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Jet differentials, holomorphic Morse inequalities and hyperboliticity

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Jet differentials, holomorphic Morse inequalities and hyperbolicity

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... ceci n'est pas une thèse ...

Introduction

In his seminal paper [1] of 1926, A. Bloch initiated a series of investigations about properties of entire holomorphic curves traced in certain algebraic varieties whose irregularity exceeds the dimension.

Since then, the geometry of entire curves, or rather the geometry of complex algebraic varieties admitting special kinds of such curves, has attracted a lot of attention. For instance, natural questions are whether or not there are (Zariski) dense entire holomorphic curves in a given manifold or whether or not the only holomorphic mappings from the whole complex plane to a given manifold are the constant ones. The latter question leads, at least in the compact case, to the definition of Kobayashi-hyperbolic manifolds, while a certain number of conjectures have been made about the "algebraic degeneracy" of entire curves in varieties of general type.

More precisely, a complex space X is said to be Kobayashi-hyperbolic if the intrinsic Kobayashi pseudo-distance (obtained e.g. by integrating the infinitesimal Kobayashi-Royden Finsler pseudo-metric $\mathbf{k}_X$) is in fact a distance. Somehow, this pseudo-distance measures how big a complex disc can be mapped holomorphically to X, when a tangent vector at the origin is prescribed (for the exact definitions, see Chapter 1). Here is the moral: the bigger the disc is, the smaller the Kobayashi infinitesimal pseudo-metric is – leading even to a degenerate pseudo-metric if the discs can be taken to be arbitrary large. Well known examples of compact hyperbolic spaces are algebraic curves of geometric genus at least two, bounded domains of complex affine spaces and quotients of such domains, e.g. quotients of complex balls. On the other hand, the family of non-hyperbolic compact spaces includes complex affine spaces, rational and elliptic curves, abelian varieties, Calabi-Yau and hyperkähler manifolds – the above list is certainly non exhaustive. In the compact setting, thanks to the classical reparametrization result of Brody, a complex manifold X is hyperbolic if and only if there are no non-constant holomorphic entire mappings $f\colon \mathbb{C} \to X$.

In 1979, Green and Griffiths [17] generalized and formalized the concept of symmetric differentials to higher order jets of curves under the name of jet differentials, and they gave a deep insight of what could be their applications to algebraic geometry. Namely, they introduced on every complex manifold a holomorphic vector bundle of algebraic differential operators acting on jets of curves, and observed that every non-constant entire holomorphic curve is automatically a solution of its global sections whose coefficients vanish on an ample divisor. Almost twenty years later, Demailly [7] proposed a refined more geometrical version of the construction made by Green and Griffiths: namely he considered the subbundle of algebraic differential operators which are invariant under the action of an arbitrary reparametrization of the curves they act on, so that these operators just "act" on the geometric locus of the curves; this is actually the datum we are interested in, since the way the curves are parametrized is mostly irrelevant. Finally, the latter bundle appears to have better positivity properties and its study has already led to further remarkable results in complex hyperbolic geometry.

Now, we would like to explain some long-standing conjectures which have somehow served as guidelines for research in several areas of complex hyperbolic and algebraic geometry during the last decades. Let $X \subset \mathbb{P}^{n+1}$ be a complex projective hypersurface (resp. $D \subset \mathbb{P}^n$ be an irreducible divisor). In [18], Kobayashi conjectured that if $\deg X \geq 2n+1$ and X is generic (resp. $\deg D \geq 2n+1$ and D generic), then X is hyperbolic (resp. the complement $\mathbb{P}^n \setminus D$ is hyperbolic). This statement is now referred to as the Kobayashi conjecture and it dates back to 1970. Another interesting and very difficult problem concerns algebraic varieties of general type (that is, varieties possessing a big canonical divisor): the statement is that all entire curves drawn in such varieties should be algebraically degenerate; it is known as the Green-Griffiths-Lang conjecture, and was formulated as presented here in the early '80s. For the sake of completeness, it should be mentioned that there is a stronger form of the conjecture asserting that one can find an algebraic degeneration locus containing all entire curves simultaneously; this stronger form implies the following conjecture of Lang: a projective algebraic variety is hyperbolic if and only if itself and all its sub-varieties are of general type.

Classically, one way to attack this kind of problems is to study the projection to X of the base locus of certain linear series canonically associated to the bundles of jet differentials: in fact, the sheaves $\mathcal{O}(E_{k,m})$ of (invariant) jet differentials arise as direct image sheaves of some canonical invertible sheaves $\mathcal{O}_{X_k}(m)$ defined over suitable "projectivized k-jet bundles" $\pi_{0,k}\colon X_k \to X$. The corresponding k-jet bundle X_k is a tower of projective bundles which is obtained by iterating a fonctorial construction $(X,V) \mapsto (\widetilde{X},\widetilde{V})$ in the category of "directed manifolds"; by definition, a directed manifold is just a pair (X,V) where X is a complex manifold and V a holomorphic subbundle of the tangent bundle T_X (or possibly, in a more general manner, a subsheaf of $\mathcal{O}(T_X)$ such that $\mathcal{O}(T_X)/V$ has no torsion). Given a non-constant entire curve $f\colon \mathbb{C} \to X$, one then has a canonical lifting $f_{[k]}\colon \mathbb{C} \to X_k$ and, once an ample divisor $A \to X$ is fixed, this lifting must be contained in the base locus

$$B_{k,m} = \bigcap_{\sigma \in \left|\mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^* A^{-1}\right|} \sigma^{-1}(0).$$

In particular, if we define the Green-Griffiths locus of X as

$$Y = \bigcap_{k,m>0} \pi_{0,k}(B_{k,m}) \subset X,$$

one sees that every entire curve must be contained in Y. Therefore, if X is compact and Y is of dimension zero, then X is hyperbolic; more generally, if Y is a proper algebraic subset of X, then every entire curve in X is algebraically degenerate, and if Y itself is hyperbolic then X is also hyperbolic.

Unfortunately, it turns out that the Green-Griffiths locus is very difficult to compute and even the a priori more tractable problem of showing whether or not the linear series involved are non-empty is, in general, unsolved. In this thesis, we study the existence of global invariant jet differentials vanishing on an ample divisor in two classical cases, namely, the case of hypersurfaces in projective space and the case of algebraic surfaces of general type.

One of the main techniques we are going to invoke is holomorphic Morse inequalities, a theory initiated by Demailly in the '80s. Roughly speaking, suppose we have a hermitian line bundle over a compact Kähler manifold and we want to control the asymptotic behavior of the partial alternating sum of the dimensions of the successive q cohomology groups with values in powers of this line bundle; the complete sum is simply the Euler characteristic, and, in general, we are concerned with the asymptotics of such sums. Then, this behavior is controlled by an estimate involving the integral of the top wedge power of the Chern curvature of the line bundle, extended over its q-index set (that is, the open set of points of the manifold where the curvature is non-degenerate and has at most q negative eigenvalues). In particular, if one is interested in some asymptotic effectivity (in fact, bigness) of any hermitian line bundle over a compact Kähler manifold, it suffices to show that the integral of the top wedge power of its curvature over the 1-index set is positive. There is also an algebraic version of these inequalities, which were first stated by Trapani [26] and expressed in terms of intersection numbers, in the case where the line bundle is written as the difference of two nef line bundles. This is the simpler version we actually use when studying hypersurfaces of projective spaces.

In Chapter 1, we introduce most of the basic tools used in the course of this work, in the general framework of directed manifolds. More precisely, we introduce the infinitesimal Kobayashi-Royden (pseudo)metric of a pair (X,V) and define, as usual, the notion of complex hyperbolicity for such a pair in terms of the non degeneracy of the metric. In the compact case, the equivalence with the notion of Brody-hyperbolicity (non existence of non-constant entire holomorphic maps) is to be pointed out. Next, we describe the fonctorial construction of projectivization of directed manifolds, as well as the procedure allowing to lift germs (or jets) of curves to the newly constructed directed manifolds. An iteration of this construction gives rise to the so-called Demailly-Semple projectivized jet bundles, which turn out to be also "natural" relative compactifications of the quotient of the space of non-singular k-jets of curves modulo the group of k-jets of biholomorphisms of the origin $(\mathbb{C},0)$. To conclude the general picture, we introduce the bundles of jet differentials and invariant jet differentials (both in the compact and in the logarithmic setting), and we put in evidence some questions about their relative positivity, as they will be useful later. We also describe their metric aspects which eventually lead to a proof of the fundamental vanishing theorem, namely that every entire curve satisfies automatically the global differential equations whose coefficients vanish on an ample divisor.

Chapter 2 and 3 are motivated by the Kobayashi conjecture and are concerned with the case of smooth hypersurfaces in projective space. It has been known since a long time that smooth hypersurfaces of projective space have no global symmetric differentials. More recently, Rousseau [21] has observed that in order to deal with smooth hypersurfaces in $\mathbb{P}^4$, one is obliged to look for 3-jet differentials, since there are no global 2-jet differentials at all on such 3-folds. We show here that this is in fact the general picture: consider the bundle $\mathcal{J}_{k,m}T^*_X$ of jet differential of order k and weighted degree m with its natural filtration, whose composition series is given by

$$\operatorname{Gr}^\bullet \mathcal{J}_{k,m}T^*_X = \bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1}T^*_X \otimes \cdots \otimes S^{\ell_k}T^*_X \ ,$$

and suppose that X is a smooth complete intersection; then we have a theorem by Brückmann and Rackwitz which ensures the vanishing of the space of global sections of Schur powers of the cotangent bundle T^*_X, provided certain conditions on the highest weight of the Schur representation are satisfied (for a precise definition of the Schur powers of a complex vector space, we refer to Chapter 2). For example, if X is a smooth hypersurface and we consider the Schur power of T^*_X associated with the highest weight $(\lambda_1,\dots,\lambda_n)$, we get that its global sections vanish if $\lambda_n = 0$ (recall that a positive λ_n would imply the presence of λ_n copies of the canonical bundle K_X in the associated representation, and would therefore bring in positivity — at least if the degree of X is large enough). In this picture, thanks to some elementary lemmas in representation theory, we are able to exclude such highest weights in the decomposition into irreducible $\mathrm{Gl}(T_X^*)$-representation of the composition series above, provided the order of jet differentials we are looking for is less than $\dim X / \operatorname{codim} X$. Moreover, using the standard cyclic $d : 1$ covering for hypersurfaces of degree d in projective space, we can reduce the logarithmic case to the compact one, thus extending our vanishing result also to logarithmic jet differentials (for the precise statements, we refer to Theorem 2.1.1 and 2.1.3). Since invariant jet differentials form a sub-bundle of $\mathcal{J}_{k,m}T_X^*$, these vanishing theorems tell us, in particular, that for typical varieties of dimension n, we have in general to look for invariant jet differential of order at least equal to n.

Now, we come to the existence results for jet differentials on hypersurfaces of projective space. We have already explained that the sheaf of sections of invariant jet differentials on a given manifold X naturally arises as a direct image sheaf of a certain (power of a) canonical line bundle $\mathcal{O}_{X_k}(m)$ over the tower of projective bundles X_k over X. Therefore, in order to find sections of $E_{k,m}T_X^*$, one could try to use (the algebraic version of) holomorphic Morse inequalities on $\mathcal{O}_{X_k}(m)$, for $k \geq \dim X$. One problem is that these line bundles are always relatively big over X, but never relatively nef when $k \geq 2$, so that holomorphic Morse inequalities take into account too many negative vertical directions. Our early attempts showed that a positive result is hopeless in this setting. To overcome this difficulty, we showed that it is enough to twist our line bundles by a special combination of the ideal sheaves of vertical divisors occurring in their relative base locus, eventually obtaining a relatively nef line bundle which admits a non-trivial morphism into the original one. Now, it is quite straightforward to decompose these new line bundles into the difference of two global nef line bundles, using positivity coming from $\mathcal{O}(2)$ over the base (the cotangent space of a hypersurface of projective space twisted by $\mathcal{O}(2)$ is a quotient of the cotangent bundle of the ambient projective space twisted by the same multiple of the hyperplane divisor, which is indeed globally generated, hence nef). The last step is a matter of calculating intersection products in the cohomology algebra of X_k. This is a polynomial algebra over $H^\bullet(X)$, whose "generators" (free indeterminates) are the first Chern classes of the tautological line bundles occurring at each intermediate floor of the tower. The computations here are quite involved, but finally one gets the desired positivity of the intersection product required for the application of algebraic holomorphic Morse inequalities, provided the degree of X is large enough. Summarizing, we get the following result: for any smooth projective hypersurface $X \subset \mathbb{P}^{n+1}$ of high degree, the space of global holomorphic sections of $E_{n,m}T_X^*$ growths like m^{n^2} (and we obtain a completely analogous result for the invariant logarithmic jet differentials over $\mathbb{P}^n \setminus D$, where D is a smooth irreducible divisor of large degree). These results are the content of our Theorem 3.1.1 and 3.1.2.

A first remark is that our method is completely effective in principle and we get, in fact, lower bounds for the degree of the hypersurfaces, at least in low dimension (we pursued our calculations up to five), improving substantially the previously known bounds. Second, and this may look somewhat curious, it is crucial in our proof of the existence of sections of order n, to use the non-existence of sections of lower order. Third, as long as one is concerned with the order of jet differentials (but not with the degree), our results are sharp, as our vanishing theorem actually shows. Last, and unfortunately for the moment, we are not able to say anything about the "algebraic independence" of the sections we produce, so that nothing can be said about the codimension of their base locus, which would be a crucial step to reach hyperbolicity-type results.

We finally come to the contents of Chapter 4, which is the more differential geometric part of this work, mainly motivated by the Green-Griffiths-Lang conjecture. The idea here is to construct a natural smooth hermitian metric on the tautological line bundles associated with the tower, in order to perform holomorphic Morse inequalities type computations. To this aim, we start with a smooth compact Kähler directed manifold (X, V, ω). The restriction of the Kähler form ω to V gives a smooth hermitian metric on the tautological line bundle $\mathcal{O}_{X_1}(-1)$ over the projectivized 1-jet space, and its Chern curvature on the dual bundle $\Theta(\mathcal{O}_{X_1}(1))$ is positive in the fiber direction (it is the Fubini-Study metric, after all!). Then, for all positive ε_1 small enough, $\omega_{1,\varepsilon_1} = \pi_{0,1}^*\omega + \varepsilon_1^2\,\Theta(\mathcal{O}_{X_1}(1))$ is a Kähler form on X_1. At this point, we take the restriction of ω_{1,ε_1} to V_1, and iterate this procedure. Thus, we finally obtain a family h_k^ε of smooth hermitian metrics on $\mathcal{O}_{X_k}(1)$ depending on $\varepsilon_1, \dots, \varepsilon_{k-1}$, whose Chern curvature depends a priori on $2k$ derivatives of ω. Of course this would seem at first to be completely impractical for calculations, since the relevant geometrical data on X depend only on the curvature and the Chern forms, that is, in the derivatives of second order of ω, but not on the higher order derivatives. The solution to overcome this difficulty, following suggestions made by Demailly in recent years, is to consider the asymptotics of the curvature of such a family of metrics when ε goes to zero. We have then been able to check that the higher order derivatives only appear in the "ε error terms" (see Theorem 4.1.1). Moreover, in case X is a surface, we get an explicit expression of this curvature in terms of rather simple products of 2×2 real matrices, As an application, we investigate invariant jet differentials, using the above curvature formula in the case of smooth surfaces of general type (cf. Theorem 4.1.2). We start with a surface X with positive canonical bundle, and then take the metric of X to be the Kähler-Einstein metric (later on, we will relax this hypothesis so as to treat the case of arbitrary surfaces of general type. This can be made e.g. by means of standard techniques for finding approximate solutions of Monge-Ampére equations. Then, we show through the example of compact quotients of the unit ball in $\mathbb{C}^2$, that we are forced to use certain linear combination of tautological line bundles coming from different floors of the tower, if we want to overcome the negative contribution of the vertical eigenvalues, while computing Morse-type integrals. Actually, let us consider the weighted line bundle $\mathcal{O}_{X_k}(a_1,\dots,a_k)=\bigotimes_{j=1}^k \pi_{j,k}^*\mathcal{O}_{X_j}(a_j)$; we write down explicitly its (ε-limit) Chern curvature with respect to the metric defined above and give sufficient conditions on the a_j's to have "vertical" positivity, showing also which kind of "horizontal" positivity these conditions ensure. Next, we observe that the integral of the top wedge power of the curvature extended to its 1-index set is definitely an intersection product, since the 1-index set is in fact the whole X_k. This intersection product can be ultimately expressed as a combination with polynomial coefficients in the a_j's, in terms of the Chern classes $c_1(X)^2$ and $c_2(X)$ of X. By holomorphic Morse inequalities, the space of global sections of invariant k-jet differentials on X is non zero as soon as the intersection product is positive. This easy observation allows us to find a sufficient condition which gives the existence of global sections of invariant jet differentials of some order as follows: we define a non-decreasing sequence of positive numbers $\{m_k\}$, obtained by taking the supremum of the ratio of the polynomial coefficients above (on a suitable closed convex cone) and we consider the limit term, say m_∞. Then, we show that if $c_2(X)/c_1(X)^2$ is smaller than m_∞, there exists a positive integer k_0 such that $\mathcal{O}_{X_{k_0}}(1)$ is big.

In particular, through explicit integral computations, we are able to obtain suitable lower bounds for m_∞, thus providing new proofs of recent results about the existence of invariant jet differentials of low order on algebraic surfaces of general type, without using deeper results such as the powerful vanishing theorem of Bogomolov.

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This is a joint research doctoral thesis in mathematics, born, developed and written up in English between Rome and Grenoble. So, please don't blame me if I write these lines in three different languages, according to the person the acknowledgments are addressed to.

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Finally, most of you know that this is not an easy task, but rather a possible embarrassment: am I forgetting anyone? This should not be a big deal... After all, I am (almost) a mathematician and for that, everyone expects me to be distracted, careless. Nevertheless, I sure hope that any omission will offend no one.

Baci e abbracci...

Contents

Chapter 1

Kobayashi-hyperbolic manifolds and jet differentials

Abstract. In this chapter we introduce the basic concepts of directed manifolds and Kobayashi-hyperbolicity and we make the link between Kobayashi-hyperbolicity and Brody-hyperbolicity in the compact case. Next, following very closely [7], we explain the construction of jet differentials, invariant jet differentials and of Demailly-Semple projectivized jet bundles. This will be done quite accurately in the compact case and in a more sketchy way in the logarithmic setting. As this intends to be an introductory chapter, we shall skip all the proofs here, sending back the reader to several references cited along these lines.

1.1 Directed manifolds and hyperbolicity

Let X be a complex manifold equipped with a holomorphic subbundle $V \subset T_X$ of the tangent bundle of rank $1 \leq r = \operatorname{rank} V \leq n = \dim X$. We will call the pair (X, V) a *complex directed manifold*.

A morphism $\Phi\colon (X,V) \to (Y,W)$ in this category is a holomorphic map such that its differential Φ_* maps V to W, $\Phi_*(V) \subset W$. Of course, the case $V = T_X$ is included in the definition, and we shall refer to it as the "absolute" case.

Now, we define the notion of hyperbolicity in this context.

Definition 1.1.1. Let (X,V) be a complex directed manifold. The *Kobayashi-Royden infinitesimal (pseudo)metric* of (X,V) is the Finsler metric on V defined for any $x \in X$ and $v \in V_x$ by

$$\mathbf{k}_{(X,V)}(v) \stackrel{\text{def}}{=} \inf\{\lambda > 0 \mid \exists f\colon \Delta \to X, f(0) = x, \lambda f'(0) = v, f'(\Delta) \subset V\},$$

where $\Delta \subset \mathbb{C}$ is the unit disc and f is holomorphic.

Next, we fix an hermitian metric ω on V and we give the definition of "infinitesimally hyperbolic" for the directed manifold (X,V).

Definition 1.1.2. We say that (X,V) is *infinitesimally hyperbolic* if $\mathbf{k}_{(X,V)}$ is positive definite on every fiber V_x and, moreover, satisfies a uniform lower bound $\mathbf{k}_{(X,V)}(v) \geq \varepsilon\, ||v||_\omega$, when x varies in a compact subset of X.

Of course, this definition does not depend on the choice of the hermitian metric ω. Moreover, it follows immediately that if $\Phi\colon (X,V) \to (Y,W)$ is a morphism of directed manifold, then

$$\mathbf{k}_{(Y,W)}(\Phi_* v) \leq \mathbf{k}_{(X,V)}(v), \quad v \in V.$$

Here is an easy example which shows that the existence of a holomorphic copy of $\mathbb{C}$ in a complex manifold implies the degeneracy of the infinitesimal Kobayashi-Royden pseudo-metric.

Example 1.1.1. Let (X,V) be a complex directed manifold. Suppose there exist a non-constant entire holomorphic map $f\colon \mathbb{C} \to X$ tangent to V, that is $f'(t) \in V_{f(t)}$, for each $t \in \mathbb{C}$. We claim that if $f(0) = x$ and $f'(0) = v \in V_x \setminus \{0\}$, then $\mathbf{k}_{(X,V)}(v) = 0$.

To see this, define a sequence of holomorphic maps $f_n\colon \Delta \to X$ by $f_n(z) = f(n\,z)$. Then, of course, f_n is tangent to V, $f_n(0) = x$ and $f_n'(0) = nf'(0) = nv$. Thus, it is clear that, from $1/n\, f_n'(0) = v$, it follows $\mathbf{k}_{(X,V)}(v) = 0$ so that $\mathbf{k}_{(X,V)}$ is not positive definite on V_x.

The latter example leads naturally to another notion of hyperbolicity for complex manifold, concerning the behavior of entire curves of the manifold. Namely, we say that a complex directed manifold (X,V) is *Brody-hyperbolic* if there are no non-constant entire holomorphic curve $f\colon \mathbb{C} \to X$ tangent to V.

The next proposition, which is essentially due to Brody, shows that, in the compact case, these two notions are equivalent.

Proposition 1.1.1. *If X is compact, then (X,V) is infinitesimally hyperbolic if and only if there are no non-constant entire holomorphic curves $f\colon \mathbb{C} \to X$ tangent to V. In this case, $\mathbf{k}_{(X,V)}$ is a continuous (and positive definite) Finsler metric on V.*

A proof of this fact, which relies on a reparametrization lemma by Brody for sequences of holomorphic maps from the unit disc to X whose derivative in 0 blows-up, can be found, for example, in [7].

1.2 Projectivization of directed manifolds

We now introduce a functorial construction in the category of directed manifold in order to produce the so-called space of 1-jets over X.

So, let (X, V) be a complex directed manifold and set $\widetilde{X} = P(V)$. Here, $P(V)$ is the projectivized bundle of lines of V and there is a natural projection $\pi\colon \widetilde{X} \to X$; moreover, if $\dim X = n$, then $\dim \widetilde{X} = n + r - 1$. On $\widetilde{X}$, we consider the tautological line bundle $\mathcal{O}_{\widetilde{X}}(-1) \subset \pi^* V$ which is defined fiberwise as

$$\mathcal{O}_{\widetilde{X}}(-1)_{(x,[v])} \stackrel{\text{def}}{=} \mathbb{C}\, v,$$

for $(x, [v]) \in \widetilde{X}$, with $x \in X$ and $v \in V_x \setminus \{0\}$. We also have the following short exact sequence which comes from the very definition of $\widetilde{X}$:

$$0 \longrightarrow T_{\widetilde{X}/X} \longrightarrow T_{\widetilde{X}} \longrightarrow \pi^* T_X \longrightarrow 0.$$

Of course, the surjection here is given by the differential π_* and $T_{\widetilde{X}/X} = \ker \pi_*$ is the relative tangent bundle.

Now, in the above exact sequence, we want to replace $\pi^* T_X$ by $\mathcal{O}_{\widetilde{X}}(-1) \subset \pi^* V \subset \pi^* T_X$, in order to build a subbundle of $T_{\widetilde{X}}$ which takes into account just one "horizontal" direction and the "vertical" ones; namely we define $\widetilde{V}$ to be the inverse image $\pi_*^{-1} \mathcal{O}_{\widetilde{X}}(-1)$ so that we have a short exact sequence

$$0 \longrightarrow T_{\widetilde{X}/X} \longrightarrow \widetilde{V} \longrightarrow \mathcal{O}_{\widetilde{X}}(-1) \longrightarrow 0$$

and $\operatorname{rank} \widetilde{V} = \operatorname{rank} V = r$. There is another short exact sequence attached to this projectivization, which is the relative version of the usual Euler exact sequence of projective spaces:

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \pi^* V \otimes \mathcal{O}_{\widetilde{X}}(1) \longrightarrow T_{\widetilde{X}/X} \longrightarrow 0.$$

By definition, $(\widetilde{X}, \widetilde{V})$ is a new complex directed manifold, which is compact as soon as X is compact and such that $\pi\colon (\widetilde{X}, \widetilde{V}) \to (X, V)$ is a morphism of complex directed manifolds.

1.2.1 Lifting of curves

Let $\Delta_R \subset \mathbb{C}$ be the open disc $\{|z| < R\}$ of radius $R > 0$ and center $0 \in \mathbb{C}$ and $f\colon \Delta_R \to X$ a holomorphic map. Suppose moreover that $f(0) = x$ for some $x \in X$ and that f is a non-constant tangent trajectory of the directed manifold, that is $f'(t) \in V_{f(t)}$ for each $t \in \Delta_R$.

In this case, there is a well-defined and unique tangent line $[f'(t)] \subset V_{f(t)}$ for every $t \in \Delta_R$ even at the stationary points of f: if $f'(t_0) = 0$ for some $t_0 \in \Delta_R$, write $f'(t) = (t - t_0)^m u(t)$ with $m \in \mathbb{N} \setminus \{0\}$ and $u(t_0) \neq 0$ and define the tangent line at t_0 to be $[u(t_0)]$.

We define the lifting $\widetilde{f}$ of f as the map

$$\widetilde{f}\colon \Delta_R \to \widetilde{X}$$

which sends $t \mapsto \widetilde{f}(t) = (f(t), [f'(t)])$. It is clearly holomorphic and the derivative f' gives rise to a section

$$f'\colon T_{\Delta_R} \to \widetilde{f}^* \mathcal{O}_{\widetilde{X}}(-1).$$

Observe moreover that, as $\pi \circ \widetilde{f} = f$, one has $\pi_* \widetilde{f}'(t) = f'(t)$, so that $\widetilde{f}'(t)$ belongs to $\widetilde{V}_{(f(t),[f'(t)])} = \widetilde{V}_{\widetilde{f}(t)}$. Thus, if f is a tangent trajectory of (X, V) then $\widetilde{f}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$.

On the other hand, if $g\colon \Delta_R \to \widetilde{X}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$, then $f \overset{\text{def}}{=} \pi \circ g$ is a tangent trajectory of (X, V) and g coincides with $\widetilde{f}$ unless g is contained in a vertical fiber $P(V_x)$: in this case f is constant.

1.2.2 Jets of curves

Let X be a complex n-dimensional manifold. Here, we follow [17] to define the bundle $J_k \to X$ of k-jets of germs of parametrized holomorphic curves in X.

It is the set of equivalence classes of holomorphic maps $f\colon (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coincide, for $0 \leq j \leq k$, in some (and hence in all) holomorphic coordinates system of X near x. Here, the projection is simply $f \mapsto f(0)$.

These are not vector bundles, unless $k = 1$: in this case J_1 is simply the holomorphic tangent bundle T_X. However, in general, the J_k's are holomorphic fiber bundles, with typical fiber $(\mathbb{C}^n)^k$ (in fact the elements of the fiber $J_{k,x}$ are uniquely determined by the Taylor expansion up to order k of a germ of curve f, once a system of coordinate is fixed).

Now, we translate this concepts to the setting of complex directed manifolds.

Definition 1.2.1. Let (X, V) be a complex directed manifold. We define the bundle $J_k V \to X$ to be the set of k-jets of curves $f\colon (\mathbb{C}, 0) \to X$ which are tangent to V, together with the projection map $f \mapsto f(0)$.

To check that this is in fact a subbundle of J_k we shall describe a special choice of local coordinates: for any point $x_0 \in X$, there are local coordinates $(z_1, \ldots, z_n)$ on a neighborhood Ω of x_0 such that the fibers V_x, for $x \in \Omega$, can be defined by linear equations

$$V_x = \left\{ v = \sum_{j=1}^{n} v_j \frac{\partial}{\partial z_j} \quad \text{such that} \quad v_j = \sum_{j=1}^{r} a_{jk}(x)\, v_j, \quad j = r+1, \ldots, n \right\},$$

where $(a_{jk}(x))$ is a holomorphic $(n-r) \times r$ matrix. From this description of the fibers, it follows that to determine a vector $v \in V_x$ it is sufficient to know its first r components $v_1, \ldots, v_r$, and the affine chart $v_r \neq 0$ of $P(V_x)$ can be endowed of the coordinates system $(z_1, \ldots, z_n, \xi_1, \ldots, \xi_{r-1})$, where $\xi_j = v_j/v_r$, $j = 1, \ldots, r-1$ (and in an analogous way for the other affine charts).

