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Groups, Probability and Combinatorics: different aspects in Gelfand Pairs Theory

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Groups, Probability and Combinatorics: different aspects in Gelfand Pairs Theory

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Introduction

This thesis concerns with essentially various and different aspects linked to the general and huge theory of the finite **Gelfand Pairs**.

More precisely we will explore this theory from different points of view: the algebraic approach enriched by the introduction of the class of self-similar groups acting on trees that show very interesting and surprising properties, the probabilistic approach suggested by the introduction of some particular Markov chains and the combinatoric approach linked mainly to the classical theory of the association schemes and to the introduction of poset block structures on which the generalized wreath products act.

The theory of the Gelfand pairs has found fundamental relations with many mathematical fields as group theory, representation theory, harmonic analysis, coding theory, combinatorics, the theory of special functions, probability and statistics.

Clearly, there exists also a very huge literature about Gelfan Pairs theory (finite and infinite). We can mention among them the works by Ceccherini-Silberstein, Tolli and Scarabotti [**CST1, CST2**] for general settings, Letac [**Let1, Let2**], Delsarte [**Del1, Del2**] with an approach also to coding theory , Dunkl [**Dun1, Dun2, Dun4**] using special and orthogonal functions, Faraut [**Far**] for the infinite case, Figà-Talamanca [**F-T2, F-T1**] linked to Markov chains, Stanton [**Stan**] and the pioneering book by Diaconis [**Dia2**] that associates the theory to probabilistic and statistical themes.

After a brief introduction in which general settings of the theory are discussed the thesis proposes to approach some topics to which the Gelfand pairs theory can be applied: the theory of self-similar groups and the Markov chains theory.

This frame contains as integrated part the theory of the association schemes that are strictly linked to Gelfand pairs and to which we have dedicated a small Appendix (see also [**Bai**] for mare details).

Given a group G (that in this thesis we suppose finite) and a subgroup K, we denote $L(K\backslash G/K)$ the space of the bi$-K-$invariant functions which has a structure of algebra with respect to the operation of convolution. We will say that (G, K) is a Gelfand pair, if $L(K\backslash G/K)$ is commutative. Equivalently if one considers the *homogeneous space* $X = G/K$ and the action of G on the space of functions on X (denoted $L(X)$) defined as follows: $gf(x) := f(g^{-1}x)$ for every $x \in X$, $g \in G$ and

$f \in L(X)$ we get a *representation* of G in $L(X)$ (the regular representation λ_G). This will admit a decomposition into irreducible submodules, then (G, K) is a Gelfand pair if this decomposition is *multiplicity free* (i.e. if V_i and V_j are two distinct irreducible subspaces of $L(X)$ under the action of G then V_i is not isomorphic to V_j). Each module contains a special function, the *spherical function* that is an eigenvalue of the action by convolution. The set of all spherical functions (whose number equals the number of irreducible submodules) constitutes a basis for the space of $K-$invariant functions.

A special case of Gelfand pairs is the case of symmetric Gelfand pairs. This is the case when for each $g \in G$ the inverse g^{-1} belongs to the double $K-$cosets KgK. This yields many examples of Gelfand pairs, also in the case that X has a *metric* structure. In effect, we get a symmetric Gelfand pair when the action of G on X is *2-points homogeneous*, namely any two pairs of points in X with same distance can be overlapped by the action of G. This criterion allows to treat the case of the action of the full automorphism group of a rooted tree $Aut(T_n)$ on the $n-$th level L_n $(= Aut(T_n)/K$, where K is the stabilizer of a fixed vertex belonging to the $n-$th level), that presents the structure of an *ultrametric space*. In effect the richness of the automorphism that such a group presents, produces an action 2-points homogeneous on L_n (see for example [**CST2**] Cap. 7).

The first idea developed in this thesis is to see if this construction can be generalized to some families of finitely generated, non dense *discrete subgroups* of $Aut(T)$.

R. I. Grigorchuk in [**BHG**] has proven an analogous result for is celebrated group, looking to the action restricted to each level of the binary rooted tree (see, for example [**Gri1**] and [**Gri2**]), showing, in particular, that the parabolic subgroup K acts transitively on each sphere around the fixed vertex in L_n. For an approach to the infinite case for groups acting on trees (the action on the bound ∂T of the tree) see [**BG2**] and [**BG1**].

We have given the same results ([**DD1**]) for three interesting examples of self-similar groups: the Basilica group, introduced by Grigorchuk and Żuk in [**GrŻu**], the Hanoi Tower Group introduced with a self-similar presentation by Grigorchuk and Šunik in [**GrŠ1**] and the group $IMG(z^2 + i)$ introduced by Grigorchuk, Savchuk and Šunik in [**GSŠ**].

These groups have the important property to be *Iterated Monodromy Group* of complex valued functions. This relation has led to many spectacular results due to V. Nekrashevych and linked to the theory of dynamical systems, Julia sets and limit spaces (see [**Nek2**] and [**BGN**]).

For two of these groups we have found a non standard proof of the fact that they give rise to Gelfand pairs, namely that the *rigid vertex* *stabilizer* of the vertices of the first level (i.e. the set of the automorphisms acting non trivially only on the subtree rooted in a vertex) acts transitively on the respective subtrees. This yields a proof analogue to that given in the case of $Aut(T_n)$ and for Grigorchuk group, for example, does not work. Moreover the decomposition into irreducible submodules given by the action of these groups on $L(L_n)$ is the same that the decomposition gotten by $Aut(T_n)$ (easy consequence of Wielandt Lemma). These groups have the property of being *weakly branch*. Can be this result true for every weakly branch group?

The decomposition obtained is constituted by irreducible modules that are the *eigenspaces* of a particular Markov operator on $L(L_n)$ associated with a Markov chain on L_n introduced by Figà-Talamanca called Insect (see [**F-T1**]). Each state of this chain is given when an insect starting from the leftmost vertex of L_n (by homogeneity this is not important) and moving a simple random walk on the tree reaches again the level L_n. Effectively this Markov chain is invariant under the action of $Aut(T_n)$ and this fact produces the correspondence of the subspaces.

In this thesis we have shown that in this Markov chain does not appear the *cut-off phenomenon* (see [**Dia1**] and [**DD4**]), this means that the distance of the probability measure associated with the Insect Markov chain from the stationary distribution does not decay in an exponentially fast way.

On the other hand, we have generalized this Markov chain to some more general and complicate structures, namely the poset and the orthogonal block structures.

This structures constitute a generalization of the tree, i.e. given a poset $(I, \leq)$, we can associate to it a combinatoric structure and a relative group of automorphisms. In the case of the tree the poset I becomes a vertical line and the associated automorphisms group is naturally given by the *wreath product* of symmetric groups.

In the general case the mentioned group has a more complicate form, something that is between the direct product and the wreath product. This group is the *generalized wreath product* F introduced by R. A. Bailey, Cheryl E. Praeger, C. A. Rowley and T. P. Speed in [**B&al**].

These groups act on the space of functions given by the product of finite spaces indexed by the vertices of the poset I. This space is the homogeneous space obtained by considering the action of the whole group F and a relative subgroup K fixing a singleton.

The pair (F, K) is effectively a Gelfand pair. This result can be directly proven by [**B&al**], but we have used a more general method, valid in a more general context. The Markov chain that generalizes the Insect can be defined in structures that are not linked to group theory (the orthogonal blocks) but in the case of the action of F (the poset blocks) we have the correspondence of the relative irreducible submodules and eigenspaces (see [**DD2**]).

We have already said that a Gelfand pair (G, K) produces a decomposition in irreducible submodules given by the action of G on the space of function $L(G/K)$.

The last part of the third chapter moves completely from an algebraic to a probabilistic point of view. The decompositions of the permutation representation λ_G can be totally derived using particular convex combinations of Markov chains on finite sets. Starting from the case of the direct and wreath products we can construct Markov chains which are the *crossed and nested products* of single Markov chains whose decompositions are the same of those given by the groups (the terminology comes from association schemes, that showing an combinatorial analog of this situation).

Generalizing this construction, for any partition $\{1, 2, \ldots, n\} = C \sqcup N$ and any Markov chain P_i on a finite space X_i, with $i \in \{1, 2, \ldots, n\}$ we can define a new Markov chain on the product $X = X_1 \times \cdots \times X_n$ whose behaviour is crossed for the indices belonging to C and nested for the indices belonging to N. *First crested product* is the name that we have given to this intermediate Markov chain P (see [**DD3**]). The name has been inspired by a similar product introduced in [**BaCa**] for association schemes.

We have given an explicit description of the eigenspaces and the eigenvalues of P. For example, choosing $\{1, 2, \ldots, n\} = N$ and every P_i the *uniform operator* (every element in X_i can be reached in one step with same probability) gets the Insect Markov chain.

Many topics that we have treated concern with the study of a rooted tree with some branching indices.

The idea developed in the last section of chapter 3, has been inspired by the work by Ceccherini-Silberstein, Scarabotti and Tolli [**CST3**]: every vertex in the $n-$th level of a rooted tree can be regarded as a *subtree* with branching indices equal to 1 inside the whole tree T. Then one can consider, in general, the *variety* $\mathcal{V}(r, s)$ of the subtrees with branching indices $r = (r_1, \ldots, r_n)$ inside a tree with branching indices $s = (s_1, \ldots, s_n)$, where $r_i \leq s_i$ for each $i = 1, \ldots, n$. This space is the quotient of the group $Aut(T_n)$ on the stabilizer $K(r, s)$ of a particular substructure. It is known that $(Aut(T_n), K(r, s))$ is a Gelfand pair and the irreducible submodules and the relative spherical functions are given (see [**CST3**]). Our starting point has been the following question: can we deduce the analogous decomposition using as before only Markov chains? That is what we have proved in a more general contest in which for the space we can forget the ultrametric structure.

Generalizing more and more, there exists an analogous construction in poset block structures (with the tree as particular case). Do they give rise to Gelfand pairs? What is the decomposition associated? And what are the relative spherical functions?

To the first question we have given a positive answer, the others are still open.

The thesis is structured in the following order.

The first chapter constitutes a sort of survey to the general theory, where some basic theorems and fundamental tools occurring many times in the following are introduced. In Section 6 we present the Gelfand Pairs associated with the full automorphism group of the $q-$ary rooted tree of depth n and the stabilizer of a single leaf, namely $(Aut(T_n), K)$.

The second chapter gives an overview of the groups acting on rooted trees, and shows that one can get Gelfand pairs by considering particular (and well known in literature) examples of $Aut(T)$, whose action is restricted to finite levels. On the other hand, sections 4 and 5 introduce a generalization of the standard crossed and wreath products (the last one corresponds to $Aut(T_n)$), the generalized wreath product linked to more complicated structure (poset blocks). Also in this case we show that we get Gelfand pairs.

The third chapter studies the so called Insect and shows that what we have obtained by using group actions can be derived from Markov chains. Here we define a very general Markov chain on some combinatoric structures called orthogonal blocks. Section 5 reflects essentially the article [**DD3**].

CHAPTER 1

Gelfand pairs

In this chapter we introduce the general theory of finite Gelfand pairs. More precisely we give the classical definition and a characterization in terms of representation theory. Spherical functions and their interesting properties will be investigated. When it will not be specified G will denote a finite group. The source is [**CST2**].

1. First definitions

Let G be a finite group and $K \leq G$ a subgroup, denote $X = G/K$ the corresponding homogeneous space constituted by the right cosets of K in G. Then G acts on X as follows: given $g \in G$ and $hK \in X$, $g \cdot hK = ghK$, i.e. G acts by left translation on X. Equivalently, if X is a finite space on which G acts transitively and $x \in X$ is a fixed element, then we can naturally identify X with the quotient group G/K, where $K = Stab_G(x)$ is the subgroup of G that stabilizes the element x, via the map $g \to gx$.

We set $L(G) = \{f : G \to \mathbb{C}\}$ the vector space of the complex functions defined on G. Actually this space has a richer structure, in fact it is an algebra with respect the following operation $*$ of convolution: if $f_1, f_2 \in L(G)$ then

$$f_1 * f_2(g) = \sum_{h \in G} f_1(gh) f_2(h^{-1}).$$

We denote $L(X) = L(G/K)$ the set of functions defined on X (i.e. $K-$invariant on the right) and $L(K \backslash G/K)$ the set of functions defined on G that are bi-$K-$invariant, i.e.

$$L(K \backslash G/K) = \{f \in L(G) : f(kgk') = f(g)\ \forall g \in G \text{ and } \forall k, k' \in K\}.$$

Both $L(X)$ and $L(K \backslash G/K)$ are algebras with the convolution $*$.

Definition 1.1. *Let G be a finite group and $K \leq G$. The pair (G, K) is a **Gelfand pair** if the algebra $L(K \backslash G/K)$ is commutative with respect to the operation of convolution.*

The following lemma is very easy.

Lemma 1.2. *If G is commutative and $K \leq G$, then (G, K) is a Gelfand pair.*

Proof. By definition

$$\begin{aligned} f_1 * f_2(g) &= \sum_{h\in G} f_1(gh)f_2(h^{-1}) = \\ &= \sum_{h\in G} f_1(gh)f_2(gh^{-1}g^{-1}) = \\ &= \sum_{t\in G} f_1(t^{-1})f_2(gt) = \\ &= \sum_{t\in G} f_2(gt)f_1(t^{-1}) = f_2 * f_1(g). \end{aligned}$$

□

Suppose that for each $g \in G$ we get $g^{-1} \in KgK$, then for any $f \in L(K\backslash G/K)$ one has $f(g^{-1}) = f(g)$ and

$$\begin{aligned} f_1 * f_2(g) &= \sum_{h\in G} f_1(gh)f_2(h^{-1}) = \\ &= \sum_{h\in G} f_1(gh)f_2(h) = \\ &= \sum_{t\in G} f_1(t)f_2(g^{-1}t) = \\ &= \sum_{t\in G} f_2(g^{-1}t)f_1(t) = \\ &= \sum_{t\in G} f_2(g^{-1}t)f_1(t^{-1}) = \\ &= f_2 * f_1(g^{-1}) = f_2 * f_1(g), \end{aligned}$$

that implies the commutativity of the algebra $L(K\backslash G/K)$.

Definition 1.3. *Let G be a finite group and $K \leq G$ such that for any $g \in G$ one has $g^{-1} \in KgK$, then the Gelfand pair (G, K) is called symmetric Gelfand pair.*

The following lemma will give an interesting characterization of symmetric Gelfand pairs, this will be useful later. Observe that G acts on the space $X \times X$ by diagonal action (i.e. $g \cdot (x, y) = (gx, gy)$ for all $x, y \in X$ and $g \in G$).

Lemma 1.4 (Gelfand Condition). *Let $X \simeq G/K$ be a finite space with a transitive action of G and $K = Stab_G(x_0)$, where $x_0 \in X$. The pair (G, K) is a symmetric Gelfand pair if and only if for all $x, y \in X$ there exists $g \in G$ such that $g(x, y) = (y, x)$.*

Proof. We use the notation $(x, y) \sim (y, x)$ for $g(x, y) = (y, x)$. If (G, K) is symmetric, let $t, s \in G$ such that $x = tx_0$ and $y = tsx_0$ and let $k_1, k_2 \in K$ such that $s^{-1} = k_1 s k_2$. Then

$$(x, y) = t(x_0, t^{-1}y) \sim (x_0, t^{-1}y) = (x_0, sx_0).$$

Moreover

$$(x, y) = s(s^{-1}x_0, x_0) \sim (s^{-1}x_0, x_0) = (k_1 s k_2 x_0, x_0).$$

But $k_1, k_2 \in K$, so

$$(k_1 s k_2 x_0, x_0) = k_1(sx_0, x_0) \sim (sx_0, x_0) = (t^{-1}y, x_0) \sim (y, x).$$

On the other hand, as $(x_0, g^{-1}x_0) \sim (x_0, gx_0)$ there exists $k \in G$ such that $k(x_0, g^{-1}x_0) = (x_0, gx_0)$. I.e. $kx_0 = x = 0$ and $kg^{-1}x_0 = gx_0$. This implies $k \in K$ and $g^{-1}kg^{-1} \in K$. $\square$

Example 1.5.

Let G be a finite group acting by isometries on a metric space (X, d). The action of G is said *2-points homogeneous* if for all $x_1, x_2, y_1, y_2 \in X$ such that $d(x_1, x_2) = d(y_1, y_2)$ there exists $g \in G$ such that $g(x_1, x_2) = (y_1, y_2)$. Then, if the action of G on a metric space (X, d) is 2-points homogeneous and $K = Stab_G(x_0)$, with $x_0 \in X$ fixed, (G, K) is a symmetric Gelfand pair. In this case it easy to show that the $K-$orbits of K on X are the spheres of center x_0 and a function $f \in L(X)$ is $K-$invariant (i.e. bi$-K-$invariant) if and only if it is constant on the spheres.

2. Decomposition of the space $L(X)$

We have already introduced the space $L(X)$ of the complex functions on X. This space (as well as $L(G)$) is an Hilbert space with respect to the inner product $\langle\ ,\ \rangle$ defined by setting, for every $f_1, f_2 \in L(X)$:

$$\langle f_1, f_2 \rangle = \sum_{x \in X} f_1(x)\overline{f_2}(x).$$

This space is so endowed by the usual metric $\| \cdot \|_2$. The group G acts on the space $L(X)$ as follows: if $f \in L(X)$ and $g \in G$, we set $g \cdot f(x) = f(g^{-1}x)$. It is easy to verify that this is effectively an action. One can ask what is the decomposition into irreducible submodules of this representation. The answer will give a characterization of the Gelfand pairs in terms of representation of groups theory.

First of all we want to recall the following celebrate lemma

Lemma 2.1 (Schur). *Let U and V irreducible representations of a group G. Then the space $Hom_G(U, V)$ of the homomorphisms $G-$invariant intertwining U and V is trivial if U is not equivalent to V and is $\mathbb{C}u$ if U is equivalent to V by the homomorphism u.*

The first step connecting representation theory to Gelfand pairs theory is given by the following proposition

PROPOSITION 2.2. *$Hom_G(L(X), L(X)) \simeq L(K\backslash G/K)$.*

Proof. Each operator $T : L(X) \longrightarrow L(X)$ can be represented by a matrix $(r(x,y))_{x,y\in X}$ such that $Tf(x) = \sum_{x\in X} r(x,y)f(y)$. The $G-$invariance of T implies that $r(gx, gy) = r(x,y)$ for every $g \in G$. If x_0 is the point stabilized by K, there exist $g, h \in G$ such that $x = gx_0$ and $y = hx_0$. Set $z = h^{-1}gx_0 \in X$, we note that $z = z(x,y)$ is well defined modulo its $K-$orbit. In fact it is easy to verify that $z \in Kh^{-1}gx_0$. Called $\varrho(z) = r(x,y)$, ϱ is $K-$invariant. Moreover

$$\begin{aligned} Tf(x) &= \frac{1}{|K|}\sum_{h\in G} r(x, hx_0)f(hx_0) = \\ &= \frac{1}{|K|}\sum_{h\in G} r(h^{-1}gx_0, x_0)f(hx_0) = \\ &= \frac{1}{|K|}\sum_{h\in G} \varrho(h^{-1}gx_0)f(hx_0). \end{aligned}$$

Then the correspondence is given by $\varrho \longleftrightarrow (r(x,y))_{x.y\in X}$ that is algebra homomorphism. $\square$

This allows us to give an analogous definition of Gelfand pairs in terms of the representation of the group G onto the space $L(X)$.

THEOREM 2.3. *Let G be a finite group, $K \leq G$ and $X = G/K$. Suppose that $L(X) = \oplus_{i=1}^n V_i$ is the decomposition of the space into irreducible submodules under the action of G. Then $V_i \ncong V_j$ for $i \neq j$ (multiplicity free) if and only if (G, K) is a Gelfand pair.*

Proof. From Proposition 2.2 we have to show the commutativity of the algebra $Hom_G(L(X), L(X))$. But in this case Schur's Lemma implies that an homomorphism T that is $G-$invariant has the form $T = \oplus_{i=1}^n T_i$, where $T_i = c_i Id_{V_i}$. This gives the assertion. $\square$

The previous criterion is very useful for studying Gelfand pairs.

For each representation V of the group G, denote $V^K = \{v \in V : k \cdot v = v\}$ the space of $K-$invariant vectors in V.

PROPOSITION 2.4. *$Hom_G(V, L(X)) \simeq Hom_K(V, \mathbb{C}) \simeq V^K$.*

Proof. It suffices, for the first isomorphism, to define an operator $\Theta : Hom_G(V, L(X)) \longrightarrow Hom_K(V, \mathbb{C})$ as $\Theta(T)(v) = T(v)(x_0)$, where x_0 is the point stabilized by K. For the second one set $\Upsilon : Hom_K(V, \mathbb{C}) \longrightarrow V^K$ such that $\Upsilon(S) = v_0$ where, $S(v) = \langle v, v_0\rangle$ for every $v \in V$. $\square$

We have a new characterization.

THEOREM 2.5. *(G, K) is a Gelfand pairs if and only if, given an irreducible representation V of G,* $\dim(V^K) \leq 1$.

Proof. (G, K) is a Gelfand pair if and only if $L(X)$ has a multiplicity free decomposition. Now $Hom_G(V, L(X)) \simeq V^K$ is the multiplicity of the representation V in $L(X)$. □

3. Spherical functions

In this section we want to study some particular bi$-K-$invariant functions called spherical functions.

DEFINITION 3.1. *A spherical function ϕ is a bi$-K-$invariant function satisfying*

- *$f * \phi = [(\phi * f)(1_G)]\phi$ for every $f \in L(K\backslash G/K)$;*
- *$\phi(1_G) = 1$.*

It is clear from definition that the constant function $\phi \equiv 1$ on G is spherical. Actually, the number of the spherical functions is the number of the irreducible representations in the decomposition of the space $L(X)$ under the action of G. More precisely there exists a spherical functions in each of such a space.

Suppose to have different spherical functions, the following lemma specifies their mutual properties.

LEMMA 3.2. *Let ϕ and φ two distinct spherical functions. Then*

(1) $\phi(g^{-1}) = \overline{\phi(g)}$ for all $g \in G$;
(2) $\langle \phi, \varphi \rangle = 0$;

Proof.

