end{cases}$$

and the explicit expression for the second bracket is

$$\hat{q}_2(x\odot y) = Iq_2(E(x)\odot E(y)) = (-1)^{k+h}k!p! \sum_{\substack{I\in I(k,n)\\ J\in I(p,n)}} \int_{\Delta^{k+p}} \omega_I\wedge\omega_J\otimes[\partial^{\bar{I}}x,\partial^{\bar{J}}y] \in \mathfrak{g}_{k+p}^{h+q}.$$

Now let $x\in\mathfrak{g}_{-1}^h$ and $y\in\mathfrak{g}_{-1}^q$, then $\partial_{0,0}(x)\in\mathfrak{g}_0^h$ and $\partial_{0,0}(y)\in\mathfrak{g}_0^q$, thus, applying the above formula of $\hat{q}_2$ for $k=p=0$, we have $\hat{q}_2(\partial_{0,0}(x)\odot\partial_{0,0}(y)) = (-1)^{\deg x}[\partial_{0,0}(x),\partial_{0,0}(y)]$ as we want.

For $n\geq 3$, condition (5.4) becomes $\hat{q}_n(\alpha(x_1)\odot\ldots\odot\alpha(x_n)) = \alpha(q_{n,\mathfrak{g}_{-1}}(x_1\odot\ldots\odot x_n)) = 0$, because the superior brackets $q_{n,\mathfrak{g}_{-1}}$ of the L_∞ structure of $\mathfrak{g}_{-1}$ are equal to zero.
The formulas of the brackets $\hat{q}_n$ defined on $\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)$ involve the evaluation over the rooted trees with n tails in each of those every tail edge is decorated by the operator E, every internal edge is decorated by the operator h, every rooted edge is decorated by the operator I and every internal vertex carries the operator q_r, where r is the arity of the vertex.
Since $q_n=0$ for all $n\geq 3$, no rooted trees with an internal vertex with arity grater than 2 give a non zero contribution. Thus only trees with internal vertices of arity 2 are involved. For $n\geq 3$ they have more than one internal tail and so they all contain the subgraph



We note that $hq_2(E(\partial_{0,0})\odot E(\partial_{0,0})) = 0$, because

$$\partial_{0,0}\colon\ \mathfrak{g}_{-1}^h\to\mathfrak{g}_0^h, \qquad E\colon\ \mathfrak{g}_0^j\to C_{TW}^{0,j},$$
$$q_2\colon C_{TW}^{0,h}\otimes C_{TW}^{0,q}\to C_{TW}^{0,h+q} \qquad h\colon C_{TW}^{0,h+q}\to 0.$$

Then, for $n\geq 3$, all rooted trees involved in the formula for $\hat{q}_n$ give zero contribution and $\hat{q}_n(\partial_{0,0}(x_1)\odot\cdots\odot\partial_{0,0}(x_n)) = 0$, as we want. $\square$

Recall two basic definitions in homological algebra (see, e.g. [38]):

Definition 5.4.4. *Let $\mathcal{C}^\cdot$ be a complex of sheaves of Lie algebras on a topological space X. An* acyclic resolution *of $\mathcal{C}^\cdot$ is the data of:*

- *$\mathcal{F}^\cdot$ a complex of sheaves of Lie algebras, with $\mathcal{F}^k$ acyclic sheaf, for all $k \in \mathbb{Z}$,*
- *$\varphi : \mathcal{C}^\cdot \to \mathcal{F}^\cdot$ a morphism of complexes of sheaves of Lie algebras, such that it is a quasi-isomorphism of complexes and $\varphi^k : \mathcal{C}^k \to \mathcal{F}^k$ is injective, for all $k \in \mathbb{Z}$.*

Definition 5.4.5. *Let $\mathcal{C}^\cdot$ be a complex of sheaves of Lie algebras on a topological space X. The k-*th hypercohomology space *of $\mathcal{C}^\cdot$ on X is*

$$\mathbb{H}^k(X, \mathcal{C}^\cdot) = R^k\Gamma(\mathcal{C}^\cdot)$$

the k-th derived functor of the functor Γ of global sections. It is defined as follows: let $\varphi : \mathcal{C}^\cdot \to \mathcal{F}^\cdot$ be an acyclic resolution of $\mathcal{C}^\cdot$, then $R^k\Gamma(\mathcal{C}^\cdot) = H^k(\Gamma(\mathcal{F}^\cdot))$, this space is determined up to isomorphism by the choice of an acyclic resolution.

Example 5.4.6. Let X be a tolopogical space, let $\mathcal{C}^\cdot$ be a complex of sheaves of Lie algebras on X and let $\mathcal{U}$ be an open covering of X acyclic with respect to $\mathcal{C}^\cdot$. Let analyse a basic example of acyclic resolution of $\mathcal{C}^\cdot$.