Now, if $f \simeq (f_1, \ldots, f_n)$ is a holomorphic tangent trajectory to (X, V) contained in Ω, then by a simple Cauchy problem argument, we see that f is uniquely determined by its initial value x_0 and its first r components: as $f'(t) \in V_{f(t)}$, we can recover the remaining components by integrating the differential system

$$f'_j(t) = \sum_{k=1}^{r} a_{jk}(f(t))\, f'_k(t),$$

where $j = r+1, \ldots, n$, and initial data $f(0) = x_0$. This shows that the fibers $J_k V_x$ are locally parametrized by

$$\big((f'_1, \ldots, f'_r), \ldots, (f_1^{(k)}, \ldots, f_r^{(k)})\big),$$

for all $x \in \Omega$, hence $J_k V$ is a locally trivial $(\mathbb{C}^r)^k$-subbundle of J_k.

1.3 Projectivized jet bundles

In this section, we iterate the construction of the projectivization of a complex directed manifold, in order to obtain a projectivized version of the jet bundles. This construction is essentially due to Jean-Pierre Demailly.

We start with a complex directed manifold (X, V), with $\dim X = n$ and $\operatorname{rank} V = r$. We also suppose that $r \geq 2$, otherwise the projectivization of V is trivial. Now, we start the inductive process in the directed manifold category by setting

$$(X_0, V_0) = (X, V), \quad (X_k, V_k) = (\widetilde{X}_{k-1}, \widetilde{V}_{k-1}).$$

In other words, (X_k, V_k) is obtained from (X, V) by iterating k times the projectivization construction $(X, V) \mapsto (\widetilde{X}, \widetilde{V})$ described above.

In this process, the rank of V_k remains constantly equal to r while the dimension of X_k growths linearly with k: $\dim X_k = n + k(r-1)$. Let us call $\pi_k \colon X_k \to X_{k-1}$ the natural projection. Then we have, as before, a tautological line bundle $\mathcal{O}_{X_k}(-1) \subset \pi_k^* V_{k-1}$ over X_k which fits into short exact sequences

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \xrightarrow{(\pi_k)_*} \mathcal{O}_{X_k}(-1) \longrightarrow 0 \tag{1.1}$$

and

$$0 \longrightarrow \mathcal{O}_{X_k} \longrightarrow \pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \longrightarrow T_{X_k/X_{k-1}} \longrightarrow 0. \tag{1.2}$$

Now we come back to the lifting of curves. Our precedent discussion has shown that given a non-constant tangent trajectory $f \colon \Delta_R \to X$ to (X, V) we have a well-defined non-constant tangent trajectory $\widetilde{f} \colon \Delta_R \to \widetilde{X} = X_1$ to $(\widetilde{X}, \widetilde{V}) = (X_1, V_1)$. Now, set inductively

$$f_{[0]} = f, \quad f_{[k]} = \widetilde{f}_{[k-1]}.$$

Then, for each k, we get a tangent trajectory $f_{[k]} \colon \Delta_R \to X_k$ to (X_k, V_k) and the derivative $f'_{[k-1]}$ gives rise to a section

$$f'_{[k-1]} \colon T_{\Delta_R} \to f^*_{[k]} \mathcal{O}_{X_k}(-1).$$

1.3.1 Regular and singular loci

By construction, there exist a canonical injection $\mathcal{O}_{X_k}(-1) \hookrightarrow \pi_k^* V_{k-1}$ and, a composition with the projection $(\pi_{k-1})_*$ gives for all $k \geq 2$ a line bundle morphism

$$\begin{tikzcd} \mathcal{O}_{X_k}(-1) \arrow[r, hook] \arrow[rr, bend right] & \pi_k^* V_{k-1} \arrow[r] & \pi_k^* \mathcal{O}_{X_{k-1}}(-1). \end{tikzcd}$$

The zero divisor of this morphism is clearly the projectivization of $T_{X_{k-1}/X_{k-2}}$, which is, of course, (fiber-wise, with respect to $\pi_k \colon X_k \to X_{k-1}$) a hyperplane subbundle of X_k. Thus, if we set

$$D_k = P(T_{X_{k-1}/X_{k-2}}) \subset P(V_{k-1}) = X_k, \quad k \geq 2,$$

we find

$$\mathcal{O}_{X_k}(-1) \simeq \pi_k^* \mathcal{O}_{X_{k-1}}(-1) \otimes \mathcal{O}_{X_k}(-D_k). \tag{1.3}$$

Now, take a regular germ of curve $f \colon (\mathbb{C}, 0) \to (X, x)$ tangent to V, that is $f'(0) \neq 0$, and consider, for $j = 2, \dots, k$, its j-th lifting $f_{[j]}$: we claim that then $f_{[j]}(0) \notin D_j$. In this case, in fact, all lifting of f are regular and $f_{[j]}(0) \in D_j$ if and only if $(\pi_{j-1})_* f'_{[j-1]}(0) = f'_{[j-2]}(0) = 0$.

On the other hand, if f is a non-constant germ of curve tangent to V such that, for all $j = 2, \ldots, k$, $f_{[j]}(0) \notin D_j$ then $f'(0) \neq 0$.

Summarizing, if we define

$$\pi_{j,k} \stackrel{\text{def}}{=} \pi_{j+1} \circ \cdots \circ \pi_k \colon X_k \to X_j,$$

then a point $w \in X_k$ can be reached by a lifting of some regular germ of curve (if and) only if $\pi_{j,k}(w) \notin D_j$, for all $j = 2, \ldots, k$. It is then natural to define

$$X_k^{\text{reg}} \stackrel{\text{def}}{=} \bigcap_{j=2}^{k} \pi_{j,k}^{-1}(X_j \setminus D_j)$$

and

$$X_k^{\text{sing}} \stackrel{\text{def}}{=} \bigcup_{j=2}^{k} \pi_{j,k}^{-1}(D_j) = X_k \setminus X_k^{\text{reg}}.$$

This singular locus comes out also if one studies the base locus of the linear system associated to the anti-tautological line bundle $\mathcal{O}_{X_k}(1)$. In fact, we have the following proposition:

Proposition 1.3.1 ([7]). *For every $m > 0$, the base locus of the linear system associated to the restriction of $\mathcal{O}_{X_k}(m)$ to every fiber $\pi_{0,k}^{-1}(x)$, $x \in X$, is exactly $X_k^{\text{sing}} \cap \pi_{0,k}^{-1}(x)$. In other words, X_k^{sing} is the "relative" base locus of $|\mathcal{O}_{X_k}(m)|$. Moreover, $\mathcal{O}_{X_k}(1)$ is relatively big.*

This proposition also shows that $\mathcal{O}_{X_k}(1)$ cannot be relatively ample, unless $k = 1$.

1.4 Jet differentials

Let (X, V) be a complex directed manifold. Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1\, t + a_2\, t^2 + \cdots + a_k\, t^k, \quad a_1 \in \mathbb{C}^*, a_j \in \mathbb{C}, j \geq 2,$$

in which the composition law is taken modulo terms t^j of degree $j > k$. Then $\mathbb{G}_k$ admits a natural fiberwise right action on $J_k V$ consisting of reparametrizing k-jets of curves by a biholomorphic change of parameter. Moreover the subgroup $\mathbb{H} \simeq \mathbb{C}^*$ of homotheties $\varphi(t) = \lambda\, t$ is a (non normal) subgroup of $\mathbb{G}_k$ and we have a semidirect decomposition $\mathbb{G}_k = \mathbb{G}'_k \ltimes \mathbb{H}$, where $\mathbb{G}'_k$ is the group of k-jets of biholomorphisms tangent to the identity. The corresponding action on k-jets is described in coordinates by

$$\lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).$$

As in [17], we introduce the vector bundle $\mathfrak{J}_{k,m}V^* \to X$ whose fibres are complex valued polynomials $Q(f', f'', \ldots, f^{(k)})$ on the fibres of J_kV, of weighted degree m with respect to the $\mathbb{C}^*$ action defined by $\mathbb{H}$, that is, such that

$$Q(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots, f^{(k)}),$$

for all $\lambda \in \mathbb{C}^*$ and $(f', f'', \ldots, f^{(k)}) \in J_kV$.

Next, we define the bundle of Demailly-Semple jet differentials (or invariant jet differentials) as a subbundle of the Green-Griffiths bundle.

Definition 1.4.1 ([7])**.** The *bundle of invariant jet differentials of order k and weighted degree m* is the subbundle $E_{k,m}V^* \subset \mathfrak{J}_{k,m}V^*$ of polynomial differential operators $Q(f', f'', \ldots, f^{(k)})$ which are equivariant under arbitrary changes of reparametrization, that is, for every $\varphi \in \mathbb{G}_k$

$$Q((f\circ\varphi)', (f\circ\varphi)'', \ldots, (f\circ\varphi)^{(k)}) = \varphi'(0)^m\, Q(f', f'', \ldots, f^{(k)}).$$

Alternatively, $E_{k,m}V^* = (\mathfrak{J}_{k,m}V^*)^{\mathbb{G}'_k}$ is the set of invariants of $\mathfrak{J}_{k,m}V^*$ under the action of $\mathbb{G}'_k$.

Remark 1.4.1. From the hyperbolicity point of view, it is of course more natural to consider the invariant jet differentials. In fact, we are only interested in the geometry of the entire curves in a given manifold. For this reason, it is redundant how the entire curves are parametrized: we just want to look at their conformal class.

We now define a filtration on $\mathfrak{J}_{k,m}V^*$: a coordinate change $f \mapsto \Psi\circ f$ transforms every monomial $(f^{(\bullet)})^\ell = (f')^{\ell_1}(f'')^{\ell_2}\cdots(f^{(k)})^{\ell_k}$ of partial weighted degree $|\ell|_s := \ell_1 + 2\ell_2 + \cdots + s\ell_s$, $1 \le s \le k$, into a polynomial $((\Psi\circ f)^{(\bullet)})^\ell$ in $(f', f'', \ldots, f^{(k)})$, which has the same partial weighted degree of order s if $\ell_{s+1} = \cdots = \ell_k = 0$ and a larger or equal partial degree of order s otherwise. Hence, for each $s = 1, \ldots, k$, we get a well defined decreasing filtration $F_s^\bullet$ on $\mathfrak{J}_{k,m}V^*$ as follows:

$$F_s^p(\mathfrak{J}_{k,m}V^*) = \left\{ \begin{matrix} Q(f', f'', \ldots, f^{(k)}) \in \mathfrak{J}_{k,m}V^* \text{ involving} \\ \text{only monomials } (f^{(\bullet)})^\ell \text{ with } |\ell|_s \ge p \end{matrix} \right\}, \quad \forall p \in \mathbb{N}.$$

The graded terms $\mathrm{Gr}^p_{k-1}(\mathfrak{J}_{k,m}V^*)$, associated with the filtration $F^p_{k-1}(\mathfrak{J}_{k,m}V^*)$, are precisely the homogeneous polynomials $Q(f', f'', \ldots, f^{(k)})$ whose all monomials $(f^{(\bullet)})^\ell$ have partial weighted degree $|\ell|_{k-1} = p$; hence, their degree ℓ_k

in $f^{(k)}$ is such that $m-p=k\ell_k$ and $\mathrm{Gr}^p_{k-1}(\mathfrak{J}_{k,m}V^*)=0$ unless $k|m-p$. Looking at the transition automorphisms of the graded bundle induced by the coordinate change $f\mapsto\Psi\circ f$, it turns out that $f^{(k)}$ behaves as an element of $V\subset T_X$ and, as a simple computation shows, we find

$$\mathrm{Gr}^{m-k\ell_k}_{k-1}(\mathfrak{J}_{k,m}V^*)=\mathfrak{J}_{k-1,m-k\ell_k}V^*\otimes S^{\ell_k}V^*.$$

Combining all filtrations $F^\bullet_s$ together, we find inductively a filtration $F^\bullet$ on $\mathfrak{J}_{k,m}V^*$ such that the graded terms are

$$\mathrm{Gr}^\ell(\mathfrak{J}_{k,m}V^*)=S^{\ell_1}V^*\otimes S^{\ell_2}V^*\otimes\cdots\otimes S^{\ell_k}V^*,\quad \ell\in\mathbb{N}^k,|\ell|_k=m.$$

Moreover there are natural induced filtrations $F^p_s(E_{k,m}V^*)=E_{k,m}V^*\cap F^p_s(\mathfrak{J}_{k,m}V^*)$ in such a way that

$$\mathrm{Gr}^\bullet(E_{k,m}V^*)=\left(\bigoplus_{|\ell|_k=m}S^{\ell_1}V^*\otimes S^{\ell_2}V^*\otimes\cdots\otimes S^{\ell_k}V^*\right)^{\mathbb{G}'_k}.$$

Let's see more concretely which are the elements of the bundles we have introduced above in the following examples. For the sake of simplicity we shall consider here only the "absolute" case, that is $V=T_X$.

Example 1.4.2. Let us first look at the Green-Griffiths jet differentials. So, we fix a point $x\in X$ and look at the elements of the fiber $\mathfrak{J}_{k,m}T^*_{X,x}$.

- For $k=1$, we simply have $\mathfrak{J}_{1,m}T^*_X=S^mT^*_X$. This is the usual bundle of symmetric differentials.

- For another example, when $k=3$, we have that a typical element of the fiber is

$$\begin{aligned}&\sum a_if'_i,\quad\text{for } m=1,\\&\sum a_{ij}f'_if''_j+b_if''_i,\quad\text{for } m=2,\\&\sum a_{ijk}f'_if'_jf'_k+b_{ij}f'_if''_j+c_if'''_i,\quad\text{for } m=3,\end{aligned}$$

  where the coefficients are holomorphic functions.

In conclusion, sections of $\mathfrak{J}_{k,m}T^*_X$ are locally given by homogeneous polynomials with holomorphic coefficients in the variables $f',\dots,f^{(k)}$, of total weight m, where $f^{(l)}_i$ is assigned weight l.

Example 1.4.3. For the invariant jet differentials, we still have $E_{1,m}T_X^* = S^m T_X^*$. But, for $k \geq 2$ things become much more complicated.

For instance, we shall see in the next chapters that, for X a complex surface, local section of $E_{2,m}T_X^*$ are given by

$$\sum_{\alpha_1+\alpha_2+3\beta=m} a_{\alpha_1\alpha_2\beta}(f_1')^{\alpha_1}(f_2')^{\alpha_2}(f_1'f_2'' - f_1''f_2')^{\beta}.$$

In general, it is still an unsolved (and probably very difficult) problem to determine the structure of the algebra

$$\mathcal{A}_k = \bigoplus_{m\geq 0} E_{k,m}T_{X,x}^*.$$

In this direction, we want to cite here the few results: the structure of $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ for $\dim X = 2$ was found by Demailly, Rousseau [22] understood $\mathcal{A}_3$ for $\dim X = 3$ and, recently, Merker [20] found $\mathcal{A}_5$ for $\dim X = 2$.

The general structure of $\mathcal{A}_k$ appears to be far from being understood even in the surface case.

1.4.1 Invariant jet differentials and projectivized jet bundles

Associated to the graded algebra bundle $\mathcal{J}_{k,\bullet}V^* = \bigoplus_{m\geq 0}\mathcal{J}_{k,m}V^*$, there is an analytic fiber bundle, namely $\mathrm{Proj}(\mathcal{J}_{k,\bullet}V^*) = J_kV^{\mathrm{nc}}/\mathbb{C}^*$, where J_kV^{nc} is the bundle of non-constant k-jets tangent to V, whose fibers are weighted projective spaces $\mathbb{P}(r,\dots,r;1,2,\dots,k)$ (see [14]).

However, we would be mostly interested in a more "geometric" quotient, for instance something like $J_kV^{\mathrm{nc}}/\mathbb{G}_k$.

In [7], it has been constructed something similar, that is the quotient space of $J_kV^{\mathrm{reg}}/\mathbb{G}_k$ of regular (i.e. with non-vanishing first derivative) k-jets tangent to V and we shall see how the projectivized jet bundles can be seen as a relative compactification of this quotient space.

This is exactly the content of the next theorem.

Theorem 1.4.1 ([7]). *Suppose* $\operatorname{rank} V \geq 2$ *and let* $\pi_{0,k}\colon X_k \to X$ *be the projectivized* k*-th jet bundle of* (X,V)*. Then*

- *the quotient* $J_kV^{\mathrm{reg}}/\mathbb{G}_k$ *has the structure of a locally trivial bundle over* X *and there is a holomorphic embedding* $J_kV^{\mathrm{reg}}/\mathbb{G}_k \hookrightarrow X_k$ *over* X*, which identifies* $J_kV^{\mathrm{reg}}/\mathbb{G}_k$ *with* X_k^{reg}*.*

- *The direct image sheaf*
$$(\pi_{0,k})_*\mathcal{O}_{X_k}(m) \simeq \mathcal{O}(E_{k,m}V^*)$$
can be identified with the sheaf of holomorphic section of $E_{k,m}V^*$.

Let us say a few words about this result. First of all, one needs to use J_kV^{reg} instead of J_kV^{nc} in order to lift a k-jet of curve f by taking the derivative $(f,[f'])$ without any cancellation of zeroes in f': in this way one gets a uniquely defined $(k-1)$-jet $\widetilde{f}$ so that, inductively, $f_{[k]}(0)$ is independent of the choice of the representative f.

Moreover, as the reparametrization commutes with the lifting process, that is $\widetilde{(f\circ\varphi)} = \widetilde{f}\circ\varphi$, we get a well defined map
$$J_kV^{\mathrm{reg}}/\mathbb{G}_k \to X_k^{\mathrm{reg}}.$$
This is the embedding of the first part of the theorem.

Next, part two of the theorem says that, for $x\in X$, we have an identification $H^0(\pi_{0,k}^{-1}(x),\mathcal{O}_{X_k}(m)) \simeq E_{k,m}V^*_{X,x}$: we want to describe briefly what this identification is. Fix a section $\sigma \in H^0(\pi_{0,k}^{-1}(x),\mathcal{O}_{X_k}(m))$. Recall that given regular k-jet of curve at $x\in X$, the derivative $f'_{[k-1]}(0)$ defines an element of the fiber of $\mathcal{O}_{X_k}(-1)$ at $f_{[k]}(0)$. Then we get a well-defined complex valued operator
$$Q(f',f'',\dots,f^{(k)}) = \sigma(f_{[k]}(0))\cdot(f'_{[k-1]}(0))^m,$$
and this definition extends to singular jets by an easy Riemann's extension theorem argument. This correspondence is easily shown to be bijective and is in fact the one given in the theorem.

1.4.2 Sufficient conditions for relative positivity

The relative structure of the fibration $\pi_{0,k}\colon X_k \to X$ is completely universal and its fibers are smooth rational varieties which depend only on k and on the rank of V.

Moreover, as X_k arises as a sequence of successive compactifications of vector bundles, its Picard group has a quite simple structure, namely we have
$$\mathrm{Pic}(X_k) \simeq \mathrm{Pic}(X)\oplus\mathbb{Z}u_1\cdots\oplus\mathbb{Z}u_k,$$
where u_j, $j=1,\dots,k$, is the class of $\mathcal{O}_{X_j}(1)$.

As we already observed, the line bundle $\mathcal{O}_{X_k}(1)$ is never relatively ample over X for $k\geq 2$. Now, for each $\mathbf{a}=(a_1,\dots,a_k)\in\mathbb{Z}^k$, we define a line bundle $\mathcal{O}_{X_k}(\mathbf{a})$ as
$$\mathcal{O}_{X_k}(\mathbf{a}) \stackrel{\mathrm{def}}{=} \pi_{1,k}^*\mathcal{O}_{X_1}(a_1)\otimes\pi_{2,k}^*\mathcal{O}_{X_2}(a_2)\otimes\cdots\otimes\mathcal{O}_{X_k}(a_k).$$

By formula (1.3), we get inductively

$$\pi_{j,k}^* \mathcal{O}_{X_j}(1) = \mathcal{O}_{X_k}(1) \otimes \mathcal{O}_{X_k}(-\pi_{j+1,k}^* D_{j+1} - \cdots - D_k).$$

Set, for $j = 1, \ldots, k-1$, $D_j^\star = \pi_{j+1,k}^* D_{j+1}$ and $D_k^\star = 0$. Then, if we define the weight $\mathbf{b} = (b_1, \ldots, b_k) \in \mathbb{Z}^k$ by $b_j = a_1 + \cdots + a_j$, $j = 1, \ldots, k$, we find an identity

$$\mathcal{O}_{X_k}(\mathbf{a}) \simeq \mathcal{O}_{X_k}(b_k) \otimes \mathcal{O}_{X_k}(-\mathbf{b} \cdot D^\star),$$

where

$$\mathbf{b} \cdot D^\star \stackrel{\text{def}}{=} \sum_{j=1}^{k-1} b_j \, \pi_{j+1,k}^* D_{j+1}.$$

In particular, as all the D_j's are effective, if $\mathbf{b} \in \mathbb{N}^k$, that is $a_1 + \cdots + a_j \geq 0$ for all $j = 1, \ldots, k$, we get a non-trivial bundle morphism

$$\mathcal{O}_{X_k}(\mathbf{a}) \simeq \mathcal{O}_{X_k}(b_k) \otimes \mathcal{O}_{X_k}(-\mathbf{b} \cdot D^\star) \to \mathcal{O}_{X_k}(b_k).$$

Set theoretically, we have seen that the relative base locus of the complete linear system $|\mathcal{O}_{X_k}(m)|$ is exactly $X_k^{\text{sing}} = \bigcup_{j=2}^k \pi_{j,k}^{-1}(D_j)$.

Now, we would like to twist the line bundle $\mathcal{O}_{X_k}(m)$ by an ideal sheaf $\mathcal{I}$, possibly supported on X_k^{sing}, in order to get rid of this base locus. If one wants to remain in the category of invertible sheaves, then this ideal sheaf should be something of the form $\mathcal{O}_{X_k}(-\mathbf{b} \cdot D^\star)$, for $\mathbf{b} \in \mathbb{N}^k$.

Next proposition gives sufficient conditions to solve this problem.

Proposition 1.4.2 ([7]). *Let $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ be a weight and $m = b_k = a_1 + \cdots + a_k$. Then*

- *we have the direct image formula*

$$(\pi_{0,k})_* \mathcal{O}_{X_k}(\mathbf{a}) \simeq \mathcal{O}(\overline{F}^{\mathbf{a}} E_{k,m} V^*) \subset \mathcal{O}(E_{k,m} V^*)$$

  where $\overline{F}^{\mathbf{a}} E_{k,m} V^*$ is the subbundle of polynomials $Q(f', f'', \ldots, f^{(k)}) \in E_{k,m} V^*$ involving only monomials $(f^{(\bullet)})^\ell$ such that

$$\ell_{s+1} + 2\ell_{s+2} + \cdots + (k-s)\ell_k \leq a_{s+1} + \cdots + a_k$$

  for all $s = 0, \ldots, k-1$.

- *if $a_1 \geq 3a_2, \ldots, a_{k-2} \geq 3a_{k-1}$ and $a_{k-1} \geq 2a_k \geq 0$, then the line bundle $\mathcal{O}_{X_k}(\mathbf{a})$ is relatively nef over X.*

- *if $a_1 \geq 3a_2, \ldots, a_{k-2} \geq 3a_{k-1}$ and $a_{k-1} > 2a_k > 0$, then the line bundle $\mathcal{O}_{X_k}(\mathbf{a})$ is relatively ample over X.*

1.5 Logarithmic jet bundles

There is a completely similar construction of jet differentials and invariant jet differentials in the setting of logarithmic manifolds. This construction has been done by Dethloff and Lu [11], with the intent of studying hyperbolicity problems of open manifolds of the form $X \setminus D$, where X is a compact manifold and D a simple normal crossing divisor.

1.5.1 Logarithmic manifolds

Let $D \subset X$ be a normal crossing divisor in a compact complex manifold X, that is for each $x \in X$ there exist local holomorphic coordinates $(z_1, \dots, z_n)$ for X, centered at x, such that locally $D = \{z_1 \cdots z_l = 0\}$, $0 \leq l \leq n$. Then $T^*_X\langle D\rangle$, the *logarithmic cotangent space* to X relative to D, is well defined and locally free: it is the subsheaf of the sheaf of meromorphic differential forms of the form

$$\sum_{j=1}^{l} f_j \frac{dz_j}{z_j} + \sum_{k=l+1}^{n} f_k \, dz_k,$$

where $f_i \in \mathcal{O}_{X,x}$ are germs of holomorphic functions in x and the local coordinates are chosen as above. In the case when the divisor D is irreducible (which actually will be the case we are interested in), we have the following short exact sequence

$$0 \longrightarrow T^*_X \longrightarrow T^*_X\langle D\rangle \longrightarrow \mathcal{O}_D \longrightarrow 0 \tag{1.4}$$

and also $T^*_X\langle D\rangle|_{X \setminus D} = T^*_X$.

The dual sheaf $T_X\langle D\rangle$, called the *logarithmic tangent bundle* relative to D, is easily checked to be the (locally free) sheaf of germs of holomorphic vector fields in $\mathcal{O}(T_X)$ which are tangent to (each component of) D.

1.5.2 Projectivization of logarithmic jet bundles

Here, we recall the construction which can be found in [11] of logarithmic jet bundles, which is in fact completely analogous to the "standard" one.

So, we start with a triple (X, D, V) where (X, V) is a compact directed manifold and $D \subset X$ is a simple normal crossing divisor whose components $D_{(j)}$ are everywhere transversal to V (that is $T_{D_{(j)}} + V = T_X$ along $D_{(j)}$).

Let $\mathcal{O}(V\langle D\rangle)$ be the (locally free in this setting) sheaf of germs of holomorphic vector fields which are tangent to each component of D.

Now, we define a sequence (X_k, D_k, V_k) of logarithmic k-jet projectivized bundles: if $(X_0, D_0, V_0) = (X, D, V\langle D\rangle)$, set $X_k = P(V_{k-1})$, $D_k = \pi_{0,k}^{-1} D$ and

V_k is the set of logarithmic tangent vectors in $T_{X_k}\langle D_k\rangle$ which project onto the line defined by the tautological line bundle $\mathcal{O}_{X_k}(-1) \subset \pi_k^* V_{k-1}$.

All the above constructions work almost in the same way in the logarithmic case. For instance, we have short exact sequences

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \xrightarrow{(\pi_k)_*} \mathcal{O}_{X_k}(-1) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_{X_k} \longrightarrow \pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \longrightarrow T_{X_k/X_{k-1}} \longrightarrow 0.$$

and, for $f\colon \Delta_R \to X \setminus D$ a non-constant trajectory tangent to V, we have a well-defined lifting $f_{[k]}\colon \Delta_R \to X_k \setminus D_k$ tangent to V_k. Moreover, the derivative $f'_{[k-1]}$ gives a section

$$f'_{[k-1]}\colon T_{\Delta_R} \to \mathcal{O}_{X_k}(-1).$$

With any section σ of $\mathcal{O}_{X_k}(m)$ on any open set $\pi_{0,k}^{-1}(U)$, $U \subset X \setminus D$, one can associate a holomorphic differential operator Q of order k acting on k-jets of germs of curve $f\colon (\mathbb{C},0) \to U$ tangent to V, by setting, as before,

$$Q(f',\dots,f^{(k)}) = \sigma(f_{[k]}) \cdot (f'_{[k-1]})^m$$

and this correspondence is again bijective.

Then, in this situation, the direct image formula of Theorem 1.4.1 becomes

$$(\pi_{0,k})_* \mathcal{O}_{X_k}(m) = \mathcal{O}(E_{k,m} V^* \langle D\rangle),$$

where $\mathcal{O}(E_{k,m} V^*\langle D\rangle)$ is the sheaf generated by all invariant (by the action of $\mathbb{G}_k$) polynomial differential operators in the derivatives of order $1,2,\dots,k$ of the components $f_1,\dots,f_r$ of a germ of holomorphic curve $f\colon (\mathbb{C},0) \to X \setminus D$ tangent to V, together with the derivatives of the extra functions $\log s_j(f)$ along the j-th components $D_{(j)}$ of D, where s_j is a local equation for $D_{(j)}$.

Remark 1.5.1. Of course, the second and the third part of Proposition 1.4.2 continue to hold in the setting of logarithmic jets, since the "relative" situation is the same as in the compact case.

1.6 Negative k-jet metrics

In this section we shall make the link between the theory of Kobayashi-hyperbolicity and invariant jet differentials.

The philosophical idea which is behind, is that global invariant differential operators on compact complex manifold vanishing on an ample divisor provide algebraic differential equations which every non-constant entire holomorphic curve of the manifold must satisfy. Then, morally, the existence of "lots" of differential equations should imply the existence of "few" curves.

1.6.1 Metrics on k-jets

We have seen that every regular k-jet of curve f, defines an element $f'_{[k-1]} \in \mathcal{O}_{X_k}(-1)$. Thus, in some sense, taking an hermitian metric on the line bundle $\mathcal{O}_{X_k}(-1)$ is a way to measure the "norm" of k-jets (this ideas date back to Green and Griffiths [17]).

Definition 1.6.1. A *smooth (or singular) k-jet metric* on a complex directed manifold (X, V) is an hermitian metric h_k on $\mathcal{O}_{X_k}(-1)$ (or, which is the same, a Finsler metric on V_{k-1}) such that the weight functions φ which represent locally the metric are smooth (or L^1_{loc}).

We define $\Sigma_{h_k} \subset X_k$ to be the singularity set of the metric, that is the closed subset of points in a neighborhood of which the weight φ is not locally bounded. Clearly, $\Sigma_{h_k} = \emptyset$ if h_k is smooth.