(1) Set $\phi^*(g) = \overline{\phi(g^{-1})}$ and observe that $\phi^* * \phi = \phi^* * \phi(1_G)\phi = \|\phi\|_2^2\phi$. Since $\phi^* * \phi(g^{-1}) = \overline{\phi^* * \phi(g)}$ we get the thesis.
(2) $\phi * \varphi(g) = \phi * \varphi(1_G)\phi(g)$. On the other hand it must be equal to $\varphi * \phi(g) = \varphi * \phi(1_G)\varphi(g)$. This implies the equality of the coefficients that must be trivial, that implies the ortogonality. □

The following property will be useful later.

PROPOSITION 3.3. *A bi$-K-$invariant non trivial function is spherical if and only if*

$$\frac{1}{|K|}\sum_{k\in K}\phi(gkh) = \phi(g)\phi(h), \tag{1}$$

for all $g, h \in G$.

Proof. Suppose that (1) is satisfied by a function ϕ. First of all $\phi(1_G) = 1$ as one can verify taking $h = 1$. Moreover if $f \in L(K\backslash G/K)$ and $k \in K$

$$\begin{aligned}
\phi * f(g) &= \sum_{h\in G} \phi(gh) f(h^{-1}) = \\
&= \sum_{h\in G} \phi(gh) f(h^{-1}k) = \\
&= \sum_{t\in G} \phi(gkt) f(t^{-1}) = \\
&= \frac{1}{|K|} \sum_{t\in G} \sum_{k\in K} \phi(gkt) f(t^{-1}) = \\
&= \phi(g) \sum_{t\in G} \phi(t) f(t^{-1}) = \\
&= (\phi * f(1_G))\phi(g).
\end{aligned}$$

Viceversa suppose that ϕ is a spherical function and g and h elements of G. Set

$$F_g(h) = \sum_{k\in K} \phi(gkh).$$

Then, if $f \in L(K\backslash G/K)$ and $g' \in G$ we have

$$\begin{aligned}
F_g * f(g') &= \sum_{h\in G} \sum_{k\in K} \phi(gkg'h) f(h^{-1}) = \\
&= \sum_{k\in K} \sum_{h\in G} \phi(gkg'h) f(h^{-1}) = \\
&= \sum_{k\in K} \phi * f(gkg') = \\
&= (\phi * f)(1_G) \sum_{k\in K} \phi(gkg') = \\
&= (\phi * f)(1_G) F_g(g').
\end{aligned}$$

Analogously, if

$$G_g(h) = \sum_{k\in K} f(hkg)$$

we get

$$\begin{aligned}
F_g * f(g') &= \sum_{h\in G}\sum_{k\in K}\phi(gkg'h)f(h^{-1}) = \\
&= \sum_{h\in G}\sum_{k\in K}\phi(gh)f(h^{-1}kg') = \\
&= \phi * G_{g'}(g) = \\
&= \phi * G_{g'}(1_G)\phi(g) = \\
&= |K|(\phi * f)(g')\phi(g) = \\
&= |K|(\phi * f)(1_G)\phi(g')\phi(g).
\end{aligned}$$

This implies $\phi * f(1_G) \neq 0$ and $F_g(g') = |K|\phi(g')\phi(g)$, that is the thesis. $\square$.

Now we can prove the following

THEOREM 3.4. *Let $X = G/K$ and $L(X) = \oplus_{i=0}^{n} V_i$ be the decomposition in irreducible submodules. Each V_i contains a spherical function ϕ_i, and coincides with the space spanned by ϕ_i.*

Proof. The space V_i contains a $K-$invariant vector v_i. Assume that $\|v_i\|_2 = 1$. Set $\phi(g) = \langle\lambda(g)v_i, v_i\rangle$, where λ is the representation associated with G. By definition $\phi(1_G) = 1$ and so we have to prove that

$$\sum_{k\in K}\phi(gkh) = \phi(g)\phi(h).$$

We have

$$\begin{aligned}
\sum_{k\in K}\phi(gkh) &= \sum_{k\in K}\langle\lambda(gkh)v_i, v_i\rangle = \\
&= \sum_{k\in K}\langle\lambda(g)v_i, \lambda(h^{-1}k^{-1})v_i\rangle = \\
&= \langle\lambda(g)v_i, \sum_{k\in K}\lambda(h^{-1}k^{-1})v_i\rangle = \\
&= \langle\lambda(g)v_i, v_i(h)\rangle.
\end{aligned}$$

Since v_i' is $K-$invariant $v_i(h) = c(h)v_i$ we get

$$\sum_{k\in K}\phi(gkh) = \phi(g)c(h).$$

Set $g = 1_g$ it follows $c(h) = \phi(h)$.

We call S_i the space spanned by ϕ_i under the action of G. Evidently, $L(X) \leq \oplus_{i=0}^{n} S_i$. But S_i is a sub-representation of $L(X)$ and so we have the assert. $\square$

Observe that we can now define a spherical function ϕ as a function in $L(X)$ that is $K-$invariant, belonging to an irreducible $G-$invariant subspace and such that $\phi(x_0) = 1$.

4. Irreducible submodules

The following fundamental fact is well known

LEMMA 4.1 (Wielandt). *Let G be a finite group, $K \leq G$ and $X = G/K$ the corresponding homogeneous space. If $L(X) = \oplus_{i=0}^{n} m_i V_i$ is the decomposition into irreducible submodules and m_i the multiplicity of the representation V_i, then $\sum_{i=0}^{n} m_i^2$ equals the number of $K-$orbits on X.*

COROLLARY 4.2. *Suppose G, K and X as before. If $L(X) = \oplus_{i=0}^{m} V_i$ and $m+1$ is the number of the $K-$orbits on X, then the V_i's are irreducible and (G, K) is a Gelfand pair.*

Proof. Considering the decomposition into irreducible submodules we have $m+1 \leq \sum_{i=0}^{n} m_i \leq \sum_{i=0}^{n} m_i^2$. Then by Wielandt's Lemma $m = n$ and $m_i = 1$ for each i. $\square$

5. The spherical Fourier formula and Garsia theorem

Let ϕ_i be the spherical function belonging to V_i and set $\dim V_i = d_i$.

DEFINITION 5.1. *The spherical transform $\mathcal{F}f$ of a $K-$invariant function in $L(X)$ is the function*

$$(\mathcal{F}f)(i) = \sum_{x \in X} f(x)\overline{\phi_i(x)} = \langle \mathcal{F}f, \phi_i \rangle.$$

If one knows the spherical Fourier transform of a function can find the function by the following inverse formula

$$f(x) = \frac{1}{|X|} \sum_{i=0}^{n} d_i (\mathcal{F}f)(i) \phi_i(x).$$

This notion is connected to the spectral analysis of a $G-$invariant operator defined on $L(X)$

DEFINITION 5.2. *Let $T : L(X) \longrightarrow L(X)$ be an operator and $T(f) = \frac{1}{|K|} f' * \psi'$, where f' and ψ' are the lifting function on G corresponding to f and ψ. Then $\psi : X \longrightarrow \mathbb{C}$ is called convolution kernel.*

PROPOSITION 5.3. *Suppose $T \in Hom(L(X), L(X))$ with corresponding convolution kernel ψ. Then V_i is an eigenspace of T with associated eigenvalue $(\mathcal{F}\psi)(i)$.*

Proof. From Schur's Lemma T has V_i as eigenspace, moreover

$$(T\phi_i)(gx_0) = \frac{1}{|K|}(\phi_i' * \psi')(g) = \frac{1}{|K|}(\phi_i' * \psi')(1_G)\phi_i'(g) = (\mathcal{F}\psi)(i)\phi_i(gx_0).$$

□

The spherical Fourier transform allows to characterize symmetric Gelfand pairs in terms of spherical functions

THEOREM 5.4 (Garcia). *A Gelfand pair (G, K) is symmetric if and only if the sparical functions are real valued.*

Proof. Let χ_{KgK} the characteristic function of the set KgK. Then

$$\mathcal{F}(\chi_{KgK})(i) \sum_{x \in KgK} \overline{\phi_i(x)} = |KgK|\overline{\phi_i(g)}.$$

On the other hand

$$\mathcal{F}(\chi_{Kg^{-1}K})(i) \sum_{x \in Kg^{-1}K} \overline{\phi_i(x)} = |K^g{-}1K|\overline{\phi_i(g^{-1})} = |KgK|\phi_i(g).$$

This implies from inversion formula that (G, K) is symmetric $(KgK = Kg^{-1}K$ for every $g \in G)$ if and only if ϕ_i is real valued. □

6. The case of the full automorphisms group

In this section we study the group of the automorphisms of a $q-$ary rooted tree in relation with the theory of Gelfand pair.

Consider the infinite $q-$ary rooted tree, i.e. the rooted tree in which each vertex has q children. We will denote this tree by T. If $X = \{0, 1, \dots, q-1\}$ is an alphabet of q elements, X^* is the set of all finite words in X. Moreover, we can identify the set of infinite words in X with the elements of the boundary of T. Each vertex in the n-th level L_n of T will be identified with a word of length n in the alphabet X.

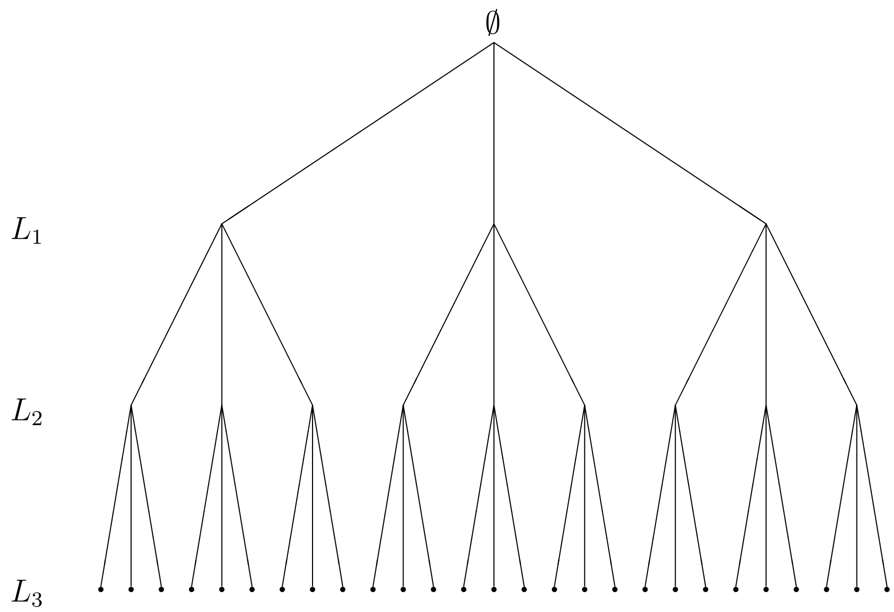


Fig.1. The ternary rooted tree of depth 3.

The set L_n has a particular metric structure.

Definition 6.1. *Let X be a set and $d : X \times X \longrightarrow [0, +\infty)$ a function. Then (X, d) is an ultrametric space if*

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$, for every $x, y \in X$;
(3) $d(x, z) \leq \max d(x, y), d(y, z)$, for every $x, y, z \in X$.

Observe that each ultrametric space is, in particular, a metric space.

The set L_n can be endowed with an ultrametric distance d, defined in the following way: if $x = x_0 \ldots x_{n-1}$ and $y = y_0 \ldots y_{n-1}$, then

$$d(x, y) = n - \max\{i : x_k = y_k,\ \forall k \leq i\}.$$

We observe that $d = d'/2$, where d' denotes the usual geodesic distance. Moreover it is clear that T is a poset with respect to the relation $>$ of *being ancestor*.

Definition 6.2. *An automorphism g of T, is a bijection $g : T \longrightarrow T$ such that if $x > y$ then $g(x) > g(y)$, for every $x, y \in T$.*

The whole group of the automorphisms of T will be denoted by $Aut(T)$. From the definition it is clear that $Aut(T)$ preserves each level L_n.

In this way (L_n, d) becomes an ultrametric space on which the automorphisms group $Aut(T)$ acts isometrically. Note that the diameter of (L_n, d) is exactly n.

To indicate the action of an automorphism $g \in Aut(T)$ on a vertex x, we will use also the notation x^g.

Every automorphism $g \in Aut(T)$ can be represented by its *labelling*. The labelling of $g \in Aut(T)$ is realized as follows: given a vertex $x = x_0 \ldots x_{n-1} \in T$, we associate with x a permutation $g_x \in S_q$ giving the action of g on the children of x. Formally, the action of g on the vertex labelled by the word $x = x_0 \ldots x_{n-1}$ is

$$x^g = x_0^{g_\emptyset} x_1^{g_{x_0}} \ldots x_{n-1}^{g_{x_0 \ldots x_{n-2}}}.$$

The group $Stab_{Aut(T)}(n)$ denotes the subgroup of the automorphism fixing all the vertices of the $n-$th level (and so of the levels L_k, with $k \leq n$). If one considers the action of the full automorphisms group of the $q-$ary rooted tree

$$Aut(T_n) = Aut(T)/Stab_{Aut(T)}(n)$$

on L_n one gets, for every n, a 2-points homogeneous action, giving rise to the symmetric Gelfand pair $(Aut(T_n), K_n)$, with $K_n = Stab_{Aut(T_n)}(0^n)$ is, as usual, the subgroup stabilizing the vertex 0^n. Observe that the K_n orbits coincide, in this case, with the sets $\Lambda_k = \{x \in L_n : d(x_0, x) = k\}$, for $k = 0, 1, \ldots, n$, i.e. the spheres of center x_0 of ray k.

Theorem 6.3. *The action of $Aut(T_n)$ on (L_n, d) is 2-points homogeneous.*

Proof. We use induction on the depth n of the tree T.

$n = 1$. The assertion follows from the 2-transitivity of the group S_q.

$n > 1$. Let (x, y) and (x', y') be pairs of vertices in L_n with $d(x, y) = d(x', y')$. If $d(x, y) < n$, then vertices x and y belong to the same subtree of T and so $x_1 = y_1$. Analogously for x' and y'. Applying, if necessary, the transposition $(x_1 x_1') \in S_q$, we can suppose $x_1 = y_1 = x_1' = y_1'$, so that x, x', y and y' belong to the same subtree of depth less or equal to $n - 1$, and then induction works.

Finally, consider the case $d(x, y) = d(x', y') = n$. Consider the automorphism $g \in Aut(T)$ such that $g(x_1) = x_1'$ and $g(y_1) = y_1'$ and which acts trivially on the other vertices of L_1. Now we have that x and x' belong to the same subtree T'. Analogously y and y' belong to the same subtree T'', with $T' \neq T''$. The restriction of $Aut(T_n)$ to T' and T'' respectively acts transitively on each level. So there is an automorphism g' of T' carrying x to x' and acting trivially on T'' and analogously there is an automorphism g'' of T'' carrying y to y' and trivial on T'. The assertion is proved. $\square$

The decomposition of the space $L(L_n)$ under the action of $Aut(T_n)$ is known.

Denote $W_0 \cong \mathbb{C}$ the space of the constant functions and for every $j = 1, \ldots, n$, define the following subspace

$$W_j = \{f \in L(L_n) : f = f(x_1, \ldots, x_j), \ \ \sum_{x=0}^{q-1} f(x_1, x_2, \ldots, x_{j-1}, x) \equiv 0\}$$

of dimension $q^{j-1}(q-1)$.

PROPOSITION 6.4. *The spaces W_j's are $Aut(T_n)$-invariant, pairwise orthogonal and the following decomposition holds*

$$L(L_n) = \bigoplus_{j=0}^{n} W_j.$$

Proof. First of all we prove that if $f \in W_j$ then $g \cdot f \in W_j$. In effect

$$g \cdot f(x_1, \ldots, x_n) = f(g_\emptyset^{-1}(x_1), \ldots, g_{x_1,\ldots,x_{n-1}}^{-1}(x_n))$$

and so

$$\begin{aligned}\sum_{x \in X} g \cdot f(x_1, \ldots, x_{j-1}, x) &= \sum_{x \in X} f(g_\emptyset^{-1}(x_1), \ldots, g_{x_1,\ldots,x_{j-2}}^{-1}(x_{j-1}), g_{x_1,\ldots,x_{j-1}}^{-1}(x)) = \\ &= \sum_{y \in X} f(g_\emptyset^{-1}(x_1), \ldots, g_{x_1,\ldots,x_{j-2}}^{-1}(x_{j-1}), y) = 0.\end{aligned}$$

Let f be in W_j and f' in $W_{j'}$, with $j < j'$.

$$\begin{aligned}\langle f, f' \rangle &= \sum_{x_1=0}^{q-1} \cdots \sum_{x_n=0}^{q-1} f(x_1, \ldots, x_n)\overline{f'(x_1, \ldots, x_n)} = \\ &= q^{n-j'} \sum_{x_1=0}^{q-1} \cdots \sum_{x_{j'-1}=0}^{q-1} f(x_1, \ldots, x_j) \sum_{k=0}^{q-1} \overline{f'(x_1, \ldots, x_{j'-1}, k)} = 0.\end{aligned}$$

This gives that the W_j's are orthogonal. Moreover

$$\dim L_n = q^n = \sum_{j=0}^{n} q^{j-1}(q-1)$$

and so these spaces fill all $L(L_n)$. $\square$

Since the spheres centered at $x_0 = 0^n$ (and so the K_n-orbits) are exactly $n+1$, we have from Lemma 4.2 that the subspaces W_j's are irreducible.

There exists a complete description of the corresponding spherical functions.

PROPOSITION 6.5. *For every $j = 0, \ldots, n$ we the spherical function ϕ_j in the space W_j is given by*

$$\phi_j(x) = \begin{cases} 1 & d(x, x_0) < n - j + 1; \\ \frac{1}{1-q} & d(x, x_0) = n - j + 1; \\ 0 & d(x, x_0) > n - j + 1. \end{cases}$$

Proof. Since each ϕ_j is defined in terms of distance and the spheres of center x_0 and ray k are the K_n-orbits we have that the ϕ_j's are K_n-invariant. Moreover $\phi_j(x_0) = 1$ by definition. We have to prove that $\phi_j \in W_j$. But $\phi_j(x_1, \ldots, x_j, x) = \phi_j(x_1, \ldots, x_j, y)$, for every $\underline{x}$ and $\underline{y}$ words of length $n - j - 1$, because the condition $d(x, x_0) < n - j + 1$ is equivalent to $x_1 = \ldots = x_j = 0$, $d(x, x_0) = n - j + 1$ to $x_1 = \ldots = x_{j-1} = 0, x_j \neq 0$ and $d(x, x_0) > n - j + 1$ to the resting cases. Moreover

$$\sum_{x \in X} \phi_j(x_1, \ldots, x_{j-1}, x) = \sum_{i=0}^{j-1} |\Lambda_i| - \frac{1}{q-1} |\Lambda_j| = 0.$$

□

7. Some constructions

Let (G, K) and (F, H) be finite Gelfand pairs on the homogeneous spaces $X \simeq G/K$ and $Y = F/H$, we can ask if it is possible to combine the two constructions to get another Gelfand pair.

The following constructions are well known.

If we denote $G \times F$ the direct product of G and F, and $K \times H$ the direct product of the respective stabilizers subgroups, it is easy to prove that $(G \times F, K \times H)$ is a Gelfand pair.

The decomposition associated with the action of $\mathcal{G}$ on $L(X \times Y)$ is

$$L(X \times Y) = \left(\bigoplus_{i=0}^{n} V_i \right) \otimes \left(\bigoplus_{j=0}^{m} W_j \right),$$

where $L(X) = \oplus_{i=0}^{n} V_i$ and $L(Y) = \oplus_{j=0}^{m} W_j$ are the decompositions into irreducibles submodules under the actions of G and F respectively. The spherical functions will be given by the tensorial product of the spherical functions of each pair. This construction is called direct product of Gelfand pairs.

Analogously, if we perform the wreath product of the groups G and F we get a the pair $(G \wr F, J)$, where

$$J = \{(k, f) \in G \wr F : \ k \in K, f(x_0) \in H\}$$

is the stabilizer of the vertex $(x_0, y_0) \in X \times Y$ under the action of $G \wr F$. Recall that $G \wr F = G \ltimes F^X = \{(g, f) : \ f : X \longrightarrow F, \ g \in G\}$.

LEMMA 7.1. *The orbits of $X \times Y$ under the action of J are*

$$X \times Y = \left[\bigsqcup_{i=1}^{n} (\Lambda_i \times Y) \right] \sqcup \left[\bigsqcup_{j=0}^{m} \{x_0\} \times \Upsilon_j \right],$$

where $X = \{x_0\} \sqcup_{i=1}^{n} \Lambda_i$ and $Y = \{y_0\} \sqcup_{J=1}^{m} \Upsilon_j$ are the decompositions of X and Y under K and H respectively.

Proof. We have $J(x_0, y) = \{(x_0, f(x_0)y),\ f(x_0) \in H\} = \{x_0\} \times \Upsilon_j$. Analogously if $x \neq x_0$, we have $J(x, y) = \{(kx, f(x)y),\ k \in K \text{ and } f(x_0) \in F\} = \Lambda_i \times Y$. $\square$

We have the following theorem

THEOREM 7.2. *(1) The decomposition into irreducibles submodules is*

$$L(X \times Y) = \left[\bigoplus_{i=0}^{n} (V_i \otimes W_0) \right] \oplus \left[\bigoplus_{j=1}^{m} (L(X) \otimes W_j) \right];$$

(2) the spherical functions have the form

$$\{\phi_i \otimes \psi_0, \delta_{x_0} \otimes \psi_j : i = 0, 1, \ldots, n;\ j = 1, \ldots, m\},$$

where ϕ_i and ψ_j are the spherical functions of the initial Gelfand pairs and δ_{x_0} is the Dirac function at the vertex x_0.

Proof. 1) Consider the element $(g, f) \in G \wr F$ acting on the function $\mathcal{G} \otimes \mathcal{F} \in L(X \times Y)$ as

$$\mathcal{G} \otimes \mathcal{F}(x, y) = \mathcal{G}(x)\mathcal{F}(y).$$

Then

$$\begin{aligned} (g, f)(\mathcal{G} \otimes \mathcal{F})(x, y) &= (\mathcal{G} \otimes \mathcal{F}) \left[(g, f)^{-1}(x, y) \right] = \\ &= (g\mathcal{G})(x) \left[f(x)\mathcal{F} \right](y). \end{aligned}$$

Now let $v \otimes \mathbf{1} \in V_i \otimes W_0$, then $(g, f)(v \otimes \mathbf{1}) = (gv \otimes \mathbf{1}) \in V_i \otimes W_0$.