For all $i \in \mathbb{Z}$, consider the Čech sheaf resolution of the sheaf $\mathcal{C}^i$ relative to the covering $\mathcal{U}$, i.e. the exact sequence of sheaves:

$$0 \to \mathcal{C}^i \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{C}^i) \to \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{C}^i) \to \dots$$

and obtain the double complex of sheaves of Lie algebras $\{\check{\mathcal{C}}^j(\mathcal{U}, \mathcal{C}^i)\}_{ij}$. The spectral sequence of this bicomplex has first sheet made up by the cohomology of the rows:

$$0 \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{C}^i) \to \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{C}^i) \to \check{\mathcal{C}}^2(\mathcal{U}, \mathcal{C}^i) \to \dots,$$

then it is given by:

$$\begin{array}{llll}
E_1^{0,0} = \mathcal{C}^0 & E_1^{0,1} = 0 & E_1^{0,2} = 0 & \dots \\
\downarrow & \downarrow & \downarrow & \\
E_1^{1,0} = \mathcal{C}^1 & E_1^{1,1} = 0 & E_1^{1,2} = 0 & \dots \\
\downarrow & \downarrow & \downarrow & \\
E_1^{2,0} = \mathcal{C}^2 & E_1^{2,1} = 0 & \dots & \\
\vdots & & &
\end{array}$$

and it has second sheet made up by the cohomology spaces $\mathcal{H}^*(\mathcal{C}^\cdot)$ of the complex $\mathcal{C}^\cdot$. This spectral sequence obviously abutts at degree 2 to $\mathcal{H}^*(\mathcal{C}^\cdot)$; on the other hand it abutts to the cohomology of the total complex of the double complex $\{\check{\mathcal{C}}^j(\mathcal{U}, \mathcal{C}^i)\}_{ij}$, then these cohomologies coincide.

Consider the morphism of complex sheaves

$$\varphi : \mathcal{C}^\cdot \to \mathrm{Tot}^\cdot(\check{\mathcal{C}}^j(\mathcal{U}, \mathcal{C}^i));$$

the sheaves $\check{\mathcal{C}}^j(\mathcal{U}, \mathcal{C}^i)$ are acyclic, because the covering $\mathcal{U}$ is $\mathcal{C}^\cdot$-acyclic, the morphism φ is a quasi-isomorphism of complexes, by the above discussion, and, for all $k \in \mathbb{Z}$, $\varphi^k : \mathcal{C}^k \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{C}^k) \subset \mathrm{Tot}^k(\check{\mathcal{C}}^j(\mathcal{U}, \mathcal{C}^i))$ is injective, because of definition of sheaf. Then φ is an acyclic resolution of the complex of sheaves of Lie algebras $\mathcal{C}^\cdot$ and the hypercohomology of $\mathcal{C}^\cdot$ on X can be calculated using this resolution.

Remark 5.4.7. Obviously a complex of sheaves of Lie algebras can be considered also as a sheaf of differential graded Lie algebras.

Theorem 5.4.8. *Let X be a paracompact Hausdorff topological space and let $\mathcal{C}^{\cdot}$ be a sheaf of differential graded Lie algebras on X. Let $\varphi\colon \mathcal{C}^{\cdot} \to \mathcal{F}^{\cdot}$ be an acyclic resolution and let $F^{\cdot} = \mathcal{F}^{\cdot}(X)$ be the DGLA of global sections of $\mathcal{F}^{\cdot}$. Then, if $\mathcal{U}$ is an open covering of X which is acyclic with respect to both $\mathcal{C}^{\cdot}$ and $\mathcal{F}^{\cdot}$, the L_∞-algebra $\widetilde{\mathrm{Tot}}(\mathcal{C}^{\cdot}(\mathcal{U}))$ is naturally quasi-isomorphic to the DGLA $F^{\cdot}$.*

Proof. The morphism of sheaves $\varphi\colon \mathcal{C}^{\cdot} \to \mathcal{F}^{\cdot}$ induces a morphism of semicosimplicial DGLAs

$$\varphi\colon \mathcal{C}^{\cdot}(\mathcal{U}) \to \mathcal{F}^{\cdot}(\mathcal{U}),$$

and so, by functoriality, it induces a linear morphism of L_∞-algebras

$$\varphi\colon \widetilde{\mathrm{Tot}}(\mathcal{C}^{\cdot}(\mathcal{U})) \to \widetilde{\mathrm{Tot}}(\mathcal{F}^{\cdot}(\mathcal{U})).$$