Definition 1.6.2. Let h_k be a k-jet metric on (X, V). We say that h_k has *negative jet curvature* if the curvature $\Theta_{h_k}(\mathcal{O}_{X_k}(-1))$ is negative definite along the subbundle $V_k \subset T_{X_k}$.

We recall here that the curvature $(1,1)$-form $\Theta_{h_k}(\mathcal{O}_{X_k}(-1))$ is locally defined by $\frac{i}{2\pi}\partial\bar{\partial}\varphi$. As φ is at least L^1_{loc}, we can always compute the curvature in the weak sense: in case that φ is not smooth, the curvature will be a $(1,1)$-current on X_k (see for example [6] for more details).

Thus, being negative definite in Definition 1.6.2, means that there exist a positive continuous function $\varepsilon > 0$ on X_k and a smooth hermitian metric ω_k on V_k such that

$$\langle \Theta_{h_k^{-1}}(\mathcal{O}_{X_k}(1)) \rangle(\xi) \geq \varepsilon ||\xi||^2_{\omega_k}, \quad \forall \xi \in V_k,$$

in the sense of distributions.

Here, by $\langle \Theta_{h_k^{-1}}(\mathcal{O}_{X_k}(1)) \rangle$, we mean the quadratic form naturally associated to the curvature.

Example 1.6.1. If (X, V) is a compact complex directed manifold such that V^* is ample, then by definition, $\mathcal{O}_{X_1}(1)$ is ample. By the celebrated Kodaira embedding theorem, $\mathcal{O}_{X_1}(1)$ admits a smooth hermitian metric with positive definite curvature. In particular, if V^* is ample, then (X, V) has a smooth 1-jet metric h_1 with negative jet curvature.

More generally, if $\mathcal{O}_{X_1}(1)$ is big, then there exist a 1-jet singular metric h'_1, a positive constant $\varepsilon > 0$ and a smooth hermitian metric ω_1 on T_{X_1}, such that

$$\langle \Theta_{h_1'^{-1}}(\mathcal{O}_{X_1}(1)) \rangle(\xi) \geq \varepsilon ||\xi||^2_{\omega_1},$$

for all $\xi \in T_{X_1}$ (cf. [6]). Then (X, V) has a singular 1-jet metric h'_1 with negative jet curvature.

As we shall see, this is the case, for example, when X is an algebraic surface of general type, $V = T_X$ and $c_1^2(X) - c_2(X) > 0$ (here $c_1(X)$ and $c_2(X)$ are the Chern classes of the tangent bundle of X).

Here is the precise statement which puts in relation (invariant) jet differentials and the geometry of non-constant entire holomorphic curves.

Theorem 1.6.1 ([7], compare with [17]). *Let (X, V) be a compact complex directed manifold. If (X, V) admits a k-jet metric h_k with negative jet curvature, then every non-constant entire holomorphic curve $f\colon \mathbb{C} \to X$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset \Sigma_{h_k}$.*

In particular, if $\Sigma_{h_k} \subset X_k^{\text{sing}}$, then (X, V) is hyperbolic.

A very important special case when the above theorem can be applied is when $\mathcal{O}_{X_k}(m)$ admits non-trivial holomorphic sections vanishing on an ample divisor coming from the base.

Namely, we have the following corollary.

Corollary 1.6.2. *Assume that there exist integers $k, m > 0$ and an ample line bundle $A \to X$ such that*

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^* A^{-1}) \simeq H^0(X, E_{k,m}V^* \otimes A^{-1})$$

has nonzero sections $\sigma_1, \dots, \sigma_N$. Let $Z = \bigcap_{j=1}^N \sigma_j^{-1}(0) \subset X_k$ be the base locus of these sections.

Then every non-constant entire holomorphic curve $f\colon \mathbb{C} \to X$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset Z$.

Remark 1.6.2. This corollary says, in other words, that for every global $\mathbb{G}_k$-invariant polynomial differential operator P which takes values in A^{-1}, every entire curve f must satisfy the corresponding algebraic differential equation $P(f) = 0$.

Remark 1.6.3. The same type of result holds true, but only in the "absolute" case $V = T_X$, if we replace $E_{k,m}T_X^*$ with $\mathcal{J}_{k,m}T_X^*$. See [17] and [25] for more details.

Remark 1.6.4. The k-jet negativity property of the curvature becomes actually weaker and weaker as k increases. In fact, one can show that if (X, V) has a $(k-1)$-jet metric h_{k-1} with negative jet curvature, then there exists a k-jet metric h_k with negative jet curvature and such that $\Sigma_{h_k} \subset \pi_k^{-1}(\Sigma_{h_{k-1}}) \cup D_k$.

1.6.2 The logarithmic case

In [11], Theorem 1.6.1 is stated in the setting of logarithmic jet differentials. For the sake of completeness, we shall give this theorem and its corollary here.

Theorem 1.6.3 ([11]). *Let (X, D, V) be a compact complex logarithmic directed manifold. If (X, D, V) admits a k-jet metric h_k with negative jet curvature, then every non-constant entire holomorphic curve $f \colon \mathbb{C} \to X \setminus D$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset \Sigma_{h_k}$.*

As above, by subtracting a little bit of positivity to $E_{k,m}V^*\langle D\rangle$, we get a more geometric version of the theorem.

Corollary 1.6.4. *Assume that there exist integers $k, m > 0$ and an ample line bundle $A \to X$ such that*

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^* A^{-1}) \simeq H^0(X, E_{k,m}V^*\langle D\rangle \otimes A^{-1})$$

has nonzero sections $\sigma_1, \ldots, \sigma_N$. Let $Z = \bigcap_{j=1}^N \sigma_j^{-1}(0) \subset X_k$ be the base locus of these sections.

Then every non-constant entire holomorphic curve $f \colon \mathbb{C} \to X \setminus D$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset Z$.

Chapter 2

Vanishing of jet differentials for projective hypersurfaces

Abstract. This chapter is devoted to the study of low order jet differentials on smooth projective hypersurfaces. It was already observed in [10] for the surface case and in [21] for the threefold case that, in general, one has to look for jet differentials of order equal to the dimension of the manifold, because of their vanishing in lower order. Using some result of Brückmann and Rackwitz contained in [4], we generalize the vanishing of global invariant jet differentials of order less than the dimension of the ambient manifold to any dimension, provided the manifold is a smooth projective hypersurface. We also give in all dimension a logarithmic version of this theorem, for the log-pair $(\mathbb{P}^n, D)$, where D is a smooth irreducible divisor, hence extending a result of [15]. Most of the material of the present chapter comes from our paper [12].

2.1 Statement of the theorems

Classically, when one wants to show some result on the algebraic degeneracy of entire curves in a given manifold, the first step is to look for global jet differentials in the given manifold (these techniques trace back to the works of Bloch [1] and Green and Griffiths [17]).

Thus, it is quite meaningful to determine which is the smallest order of jet differentials we are obliged to deal with.

In 1970, Kobayashi [18] conjectured that if $X \subset \mathbb{P}^{n+1}$ is a generic complex projective hypersurface of degree $\deg X = d$, then X is hyperbolic provided $d \geq 2n+1$. Again in [18], he also conjectured the logarithmic version of the previous statement: if $D \subset \mathbb{P}^n$ is a generic irreducible divisor of degree $\deg D = d$, then $\mathbb{P}^n \setminus D$ is hyperbolic provided $d \geq 2n+1$.

In this section we show that for the "first step" when one tries to prove these conjectures, one has to look for jet differentials of order n, since there are no global jet differentials of order less then the dimension of the ambient manifold.

More precisely, we shall show the following theorem.

Theorem 2.1.1 ([12]). *Let $X \subset \mathbb{P}^N$ be a smooth complete intersection. Then*

$$H^0(X, \mathfrak{J}_{k,m} T_X^*) = 0$$

for all $m \geq 1$ and $1 \leq k < \dim X / \operatorname{codim} X$.

The idea of the proof is to exclude, in the direct sum decomposition into irreducible $\mathrm{Gl}(T_X^*)$-representation of the graded bundle $\mathrm{Gr}^\bullet \mathfrak{J}_{k,m} T_X^*$, the existence of Schur powers $\Gamma^{(\lambda_1,\dots,\lambda_n)} T_X^*$, with $\lambda_n > 0$, and then to use a vanishing theorem due to Brückmann and Rackwitz for Schur powers of the cotangent bundle of smooth projective complete intersections.

Corollary 2.1.2. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex projective hypersurface and $A \to X$ any ample line bundle. Then*

$$H^0(X, E_{k,m} T_X^* \otimes A^{-1}) = 0$$

for all $m \geq 1$ and $1 \leq k \leq n-1$.

Of course, the corollary is an immediate consequence of our theorem, since $E_{k,m} T_X^* \subset \mathfrak{J}_{k,m} T_X^*$ and we are moreover subtracting some positivity.

We also have the logarithmic version of this theorem and of its corollary.

Theorem 2.1.3. *Let $D \subset \mathbb{P}^n$ be a smooth irreducible divisor of degree $\deg D = \delta$. Then*

$$H^0(\mathbb{P}^n, \mathfrak{J}_{k,m} T_{\mathbb{P}^n}^* \langle D \rangle) = 0$$

for all $m \geq 1$ and $1 \leq k \leq n-1$, provided $\delta \geq 3$.

Corollary 2.1.4. *Let $D \subset \mathbb{P}^n$ be a smooth irreducible divisor of degree $\deg D = \delta$ and $A \to \mathbb{P}^n$ any ample line bundle. Then*

$$H^0(\mathbb{P}^n, E_{k,m} T_{\mathbb{P}^n}^* \langle D \rangle \otimes A^{-1}) = 0$$

for all $m \geq 1$ and $1 \leq k \leq n-1$, provided $\delta \geq 3$.

The proof of this theorem is achieved by reduction to the compact case thanks to the standard ramified covering $\mathbb{P}^{n+1} \supset \widetilde{D} \to \mathbb{P}^n$ associated to D.

We want to make here just a few remark to end this section. First of all, these are "negative" result, since they deal with the non-existence of something which we are interested in.

Nevertheless, as we shall see in the next chapter, both are essential in proving the *existence* of global invariant jet differentials of order n.

Finally, these theorems are the natural generalization of analogous result which can be found, for the compact case, in [21] up to dimension three and, for the logarithmic case, in [15] up to dimension two.

2.2 Schur powers of a complex vector space

Here, we recall the notation and a possible construction of Schur powers of a complex vector space. Let V be a complex vector space of dimension r. With every set of non-increasing r-tuples $(\lambda_1, \dots, \lambda_r) \in \mathbb{Z}^r$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, one associates, in a functorial way, a collection of vector spaces $\Gamma^{(\lambda_1,\dots,\lambda_r)}V$, which provide the list of all irreducible representations of the general linear group $\mathrm{Gl}(V)$, up to isomorphism (in fact $(\lambda_1, \dots, \lambda_r)$ is the highest weight of the action of a maximal torus $(\mathbb{C}^*)^r \subset \mathrm{Gl}(V)$). The Schur functors can be defined in an elementary way as follows. Let

$$\mathbb{U}_r = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}$$

be the group of lower triangular unipotent $r \times r$ matrices. If all λ_j are nonnegative, one defines

$$\Gamma^{(\lambda_1,\dots,\lambda_r)}V \subset S^{\lambda_1}V \otimes \cdots \otimes S^{\lambda_r}V$$

as the set of polynomials $P(\xi_1, \dots, \xi_r)$ on $(V^*)^r$ which are homogeneous of degree λ_j with respect to ξ_j and which are invariant under the left action of $\mathbb{U}_r$ on $(V^*)^r = \mathrm{Hom}(V, \mathbb{C}^r)$, that is

$$P(\xi_1, \dots, \xi_{j-1}, \xi_j + \xi_k, \xi_{j+1}, \dots, \xi_r) = P(\xi_1, \dots, \xi_r), \quad \forall k < j.$$

We agree that $\Gamma^{(\lambda_1,\dots,\lambda_r)}V = 0$ unless $(\lambda_1, \dots, \lambda_r)$ is non-increasing. As a special case, we recover symmetric and exterior powers

$$S^m V = \Gamma^{(m,0,\dots,0)}V,$$
$$\bigwedge^k V = \Gamma^{(1,\dots,1,0,\dots,0)}V, \quad \text{with } k \text{ indices } 1.$$

The Schur functors satisfy the well-known formula

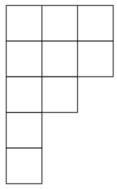
$$\Gamma^{(\lambda_1+\ell,\dots,\lambda_r+\ell)}V = \Gamma^{(\lambda_1,\dots,\lambda_r)}V \otimes (\det V)^\ell,$$

which can be used to define $\Gamma^{(\lambda_1,\dots,\lambda_r)}V$ if any of the λ_j's happens to be negative.

2.2.1 Young tableaux

Here, we give an alternative (and maybe more classical) description of Schur functors in terms of Young tableau.

To every partition $d = \lambda_1 + \cdots + \lambda_k$, with $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$, it is associated a *Young diagram*



with λ_i boxes in the i-th row, the rows of boxes lined up on the left. For a given Young diagram, number the boxes, say consecutively as shown:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5\\ \cline{1-2} 6&7\\ \cline{1-2} 8\\ \cline{1-1}\end{array}$$

More generally, define a *tableau* on a given Young diagram to be a numbering of the boxes by the integers $1, \ldots, d$. The *length* of a given Young tableau is defined to be the maximum, over the rows of its associated Young diagram, of the number of boxes in its rows.

Given a tableau, say the canonical one shown, define two subgroups of the symmetric group

$$P = P_\lambda = \{g \in \mathfrak{S}_d \mid g \text{ preserves each row}\}$$

and

$$Q = Q_\lambda = \{g \in \mathfrak{S}_d \mid g \text{ preserves each column}\}.$$

In the group algebra $\mathbb{C}\mathfrak{S}_d$, we introduce two elements corresponding to these subgroups: we set

$$a_\lambda = \sum_{g \in P} e_g \quad \text{and} \quad b_\lambda = \sum_{g \in Q} \operatorname{sgn}(g) \cdot e_g.$$

To see what a_λ and b_λ do, observe that if V is any vector space and $\mathfrak{S}_d$ acts on the d-th tensor power $V^{\otimes d}$ by permuting factors, the image of the element $a_\lambda \in \mathbb{C}\mathfrak{S}_d \to \operatorname{End}(V^{\otimes d})$ is just the subspace

$$\operatorname{Im}(a_\lambda) = S^{\lambda_1} V \otimes \cdots \otimes S^{\lambda_k} V \subset V^{\otimes d}.$$

Similarly, the image of b_λ on this tensor power is

$$\mathrm{Im}(b_\lambda) = \wedge^{\mu_1} V \otimes \cdots \otimes \wedge^{\mu_l} V \subset V^{\otimes d},$$

where μ is the conjugate partition of λ, that is the partition defined by interchanging rows and column in the Young diagram.

Finally, we set

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}\mathfrak{S}_d;$$

this is called a *Young symmetrizer.*

We then recover our previously defined Schur functors as the image of the Young symmetrizer associated to the partition λ:

$$\Gamma^{(\lambda)} V \stackrel{\text{def}}{=} \mathrm{Im}(c_\lambda).$$

2.3 Proof of Theorem 2.1.1

Our starting point is a theorem by Brückmann and Rackwitz. We cite it here as it appears in [4].

Theorem 2.3.1 ([4]). *Let X be a n-dimensional smooth complete intersection in $\mathbb{P}^N$. Let T be any Young tableau and t_i be the number of cells inside the i-th column of T. Set*

$$t := \sum_{i=1}^{N-n} t_i, \quad t_i = 0 \text{ if } i > \operatorname{length} T.$$

Then, if $t < n$ one has the vanishing

$$H^0(X, \Gamma^T T_X^*) = 0,$$

that is, the smooth complete intersection X has no global T-symmetrical tensor forms different from zero if the Young tableau T has less than $\dim X$ *cells inside its* $\operatorname{codim} X$ *front columns.*

In our notation, the irreducible $\mathrm{Gl}(T_X^*)$-representation, given for $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ by $\Gamma^{(\lambda_1,\ldots,\lambda_n)} T_X^*$, corresponds to $\Gamma^{T_\lambda} T_X^*$ where the tableau T_λ is obtained from the partition $\lambda_1 + \cdots + \lambda_n$. Thus, for example, the tableau with only one row of length m corresponds to $S^m T_X^*$ and the tableau with only one column of depth k corresponds to $\Lambda^k T_X^*$.

Now, we need a simple lemma.

Lemma 2.3.2. *Let V be a complex vector space of dimension n and $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. Then*

$$\Gamma^\lambda V \otimes S^m V \simeq \bigoplus_\mu \Gamma^\mu V$$

as $\mathrm{Gl}(V)$*-representations, the sum being over all μ whose Young diagram is obtained by adding m boxes to the Young diagram of λ, with no two in the same column.*

Proof. This follows immediately by Pieri's formula, see e.g. [16]. □

Note that this implies that among all the irreducible $\mathrm{Gl}(V)$-representations of $S^l V \otimes S^m V$, we cannot find terms of type $\Gamma^{(\lambda_1,\dots,\lambda_n)}V$ with $\lambda_i > 0$ for $i > 2$ (they are all of type $\Gamma^{(l+m-j,j,0,\dots,0)}V$ for $j = 0, \dots, \min\{m, l\}$).

Thus, by induction on the number of factor in the tensor product of symmetric powers, we easily find:

Proposition 2.3.3. *If $k \leq n$ then we have a direct sum decomposition into irreducible* $\mathrm{Gl}(V)$*-representations*

$$S^{l_1}V \otimes S^{l_2}V \otimes \dots \otimes S^{l_k}V = \bigoplus_\lambda \nu_\lambda \, \Gamma^\lambda V,$$

where $\nu_\lambda \neq 0$ only if $\lambda = (\lambda_1, \dots, \lambda_n)$ is such that $\lambda_i = 0$ for $i > k$.

Proof of the theorem

The bundle $\mathcal{J}_{k,m}T_X^*$ admits a filtration whose associated graded bundle is given by

$$\mathrm{Gr}^\bullet \mathcal{J}_{k,m}T_X^* = \bigoplus_{\ell_1+2\ell_2+\dots+k\ell_k=m} S^{\ell_1}T_X^* \otimes S^{\ell_2}T_X^* \otimes \dots \otimes S^{\ell_k}T_X^*.$$

The addenda in the direct sum decomposition into irreducible $\mathrm{Gl}(T_X^*)$-representation of $\mathrm{Gr}^\bullet \mathcal{J}_{k,m}T_X^*$ are all of type $\Gamma^{(\lambda_1,\dots,\lambda_n)}T_X^*$ with $\lambda_i = 0$ for $i > k$. This means that in the statement of Theorem 2.3.1, each t_i is less than or equal to k. Then, if $\dim X = n$, we have

$$\begin{aligned} t &= \sum_{i=1}^{N-n} t_i \leq \sum_{i=1}^{N-n} k \\ &= k(N-n) < \frac{n}{N-n}(N-n) = n. \end{aligned}$$

Thus, by Theorem 2.3.1, we have vanishing of global section of each graded piece. We now only need to link the vanishing of the cohomology of a filtered vector bundle to the vanishing of his graded bundle. This is done in the next lemma and the proof is achieved.

Lemma 2.3.4. *Let $E \to X$ be a holomorphic filtered vector bundle with filtered pieces $\{0\} = E_r \subset \cdots \subset E_{p+1} \subset E_p \subset \cdots \subset E_0 = E$. If $H^q(X, \operatorname{Gr}^\bullet E) = 0$ then $H^q(X, E) = 0$.*

Proof. Consider the short exact sequence

$$0 \longrightarrow \operatorname{Gr}^p E \longrightarrow E/E_{p+1} \longrightarrow E/E_p \longrightarrow 0$$

and the associated long exact sequence in cohomology

$$\cdots \longrightarrow \underbrace{H^q(X, \operatorname{Gr}^p E)}_{=0} \longrightarrow H^q(X, E/E_{p+1}) \longrightarrow H^q(X, E/E_p) \longrightarrow \cdots .$$

For $p = 1$, we get $H^q(X, E/E_1) = H^q(X, \operatorname{Gr}^0 E) = 0$ by hypothesis. Thus, $H^q(X, E/E_2) = 0$ and, by induction on p, we find $H^q(X, E/E_p) = 0$. Therefore, for $p = r$ we get the desired result. □

2.4 Logarithmic vanishing: proof of Theorem 2.1.3

Let (X, D, V) be a logarithmic directed manifold as in §1.5.2. As in the compact case, one can equip the (locally free) sheaf $\mathcal{O}(\mathcal{J}_{k,m} V^*\langle D\rangle)$ of logarithmic jet differentials (that is the sheaf generated by all polynomial differential operators in the derivatives of order $1, 2, \ldots, k$, of f together with the derivatives of the extra functions $\log s_j(f)$ along the j-th components $D_{(j)}$ of D) with a filtration whose associated graded bundle is

$$\operatorname{Gr}^\bullet \mathcal{J}_{k,m} V^*\langle D\rangle = \bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1}V^*\langle D\rangle \otimes \cdots \otimes S^{\ell_k}V^*\langle D\rangle.$$

2.4.1 Reduction to the compact case

Now, we fix our attention on the triple $(\mathbb{P}^n, D, T_{\mathbb{P}^n})$, where $D \subset \mathbb{P}^n$ is a smooth divisor of degree $\deg D = \delta$.

Consider the standard ramified covering $\mathbb{P}^{n+1} \supset \widetilde{D} \to \mathbb{P}^n$ associated to D: if D is given by a homogeneous equation $P(z_0, \ldots, z_n) = 0$ of degree δ, then $\widetilde{D} \subset \mathbb{P}^{n+1}$ is cut out by the single equation $z_{n+1}^\delta = P(z_0, \ldots, z_n)$.

If we take pullbacks of logarithmic differential forms on $\mathbb{P}^n$, we obtain an injection $H^0(\mathbb{P}^n, T^*_{\mathbb{P}^n}\langle D\rangle) \hookrightarrow H^0(\widetilde{D}, T^*_{\widetilde{D}} \otimes \mathcal{O}_{\widetilde{D}}(1))$. This is easily seen, as on $\widetilde{D}$ one has

$$\frac{dP}{P}\bigg|_{\widetilde{D}} = \frac{dz_{n+1}^{\delta}}{z_{n+1}^{\delta}}\bigg|_{\widetilde{D}} = \delta\,\frac{dz_{n+1}}{z_{n+1}}\bigg|_{\widetilde{D}},$$

so that pullbacks of logarithmic forms downstairs give rise to forms with one simple pole along the hyperplane section $\{z_{n+1} = 0\} \cap \widetilde{D}$. We now need the “twisted” version of Theorem 2.3.1.

Let $Y = H_1 \cap \cdots \cap H_{N-n} \subset \mathbb{P}^N$ be an n-dimensional smooth complete intersection by the hypersurfaces $H_i \subset \mathbb{P}^N$, with $d_i = \deg H_i$.

Theorem 2.4.1 ([4]). *If $p < r + \min\{\text{length}(T), d_1 - 2, \ldots, d_{N-n} - 2\}$ and $t < n$, then*

$$H^0(Y, \Gamma^{(\lambda)} T^*_Y \otimes \mathcal{O}_Y(p)) = 0.$$

In particular, if $Y \subset \mathbb{P}^{n+1}$ is a smooth projective hypersurface of degree $\deg Y = d$, *then*

$$H^0(Y, \Gamma^{(\lambda)} T^*_Y \otimes \mathcal{O}_Y(r + p)) = 0$$

if $\lambda_n = 0$ and $p < \min\{\lambda_1, d - 2\}$.

Being this theorem available, now we just have to apply, given the weight λ, the Schur functors to the injection $H^0(\mathbb{P}^n, T^*_{\mathbb{P}^n}\langle D\rangle) \hookrightarrow H^0(\widetilde{D}, T^*_{\widetilde{D}} \otimes \mathcal{O}_{\widetilde{D}}(1))$, in order to obtain the new injection

$$H^0(\mathbb{P}^n, \Gamma^{(\lambda)} T^*_{\mathbb{P}^n}\langle D\rangle) \hookrightarrow H^0(\widetilde{D}, \Gamma^{(\lambda)} T^*_{\widetilde{D}} \otimes \mathcal{O}_{\widetilde{D}}(|\lambda|)),$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_n$. Our Theorem 2.1.3 now follows from Theorem 2.4.1 with $r = |\lambda|$ and $p = 0$.

Chapter 3

Existence of invariant jet differentials on projective hypersurfaces

Abstract. In this chapter we consider the problem of finding global invariant jet differentials vanishing on an ample divisor, on smooth projective hypersurfaces in $\mathbb{P}^{n+1}$. We shall prove the existence of such global invariant jet differentials of order n, provided the degree of the hypersurface is large enough, using an algebraic version of Demailly's holomorphic Morse inequalities, first found by Trapani [26]. Moreover, we obtain effective estimates for the lower bound on the degree, at least in low dimension. Similar result are also found in the logarithmic case. A large part of the content of this chapter comes from our papers [12] and [13].

3.1 Introduction and statement of the theorems

We start this chapter with a brief (and certainly incomplete) account of the developments of the research about invariant (and non necessarily invariant) jet differentials.

Several decades after the pioneering work of Bloch [1] in 1926, it has been realized that an essential tool for controlling the geometry of entire curves on a manifold X is to produce differential equations on X that every entire curve must satisfy. For instance, in 1979, Green and Griffiths [17] constructed the sheaf $\mathcal{J}_{k,m}$ of jet differentials of order k and weighted degree m and were thus able to prove the Bloch conjecture (that is, that every entire holomorphic curve in a projective variety is algebraically degenerate as soon as its irregularity is greater than its dimension).

Several years later, Siu outlined new ideas for proving Kobayashi's conjecture, by making use of jet differentials and by generalizing some techniques due to Clemens, Ein and Voisin (see [24]). However many details are missing, and also it seems to be hard to derive effective results from Siu's approach.

It is known by [10] that every smooth surface in $\mathbb{P}^3$ of degree greater or equal to 15 has got many global invariant jet differential equations. For the dimension three case, Rousseau [21] observed that one needs to look for order three equations since one has in general the vanishing of symmetric differentials and invariant 2-jet differentials for smooth hypersurfaces in projective 4-space. On the other hand [21] shows the existence of global invariant 3-jet differentials vanishing on an ample divisor on every smooth hypersurface X in $\mathbb{P}^4$, provided that $\deg X \geq 97$.

Recently, in [12], we improved the bound for the degree obtained in [21] and found the existence of invariant jet differentials for smooth projective hypersurfaces of dimension at most 8 (with an explicit effective lower bound for the degree of the hypersurface up to dimension 5).

Until our paper [12], the existence was usually obtained by showing first that the Euler characteristic $\chi(E_{k,m}T^*_X)$ of the bundle of invariant jet differentials is positive for m large enough. Then, with a delicate study of the even cohomology groups of such bundles – which usually involves the rather difficult investigation of the composition series of $E_{k,m}T^*_X$ – one could obtain in principle a positive lower bound for $h^0(X, E_{k,m}T^*_X)$ in terms of the Euler characteristic.

Here we generalize the result of [10], [21] and [12] to arbitrary dimension, thus completely solving the problem of finding *invariant* jet differentials on complex projective hypersurfaces of high degree. Namely, we get the following

Theorem 3.1.1 ([12], [13]). *Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex projective hypersurface and let $A \to X$ be an ample line bundle. Then there exists a positive integer d_n such that*

$$H^0(X, E_{k,m}T^*_X \otimes A^{-1}) \neq 0, \quad k \geq n,$$

provided that $\deg(X) \geq d_n$ *and m is large enough.*

Moreover, we have the effective lower bounds for the degree d of X as shown in Table 3.1 (depending on the values of n and k).

In other words, on every smooth n-dimensional complex projective hypersurface of sufficiently high degree, there exist global invariant jet differentials of order n vanishing on an ample divisor, and every entire curve must satisfy the corresponding differential equation.

Table 3.1: Effective lower bound for degree d in the compact case.

	k				
n	1	2	3	4	5
2		18	16	16	16
3			82	74	74
4				329	298
5					1222

Table 3.2: Effective lower bound for degree δ in the logarithmic case.

	k				
n	1	2	3	4	5
2		15	14	14	14
3			75	67	67
4				306	280
5					1154

Unfortunately, the lower bound for the degree of X are really effective only for low values of n, being in general effective just theoretically, and one can compute a reasonable explicit value just for low dimensions. Nevertheless, the result is sharp as far as the order k of jets is concerned since, by our Theorem 2.1.1, there are no jet differentials of order $k < n$ on a smooth projective hypersurface of dimension n.

Here is the logarithmic counterpart of the above theorem, which is a generalization and somehow an improvement of the corresponding result contained in [21].

Theorem 3.1.2 ([13]). *Let $D \subset \mathbb{P}^n$ be a smooth irreducible divisor and let $A \to \mathbb{P}^n$ be an ample line bundle. Then there exists a positive integer δ_n such that*

$$H^0(\mathbb{P}^n, E_{k,m}T^*_{\mathbb{P}^n}\langle D\rangle \otimes A^{-1}) \neq 0, \quad k \geq n,$$

provided that $\deg(D) \geq \delta_n$ *and m is large enough.*

Moreover, we have the effective lower bounds for the degree δ of D as shown in Table 3.2 (depending on the values of n and k).