Let $\delta_x \otimes w \in L(X) \otimes W_j$, then $(g, f)(\delta_x \otimes w) = \delta_{gx} \otimes f(x)w \in L(X) \otimes W_j$. This implies the invariance of the subspaces. Their number coincides with the number on the $J-$orbits on $X \times Y$, so these spaces are irreducibles.

2) Follows from the trivial $J-$invariance of the functions in the statement. $\square$

We observe that $(x, y) \in X \times Y$ can be identified with a leaf of the tree of depth 2. We have already considered the Gelfand pair associated with the action of the wreath product on a vertex of the tree.

One can consider the following generalizing construction (generalized Johnson scheme) due to Ceccherini-Silberstein, Scarabotti and Tolli in [**CST3**]: let Y be an homogeneous space associated with a Gelfand pair (F, H). The space X is finite of cardinality n, say

$X = \{1, 2, \ldots, n\}$. For every $h = 1, \ldots, n$, denote by Ω_h the set of $h-$subsets of X, so that $|\Omega_h| = \binom{n}{h}$.

The decomposition of $L(Y)$ into irreducible submodules, under the action F is

$$L(Y) = \bigoplus_{j=0}^{m} W_j.$$

For every $h = 1, \ldots, n$, consider the space

$$\Theta_h = \{(A, \theta) : A \in \Omega_h \text{ and } \theta \in Y^A\},$$

i.e. the space of functions whose domain is a $k-$subset of X and which take values in Y.

On Θ_h acts the group $S_n \wr F$. Given $\theta \in \Theta_h$ and $(\pi, f) \in S_n \wr F$ we have

$$[(\pi, f)\theta]\,(j) = f(j)\theta(\pi^{-1}j),$$

for every $j \in \pi dom\theta$.

Let us denote $C(h, m+1)$ the set of the weak $(m+1)-$composition of h, i.e. the elements $\mathbf{a} = (a_0, a_1, \ldots, a_m)$ such that $a_0 + \cdots + a_m = h$.

In order to get a basis for the space $L(Y^A)$, for every $A \in \Omega_h$, we introduce some special functions that we will call fundamental functions.

Definition 7.3. *Suppose that $A \in \Omega_h$ and that $\mathcal{F}^j \in L(Y)$ for every $j \in A$. Suppose also that each $\mathcal{F}^j$ belongs to an irreducible submodules of the action of F and set $a_i = |\{j \in A : \mathcal{F}^j \in W_i\}|$. Then the tensor product $\bigotimes_{j \in A} \mathcal{F}^j$ will be called a fundamental function of type $\underline{a} = (a_0, a_1, \ldots, a_m)$ in $L(Y^A)$.*

In other words, we have

$$(\bigotimes_{j \in A} \mathcal{F}^j)(\theta) = \prod_{j \in A} \mathcal{F}^j(\theta(j)),$$

for every $\theta \in Y^A$. We also set $\ell(\underline{a}) = a_1 + \cdots + a_m = h - a_0$.

Given $\mathbf{a} \in C(h, m+1)$ and $A \in \Omega_h$ a composition of A of type $\mathbf{a}$ is a sequence $\mathbf{A} = (A_0, A_1, \ldots, A_m)$ of subsets that are a partitions of A and such that $|A_i| = a_i$ for every $i = 0, 1, \ldots, m$. The set of the compositions of type $\mathbf{a}$ is denoted $\Omega_{\mathbf{a}}(A)$.

It is known that the action of $S_n \wr F$ on $L(\Theta_h)$ gives rise to a Gelfand pair. To give the associated decomposition we need the following definitions.

The subspace of $L(Y^A)$ spanned by the tensor products $\bigotimes_{j \in A} \mathcal{F}^j$ such that $\mathcal{F}^j \in W_i$ for every $j \in A_i, i = 0, 1, \ldots, m$ is denoted by $W_{\mathbf{a}}(\mathbf{A})$.

Define

$$W_{h,\mathbf{a}} = \bigoplus_{A\in\Omega_h} \bigoplus_{\mathbf{A}\in\Omega_{\mathbf{a}}(A)} W_{\mathbf{a}}(\mathbf{A}).$$

We denote $S^{n-h,h}$ the irreducible representation of the symmetric group S_n acting on the space $L(\Omega_h)$, given by

$$S^{n-k,k} = L(\Omega_k) \cap ker d,$$

where $d : L(\Omega_h) \longrightarrow L(\Omega_{h-1})$ is the Radon transform defined as

$$(d\gamma)(B) = \sum_{A\in\Omega_h : A=B\cup\{j\}} \gamma(A).$$

Definition 7.4. *For* $0 \le k \le \frac{n-\ell(\boldsymbol{a})}{2}$ *define*

$$W_{h,\boldsymbol{a},k} = Ind_{S_{n-\ell(\boldsymbol{a})}\wr\times S_{a_1}\wr F\times\ldots\times S_{a_m}\wr F}^{S_n\wr F} S^{n-\ell(\boldsymbol{a})-k,k} \otimes W_1^{\otimes a_1} \otimes \cdots \otimes W_m^{\otimes a_m}.$$

In [**CST3**] is proven the following theorem

Theorem 7.5. *The decomposition of* $L(\Theta_h)$ *into irreducible representations under the action* $S_n \wr F$ *is given by*

$$L(\Theta_h) = \bigoplus_{\boldsymbol{a}\in C(h,m+1)} \bigoplus_{k=0}^{\min\{n-h,h-\ell(\boldsymbol{a})\}} W_{h,\boldsymbol{a},k}.$$

Remark 7.6.

The starting point for the previous version is the consideration of *substructures* in a discrete space, i.e. consider subtrees with assigned ramification indices in a rooted homogeneous tree (see [**CST3**]). In effect if we work on the $n-$th level L_n of a rooted tree, we can identify each vertex $x \in L_n$ with the geodesic path that connects it to the root. This is a *subtree* with branching indices $(1, 1, \ldots, 1)$ in the whole tree.

If we choose different branching indices $r = (r_1, \ldots, r_n)$ in a rooted tree with branching indices $m = (m_1, \ldots, m_n)$, where $0 < r_i \le m_i$ for every $i = 1, \ldots, n$ we can consider the variety $\mathcal{V}(m, r)$ of such a subtrees.

This space is the quotient of the full automorphism group $Aut(T_n)$ by the stabilizer $K(m, r)$ of a fixed subtree $\mathcal{T}$.

This is a Gelfand pair. But, looking on the first level, we can think each of the s_1's indices in the subtree as the domain of a function whose image is a subtree. This means that in this case Θ is defined on all the r_1 subsets of m_1 and the image of every vertex is a subtree again.

This recurrence justifies the utilization of the space Y, that has, in this case, the same ultrametric structure.

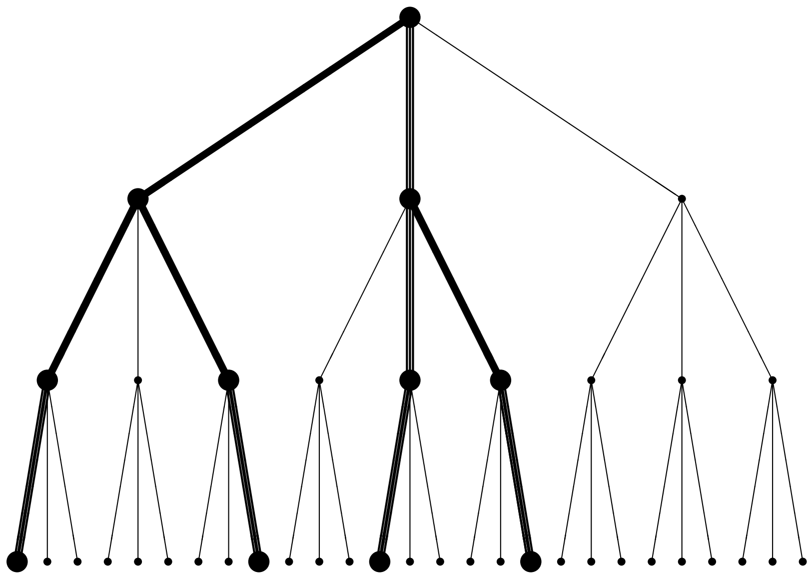


Fig. 2: A tree of type (3,3,3) with a subtree of type (2,2,1).

There exists a generalization of that, considering different structures that have the ultrametric space as particular case. The idea developed in the end of the following chapter is to study this structures in relation with the Gelfand pairs.

CHAPTER 2

Self-similar groups and generalized wreath products

In this chapter we will study a particular class of subgroups of the whole automorphism group of the rooted tree: the class of self-simlar groups. The famous Grigorchuk group, for example, belongs to this class as well as other groups having interesting and exotic properties. A new course is the realization of such a groups as Iterated Monodromy Groups (IMG) of some complex rational functions (see [**Nek1**] or [**Nek2**] for further suggestions).

1. General settings

If we consider a countable subgroup of $Aut(T)$ and the relative action on L_n, we can ask if it is possible to find the same results about Gelfand pairs obtained for the full automorphisms group. In some cases the answer is positive. In what follows we will investigate this problem.

Recall that a group G is spherically transitive on the rooted tree T if it is transitive on each level L_n of T.

The fundamental tool will be the following easy lemma.

LEMMA 1.1. *Let G act spherically transitively on T. Denote by G_n the quotient group $G/Stab_G(n)$ and by K_n the stabilizer in G_n of a fixed leaf $x_0 \in L_n$. Then the action on L_n is 2-points homogeneous if and only if K_n acts transitively on each sphere of L_n.*

Proof. Suppose that K_n acts transitively on each sphere of L_n and consider the elements x, y, x' and y' such that $d(x, y) = d(x', y')$. Since the action of G_n is transitive, there exists an automorphism $g \in G_n$ such that $g(x) = x'$. Now $d(x', g(y)) = d(x', y')$ and so $g(y)$ and y' are in the same sphere of center x' and radius $d(x', y')$. But K_n is conjugate with $Stab_{G_n}(x')$ and so there exists an automorphism $g' \in Stab_{G_n}(x')$ carrying $g(y)$ to y'. The composition of g and g' is the required automorphism.

Suppose now that the action of G_n on L_n is 2-points homogeneous and consider two elements x and y in the sphere of center x_0 and radius i. Then $d(x_0, x) = d(x_0, y) = i$. So there exists an automorphism $g \in Stab_{G_n}(x_0)$ such that $g(x) = y$. This completes the proof. $\square$

We introduce some definitions for the rest of the theory. Recall that if $G \leq Aut(T)$ acts on the tree T, we can study the action on the first level L_1 and consider the action of G restricted to each subtree T_x, $x \in X$ (rooted at x). The automorphism induced on T_x can be regarded as an automorphism of the whole tree, via the identification of T_x with T. Is this restricted automorphism still in G?

Definition 1.2. *A group G acting on T is* self-similar *if for every $g \in G$, $x \in X$, there exist $g_x \in G$, $x' \in X$ such that $g(xw) = x' g_x(w)$ for all $w \in X^*$.*

Moreover, a self-similar group G can be embedded into the wreath product $G \wr X = (G^q) \rtimes S_q$, where S_q is the symmetric group on q elements.

The self-similar groups are strictly linked to the theory of automata, see [**Nek2**]

We recall now that, for an automorphisms group $G \leq Aut(T)$, the *stabilizer* of the vertex $x \in T$ is the subgroup of G defined as $Stab_G(x) = \{g \in G : g(x) = x\}$ and the stabilizer of the n-th level is $Stab_G(n) = \bigcap_{x \in L_n} Stab_G(x)$. Observe that $Stab_G(n)$ is a normal subgroup of G of finite index for all $n \geq 1$. In particular, an automorphism $g \in Stab_G(1)$ can be identified with the elements $g_i, i = 0, 1, \ldots, q-1$ that describe the action of g on the respective subtrees T_i rooted at the vertex i of the first level. So we get the following embedding

$$\varphi : Stab_G(1) \longrightarrow \underbrace{Aut(T) \times Aut(T) \times \cdots \times Aut(T)}_{q \text{ times}}$$

that associates with g the $q-$ple $(g_0, g_1, \ldots, g_{q-1})$.

Definition 1.3. *G is said to be* fractal *if the map*

$$\varphi : Stab_G(1) \longrightarrow G \times G \times \cdots \times G$$

is a subdirect embedding, that is it is surjective on each factor.

Lemma 1.4. *If G is transitive on L_1 and fractal then G is spherically transitive (i.e. it acts transitively on each level).*

Proof. Suppose that T is the $q-$ary rooted tree. We can switch the subtrees $T_0, \ldots, T_{q-1}$. The restriction of $Stab_G(1)$ on T_i is G and so by an inductive recurrence we have the claim. $\square$

In what follows we will often use the notion of rigid stabilizer. For a group G acting on T and a vertex $x \in T$, the *rigid vertex stabilizer* $Rist_G(x)$ is the subgroup of $Stab_G(x)$ consisting of the automorphisms acting trivially on the complement of the subtree T_x rooted at x. Equivalently, they have a trivial labelling at each vertex outside T_x. The rigid stabilizer of the n-th level is defined as $Rist_G(n) = \prod_{x \in L_n} Rist_G(x)$.

In contrast to the level stabilizers, the rigid level stabilizers may have infinite index and may even be trivial. We observe that if the action of G on T is spherically transitive, then the subgroups $Stab_G(x)$, $x \in L_n$ are all conjugate, as well as the subgroups $Rist_G(x)$, $x \in L_n$.

We recall the following definitions for spherically transitive groups (see, for more details, [**BGŠ**]).

Definition 1.5. *G is* regular weakly branch *on K if there exists a normal subgroup $K \neq \{1\}$ in G, with $K \leq Stab_G(1)$, such that $\varphi(K) > K \times K \times \cdots \times K$. In particular G is* regular branch *on K if it is regular weakly branch on K and K has finite index in G.*

We observe that this property for the subgroup K is stronger than fractalness, since the map φ is surjective on the whole product $K \times K \times \cdots \times K$.

Definition 1.6. *G is* weakly branch *if $Rist_G(x) \neq \{1\}$, for every $x \in T$ (this automatically implies $|Rist_G(x)| = \infty$ for every x). In particular, G is* branch *if $[G : Rist_G(n)] < \infty$ for every $n \geq 1$.*

Example 1.7 (Adding Machine).

Let G be the self-similar group acting on the binary rooted tree generated by the automorphism $a = (a, 1)\varepsilon$, where ε denotes the nontrivial permutation of the group S_2.

It is easy to check that the following identities hold:

$$a^{2k} = (a^k, a^k), \qquad a^{2k+1} = (a^k, a^{k+1})\varepsilon. \tag{2}$$

In particular, the first level stabilizer is given by $Stab_G(1) =< a^2 >$, with $a^2 = (a, a)$.

From (2) it follows that

$$Stab_G(n) =< a^{2^n} > .$$

Moreover, since G is abelian, one has $Stab_G(n) = Stab_G(x)$ for all $x \in L_n$. Formulas (2) tells us that the element a^{2^n} has the labelling $g_x = \varepsilon$ at each vertex $x \in L_n$ and the labelling $g_y = 1$ at each vertex $y \in L_i$, for $i < n$. Therefore $a^{2^n} \notin Rist_G(n)$ and all its powers do not belong to $Rist_G(n)$ too. So $Rist_G(n) = \{1\}$ for every $n \geq 1$. So this is an example where the subgroups $Stab_G(n)$ and $Rist_G(n)$ do not coincide, showing that $Rist_G(n)$ can also be trivial.

G is fractal, in fact $Stab_G(1) =< a^2 >$, where $a^2 = (a, a)id$ and so the application from $Stab_G(1)$ to $G \times G$ is surjective on each factor. This implies that G is spherically transitive. Observe that, for each $n \in \mathbb{N}$, we have $[G : Stab_G(n)] = 2^n$, on the other hand $[G : Rist_G(n)] = \infty$.

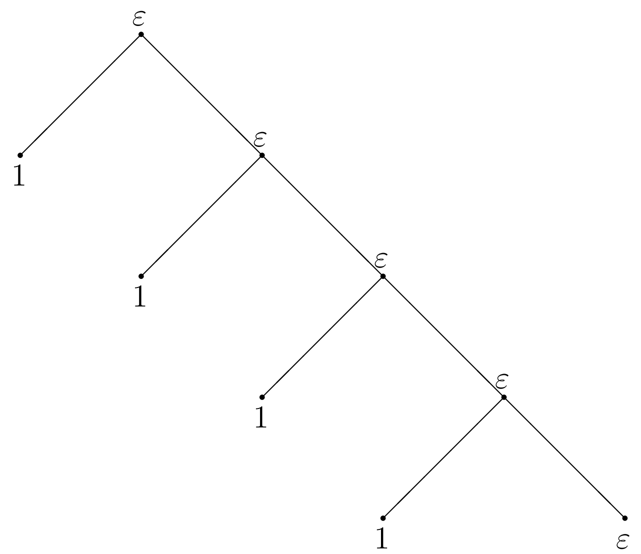


Fig.3. Labelling of a.

From this we can deduce the isomorphism $G \simeq \mathbb{Z}$. The group quotient $G/Stab_G(n) \simeq \mathbb{Z}_{2^n}$. In this case the parabolic group K_n is trivial, since $|L_n| = 2^n$ and so we get for every n a Gelfand pair $(\mathbb{Z}_{2^n}, 1)$ (the group is commutative) that is not symmetric (the spherical functions correspond to the characters that, in general, are not real).

2. The Basilica group

The Basilica group B was introduced by R. I. Grigorchuk and A. Żuk in [**GrŻu**] This group was the first example of an amenable group of exponential growth that cannot be obtained as limit of groups of sub-exponential groups (see [**BaVi**] or the interesting paper [**Kai**]).

The Basilica group is generated by the automorphisms a and b having the following self-similar form:

$$a = (b, 1), \quad b = (a, 1)\varepsilon$$

where ε denotes the nontrivial permutation of the group S_2. In the following figure the labelling of the automorphisms a and b are presented. Observe that the labelling of each vertex not contained in the leftmost branch of the tree is trivial.

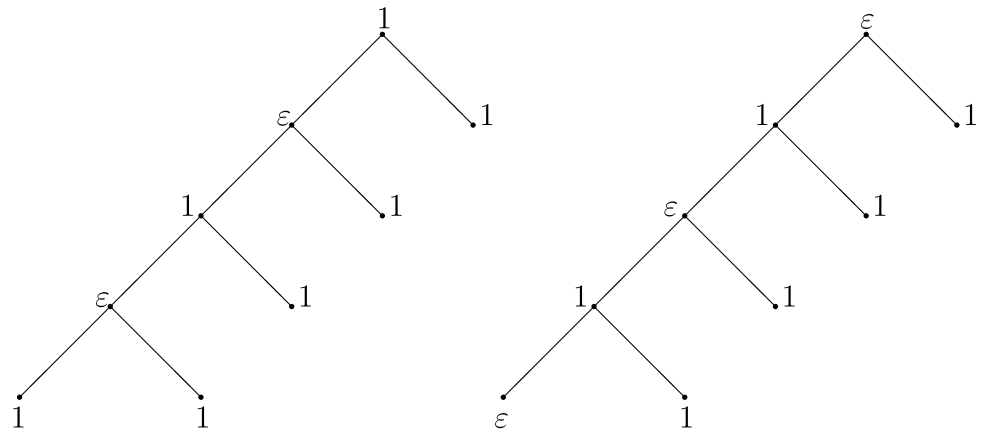


Fig.4. Labelling of the generators a and b.

EXAMPLE 2.1.

Consider $x_0 = 000...$ and let us study the action of the generators of B on x_0:

$$a(000...) = 0b(00...) = 01a(0...) = 010...$$

and

$$b(000...) = 1a(00...) = 10b(0...) = 101...$$

The product can be performed in according with the embedding into the wreath product as:

$$ab = (b,1)id(a,1)\varepsilon = (ba,1)\varepsilon = ((a,b)\varepsilon,(1,1)id)\varepsilon = ..,$$

and

$$ba = (a,1)\varepsilon(b,1)id = (a,b)\varepsilon = ((b,1)id,(a,1)\varepsilon)\varepsilon...$$

So

$$ab(000...) = 110... = b(a(000...)),$$

i.e. the action is at right (we can use the exponential action x_0^g).

It is a remarkable fact due to V. Nekrashevich that B can be obtained as Iterated Monodromy Group (IMG) of the complex polynomial $f(z) = z^2 - 1$. The same author found interesting links between fractal sets viewed as Julia set of such a polynomials and Schrier graphs of the action of the corresponding groups on the levels of the tree. See [**Nek1**].

It can be easily proved that the Basilica group is a fractal group. In fact, the stabilizer of the first level is

$$Stab_B(1) = < a, a^b, b^2 >,$$

with $a = (b,1)$, $a^b = (1,b^a)$ and $b^2 = (a,a)$.

It is obvious that the action of the Basilica group on the first level of T is transitive. Since this group is fractal, it easily follows that the action is also spherically transitive, i.e. transitive on each level of the tree. Moreover, it is known (see [**GrŻu**]) that the Basilica group is weakly regular branch over its commutator subgroup B'.

Theorem 2.2. *The action of the Basilica group B on L_n is 2-points homogeneous for all n.*

Proof. From Lemma 1.1 it suffices to show that the action of the parabolic subgroup $K_n = Stab_{B_n}(0^n)$ is transitive on each sphere.

Denote by u_j the vertex $0^{j-1}1$ for every $j = 1, \dots, n$. Observe that the automorphisms

$$(b^2)^a = a^{-1}b^2a = (b^{-1}, 1)(a, a)(b, 1) = (a^b, a) = ((1, b^a), a)$$

and

$$b^a b^{-1} a = (b^{-1}, 1)(a, 1)\varepsilon(b, 1)(1, a^{-1})\varepsilon(b, 1) = (1, b)$$

belong to K_n for each n. Moreover, using the fractalness of B, it is possible to find elements $g_j \in K_n$ such that the restriction $g_j|T_{0^{j-1}}$ is $(b^2)^a = ((1, b^a), a)$ or $b^a b^{-1} a = (1, b)$. So, the action of such automorphisms on the subtree T_{u_j} corresponds to the action of the whole group $B =< a, b >$ on T. We can regard this action as the action of K_n on the spheres of center $x_0 = 0^n$, and so we get that K_n acts transitively on these spheres. This implies that the action of B is 2-points homogeneous on L_n. $\square$

Corollary 2.3. *For every $n \geq 1$, (B_n, K_n) is a symmetric Gelfand pair.*

The number of K_n-orbits is exactly the number of the irreducible submodules occurring in the decomposition of $L(L_n)$ under the action of B_n. Since the submodules W_j's described in the previous section are $n + 1$ as the K_n-orbits, it follows that the Basilica group admits the same decomposition into irreducible submodules and the same spherical functions that we get for $Aut(T_n)$.