The cohomology of the total complex $\mathrm{Tot}(\mathcal{C}^{\cdot}(\mathcal{U}))$ is equal to the hypercohomology of $\mathcal{C}^{\cdot}$ on X:

$$H^*(\mathrm{Tot}(\mathcal{C}(\mathcal{U}))) \simeq \mathbb{H}^*(X;\mathcal{C}),$$

in fact, by Example 5.4.6, $\mathbb{H}^*(X,\mathcal{C}^{\cdot}) = H^*(\Gamma(\mathrm{Tot}(\check{\mathcal{C}}^j(\mathcal{U},\mathcal{C}^i)))) = H^*(\mathrm{Tot}(\mathcal{C}^{\cdot}(\mathcal{U})))$ and in the same way the cohomology of the total complex $\mathrm{Tot}(\mathcal{F}^{\cdot}(\mathcal{U}))$ is equal to the hypercohomology of $\mathcal{F}^{\cdot}$ on X:

$$H^*(\mathrm{Tot}(\mathcal{F}^{\cdot}(\mathcal{U}))) \simeq \mathbb{H}^*(X,\mathcal{F}^{\cdot}).$$

Moreover the hypercohomologies of $\mathcal{C}^{\cdot}$ and $\mathcal{F}^{\cdot}$ coincide, because $\varphi : \mathcal{C}^{\cdot} \to \mathcal{F}^{\cdot}$ is an acyclic resolution. Then

$$\varphi\colon \widetilde{\mathrm{Tot}}(\mathcal{C}(\mathcal{U})) \to \widetilde{\mathrm{Tot}}(\mathcal{F}(\mathcal{U}))$$

is a quasi-isomorphism of L_∞-algebras.

The natural inclusion $F^{\cdot} \to \mathcal{F}^{\cdot}(\mathcal{U})$ gives an augmented semicosimplicial DGLA, and so, by Theorem 5.4.3, it induces a morphism $F^{\cdot} \to \widetilde{\mathrm{Tot}}(\mathcal{F}^{\cdot}(\mathcal{U}))$ of L_∞-algebras. To calculate the cohomology of the total complex $\mathrm{Tot}(\mathcal{F}^{\cdot}(\mathcal{U}))$, let analyse of the spectral sequence associated to the double complex $\mathcal{F}^{\cdot}(\mathcal{U})$. Its first sheet is made up by the cohomology of the rows:

$$0 \to C^0(\mathcal{U},\mathcal{F}^i) \to C^1(\mathcal{U},\mathcal{F}^i) \to C^2(\mathcal{U},\mathcal{F}^i) \to \dots,$$

then, since $\mathcal{F}^i$ are acyclic sheaves, it is given by:

$$\begin{array}{lll}
E_1^{0,0} = H^0(\mathcal{U},\mathcal{F}^0) & E_1^{0,1} = 0 & E_1^{0,2} = 0 \quad \dots \\
\downarrow & \downarrow & \downarrow \\
E_1^{1,0} = H^0(\mathcal{U},\mathcal{F}^1) & E_1^{1,1} = 0 & E_1^{1,2} = 0 \quad \dots \\
\downarrow & \downarrow & \downarrow \\
E_1^{2,0} = H^0(\mathcal{U},\mathcal{F}^2) & E_1^{2,1} = 0 & \dots \\
\vdots & &
\end{array}$$

where $H^0(\mathcal{U},\mathcal{F}^i) = H^0(X,\mathcal{F}^i) = F^i$, since the covering $\mathcal{U}$ is $\mathcal{F}^{\cdot}$-acyclic, then its second sheet is made up by the cohomology of the DGLA F. This spectral sequence obviously abutts at degree 2 to $H^*(F^\cdot)$, on the other hand it abutts to the cohomology of the total complex $\mathrm{Tot}(\mathcal{F}^\cdot(\mathcal{U}))$, then $H^*(\mathrm{Tot}(\mathcal{F}^\cdot(\mathcal{U}))) \cong H^*(F^\cdot)$ and

$$F^\cdot \to \widetilde{\mathrm{Tot}}(\mathcal{F}^\cdot(\mathcal{U}))$$

is a quasi-isomorphism of L_∞-algebra.