Finally, we would like to stress that our proof is based on an algebraic version of Demailly's holomorphic Morse inequalities, first obtained by Trapani [26], so that we can deal directly with the dimension of the space of global sections: we are able in this way to skip entirely the arduous study of the higher cohomology and of the graded bundle associated to $E_{k,m}T_X^*$.

3.2 An example: existence of invariant 2-jet differentials

As seen in Chapter 1, the group $\mathbb{G}'_k$ of k-jets of biholomorphic reparametrization tangent to the identity

$$\begin{aligned}\varphi\colon (\mathbb{C},0) &\to (\mathbb{C},0)\\ t &\mapsto t + \frac{a_2}{2!}\,t^2 + \cdots + \frac{a_k}{k!}\,t^k + O(t^{k+1}),\end{aligned}$$

where $a_j \in \mathbb{C}$, $j = 2,\dots,k$, acts on k-tuples $(f',\dots,f^{(k)})$ of derivative at 0 of a k-jet of holomorphic curve. The action is given by the formula

$$(f\circ\varphi)' = f', \quad (f\circ\varphi)'' = f'' + a_2\, f', \quad (f\circ\varphi)''' = f''' + 3a_2\, f'' + a_3\, f', \quad \dots$$

This is a unipotent action, induced by the action of $\mathbb{U}_k$ (that is the group of lower triangular unipotent $k\times k$ matrices) through an embedding $\mathbb{G}'_k \hookrightarrow \mathbb{U}_k$ given by

$$\varphi \mapsto \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0\\ a_2 & 1 & 0 & \cdots & 0\\ a_3 & 3a_2 & 1 & 0 & \vdots\\ \vdots & * & \ddots & \ddots & 0\\ a_k & * & * & \frac{k(k-1)}{2}a_2 & 1\end{pmatrix}.$$

We also recall that we have the following decomposition into graded terms:

$$\mathrm{Gr}^\bullet E_{k,m}V^* = \left(\bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1}V^*\otimes S^{\ell_2}V^*\otimes\cdots\otimes S^{\ell_k}V^*\right)^{\mathbb{G}'_k}.$$

Now, in general, the action of $\mathbb{G}'_k$, does not preserve each individual component in the summation; if $k=1$ we have already seen that $E_{1,m}V^* = S^mV^*$. If $k=2$, the embedding $\mathbb{G}'_k \hookrightarrow \mathbb{U}_k$ is in fact an isomorphism and, moreover, the effect of a parameter change $(f',f'')\mapsto(f',f''+\lambda f')$ on a weighted monomial cannot produce a polynomial of different weighted multi-degree. Thus the various components do not mix up and we get

$$\mathrm{Gr}^\bullet E_{2,m}V^* = \bigoplus_{\ell_1+2\ell_2=m}(S^{\ell_1}V^*\otimes S^{\ell_2}V^*)^{\mathbb{U}_2} = \bigoplus_{\ell_1+2\ell_2=m}\Gamma^{(\ell_1,\ell_2,0\dots,0)}V^*;$$

in particular, if $r = \operatorname{rank} V = 2$, we have that $\Gamma^{(\ell_1,\ell_2)}V^* = S^{\ell_1-\ell_2}V^* \otimes (\det V^*)^{\ell_2}$ and so we derive the very simple formula

$$\operatorname{Gr}^\bullet E_{2,m}V^* = \bigoplus_{j=0}^{m/3} S^{m-3j}V^* \otimes (\det V^*)^j.$$

As a byproduct, we obtain – as announced in Chapter 1 – that the elements of the fibers $E_{2,m}V_x^*$ of $E_{2,m}V^*$ over a point $x \in X$ are of the form

$$\sum_{\alpha_1+\alpha_2+3\beta=m} a_{\alpha_1\alpha_2\beta}(f_1')^{\alpha_1}(f_2')^{\alpha_2}(f_1'f_2''-f_1''f_2')^{\beta}.$$

Remark 3.2.1. As already observed, similar computation have been done for low values of k and r, but it is a major unsolved problem to compute this composition series for arbitrary values of k and r.

3.2.1 Minimal surfaces of general type

Now, suppose that X is an algebraic surface and consider the "absolute" case $V = T_X$.

Since the Euler characteristics of the bundle of invariant jet differentials and of its associated graded bundle are equal, $\chi(E_{2,m}T_X^*) = \chi(\operatorname{Gr}^\bullet E_{2,m}T_X^*)$, and since we know the composition series, a quite easy Riemann-Roch computation gives

$$\chi(E_{2,m}T_X^*) = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3),$$

where c_1 and c_2 are the Chern classes of X.

Next, suppose X is minimal and of general type (that is, its canonical bundle K_X is big and nef). The following theorem can be obtained both as a combination of results of Bogmolov and Tsuji on semi-stability of the tangent bundle of a minimal nonsingular projective variety of general type (cf. [3] and [27]) and as a direct consequence of the existence of approximate Kähler-Einstein metrics combined with a Bochner-type formula (see [7] for a direct proof based on the second kind of approach).

Theorem 3.2.1. *Let X be a n-dimensional projective algebraic manifold admitting a smooth minimal model and let $L \to X$ be a holomorphic line bundle over X. Assume that X is minimal and of general type and let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ be a non-increasing weight. If either L is pseudo-effective (that is, a limit of effective divisors) and $|\lambda| = \sum_{j=1}^n \lambda_j > 0$, or L is big and $|\lambda| \geq 0$, then*

$$H^0(X, \Gamma^{(\lambda)}T_X \otimes L^{-1}) = 0.$$

Using the above theorem in combination with a Serre duality argument, one obatins the following corollary.

Corollary 3.2.2 ([10]). *If X is a minimal algebraic surface of general type and $A \to X$ any ample line bundle over X, then*

$$h^0(X, E_{2,m}T_X^* \otimes A^{-1}) \geq \frac{m^4}{648}(13c_1^2 - 9c_2) - O(m^3).$$

In particular every smooth surface $X \subset \mathbb{P}^3$ of degree greater than or equal to 15 *admits nonzero invariant* 2*-jet differentials vanishing on an ample divisor.*

Of course, everything is easier here because to get a lower bound for h^0 we just have to control h^2, which is done by Serre duality and the vanishing theorem above.

In general (that is in higher dimension), in order to get a lower bound for h^0 starting from the (eventual) positivity of the Euler characteristic, one needs to control the entire even cohomology. The top cohomology can be still shown to be zero by Serre duality and the vanishing theorem above; it is anyway a hard problem to control the intermediate even cohomology which, in general, does not vanish at all.

This has been extensively studied by Rousseau in his papers [22] and [21] for threefold in $\mathbb{P}^4$ of degree greater than or equal to 97, giving then a positive answer for the existence of nonzero invariant 3-jet differentials vanishing on an ample divisor.

3.3 Preliminary material

In this section we shall spend a few word about the cohomology ring of X_k and, for the convenience of the reader, we shall recall the statement of the algebraic version of Demailly's holomorphic Morse inequalities we are interested in.

3.3.1 Cohomology ring of X_k

Denote by $c_\bullet(E)$ the total Chern class of a vector bundle E. The short exact sequences (1.1) and (1.2) give us, for each $k > 0$, the following formulae:

$$c_\bullet(V_k) = c_\bullet(T_{X_k/X_{k-1}})c_\bullet(\mathcal{O}_{X_k}(-1))$$

and

$$c_\bullet(\pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1)) = c_\bullet(T_{X_k/X_{k-1}}),$$

so that

$$c_\bullet(V_k) = c_\bullet(\mathcal{O}_{X_k}(-1))c_\bullet(\pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1)). \tag{3.1}$$

Let us call $u_j = c_1(\mathcal{O}_{X_j}(1))$ and $c_l^{[j]} = c_l(V_j)$. With these notations, (3.1) becomes

$$c_l^{[k]} = \sum_{s=0}^{l} \left[\binom{n-s}{l-s} - \binom{n-s}{l-s-1} \right] u_k^{l-s} \cdot \pi_k^* c_s^{[k-1]}, \quad 1 \le l \le r. \tag{3.2}$$

Since X_j is the projectivized bundle of line of V_{j-1}, we also have the polynomial relations

$$u_j^r + \pi_j^* c_1^{[j-1]} \cdot u_j^{r-1} + \cdots + \pi_j^* c_{r-1}^{[j-1]} \cdot u_j + \pi_j^* c_r^{[j-1]} = 0, \quad 1 \le j \le k. \tag{3.3}$$

After all, the cohomology ring of X_k is defined in terms of generators and relations as the polynomial algebra $H^\bullet(X)[u_1, \dots, u_k]$ with the relations (3.3) in which, utilizing recursively (3.2), we have that $c_l^{[j]}$ is a polynomial with integral coefficients in the variables $u_1, \dots, u_j, c_1(V), \dots, c_l(V)$.

In particular, for the first Chern class of V_k, we obtain the very simple expression

$$c_1^{[k]} = \pi_{0,k}^* c_1(V) + (r-1) \sum_{s=1}^{k} \pi_{s,k}^* u_s. \tag{3.4}$$

3.3.2 Algebraic holomorphic Morse inequalities

Let $L \to X$ be a holomorphic line bundle over a compact Kähler manifold of dimension n and $E \to X$ a holomorphic vector bundle of rank r. Suppose that L can be written as the difference of two nef line bundles, say $L = F \otimes G^{-1}$, with $F, G \to X$ numerically effective. Then we have the following asymptotic estimate for the partial alternating sum of the dimension of cohomology groups of powers of L with values in E.

Theorem 3.3.1 (see e.g. [8], [23]). *With the previous notation, we have (strong algebraic holomorphic Morse inequalities) :*

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X, L^{\otimes m} \otimes E) \le r \frac{m^n}{n!} \sum_{j=0}^{q} (-1)^{q-j} \binom{n}{j} F^{n-j} \cdot G^j + o(m^n).$$

In particular [26], $L^{\otimes m} \otimes E$ has a global section for m large as soon as $F^n - n\, F^{n-1} \cdot G > 0$.

3.4 Proof of Theorem 3.1.1

The idea of the proof is to apply the algebraic holomorphic Morse inequalities to a particular relatively nef line bundle over X_n which admits a nontrivial morphism to (a power of) $\mathcal{O}_{X_n}(1)$ and then to conclude by the direct image argument of Theorem 1.4.1, and Proposition 1.4.2.

From now on, we will set in the "absolute" case $V = T_X$.

3.4.1 Choice of the appropriate subbundle

Recall that, for $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}^k$, we have defined a line bundle $\mathcal{O}_{X_k}(\mathbf{a})$ on X_k as

$$\mathcal{O}_{X_k}(\mathbf{a}) = \pi_{1,k}^* \mathcal{O}_{X_1}(a_1) \otimes \pi_{2,k}^* \mathcal{O}_{X_2}(a_2) \otimes \cdots \otimes \mathcal{O}_{X_k}(a_k)$$

and an associated weight $\mathbf{b} = (b_1, \dots, b_k) \in \mathbb{Z}^k$ such that $b_j = a_1 + \cdots + a_j$, $j = 1, \dots, k$. Moreover, if $\mathbf{b} \in \mathbb{N}^k$, that is if $a_1 + \cdots + a_j \geq 0$, we had a nontrivial morphism

$$\mathcal{O}_{X_k}(\mathbf{a}) = \mathcal{O}_{X_k}(b_k) \otimes \mathcal{O}_{X_k}(-\mathbf{b} \cdot D^\star) \to \mathcal{O}_{X_k}(b_k)$$

and, if

$$a_1 \geq 3a_2, \dots, a_{k-2} \geq 3a_{k-1} \quad \text{and} \quad a_{k-1} \geq 2a_k > 0, \tag{3.5}$$

then $\mathcal{O}_{X_k}(\mathbf{a})$ is relatively nef over X.

Now, let $X \subset \mathbb{P}^{n+1}$ be a smooth complex projective hypersurface. Then it is always possible to express $\mathcal{O}_{X_k}(\mathbf{a})$ as the difference of two globally nef line bundles, provided condition (3.5) is satisfied.

Lemma 3.4.1. *Let $X \subset \mathbb{P}^{n+1}$ be a projective hypersurface. Set $\mathcal{L}_k = \mathcal{O}_{X_k}(2 \cdot 3^{k-2}, \dots, 6, 2, 1)$. Then $\mathcal{L}_k \otimes \pi_{0,k}^* \mathcal{O}_X(\ell)$ is nef if $\ell \geq 2 \cdot 3^{k-1}$. In particular,*

$$\mathcal{L}_k = \mathcal{F}_k \otimes \mathcal{G}_k^{-1},$$

where $\mathcal{F}_k \stackrel{\text{def}}{=} \mathcal{L}_k \otimes \pi_{0,k}^* \mathcal{O}_X(2 \cdot 3^{k-1})$ and $\mathcal{G}_k \stackrel{\text{def}}{=} \pi_{0,k}^* \mathcal{O}_X(2 \cdot 3^{k-1})$ are nef.

Proof. Of course, as a pull-back of an ample line bundle,

$$\mathcal{G}_k = \pi_{0,k}^* \mathcal{O}_X(2 \cdot 3^{k-1})$$

is nef. It is well known that the cotangent space of the projective space twisted by $\mathcal{O}(2)$ is globally generated. Hence, $T_X^* \otimes \mathcal{O}_X(2)$ is globally generated as a quotient of $T_{\mathbb{P}^{n+1}}^*|_X \otimes \mathcal{O}_X(2)$, so that $\mathcal{O}_{X_1}(1) \otimes \pi_{0,1}^* \mathcal{O}_X(2) = \mathcal{O}_{\mathbb{P}(T_X^* \otimes \mathcal{O}_X(2))}(1)$ is nef.

Next, we construct by induction on k, a nef line bundle $A_k \to X_k$ such that $\mathcal{O}_{X_{k+1}}(1) \otimes \pi_k^* A_k$ is nef. By definition, this is equivalent to say that the vector bundle $V_k^* \otimes A_k$ is nef. By what we have just seen, we can take $A_0 = \mathcal{O}_X(2)$ on $X_0 = X$. Suppose $A_0, \ldots, A_{k-1}$ as been constructed. As an extension of nef vector bundles is nef, dualizing the short exact sequence (1.1) we find

$$0 \longrightarrow \mathcal{O}_{X_k}(1) \longrightarrow V_k^* \longrightarrow T^*_{X_k/X_{k-1}} \longrightarrow 0,$$

and so we see, twisting by A_k, that it suffices to select A_k in such a way that both $\mathcal{O}_{X_k}(1) \otimes A_k$ and $T^*_{X_k/X_{k-1}} \otimes A_k$ are nef. To this aim, considering the second wedge power of the central term in (1.2), we get an injection

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow \bigwedge^2(\pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1))$$

and so dualizing and twisting by $\mathcal{O}_{X_k}(2) \otimes \pi_k^* A_{k-1}^{\otimes 2}$, we find a surjection

$$\pi_k^* \bigwedge^2(V_{k-1}^* \otimes A_{k-1}) \longrightarrow T^*_{X_k/X_{k-1}} \otimes \mathcal{O}_{X_k}(2) \otimes \pi_k^* A_{k-1}^{\otimes 2} \longrightarrow 0.$$

By induction hypothesis, $V_{k-1}^* \otimes A_{k-1}$ is nef so the quotient $T^*_{X_k/X_{k-1}} \otimes \mathcal{O}_{X_k}(2) \otimes \pi_k^* A_{k-1}^{\otimes 2}$ is nef, too. In order to have the nefness of both $\mathcal{O}_{X_k}(1) \otimes A_k$ and $T^*_{X_k/X_{k-1}} \otimes A_k$, it is enough to select A_k in such a way that $A_k \otimes \pi_k^* A_{k-1}^*$ and $A_k \otimes \mathcal{O}_{X_k}(-2) \otimes \pi_k^* {A_{k-1}^*}^{\otimes 2}$ are both nef: therefore we set

$$A_k = \mathcal{O}_{X_k}(2) \otimes \pi_k^* A_{k-1}^{\otimes 3} = \big(\mathcal{O}_{X_k}(1) \otimes \pi_k^* A_{k-1}\big)^{\otimes 2} \otimes \pi_k^* A_{k-1},$$

which, as a product of nef line bundles, is nef and satisfies the two conditions above. This gives A_k inductively, and the resulting formula for $\mathcal{O}_{X_k}(1) \otimes \pi_k^* A_{k-1}$ is

$$\begin{aligned}\mathcal{O}_{X_k}(1) \otimes \pi_k^* A_{k-1} &= \mathcal{L}_k \otimes \pi_{0,k}^* \mathcal{O}_X(2 \cdot (1 + 2 + \cdots + 2 \cdot 3^{k-2})) \\ &= \mathcal{L}_k \otimes \pi_{0,k}^* \mathcal{O}_X(2 \cdot 3^{k-1}).\end{aligned}$$

The lemma is proved. □

We now use the above lemma to deal with general weights satisfying condition (3.5).

Proposition 3.4.2. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface and $\mathcal{O}_X(1)$ be the hyperplane divisor on X. If condition (3.5) holds, then $\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^* \mathcal{O}_X(\ell)$ is nef provided that $\ell \geq 2|\mathbf{a}|$, where $|\mathbf{a}| = a_1 + \cdots + a_k$.*

In particular $\mathcal{O}_{X_k}(\mathbf{a}) = \big(\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}|)\big) \otimes \pi_{0,k}^* \mathcal{O}_X(-2|\mathbf{a}|)$ and both $\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}|)$ and $\pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}|)$ are nef.

Proof. By Lemma 3.4.1, we know that the line bundle

$$\mathcal{O}_{X_k}(2\cdot 3^{k-2}, 2\cdot 3^{k-3}, \dots, 6, 2, 1)\otimes \pi_{0,k}^*\mathcal{O}_X(\ell)$$

is nef as soon as $\ell \geq 2\cdot(1+2+6+\cdots+2\cdot 3^{k-2}) = 2\cdot 3^{k-1}$. Now we take $\mathbf{a}=(a_1,\dots,a_k)\in\mathbb{N}^k$ such that $a_1\geq 3a_2,\dots,a_{k-2}\geq 3a_{k-1}, a_{k-1}\geq 2a_k>0$ and we proceed by induction, the case $k=1$ being obvious. Write

$$\begin{aligned}
&\mathcal{O}_{X_k}(a_1,a_2,\dots,a_k)\otimes\pi_{0,k}^*\mathcal{O}_X(2\cdot(a_1+\cdots+a_k))\\
&\quad=\big(\mathcal{O}_{X_k}(2\cdot 3^{k-2},\dots,6,2,1)\otimes\pi_{0,k}^*\mathcal{O}_X(2\cdot 3^{k-1})\big)^{\otimes a_k}\\
&\qquad\otimes\pi_k^*\bigg(\mathcal{O}_{X_{k-1}}(a_1-2\cdot 3^{k-2}a_k,\dots,a_{k-2}-6a_k,a_{k-1}-2a_k)\\
&\qquad\otimes\pi_{0,k-1}^*\mathcal{O}_X\big(2\cdot(a_1+\cdots+a_k-3^{k-1}a_k)\big)\bigg).
\end{aligned}$$

Therefore, we have to prove that

$$\begin{gathered}
\mathcal{O}_{X_{k-1}}(a_1-2\cdot 3^{k-2}a_k,\dots,a_{k-2}-6a_k,a_{k-1}-2a_k)\\
\otimes\pi_{0,k-1}^*\mathcal{O}_X\big(2\cdot(a_1+\cdots+a_k-3^{k-1}a_k)\big)
\end{gathered}$$

is nef. Our chain of inequalities gives, for $1\leq j\leq k-2$, $a_j\geq 3^{k-j-1}a_k$ and $a_{k-1}\geq 2a_k$. Thus, condition (3.5) is satisfied by the weights of

$$\mathcal{O}_{X_{k-1}}(a_1-2\cdot 3^{k-2}a_k,\dots,a_{k-2}-6a_k,a_{k-1}-2a_k)$$

and $2\cdot(a_1+\cdots+a_k-3^{k-1}a_k)$ is exactly twice the sum of these weights. □

Remark 3.4.1. At this point it should be clear that to prove Theorem 3.1.1 is sufficient to show the existence of an n-tuple $(a_1,\dots,a_n)$ satisfying condition (3.5) and such that

$$\begin{aligned}
&\big(\mathcal{O}_{X_n}(\mathbf{a})\otimes\pi_{0,n}^*\mathcal{O}_X(2|\mathbf{a}|)\big)^{n^2}\\
&\quad-n^2\big(\mathcal{O}_{X_n}(\mathbf{a})\otimes\pi_{0,n}^*\mathcal{O}_X(2|\mathbf{a}|)\big)^{n^2-1}\cdot\pi_{0,n}^*\mathcal{O}_X(2|\mathbf{a}|)>0
\end{aligned}\tag{3.6}$$

for $d=\deg X$ large enough, where $n^2=n+n(n-1)=\dim X_n$.

In fact, this would show the bigness of $\mathcal{O}_{X_n}(\mathbf{a})\hookrightarrow\mathcal{O}_{X_n}(|\mathbf{a}|)$ and so the bigness of $\mathcal{O}_{X_n}(1)$.

3.4.2 Evaluation in terms of the degree

For $X \subset \mathbb{P}^{n+1}$ a smooth projective hypersurface of degree $\deg X = d$, we have a short exact sequence

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^{n+1}}|_X \longrightarrow \mathcal{O}_X(d) \longrightarrow 0;$$

so we get the following relation for the total Chern class of X:

$$(1+h)^{n+2} = (1+d\,h)c_\bullet(X),$$

where $h = c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))$ and $(1+h)^{n+2}$ is the total Chern class of $\mathbb{P}^{n+1}$. Thus, an easy computation shows that

$$c_j(X) = c_j(T_X) = (-1)^j h^j \sum_{k=0}^{j} (-1)^k \binom{n+2}{k} d^{j-k},$$

where $h \in H^2(X,\mathbb{Z})$ is the hyperplane class. In particular

$$c_j(X) = h^j\big((-1)^j d^j + o(d^j)\big), \quad j = 1,\dots,n,$$

and $o(d^j)$ is a polynomial in d of degree at most $j-1$.

Proposition 3.4.3. *The quantities*

$$\begin{aligned}&\big(\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|)\big)^{n+k(n-1)}\\ &-[n+k(n-1)]\big(\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|)\big)^{n+k(n-1)-1} \cdot \pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|)\end{aligned}$$

and

$$\mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)}$$

are both polynomials in the variable d with coefficients in $\mathbb{Z}[a_1,\dots,a_k]$ of degree at most $n+1$ and the coefficients of d^{n+1} of the two expressions are equal.

Moreover this coefficient is a homogeneous polynomial in $a_1,\dots,a_k$ of degree $n+k(n-1)$ or identically zero.

Proof. Set $\mathcal{F}_k(\mathbf{a}) = \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|)$ and $\mathcal{G}_k(\mathbf{a}) = \pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|)$. Then we have

$$\begin{aligned}&\mathcal{F}_k(\mathbf{a})^{n+k(n-1)} + [n+k(n-1)]\mathcal{F}_k(\mathbf{a})^{n+k(n-1)-1} \cdot \mathcal{G}_k(\mathbf{a})\\ &\qquad = \mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)} + \text{terms which have } \mathcal{G}_k(\mathbf{a}) \text{ as a factor.}\end{aligned}$$

Now we use relations (3.2) and (3.3) to observe that

$$\mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)} = \sum_{j_1+2j_2+\cdots+nj_n=n} P^{[k]}_{j_1\cdots j_n}(\mathbf{a})\, c_1(X)^{j_1}\cdots c_n(X)^{j_n},$$

where the $P^{[k]}_{j_1\cdots j_n}(\mathbf{a})$'s are homogeneous polynomial of degree $n+k(n-1)$ in the variables $a_1,\ldots,a_k$ (or possibly identically zero). Thus, substituting the $c_j(X)$'s with their expression in terms of the degree, we get

$$\mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)} = (-1)^n \left(\sum_{j_1+2j_2+\cdots+nj_n=n} P^{[k]}_{j_1\cdots j_n}(\mathbf{a})\right) d^{n+1} + o(d^{n+1}),$$

since $h^n=d$. On the other hand, utilizing relations (3.2) and (3.3) on terms which have $\mathcal{G}_k(\mathbf{a})$ as a factor, gives something of the form

$$\sum_{\substack{j_1+2j_2+\cdots+nj_n+i=n\\ i>0}} Q^{[k]}_{j_1\cdots j_n i}(\mathbf{a})\, h^i\cdot c_1(X)^{j_1}\cdots c_n(X)^{j_n},$$

since $c_1(\mathcal{G}_k(\mathbf{a}))=|\mathbf{a}|h$ and $\mathcal{G}_k(\mathbf{a})$ is always a factor. Substituting the $c_j(X)$'s with their expression in terms of the degree, we get here

$$h^i\cdot c_1(X)^{j_1}\cdots c_n(X)^{j_n} = (-1)^{j_1+\cdots+j_n}\underbrace{h^n}_{=d}\cdot d^{j_1+\cdots+j_n} = o(d^{n+1}).$$

□

At this point, we need an elementary lemma to deal with "generic" weights.

Lemma 3.4.4. *Let $\mathfrak{C}\subset\mathbb{R}^k$ be a cone with nonempty interior. Let $\mathbb{Z}^k\subset\mathbb{R}^k$ be the canonical lattice in $\mathbb{R}^k$. Then $\mathbb{Z}^k\cap\mathfrak{C}$ is Zariski dense in $\mathbb{R}^k$.*

Proof. Since $\mathfrak{C}$ is a cone with nonempty interior, it contains cubes of arbitrary large edges, so $\mathbb{Z}^k\cap\mathfrak{C}$ contains a product of integral intervals $\prod[\alpha_i,\beta_i]$ with $\beta_i-\alpha_i>N$. By using induction on dimension, this implies that a polynomial P of degree at most N vanishing on $\mathbb{Z}^k\cap\mathfrak{C}$ must be identically zero. As N can be taken arbitrary large, we conclude that $\mathbb{Z}^k\cap\mathfrak{C}$ is Zariski dense. □

Corollary 3.4.5. *If the top self-intersection $\mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)}$ has degree exactly equal to $n+1$ in d for some choice of $\mathbf{a}$, then $\mathcal{O}_{X_k}(m)\otimes\pi^*_{0,k}A^{-1}$ has a global section for all line bundle $A\to X$ and for all d,m sufficiently large.*

Proof. The real k-tuples which satisfy condition (3.5), form a cone with non-empty interior in $\mathbb{R}^k$. Thus, by Lemma 3.4.4, there exists an integral $\mathbf{a}'$ satisfying condition (3.5) and such that $\mathcal{O}_{X_k}(\mathbf{a}')^{n+k(n-1)}$ has degree exactly $n+1$ in d. For reasons similar to those in the proof of Proposition 3.4.3, the coefficient of degree $n+1$ in d of $\mathcal{O}_{X_k}(\mathbf{a}')^{n+k(n-1)}$ and $\big(\mathcal{O}_{X_k}(\mathbf{a}')\otimes\pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}'|)\big)^{n+k(n-1)}$ are the same; the second one being nef, this coefficient must be positive.

Now, by Proposition 3.4.3, this coefficient is the same as the coefficient of degree $n+1$ in d of

$$\begin{aligned}&\big(\mathcal{O}_{X_k}(\mathbf{a}')\otimes\pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}'|)\big)^{n+k(n-1)}\\ &-[n+k(n-1)]\big(\mathcal{O}_{X_k}(\mathbf{a}')\otimes\pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}'|)\big)^{n+k(n-1)-1}\cdot\pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}'|).\end{aligned}$$

But then this last quantity is positive for d large enough, and the Corollary follows by an application of algebraic holomorphic Morse inequalities. □

Corollary 3.4.6. *For $k<n$, the coefficient of d^{n+1} in the expression of*

$$\mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)}$$

is identically zero.

Proof. Otherwise, we would have global sections of $\mathcal{O}_{X_k}(m)$ for m large and $k<n$, which is impossible by Theorem 2.1.1. □

3.4.3 Bigness of $\mathcal{O}_{X_n}(1)$

Thanks to the results of the previous subsection, to show the existence of a global section of $\mathcal{O}_{X_n}(m)\otimes\pi_{0,n}^*A^{-1}$ for m and d large, we just need to show that $\mathcal{O}_{X_n}(\mathbf{a})^{n^2}$ has degree exactly $n+1$ in d for some n-tuple $(a_1,\dots,a_n)$.

The multinomial theorem gives

$$\begin{aligned}&(a_1\pi_{1,k}^*u_1+\cdots+a_ku_k)^{n+k(n-1)}\\ &\qquad=\sum_{j_1+\cdots+j_k=n+k(n-1)}\frac{(n+k(n-1))!}{j_1!\cdots j_k!}\,a_1^{j_1}\cdots a_k^{j_k}\,\pi_{1,k}^*u_1^{j_1}\cdots u_k^{j_k}.\end{aligned}$$

We need two technical lemmas.

Lemma 3.4.7. *The coefficient of degree $n+1$ in d of the two following intersections is zero:*

- $\pi_{1,k}^*u_1^{j_1}\cdot\pi_{2,k}^*u_2^{j_2}\cdots u_k^{j_k}$ *for all* $1\le k\le n-1$ *and* $j_1+\cdots+j_k=n+k(n-1)$

- $\pi_{1,n-i-1}^* u_1^{j_1} \cdot \pi_{2,n-i-1}^* u_2^{j_2} \cdots u_{n-i-1}^{j_{n-i-1}} \cdot \pi_{0,n-i-1}^* c_1(X)^i$ *for all* $1 \leq i \leq n-2$ *and* $j_1 + \cdots + j_{n-i-1} = (n-i-1)n+1$.