We can observe that in the proof of the Theorem 6.3 of Chapter 1 the fundamental tool is that the automorphisms g' and g'' act transitively on the subtrees T' and T'', respectively, and trivially elsewhere. Moreover, the only fractalness does not guarantee that the action is 2-points homogeneous, as one can easily verify in the case of the Adding Machine, for which one gets symmetric Gelfand pairs only for $n = 1, 2$. On the other hand, if a fractal group G acts 2-transitively on L_1 and if it has the property that the rigid stabilizers of the vertices of the first level $Rist_G(i), i = 0, 1, \dots, q - 1$ are spherically transitive for each i, the proof of the Theorem 6.3 of Chapter 1 works again by taking the automorphisms g' and g'' in the rigid vertex stabilizers. But this is not a necessary condition, as the example of the Grigorchuk group shows.

In fact, one can verify (see [**BG2**]) that, in this case, $Rist_G(0) =< d^a, d^{ac} >$, with $d^a = (b, 1)$ and $d^{ac} = (b^a, 1)$. So $Rist_G(0)$ fixes the vertices 00 and 01, and then it does not act transitively on the subtree T_0. This shows, for instance, that a fractal regular branch group could not have this property, which appears to be very strong.

On the other hand, a direct computation shows that Basilica group has this property, what gives another proof that the action on each level L_n is 2-points homogeneous.

THEOREM 2.4. *Let B be the Basilica group. Then the rigid vertex stabilizers $Rist_B(i)$, $i = 0, 1$, act spherically transitively on the corresponding subtrees T_i.*

Proof. Since B is spherically transitive and so $Rist_B(0) \simeq Rist_B(1)$, it suffices to prove the assertion only for $Rist_B(0)$. Consider the automorphisms $a = (b, 1)$ and $a^{b^2} = (b^a, 1)$ in $Rist_B(0)$. We want to show that the subgroup $< a, a^{b^2} >$ is spherically transitive on T_0, equivalently we will prove that the group $< b, b^a >$ is spherically transitive on T.

The latter is clearly transitive on the first level. To complete it suffices to prove its fractalness. We have

$$b^{-1}b^a = (1, a^{-1})\varepsilon(b^{-1}, 1)(a, 1)\varepsilon(b, 1) = (1, a^{-1}b^{-1})\varepsilon(a, b)\varepsilon = (b, (b^{-1})^a)$$

and

$$(b^{-1}b^a)^{b^2} = (a^{-1}, a^{-1})(b, (b^{-1})^a)(a, a) = (b^a, (b^{-1})^{a^2}),$$

and so the projection on the first factor gives both the generators b and b^a. The elements

$$(b^{-1}b^a)^{-1} = (b^{-1}, b^a), \quad ((b^{-1}b^a)^{-1})^{b^{-2}} = ((b^{-1})^{a^{-1}}, b)$$

fulfill the requirements for the projection on the second factor and this completes the proof. □

3. The Grigorchuk group

The Grigorchuk group G was introduced by R. I. Grigorchuk in 1980 (see [**Gri1**]) to solve the problem of the existence of groups of intermediate growth. This group acts on the rooted binary tree and it is a fractal, regular branch group, generated by the automorphisms

$$a = (1, 1)\varepsilon, \quad b = (a, c), \quad c = (a, d), \quad d = (1, b).$$

LEMMA 3.1. *G is regular branch on the normal subgroup*

$$P =< (ab)^2 >^G .$$

Proof. We have $(ab)^2 = (ca, ac)$. From direct computation we get

$$[(ab)^{-2}, d] = (ab)^2d^{-1}(ab)^{-2}d = (ca, ac)(1, b)(ac, ca)(1, b) = \tag{3}$$

$$= (1, ab^cab) = (1, (ab)^2). \tag{4}$$

(5)

Analogously $[(ab)^{-2}, d]^a = ((ab)^2, 1)$. By performing conjugations we conclude $P > P \times P$. $\square$

LEMMA 3.2. *For every $n \geq 1$, the action of P on L_n has two orbits given by the sets*

$$\{x = x_1 \dots x_n \in L_n : x_1 = 0\} \text{ and } \{x = x_1 \dots x_n \in L_n : x_1 = 1\}$$

Proof. The subgroup $M = Stab_K(0)|_{T_0}$ is generated by the elements ca and $(ab)^2$, analogously $M' = Stab_K(1)|_{T_1}$ is generated by ac and $(ab)^2$. This implies that $M = M'$ since $ac = (ca)^{-1}$. The thesis follows if we show that M is transitive on L_n for each $n \geq 1$.

First of all observe that M is transitive on L_1 because $ca \in M$. Consider the subgroup $Stab_M(0)$. This group contains the elements $(ab)^2 = (ca, ac)$, $ca(ab)^2ac = (ca, bad)$ and $(ca)^2 = (ad, da)$ that generate it. Let N and N' be the restrictions of $Stab_M(0)$ to T_0 and T_1 respectively. Then

$$N = \langle ca, ad \rangle = \langle ca, b \rangle \quad \text{and} \quad N' = \langle ac, bad, da \rangle = N.$$

This implies that M is transitive on L_2 because N contains ca. Moreover $Stab_N(0)$ contains the elements $(ca)^2 = (ad, da)$, $cabac = (aca, dad)$ and $b = (ad)$. This implies that the restriction of $Stab_N(0)$ and $Stab_N(1)$ to T_0 and T_1 is isomorphic to G. From this M is transitive on L_n, for $n \geq 3$ since G is transitive on each level. $\square$

THEOREM 3.3 (Grigorchuk). *The action of the Grigorchuk group G on L_n is $2-$points homogeneos for every $n \in \mathbb{N}$.*

Proof. Set $\omega_j = 1^j \in L_j$ and denote $u_j = 1^{j-1}0$ for every $j \leq n$. Since G is fractal, there exists an element $g_j \in G$ such that $g_j|_{T_{\omega_{j-1}}} = b$. Observe that $g_j \in K_n$ for each n, as one can check considering the labelling of $b = (a, c)$. This implies $g_j(u_j) = u_j$ and $g_j|_{T_{u_j}} = a$. Since G is regular branch on P, we get that K_n contains, for every $j \geq 1$, a subgroup P_j such that $P_j|_{T_{u_j}} = P$. This gives that $< P_j, g_j >$ acts on T_{u_j} as $< P, a >$ and, we have seen that the action of K_n is transitive on each level of T_{u_j}. But, for every n, the vertices of L_n belonging to T_{u_j} constitute the elements of the sphere of distance $n - j + 1$ from the center ω_n. The transitivity of K_n on the spheres implies that the action is 2-points homogeneous. $\square$

COROLLARY 3.4. *(G_n, K_n) is a symmetric Gelfand pair.*

As a consequence, the decomposition of $L(L_n)$ under the action of this group into irreducible submodules is still $L(L_n) = \bigoplus_{j=0}^{n} W_j$, where the W_j's are the subspaces defined above. See [**Gri2**] and [**BHG**] for more details.

4. $I = IMG\{z^2 + i\}$

Consider now the group $I = IMG(z^2 + i)$, i.e. the iterated monodromy group defined by the map $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ given by $f(z) = z^2 + i$. The generators of this group have the following self-similar form:

$$a = (1, 1)\varepsilon, \quad b = (a, c), \quad c = (b, 1),$$

where ε denotes, as usual, the nontrivial permutation in S_2. In the following figure we present the corresponding labellings.

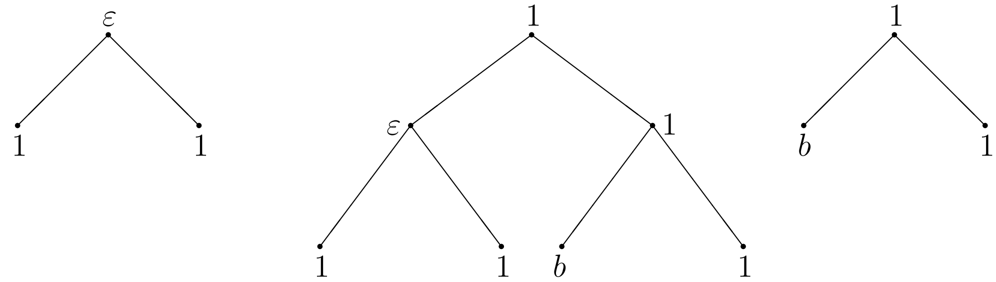


Fig.5. Labelling of the generators a, b and c.

One can easily prove the following relations:

$$a^2 = b^2 = c^2 = (ac)^4 = (ab)^8 = (bc)^8 = 1.$$

Moreover, the stabilizer of the first level is $Stab_I(1) =< b, c, b^a, c^a >$. In particular, since

$$b^a = (c, a), \qquad c^a = (1, b),$$

I is a fractal group. It is obvious that I acts transitively on the first level of the rooted binary tree. Since this group is fractal, it follows that this action is also spherically transitive.

Moreover, it is known (see [**GSŠ**]) that I is a regular branch group over its subgroup N defined by

$$N =< [a, b], [b, c] >^I .$$

Also for the group I it is possible to prove the same result proven for the Basilica group in Theorem 2.4. So consider the $n-$th level L_n of the tree and the group $I_n = I/Stab_I(n)$. In order to get an easy computation, we choose the vertex $x_0 = 1^n \in L_n$ and we set $K_n = Stab_{I_n}(1^n)$. In the following theorem we will prove that the action of the parabolic subgroup K_n is transitive on each sphere.

THEOREM 4.1. *The action of the group I on L_n is 2-points homogeneous for all n.*

Proof. Denote by u_j the vertex $1^{j-1}0$ for every $j = 1, \ldots, n$. Using the fractalness of I, it is possible to find an element $g_j \in K_n$ such that the restriction $g_j|T_{1^{j-1}}$ is b and an element $h_j \in K_n$ such that the restriction $h_j|T_{1^{j-1}}$ is c. Consider now the automorphism $b^a bb^a = (c, a)(a, c)(c, a) = (a^c, c^a)$. By fractalness it is possible to find an element $k_j \in K_n$ such that the restriction $k_j|T_{1^{j-1}}$ is $b^a bb^a$. The action of the subgroup generated by the automorphisms g_j, h_j, k_j on the subtree T_{u_j} corresponds to the action of the subgroup $H =< a, b, a^c >$ on T. It is easy to prove that this action is spherically transitive. In fact it is obvious that H acts transitively on the first level, so it suffices to show that H is fractal. To show this consider, for instance, the elements

$$b = (a, c), \quad a^c a = (b, b), \quad b^a bb^a = (a^c, c^a)$$

and

$$b^a = (c, a), \quad a^c a = (b, b), \quad bb^a b = (c^a, a^c).$$

Now, the action of H on T_{u_j} can be regarded as the action of K_n on the spheres of center x_0, and so we get that K_n acts transitively on these spheres. This implies that the action of I on L_n is 2-points homogeneous, as required. □

COROLLARY 4.2. *For every* $n \geq 1$, (I_n, K_n) *is a symmetric Gelfand pair.*

As in the case of the Basilica group, it follows that the group I_n admits the same decomposition into irreducible submodules and the same spherical functions that we get for $Aut(T_n)$.

It is possible to show that the rigid stabilizers of the vertices of the first level of T do not act spherically transitively on the corresponding subtrees T_0 and T_1. In fact, the rigid stabilizer of the first level is $Rist_I(1) =< c >^G$, so every automorphism in $Rist_I(1)$ is the product of elements of the form c^g, where $g = w(a, b, c)$ is a word in a, b and c, and of their inverses. Set $\varphi(c^g) = (g_0, g_1)$. We want to show, by induction on the length of the word $w(a, b, c)$, that we suppose reduced, that in both g_0 and g_1 the number of occurrences of a is even. This will imply that the action of $Rist_I(1)$ on the first level of the subtrees T_0 and T_1 cannot be transitive and will prove the assertion.

If $|w(a, b, c)| = 0$, then $c^g = c = (b, 1)$. If $|w(a, b, c)| = 1$, then we can have $c^a = (1, b)$, $c^b = (b^a, 1)$ or $c^c = c = (b, 1)$. Let us suppose the result to be true for $|w'(a, b, c)| = n - 1$. Then we have $c^{w(a,b,c)} = c^{w'(a,b,c)x}$, with $x \in \{a, b, c\}$ and $c^{w'(a,b,c)} = (g_0', g_1')$ such that in both g_0' and g_1' the number of occurrences of a is even. If $x = a$, we get $c^{w(a,b,c)} = (g_1', g_0')$, if $x = b$, we get $c^{w(a,b,c)} = ((g_0')^a, (g_1')^b)$ and if $x = c$ then we get $c^{w(a,b,c)} = ((g_0')^b, g_1')$. In all cases, we get a pair (g_0, g_1) satisfying the condition that in both g_0 and g_1 the number of occurrences of a is even, as required.

5. The Hanoi Tower group H

The Hanoi Towers group H is a group of automorphisms of the rooted ternary tree. For the rooted ternary tree all the definitions of level stabilizer, rigid level stabilizer, fractalness, spherically transitive action, given in the binary case, hold.

The generators of H have the following self-similar form:

$$a = (1, 1, a)(01), \quad b = (1, b, 1)(02), \quad c = (c, 1, 1)(12),$$

where $(01), (02)$ and (12) are transpositions in S_3. In the following figures we present the corresponding labellings.

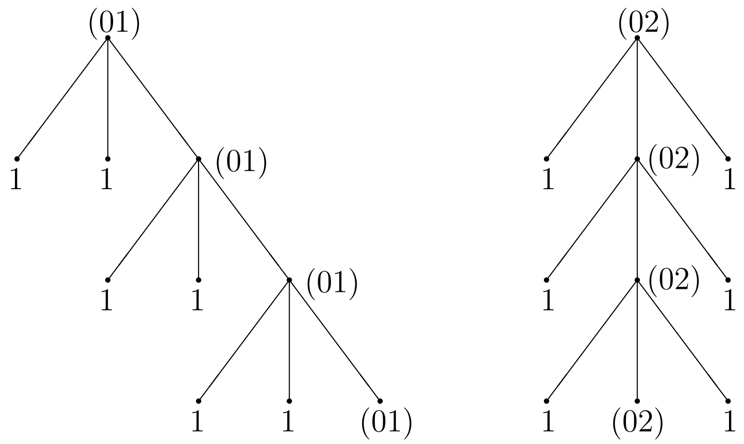


Fig.6. Labelling of the generators a and b.

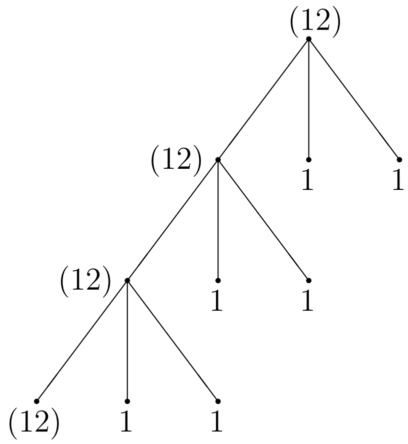


Fig.7. Labelling of the generator c.

From the definition it easily follows that $a^2 = b^2 = c^2 = 1$. Considering the following elements belonging to $Stab_H(1)$

$$acab = (a, cb, a), \quad bcba = (b, b, ca), \quad cacb = (c, ab, c),$$

$$caba = (cb, a, a), \quad (ac)^2ba = (ab, c, c), \quad cbab = (ca, b, b),$$

one can deduce that H is a fractal group. It is obvious that H acts transitively on the first level of the rooted ternary tree. Since this group is fractal, it follows that this action is also spherically transitive.

Moreover, it is known (see [**GrŠ1**]) that H is a regular branch group over its commutator subgroup H'. We observe that we have not the inclusion $H' \leq Stab_H(1)$ that we have in the case of the Basilica group and in the case of $IMG(z^2 + i)$.

Also for the group H it is possible to prove that its action on L_n, $n \geq 1$, gives rise to symmetric Gelfand pairs as it has been proven for B and I. So consider the $n-$th level L_n of the tree and the group $H_n = H/Stab_H(n)$. Fix the vertex $x_0 = 0^n \in L_n$ and set $K_n = Stab_{H_n}(x_0)$. In the following theorem we will prove that the action of the parabolic subgroup K_n is transitive on each sphere.

Theorem 5.1. *The action of the group H on L_n is 2-points homogeneous for all n.*

Proof. Denote by u_j the vertex $0^{j-1}1$ and by v_j the vertex $0^{j-1}2$, for every $j = 1, \ldots, n$. Consider the element

$$acb = (1, c, ab)(12).$$

Using the fractalness of H, it is possible to find an element $g_j \in K_n$ such that the restriction $g_j|T_{0^{j-1}}$ is acb. Since H is regular branch over H', there exists a subgroup H_j of K_n such that $H_j|_{T_{u_j}} = H'$ and which fixes any vertex of the tree whose u_j is not an ancestor. Let us prove that the action of H' on the whole tree is spherically transitive. Considering, for example, the element $[c, b] = cbcb = (cb, c, b)(012)$, one gets that this action is transitive on the first level. Since $H' \geq H' \times H' \times H'$, the action is transitive on each level of the tree. So the action of the subgroup $K =< H_j, g_j >$ on the subtree $T_{0^{j-1}}$ is transitive on the vertices of L_n belonging to the subtrees T_{u_j} and T_{v_j}. This action can be regarded as the action of K_n on the spheres of center x_0, and so we get that K_n acts transitively on these spheres. This implies that the action of H is 2-points homogeneous on L_n, as required. □

Corollary 5.2. *For every $n \geq 1$, (H_n, K_n) is a symmetric Gelfand pair.*

As in the case of the Basilica group and of $IMG(z^2 + i)$, the group H_n admits the same decomposition into irreducible submodules and the same spherical functions that we get for $Aut(T_n)$.

Now we want to prove that the action of the rigid vertex stabilizers $Rist_H(0), Rist_H(1)$ and $Rist_H(2)$ is spherically transitive on the subtrees T_0, T_1 and T_2, respectively. Since these subgroups are conjugate, is suffices to prove the result for $Rist_H(0)$. We use again the fact that H is regular branch over its commutator subgroup H'. So there exists a subgroup $L \leq H'$ such that $L|_{T_0} = H'$ and $L|_{T_1} = L|_{T_2} = 1$. In particular, L is a subgroup of $Rist_H(0)$. Since H' is spherically transitive on T, it follows that $Rist_H(0)$ is spherically transitive on T_0, as required.

This property of the rigid vertex stabilizers, together with the fractalness of H and with the fact that the action of H on the first level is 2-transitive, gives a second proof of the fact that the action of H_n on L_n is 2-points homogeneous, following the same idea that we used for the Basilica group.

6. Generalized wreath products of permutation groups

The generalized wreath product has been introduced by R. A. Bailey, Cheryl E. Praeger, C. A. Rowley and T. P. Speed in [**B&al**]. This is a construction that generalizes the classical direct and wreath product of groups. On the obtained structure one can apply the theory of Gelfand pairs.

6.1. Preliminaries. Let $(I, \leq)$ be a finite poset, with $|I| = n$. First of all, we need some definitions (see, for example, [**B&al**]).

DEFINITION 6.1. *A subset $J \subseteq I$ is said*

- **ancestral** *if, whenever $i > j$ and $j \in J$, then $i \in J$;*
- **hereditary** *if, whenever $i < j$ and $j \in J$, then $i \in J$;*
- **a chain** *if, whenever $i, j \in J$, then either $i \leq j$ or $j \leq i$;*
- **an antichain** *if, whenever $i, j \in J$ and $i \neq j$, then neither $i \leq j$ nor $j \leq i$.*

In particular, for every $i \in I$, the following subsets of I are ancestral:

$$A(i) = \{j \in I : j > i\} \text{ and } A[i] = \{j \in I : j \geq i\},$$

and the following subsets of I are hereditary:

$$H(i) = \{j \in I : j < i\} \text{ and } H[i] = \{j \in I : j \leq i\}.$$

Given a subset $J \subseteq I$, we set

- $A(J) = \bigcup_{i \in J} A(i)$;
- $A[J] = \bigcup_{i \in J} A[i]$;
- $H(J) = \bigcup_{i \in J} H(i)$;
- $H[J] = \bigcup_{i \in J} H[i]$.

In what follows we will use the notation in [**B&al**].

For each $i \in I$, let $\Delta_i = \{\delta_0^i, \ldots, \delta_{m-1}^i\}$ be a finite set, with $m \geq 2$. For $J \subseteq I$, put $\Delta_J = \prod_{i \in J} \Delta_i$. In particular, we put $\Delta = \Delta_I$.

If $K \subseteq J \subseteq I$, let π^J_K denote the natural projection from Δ_J onto Δ_K. In particular, we set $\pi_J = \pi^I_J$ and $\delta_J = \delta\pi_J$. Moreover, we will use Δ^i for $\Delta_{A(i)}$ and π^i for $\pi_{A(i)}$.

Let $\mathcal{A}$ be the set of ancestral subsets of I. If $J \in \mathcal{A}$, then the equivalence relation $\sim_J$ on Δ associated with J is defined as

$$\delta \sim_J \epsilon \quad \Leftrightarrow \quad \delta_J = \epsilon_J,$$

for each $\delta, \epsilon \in \Delta$. We denote $| \sim_J |$ the cardinality of an equivalence class of $\sim_J$.

Definition 6.2. *A* **poset block structure** *is a pair* $(\Delta, \sim_{\mathcal{A}})$, *where*

(1) $\Delta = \prod_{(I,\leq)} \Delta_i$, *with* $(I, \leq)$ *a finite poset and* $|\Delta_i| \geq 2$, *for each* $i \in I$*;*

(2) $\sim_{\mathcal{A}}$ *denotes the set of equivalence relations on* Δ *defined by the ancestral subsets of* I*.*

Remark 6.3.