We therefore have the chain of quasi-isomorphisms of L_∞-algebras

$$\widetilde{\mathrm{Tot}}(\mathcal{C}^\cdot(\mathcal{U})) \xrightarrow{\sim} \widetilde{\mathrm{Tot}}(\mathcal{F}^\cdot(\mathcal{U})) \xleftarrow{\sim} F^\cdot.$$

□

Corollary 5.4.9. *Under the same hypothesis of Theorem 5.4.8, there is an isomorphism of functors:*

$$\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{C}^\cdot(\mathcal{U}))} \cong \mathrm{Def}_F\,.$$

Proof. Apply Theorem 5.4.8, remembering that quasi-isomorphic L_∞-algebras have isomorphic deformation functors and that the deformation functor associated to F as DGLA coincide with the deformation functor associated to F as L_∞-algebra. □

Corollary 5.4.10. *Let $\mathcal{L}$ be a sheaf of Lie algebras over a paracompact Hausdorff topological space X, $\varphi\colon \mathcal{L} \to \mathcal{F}^\cdot$ a morphism of sheaves of DGLAs which is an acyclic resolution, and F the DGLA of global sections of $\mathcal{F}$. If acyclic open coverings for both $\mathcal{L}$ and $\mathcal{F}^\cdot$ are cofinal in the directed family of open coverings of X, then there is a natural isomorphism of deformation functors*

$$H^1(X; \exp \mathcal{L}) \simeq \mathrm{Def}_F\,.$$

Proof. Applying Corollary 5.4.9, we have

$$\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{U}))} \cong \mathrm{Def}_F,$$

for $\mathcal{U}$ acyclic open covering with respect to both $\mathcal{L}$ and $\mathcal{F}$. Moreover, under these hypothesis we can apply Theorem 5.3.21 to obtain

$$\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathcal{L}(\mathcal{U}))} = \mathrm{Def}_{[\mathcal{L}]} \cong H^1(X; \exp \mathcal{L}).$$

□

These constructions and results have strong geometric motivations, in fact they allow us to find a concrete and rigorous links between the two major different approach to deformation theory.

The classical approach to deformation theory in several cases identifies a sheaf of Lie algebras $\mathcal{L}$ on a topological space X, which controls deformations via the Čech functor $H^1(X, \exp \mathcal{L})$. On the other hand, the theory of deformation via DGLAs is based on the principle that in characteristic zero, every deformation problem is governed by a differential graded Lie algebra, via the deformation functor associated to it.

These two approaches to deformations suggest that there should exists a canonical isomorphism between the Čech functor of the sheaf of Lie algebras identified by the problem and the deformation functor associated to the DGLA that governs it. Corollary 5.4.10 gives this isomorphism, let apply it to some well-known geometric cases.

Example 5.4.11. Let X be a complex manifold and let $\mathcal{L}$ be a locally free sheaf of $\mathcal{O}_X$-Lie algebras. Let $\mathcal{F} = \mathcal{A}^{0,*}(\mathcal{L}) = \mathcal{A}^{0,*} \otimes_{\mathcal{O}_X} \mathcal{L}$ be the sheaf of the $(0,*)$-forms with values in the sheaf $\mathcal{L}$ and let $A^{0,*}_X(\mathcal{L}) = \Gamma(X, \mathcal{A}^{0,*}(\mathcal{L}))$ be the space of its global sections. The natural inclusion $\mathcal{L} \hookrightarrow \mathcal{F}$ is a fine resolution and therefore Corollary 5.4.10 gives a natural isomorphism

$$H^1(X; \exp \mathcal{L}) \simeq \operatorname{Def}_{A^{0,*}_X(\mathcal{L})}.$$

Example 5.4.12. Let X be a complex manifold and let $\mathcal{T}_X$ the holomorphic tangent sheaf. Then we have the Kodaira identification

$$\operatorname{Def}_X \simeq H^1(X; \exp \mathcal{T}_X)$$

and so Corollary 5.4.10 gives a natural isomorphism

$$\operatorname{Def}_X \simeq \operatorname{Def}_{A^{0,\bullet}_X(\mathcal{T}_X)},$$

recovering the well-known statement that the infinitesimal deformations of X are governed by the Kodaira-Spencer DGLA $A^{0,\bullet}_X(\mathcal{T}_X)$ (see Example 3.2.14).

Example 5.4.13. Let X be a complex manifold, let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules and let $\operatorname{End}(\mathcal{E})$ and $D^1(\mathcal{E})$ be the sheaves of endomorphisms of $\mathcal{E}$ and of first order differential operators on $\mathcal{E}$ with scalar principal symbol. Then we recover the natural isomorphisms (see Examples 3.2.15 and 3.2.16)

$$\operatorname{Def}_{\mathcal{E}} \simeq H^1(X; \exp \operatorname{End}(\mathcal{E})) \simeq \operatorname{Def}_{A^{0,*}_X(\operatorname{End}(\mathcal{E}))}$$

and

$$\operatorname{Def}_{(X,\mathcal{E})} \simeq H^1(X; \exp D^1(\mathcal{E})) \simeq \operatorname{Def}_{A^{0,*}_X(D^1(\mathcal{E}))}.$$

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