Proof. The first statement is straightforward: if it fails to be true, we would find an $\mathbf{a}$ which satisfies the hypothesis of Corollary 3.4.5 for $k < n$, contradicting Corollary 3.4.6.

For the second statement we proceed by induction on i. Let us start with $i = 1$. By the first part of the present lemma, we have that

$$\pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \cdots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \cdot u_{n-1}^n = o(d^{n+1}).$$

On the other hand, relation (3.3) gives

$$\begin{aligned} &\pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \cdots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \cdot u_{n-1}^n \\ &\quad = \pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \cdots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \\ &\qquad \cdot \left(-\pi_{n-1}^* c_1^{[n-2]} \cdot u_{n-1}^{n-1} - \cdots - \pi_{n-1}^* c_{n-1}^{[n-2]} \cdot u_{n-1} - \pi_{n-1}^* c_n^{[n-2]}\right) \\ &\quad = -\pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \cdots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \cdot \pi_{n-1}^* c_1^{[n-2]} \cdot u_{n-1}^{n-1} \end{aligned}$$

and the second equality is true for degree reasons:

$$u_1^{j_1} \cdot u_2^{j_2} \cdots u_{n-2}^{j_{n-2}} \cdot c_l^{[n-2]}, \quad l = 2, \dots, n,$$

"lives" on X_{n-2} and has total degree $n+(n-2)(n-1)-1+l$ which is strictly greater than $n+(n-2)(n-1) = \dim X_{n-2}$, so that $u_1^{j_1} \cdot u_2^{j_2} \cdots u_{n-2}^{j_{n-2}} \cdot c_l^{[n-2]} = 0$. Now, we use relation (3.4) and obtain in this way

$$\begin{aligned} &\pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \cdots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \cdot u_{n-1}^n \\ &\quad = -\pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \cdots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \cdot u_{n-1}^{n-1} \\ &\qquad \cdot \left(\pi_{0,n-1}^* c_1(X) + (n-1) \sum_{s=1}^{n-2} \pi_{s,n-1}^* u_s\right) \\ &\quad = -\pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \cdots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \cdot u_{n-1}^{n-1} \cdot \pi_{0,n-1}^* c_1(X) \\ &\qquad - (n-1) u_{n-1}^{n-1} \cdot \sum_{s=1}^{n-2} \pi_{1,n-1}^* u_1^{j_1} \cdots \pi_{s,n-1}^* u_s^{j_s+1} \cdots \pi_{n-1}^* u_{n-2}^{j_{n-2}}. \end{aligned}$$

An integration along the fibers of $X_{n-1} \to X_{n-2}$ then gives

$$\begin{aligned} &\pi_{1,n-2}^* u_1^{j_1} \cdot \pi_{2,n-2}^* u_2^{j_2} \cdots u_{n-2}^{j_{n-2}} \cdot \pi_{0,n-2}^* c_1(X) \\ &\quad = -(n-1) \cdot \sum_{s=1}^{n-2} \underbrace{\pi_{1,n-2}^* u_1^{j_1} \cdots \pi_{s,n-2}^* u_s^{j_s+1} \cdots u_{n-2}^{j_{n-2}}}_{=o(d^{n+1}) \text{ by the first part of the lemma}} \\ &\qquad + o(d^{n+1}) \end{aligned}$$

and so $\pi^*_{1,n-2}u_1^{j_1} \cdot \pi^*_{2,n-2}u_2^{j_2} \cdots u_{n-2}^{j_{n-2}} \cdot \pi^*_{0,n-2}c_1(X) = o(d^{n+1})$.

To complete the proof, observe that – as before – relations (3.3) and (3.4) together with a completely similar degree argument give

$$\begin{aligned}
&\pi^*_{1,n-i}u_1^{j_1} \cdot \pi^*_{2,n-i}u_2^{j_2} \cdots u_{n-i}^{j_{n-i-1}} \cdot \pi^*_{0,n-i}c_1(X)^i \cdot u_{n-i}^n \\
&\quad = -\pi^*_{1,n-i}u_1^{j_1} \cdot \pi^*_{2,n-i}u_2^{j_2} \cdots \pi^*_{n-i}u_{n-i-1}^{j_{n-i-1}} \cdot u_{n-i}^{n-1} \cdot \pi^*_{0,n-i}c_1(X)^{i+1} \\
&\qquad - (n-1)u_{n-i}^{n-1} \cdot \sum_{s=1}^{n-i-1} \pi^*_{1,n-i}u_1^{j_1} \cdots \pi^*_{s,n-i}u_s^{j_s+1} \cdots \pi^*_{n-i}u_{n-i-1}^{j_{n-i-1}}.
\end{aligned}$$

But
$$\pi^*_{1,n-i}u_1^{j_1} \cdot \pi^*_{2,n-i}u_2^{j_2} \cdots u_{n-i}^{j_{n-i-1}} \cdot \pi^*_{0,n-i}c_1(X)^i \cdot u_{n-i}^n = o(d^{n+1})$$
by induction, and
$$\pi^*_{1,n-i}u_1^{j_1} \cdots \pi^*_{s,n-i}u_s^{j_s+1} \cdots \pi^*_{n-i}u_{n-i-1}^{j_{n-i-1}} = o(d^{n+1}),$$
$1 \le s \le n-i-1$, thanks to the first part of the lemma. □

Lemma 3.4.8. *The coefficient of degree $n+1$ in d of $\pi^*_{1,n}u_1^n \cdot \pi^*_{2,n}u_2^n \cdots u_n^n$ is the same of the one of $(-1)^n c_1(X)^n$, that is 1.*

Proof. An explicit computation yields:

$$\begin{aligned}
\pi^*_{1,n}u_1^n \cdot \pi^*_{2,n}u_2^n \cdots u_n^n &\overset{(i)}{=} \pi^*_{1,n}u_1^n \cdot \pi^*_{2,n}u_2^n \cdots \pi^*_n u_{n-1}^n \big(-\pi^*_n c_1^{[n-1]} \cdot u_n^{n-1} \\
&\qquad - \cdots - \pi^*_n c_{n-1}^{[n-1]} \cdot u_n - \pi^*_n c_n^{[n-1]}\big) \\
&\overset{(ii)}{=} -\pi^*_{1,n}u_1^n \cdot \pi^*_{2,n}u_2^n \cdots \pi^*_n u_{n-1}^n \cdot u_n^{n-1} \cdot \pi^*_n c_1^{[n-1]} \\
&\overset{(iii)}{=} -\pi^*_{1,n}u_1^n \cdot \pi^*_{2,n}u_2^n \cdots \pi^*_n u_{n-1}^n \cdot u_n^{n-1} \\
&\qquad \cdot \pi^*_n \left(\pi^*_{0,n-1}c_1(X) + (n-1)\sum_{s=1}^{n-1} \pi^*_{s,n-1}u_s\right) \\
&\overset{(iv)}{=} -\pi^*_{1,n}u_1^n \cdot \pi^*_{2,n}u_2^n \cdots \pi^*_n u_{n-1}^n \cdot u_n^{n-1} \cdot \pi^*_{0,n}c_1(X) \\
&\qquad + o(d^{n+1}) \\
&= \cdots \\
&\overset{(v)}{=} (-1)^n \pi^*_{0,k}c_1(X)^n \cdot \pi^*_{1,k}u_1^{n-1} \cdots u_n^{n-1} + o(d^{n+1}) \\
&\overset{(vi)}{=} (-1)^n c_1(X)^n + o(d^{n+1}).
\end{aligned}$$

Let us say a few words about the previous equalities. Equality (i) is just relation (3.3). Equality (ii) is true for degree reasons: $u_1^n \cdot u_2^n \cdots u_{n-1}^n \cdot c_l^{[n-1]}$,

$l = 2, \dots, n$, "lives" on X_{n-1} and has total degree $n(n-1)+l$ which is strictly greater than $n+(n-1)(n-1) = \dim X_{n-1}$, so that $u_1^n \cdot u_2^n \cdots u_{n-1}^n \cdot c_l^{[n-1]} = 0$. Equality (iii) is just relation (3.4). Equality (iv) follows from the first part of Lemma 3.4.7: $u_1^n \cdots u_s^{n+1} \cdots u_{n-1}^n = o(d^{n+1})$. Equality (v) is obtained by applying repeatedly the second part of Lemma 3.4.7. Finally, equality (vi) is simply integration along the fibers. The lemma is proved. $\square$

Now, look at the coefficient of degree $n+1$ in d of the expression

$$\mathcal{O}_{X_n}(\mathbf{a})^{n^2} = \big(a_1 \pi_{1,n}^* u_1 + \cdots + a_n u_n\big)^{n^2},$$

where we consider the a_j's as variables: we claim that it is a non identically zero homogeneous polynomial of degree n^2. To see this, we just observe that, thanks to Lemma 3.4.8, the coefficient of the monomial $a_1^n \cdots a_n^n$ is $(n^2)!/(n!)^n$.

Hence there exists an $\mathbf{a}$ which satisfies the hypothesis of Corollary 3.4.5 for $k = n$, and Theorem 3.1.1 is proved (we postpone the proof of the effective part of Theorem 3.1.1 and of Theorem 3.1.2 to §3.6).

3.5 Proof of Theorem 3.1.2

We begin with the following not completely standard computations of logarithmic Chern classes.

3.5.1 Chern classes computations

Here, we compute Chern classes for the logarithmic (co)tangent bundle of the pair $(\mathbb{P}^n, D)$, when D is a smooth projective hypersurface of degree $\deg D = d$. In this case (1.4) becomes

$$\begin{tikzcd}
 & & & 0 & \\
0 \arrow[r] & T^*_{\mathbb{P}^n} \arrow[r] & T^*_{\mathbb{P}^n}\langle D\rangle \arrow[r] & \mathcal{O}_D \arrow[r] \arrow[u] & 0, \\
 & & & \mathcal{O}_{\mathbb{P}^n} \arrow[u] & \\
 & & & \mathcal{O}_{\mathbb{P}^n}(-D) \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

where the vertical arrows are the usual locally free resolution of the structure sheaf of a divisor in $\mathbb{P}^n$; then

$$c_\bullet(T^*_{\mathbb{P}^n}\langle D\rangle) = c_\bullet(T^*_{\mathbb{P}^n})c_\bullet(\mathcal{O}_D) \quad \text{and} \quad 1 = c_\bullet(\mathcal{O}_{\mathbb{P}^n}(-D))c_\bullet(\mathcal{O}_D),$$

so that if $h \in H^2(\mathbb{P}^n, \mathbb{Z})$ is the hyperplane class, we have

$$1 = (1 - d\,h)c_\bullet(\mathcal{O}_D)$$

and thus $c_\bullet(\mathcal{O}_D) = 1 + d\,h + (d\,h)^2 + \cdots + (d\,h)^n$. Now, recalling that $c_\bullet(T_{\mathbb{P}^n}) = (1+h)^{n+1}$ and that, for a vector bundle E, $c_j(E^*) = (-1)^j c_j(E)$, we get the following:

Proposition 3.5.1. *Let $D \subset \mathbb{P}^n$ be a smooth hypersurface of degree $\deg D = d$. Then the Chern classes of the logarithmic tangent bundle $T_{\mathbb{P}^n}\langle D\rangle$ are given by*

$$c_j(T_{\mathbb{P}^n}\langle D\rangle) = (-1)^j h^j \sum_{k=0}^{j} (-1)^k \binom{n+1}{k} d^{j-k}, \tag{3.7}$$

for $j = 1, \dots, n$.

In particular, $c_j(T_{\mathbb{P}^n}\langle D\rangle) = (-1)^j h^j (d^j + o(d^j))$, $j = 1, \dots, n$.

3.5.2 Strategy of the proof and logarithmic case

Recall the content of Remark 1.5.1: the construction of logarithmic jet bundles is, from the "relative" point of view, exactly the same as in the compact case.

This means that the short exact sequences which determine the relations on Chern classes and thus the relative structure of the cohomology algebra are, in the logarithmic case, the same as in the compact case.

We summarize here the main points of the proof in the compact case:

- For $\dim X = n$, go up to the n-th projectivized jet bundle, and find a (class of) relatively nef line bundle $\mathcal{O}_{X_n}(\mathbf{a}) \to X_n$, with a nontrivial morphism into $\mathcal{O}_{X_n}(m)$ for some large m.
- Write $\mathcal{O}_{X_n}(\mathbf{a})$ as the difference of two globally nef line bundle, namely

$$\big(\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi^*_{0,k}\mathcal{O}_X(2|\mathbf{a}|)\big) \otimes \pi^*_{0,k}\mathcal{O}_X(-2|\mathbf{a}|).$$

- Compute the "Morse" intersection $F^n - nF^{n-1} \cdot G$ for $\mathcal{O}_{X_n}(\mathbf{a})$ and show that, once expressed in term of the degree of X, the leading term is the same of $\mathcal{O}_{X_n}(\mathbf{a})^{n^2}$.

- Use the vanishing theorem 2.1.1 to conclude that the term of maximal possible degree in $\mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)}$ vanishes for $k < n$.

- Find a particular non-vanishing monomial in the variables $\mathbf{a}$, in the expression on maximal possible degree of $\mathcal{O}_{X_n}(\mathbf{a})^{n^2}$.

From this discussion, it follows that to prove Theorem 3.1.2, one has just to replace the vanishing theorem 2.1.1 with Theorem 2.1.3 and to make the following remark.

Remark 3.5.1. The starting point to write $\mathcal{O}_{X_n}(\mathbf{a})$ as the difference of two globally nef line bundles is, for X a smooth projective hypersurface, that $T^*_X \otimes \mathcal{O}(2)$ is nef as a quotient of $T^*_{\mathbb{P}^{n+1}} \otimes \mathcal{O}(2)$ (cf. Lemma 3.4.1). Thanks to the short exact sequence (1.4), one can proceed in a similar way in the logarithmic case:

$$0 \longrightarrow T^*_{\mathbb{P}^n} \otimes \mathcal{O}(2) \longrightarrow T^*_{\mathbb{P}^n}\langle D\rangle \otimes \mathcal{O}(2) \longrightarrow \mathcal{O}_D(2) \longrightarrow 0,$$

and $T^*_{\mathbb{P}^n}\langle D\rangle \otimes \mathcal{O}(2)$ in nef as an extension of a nef vector bundle by a nef line bundle.

Theorem 3.1.2 is then proved.

3.6 Effective results

The get the effective results announced in the statements of Theorem 3.1.1 and of Theorem 3.1.2, we just compute the algebraic holomorphic Morse inequalities, for $\mathbf{a} = (2 \cdot 3^{n-2}, \dots, 6, 2, 1) \in \mathbb{N}^n$. Hence we get an explicit polynomial in the variable d, which has positive leading coefficient, and we compute its largest positive root.

All this is done by implementing a quite simple code on GP/PARI CALCULATOR Version 2.3.2. The computation complexity blows-up rapidly and, starting from dimension 6, our computers were not able to achieve any result in a finite time.

Remark 3.6.1. Although very natural, we don't know if the weight $\mathbf{a} = (2 \cdot 3^{n-2}, \dots, 6, 2, 1) \in \mathbb{N}^n$ we utilize is the best possible.

3.6.1 GP/PARI CALCULATOR computations

Here we reproduce the code we utilised to perform all computations with GP/PARI CALCULATOR Version 2.3.2.

Compact case

```
/*scratch variable*/
X

/*main formal variables*/
c=[c1,c2,c3,c4,c5,c6,c7,c8,c9] /*Chern classes of V on X*/
u=[u1,u2,u3,u4,u5,u6,u7,u8,u9] /*Chern classes of OXk(1)*/
v=[v1,v2,v3,v4,v5,v6,v7,v8,v9] /*Chern classes of Vk on Xk*/
w=[w1,w2,w3,w4,w5,w6,w7,w8,w9] /*formal variables*/
e=[0,0,0,0,0,0,0,0,0] /*empty array for hypersurfaces Chern classes*/
q=[0,0,0,0,0,0,0,0,0] /*empty array for Chern equations*/

/*main*/
Calcul(dim,order)=
{
local(j,n,N);
n=dim;
r=dim;
k=order;
N=n+k*(r-1);
H(n+1);
Chern();
B=2*h*3^(k-1);
A=B+u[k];
for(j=1,k-1,A=A+2*3^(k-j-1)*u[j]);
R=Reduc((A-N*B)*A^(N-1));
C=Integ(R);
print("Calculation for order ", k, " jets on a ", n, "-fold");
print("Line bundle A= ", A);
print("Line bundle B= ", B);
print("Chern class of A^", N, "-", N, "*A^", N-1, "*B :");
print(C);
E=Eval(C);
print("Evaluation for degree d hypersurface in P^", n+1, " :");
print(E)
}

/*compute Chern relations*/
Chern()=
```

```
{
local(j,s,t);
q[1]=X^r; for(j=1,r,q[1]=q[1]+c[j]*X^(r-j));
for(s=1,r,v[s]=c[s]);
for(s=r+1,9,v[s]=0);
for(t=1,k-1,
for(s=1,r,w[s]=v[s]+(binomial(r,s)-binomial(r,s-1))*u[t]^s;
for(j=1,s-1,w[s]=w[s]+
(binomial(r-j,s-j)-binomial(r-j,s-j-1))*v[j]*u[t]^(s-j)));
for(s=1,r,v[s]=w[s]);
q[t+1]=X^r; for(j=1,r,q[t+1]=q[t+1]+v[j]*X^(r-j)))
}

/*reduction to Chern classes of X*/
Reduc(p)=
{
local(j,a);
a=p;
for(j=0,k-1,
a=subst(a,u[k-j],X);
a=subst(lift(Mod(a,q[k-j])),X,u[k-j]));
a
}

/*integration along fibers */
Integ(p)=
{
local(j,a);
a=p;
for(j=0,k-1,
a=polcoeff(a,r-1,u[k-j]));
a
}

/*compute Chern classes of hypersurface of degree d in P^n*/
H(n)=
{
local(j,s);
for(s=1,n-1,
e[s]=binomial(n+1,s);
for(j=1,s,e[s]=e[s]+(-d)^j*binomial(n+1,s-j)))
```

```
}

/*evaluation in terms of the degree*/
Eval(p)=
{
local(a,s);
a=p;
for(s=1,r,a=subst(a,c[s],e[s]));
subst(a,h,1)*d
}
```

Logarithmic case

```
/*scratch variable*/
X

/*main formal variables*/
c=[c1,c2,c3,c4,c5,c6,c7,c8,c9] /*Chern classes of V<D> on P^n*/
u=[u1,u2,u3,u4,u5,u6,u7,u8,u9] /*Chern classes of OXk(1)*/
v=[v1,v2,v3,v4,v5,v6,v7,v8,v9] /*Chern classes of Vk on Xk*/
w=[w1,w2,w3,w4,w5,w6,w7,w8,w9] /*formal variables*/
e=[0,0,0,0,0,0,0,0,0] /*empty array for logarithmic Chern classes*/
q=[0,0,0,0,0,0,0,0,0] /*empty array for Chern equations*/

/*main*/
Calcul(dim,order)=
{
local(j,n,N);
n=dim;
r=dim;
k=order;
N=n+k*(r-1);
H(n);
Chern();
B=2*h*3^(k-1);
A=B+u[k];
for(j=1,k-1,A=A+2*3^(k-j-1)*u[j]);
R=Reduc((A-N*B)*A^(N-1));
C=Integ(R);
print("Calculation for order ", k, " jets on logarithmic projective
```

```
", n, "-space");
print("Line bundle A= ", A);
print("Line bundle B= ", B);
print("Chern class of A^", N, "-", N, "*A^", N-1, "*B :");
print(C);
E=Eval(C);
print("Evaluation for degree d logarithmic projective ", n,"-space:");
print(E)
}

/*compute Chern relations*/
Chern()=
{
local(j,s,t);
q[1]=X^r; for(j=1,r,q[1]=q[1]+c[j]*X^(r-j));
for(s=1,r,v[s]=c[s]);
for(s=r+1,9,v[s]=0);
for(t=1,k-1,
for(s=1,r,w[s]=v[s]+(binomial(r,s)-binomial(r,s-1))*u[t]^s;
for(j=1,s-1,w[s]=w[s]+
(binomial(r-j,s-j)-binomial(r-j,s-j-1))*v[j]*u[t]^(s-j)));
for(s=1,r,v[s]=w[s]);
q[t+1]=X^r; for(j=1,r,q[t+1]=q[t+1]+v[j]*X^(r-j)))
}

/*reduction to Chern classes of (P^n,D)*/
Reduc(p)=
{
local(j,a);
a=p;
for(j=0,k-1,
a=subst(a,u[k-j],X);
a=subst(lift(Mod(a,q[k-j])),X,u[k-j]));
a
}

/*integration along fibers*/
Integ(p)=
{
local(j,a);
a=p;
```

```
for(j=0,k-1,
a=polcoeff(a,r-1,u[k-j]));
a
}

/*compute Chern classes of degree d logarithmic projective n-space*/
H(n)=
{
local(j,s);
for(s=1,n,
e[s]=d^s;
for(j=1,s,e[s]=e[s]+(-1)^j*(d)^(s-j)*binomial(n+1,j));
e[s]=(-1)^s*e[s])
}

/*evaluation in terms of the degree*/
Eval(p)=
{
local(a,s);
a=p;
for(s=1,r,a=subst(a,c[s],e[s]));
subst(a,h,1)*d
}
```

Chapter 4

Curvature of jet bundles and applications

Abstract. This last chapter has a more differential-geometric flavor. For each $k \geq 1$, we endow, by an induction process, the tautological line bundle $\mathcal{O}_{X_k}(1)$ of an arbitrary complex directed manifold (X, V) with a natural smooth hermitian metric. Then, we compute recursively the Chern curvature form for this metric, and we show that it depends (asymptotically – in a sense to be specified later) only on the curvature of V and on the structure of the fibration $X_k \to X$. When X is a surface and $V = T_X$, we give explicit formulae to write down the above curvature as a product of matrices. Finally, we obtain a new proof of the existence of global invariant jet differentials vanishing on an ample divisor, for X a minimal surface of general type whose Chern classes satisfy certain inequalities, without using the vanishing theorem [2] of Bogomolov (cf. §3.2).

4.1 Introduction

In [17], Green and Griffiths showed, among other things, that if X is an algebraic surface of general type, then there exist $m \gg k \gg 1$, such that $H^0(X, \mathcal{J}_{k,m}T^*X) \neq 0$. Their proof is based on an asymptotic computation of the Euler characteristic $\chi(\mathcal{J}_{k,m}T^*X)$ of the bundle of jet differentials (which has been possible thanks to the full knowledge of the composition series of this bundle) together with a vanishing theorem of Bogomolov [2].

More precisely, let X be a n-dimensional smooth projective variety, and $\mathcal{J}_{k,m}T^*X \to X$ the bundle of jet differentials of order k and weighted degree

m. Then

$$\chi(\mathcal{J}_{k,m}T_X^*) = \frac{m^{(k+1)n-1}}{(k!)^n((k+1)n-1)!} \times \left(\frac{(-1)^n}{n!}c_1(X)^n(\log k)^n + O\big((\log k)^{n-1}\big)\right) + O\big(m^{(k+1)n-2}\big).$$

In particular, if X is a surface of general type, then the Bogomolov vanishing theorem applies and, having cancelled the h^2 term, they get a positive lower bound for $h^0(X, \mathcal{J}_{k,m}T_X^*)$ when $m \gg k \gg 1$.

Nowadays, there are no general results about the existence of global *invariant* jet differentials on a surface of general type neither, of course, for varieties of general type in arbitrary dimension.

Nevertheless, thanks to a beautiful and relatively simple argument of Demailly [9], their asymptotic existence on every variety of general type should potentially lead to solve the following celebrated conjecture.

Conjecture (Green and Griffiths [17], Lang)**.** *Let X be an algebraic variety of general type. Then there exist a proper algebraic sub-variety $Y \subsetneq X$ such that every non-constant holomorphic entire curve $f\colon \mathbb{C} \to X$, has image $f(\mathbb{C})$ contained in Y.*

Remark 4.1.1. A positive answer to this conjecture in dimension 2 has been given by McQuillan in [19], when the second Segre number $c_1(X)^2 - c_2(X)$ of X is positive (this hypothesis ensures the existence of an algebraic (multi)foliation on X, whose parabolic leaves are shown to be algebraically degenerate: this is the very deep and difficult part of the proof).

4.1.1 Main ideas, statement of the results and holomorphic Morse inequalities

Let (X, V) be a complex directed manifold with $\dim X = n$ and $2 \le \operatorname{rank} V = r \le n$. Let ω be a hermitian metric on V. Such a metric naturally induces a smooth hermitian metric on the tautological line bundle $\mathcal{O}_{\widetilde{X}}(-1)$ on the projectivized bundle of line of V.

Now, the Chern curvature of its dual $\mathcal{O}_{\widetilde{X}}(1)$, is a $(1,1)$-form on $\widetilde{X}$ whose restriction to the fiber over a point $x \in X$ coincides with the Fubini-Study metric of $P(V_x)$ with respect to $\omega|_{V_x}$. Thus, it is positive in the fibers direction. Next, consider the pullback $\pi^*\omega$ on $\widetilde{X}$: this is a $(1,1)$-form which is zero in the fibers direction and, of course, positive in the base direction.

If X is compact so is $\widetilde{X}$ and hence, for all $\varepsilon > 0$ small enough, the restriction to $\widetilde{V}$ of the $(1,1)$-form given by

$$\pi^*\omega + \varepsilon^2\,\Theta(\mathcal{O}_{\widetilde{X}}(1))$$

gives rise to a hermitian metric on $\widetilde{V}$. Moreover, this metric depends on two derivatives of the metric ω.

Of course, we can repeat this process for the compact directed manifold $(\widetilde{X},\widetilde{V})$, and by induction, for each $k \geq 1$ for the tower of projectivized bundles (X_k, V_k). *A priori*, the hermitian metric we obtain in this fashion on $\mathcal{O}_{X_k}(-1)$, depends on $2k$ derivatives of the starting metric ω and on the choice of $\varepsilon^{(k)} = (\varepsilon_1, \dots, \varepsilon_{k-1})$.

However, from a philosophical point of view, we would like to avoid the dependence on the last $2k-2$ derivatives of ω, since the relevant geometrical data for X lies in the first two derivatives of ω, namely on its Chern curvature. As $\varepsilon^{(k)}$ has to be small enough, it is quite natural to look for an asymptotic expression of the Chern curvature of the metric on $\mathcal{O}_{X_k}(-1)$ we have constructed, when $\varepsilon^{(k)}$ tends to zero: this is the content of the first theorem of this chapter.

Theorem 4.1.1. *The vector bundle V_k can be endowed inductively with a smooth hermitian metric*

$$\omega^{(k)} = \big(\pi_k^*\omega^{(k-1)} + \varepsilon_k^2\,\Theta(\mathcal{O}_{X_k}(1))\big)\big|_{V_k},$$

where the metric on $\mathcal{O}_{X_k}(1)$ is induced by $\omega^{(k-1)}$, depending on $k-1$ positive real numbers $\varepsilon^{(k)} = (\varepsilon_1, \dots, \varepsilon_{k-1})$, such that the asymptotic of its Chern curvature with respect to this metric depends only on the curvature of V and on the (universal) structure of the fibration $X_k \to X$, as $\varepsilon^{(k)} \to 0$.

As a byproduct of the proof of the above theorem, we also obtain induction formulae for an explicit expression of the curvature in terms of the curvature coefficients of V. These formulae, which are quite difficult to handle in higher dimension, are reasonably simple for X a smooth surface: in this case, it turns out that the curvature coefficients of $\mathcal{O}_{X_k}(-1)$ are given by a sequence of products of 2×2 real matrices.

A general remark in analytic geometry is that the existence of global sections of a hermitian line bundle is strictly correlated with the positivity properties of its Chern curvature form. One of the countless correlations, is given by the theory of Demailly's holomorphic Morse inequalities [5]. We summarize his main result here below.

Holomorphic Morse inequalities

Let X be a compact Kähler manifold of dimension n, E a holomorphic vector bundle of rank r and L a line bundle over X. If L is equipped with a smooth metric of curvature form $\Theta(L)$, we define the q-index set of L to be the open subset

$$X(q,L)=\left\{x\in X \mid i\,\Theta(L) \text{ has } \begin{matrix} q \text{ negative eigenvalues} \\ n-q \text{ positive eigenvalues}\end{matrix}\right\},$$

for $q=0,\dots,n$. Hence X admits a partition $X=\Delta\cup\bigcup_{q=0}^{n}X(q,L)$, where $\Delta=\{x\in X \mid \det(i\,\Theta(L))=0\}$ is the degeneracy set. We also introduce

$$X(\leq q,L)\stackrel{\text{def}}{=}\bigcup_{j=0}^{q}X(q,L).$$

It was shown by Demailly in [5], that the partial alternating sums of the dimension of the cohomology groups of tensor powers of L with values in E satisy the following asymptotic *strong Morse inequalities* as $k\to+\infty$:

$$\sum_{j=0}^{q}(-1)^{q-j}h^j(X,L^{\otimes k}\otimes E)\leq r\frac{k^n}{n!}\int_{X(\leq q,L)}(-1)^q\left(\frac{i}{2\pi}\,\Theta(L)\right)^n+O(k^{n-1}).$$

In particular, if

$$\int_{X(\leq 1,L)}\left(\frac{i}{2\pi}\,\Theta(L)\right)^n>0,$$

then some high power of L twisted by E has a (many, in fact) nonzero section.