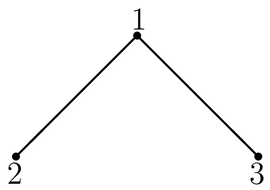
Observe that the set $\sim_{\mathcal{A}}$ is a poset and $\sim_J \leq \sim_K$ if and only if $J \supseteq K$. We will call it the *ancestral poset* associated with I. Moreover, all the maximal chains in $\sim_{\mathcal{A}}$ have the same length n. In fact, the empty set is always ancestral. A singleton $\{i\}$ constituted by a maximal element in I is still an ancestral set. Inductively, if $J \in \mathcal{A}$ is an ancestral set, then $J \sqcup \{i\}$ is an ancestral set if i is a maximal element in $I \setminus J$. So every maximal chain in the poset of ancestral subsets has length n.

To have a representation of a poset block structure, we can perform the following construction (see [**DD3**]). Let $C = \{\sim_I, \sim_J, \ldots, \sim_\emptyset\}$ be a maximal chain of ancestral relations such that $\sim_{J_i} \leq \sim_{J_{i+1}}$ for all $i = 0, \ldots, n-1$. Let us define a rooted tree of depth n as follows: the $n-$th level is constituted by $|\Delta|$ vertices; the $(n-1)-$st by $\frac{|\Delta|}{|\sim_{J_1}|}$ vertices. Each of these vertices is a father of $| \sim_{J_1} |$ sons that are in the same $\sim_{J_1}$ $-$class. Inductively, at the $i-$th level there are $\frac{|\Delta|}{|\sim_{J_{n-i}}|}$ vertices fathers of $| \sim_{J_{n-i}} |$ vertices of the $(i+1)-$st level belonging to the same $\sim_{J_{n-i}}$ $-$class.

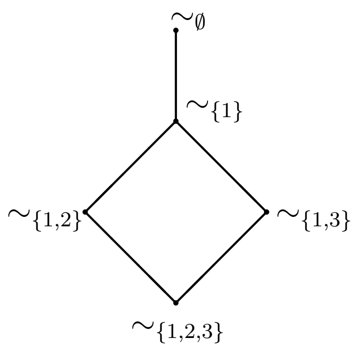
We can perform the same construction for every maximal chain C in $\sim_{\mathcal{A}}$. The next step is to glue the different structures identifying the vertices associated with the same equivalence. The resulting structure is the poset block structure associated with I.

Example 6.4.

Consider the case of the following poset $(I, \leq)$:



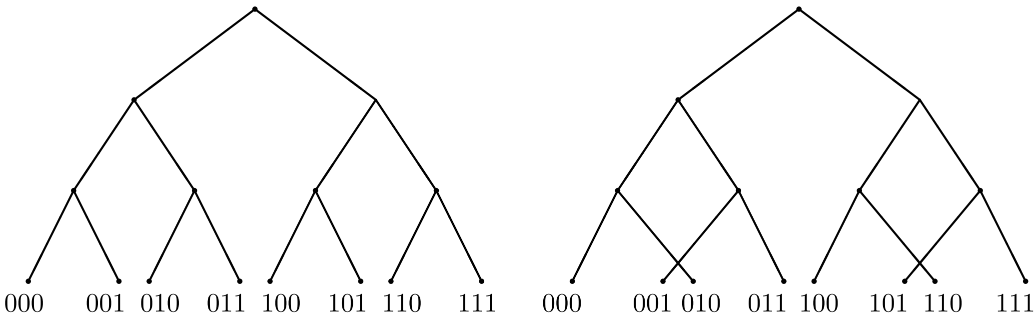
One can easily check that, in this case, the ancestral poset $(\sim_{\mathcal{A}}, \leq)$ is the following:



Suppose $m = 2$ and $\Delta_1 = \Delta_2 = \Delta_3 = \{0, 1\}$, so that we can think of Δ as the set of words of length 3 in the alphabet $\{0, 1\}$. The partitions of Δ given by the equivalences $\sim_J$, with $J \subseteq I$ ancestral, are:

- $\Delta = \{000, 001, 010, 011, 100, 101, 110, 111\}$ by the equivalence $\sim_{\emptyset}$;
- $\Delta = \{000, 001, 010, 011\} \coprod \{100, 101, 110, 111\}$ by the equivalence $\sim_{\{1\}}$;
- $\Delta = \{000, 001\} \coprod \{010, 011\} \coprod \{100, 101\} \coprod \{110, 111\}$ by the equivalence $\sim_{\{1,2\}}$;
- $\Delta = \{000, 010\} \coprod \{001, 011\} \coprod \{100, 110\} \coprod \{101, 111\}$ by the equivalence $\sim_{\{1,3\}}$;
- $\Delta = \{000\} \coprod \{001\} \coprod \{010\} \coprod \{011\} \coprod \{100\} \coprod \{101\} \coprod \{110\} \coprod \{111\}$ by the equivalence $\sim_I$.

Consider the chains $C_1 = \{\sim_I, \sim_{\{1,2\}}, \sim_{\{1\}}, \sim_{\emptyset}\}$ and $C_2 = \{\sim_I, \sim_{\{1,3\}}, \sim_{\{1\}}, \sim_{\emptyset}\}$ in $(\sim_{\mathcal{A}}, \leq)$. The associated trees T_1 and T_2 are, respectively,



Assembling these trees, we get the following poset block structure.

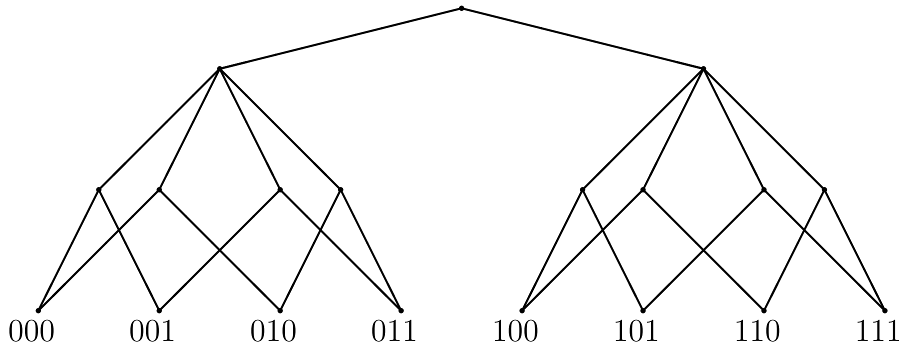


Fig. 8. The poset block structure

6.2. The generalized wreath product. We present here the definition of generalized wreath product given in [**B&al**]. We will follow the same notation of the action to the right presented there. For each $i \in I$, let G_i be a permutation group on Δ_i and let F_i be the set of all functions from Δ^i into G_i. For $J \subseteq I$, we put $F_J = \prod_{i\in J} F_i$ and set $F = F_I$. An element of F will be denoted $f = (f_i)$, with $f_i \in F_i$.

Definition 6.5. *For each $f \in F$, the action of f on Δ is defined as follows: if $\delta = (\delta_i) \in \Delta$, then*

$$\delta f = \varepsilon, \quad \text{where } \varepsilon = (\varepsilon_i) \in \Delta \text{ and } \varepsilon_i = \delta_i(\delta\pi^i f_i). \tag{6}$$

It is easy to verify that this is a faithful action of F on Δ. If $(I, \leq)$ is a finite poset, then (F, Δ) is a permutation group, which is called the *generalized wreath product of the permutation groups* $(G_i, \Delta_i)_{i\in I}$ and denoted $\prod_{(I,\leq)}(G_i, \Delta_i)$.

Definition 6.6. *An automorphism of a poset block structure $(\Delta, \sim_{\mathcal{A}}$ $)$ is a permutation σ of Δ such that, for every equivalence $\sim_J$ in $\sim_{\mathcal{A}}$,*

$$\delta \sim_J \varepsilon \qquad \Leftrightarrow \qquad (\delta\sigma) \sim_J (\varepsilon\sigma),$$

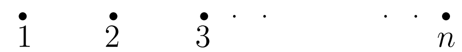
for all $\delta, \varepsilon \in \Delta$.

The following fundamental theorems are proven in [**B&al**]. We denote by $Sym(\Delta_i)$ the symmetric group acting on the set Δ_i. Sometimes we denote it by $Sym(m)$, where $m = |\Delta_i|$.

Theorem 6.7. *The generalized wreath product of the permutation groups* $(G_i, \Delta_i)_{i \in I}$ *is transitive on* Δ *if and only if* (G_i, Δ_i) *is transitive for each* $i \in I$.

Theorem 6.8. *Let* $(\Delta, \sim_{\mathcal{A}})$ *be a poset block structure with associated poset* $(I, \leq)$. *Let* F *be the generalized wreath product* $\prod_{(I,\leq)} Sym(\Delta_i)$. *Then* F *is the group of automorphisms of* $(\Delta, \sim_{\mathcal{A}})$.

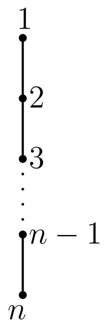
Remark. If $(I, \leq)$ is a finite poset, with $\leq$ the identity relation, then the generalized wreath product becomes the *permutation direct product.*



In this case, we have $A(i) = \emptyset$ for each $i \in I$ and so an element f of F is given by $f = (f_i)_{i \in I}$, where f_i is a function from a singleton $\{*\}$ into G_i and so its action on δ_i does not depend from any other components of δ.

Remark. If $(I, \leq)$ is a finite chain, then the generalized wreath product becomes the *permutation wreath product*

$$(G_n, \Delta_n) \wr (G_{n-1}, \Delta_{n-1}) \wr \cdots \wr (G_1, \Delta_1).$$



In this case, we have $A(i) = \{1, 2, \ldots, i-1\}$ for each $i \in I$ and so an element f of F is given by $f = (f_i)_{i \in I}$, with

$$f_i : \Delta_1 \times \cdots \times \Delta_{i-1} \longrightarrow G_i$$

and so its action on δ_i depends on all the previous components of δ.

6.3. Gelfand pairs. In what follows we suppose $G_i = Sym(m)$ where $m = |\Delta_i|$. Fixed an element $\delta_0 = (\delta_0^1, \ldots, \delta_0^n)$ in Δ, the stabilizer $Stab_F(\delta_0)$ is the subgroup of F acting trivially on δ_0. If we represent

$f \in F$ as the $n-$tuple $(f_1, \ldots, f_n)$, with $f_i : \Delta^i \longrightarrow Sym(m)$ and set $\Delta_0^i = \prod_{j \in A(i)} \delta_0^j$, we have the following lemma.

LEMMA 6.9. *The stabilizer of* $\delta_0 = (\delta_0^1, \ldots, \delta_0^n) \in \Delta$ *in* F *is the subgroup*

$$\begin{aligned} K := Stab_F(\delta_0) &= \{g = (f_1, \ldots, f_n) \in G \;:\; f_i|_{\Delta_0^i} \in Stab_{Sym(m)}(\delta_0^i) \\ &\qquad \textit{whenever } \Delta^i = \Delta_0^i \textit{ or } A(i) = \emptyset\}. \end{aligned}$$

Proof. One can easily verify that K is a subgroup of F. If $i \in I$ is such that $A(i) = \emptyset$ then, by definition of generalized wreath product, it must be $f_i \in Stab_{Sym(m)}(\delta_0^i)$. For all i we have

$$\begin{aligned} \delta_0^i f = \delta_0^i &\iff \delta_0^i(\delta_0^{A(i)}) f_i = \delta_0^i \\ &\iff (\delta_0^{A(i)}) f_i \in \; Stab_{Sym(m)}(\delta_0^i) \\ &\iff f_i|_{\Delta_0^i} \in Stab_{Sym(m)}(\delta_0^i). \end{aligned}$$

□

Now we want to study the $K-$orbits on Δ. We recall that the action of $Sym(m-1) \equiv Stab_{Sym(m)}(\delta_0^i)$ on Δ_i has two orbits, i.e. $\Delta_i = \{\delta_0^i\} \coprod (\Delta_i \setminus \{\delta_0^i\})$. Set $\Delta_i^0 = \{\delta_0^i\}$ and $\Delta_i^1 = \Delta_i \setminus \{\delta_0^i\}$.

LEMMA 6.10. *The* $K-$*orbits on* Δ *have the following form:*

$$\left(\prod_{i \in I \setminus H[S]} \Delta_i^0 \right) \times \left(\prod_{i \in S} \Delta_i^1 \right) \times \left(\prod_{i \in H(S)} \Delta_i \right),$$

where S *is any antichain in* I.

Proof. First of all suppose that $\delta, \epsilon \in \left(\prod_{i \in I \setminus H[S]} \Delta_i^0\right) \times \left(\prod_{i \in S} \Delta_i^1\right) \times \left(\prod_{i \in H(S)} \Delta_i\right)$, for some antichain S. Then $\delta_{I \setminus H[S]} = \epsilon_{I \setminus H[S]} = \delta_0^{I \setminus H[S]}$. If $s \in S$ we have $A(s) \subseteq I \setminus H[S]$ and this implies $(A(s)) f_s \in Stab_{Sym(m)}(\delta_0^s)$. So $\epsilon_s = \delta_s(\delta_0^{A(s)} f_s)$. If $i \in H(S)$ then $A(i) \neq \emptyset$ and $\Delta^i \neq \Delta_0^i$. This implies $(A(i)) f_i \in Sym(m)$ and so $\epsilon_i = \delta_i(\delta_0^{A(i)} f_i)$. This shows that K acts transitively on each orbit.

On the other hand, let $S \neq S'$ be two distinct antichains and $\delta \in \left(\prod_{i \in I \setminus H[S]} \Delta_i^0\right) \times \left(\prod_{i \in S} \Delta_i^1\right) \times \left(\prod_{i \in H(S)} \Delta_i\right)$ and $\epsilon \in \left(\prod_{i \in I \setminus H[S']} \Delta_i^0\right) \times \left(\prod_{i \in S'} \Delta_i^1\right) \times \left(\prod_{i \in H(S')} \Delta_i\right)$. Suppose $s \in S \setminus (S \cap S')$ and so $I \setminus H[S] \neq I \setminus H[S']$. If $s \in I \setminus H[S']$ then $\delta_s \neq \delta_0^s = \epsilon_s$. But $(A(S)) f_s \in Stab_{Sym(m)}(\delta_0^s)$ and so $\delta_s(A(S) f_s) \neq \epsilon_s$. If $s \in H(S')$ there exists $s' \in S' \setminus (S \cap S')$ such that $s < s'$. This implies that $s' \in I \setminus H[S]$ and we can proceed as above.

The proof follows from the fact that the orbits are effectively a partition of Δ. $\square$

Finally, we want to prove that the group $F = \prod_{i\in I} G_i$ acting on Δ and the stabilizer K of the element $\delta_0 = (\delta_0^1, \dots, \delta_0^n)$ yield a Gelfand pair. To show this, we use the Gelfand condition.

PROPOSITION 6.11. *Given $\delta, \epsilon \in \Delta$, there exists an element $g \in F$ such that $\delta g = \epsilon$ and $\epsilon g = \delta$.*

Proof. Let i be in I such that $A(i) = \emptyset$. Then, by the $m-$transitivity of the symmetric group, there exists $g_i \in Sym(\Delta_i)$ such that $\delta_i g_i = \epsilon_i$ and $\epsilon_i g_i = \delta_i$. For every index i such that $A(i) \neq \emptyset$ define $f_i : \Delta^i \longrightarrow Sym(\Delta_i)$ as $\delta_{\Delta^i} f_i = \epsilon_{\Delta^i} f_i = \sigma_i$ where $\sigma_i \in Sym(\Delta_i)$ is a permutation such that $\delta_i \sigma_i = \epsilon_i$ and $\epsilon_i \sigma_i = \delta_i$. So the element $g \in F$ that we get is the requested automorphism. $\square$

From this we get the following corollary.

COROLLARY 6.12. *(G, K) is a symmetric Gelfand pair.*

Set $L(\Delta) = \{f : \Delta \longrightarrow \mathbb{C}\}$. It is known ([**B&al**]) that the decomposition of $L(\Delta)$ into $G-$irreducible submodules is given by

$$L(\Delta) = \bigoplus_{S\subseteq I \ \ antichain} W_S$$

with

$$W_S = \left(\bigotimes_{i\in A(S)} L(\Delta_i)\right) \otimes \left(\bigotimes_{i\in S} V_i^1\right) \otimes \left(\bigotimes_{i\in I\setminus A[S]} V_i^0\right), \tag{7}$$

where, for each $i = 1, \dots, n$, we denote $L(\Delta_i)$ the space of the real valued functions on Δ_i, whose decomposition into G_i-irreducible submodules is

$$L(\Delta_i) = V_i^0 \bigoplus V_i^1,$$

with V_i^0 the subspace of the constant functions on Δ_i and $V_i^1 = \{f : \Delta_i \to \mathbb{C} : \sum_{x\in\Delta_i} f(x) = 0\}$.

PROPOSITION 6.13. *The spherical function associated with W_S is*

$$\phi_S = \bigotimes_{i\in A(S)} \varphi_i \bigotimes_{i\in S} \psi_i \bigotimes_{i\in I\setminus A[S]} \varrho_i, \tag{8}$$

where φ_i is the function defined on Δ_i as

$$\varphi_i(x) = \begin{cases} 1 & x = \delta_0^i \\ 0 & otherwise \end{cases}$$

and ψ_i is the function defined on Δ_i as

$$\psi_i(x) = \begin{cases} 1 & x = \delta_0^i \\ -\frac{1}{m-1} & \textit{otherwise} \end{cases}$$

and ϱ_i is the function on Δ_i such that $\varrho_i(x) = 1$ for every $x \in \Delta_i$.

Proof. It is clear that $\phi_S \in W_S$ and $(\delta_0)\phi_S = 1$, so we have to show that each ϕ_S is $K-$invariant.

Set $B_1 = \{i \in A(S) : A(i) = \emptyset\}$. If there exists $i \in B_1$ such that $\delta_i \neq \delta_0^i$ then $(\delta)\phi_S = (\delta)\phi_S^k = 0$ for every $k \in K$, since $\delta_i\varphi_i = (\delta_i k^{-1})\varphi_i = 0$. Hence ϕ and ϕ^k coincide on $\delta \in \Delta$ satisfying this property. So we can suppose that $\delta_i = \delta_0^i$ for each $i \in B_1$.

Let B_2 be the set of maximal elements in $A(S) \setminus B_1$. If there exists $j \in B_2$ such that $\delta_j \neq \delta_0^j$ then $(\delta)\phi_S = (\delta)\phi_S^k = 0$ for every $k \in K$, since $\delta_j\varphi_j = (\delta_j k^{-1})\varphi_j = 0$. Hence ϕ and ϕ^k coincide on $\delta \in \Delta$ satisfying this property. So we can suppose that $\delta_j = \delta_0^j$ for each $j \in B_2$. Inductively it remains to show that $(\delta)\phi_S = (\delta)\phi_S^k$ only for the elements δ such that $\delta_{A(S)} = \delta_0^{A(S)}$, i.e. $(\delta_i)\psi_i = (\delta_i)\psi_i^k$ for every $i \in S$. This easily follows from the definition of K and of the function ψ_i. □

Remark 6.14.

In [**B&al**] the authors give the decomposition of the space $L(\Delta)$ into irreducible submodules under the action of F and they prove that W_S is not isomorphic to W_T if $S \neq T$ and so this decomposition is multiplicity-free. Although this implies that one gets a Gelfand pair, they do not deal with Gelfand pairs theory. Actually, Proposition 6.11 is a stronger result, valid in the more general case of more complex substructures of the poset block structure, that implies that the Gelfand pair is also symmetric.

7. Substructures

Consider the rooted tree of depth n denoted by T_n, with ramification indices $(m_1, \ldots, m_n)$, we have associated with it the homogeneous space obtained by considering its full automorphism group and the stabilizer of a fixed vertex (a leaf) of the $n-$th level. But fixing new indices $(r_1, \ldots, r_n)$ such that $r_i \leq m_i$ fer every $i = 1, \ldots, n$ we can consider the *variety* of the subtrees in the whole tree T_n. The full automorphism group $Aut(T_n)$ acts transitively on the variety of subtrees and associated with the stabilizer of a particular subtree gives rise to a Gelfand pair as shown in [**CST3**].

We have noted that the tree and its group of automorphisms are a specific case in the theory of the poset block structures (as well as the case of the direct product). Then we can ask: is this result in general true in the context of poset block structures? I.e. if we choose a r_i − *subset* of elements in the sets Δ_i with $i \in I$ according with the structure of the poset and its group of automorphisms, we can get a Gelfand pairs considering the subgroup stabilizer a particular one?

Consider the poset block structure associated with the poset $(I, \leq)$, with $|I| = n$.

For each $i \in I$, let $\Delta_i = \{\delta_0^i, \ldots, \delta_{m_i-1}^i\}$ be a finite set, with $m_i \geq 2$ for all $i = 1, \ldots, n$.

We can represent Δ by a rooted tree of depth n and whose branch indices are $\mathbf{m} = (m_1, \ldots, m_n)$.

Consider the indices $\mathbf{r} = (r_1, \ldots, r_n)$ as the indices of the substructure that we want to define. If $i \in \{1, \ldots, n\}$ is an index such that $A(i) = \emptyset$, then the choice of r_i elements in Δ_i does not depend from any other index.

If $i \in \{1, \ldots, n\}$ is an index such that $\emptyset \neq A(i) = \{i_1, \ldots, i_k\}$, then the choice of r_i elements in Δ_i depends on the choose performed for the indices $i_1, \ldots, i_k$. In other words, the $i-$th choice is the same for those substructures that coincide on the chooses given for the indices belonging to the anchestral set $A(i)$.

It is easy to check that the number of the substructures defined above is exactly

$$\prod_{i \in I : A(i) = \emptyset} \binom{m_i}{r_i} \cdot \prod_{i \in I : A(i) \neq \emptyset} \binom{m_i}{r_i}^{\prod_{j \in A(i)} r_j}.$$

In fact, for those indices $i \in I$ such that $A(i) = \emptyset$, we have $\binom{m_i}{r_i}$ possible choices; for those indices $i \in I$ such that $A(i) \neq \emptyset$, we have $\binom{m_i}{r_i}$ possible choices for each of the $\prod_{j \in A(i)} r_j$ vertices corresponding to (eventually) different choices for the coordinates in $A(i)$.

It is not difficult to verify that the action of the generalized wreath product F of the symmetric groups of the sets Δ_i transitively acts on the variety of the substructures of a poset block structure.

We can also prove, using Gelfand's Condition (Lemma 1.4 Chapter 1), that (F, K) is a symmetric Gelfand pair, where K denotes the stabilizer of a fixed substructure. In fact, the following theorem holds.

THEOREM 7.1. *Let $(I, \leq)$ be a finite poset and let Δ be the associated poset block structure. Let F be the respective generalized wreath product, with $|\Delta_i| = m_i \geq 2$ for all $i \in I$. Let $\boldsymbol{r}$=$(r_1, \ldots, r_n)$ be an $n-$tuple of integers such that $1 \leq r_i \leq m_i$. If A and B are two substructures of type $\boldsymbol{r}$ in Δ, then there exists an automorphism $f \in F$ of Δ such that $f(A) = B$ and $f(B) = A$.*

Proof. We can suppose, without loss of generality, that $A(1) = \emptyset$. We want to get an automorphism $f = (f_i)_{i\in I} \in F$ such that $f(A) = B$ and $f(B) = A$. We will proceed by induction on the depth of the substructure.

Set $\pi_1(A) = \{i_1^A, \ldots, i_{r_1}^A\}$ and $\pi_1(B) = \{i_1^B, \ldots, i_{r_1}^B\}$.
By the m_1-transitivity of $Sym(\Delta_1)$, we can choose a permutation $f_1 \in Sym(\Delta_1)$ fixing $\pi_1(A) \cap \pi_1(B)$ such that $f_1(\pi_1(A) \setminus (\pi_1(A) \cap \pi_1(B))) = \pi_1(B) \setminus (\pi_1(A) \cap \pi_1(B))$ and $f_1(\pi_1(B) \setminus (\pi_1(A) \cap \pi_1(B))) = \pi_1(A) \setminus (\pi_1(A) \cap \pi_1(B))$.

Now let $2 \leq j \leq n$ and $A(j) = \{j_1, \ldots, j_k\}$, with $j_1 < \ldots < j_k < j$ in $\mathbb{N}$. Suppose that we have found an automorphism $f' \in F$ such that $f'(\pi_{\{1,\ldots,j-1\}}(A)) = \pi_{\{1,\ldots,j-1\}}(B)$ and $f'(\pi_{\{1,\ldots,j-1\}}(B)) = \pi_{\{1,\ldots,j-1\}}(A)$. We want to show that this result can be extended to the $j-$th level. For both A and B, the vertices at the $(j-1)-$st level are exactly $r_1 r_2 \cdots r_{j-1}$. Moreover f' maps vertices of the $(j-1)-$st level having the same choices for the coordinates in $A(j)$ into vertices that still have the same choices for the coordinates in $A(j)$, since f' is an automorphism of the poset block structure. Now for each possible ancestral situation $a_j \in \Delta^j$ for the vertices of the $(j-1)-$st level of A, we put $f_j(a_j) = g_j^A \in Sym(\Delta_j)$, where g_j^A maps the r_j elements starting from those vertices into the r_j elements in B starting from the image of those vertices by f'.

Analogously for each possible ancestral situation $b_j \in \Delta^j$ for the vertices of the $(j-1)-$st level of B.

If $a_j = b_j$, then f_j has to be defined has $f_j(a_j) = g_j^{AB} \in Sym(\Delta_j)$, where g_j^{AB} maps the r_j elements in A into the r_j elements of B and viceversa.

If we put $f'' = (1, \ldots, 1, f_j, 1, \ldots, 1)$, then the composition of f' and f'' gives the automorphism f required. $\square$

Now let K be the stabilizer of a fixed substructure. We get the following corollary.

COROLLARY 7.2. (F, K) *is a symmetric Gelfand pair.*

The question about the decompositions into irreducible submodules, and the corresponding spherical functions is still open.

CHAPTER 3

Markov Chains

This chapter is devoted to the study of particular Markov chains linked with the theory of Gelfand Pairs. The Insect is studied in relation with the cut-off theory and it is generalized as Markov chain on more general posets. Finally the first and the second crested products are defined, as a generalization giving the same decomposition obtained by the group theory.

1. General properties

The following topics about finite Markov chains can be found in [**CST2**].

Consider a finite set X, with $|X| = m$. Let P be a stochastic matrix of size m whose rows and columns are indexed by the elements of X, so that
$$\sum_{x\in X} p(x_0, x) = 1,$$
for every $x_0 \in X$. Consider the Markov chain on X with transition matrix P.

Definition 1.1. *The Markov chain P is reversible if there exists a strict probability measure π on X such that*
$$\pi(x)p(x, y) = \pi(y)p(y, x),$$
for all $x, y \in X$.

We will say that P and π are in detailed balance. For a complete treatment about these and related topics see [**AlFi**].

Define on $L(X) = \{f : X \longrightarrow \mathbb{C}\}$ a scalar product in the following way:
$$\langle f_1, f_2\rangle_\pi = \sum_{x\in X} f_1(x)\overline{f_2(x)}\pi(x),$$
for all $f_1, f_2 \in L(X)$ and the linear operator $P : L(X) \longrightarrow L(X)$ by
$$(Pf)(x) = \sum_{y\in X} p(x, y)f(y).$$
It is easy to verify that π and P are in detailed balance if and only if P is self-adjoint with respect to the scalar product $\langle\cdot, \cdot\rangle_\pi$. Under these hypothesis, it is known that the matrix P can be diagonalized over the reals. Moreover 1 is always an eigenvalue of P and, if λ is another

eigenvalue, one has $|\lambda| \leq 1$.

Let λ_z be the eigenvalues of the matrix P, for every $z \in X$, with $\lambda_{z_0} = 1$. Then there exists an invertible unitary real matrix $U = (u(x,y))_{x,y\in X}$ such that $PU = U\Delta$, where $\Delta = (\lambda_x \delta_x(y))_{x,y\in X}$ is the diagonal matrix whose entries are the eigenvalues of P. This equation gives, for all $x, z \in X$,

$$\sum_{y\in X} p(x,y)u(y,z) = u(x,z)\lambda_z. \tag{9}$$

Moreover, we have $U^T DU = I$, where $D = (\pi(x)\delta_x(y))_{x,y\in X}$ is the diagonal matrix of coefficients of π. This second equation gives, for all $y, z \in X$,

$$\sum_{x\in X} u(x,y)u(x,z)\pi(x) = \delta_y(z). \tag{10}$$

Hence, the first equation tells us that each column of U is an eigenvector of P, the second one tells us that these columns are orthogonal with respect to the product $\langle \cdot, \cdot \rangle_\pi$.

Let μ and ν two probability distributions on X. Then their total variation distance is defined as

$$\|\mu - \nu\|_{TV} = \max_{A\subseteq X} \left| \sum_{x\in A} \mu(x) - \nu(x) \right| \equiv \max_{A\subseteq X} |\mu(A) - \nu(A)|.$$

It is easy to prove that $\|\mu - \nu\|_{TV} = \frac{1}{2}\|\mu - \nu\|_{L^1}$, where

$$\|\mu - \nu\|_{L^1} = \sum_{x\in X} |\mu(x) - \nu(x)|.$$

Proposition 1.2. *The $k-th$ step transition probability is given by*

$$p^{(k)}(x,y) = \pi(y) \sum_{z\in X} u(x,z)\lambda_z^k u(y,z), \tag{11}$$

for all $x, y \in X$.

Proof. The proof is a consequence of (9) and (10). In fact, the matrix $U^T D$ is the inverse of U, so that $UU^T D = I$. In formulæ, we have

$$\sum_{y\in X} u(x,y)u(z,y) = \frac{1}{\pi(z)} \Delta_z(x).$$

From the equation $PU = U\Delta$ we get $P = U\Delta U^T D$, which gives

$$p(x,y) = \pi(y) \sum_{z\in X} u(x,z)\lambda_z u(y,z).$$

Iterating this argument we get

$$P^k = U\Delta^k U^T D,$$

which is the assertion. $\square$

Recall that there exists a correspondence between reversible Markov chains and weighted graphs.

Definition 1.3. *A weight on a graph $\mathcal{G} = (X, E)$ is a function $w : X \times X \longrightarrow [0, +\infty)$ such that*

(1) $w(x, y) = w(y, x)$;
(2) $w(x, y) > 0$ if and only if $x \sim y$.

If $\mathcal{G}$ is a weighted graph, it is possible associate with w a stochastic matrix $P = (P(x, y))_{x,y \in X}$ on X by setting

$$p(x, y) = \frac{w(x, y)}{W(x)},$$

with $W(x) = \sum_{z \in X} w(x, z)$. The corresponding Markov chain is called the random walk on $\mathcal{G}$. It is easy to prove that the matrix P is in detailed balance with the distribution π defined, for every $x \in X$, as

$$\pi(x) = \frac{W(x)}{W},$$

with $W = \sum_{z \in X} W(z)$. Moreover, π is strictly positive if X does not contain isolated vertices. The inverse construction can be done. So, if we have a transition matrix P on X which is in detailed balance with the probability π, then we can define a weight w as $w(x, y) = \pi(x)p(x, y)$. This definition guarantees the symmetry of w and, by setting $E = \{\{x, y\} : w(x, y) > 0\}$, we get a weighted graph.

There are some important relations between the weighted graph associated with a transition matrix P and its spectrum. In fact, it is easy to prove that the multiplicity of the eigenvalue 1 of P equals the number of connected components of $\mathcal{G}$. Moreover, the following propositions hold.

Proposition 1.4. *Let $\mathcal{G} = (X, E, w)$ be a finite connected weighted graph and denote P the corresponding transition matrix. Then the following are equivalent:*

(1) $\mathcal{G}$ is bipartite;
(2) the spectrum $\sigma(P)$ is symmetric;
(3) $-1 \in \sigma(P)$.

Proof. 1) $\Rightarrow$ 2) Suppose that $Pf = \lambda f$, we have to show that exists $f' \in L(X)$ such that $Pf' = -\lambda f'$. Since $\mathcal{G}$ is bipartite we can write $X = X_0 \sqcup X_1$. If $x \in X_j$ set $f'(x) = (-1)^j f(x)$. So

$$\begin{aligned} Pf'(x) &= \sum_{y\sim x} p(x,y)f'(y) = \\ &= \sum_{y\sim x}(-1)^{j+1}p(x,y)f(y) = \\ &= (-1)^{j+1}\lambda f(x) = -\lambda f'(x). \end{aligned}$$

2) $\Rightarrow$ 3) Trivial.

3) $\Rightarrow$ 1) There exists $f \in L(X)$ such that $Pf = -f$. Suppose that $x_0 \in X$ is a point of maximum for $|f|$ and that $f(x_0) > 0$. From $-f(x_0) = \sum_{y\sim x_0} p(x_0,y)f(y)$ we get $f(x_0) = -f(y)$ for each $y \sim x_0$. Set $X_j = \{y \in X : f(y) = (-1)^j f(x_0)\}$ for $j = 0, 1$. This gives the bipartition of the graph $\mathcal{G}$. $\square$

Definition 1.5. *Let P be a stochastic matrix. P is ergodic if there exists $n_0 \in \mathbb{N}$ such that*

$$p^{(n_0)}(x,y) > 0, \quad \textit{for all } x, y \in X.$$

Proposition 1.6. *Let $\mathcal{G} = (X,E)$ be a finite graph. Then the following conditions are equivalent:*

(1) $\mathcal{G}$ is connected and not bipartite;
(2) for every weight function on X, the associated transition matrix P is ergodic.

Proof. 2) $\Rightarrow$ 1) By hypothesis there exists k_0 such that $p^{(k_0)}(x,y) > 0$ for every $x, y \in X$. This implies that, for $k > k_0$ we get $p^k(x,y) = \sum_{z\in X} p^{(n-n_0)}(x,z)p^{k_0}(z,y) > 0$. This assures the existence of paths of even and odd length from x and y, i.e. X is not bipartite.

1) $\Rightarrow$ 2) It is clear that $\mathcal{G}$ is bipartite if and only if the length of a path connecting a vertex x with itself is even. This implies that there exists a path of odd length from x to x. For every $x \in X$ denote it $l(x)$. Set $2M + 1 = \max_{x\in X} |l(x)|$. We can construct paths starting and ending at x of length $\geq 2M$. If m is even we choose $z \sim x$ and the path $q_{2t} = (x, z, x, \ldots, z, x)$, if m is odd we consider $l(x)$ composed with q_{2t}. Set $\delta = \max_{x,y\in X} d(x,y)$. We can conclude from this that for any $x, y \in X$ there exists a path joining them after n steps, where $n \geq 2M + \delta$. In fact, denote $l(x,y)$ the minimal path connecting x and y and choose $m = n - d(x,y) \geq 2M$. We have seen that we can construct a path starting and ending at x of length $\geq 2M$, compose it with $l(x,y)$ of length $\leq \delta$ and this gives the path connecting x and y in n steps. $\square$

So we can conclude that a reversible transition matrix P is ergodic if and only if the eigenvalue 1 has multiplicity one and -1 is not an eigenvalue.

This allows to prove the following fundamental theorem that is true in a more general settings.

THEOREM 1.7. *Let P a probability on X in detailed balance with the distribution π, then*

$$\lim_{k\to\infty} p^{(k)}(x,y) = \pi(y), \quad \forall\ x,y \in X.$$

Proof. We have from Proposition 1.2

$$p^{(k)}(x,y) = \pi(y)\sum_{z\in X} u(x,z)\lambda_z^k u(y,z) = \pi(y)\left(1+\sum_{z\neq z_0} u(x,z)\lambda_z^k u(y,z)\right),$$

the second summand goes to 0 since $|\lambda_z|_{z\neq z_0} < 1$. $\square$

In what follows we always suppose that the eigenvalue 1 has multiplicity one, so that the graph associated with the probability P is connected. This is equivalent to require that the probability P is irreducible, according with the following definition.

DEFINITION 1.8. *A stochastic matrix P on a set X is irreducible if, for every $x_1, x_2 \in X$, there exists $n = n(x_1,x_2)$ such that $p^{(n)}(x_1,x_2) > 0$.*

2. Insect Markov chain

In [**F-T1**] the following Markov chain on the space L_n is defined. Suppose that at time zero we start from the vertex $x_0 = 0^n \in L_n$. Let ξ_i denote the vertex 0^{n-i} and α_i the probability to reach ξ_{i+1} from staying at ξ_i. It is clear that $\alpha_0 = 1$, $\alpha_1 = \frac{1}{q+1}$ and $\alpha_n = 0$. This leads to the following recursive expression

$$\alpha_j = \frac{1}{q+1} + \alpha_{j-1}\alpha_j\frac{1}{q+1}.$$

Solving the equation we get

$$\alpha_j = \frac{q^j-1}{q^{j+1}-1}, \quad 1 \le j \le n-1.$$

Hence we can define $P = (p(x,y))_{x,y\in L_n}$, as the stochastic matrix whose entry $p(x,y)$ is the probability that y is the first vertex in L_n reached from x in the Markov chain defined above. It is clear that if $d(x,y) = d(x,z)$ (i.e. y and z are in the same sphere of center x) we have $p(x,y) = p(x,z)$. Fixed the vertex $x_0 = 0^n$, we can compute, recalling the significance of the α_j's

$$\begin{aligned} p(x_0,x_0) &= q^{-1}(1-\alpha_1) + q^{-2}\alpha_1(1-\alpha_2) + \cdots + \\ &+ q^{-n+1}\alpha_1\alpha_2\cdots\alpha_{n-2}(1-\alpha_{n-1}) + q^{-n}\alpha_1\alpha_2\cdots\alpha_{n-1}. \end{aligned}$$

It is clear that, if $d(x_0, x) = 1$, then $p(x_0, x) = p(x_0, x_0)$.

More generally, if $d(x_0, x) = j > 1$, we have

$$\begin{aligned} p(x_0, x) &= q^{-j}\alpha_1\alpha_2\cdots\alpha_{j-1}(1-\alpha_j) + \cdots + \\ &+ q^{-n+1}\alpha_1\alpha_2\cdots\alpha_{n-2}(1-\alpha_{n-1}) + q^{-n}\alpha_1\alpha_2\cdots\alpha_{n-1}. \end{aligned}$$

In order to compute the eigenvalues $\lambda_j, j = 0, 1, \ldots, n$ of the associated operator P one can observe that by equivalence between $Aut(T_{q,n})-$invariant operators and bi$-K_{q,n}-$invariant functions it is enough to consider the spherical Fourier transform of the convolver representing P (see [**CST1**]), namely

$$\lambda_j = \sum_{x\in L_n} p(x_0, x)\phi_j(x), \quad j = 0, 1, \ldots, n.$$

Using the expressions given for P and the ϕ_j's we get the following eigenvalues.

For $j = 0$, we get

$$\lambda_0 = \sum_{x\in L_n} p(x_0, x) = 1.$$

For $j = n$, we have

$$\lambda_n = p(x_0, x_0)\cdot 1 + p(x_0, x)\left(-\frac{1}{q-1}\right)\cdot(q-1) = 0.$$

For $1 \le j < n$, we get

$$\begin{aligned} \lambda_j &= qp(x_0, x_1) + (q^2 - q)p(x_0, x_2) + \cdots + (q^{n-j} - q^{n-j-1})p(x_0, x_{n-j}) \\ &+ (1-q)^{-1}(q^{n-j+1} - q^{n-j})p(x_0, x_{n-j+1}) \\ &= q(p(x_0, x_1) - p(x_0, x_2)) + q^2(p(x_0, x_2) - p(x_0, x_3)) + \cdots \\ &+ q^{n-j-1}(p(x_0, x_{n-j-1}) - p(x_0, x_{n-j})) + q^{n-j}p(x_0, x_{n-j}) \\ &+ (1-q)^{-1}(q^{n-j+1} - q^{n-j})p(x_0, x_{n-j+1}) \\ &= \sum_{h=1}^{n-j} q^h(p(x_0, x_h) - p(x_0, x_{h+1})) \\ &= (1-\alpha_1) + \alpha_1(1-\alpha_2) + \cdots + \alpha_1\alpha_2\cdots\alpha_{n-j-1}(1-\alpha_{n-j}) \\ &= 1 - \alpha_1\alpha_2\cdots\alpha_{n-j} \\ &= 1 - \frac{q-1}{q^{n-j+1}-1}. \end{aligned}$$

Observe that, by Proposition 1.6 of this section, the Insect Markov chain is ergodic. Moreover, it is clear that P is in detailed balance with the uniform distribution π on L_n given by $\pi(x) = \frac{1}{q^n}$ for all $x \in L_n$.

3. Cut-off phenomenon

Let $m_x^{(k)}(y) = p^{(k)}(x,y)$ be the distribution probability after k steps. The total variation distance allows to estimate how $m^{(k)}$ converges to the stationary distribution π.

There are interesting cases in which the total variation distance remains close to 1 for a long time and then tends to 0 in a very fast way (see, for some examples, [**Dia1**] and [**DSC2**]). This suggests the following definition (see [**CST2**]).

Suppose that X_n is a sequence of finite sets. Let m_n and p_n be a probability measure on X_n and an ergodic transition probability on X_n, respectively. Denote π_n the corresponding stationary measure and $m_n^{(k)}$ the distribution of (X_n, m_n, p_n) after k steps.

Now let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of positive real numbers such that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

Definition 3.1. *The sequence of Markov chains (X_n, m_n, p_n) has a $(a_n, b_n)-$cut-off if there exist two functions $f_1, f_2 : [0, +\infty) \longrightarrow \mathbb{R}$ with*

- $\lim_{c\to+\infty} f_1(c) = 0$
- $\lim_{c\to+\infty} f_2(c) = 1$

such that, for any fixed $c > 0$, one has

$$\|m_n^{(a_n+cb_n)} - \pi_n\|_{TV} \leq f_1(c) \quad \text{and} \quad \|m_n^{(a_n-cb_n)} - \pi_n\|_{TV} \geq f_2(c)$$

for sufficiently large n.

The following proposition gives a necessary condition for the cut-off phenomenon.

Proposition 3.2. *If (X_n, m_n, p_n) has an $(a_n, b_n)-$cut-off, then for any $0 < \epsilon_1 < \epsilon_2 < 1$ there exist $k_2(n) \leq k_1(n)$ such that*

(1) $k_2(n) \leq a_n \leq k_1(n)$;
(2) for n large, $k \geq k_1(n) \Rightarrow \|m_n^{(k)} - \pi_n\|_{TV} \leq \epsilon_1$;
(3) for n large, $k \leq k_2(n) \Rightarrow \|m_n^{(k)} - \pi_n\|_{TV} \geq \epsilon_2$;
(4) $\lim_{n\to\infty} \frac{k_1(n)-k_2(n)}{a_n} = 0$.

Proof. By definition there exist c_1 and c_2 such that $f_2(c) \geq \epsilon_2$ for $c \geq c_2$ and $f_1(c) \leq \epsilon_1$ for $c \geq c_1$. So it suffices to take $k_1(n) = a_n + c_1 b_n$ and $k_2(n) = a_n - c_2 b_n$ to get the assertion. $\square$

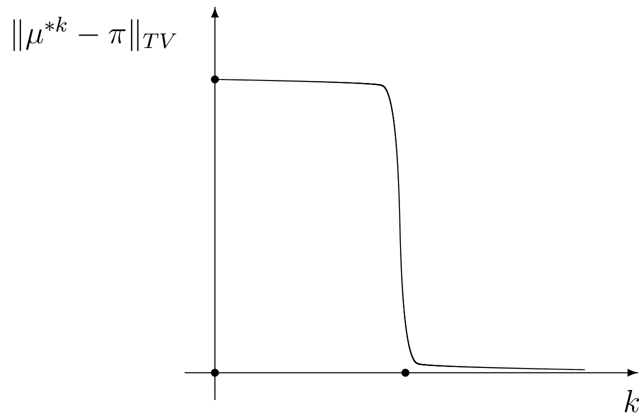


Fig.9: The cut–off phenomenon

The cut-off phenomenon occurs in several examples of Markov chains. In general it can be detected thanks to a careful spectral analysis, as we will do in the proof of the following theorem. In what follows suppose $n \geq 2$.