The idea is now to apply holomorphic Morse inequalities to the anti-tautological line bundle $\mathcal{O}_{X_k}(1)$ together with the asymptotic hermitian metric constructed above, to find global sections of invariant k-jet differentials on a surface X: we shall deal with the absolute case $V=T_X$. Our first geometrical hypothesis is to suppose X to be Kähler-Einstein, that is with ample canonical bundle. Nevertheless, standard arguments coming from the theory of Monge-Ampère equations, will show that we just need to assume X to be minimal and of general type, that is K_X big and numerically effective. Finally, once sections are found, we can drop the hypothesis of nefness, since the dimension of the space of global section of jet differentials is a birational invariant (see, for instance, [17] and [7]).

For each $k\geq 1$, in $\mathbb{R}^k$ define the closed convex cone $\mathfrak{N}=\{\mathbf{a}=(a_1,\dots,a_k)\in\mathbb{R}^k \mid a_j\geq 2\sum_{\ell=j+1}^{k}a_\ell$ for all $j=1,\dots,k-1$ and $a_k\geq 0\}$. For X a smooth compact surface, set

$$\mathcal{O}_{X_k}(\mathbf{a})^{k+2} = F_k(\mathbf{a})\, c_1(X)^2 - G_k(\mathbf{a})\, c_2(X)$$

and

$$m_k = \sup_{\mathbf{a}\in\mathfrak{N}\setminus\Sigma_k} \frac{F_k(\mathbf{a})}{G_k(\mathbf{a})},$$

where Σ_k is the zero locus of G_k. Finally, call m_∞ the supremum of the sequence $\{m_k\}$.

Theorem 4.1.2. *Notations as above, the two following facts can occur: either*

- *there exist a $k_0 \geq 1$ such that for every surface X of general type, $\mathcal{O}_{X_{k_0}}(1)$ is big,*

or

- *the sequence $\{m_k\}$ is positive non-decreasing and for X a surface of general type, there exists a positive integer k such that $\mathcal{O}_{X_k}(1)$ is big as soon as $m_\infty > c_2(\widehat{X})/c_1(\widehat{X})^2$, where $\widehat{X}$ is the minimal model of X.*

As a corollary, we obtain the existence of low order jet differentials, for X a minimal surface of general type whose Chern classes satisfy certain inequalities. This will be done in §4.5.2.

4.2 From (X,V) to $(\widetilde{X},\widetilde{V})$

Let X be a compact complex manifold of complex dimension n and $V \subset T_X$ a holomorphic (non necessarily integrable) subbundle of T_X of rank r. In this section, given a hermitian metric ω on V, we construct a (family of) metric on $\widetilde{V}$ depending on a "small" positive constant ε, and we compute the curvature of $\widetilde{V}$ with respect to this metric, letting ε tend to zero.

So, fix a hermitian metric ω on V, a point $x_0 \in X$ and a unit vector $v_0 \in V_{x_0}$ with respect to ω. Then there exist coordinates $(z_1,\dots,z_n)$ centered at x_0 and a holomorphic normal local frame $e_1,\dots,e_r$ for V such that $e_r(x_0) = v_0$ and

$$\omega(e_\lambda, e_\mu) = \delta_{\lambda\mu} - \sum_{j,k=1}^{n} c_{jk\lambda\mu}\, z_j \overline{z}_k + O(|z|^3).$$

Remark that, as V is a holomorphic subbundle of the holomorphic tangent space of X, then there exists a holomorphic matrix $(g_{i\lambda}(z))$ such that $e_\lambda(z) = \sum_{i=1}^n g_{i\lambda}(z)\frac{\partial}{\partial z_i}$.

Moreover, the Chern curvature at x_0 of V is expressed by

$$\Theta(V)_{x_0} = \sum_{j,k=1}^{n} \sum_{\lambda,\mu=1}^{r} c_{jk\lambda\mu}\, dz_j \wedge d\overline{z}_k \otimes e_\lambda^* \otimes e_\mu.$$

Now consider the projectivized bundle $\pi\colon P(V) = \widetilde{X} \to X$ of lines in V: its points can be seen as pairs $(x,[v])$ where $x \in X$, $v \in V_x \setminus \{0\}$ and $[v] = \mathbb{C}v$. In a neighborhood of $(x_0,[v_0]) \in \widetilde{X}$ we have local holomorphic coordinates given by $(z,\xi_1,\dots,\xi_{r-1})$ where ξ corresponds to the direction $[\xi_1 e_1(z) + \cdots + \xi_{r-1}e_{r-1}(z) + e_r(z)]$ in V_z.

On $\widetilde{X}$ we have a tautological line bundle $\mathcal{O}_{\widetilde{X}}(-1) \subset \pi^* V$ such that the fiber over $(x,[v])$ is simply $[v]$: then $\mathcal{O}_{\widetilde{X}}(-1) \subset \pi^* V$ inherits a metric from V in such a way that its local non vanishing section $\eta(z,\xi) = \xi_1 e_1(z) + \cdots + \xi_{r-1}e_{r-1}(z) + e_r(z)$ has squared length

$$\begin{aligned}|\eta|^2_\omega = & 1 + |\xi|^2 - \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\, z_j \overline{z}_k \xi_\lambda \overline{\xi}_\mu - \sum_{j,k,\lambda} c_{jk\lambda r}\, z_j \overline{z}_k \xi_\lambda \\ & - \sum_{j,k,\mu} c_{jkr\mu}\, z_j \overline{z}_k \overline{\xi}_\mu - \sum_{j,k} c_{jkrr}\, z_j \overline{z}_k + O(|z|^3).\end{aligned}$$

So we have

$$\begin{aligned}
\overline{\partial}|\eta|^2_\omega = & \sum_\mu \xi_\mu\, d\overline{\xi}_\mu - \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} z_j \overline{z}_k \xi_\lambda\, d\overline{\xi}_\mu \\
& - \sum_{j,k} c_{jkrr} z_j\, d\overline{z}_k + O((|z|+|\xi|)^2|dz| + |z|^2|d\xi|), \\
\partial|\eta|^2_\omega = & \sum_\lambda \overline{\xi}_\lambda\, d\xi_\lambda - \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} z_j \overline{z}_k \overline{\xi}_\mu\, d\xi_\lambda \\
& - \sum_{j,k} c_{jkrr} \overline{z}_k\, dz_j + O((|z|+|\xi|)^2|dz| + |z|^2|d\xi|), \\
\partial\overline{\partial}|\eta|^2_\omega = & \sum_\lambda d\xi_\lambda \wedge d\overline{\xi}_\lambda - \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} z_j \overline{z}_k\, d\xi_\lambda \wedge d\overline{\xi}_\mu - \sum_{j,k=1}^{n} c_{jkrr}\, dz_j \wedge d\overline{z}_k \\
& + O((|z|+|\xi|)|dz|^2 + |z|\,|dz|\,|d\xi| + (|z|+|\xi|)^3|d\xi|^2),
\end{aligned}$$

where all the summations here are taken with $j,k = 1,\dots,n$ and $\lambda,\mu = 1,\dots,r-1$. We remark that inside the O's there are hidden terms which are useless for our further computations. We finally obtain

$$\begin{aligned}\Theta(\mathcal{O}_{\widetilde{X}}(1)) = \partial\overline{\partial}\log|\eta|^2_\omega &= -\frac{1}{|\eta|^4_\omega}\partial|\eta|^2_\omega\wedge\overline{\partial}|\eta|^2_\omega + \frac{1}{|\eta|^2_\omega}\partial\overline{\partial}|\eta|^2_\omega \\ &= \sum_{\lambda,\mu}\Big(-\xi_\mu\overline{\xi}_\lambda - \sum_{j,k}c_{jk\lambda\mu}z_j\overline{z}_k \\ &\quad + \delta_{\lambda\mu}\big(1-|\xi|^2+\sum_{j,k}c_{jkrr}z_j\overline{z}_k)\big)\Big)\,d\xi_\lambda\wedge d\overline{\xi}_\mu \\ &\quad -\sum_{j,k}c_{jkrr}\,dz_j\wedge d\overline{z}_k \\ &\quad + O((|z|+|\xi|)|dz|^2+|z|\,|dz|\,|d\xi|+(|z|+|\xi|)^3|d\xi|^2).\end{aligned}$$

So we get in particular

$$\Theta(\mathcal{O}_{\widetilde{X}}(1))_{(x_0,[v_0])} = \sum_{\lambda=1}^{r-1}d\xi_\lambda\wedge d\overline{\xi}_\lambda - \sum_{j,k=1}^{n}c_{jkrr}\,dz_j\wedge d\overline{z}_k,$$

which shows that

$$\Theta(\mathcal{O}_{\widetilde{X}}(1))_{(x_0,[v_0])} = |\bullet|^2_{\mathrm{FS}} - \theta_{V,x_0}(\bullet\otimes v_0,\bullet\otimes v_0),$$

where FS denotes the Fubini-Study metric along the vertical tangent space $\ker\pi_*$ and θ_{V,x_0} is the natural hermitian form on $T_X\otimes V$ corresponding to $i\,\Theta(V)$, at the point x_0.

Now consider the rank r holomorphic subbundle $\widetilde{V}$ of $T_{\widetilde{X}}$ whose fiber over a point $(x,[v])$ is given by

$$\widetilde{V}_{(x,[v])} = \{\tau\in T_{\widetilde{X}} \mid \pi_*\tau\in\mathbb{C}v\}.$$

To start with, let's consider the holomorphic local frame of $\widetilde{V}$ given by $\frac{\partial}{\partial\xi_1},\dots,\frac{\partial}{\partial\xi_{r-1}},\widetilde{\eta}$, where

$$\widetilde{\eta}(z,\xi) = \sum_{i=1}^{n}\Big(g_{ir}(z)+\sum_{\lambda=1}^{r-1}g_{i\lambda}(z)\xi_\lambda\Big)\frac{\partial}{\partial z_i},$$

so that $\widetilde{\eta}$ formally is equal to η but here, with a slight abuse of notations, the $\frac{\partial}{\partial z_i}$ are regarded as tangent vector fields to $\widetilde{X}$ (so, $\widetilde{\eta}$ actually means a lifting of η from $\mathcal{O}_{\widetilde{X}}(-1)\subset\pi^*V\subset\pi^*T_X$ to $T_{\widetilde{X}}$, which admits π^*T_X as a quotient).

For all sufficiently small $\varepsilon > 0$ we get an hermitian metric on $\widetilde{V}$ by restricting $\widetilde{\omega}_\varepsilon = \pi^*\omega + \varepsilon^2\,\Theta(\mathcal{O}_{\widetilde{X}}(1))$ to $\widetilde{V}$; at the point $(x_0, [v_0]) = (0,0)$ we have

$$\begin{aligned}
\widetilde{\omega}_\varepsilon\left(\frac{\partial}{\partial\xi_\lambda}, \frac{\partial}{\partial\xi_\mu}\right) &= \underbrace{\pi^*\omega\left(\frac{\partial}{\partial\xi_\lambda}, \frac{\partial}{\partial\xi_\mu}\right)}_{=0} + \varepsilon^2\,\Theta(\mathcal{O}_{\widetilde{X}}(1))\left(\frac{\partial}{\partial\xi_\lambda}, \frac{\partial}{\partial\xi_\mu}\right)\\
&= \delta_{\lambda\mu}\varepsilon^2,
\end{aligned}$$

$$\begin{aligned}
\widetilde{\omega}_\varepsilon\left(\frac{\partial}{\partial\xi_\lambda}, \widetilde{\eta}\right) &= \underbrace{\pi^*\omega\left(\frac{\partial}{\partial\xi_\lambda}, \widetilde{\eta}\right)}_{=0} + \varepsilon^2\,\Theta(\mathcal{O}_{\widetilde{X}}(1))\left(\frac{\partial}{\partial\xi_\lambda}, \widetilde{\eta}\right)\\
&= 0, \quad \text{since } \Theta(\mathcal{O}_{\widetilde{X}}(1))_{(x_0,[v_0])}\left(\frac{\partial}{\partial\xi_\lambda}, \frac{\partial}{\partial z_i}\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\omega}_\varepsilon(\widetilde{\eta}, \widetilde{\eta}) &= \pi^*\omega(\widetilde{\eta}, \widetilde{\eta}) + \varepsilon^2\,\Theta(\mathcal{O}_{\widetilde{X}}(1))(\widetilde{\eta}, \widetilde{\eta})\\
&= |\eta(0,0)|^2_\omega + O(\varepsilon^2) = 1 + O(\varepsilon^2).
\end{aligned}$$

We now renormalize this local frame of $\widetilde{V}$ by setting

$$f_1 = \frac{1}{\varepsilon}\frac{\partial}{\partial\xi_1}, \ldots, f_{r-1} = \frac{1}{\varepsilon}\frac{\partial}{\partial\xi_{r-1}}, f_r = C_\varepsilon\,\widetilde{\eta},$$

where

$$C_\varepsilon = \frac{1}{\sqrt{\widetilde{\omega}_\varepsilon(\widetilde{\eta}, \widetilde{\eta})}} = 1 + O(\varepsilon).$$

Then (f_λ) is unitary at $(x_0, [v_0])$ with respect to $\widetilde{\omega}_\varepsilon$ and we have

$$\widetilde{\omega}_\varepsilon(f_\lambda, f_\mu) = \begin{cases} -\xi_\mu\overline{\xi}_\lambda - \sum_{j,k} c_{jk\lambda\mu} z_j\overline{z}_k & \text{if } 1 \le \lambda, \mu \le r-1\\ +\delta_{\lambda\mu}(1 - |\xi|^2 + \sum_{j,k} c_{jkrr} z_j\overline{z}_k) & \\ 0 & \text{if } 1 \le \lambda \le r-1 \text{ and } \mu = r\\ & \text{or } 1 \le \mu \le r-1 \text{ and } \lambda = r\\ |\eta|^2_\omega & \text{if } \lambda = \mu = r, \end{cases}$$

modulo ε and terms of order three in z and ξ.

Next, we compute the curvature

$$\Theta(\widetilde{V})_{(x_0,[v_0])} = \sum_{j,k=1}^{n+r-1} \sum_{\lambda,\mu=1}^{r} \gamma_{jk\lambda\mu}\, dz_j \wedge d\overline{z}_k \otimes f^*_\lambda \otimes f_\mu$$

for $\varepsilon \to 0$, where we have set $z_{n+\lambda} = \xi_\lambda$. Recall that for a hermitian vector bundle $E \to Y$, given a holomorphic trivialization, the curvature operator at a point $0 \in Y$ is given by

$$\Theta(E)_0 = \overline{\partial}(\overline{H}^{-1}\partial\overline{H})(0) = (\overline{\partial}\,\overline{H}^{-1})(0) \wedge (\partial\overline{H})(0) + \overline{H}^{-1}(0)(\overline{\partial}\partial\overline{H})(0),$$

where H is the hermitian matrix of hermitian products between the elements of the local frame. If the local holomorphic frame is unitary in 0, so that $H(0) = \mathrm{Id}$, observing that $0 = \overline{\partial}(\overline{H}^{-1}\overline{H}) = (\overline{\partial}\,\overline{H}^{-1})(0)\,\overline{H}(0) + \overline{H}^{-1}(0)(\overline{\partial}\,\overline{H})(0)$, we obtain

$$\Theta(E)_0 = -\overline{\partial}\,\overline{H}(0) \wedge \partial\overline{H}(0) + \overline{\partial}\partial\overline{H}(0). \tag{4.1}$$

Thus, in our case, it suffices to compute the part with second derivatives in (4.1) to get the following proposition.

Proposition 4.2.1. *Notations as given, the Chern curvature of $\widetilde{V}$ has the following expression:*

$$\begin{aligned}\Theta(\widetilde{V}) = \sum_{\lambda,\mu=1}^{r-1} \Bigg(& d\xi_\mu \wedge d\overline{\xi}_\lambda + \sum_{j,k=1}^{n} c_{jk\lambda\mu}\, dz_j \wedge d\overline{z}_k \\ & + \delta_{\lambda\mu}\left(\sum_{\nu=1}^{r-1} d\xi_\nu \wedge d\overline{\xi}_\nu - \sum_{j,k=1}^{n} c_{jkrr}\, dz_j \wedge d\overline{z}_k\right)\Bigg) \otimes f_\lambda^* \otimes f_\mu \\ & + \left(\sum_{j,k=1}^{n} c_{jkrr}\, dz_j \wedge d\overline{z}_k - \sum_{\nu=1}^{r-1} d\xi_\nu \wedge d\overline{\xi}_\nu\right) \otimes f_r^* \otimes f_r + O(\varepsilon).\end{aligned} \tag{4.2}$$

In particular, we get the following identities modulo ε:

$$\gamma_{jk\lambda\mu} = \begin{cases} c_{jk\lambda\mu} - \delta_{\lambda\mu} c_{jkrr} & \text{if } 1 \le j,k \le n \text{ and } 1 \le \lambda,\mu \le r-1 \\ \delta_{\lambda\mu}\delta_{jk} + \delta_{(j-n)\mu}\delta_{(k-n)\lambda} & \text{if } n+1 \le j,k \le n+r-1 \\ & \text{and } 1 \le \lambda,\mu \le r-1, \end{cases}$$

$$\gamma_{jkrr} = \begin{cases} c_{jkrr} & \text{if } 1 \le j,k \le n \\ -1 & \text{if } n < j = k \le n+r-1, \end{cases}$$

the remaining coefficients being zero.

4.3 A special choice of coordinates and local frames

As usual, we now construct the tower of projective bundles (X_k, V_k) over (X, V). We recall that we simply set $(X, V) = (X_0, V_0)$ and, for all integer

$k > 0$, $(X_k, V_k) = (\widetilde{X}_{k-1}, \widetilde{V}_{k-1})$ together with the projection $\pi_{k-1,k}\colon X_k \to X_{k-1}$ so that the total fibration is given by $\pi_{0,k} = \pi_{0,1}\circ\pi_{1,2}\circ\cdots\circ\pi_{k-1,k}\colon X_k \to X$.

For all k, we also have a tautological line bundle $\mathcal{O}_{X_k}(-1)$ and a metric $\omega^{(k)} = \omega^{(k)}(\varepsilon_1, \ldots, \varepsilon_k)$ on V_k, with the ε_l's positive and small enough, obtained recursively by setting $\omega^{(k)} = \big(\pi^*_{k-1,k}\omega^{(k-1)} + \varepsilon_k^2\, \Theta(\mathcal{O}_{X_k}(1))\big)|_{V_k}$, $\omega^{(0)} = \omega$.

To start with, fix a point $x_0 \in X$, a ω-unitary vector $v_0 \in V$ and a holomorphic local normal frame $(e_\lambda^{(0)})$ for (V, ω) such that $e_r(x_0) = v_0$.

First step

On X_1, we have local holomorphic coordinates centered at $(x_0, [v_0])$ given by $(z, \xi^{(1)})$ where, $(z, \xi) \mapsto [\xi_1^{(1)} e_1^{(0)}(z) + \cdots + \xi_{r-1}^{(1)} e_{r-1}^{(0)}(z) + e_r^{(0)}(z)] \in P(V_z)$. Recall that we have, as before, a "natural" local section η_1 of $\mathcal{O}_{X_1}(-1)$ given by

$$\eta_1(z, \xi^{(1)}) = \xi_1^{(1)} e_1(z) + \cdots + \xi_{r-1}^{(1)} e_{r-1}(z) + e_r(z)$$

and for all $\varepsilon_1 > 0$ small enough, a holomorphic local frame $(f_\lambda^{(1)})$ for V_1 near $(x_0, [v_0])$ which is a $\omega^{(1)}$-unitary basis for $V_{1(x_0,[v_0])}$.

Now, choose a $\omega^{(1)}$-unitary vector $v_1 \in V_{1(x_0,[v_0])}$ and a holomorphic local normal frame $(e_\lambda^{(1)})$ for V_1 such that $e_r^{(1)}(x_0, [v_0]) = v_1$. Then there exist a unitary $r \times r$ matrix $U_1 = (a_{\lambda\mu}^{(1)})$ such that at $(x_0, [v_0])$ we have

$$f_\mu^{(1)} = \sum_{\lambda=1}^{r} a_{\lambda\mu}^{(1)}\, e_\lambda^{(1)}.$$

So, if we call respectively $\gamma_{ij\lambda\mu}^{(1)}$ and $c_{ij\lambda\mu}^{(1)}$ the coefficients of curvature of V_1 at $(x_0, [v_0])$ with respect to the basis $(f_\lambda^{(1)})$ and $(e_\lambda^{(1)})$ we have

$$c_{ij\lambda\mu}^{(1)} = \sum_{\alpha,\beta=1}^{r} \gamma_{ij\alpha\beta}^{(1)}\, \overline{a}_{\lambda\alpha}^{(1)} a_{\mu\beta}^{(1)},$$

with $i, j = 1, \ldots, n + (r-1)$ and $\lambda, \mu = 1, \ldots, r$.

General step

For the general case, suppose for all $\varepsilon_1, \ldots, \varepsilon_{k-1} > 0$ small enough we have built a system of holomorphic local coordinates $(z, \xi^{(1)}, \ldots, \xi^{(k-1)})$ for X_{k-1} and a holomorphic local normal frame $(e_\lambda^{(k-1)})$ for $(V_{k-1}, \omega^{(k-1)})$, $k \geq 2$, such that $e_r^{(k-1)}(x_0,[v_0],\dots,[v_{k-2}]) = v_{k-1}$ where $v_{k-1} \in V_{k-1(x_0,[v_0],\dots,[v_{k-2}])}$ is a $\omega^{(k-1)}$-unitary vector. Our procedure gives us also a holomorphic local frame $(f_\lambda^{(k-1)})$ for V_{k-1} near $(x_0,[v_0],\dots,[v_{k-2}])$ which is a $\omega^{(k-1)}$-unitary basis for $V_{k-1(x_0,[v_0],\dots,[v_{k-2}])}$ and a unitary $r\times r$ matrix $U_{k-1} = (a_{\lambda\mu}^{(k-1)})$ such that

$$f_\mu^{(k-1)} = \sum_{\lambda=1}^{r} a_{\lambda\mu}^{(k-1)}\, e_\lambda^{(k-1)}.$$

Then, we put holomorphic local coordinates $(z,\xi^{(1)},\dots,\xi^{(k)})$ on X_k centered at the point $(x_0,[v_0],\dots,[v_{k-1}])$ where

$$\begin{aligned}(z,\xi^{(1)},\dots,\xi^{(k)}) \mapsto [&\xi_1^{(k)} e_1^{(k-1)}(z,\dots,\xi^{(k-1)}) + \dots + \xi_{r-1}^{(k)} e_{r-1}^{(k-1)}(z,\dots,\xi^{(k-1)}) \\ &+ e_r^{(k-1)}(z,\dots,\xi^{(k-1)})] \in P(V_{k-1(z,\dots,\xi^{(k-1)})})\end{aligned}$$

and also

$$\begin{aligned}\eta_k(z,\xi^{(1)},\dots,\xi^{(k)}) =&\xi_1^{(k)} e_1^{(k-1)}(z,\dots,\xi^{(k-1)}) + \dots + \xi_{r-1}^{(k)} e_{r-1}^{(k-1)}(z,\dots,\xi^{(k-1)}) \\ &+ e_r^{(k-1)}(z,\dots,\xi^{(k-1)})\end{aligned}$$

is a local nonzero section of $\mathcal{O}_{X_k}(-1)$.

As we have already done, if we call

$$f_\lambda^{(k)} = \frac{1}{\varepsilon_k}\frac{\partial}{\partial \xi_\lambda^{(k)}}, \quad \lambda = 1,\dots,r-1, \qquad f_r^{(k)} = C_{\varepsilon_k}^{(k)}\,\widetilde{\eta}_k,$$

where $C_{\varepsilon_k}^{(k)} = \frac{1}{\sqrt{\omega^{(k)}(\widetilde{\eta}_k,\widetilde{\eta}_k)}} = 1 + O(\varepsilon_k)$, then $(f_\lambda^{(k)})$ is a local holomorphic frame for V_k, unitary at $(x_0,[v_0],\dots,[v_{k-1}])$. We now fix a $\omega^{(k)}$-unitary vector $v_k \in V_{k(x_0,[v_0],\dots,[v_{k-1}])}$ and choose a holomorphic local normal frame $(e_\lambda)^{(k)}$ for $(V_k,\omega^{(k)})$ such that $e_\lambda^{(k)}(x_0,[v_0],\dots,[v_{k-1}]) = v_k$ and a $r\times r$ unitary matrix $U_k = (a_{\lambda\mu}^{(k)})$ such that $f_\mu^{(k)} = \sum_{\lambda=1}^{r} a_{\lambda\mu}^{(k)}\, e_\lambda^{(k)}$.

So, if we call respectively $\gamma_{jk\lambda\mu}^{(k)}$ and $c_{jk\lambda\mu}^{(k)}$ the coefficients of curvature of V_k at $(x_0,[v_0],\dots,[v_{k-1}])$ with respect to the basis $(f_\lambda^{(k)})$ and $(e_\lambda^{(k)})$ we have

$$c_{ij\lambda\mu}^{(k)} = \sum_{\alpha,\beta=1}^{r} \gamma_{ij\alpha\beta}^{(k)}\, \overline{a}_{\lambda\alpha}^{(k)} a_{\mu\beta}^{(k)}, \tag{4.3}$$

with $i,j = 1,\dots,n+k(r-1)$ and $\lambda,\mu = 1,\dots,r$.

4.4 Curvature of $\mathcal{O}_{X_k}(1)$ and proof of Theorem 4.1.1

We now use (4.2) and (4.3) to get the induction formulas to derive an expression for the curvature of $\mathcal{O}_{X_k}(1)$, when $\varepsilon^{(k)} = (\varepsilon_1, \dots, \varepsilon_{k-1})$ tends to zero.

We start by observing that (4.2) shows how $\gamma^{(s)}_{ij\lambda\mu}$ depends on $c^{(s-1)}_{lm\alpha\beta}$; we rewrite here the dependence modulo ε_s:

$$\gamma^{(s)}_{ij\lambda\mu} = \begin{cases} c^{(s-1)}_{ij\lambda\mu} - \delta_{\lambda\mu} c^{(s-1)}_{ijrr} & \text{if } 1 \le i, j \le n + (s-1)(r-1) \\ & \text{and } 1 \le \lambda, \mu \le r-1 \\ \delta_{jk}\delta_{\lambda\mu} + & \text{if } i, j \ge n + (s-1)(r-1) + 1, \\ \delta_{(i-n-(s-1)(r-1))\mu}\delta_{(j-n-(s-1)(r-1))\lambda} & i, j \le n + s(r-1) \\ & \text{and } 1 \le \lambda, \mu \le r-1, \end{cases}$$

$$\gamma^{(s)}_{ijrr} = \begin{cases} c^{(s-1)}_{ijrr} & \text{if } 1 \le i, j \le n + (s-1)(r-1) \\ -1 & \text{if } n + (s-1)(r-1) + 1 \le i = j \le n + s(r-1), \end{cases} \tag{4.4}$$

the remaining coefficients being zero. Recall also that, by (4.3),

$$c^{(s)}_{ij\lambda\mu} = \sum_{\alpha,\beta=1}^{r} \gamma^{(s)}_{ij\alpha\beta}\, \overline{a}^{(s)}_{\lambda\alpha} a^{(s)}_{\mu\beta}.$$

Now, we have

$$\Theta(\mathcal{O}_{X_k}(1))_{(x_0,[v_0],\dots,[v_{k-1}])} = \sum_{\lambda=1}^{r-1} d\xi^{(k)}_\lambda \wedge d\overline{\xi}^{(k)}_\lambda - \sum_{i,j=1}^{n+(k-1)(r-1)} c^{(k-1)}_{ijrr}\, dz_i \wedge d\overline{z}_j, \tag{4.5}$$

where we have set $z_{n+(s-1)(r-1)+\lambda} = \xi^{(s)}_\lambda$, $\lambda = 1, \dots, r-1$, and to get the expression of this curvature with respect to the coefficients of curvature of V it suffices to perform the recursive substitutions (4.3) and (4.4) and to stop with $c^{(0)}_{ij\lambda\mu} = c_{ij\lambda\mu}$.

Thus, Theorem 4.1.1 is proved.