THEOREM 3.3. *The probability measure associated with the Insect Markov chain converges to the stationary distribution without a cut-off behavior.*

Proof. We want to give an expression for $m^{(k)}(x) = p^{(k)}(x_0, x)$. We get

- If $x = x_0$, then

$$m^{(k)}(x_0) = \frac{1}{q^n}\left\{1 + \sum_{j=1}^{n} q^{j-1}(q-1)\left[1 - \frac{q-1}{q^{n-j+1}-1}\right]^k\right\}.$$

- If $d(x_0, x) = h$, with $1 \leq h \leq n-1$, then

$$\begin{aligned} m^{(k)}(x) &= \frac{1}{q^n}\left\{1 + \sum_{j=1}^{n-h+1} q^{j-1}(q-1)\left[1 - \frac{q-1}{q^{n-j+1}-1}\right]^k \phi_j(x)\right\} \\ &= \frac{1}{q^n}\left\{1 + \sum_{j=1}^{n-h} q^{j-1}(q-1)\left[1 - \frac{q-1}{q^{n-j+1}-1}\right]^k - q^{n-h}\left[1 - \frac{q-1}{q^h-1}\right]^k\right\} \end{aligned}$$

- If $d(x_0, x) = n$, then

$$m^{(k)}(x) = \frac{1}{q^n}\left\{1 - \left[1 - \frac{q-1}{q^n-1}\right]^k\right\}.$$

Let π be the uniform distribution on L_n. Then we have

$$\begin{aligned}\|m^{(k)}-\pi\|_{L^1} &= \frac{1}{q^n}\left\{\sum_{j=1}^{n} q^{j-1}(q-1)\lambda_j^k\right.\\ &+ \sum_{h=1}^{n-1}(q^h-q^{h-1})\left|\sum_{j=1}^{n-h} q^{j-1}(q-1)\lambda_j^k - q^{n-h}\lambda_{n-h+1}^k\right|\\ &+ \left. q^{n-1}(q-1)\lambda_1^k\right\}.\end{aligned}$$

Now observe that

$$\frac{1}{q^n}\sum_{h=1}^{n-1}(q^h-q^{h-1})\sum_{j=1}^{n-h} q^{j-1}(q-1)\lambda_j^k + \frac{1}{q^n}\sum_{j=1}^{n} q^{j-1}(q-1)\lambda_j^k =$$

$$\frac{1}{q^n}\sum_{j=1}^{n-1}\left[1+(q-1)+(q^2-q)+\cdots+(q^{n-j}-q^{n-j-1})\right]\cdot q^{j-1}(q-1)\lambda_j^k =$$

$$\frac{1}{q^n}\sum_{j=1}^{n-1} q^{n-1}(q-1)\lambda_j^k = \frac{q-1}{q}\sum_{j=1}^{n-1}\lambda_j^k$$

and

$$\frac{1}{q^n}\sum_{h=1}^{n-1}(q^h-q^{h-1})q^{n-h}\lambda_{n-h+1}^k + \frac{1}{q^n}(q^n-q^{n-1})\lambda_1^k = \frac{q-1}{q}\sum_{j=1}^{n-1}\lambda_j^k.$$

Using the trivial fact that $\sum_j |a_j-b_j| \leq \sum_j(|a_j|+|b_j|)$, we conclude

$$\|m^{(k)}-\pi\|_{L^1} \leq \frac{2(q-1)}{q}\sum_{j=1}^{n-1}\lambda_j^k.$$

On the other hand

$$\begin{aligned}\|m^{(k)}-\pi\|_{L^1} &\geq \sum_{x:d(x_0,x)=n}|m^{(k)}(x)-\pi(x)|\\ &= \frac{1}{q^n}(q^n-q^{n-1})\lambda_1^k = \frac{q-1}{q}\lambda_1^k.\end{aligned}$$

So we get the following estimate:

$$\frac{q-1}{q}\lambda_1^k \leq \|m^{(k)}-\pi\|_{L^1} \leq \frac{2(q-1)}{q}\sum_{j=1}^{n-1}\lambda_j^k,$$

or, equivalently,

$$\frac{q-1}{2q}\lambda_1^k \leq \|m^{(k)}-\pi\|_{TV} \leq \frac{(q-1)}{q}\sum_{j=1}^{n-1}\lambda_j^k.$$

In what follows the following inequalities will be used:

(1) $(1-x)^k \leq \exp(-kx)$ if $x \leq 1$.
(2) $\frac{q^n-1}{q^{n-j+1}-1} \geq q^{j-1}$, for $j \geq 1$.

(3) $q^{j-1} \geq j$, for $q \geq 2$ and $j \geq 1$.

Choose $k_2(n) = \frac{q^n-1}{q-1}$, then

$$\begin{aligned}
\frac{q-1}{q}\sum_{j=1}^{n-1}\lambda_j^k &\leq \frac{q-1}{q}\sum_{j=1}^{n-1}\exp\left(-\frac{q-1}{q^{n-j+1}-1}k\right) \leq (\text{ if } k \geq k_2(n)) \\
&\leq \frac{q-1}{q}\sum_{j=1}^{n-1}\exp\left(-\frac{q-1}{q^{n-j+1}-1}k_2(n)\right) \\
&\leq \frac{q-1}{q}\sum_{j=1}^{n-1}\exp(-q^{j-1}) \leq \frac{(q-1)}{q}\sum_{j=1}^{n-1}(e^{-j}) \\
&\leq \frac{(q-1)}{q}\sum_{j=1}^{\infty}(e^{-1})^j = \frac{q-1}{q}\cdot\frac{1}{e-1} := \epsilon_2.
\end{aligned}$$

On the other hand, if $k_1(n) = 2\frac{q^n-1}{q-1}$, we get

$$\begin{aligned}
\frac{q-1}{2q}\lambda_1^k &= \frac{q-1}{2q}\left[1-\frac{q-1}{q^n-1}\right]^k \geq (\text{ if } k \leq k_1(n)) \\
&\geq \frac{q-1}{2q}\left[1-\frac{q-1}{q^n-1}\right]^{2\frac{q^n-1}{q-1}} := \epsilon_1.
\end{aligned}$$

Now $k_1(n) > k_2(n), \epsilon_1 < \epsilon_2$ and

- for $k \geq k_2(n)$ we have $\|m^{(k)} - \pi\|_{TV} \leq \epsilon_2$,
- for $k \leq k_1(n)$ we have $\|m^{(k)} - \pi\|_{TV} \geq \epsilon_1$.

This implies that cut-off phenomenon does not occur in this case by Proposition 3.2. In fact, the sequences $k_1(n)$ and $k_2(n)$ cannot satisfy condition (4) of Proposition 3.2. This gives the assertion. □

REMARK 3.4.

Using the same strategy of Theorem 3.3 one can easily check that cut-off phenomenon does not occur also if we fix n and let $q \to +\infty$.

REMARK 3.5.

If $n = 1$ we get the simple random walk on the complete graph K_q on q vertices, in which each vertex has a loop. It is straightforward that the first is performed choosing equiprobably one of the q vertices and so the probability measure $m^{(1)}$ equals the uniform distribution π on the set of the vertices.

4. Orthogonal block structures

This section is devoted to introduce a Markov chain in a general structure. One can observe the similitude with the construction performed in Chapter 2 Section 6.

In effect here, we consider partitions and not anchestral relations. This is a generalization that does not require group theory.

4.1. Preliminaries. The following definitions can be found in [**BaCa**]. Given a partition F of a finite set Ω, let R_F be the relation matrix of F, i.e.

$$R_F(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same part of } F \\ 0 & \text{otherwise.} \end{cases}$$

If $R_F(\alpha,\beta)=1$, we usually write $\alpha \sim_F \beta$.

Definition 4.1. *A partition F of Ω is uniform if all its parts have the same size. This number is denoted k_F.*

The trivial partitions of Ω are the *universal* partition U, which has a single part and whose relation matrix is J_Ω, and the *equality partition* E, all of whose parts are singletons and whose relation matrix is I_Ω.

The partitions of Ω constitute a poset with respect to the relation $\preccurlyeq$, where $F \preccurlyeq G$ if every part of F is contained in a part of G. We use $F \lhd G$ if $F \preccurlyeq G$ and $F \preccurlyeq H \preccurlyeq G$ implies $H = F$ or $H = G$. Given any two partitions F and G, their *infimum* is denoted $F \wedge G$ and is the partition whose parts are intersections of $F-$parts with $G-$parts; their *supremum* is denoted $F \vee G$ and is the partition whose parts are minimal subject to being unions of $F-$parts and $G-$parts.

Definition 4.2. *A set $\mathcal{F}$ of uniform partitions of Ω is an orthogonal block structure if:*

(1) $\mathcal{F}$ contains U and E;
(2) for all F and $G \in \mathcal{F}$, $\mathcal{F}$ contains $F \wedge G$ and $F \vee G$;
(3) for all F and $G \in \mathcal{F}$, the matrices R_F and R_G commute with each other.

4.2. Probability. Let $\mathcal{F}$ be an orthogonal block structure on the finite set Ω. We want to associate with $\mathcal{F}$ a Markov chain on Ω. To perform this, we have to define a new poset $(P,\leq)$ starting from the partitions in $\mathcal{F}$.

Let $C = \{E = F_0, F_1, \ldots, F_n = U\}$ a maximal chain of partitions such that $F_i \lhd F_{i+1}$ for all $i = 0,\ldots,n-1$. Let us design a rooted tree of depth n as follows: the $n-$th level is constituted by $|\Omega|$ vertices; the $(n-1)-$th by $\frac{|\Omega|}{k_{F_1}}$ vertices. Each of these vertices is a father of k_{F_1} sons that are in the same F_1-class. Inductively, at the $i-$th level there are $\frac{|\Omega|}{k_{F_{n-i}}}$ vertices fathers of $k_{F_{n-i}}$ vertices of the $(i+1)-$th level belonging to the same $F_{n-i}-$class.

We can perform the same construction for every maximal chain C in $\mathcal{F}$. The next step is to glue the different structures identifying the vertices associated with the same partition. The resulting structure is the poset $(P,\leq)$.

Example 4.3.

Consider the set $\Omega = \{000, 001, 010, 011, 100, 101, 110, 111\}$ and the set of partitions of Ω given by $\mathcal{F} = \{E, F_1, F_2, F_3, U\}$ where, as usually, E denotes the equality partition and U the universal partition of Ω. The nontrivial partitions are defined as:

- $F_1 = \{000, 001, 010, 011\} \coprod \{100, 101, 110, 111\}$;
- $F_2 = \{000, 001\} \coprod \{010, 011\} \coprod \{100, 101\} \coprod \{110, 111\}$;
- $F_3 = \{000, 010\} \coprod \{001, 011\} \coprod \{100, 110\} \coprod \{101, 111\}$.

So the orthogonal block structure $\mathcal{F}$ can be represented as the following poset:

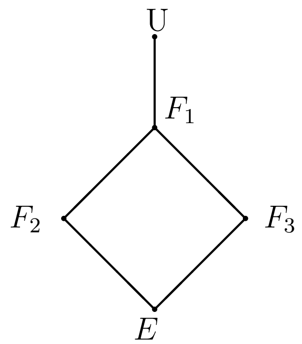


Fig.10. The orthogonal block structure $\mathcal{F} = \{E, F_1, F_2, F_3, U\}$.

The maximal chains in $\mathcal{F}$ have length 3 and they are:

- $C_1 = \{E, F_2, F_1, U\}$;
- $C_2 = \{E, F_3, F_1, U\}$.

The associated rooted trees T_1 and T_2 have depth 3 and they are, respectively,

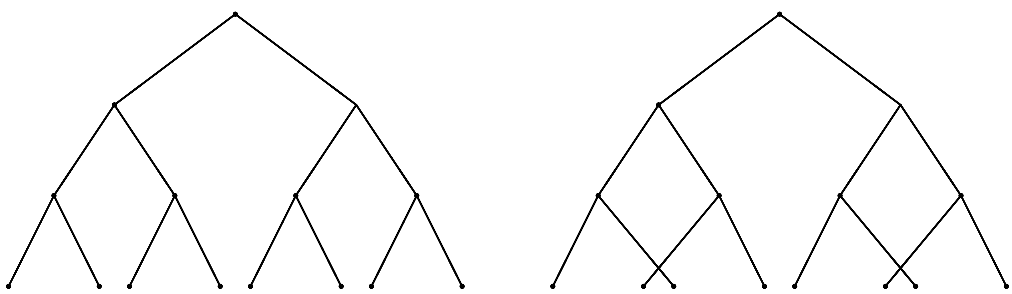


Fig.11. The rooted trees associated with C_1 and C_2.

So the poset $(P, \leq)$ associated with $\mathcal{F}$ is

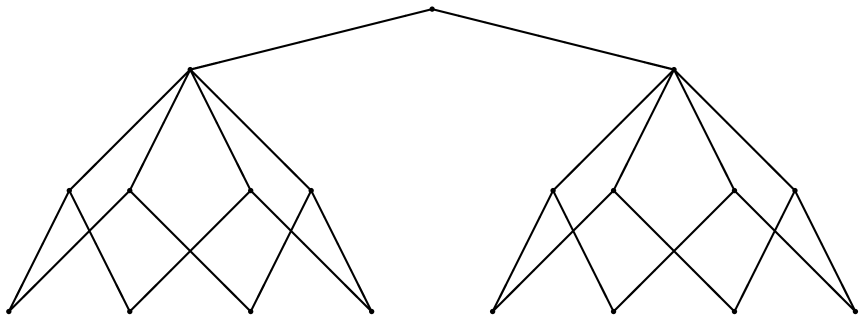


Fig.12. The poset $(P, \leq)$ associated with $\mathfrak{F} = \{E, F_1, F_2, F_3, U\}$.

Observe that, if $F_1 \lhd F_2$, then the number of F_1-classes contained in a F_2-class is k_{F_2}/k_{F_1}.

The Markov chain that we want to describe is performed on the last level of the poset $(P, \leq)$ associated with the set $\mathfrak{F}$. We can think of an insect which lies at the starting time on a fixed element ω_0 of Ω (this corresponds to the identity relation E, i.e. each element is in relation only with itself). The insect randomly moves reaching an adjacent site in $(P, \leq)$ (this corresponds, in the orthogonal block structure $\mathfrak{F}$, to move from E to another relation F such that $E \lhd F$, i.e. ω_0 is identified with all the elements in the same $F-$class) and so on. At each step in $(P, \leq)$ (that does not correspond necessarily to a step in the Markov chain on Ω) the insect could randomly move from the $i-$th level of $(P, \leq)$ either to the $(i-1)-$th level or to the $(i+1)-$th level. Going up means to pass in $\mathfrak{F}$ from a partition F to a partition L such that $F \lhd L$ (these are $|\{L \in \mathfrak{F} : F \lhd L\}|$ possibilities in $(P, \leq)$), going down means to pass in $\mathfrak{F}$ to a partition J such that $J \lhd F$ (these are $\sum_{J \in \mathfrak{F}: J \lhd F} \frac{k_F}{k_J}$ possibilities in $(P, \leq)$). The random walk on Ω stops whenever the insect reaches once again the last level in $(P, \leq)$. In order to describe this idea let us introduce the following definitions.

Let $\alpha_{F,G}$ the probability of moving from the partition F to the partition G. So the following relation is satisfied:

$$(12) \quad \alpha_{F,G} = \frac{1}{\sum_{J \in \mathfrak{F}: J \lhd F}(k_F : k_J) + |\{L \in \mathfrak{F} : F \lhd L\}|} + \sum_{J \in \mathfrak{F}: J \lhd F} \frac{(k_F : k_J)\alpha_{J,F}\alpha_{F,G}}{\sum_{J \in \mathfrak{F}: J \lhd F}(k_F : k_J) + |\{L \in \mathfrak{F} : F \lhd L\}|}.$$

In fact, the insect can directly pass from F to G with probability $\alpha_{F,G}$ or go down to any J such that $J \lhd F$ and then come back to F with probability $\alpha_{J,F}$ and one starts the recursive argument. From direct computations one gets

$$\alpha_{E,F} = \frac{1}{|\{L \in \mathcal{F} : E \lhd L\}|}. \tag{13}$$

Moreover, if $\alpha_{E,F} = 1$ we have, for all G such that $F \lhd G$

$$\alpha_{F,G} = \frac{1}{\sum_{J\in\mathcal{F}:J\lhd F}(k_F : k_J) + |\{L \in \mathcal{F} : F \lhd L\}|}; \tag{14}$$

if $\alpha_{E,F} \neq 1$, the coefficient $\alpha_{F,G}$ is defined as in (12).

Definition 4.4. *For every $\omega \in \Omega$, define*

$$p(\omega_0, \omega) = \sum_{\substack{E\neq F\in\mathcal{F} \\ \omega_0\sim_F\omega}} \sum_{\substack{C\subseteq\mathcal{F}\ chain \\ C=\{E,F_1,\dots,F',F\}}} \frac{\alpha_{E,F_1}\cdots\alpha_{F',F}\left(1-\sum_{F\lhd L}\alpha_{F,L}\right)}{k_F}.$$

The fact that p is effectively a transition probability on Ω will follow from Theorem 4.7. First define the following numbers:

$$p_F = \sum_{\substack{C\subseteq\mathcal{F}\ chain \\ C=\{E,F_1,\dots,F',F\}}} \alpha_{E,F_1}\cdots\alpha_{F',F}\left(1-\sum_{F\lhd L}\alpha_{F,L}\right). \tag{15}$$

Observe that the coefficient p_F expresses the probability of reaching the partition F but no partition L such that $F \prec L$ in $\mathcal{F}$.

Lemma 4.5. *The coefficients p_F's defined in (15) satisfy the following identity:*

$$\sum_{E\neq F\in\mathcal{F}} p_F = 1.$$

Proof. Using the definitions we have

$$\begin{aligned}
\sum_{E\neq F\in\mathcal{F}} p_F &= \sum_{E\neq F\in\mathcal{F}} \sum_{\substack{C\subseteq\mathcal{F}\ chain \\ C=\{E,F_1,\dots,F',F\}}} \alpha_{E,F_1}\cdots\alpha_{F',F}\left(1-\sum_{F\lhd L}\alpha_{F,L}\right) \\
&= \sum_{E\lhd F}\alpha_{E,F} = 1.
\end{aligned}$$

In fact, for every $F \in \mathcal{F}$ such that $E \ntriangleleft F$, given a chain $C = \{E, F_1, \dots, F', F\}$ we get the terms $\alpha_{E,F_1}\cdots\alpha_{F',F}\left(1-\sum_{F\lhd L}\alpha_{F,L}\right)$. Since $C = \{E, F_1, \dots, F', F, L\}$ is still a term of the sum one can check that only the summands $\sum_{E\lhd F}\alpha_{E,F}$ are not cancelled. The thesis follows from (13). $\square$

For every $F \in \mathfrak{F}, F \neq E$ define M_F as the Markov operator whose transition matrix is

$$M_F = \frac{1}{k_F} R_F. \tag{16}$$

DEFINITION 4.6. *Given the operators M_F's as in (16) and the coefficients p_F's as in (15), set*

$$M = \sum_{E \neq F \in \mathfrak{F}} p_F M_F. \tag{17}$$

By abuse of notation, we denote M the stochastic matrix associated with the Markov operator M.

THEOREM 4.7. *M coincides with the transition matrix of p.*

Proof. By computation we get:

$$\begin{aligned} M(\omega_0, \omega) &= \sum_{E \neq F \in \mathfrak{F}} p_F M_F(\omega_0, \omega) = \sum_{\substack{E \neq F \in \mathfrak{F} \\ \omega_0 \sim_F \omega}} p_F \cdot \frac{1}{k_F} \\ &= \sum_{\substack{E \neq F \in \mathfrak{F} \\ \omega_0 \sim_F \omega}} \sum_{\substack{C \subseteq \mathfrak{F} \text{ chain} \\ C = \{E, F_1, \ldots, F', F\}}} \frac{\alpha_{E,F_1} \cdots \alpha_{F',F} \left(1 - \sum_{F \lhd L} \alpha_{F,L}\right)}{k_F} \\ &= p(\omega_0, \omega). \end{aligned}$$

□

4.3. Spectral analysis of M. We want to give the spectral analysis of the operator M acting on the space $L(\Omega)$ of the complex functions defined on the set Ω. First of all introduce (see, for example, [**Bai**]), for every $F \in \mathfrak{F}$, the following subspaces of $L(\Omega)$:

$$V_F = \{f \in L(\Omega) : \ f(\alpha) = f(\beta) \text{ if } \alpha \sim_F \beta\}.$$

It is easy to show that the operator M_F defined in (16) is the projector onto V_F. In fact if $f \in L(\Omega)$, then $M_F f(\omega_0)$ is the average of the values that f takes on the elements ω such that $\omega \sim_F \omega_0$ and so $M_F f = f$ if $f \in V_F$ and $M_F f = 0$ if $f \in V_F^{\perp}$.

Set

$$W_G = V_G \cap (\sum_{G \prec F} V_F)^{\perp}.$$

In [**Bai**] is proven that $L(\Omega) = \bigoplus_{G \in \mathfrak{F}} W_G$. We can deduce the following proposition.

Proposition 4.8. *The W_G's are eigenspaces for the operator M with associated eigenvalue*

$$\lambda_G = \sum_{\substack{E \neq F \in \mathcal{F} \\ F \preccurlyeq G}} p_F. \tag{18}$$

Proof. By definition, $W_G \subseteq V_G$. This implies that, if $f \in W_G$,

$$M_F f = \begin{cases} f & \text{if } F \preccurlyeq G \\ 0 & \text{otherwise} \end{cases}$$

So we get

$$\begin{aligned} M \cdot W_G &= \sum_{E \neq F \in \mathcal{F}} p_F M_F \cdot W_G \\ &= \Big(\sum_{\substack{E \neq F \in \mathcal{F} \\ F \preccurlyeq G}} p_F \Big) \cdot W_G. \end{aligned}$$

Hence the eigenvalue λ_G associated with the eigenspace W_G is

$$\lambda_G = \sum_{\substack{E \neq F \in \mathcal{F} \\ F \preccurlyeq G}} p_F.$$

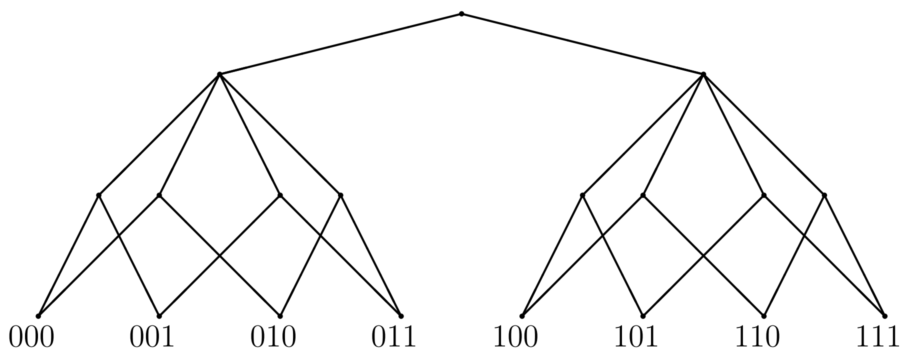
and the assertion follows. □

Example 4.9.