4.4.1 The case of surfaces

In the case $\operatorname{rank} V = \dim X = 2$, we have a nice matrix representation of these formulae. First of all, note that in this case the identities (4.4) become much simpler:

$$\gamma^{(s)}_{ij11} = \begin{cases} c^{(s-1)}_{ij11} - c^{(s-1)}_{ij22} & \text{if } 1 \le i,j \le s+1 \\ 2 & \text{if } i = j = s+2 \end{cases}$$

$$\gamma^{(s)}_{ij22} = \begin{cases} c^{(s-1)}_{ij22} & \text{if } 1 \le i,j \le s+1 \\ -1 & \text{if } i = j = s+2. \end{cases}$$

Now, for each $s \ge 1$, let $v_s = v^1_s\, f^{(s)}_1 + v^2_s\, f^{(s)}_2$, with $|v^1_s|^2 + |v^2_s|^2 = 1$. Then we have $a^{(s)}_{21} = \overline{v}^1_s$ and $a^{(s)}_{22} = \overline{v}^2_s$ and so, for instance $a^{(s)}_{11} = -v^2_s$ and $a^{(s)}_{12} = v^1_s$ would work. It follows that

$$c^{(s)}_{ij11} = \gamma^{(s)}_{ij11}|v^2_s|^2 + \gamma^{(s)}_{ij22}|v^1_s|^2, \quad c^{(s)}_{ij22} = \gamma^{(s)}_{ij11}|v^1_s|^2 + \gamma^{(s)}_{ij22}|v^2_s|^2.$$

So, if we set

$$R_s = \begin{pmatrix} |v^2_s|^2 & |v^1_s|^2 \\ |v^1_s|^2 & |v^2_s|^2 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad C^{(s)}_{ij} = \begin{pmatrix} c^{(s)}_{ij11} \\ c^{(s)}_{ij22} \end{pmatrix}$$

we have that

$$C^{(s)}_{ij} = \begin{cases} R_s \cdot T \cdot R_{s-1} \cdot T \cdots R_1 \cdot T \cdot C^{(0)}_{ij} & \text{if } 1 \le i,j \le 2 \\ R_s \cdot T \cdot R_{s-1} \cdot T \cdots R_{i-2} \cdot T \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \text{if } 3 \le i = j \le s+2 \end{cases}$$

and we are interested in the second element of the vector $C^{(k-1)}_{ij}$: in fact, in the surface absolute case, formula (4.5) can be rewritten in the form

$$\begin{aligned} \Theta(\mathcal{O}_{X_k}(1)) = \quad & d\xi^{(k)} \wedge d\overline{\xi}^{(k)} - \sum_{s=3}^{k+1} c^{(k-1)}_{ss22}\, d\xi^{(s-2)} \wedge d\overline{\xi}^{(s-2)} \\ & - \sum_{i,j=1}^{2} c^{(k-1)}_{ij22}\, dz_i \wedge d\overline{z}_j. \end{aligned} \tag{4.6}$$

We shall see in the next sections how this explicit formulae can be use to compute Morse-type integrals, in order to obtain the existence of nonzero global section of the bundle of invariant jet differentials.

4.5 Holomorphic Morse inequalities for jets

Let X be a smooth surface and $V = T_X$. From now on we will suppose that K_X is ample, so that we can take as a metric on X the Kähler-Einstein one, and we will work always modulo ε^k (this will be possible thanks to Lebesgue's dominated convergence theorem).

4.5.1 The Kähler-Einstein assumption

So, let K_X be ample. Then we have a unique hermitian metric ω on T_X, such that $\mathrm{Ricci}(\omega) = -\omega$ and, for this metric,

$$\mathrm{Vol}_\omega(X) = \frac{\pi^2}{2} c_1^2(X) > 0,$$

where

$$\mathrm{Vol}_\omega(X) \stackrel{\mathrm{def}}{=} \int_X \frac{\omega^2}{2!}.$$

Now, consider the two hermitian matrices (c_{ij11}) and (c_{ij22}). The Kähler-Einstein assumption implies that

$$(c_{ij11}) + (c_{ij22}) = (-\delta_{ij})$$

and so they are simultaneously diagonalizable. Let

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

be a diagonal form for (c_{ij11}). Then

$$\lambda + \mu = c_{1111} + c_{2211} = c_{1111} + c_{1122} = -1$$

thanks to the Kähler symmetries. If the eigenvalues of (c_{ij22}) are λ', μ' then $\lambda' + \lambda = \mu + \mu' = -1$ thus a diagonal form for (c_{ij22}) is

$$\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}.$$

As a consequence, for $\alpha, \beta \in \mathbb{C}$, the eigenvalues of the matrix $\alpha(c_{ij11}) + \beta(c_{ij22})$ are $\alpha\lambda + \beta\mu$ and $\alpha\mu + \beta\lambda$ and so

$$\begin{aligned} \det(\alpha(c_{ij11}) + \beta(c_{ij22})) &= (\alpha\lambda + \beta\mu)(\alpha\mu + \beta\lambda) \\ &= \alpha\beta(\lambda^2 + \mu^2) + \lambda\mu(\alpha^2 + \beta^2) \\ &= \alpha\beta[(\lambda + \mu)^2 - 2\lambda\mu] + \lambda\mu(\alpha^2 + \beta^2) \\ &= \alpha\beta + \lambda\mu(\alpha - \beta)^2 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \mathrm{tr}(\alpha(c_{ij11}) + \beta(c_{ij22})) &= (\alpha\lambda + \beta\mu) + (\alpha\mu + \beta\lambda) \\ &= (\alpha + \beta)(\lambda + \mu) \\ &= -(\alpha + \beta). \end{aligned} \tag{4.8}$$

4.5.2 Computations for low order jets

We now use all this to compute some "Morse" integral, in order to obtain existence of sections of (a certain linear combination of) the anti-tautological line bundles for some low value of k. For $k = 1$, we simply have

$$\Theta(\mathcal{O}_{X_1}(1)) = d\xi^{(1)} \wedge d\overline{\xi}^{(1)} - \sum_{i,j=1}^{2} c_{ij22}\, dz_i \wedge d\overline{z}_j$$

and so

$$\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_1}(1))\right)^3 = 3!\left(\frac{i}{2\pi}\right)^3 D\, dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge d\xi^{(1)} \wedge d\overline{\xi}^{(1)},$$

where we have set $D \stackrel{\text{def}}{=} \lambda\mu$, which is, of course, a function $X_1 \to \mathbb{R}$. In particular,

$$\begin{aligned}\int_{X_1}\left(\frac{i}{2\pi}\right)^3 D\, dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge d\xi^{(1)} \wedge d\overline{\xi}^{(1)} \\ &= \frac{1}{6}\int_{X_1}\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_1}(1))\right)^3 \\ &= \frac{1}{6}(c_1^2(X) - c_2(X)),\end{aligned}$$

in fact, this integral over X_1 is just the top-self intersection of $u_1 = c_1(\mathcal{O}_{X_1}(1))$, and this is easily seen to be $c_1^2(X) - c_2(X)$ (cf. relations on Chern classes in Chapter 3).

Moreover, we have that

$$\left(\frac{i}{2\pi}\right)^3 dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge d\xi^{(1)} \wedge d\overline{\xi}^{(1)} = \pi_{0,1}^*\left(\frac{1}{\pi^2}dV_\omega\right) \wedge \left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_1}(1))\right),$$

so that

$$\begin{aligned}\int_{X_1}\left(\frac{i}{2\pi}\right)^3 dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge d\xi^{(1)} \wedge d\overline{\xi}^{(1)} \\ &= \int_{X_1}\pi_{0,1}^*\left(\frac{1}{\pi^2}dV_\omega\right) \wedge \left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_1}(1))\right) \\ &= \frac{1}{2}c_1^2(X)\end{aligned}$$

by Fubini.

Existence of 1-jets

Observe that

$$\int_{X_1}\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_1}(1))\right)^3=\int_{X_1(\le 1,\mathcal{O}_{X_1}(1))}\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_1}(1))\right)^3,$$

because the "vertical" eigenvalue of $\Theta(\mathcal{O}_{X_1}(1))$ is always positive and the two "horizontal" ones (namely $-\lambda$ and $-\mu$) have positive sum. This shows the well-known fact that if $c_1^2(X)-c_2(X)>0$ then $\mathcal{O}_{X_1}(1)$ is big.

Existence of 2-jets

We use notations of §4.4.1 and set moreover $|v_1^1|^2=x$, so that $|v_1^2|^2=1-x$, $0\le x\le 1$. The matrices involved in curvature computations are

$$R_1=\begin{pmatrix}1-x & x\\ x & 1-x\end{pmatrix},\quad R_1\cdot T=\begin{pmatrix}1-x & -1+2x\\ x & -2x+1\end{pmatrix}.$$

Then, we have the following expressions for the curvature respectively of $\mathcal{O}_{X_2}(1)$ and $\mathcal{O}_{X_2}(2,1)$:

$$\begin{aligned}\Theta(\mathcal{O}_{X_2}(1))=&\,d\xi^{(2)}\wedge d\overline{\xi}^{(2)}+(1-3x)\,d\xi^{(1)}\wedge d\overline{\xi}^{(1)}\\ &+\sum_{i,j=1}^{2}\big((2x-1)\,c_{ij22}-x\,c_{ij11}\big)\,dz_i\wedge d\overline{z}_j\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}\Theta(\mathcal{O}_{X_2}(2,1))&=\Theta(\mathcal{O}_{X_2}(1))+\pi_2^*\,\Theta(\mathcal{O}_{X_1}(2))\\ &=\ d\xi^{(2)}\wedge d\overline{\xi}^{(2)}+(3-3x)\,d\xi^{(1)}\wedge d\overline{\xi}^{(1)}\\ &\quad+\sum_{i,j=1}^{2}\big((2x-3)\,c_{ij22}-x\,c_{ij11}\big)\,dz_i\wedge d\overline{z}_j.\end{aligned}$$

To compute the integral involved in holomorphic Morse inequalities for the relatively nef line bundle $\mathcal{O}_{X_2}(2,1)$, we just observe that we have two non-negative "vertical" eigenvalues and that the sum of the "horizontal" ones is $-(2x-3-x)=3-x>0$, because of (4.8) and the fact that $0\le x\le 1$;

thus we have at most one negative eigenvalue and thus

$$\begin{aligned}
\int_{X_2(\leq 1, \mathcal{O}_{X_2}(2,1))} \left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_2}(2,1))\right)^4 &= \int_{X_2} \left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_2}(2,1))\right)^4 \\
&= 4! \left(\frac{i}{2\pi}\right)^4 \int_{X_2} \big((3-3x)^3 D \\
&\qquad + x(3-2x)(3-3x)\big)\, dz_1 \wedge \cdots \wedge d\bar{\xi}^{(2)} \\
&= 4! \left(\frac{i}{2\pi}\right)^3 \int_{X_1} dz_1 \wedge \cdots \wedge d\bar{\xi}^{(1)} \int_0^1 dx \int_0^{2\pi} \big((3-3x)^3 D \\
&\qquad + x(3-2x)(3-3x)\big) \frac{d\vartheta}{2\pi} \\
&= 4! \left(\frac{i}{2\pi}\right)^3 \int_{X_1} \left(\frac{27}{4} D + 1\right) dz_1 \wedge \cdots \wedge d\bar{\xi}^{(1)} \\
&= \frac{162}{6}(c_1^2(X) - c_2(X)) + \frac{24}{2} c_1^2(X) = 39\, c_1^2(X) - 27\, c_2(X).
\end{aligned}$$

Observe that, in the third equality, we have simply done the change of variable in the complex $\xi^{(2)}$-plane

$$\frac{i}{2\pi} d\xi^{(2)} \wedge d\bar{\xi}^{(2)} \mapsto \frac{dx\, d\vartheta}{2\pi}.$$

So, we get that $\mathcal{O}_{X_2}(2,1)$ (and then $\mathcal{O}_{X_2}(1)$) is big if $13\, c_1^2(X) - 9\, c_2(X) > 0$. This is the case, for instance, if $X \subset \mathbb{P}^3$ is a smooth projective hypersurface of degree $\deg X \geq 15$ (cf. §3.2.1)

Existence of 3-jets

Now, as a third application, we do the same thing with the relatively nef line bundle $\mathcal{O}_{X_3}(6,2,1)$. Again using notations of §4.4.1 and setting moreover $|v_1^1|^2 = x$, $|v_2^1|^2 = y$ so that $|v_1^2|^2 = 1-x$, $|v_2^2|^2 = 1-y$, $0 \leq x, y \leq 1$, we obtain

$$\begin{aligned}
R_2 &= \begin{pmatrix} 1-y & y \\ y & 1-y \end{pmatrix}, \\
R_2 \cdot T &= \begin{pmatrix} 1-y & -1+2y \\ y & -2y+1 \end{pmatrix}, \\
R_2 \cdot T \cdot R_1 \cdot T &= \begin{pmatrix} (3y-2)x - y + 1 & (-6y+4)x + 3y - 2 \\ (-3y+1)x + y & (6y-2)x - 3y + 1 \end{pmatrix}.
\end{aligned}$$

Then, for the curvature, we have

$$\begin{aligned}\Theta(\mathcal{O}_{X_3}(1)) = {} & d\xi^{(3)} \wedge d\overline{\xi}^{(3)} + (1-3y)\, d\xi^{(2)} \wedge d\overline{\xi}^{(2)} \\ & + \big((9y-3)x + 1 - 4y)\big)\, d\xi^{(1)} \wedge d\overline{\xi}^{(1)} \\ & + \sum_{i,j=1}^{2} \big(((-6y+2)x + 3y - 1)\, c_{ij22} \\ & + ((3y-1)x - y)\, c_{ij11}\big)\, dz_i \wedge d\overline{z}_j \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\Theta(\mathcal{O}_{X_3}(6,2,1)) = {} & d\xi^{(3)} \wedge d\overline{\xi}^{(3)} + (3-3y)\, d\xi^{(2)} \wedge d\overline{\xi}^{(2)} \\ & + \big((9y-9)x - 4y + 9\big)\, d\xi^{(1)} \wedge d\overline{\xi}^{(1)} \\ & + \sum_{i,j=1}^{2} \big(((-6y+6)x + 3y - 9)\, c_{ij22} \\ & + ((3y-3)x - y)\, c_{ij11}\big)\, dz_i \wedge d\overline{z}_j. \end{aligned}$$

Now, it is an easy matter to show that once again the "vertical" eigenvalues of $\Theta(\mathcal{O}_{X_3}(6,2,1))$ are nonnegative and that the sum of the "horizontal" ones is $(3y-3)x - 2y + 9$, which is positive for $0 \leq x, y \leq 1$. Thus, we have that

$$\begin{aligned}\int_{X_3(\leq 1, \mathcal{O}_{X_3}(6,2,1))} & \left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_2}(6,2,1))\right)^5 = \int_{X_3} \left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_2}(6,2,1))\right)^5 \\ & = 5!\left(\frac{i}{2\pi}\right)^5 \int_{X_3} \big(P(x,y)\, D + Q(x,y)\big)\, dz_1 \wedge \cdots \wedge d\overline{\xi}^{(3)} \\ & = 5!\left(\frac{i}{2\pi}\right)^3 \int_{X_1} dz_1 \wedge \cdots \wedge d\overline{\xi}^{(1)} \int_0^1 dx \int_0^{2\pi} \frac{d\vartheta_x}{2\pi} \\ & \quad \int_0^1 dy \int_0^{2\pi} \big(P(x,y)\, D + Q(x,y)\big)\, \frac{d\vartheta_y}{2\pi} \\ & = 5!\left(\frac{i}{2\pi}\right)^3 \int_{X_1} \left(\frac{1113}{5} D + \frac{453}{10}\right) dz_1 \wedge \cdots \wedge d\overline{\xi}^{(1)} \\ & = 4452\, (c_1^2(X) - c_2(X)) + 2718\, c_1^2(X) \\ & = 7170\, c_1^2(X) - 4452\, c_2(X), \end{aligned}$$

where

$$\begin{aligned} P(x,y) = {} & (-2187y^4 + 8748y^3 - 13122y^2 + 8748y - 2187)x^3 \\ & + (2916y^4 - 15309y^3 + 28431y^2 - 22599y + 6561)x^2 \\ & + (-1296y^4 + 8424y^3 - 19521y^2 + 18954y - 6561)x \\ & + 192y^4 - 1488y^3 + 4212y^2 - 5103y + 2187 \end{aligned}$$

and

$$\begin{aligned}Q(x,y) =&(486y^4 - 1944y^3 + 2916y^2 - 1944y + 486)x^3\\&+ (-621y^4 + 3078y^3 - 5508y^2 + 4266y - 1215)x^2\\&+ (261y^4 - 1494y^3 + 2934y^2 - 2430y + 729)x\\&- 36y^4 + 225y^3 - 432y^2 + 243y.\end{aligned}$$

So, we get that $\mathcal{O}_{X_3}(6,2,1)$ (and then $\mathcal{O}_{X_3}(1)$) is big if $1195\, c_1^2(X) - 742\, c_2(X) > 0$.

This is the case, for instance, if X a smooth surface in $\mathbb{P}^3$, with $\deg(X) \geq 12$.

Comparison with lower bounds of §3.1

We would like to remark here, that, even if we are dealing with the same relatively nef bundles of §3.1, we get considerably better lower bounds for the degree in the hypersurface case.

The reason is quite subtle: from a hermitian point of view, in proving Theorem 3.1.1, we tacitly use the restriction of the Fubini-Study metric of the projective space the hypersurface is embedded in, to the tangent bundle of the hypersurfaces. This is why we had to "correct" this metric by adding some positivity coming from $\mathcal{O}(2)$ and thus loosing some effectivity.

Using instead the differential-geometric approach of the present chapter, we were able to take advantage of the full strength of the Kähler-Einstein metric, which reflects directly the strong positivity properties of varieties with ample canonical bundle.

4.5.3 A "negative" example: quotients of the ball

Here, we wish to make an example to clarify why, if we deal with smooth metrics, we have to use the relatively nef weighted line bundles introduced above.

Suppose you want to show, using just $\mathcal{O}_{X_k}(1)$, the existence of global k-jet differentials on a surface X. From our point of view, a good possible "test" case is when X is a compact unramified quotient of the unit ball $\mathbb{B}_2 \subset \mathbb{C}^2$; surfaces which arise in this way are Kähler-Einstein, hyperbolic and with ample cotangent bundle: the best one can hope (these surfaces have even lots of symmetric differentials).

So, let $\mathbb{B}_2 = \{z \in \mathbb{C}^2 \mid |z| < 1\}$ endowed with the Poincaré metric

$$\begin{aligned}\omega_P &= -\frac{i}{2}\partial\overline{\partial}\log(1-|z|^2) \\ &= \frac{i}{2}\left(\frac{dz \otimes d\overline{z}}{1-|z|^2} + \frac{|\langle dz, z\rangle|^2}{(1-|z|^2)^2}\right).\end{aligned}$$

Consider a compact unramified quotient $X = \mathbb{B}_2/\Gamma$ with the quotient metric, say ω. Then, ω has constant curvature; in particular, the function $D\colon X_1 \to \mathbb{R}$ we defined in §4.5.2 is constant.

This constant can be quite easily directly computed by hands. Here, we shall compute it as a very simple application of the celebrated Bogomolov-Miyaoka-Yau inequality $c_1^2 \leq 3\,c_2$ for surfaces of general type with ample canonical bundle, which says moreover that the equality holds if and only if the surface is a quotient of the ball $\mathbb{B}_2$.

Using computations made in §4.5.2, we have

$$\begin{aligned}\frac{1}{6}\big(c_1(X)^2 - c_2(X)\big) &= \int_{X_1}\left(\frac{i}{2\pi}\right)^3 D\, dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge d\xi^{(1)} \wedge d\overline{\xi}^{(1)} \\ &= D\int_{X_1}\left(\frac{i}{2\pi}\right)^3 dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge d\xi^{(1)} \wedge d\overline{\xi}^{(1)} \\ &= \frac{1}{2}\, c_1(X)^2\, D,\end{aligned}$$

so that, making the substitution $c_1(X)^2 = 3\,c_2(X)$, we find $D \equiv 2/9$.

Now, we compute the “Morse” integrals for $\mathcal{O}_{X_k}(1)$ and low values of k, using the new information about D.

$\boxed{k=1}$ We have already done this integral in §4.5.2: in this special case it gives $\frac{2}{3}\,c_1(X)^2 > 0$ and so the existence of 1-jet differentials.

$\boxed{k=2}$ In this case (the line bundle is no more relatively nef) we don't have the equality $(X_2, \leq 1) = X_2$ and so we have to determine the open set $(X_2, \leq 1)$. This is an easy matter: formula (4.9) gives directly

$$(X_2, \leq 1) = \left\{0 < x < \frac{2}{3}\right\},$$

since the trace of the “horizontal” part is always positive for $k = 2$.

Then we have

$$\begin{aligned}
&\int_{(X_2,\leq 1)} \left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_2}(1))\right)^4 \\
&\quad = 4!\left(\frac{i}{2\pi}\right)^4 \int_{(X_2,\leq 1)} (1-3x)\left(-\frac{1}{3}x+\frac{2}{9}\right) dz_1 \wedge \cdots \wedge d\overline{\xi}^{(2)} \\
&\quad = 4!\left(\frac{i}{2\pi}\right)^3 \int_{X_1} dz_1 \wedge \cdots \wedge d\overline{\xi}^{(1)} \int_0^{2/3} (1-3x)\left(-\frac{1}{3}x+\frac{2}{9}\right) dx \\
&\quad = 4!\left(\frac{i}{2\pi}\right)^3 \int_{X_1} \frac{2}{81}\, dz_1 \wedge \cdots \wedge d\overline{\xi}^{(1)} \\
&\quad = \frac{8}{27}\, c_1(X)^2 > 0,
\end{aligned}$$

hence the existence of 2-jet differentials (the optimal attended result should be 10/27).

$\boxed{k=3}$ Here the situation becomes much more involved (see Figure 4.1). First of all (see formula (4.10)), the trace and the determinant of the "horizontal" part are given by:

$$\text{trace} = (3y-1)x + 1 - 2y$$

and

$$\text{determinant} = \left(-y^2 + \frac{4}{3}\,y - \frac{1}{3}\right)x + \frac{5}{9}\,y^2 - \frac{7}{9}\,y + \frac{2}{9}.$$

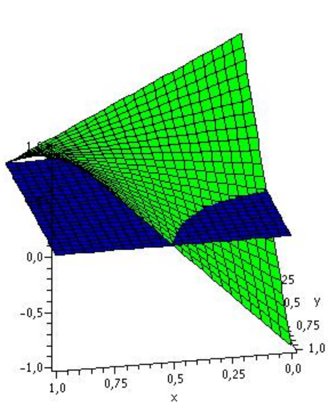
In particular, the locus of negative trace is contained in the locus of negative determinant (as usual, $0 \leq x, y \leq 1$). Secondly, the sign of the "first" eigenvalue is given, in the square $[0,1]^2$, by $(9y-3)x+1-4y \gtrless 0$ and the sign of the "second" eigenvalue by $1-3y \gtrless 0$. A somewhat tedious study of these inequalities leads to

$$X_3(0, \mathcal{O}_{X_3}(1)) = \left\{0 < y < \frac{3x-1}{9x-4},\ 0 < x < \frac{1}{3}\right\}$$

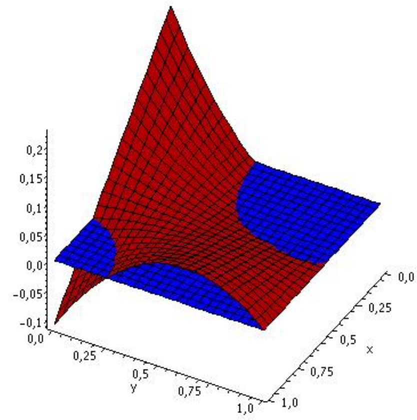
and

$$\begin{aligned}
X_3(1, \mathcal{O}_{X_3}(1)) = &\left\{\frac{3x-1}{9x-4} < y < 1,\ \frac{1}{2} < x < 1\right\} \\
&\cup \left\{\frac{3x-1}{9x-4} < y < \frac{1}{3},\ 0 < x < \frac{1}{3}\right\} \\
&\cup \left\{0 < y < \frac{1}{3},\ \frac{1}{3} < x < \frac{2}{3}\right\} \\
&\cup \left\{\frac{3x-2}{9x-5} < y < \frac{1}{3},\ \frac{2}{3} < x < 1\right\}.
\end{aligned}$$

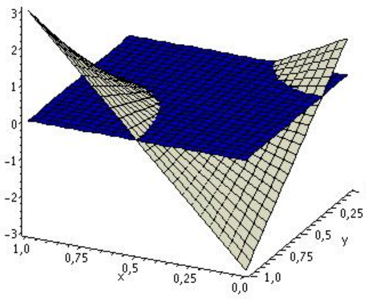
Figure 4.1: Behavior of eigenvalues for 3-jets



(a) Trace of the "horizontal" part



(b) Determinant of the "horizontal" part



(c) The first "vertical" eigenvalue

Let us call the determinant of the curvature

$$T(x,y) = (1-3y)\big((9y-3)x+1-4y\big) \cdot \left(\left(-y^2+\frac{4}{3}\,y-\frac{1}{3}\right)x+\frac{5}{9}\,y^2-\frac{7}{9}\,y+\frac{2}{9}\right),$$

so that the top-wedge is expressed by

$$\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_3}(1))\right)^5 = 5!\left(\frac{i}{2\pi}\right)^5 T(x,y).$$

Then, the "Morse" integral is given by

$$\begin{aligned}\int_{X_3(\leq 1,\mathcal{O}_{X_3}(1))} &\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_3}(1))\right)^5 \\ &= 5!\left(\frac{i}{2\pi}\right)^5 \int_{X_3(\leq 1,\mathcal{O}_{X_3}(1))} T(x,y)\,dz_1\wedge\cdots\wedge d\overline{\xi}^{(3)} \\ &= 5!(I_1+I_2+I_3+I_4+I_5),\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \left(\frac{i}{2\pi}\right)^3 \int_{X_1} dz_1\wedge\cdots\wedge d\overline{\xi}^{(1)} \int_0^{1/3} dx \int_0^{\frac{3x-1}{9x-4}} T(x,y)\,dy,\\
I_2 &= \left(\frac{i}{2\pi}\right)^3 \int_{X_1} dz_1\wedge\cdots\wedge d\overline{\xi}^{(1)} \int_{1/2}^{1} dx \int_{\frac{3x-1}{9x-4}}^{1} T(x,y)\,dy,\\
I_3 &= \left(\frac{i}{2\pi}\right)^3 \int_{X_1} dz_1\wedge\cdots\wedge d\overline{\xi}^{(1)} \int_0^{1/3} dx \int_{\frac{3x-1}{9x-4}}^{1/3} T(x,y)\,dy,\\
I_4 &= \left(\frac{i}{2\pi}\right)^3 \int_{X_1} dz_1\wedge\cdots\wedge d\overline{\xi}^{(1)} \int_{1/3}^{2/3} dx \int_0^{1/3} T(x,y)\,dy,\\
I_5 &= \left(\frac{i}{2\pi}\right)^3 \int_{X_1} dz_1\wedge\cdots\wedge d\overline{\xi}^{(1)} \int_{2/3}^{1} dx \int_{\frac{3x-2}{9x-5}}^{1/3} T(x,y)\,dy.
\end{aligned}$$

Finally we get

$$\int_{X_3(\leq 1,\mathcal{O}_{X_3}(1))} \left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_3}(1))\right)^5 = -\underbrace{\frac{715933}{1944000}}_{\simeq 0{,}37} c_1(X)^2 < 0,$$

and so we are not able to check the existence of 3-jet differentials.

Here are some considerations.

First, the value of the above integrals is, at least in these first cases, decreasing while morally one should expect an increasing sequence (the existence of k-jet differentials implies obviously the existence of $(k+1)$-jet differentials).

Second, we suspect that, in fact, this sequence continues to be non-increasing in general, since going up with k, adds more and more regions of negativity along the fiber direction ($\mathcal{O}_{X_k}(1)$ is not relatively positive over X, for $k \geq 2$). Moreover, recall that we are working here on a quotient of the ball, so that we had the most favorable "horizontal" contribution in terms of positivity: thus, the problem really relies in the fibers direction.

From these considerations, we deduce that to get a Green-Griffiths type result about asymptotic (on k) existence of section, we are naturally led to study either the smooth relatively nef line case (weighted line bundles $\mathcal{O}_{X_k}(\mathbf{a})$), or to leave the "smooth world" and to study singular hermitian metrics on $\mathcal{O}_{X_k}(1)$ which reflects the relative base locus of this bundle.

The last section of the present chapter will be devoted to the first of these two different approaches.

4.5.4 Minimal surfaces of general type

If we relax the hypothesis on the canonical bundle of our surface X, and we just take it to be big and nef, then our previous computation gives the same results.

To see this, it suffices to select an ample class A on X and, for every $\varepsilon > 0$, to solve the "approximate" Kähler-Einstein equation $\mathrm{Ricci}(\omega) = -\omega + \varepsilon\,\Theta(A)$ (the existence of such a metric ω on T_X is a well-known consequence of the theory of Monge-Ampère equations).

Once we have such a metric we just observe that, with the notations of this section, we have $\lambda + \mu = -1 + O(\varepsilon)$, so that

$$\det(\alpha(c_{ij11}) + \beta(c_{ij22})) = \alpha\beta(1 + O(\varepsilon)) + \lambda\mu(\alpha^2 - \beta^2)$$

and

$$\mathrm{tr}(\alpha(c_{ij11}) + \beta(c_{ij22})) = -(\alpha + \beta)(1 - O(\varepsilon)).$$

It is then clear that, our integral computation will now have a final error term which is in fact a $O(\varepsilon)$, and thus we obtain the same results, by letting ε tend to zero.

4.6 Proof of Theorem 4.1.2

In this section we compute explicitly the Chern curvature of the weighted line bundles $\mathcal{O}_{X_k}(\mathbf{a})$ and we find conditions for them to be relatively positive. Next, thanks to holomorphic Morse inequalities, we study the consequences of positive auto-intersection and finally we prove Theorem 4.1.2.

4.6.1 Curvature of weighted line bundles

We recall some notations and formulae. Let $v_s = v_s^1\, f_1^{(s)} + v_s^2\, f_2^{(s)} \in V_s$, with $|v_s^1|^2 + |v_s^2|^2 = 1$ and set $x_s = |v_s^1|^2$, $0 \le x_s \le 1$. Then, if

$$R_s = \begin{pmatrix} 1 - x_s & x_s \\ x_s & 1 - x_s \end{pmatrix}, \quad T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and

$$R_p \cdot T \cdots R_q \cdot T = \begin{pmatrix} \delta_{p,q} & \gamma_{p,q} \\ \beta_{p,q} & \alpha_{p,q} \end{pmatrix}, \quad p \geq q \geq 1,$$

where $\alpha_{p,q}, \beta_{p,q}, \gamma_{p,q}$ and $\delta_{p,q}$ are functions of $(x_q, \ldots, x_p)$, we have that, for $k \geq 2$,

$$\begin{aligned} \Theta(\mathcal{O}_{X_k}(1)) &= d\xi^{(k)} \wedge d\overline{\xi}^{(k)} + \sum_{s=1}^{k-1} (\alpha_{k-1,s} - \beta_{k-1,s})\, d\xi^{(s)} \wedge d\overline{\xi}^{(s)} \\ &\quad + \sum_{i,j=1}^{2} (-\beta_{k-1,1}\, c_{ij11} - \alpha_{k-1,1}\, c_{ij22})\, dz_i \wedge d\overline{z}_j. \end{aligned}$$

More generally, for $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ (or possibly $\in \mathbb{R}^k$), we have

$$\begin{aligned} \Theta(\mathcal{O}_{X_k}(a_1, \ldots, a_k)) &= a_k\, d\xi^{(k)} \wedge d\overline{\xi}^{(k)} + \sum_{s=1}^{k-1} \left(\sum_{j=s}^{k-1} a_{j+1}\, y_{j,s} + a_s \right) d\xi^{(s)} \wedge d\overline{\xi}^{(s)} \\ &\quad + \sum_{i,j=1}^{2} \left(\underbrace{\sum_{\ell=0}^{k-1} -a_{\ell+1} \beta_{\ell,1}\, c_{ij11} - a_{\ell+1} \alpha_{\ell,1}\, c_{ij22}}_{\stackrel{\text{def}}{=} A_{ij}(a_1, \ldots, a_k)} \right) dz_i \wedge d\overline{z}_j, \end{aligned} \tag{4.11}$$

where $y_{p,q}(x_q, \ldots, x_p) \stackrel{\text{def}}{=} \alpha_{p,q} - \beta_{p,q}$ (we also set formally $\alpha_{0,1} = \beta_{0,1} = 1$). Observe that, for the (2×2)-matrix (A_{ij}), we have

$$\operatorname{tr}(A_{ij}) = \sum_{s=0}^{k-1} a_{s+1}\, w_{s,1}, \quad w_{p,q}(x_q, \ldots, x_p) \stackrel{\text{def}}{=} \alpha_{p,q} + \beta_{p,q},$$

thanks to the Kähler-Einstein assumption and formula (4.8).

Now, define $\theta_s^k = \theta_s^k(x_s, \ldots, x_{k-1})$ to be the function given by

$$(x_s, \ldots, x_{k-1}) \mapsto \sum_{j=s}^{k-1} a_{j+1}\, y_{j,s} + a_s.$$

This is the s-th "vertical" eigenvalue of the weighted curvature.

Remark 4.6.1. As the θ_s^k's are linear combinations of the $y_{j,s}$'s, we have that they all are of degree one in each variable. Hence they and their restriction to each edge of the cube $[0,1]^{k-s}$ are harmonic. In particular they attain their minimum on some vertex of this cube.

In $\mathbb{R}^k$, define the closed convex cone

$$\mathfrak{N} = \left\{ \mathbf{a} \in \mathbb{R}^k \mid a_j \geq 2 \sum_{\ell=j+1}^{k} a_\ell, \, \forall j = 1, \dots, k-1 \text{ and } a_k \geq 0 \right\}.$$

We have the following three lemmas.

Lemma 4.6.1. *The functions θ_s^k are positive if (and only if) $\mathbf{a} \in \mathring{\mathfrak{N}}$.*

Proof. First of all, observe that the structure of the four functions $\alpha_{p,q}$, $\beta_{p,q}$, $\gamma_{p,q}$ and $\delta_{p,q}$ (and hence $y_{p,q}$) depends only on $p-q$. Now, it is immediate to check by induction that we have the following expression for the $\gamma_{p,q}$'s and the $\delta_{p,q}$'s:

$$\gamma_{p,q} = -\sum_{h=q}^{p} \alpha_{h,q} \quad \text{and} \quad \delta_{p,q} = 1 - \sum_{h=q}^{p} \beta_{h,q}.$$

Next, observe that, for all $s \geq 1$,

$$R_s(0) \cdot T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R_s(1) \cdot T = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},$$

so that, if $j \geq 1$, $y_{j+1,1}(\bullet, 0) = y_{j,1}(\bullet)$ and $y_{j+1,1}(\bullet, 1) = -1 - 2\, y_{j,1}(\bullet) - \sum_{h=1}^{j-1} y_{h,1}(\bullet)$; moreover, $y_{1,1}(0) = 1$ and $y_{1,1}(1) = -2$.

The lemma is clearly true for $k = 1$, so we proceed by induction on k. We have, for $s \geq 2$,

$$\begin{aligned} \theta_s^k = \theta_s^k(x_s, \dots, x_{k-1}; \mathbf{a}) &= a_s + \sum_{j=s}^{k-1} a_{j+1}\, y_{j,s} \\ &= a_s + \sum_{j=s}^{k-1} a_{j+1}\, y_{j-s+1,1} = \theta_1^{k-s+1}(x_s, \dots, x_{k-1}; \mathbf{b}), \end{aligned}$$

where $\mathbf{b} = (a_s, \dots, a_k) \in \mathbb{R}^{k-s+1}$ is again in the corresponding $\mathring{\mathfrak{N}}$: it remains then to show that, for a general $k \geq 2$, the lemma is true for θ_1^k. Recall that, by Remark 4.6.1, it suffices to check positivity on the vertices of the cube $[0,1]^{k-1}$. Let $\star$ denote an arbitrary sequence of 0 and 1 of length $k-2$: we shall treat the two cases $(\star, 0)$ and $(\star, 1)$ separately. For the first one, we have

$$
\begin{aligned}
\theta_1^k(\star\,,0;\mathbf{a}) &= a_1 + \sum_{j=1}^{k-1} a_{j+1}\, y_{j,1}(\star,0)\\
&= a_1 + \sum_{j=1}^{k-2} a_{j+1}\, y_{j,1}(\star) + a_k\, y_{k-1,1}(\star,0)\\
&= a_1 + \sum_{j=1}^{k-2} a_{j+1}\, y_{j,1}(\star) + a_k\, y_{k-2,1}(\star)\\
&= a_1 + \sum_{j=1}^{k-3} a_{j+1}\, y_{j,1}(\star) + (a_{k-1}+a_k)\, y_{k-2,1}(\star)\\
&= \theta_1^{k-1}(\star\,;\mathbf{b}')
\end{aligned}
$$

for a new $\mathbf{b}' \in \mathbb{R}^{k-1}$ which is easily seen to be in the corresponding $\mathring{\mathfrak{N}}$. Similarly, for the second case, we have

$$
\begin{aligned}
\theta_1^k(\star\,,1;\mathbf{a}) &= a_1 + \sum_{j=1}^{k-1} a_{j+1}\, y_{j,1}(\star,1)\\
&= a_1 + \sum_{j=1}^{k-2} a_{j+1}\, y_{j,1}(\star) + a_k\, y_{k-1,1}(\star,1)\\
&= a_1 + \sum_{j=1}^{k-2} a_{j+1}\, y_{j,1}(\star) + a_k\left(-\sum_{h=1}^{k-3} y_{h,1}(\star) - 2\, y_{k-2,1}(\star) - 1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (a_1 - a_k) + \sum_{j=1}^{k-3}(a_{j+1}-a_k)\, y_{j,1}(\star) + (a_{k-1}-2a_k)\, y_{k-2,1}(\star)\\
&= \theta_1^{k-1}(\star\,;\mathbf{b}''),
\end{aligned}
$$

where again $\mathbf{b}'' \in \mathbb{R}^{k-1}$ is a new weight which satisfies the (strict) inequalities defining $\mathfrak{N}$. The lemma is proved. □

The reason why we choose $\mathbf{a}$ in the interior of the cone $\mathfrak{N}$, is that with such a choice the vertical eigenvalues of the curvature $\Theta(\mathcal{O}_{X_k}(\mathbf{a}))$ are positive for all small $\varepsilon^{(k)}$.

Remark 4.6.2. The above lemma says in particular, that if $\mathbf{a} \in \mathfrak{N}$, then for all $\varepsilon > 0$, we can endow $\mathcal{O}_{X_k}(\mathbf{a})$ with a smooth hermitian metric h_k (namely, the one we are working with) such that $\Theta_{h_k}(\mathcal{O}_{X_k}(\mathbf{a})) \geq -\varepsilon\,\omega$ along the fiber of $X_k \to X$, for some hermitian metric ω on T_{X_k} (recall that we are always working modulo $\varepsilon^{(k)}$). In particular, the cone $\mathfrak{N}$ is contained in the cone of relatively nef (over X) line bundles.

Lemma 4.6.2. *If $\theta_s^k \geq 0$ for all $s = 1, \ldots, k-1$, and $\mathbf{a} \in \mathbb{N}^k$ with at least one of the a_j's is strictly positive, then* $\operatorname{tr}(A_{ij}) > 0$ *in the cube* $[0,1]^{k-1}$.

Proof. First of all, we recover the expression of the $w_{p,q}$'s in terms of the $y_{r,s}$'s. We have $w_{p,p} = (2 + y_{p,p})/3$ and

$$\begin{aligned} w_{p,j-1} &= \alpha_{p,j-1} + \beta_{p,j-1} = x_{j-1}(\beta_{p,j} - \alpha_{p,j}) + \alpha_{p,j} \\ &= \alpha_{p,j} - y_{p,j}\, x_{j-1} = \alpha_{p,j} + \frac{y_{p,j-1} + 2\beta_{p,j} - \alpha_{p,j}}{3y_{p,j}}\, y_{p,j} \\ &= \frac{y_{p,j-1} + 2w_{p,j}}{3}, \end{aligned}$$

as, $x_{j-1} = -(y_{p,j-1} + 2\beta_{p,j} - \alpha_{p,j})/3y_{p,j}$ (this is easily seen from the very definitions). Then, by induction, we obtain

$$w_{p,q} = \left(\frac{2}{3}\right)^{p-q+1} + \frac{1}{3}\sum_{\ell=q}^{p}\left(\frac{2}{3}\right)^{\ell-q} y_{p,\ell}.$$

Now, $\operatorname{tr}(A_{ij}) = \sum_{s=0}^{k-1} a_{s+1}\, w_{s,1}$ and so

$$\begin{aligned} \sum_{s=0}^{k-1} a_{s+1}\, w_{s,1} &= a_1 + \sum_{s=1}^{k-1}\left(\left(\frac{2}{3}\right)^{s} + \frac{1}{3}\sum_{\ell=1}^{s}\left(\frac{2}{3}\right)^{\ell-1} y_{s,\ell}\right) \\ &= a_1 + \sum_{s=1}^{k-1}\left(\frac{2}{3}\right)^{s} a_{s+1} + \frac{1}{3}\sum_{\ell,s=1}^{k-1}\left(\frac{2}{3}\right)^{\ell-1} a_{s+1}\, y_{s,\ell} \\ &= a_1 + \sum_{s=1}^{k-1}\left(\frac{2}{3}\right)^{s} a_{s+1} + \frac{1}{3}\sum_{\ell=1}^{k-1}\left(\frac{2}{3}\right)^{\ell-1} (\theta_\ell^k - a_\ell) \\ &= \frac{2}{3}\, a_1 + \left(\frac{2}{3}\right)^{k-1} a_k + \sum_{s=1}^{k-2}\left(\frac{2}{3}\right)^{s+1} a_{s+1} + \frac{1}{3}\sum_{\ell=1}^{k-1}\left(\frac{2}{3}\right)^{\ell-1} \theta_\ell^k. \end{aligned}$$

$\square$

Lemma 4.6.3. *Let $D\colon X_1 \to \mathbb{R}$ be as in §4.5.2. If $\theta_s^k \geq 0$ for all $s = 1, \ldots, k-1$, $\mathbf{a} \in \mathbb{N}^k$ with at least one of the a_j's strictly positive and $D \equiv 2/9$, then* $\det(A_{ij}) > 0$ *in the cube* $[0,1]^{k-1}$.

Proof. Set

$$\alpha(\mathbf{a}) \stackrel{\text{def}}{=} a_1 + \sum_{\ell=1}^{k-1} a_{\ell+1}\,\alpha_{\ell,1} \quad \text{and} \quad \beta(\mathbf{a}) \stackrel{\text{def}}{=} a_1 + \sum_{\ell=1}^{k-1} a_{\ell+1}\,\beta_{\ell,1}.$$

Then, formula (4.7) yields

$$\begin{aligned}\det(A_{ij}) &= \alpha(\mathbf{a})\beta(\mathbf{a}) + (\alpha(\mathbf{a}) - \beta(\mathbf{a}))^2 D \\ &= \frac{1}{4}(\alpha(\mathbf{a}) + \beta(\mathbf{a}))^2 - \frac{1}{36}(\alpha(\mathbf{a}) - \beta(\mathbf{a}))^2\end{aligned}$$

But now, we observe that $\alpha(\mathbf{a}) + \beta(\mathbf{a}) = \operatorname{tr}(A_{ij})$ and that $\alpha(\mathbf{a}) - \beta(\mathbf{a}) = \theta_1^k$; the end of the proof of Lemma 4.6.2 shows that $\operatorname{tr}(A_{ij}) > 1/3\,\theta_1^k$, so that $\det(A_{ij}) > 0$. □

Proposition 4.6.4. *If $\mathbf{a} \in \mathfrak{N}$, then the line bundle $\mathcal{O}_{X_k}(1)$ is big as soon as the top self-intersection $\mathcal{O}_{X_k}(\mathbf{a})^{k+2}$ is positive.*

Proof. Without loss of generality, we can suppose that $\mathbf{a}$ is integral and that $\mathbf{a} \in \mathring{\mathfrak{N}}$. Then Lemma 4.6.1 ensures that all the "vertical" eigenvalues are positive: in this case, thanks to Lemma 4.6.2, we conclude that the curvature of $\mathcal{O}_{X_k}(\mathbf{a})^{k+2}$ can have at most one negative "horizontal" eigenvalue.

Thus, $X_k(\leq 1, \mathcal{O}_{X_k}(\mathbf{a})) = X_k$ and so

$$\begin{aligned}\int_{X_k(\leq 1, \mathcal{O}_{X_k}(\mathbf{a}))} \left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_k}(\mathbf{a}))\right)^{k+2} &= \int_{X_k} \left(\frac{i}{2\pi}\Theta(\mathcal{O}_{X_k}(\mathbf{a}))\right)^{k+2} \\ &= \mathcal{O}_{X_k}(\mathbf{a})^{k+2}.\end{aligned}$$

If $\mathcal{O}_{X_k}(\mathbf{a})^{k+2} > 0$, then, by Demailly's holomorphic Morse inequalities, $\mathcal{O}_{X_k}(\mathbf{a})$ is big and so is $\mathcal{O}_{X_k}(1)$ (recall that if $\mathbf{a} \in \mathbb{N}^k$, then there is a non-trivial morphism $\mathcal{O}_{X_k}(\mathbf{a}) \to \mathcal{O}_{X_k}(|\mathbf{a}|)$). □

Remark 4.6.3. The cone $\mathfrak{N}$ contains the cone defined by Demailly in [7] (see Proposition 1.4.2). Nevertheless, one can obtain it as the cone generated by all the pull-backs of the cones defined by Demailly at the different levels of the tower of projective bundles $X_k \to X$.

4.6.2 End of the proof

Let, as usual, $u_j = c_1(\mathcal{O}_{X_j}(1))$ be the first Chern class of the anti-tautological line bundle on X_j. Define the (real) polynomials $F_k, G_k \colon \mathbb{R}^k \to \mathbb{R}$ by

$$(a_1\,u_1 + \cdots + a_k\,u_k)^{k+2} = F_k(\mathbf{a})\,c_1(X)^2 - G_k(\mathbf{a})\,c_2(X).$$

Observe that these two polynomials do not depend on the particular surface X, but only on the relative structure of the fibration $X_k \to X$, which is universal.

Lemma 4.6.5. *Suppose that for each $k \geq 1$, there exists a minimal surface of general type X such that $\mathcal{O}_{X_k}(1)$ is not big. Then, if $\mathbf{a} \in \mathfrak{N}$, we have the inequalities*

$$3F_k(\mathbf{a}) \geq G_k(\mathbf{a}) \geq 0$$

and $G_k \not\equiv 0$.

Proof. Since F_k and G_k are independent of the particular surface chosen, we can suppose X to be a compact unramified quotient of the ball $\mathbb{B}_2$. In this case, $D \equiv 2/9$ and, for $\mathbf{a} \in \mathring{\mathfrak{N}}$ rational,

$$(a_1\, u_1 + \cdots + a_k\, u_k)^{k+2} = \int_{X_k} \left(\frac{i}{2\pi} \Theta(\mathcal{O}_{X_k}(\mathbf{a})) \right)^{k+2} > 0$$

by Lemmas 4.6.1, 4.6.2 and 4.6.3; on the other hand, by Bogomolov-Miyaoka-Yau, $c_1(X)^2 = 3\, c_2(X)$ and $c_1(X)^2 > 0$. Hence, by continuity, $3F_k(\mathbf{a}) - G_k(\mathbf{a}) \geq 0$ on $\mathfrak{N}$ (with strict inequality for $\mathbf{a}$ rational in the interior of the cone).

Now, let us compute the intersection $(a_1\, u_1 + \cdots + a_k\, u_k)^{k+2}$ on a minimal surface of general type X as in the hypotheses: of course, such a surface cannot be a compact unramified quotient of the ball $\mathbb{B}_2$. In this case we have $c_1(X)^2 < 3\, c_2(X)$, and so

$$\begin{aligned}(a_1\, u_1 + \cdots + a_k\, u_k)^{k+2} &= F_k(\mathbf{a})\, c_1(X)^2 - G_k(\mathbf{a})\, c_2(X) \\ &\geq \frac{1}{3}\, G_k(\mathbf{a}) \big(\underbrace{c_1(X)^2 - 3\, c_2(X)}_{<0} \big).\end{aligned}$$

Thus, if there exists a point $\mathbf{a}' \in \mathfrak{N}$ such that $G_k(\mathbf{a}') < 0$, we would have $(a_1\, u_1 + \cdots + a_k\, u_k)^{k+2} > 0$ and hence, by Proposition 4.6.4, $\mathcal{O}_{X_k}(1)$ big, contradiction. Finally, if $G_k \equiv 0$, fix a rational point $\mathbf{a}$ in the interior of the cone $\mathfrak{N}$: such an $\mathbf{a}$ gives $F_k(\mathbf{a}) > 0$. Then, for such a point we would have $(a_1\, u_1 + \cdots + a_k\, u_k)^{k+2} = F_k(\mathbf{a})\, c_1(X)^2 > 0$, again contradiction. □

Remark 4.6.4. Call the first hypothesis of the above lemma ($\star$). If ($\star$) is not satisfied, then we would have that there exist a $k \geq 1$ such that for each minimal surface of general type X, the line bundle $\mathcal{O}_{X_k}(1)$ is big. In this case, we would already have the global sections we are looking for.

Now, let $\Sigma_k \in \mathbb{R}^k$ the zero locus of G_k. By the above lemma, $\mathfrak{N} \setminus \Sigma_k$ is dense in $\mathfrak{N}$. Set

$$m_k \stackrel{\text{def}}{=} \sup_{\mathbf{a}\in\mathfrak{N}\setminus\Sigma_k} \frac{F_k(\mathbf{a})}{G_k(\mathbf{a})}.$$

If ($\star$) holds, then $m_k < +\infty$: otherwise, for each $M > 0$ we would find an $\mathbf{a}_M \in \mathfrak{N} \setminus \Sigma_k$ such that $F_k(\mathbf{a}_M) > M\, G_k(\mathbf{a}_M)$ and so

$$\begin{aligned} F_k(\mathbf{a}_M)\, c_1(X)^2 - G_k(\mathbf{a}_M)\, c_2(X) &> M\, G_k(\mathbf{a}_M)\, c_1(X)^2 - G_k(\mathbf{a}_M)\, c_2(X) \\ &= G_k(\mathbf{a}_M)\big(M\, c_1(X)^2 - c_2(X)\big). \end{aligned}$$

We would then contradict ($\star$), by choosing $M > c_2(X)/c_1(X)^2$.

On the other hand, obviously, if $m_k > c_2(X)/c_1(X)^2$, then $\mathcal{O}_{X_k}(1)$ is big. Moreover, for each $k \geq 1$, we have $1/3 \leq m_k \leq m_{k+1}$. The inequality $m_k \geq 1/3$ follows directly from Lemma 4.6.5. To see the monotonicity, notice that $(a_1\, u_1 + \cdots + a_k\, u_k)^{k+2}|_{a_k=0} \equiv 0$ (just for dimension reasons) and so

$$(a_1\, u_1 + \cdots + a_k\, u_k)^{k+2} = a_k \cdot \underbrace{\frac{1}{a_k}(a_1\, u_1 + \cdots + a_k\, u_k)^{k+2}}_{\text{well defined for } a_k = 0}.$$

But then,

$$\begin{aligned} \frac{1}{a_k}(a_1\, u_1 + \cdots + a_k\, u_k)^{k+2}\bigg|_{a_k=0} &= \frac{\partial}{\partial a_k}(a_1\, u_1 + \cdots + a_k\, u_k)^{k+2}\bigg|_{a_k=0} \\ &= (k+2)(a_1\, u_1 + \cdots + a_{k-1}\, u_{k-1})^{k+1} \cdot u_k \\ &= (k+2)(a_1\, u_1 + \cdots + a_{k-1}\, u_{k-1})^{k+1}, \end{aligned}$$

where the last equality is simply obtained by integrating along the fibers of $X_k \to X_{k-1}$. Hence, we have that

$$\frac{F_k(a_1, \ldots, a_{k-1}, 0)}{G_k(a_1, \ldots, a_{k-1}, 0)} = \frac{F_{k-1}(a_1, \ldots, a_{k-1})}{G_{k-1}(a_1, \ldots, a_{k-1})}$$

and monotonicity follows.

Finally, if we set m_∞ to be the $\sup_{k\geq 1} m_k$, we find that, for X a given minimal surface of general type, if $m_\infty > c_2(X)/c_1(X)^2$, then there exist a $k_0 \in \mathbb{N}$ such that $\mathcal{O}_{X_{k_0}}(1)$ is big.

Remark 4.6.5. For the moment, we are not able to compute or even to estimate the limit term m_∞. Of course, a divergent sequence would imply the existence of global invariant jet differentials of some order on every surface of general type. A less ambitious aim could be, for example, to encompass the case of hypersurfaces X of $\mathbb{P}^3$ of degree greater than or equal to five (which is the minimum degree for X to be of general type). In this case, a simple Chern classes computation shows that $m_\infty > 11$ would be sufficient.

Remark 4.6.6. For the first terms of this sequence, computations made in §4.5.2 supply the estimates

$$m_1 \geq 1, \quad m_2 \geq \frac{13}{9} \simeq 1,44, \quad m_3 \geq \frac{1195}{742} \simeq 1,61, \quad m_4 \geq \frac{442243}{271697} \simeq 1,63$$

and so on: unfortunately, these first terms are still very far from being close even to 11.

Bibliography

[1] A. BLOCH, *Sur les systemes de fonctions uniformes satisfaisant a l'équation d'une variété algébrique dont l'irrégularité dépasse la dimension*, J. Math. Pures Appl., 5 (1926), pp. 9–66.

[2] F. A. BOGOMOLOV, *Families of curves on a surface of general type*, Dokl. Akad. Nauk SSSR, 236 (1977), pp. 1041–1044.

[3] ——, *Holomorphic tensors and vector bundles on projective manifolds*, Izv. Akad. Nauk SSSR Ser. Mat., 42 (1978), pp. 1227–1287, 1439.

[4] P. BRÜCKMANN AND H.-G. RACKWITZ, *T-symmetrical tensor forms on complete intersections*, Math. Ann., 288 (1990), pp. 627–635.

[5] J.-P. DEMAILLY, *Champs magnétiques et inégalités de Morse pour la d''-cohomologie*, Ann. Inst. Fourier, 35 (1985), pp. 189–229.

[6] ——, *Singular Hermitian metrics on positive line bundles*, in Complex algebraic varieties (Bayreuth, 1990), vol. 1507 of Lecture Notes in Math., Springer, Berlin, 1992, pp. 87–104.

[7] ——, *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials*, in Algebraic geometry—Santa Cruz 1995, vol. 62 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 1997, pp. 285–360.

[8] ——, *Multiplier ideal sheaves and analytic methods in algebraic geometry*, in School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), vol. 6 of ICTP Lect. Notes, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001, pp. 1–148.

[9] ——, Private communication, (2007).

[10] J.-P. DEMAILLY AND J. EL GOUL, *Hyperbolicity of generic surfaces of high degree in projective 3-space*, Amer. J. Math., 122 (2000), pp. 515–546.

[11] G.-E. Dethloff and S. S.-Y. Lu, *Logarithmic jet bundles and applications*, Osaka J. Math., 38 (2001), pp. 185–237.

[12] S. Diverio, *Differential equations on complex projective hypersurfaces of low dimension*, Compos. Math., to appear (2008).

[13] ——, *Existence of global invariant jet differentials on projective hypersurfaces of high degree*, arXiv:0802.0045v2 [math.AG], (2008).

[14] I. Dolgachev, *Weighted projective varieties*, in Group actions and vector fields (Vancouver, B.C., 1981), vol. 956 of Lecture Notes in Math., Springer, Berlin, 1982, pp. 34–71.

[15] J. El Goul, *Logarithmic jets and hyperbolicity*, Osaka J. Math., 40 (2003), pp. 469–491.

[16] W. Fulton and J. Harris, *Representation theory*, vol. 129 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[17] M. Green and P. Griffiths, *Two applications of algebraic geometry to entire holomorphic mappings*, in The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), Springer, New York, 1980, pp. 41–74.

[18] S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, vol. 2 of Pure and Applied Mathematics, Marcel Dekker Inc., New York, 1970.

[19] M. McQuillan, *Diophantine approximations and foliations*, Inst. Hautes Études Sci. Publ. Math., No. 87 (1998), pp. 121–174.

[20] J. Merker, *Demailly-Semple jets of orders 4 and 5 in dimension 2*, arXiv:0710.2393v1 [math.CV], (2007).

[21] E. Rousseau, *Équations différentielles sur les hypersurfaces de $\mathbb{P}^4$*, J. Math. Pures Appl. (9), 86 (2006), pp. 322–341.

[22] ——, *Étude des jets de Demailly-Semple en dimension 3*, Ann. Inst. Fourier (Grenoble), 56 (2006), pp. 397–421.

[23] Y.-T. Siu, *An effective Matsusaka big theorem*, Ann. Inst. Fourier (Grenoble), 43 (1993), pp. 1387–1405.

[24] ——, *Hyperbolicity in complex geometry*, in The legacy of Niels Henrik Abel, Springer, Berlin, 2004, pp. 543–566.

[25] Y.-T. Siu and S.-K. Yeung, *Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees*, Amer. J. Math., 119 (1997), pp. 1139–1172.

[26] S. Trapani, *Numerical criteria for the positivity of the difference of ample divisors*, Math. Z., 219 (1995), pp. 387–401.

[27] H. Tsuji, *Stability of tangent bundles of minimal algebraic varieties*, Topology, 27 (1988), pp. 429–442.

ABSTRACT

Let $X \subset \mathbb{P}^{n+1}$ (resp. $D \subset \mathbb{P}^n$) be a smooth projective hypersurface (resp. a smooth irreducible divisor). We show that if the degree of X (resp. of D) is large enough, then there are a lot of global sections of the bundle of invariant jet differentials of order n over X (resp. of the bundle of logarithmic invariant jet differentials of order n over the log-pair $(\mathbb{P}^n, D)$), vanishing on a fixed ample divisor. We also give effective lower bounds for the degree of X (resp. of D) in order to have these sections, at least in low dimension. Moreover, we show that these results are sharp as far as the order k of jets is concerned: there are no such global jet differentials of order $k < n$.

Finally, for (X, V) a hermitian compact complex directed manifold, given $k \geq 1$, we endow the tautological line bundle $\mathcal{O}_{X_k}(-1)$ of the k-th projectivized jet bundle, with a canonical hermitian metric whose asymptotic does depend only on the Chern curvature of V. We then apply these curvature computations to obtain a new proof of the existence of invariant jet differentials on algebraic surfaces of general type, whose Chern classes satisfy certain inequalities.

KEYWORDS

Kobayashi-hyperbolicity, directed manifold, invariant jet differential, Schur power, vanishing theorem, holomorphic global section, holomorphic Morse inequalities, projective hypersurface, logarithmic manifold, Kähler-Einstein metric, minimal surface of general type, Kobayashi's conjecture, Green-Griffiths' conjecture.

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