We want to study the transition probability p in the case of the orthogonal block structure of the Example 4.3. One can easily verify that we have:

- $\alpha_{E,F_2} = \alpha_{E,F_3} = \alpha_{F_2,F_1} = \alpha_{F_3,F_1} = \frac{1}{2}$;
- $\alpha_{F_1,U} = \frac{1}{3}$.

Let us compute the transition probability p on the last level of $(P, \leq)$:



We have:

$$\begin{aligned}
p(000,000) &= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}+2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{8}=\frac{17}{48};\\
p(000,001) &= p(000,010)\\
&= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}+2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{8}=\frac{11}{48};\\
p(000,011) &= 2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}+2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{8}=\frac{5}{48};\\
p(000,100) &= p(000,101)=p(000,110)=p(000,111)\\
&= 2\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{8}=\frac{1}{48}.
\end{aligned}$$

The corresponding transition matrix is given by

$$P=\frac{1}{48}\begin{pmatrix}
17 & 11 & 11 & 5 & 1 & 1 & 1 & 1\\
11 & 17 & 5 & 11 & 1 & 1 & 1 & 1\\
11 & 5 & 17 & 11 & 1 & 1 & 1 & 1\\
5 & 11 & 11 & 17 & 1 & 1 & 1 & 1\\
1 & 1 & 1 & 1 & 17 & 11 & 11 & 5\\
1 & 1 & 1 & 1 & 11 & 17 & 5 & 11\\
1 & 1 & 1 & 1 & 11 & 5 & 17 & 11\\
1 & 1 & 1 & 1 & 5 & 11 & 11 & 17
\end{pmatrix}$$

The coefficients P_F, with $E \neq F$, are the following (see (15)):

- $p_U = 2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{6}$;
- $p_{F_1} = 2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$;
- $p_{F_2} = \frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$;
- $p_{F_3} = \frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$.

The Markov operator M is given by (see (17) and (16)):

$$M=\frac{1}{4}M_{F_2}+\frac{1}{4}M_{F_3}+\frac{1}{3}M_{F_1}+\frac{1}{6}M_U$$

and its eigenvalues are from formula (18) the following:

- $\lambda_U = 1$;
- $\lambda_{F_1} = \frac{5}{6}$;
- $\lambda_{F_2} = \frac{1}{4}$;
- $\lambda_{F_3} = \frac{1}{4}$;
- $\lambda_E = 0$;

In the case of poset block structure, that is a particular case of the orthogonal block structure, we get the same decomposition using the following easy lemmas.

LEMMA 4.10. *There exists a one-to-one correspondence between antichains and ancestral subsets of I.*

Proof. First of all, we prove that given an antichain S the set $A_S = I \setminus H[S]$ is ancestral. Suppose $i \in A_S$ and $j > i$, then it must be $j \in A_S$. In fact, if $j \in H[S]$, then we should have $i \in H(S)$, since $i < j$; this is absurd.

Now let us show that this correspondence is injective. Suppose that, given two antichains S_1 and S_2, with $S_1 \neq S_2$, one gets $A_{S_1} = A_{S_2}$. This implies that $H[S_1] = H[S_2]$. By hypothesis we can suppose without loss of generality that there exists $s_1 \in S_1 \setminus (S_1 \cap S_2)$. Hence $s_1 \in H(S_2)$ and there exists $s_2 \in S_2$ such that $s_1 < s_2$. So $s_2 \in H[S_1]$. In particular, if $s_2 \in S_1$ we have an absurd because S_1 is an antichain, if $s_2 \in H(S_1)$ there exists $s_1' \in S_1$ such that $s_1' > s_2 > s_1$, absurd again.

So the application $S \longrightarrow I \setminus H[S]$, for each S antichain, is injective.

Given an ancestral set J, define the set of the maximal elements in $I \setminus J$ as $S_J = \{i \in I \setminus J : \; A(i) \cap (I \setminus J) = \emptyset\}$. It is easy to prove that S_J is an antichain. In fact, if $i, j \in S_J$ then if $i < j$ or $i > j$ one of i or j is not maximal.

Now we want to show that $J = I \setminus H[S_J]$, that is equivalent to show that $I \setminus J = H[S_J]$. First we have that $I \setminus J \subseteq H[S_J]$ because if i is maximal in $I \setminus J$ than it belongs to S_J, otherwise there exists j in S_J such that $i < j$, and so $i \in H[S_J]$. On the other hand, let i be in $H[S_J]$. If i is in S_J, then it is in $I \setminus J$ by definition. If i is in $H(S_J)$ there exists j in S_J such that $i < j$. Now if i is an element of J then j has the same property since J is ancestral and this is absurd and so $H[S_J] \subseteq I \setminus J$. This shows that $J = I \setminus H[S_J]$.

From this we have the equivalence $S \longleftrightarrow I \setminus H[S]$ between antichains and ancestral sets. $\square$

Remark 4.11.

Observe that, for $S = \emptyset$, one gets $A_S = I$.

Remark 4.12.

Observe that all the maximal chains in $\mathcal{A}$ have the same length n. In fact, the empty set is always ancestral. A singleton $\{i\}$ constituted by a maximal element in I is still an ancestral set. Inductively, if $J \in \mathcal{A}$ is an ancestral set, then $J \sqcup \{i\}$ is an ancestral set if i is a maximal element in $I \setminus J$. So every maximal chain in the poset of ancestral subsets has length n. In particular, the empty set $\emptyset$ corresponds to the universal partition U and I to the equality partition E in $\sim_{\mathcal{A}}$.

Remark 4.13.

Observe that the operator $M_{\sim_J} =: M_J$ can be obtained as follows:

$$M_J = \left(\bigotimes_{i \in I \setminus H[S_J]} I_i \right) \otimes \left(\bigotimes_{i \in H[S_J]} U_i \right), \tag{19}$$

where I_i denotes the identity operator on Δ_i and U_i is the uniform operator on Δ_i, whose adjacency matrix is $\frac{1}{m}J_i$, where

$$J_i = \frac{1}{m_i}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix}.$$

.

Considering the action of M on the spherical function ϕ_S (given in Proposition 6.13 Section 6 of Chapter 2) we get the following eigenvalue λ_S for ϕ_S:

$$\lambda_S = \sum_{\emptyset \neq S_J : S \subseteq I \setminus H[S_J]} p_{\sim_J}. \tag{20}$$

REMARK 4.14.

One can observe that the eigenspaces and the corresponding eigenvalues have been indexed by the antichains of the poset I in (7) and in (20); instead in the first part they are indexed by the relations of the orthogonal poset block $\mathcal{F}$. The correspondence is the following.

Given a relation $G \in \mathcal{F}$, it can be regarded as an ancestral relation $\sim_J$, for some ancestral subset $J \subseteq I$. Set

$$S = \{i \in J : H(i) \cap J = \emptyset\}.$$

It is clear that S is an antichain of I. From the definition it follows that

$$A(S) = J \setminus S \text{ and } I \setminus A[S] = I \setminus J.$$

The corresponding eigenspace W_S becomes:

$$W_S = \left(\bigotimes_{i \in J\setminus S} L(\Delta_i)\right) \otimes \left(\bigotimes_{i \in S} V_i^1\right) \otimes \left(\bigotimes_{i \in I \setminus J} V_i^0\right).$$

It is easy to check that the functions in W_S are constant on the equivalence classes of the relation $\sim_J$. Moreover, these functions are orthogonal to the functions which are constant on the equivalence classes of the relation $\sim_{J'}$, with $\sim_{J'} \rhd \sim_J$ (where J' is obtained from J deleting an element of S). Since the orthogonality with the functions constant on $\sim_{J'}$ implies the orthogonality with all functions constant on $\sim_L$, where $\sim_L \succ \sim_J$, then we have $W_S \subseteq W_G$. On the other hand, it is easy to verify that

$$\dim(W_S) = \dim(W_G) = m^{|J\setminus S|} \cdot (m-1)^{|S|},$$

and so we have $W_S = W_G$.

Analogously, if $G =\sim_J$, from (20) we get

$$\lambda_S = \sum_{\emptyset\neq S_K : S\subseteq I\setminus H[S_K]} p_{\sim_K} = \sum_{I\neq K : S\subseteq K} p_{\sim_K},$$

since $S_K = \{i \in I \setminus K : A(i) = \emptyset\}$ and $H[S_K] = I \setminus K$ whose consequence is $I \setminus H[S_K] = K$. Moreover, since $S \subseteq K$ if and only if $J \subseteq K$, we get

$$\lambda_S = \sum_{I\neq K : J\subseteq K} p_{\sim_K} = \sum_{E\neq\sim_K : \sim_K\preccurlyeq\sim_J} p_{\sim_K} = \lambda_G.$$

5. First and Second crested product

In this section we introduce a particular product of Markov chains defined on different sets. This idea is inspired to the definition of crested product for association schemes given in [**BaCa**].

5.1. The First Crested Product. In this subsection we introduce a particular product of Markov chains defined on different sets. This idea is inspired to the definition of crested product for association schemes given in [**BaCa**].

Let X_i be a finite set, with $|X_i| = m_i$, for every $i = 1, \dots, n$, so that we can identify X_i with the set $\{0, 1, \dots, m_i - 1\}$. Let P_i be an irreducible Markov chain on X_i and let p_i be the transition probability associated with P_i. Moreover, assume that p_i is in detailed balance with the strict probability measure σ_i on X_i, i.e.

$$\sigma_i(x)p_i(x, y) = \sigma_i(y)p_i(y, x),$$

for all $x, y \in X_i$.

Consider the product $X_1 \times \cdots \times X_n$. Let $\{1, \dots, n\} = C \coprod N$ be a partition of the set $\{1, \dots, n\}$ and let $p_1^0, p_2^0, \dots, p_n^0$ a probability distribution on $\{1, \dots, n\}$, i.e. $p_i^0 > 0$ for every $i = 1, \dots, n$ and $\sum_{i=1}^n p_i^0 = 1$.

Definition 5.1. *The first crested product of Markov chains P_i's with respect to the partition $\{1, \dots, n\} = C \coprod N$ is the Markov chain on the product $X_1 \times \cdots \times X_n$ whose transition matrix is*

$$\begin{aligned} P \;&=\; \sum_{i\in C} p_i^0 \left(I_1 \otimes \cdots \otimes I_{i-1} \otimes P_i \otimes I_{i+1} \otimes \cdots \otimes I_n\right) \\ &+\; \sum_{i\in N} p_i^0 \left(I_1 \otimes \cdots \otimes I_{i-1} \otimes P_i \otimes J_{i+1} \otimes \cdots \otimes J_n\right), \end{aligned}$$

where I_i denotes the identity matrix of size m_i and J_i denotes the uniform matrix on X_i.

In other words, we choose an index i in $\{1, \dots, n\}$ following the distribution $p_1^0, \dots, p_n^0$. If $i \in C$, then P acts on the $i-$th coordinate by the matrix P_i and fixes the remaining coordinates; if $i \in N$, then P fixes the coordinates $\{1, \dots, i-1\}$, acts on the $i-$th coordinate by the matrix P_i and changes uniformly the remaining ones.

For all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in X_1 \times \cdots \times X_n$, the transition probability p associated with P is given by

$$p((x_1, \ldots, x_n), (y_1, \ldots, y_n)) =$$

$$\sum_{i \in C} p_i^0 (\delta_1(x_1, y_1) \cdots \delta_{i-1}(x_{i-1}, y_{i-1}) p_i(x_i, y_i) \delta_{i+1}(x_{i+1}, y_{i+1}) \cdots \delta_n(x_n, y_n))$$

$$+ \sum_{i \in N} p_i^0 \left(\frac{\delta_1(x_1, y_1) \cdots \delta_{i-1}(x_{i-1}, y_{i-1}) p_i(x_i, y_i)}{\prod_{j=i+1}^n m_j} \right),$$

where δ_i is defined by

$$\delta_i(x_i, y_i) = \begin{cases} 1 & \text{if } x_i = y_i, \\ 0 & \text{otherwise.} \end{cases}$$

We want to study the spectral theory of the operator P. We recall that the following isomorphism holds:

$$L(X_1 \times \cdots \times X_n) \cong \bigotimes_{i=1}^n L(X_i),$$

with $(f_1 \otimes \cdots \otimes f_n)(x_1, \ldots, x_n) := f_1(x_1) f_2(x_2) \cdots f_n(x_n)$.

Assume that, for every $i = 1, \ldots, n$, the following spectral decomposition holds:

$$L(X_i) = \bigoplus_{j_i=0}^{r_i} V_{j_i}^i,$$

i.e. V_{j_i} is an eigenspace for P_i with associated eigenvalue λ_{j_i} and whose dimension is m_{j_i}.

Now set $N = \{i_1, \ldots, i_l\}$ and $C = \{c_1, \ldots, c_h\}$, with $h + l = n$ and such that $i_1 < \ldots < i_l$ and $c_1 < \ldots < c_h$.

Theorem 5.2. *The probability P defined above is reversible if and only if P_k is symmetric for every $k > i_1$. If this is the case, P is in detailed balance with the strict probability measure π on $X_1 \times \cdots \times X_n$ given by*

$$\pi(x_1, \ldots, x_n) = \frac{\sigma_1(x_1) \sigma_2(x_2) \cdots \sigma_{i_1}(x_{i_1})}{m_{i_1+1} \cdots m_n}.$$

Proof. Consider the elements $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ belonging to $X_1 \times \cdots \times X_n$. First, we want to prove that the condition $\sigma_k = \frac{1}{m_k}$, for every $k > i_1$, is sufficient. Let $k \in \{1, \ldots, n\}$ such that $x_i = y_i$ for every $i = 1, \ldots, k-1$ and $x_k \neq y_k$. Suppose $k < i_1$. Then we have

$$p(x, y) = p_k^0 \left(p_k(x_k, y_k) \delta_{k+1}(x_{k+1}, y_{k+1}) \cdots \delta_n(x_n, y_n) \right).$$

If $x_i = y_i$ for every $i = k+1, \dots, n$, we get

$$\begin{aligned}
\pi(x)p(x,y) &= \sigma_1(x_1)\cdots\sigma_k(x_k)\cdots\sigma_{i_1}(x_{i_1})p_k^0\frac{p_k(x_k,y_k)}{m_{i_1+1}\cdots m_n} \\
&= \sigma_1(y_1)\cdots\sigma_k(y_k)\cdots\sigma_{i_1}(y_{i_1})p_k^0\frac{p_k(y_k,x_k)}{m_{i_1+1}\cdots m_n} \\
&= \pi(y)p(y,x),
\end{aligned}$$

since $\sigma_k(x_k)p_k(x_k,y_k) = \sigma_k(y_k)p_k(y_k,x_k)$. If the condition $x_i = y_i$ is not satisfied for every $i = k+1, \dots, n$, then the equality $\pi(x)p(x,y) = \pi(y)p(y,x) = 0$ easily follows.

If $k = i_1$, then we get

$$p(x,y) = p_{i_1}^0\left(p_{i_1}(x_{i_1},y_{i_1})\frac{1}{m_{i_1+1}\cdots m_n}\right)$$

and so

$$\begin{aligned}
\pi(x)p(x,y) &= \sigma_1(x_1)\cdots\sigma_{i_1}(x_{i_1})p_{i_1}^0\frac{p_{i_1}(x_{i_1},y_{i_1})}{m_{i_1+1}^2\cdots m_n^2} \\
&= \sigma_1(y_1)\cdot\cdots\cdot\sigma_{i_1}(y_{i_1})p_{i_1}^0\frac{p_{i_1}(y_{i_1},x_{i_1})}{m_{i_1+1}^2\cdots m_n^2} \\
&= \pi(y)p(y,x),
\end{aligned}$$

since $\sigma_{i_1}(x_{i_1})p_{i_1}(x_{i_1},y_{i_1}) = \sigma_{i_1}(y_{i_1})p_{i_1}(y_{i_1},x_{i_1})$.

In the case $k > i_1$, we have

$$p(x,y) = \sum_{i\in N, i\le k} p_i^0\frac{p_i(x_i,y_i)}{m_{i+1}\cdots m_n}$$

and so

$$\begin{aligned}
\pi(x)p(x,y) &= \frac{\sigma_1(x_1)\cdots\sigma_{i_1}(x_{i_1})}{m_{i_1+1}\cdots m_n}\sum_{i\in N, i\le k} p_i^0\frac{p_i(x_i,y_i)}{m_{i+1}\cdots m_n} \\
&= \frac{\sigma_1(y_1)\cdots\sigma_{i_1}(y_{i_1})}{m_{i_1+1}\cdots m_n}\sum_{i\in N, i\le k} p_i^0\frac{p_i(y_i,x_i)}{m_{i+1}\cdots m_n} \\
&= \pi(y)p(y,x).
\end{aligned}$$

In fact, the terms corresponding to an index $i < k$ satisfy $p_i(x_i,y_i) = p_i(y_i,x_i)$ since $x_i = y_i$, the term corresponding to the index k satisfies $p_k(x_k,y_k) = p_k(y_k,x_k)$ since the equality

$$p_k(x_k,y_k) = p_k(y_k,x_k)$$

holds by hypothesis.

Now we want to prove that the condition $\sigma_k = \frac{1}{m_k}$, for every $k > i_1$, is also necessary. Suppose that the equality $\pi(x)p(x,y) = \pi(y)p(y,x)$ holds. By the hypothesis of irreducibility we can consider two elements

$x^0, y^0 \in X_1 \times \cdots \times X_n$ such that $x^0_{i_1} \neq y^0_{i_1}$ and with the property that $p_{i_1}(x^0_{i_1}, y^0_{i_1}) \neq 0$. Now we have

$$\pi(x^0)p(x^0, y^0) = \pi(y^0)p(y^0, x^0) \Leftrightarrow \pi(x^0)p_{i_1}(x^0_{i_1}, y^0_{i_1}) = \pi(y^0)p_{i_1}(y^0_{i_1}, x^0_{i_1}).$$

This gives

$$\frac{\pi(x^0)}{\pi(y^0)} = \frac{p_{i_1}(y^0_{i_1}, x^0_{i_1})}{p_{i_1}(x^0_{i_1}, y^0_{i_1})} = \frac{\sigma_{i_1}(x^0_{i_1})}{\sigma_{i_1}(y^0_{i_1})}.$$

Consider now the element $x = (x^0_1, \ldots, x^0_{i_1}, y^0_{i_1+1}, \ldots, y^0_n)$. The equality $\pi(x)p(x, y^0) = \pi(y^0)p(y^0, x)$ implies

$$\frac{\pi(x)}{\pi(y^0)} = \frac{p_{i_1}(y^0_{i_1}, x^0_{i_1})}{p_{i_1}(x^0_{i_1}, y^0_{i_1})} = \frac{\sigma_{i_1}(x^0_{i_1})}{\sigma_{i_1}(y^0_{i_1})}.$$

So we get $\pi(x^0) = \pi(x)$, i.e. the probability π does not depend from the coordinates $i_1+1, \ldots, n$. Set now $x' = (x^0_1, \ldots, x^0_{i_1}, \ldots, x^0_{k-1}, x_k, \ldots, x_n)$. The equality $\pi(x^0)p(x^0, x') = \pi(x')p(x', x^0)$ gives

$$\pi(x^0)\left(\sum_{j \in N, j \leq k} p^0_j(p_j(x^0_j, x'_j))\right) = \pi(x')\left(\sum_{j \in N, j \leq k} p^0_j(p_j(x'_j, x^0_j))\right).$$

Since the probability π does not depend from the coordinates $i_1 + 1, \ldots, n$, we get $p_k(x^0_k, x'_k) = p_k(x'_k, x^0_k)$. This implies $\sigma_k(x'_k) = \sigma_k(x^0_k)$ and so the hypothesis of irreducibility guarantees that σ_k is uniform on X_k. This completes the proof. $\square$

THEOREM 5.3. *The eigenspaces of the operator P are given by*

- $W^1 \otimes \cdots \otimes W^{k-1} \otimes V^k_{j_k} \otimes V^{k+1}_0 \otimes V^{k+2}_0 \otimes \cdots \otimes V^n_0$, *with $j_k \neq 0$, for $k \in \{i_1 + 1, \ldots, n\}$ and where*

$$W^i = \begin{cases} L(X_i) & \text{if } i \in N, \\ V^i_{j_i}, \quad j_i = 0, \ldots, r_i & \text{if } i \in C, \end{cases}$$

with eigenvalue

$$\sum_{i \in C: i < k} p^0_i \lambda_{j_i} + p^0_k \lambda_{j_k} + \sum_{i > k} p^0_i.$$

- $V^1_{j_1} \otimes \cdots \otimes V^{i_1-1}_{j_{i_1-1}} \otimes V^{i_1}_{j_{i_1}} \otimes V^{i_1+1}_0 \otimes \cdots \otimes V^n_0$, *with $j_t = 0, \ldots, r_t$, for every $t = 1, \ldots, i_1$, with eigenvalue*

$$\sum_{i=1}^{i_1} p^0_i \lambda_{j_i} + \sum_{i=i_1+1}^{n} p^0_i.$$

Proof. Fix an index $k \in \{i_1 + 1, i_1 + 2, \ldots, n\}$ and consider the function φ in the space

$$W^1 \otimes \cdots \otimes W^{k-1} \otimes V^k_{j_k} \otimes V^{k+1}_0 \otimes V^{k+2}_0 \otimes \cdots \otimes V^n_0,$$

with $j_k \neq 0$ and

$$W^i = \begin{cases} L(X_i) & \text{if } i \in N, \\ V^i_{j_i}, \quad j_i = 0, \dots, r_i & \text{if } i \in C, \end{cases}$$

so that $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_{k-1} \otimes \varphi_k \otimes \varphi_{k+1} \otimes \cdots \otimes \varphi_n$ with $\varphi_i \in W^i$ for $i = 1, \dots, k-1$, $\varphi_k \in V^k_{j_k}$ and $\varphi_l \in V^l_0$ for $l = k+1, \dots, n$. Set $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, then

$$\begin{aligned}
(P\varphi)(x) &= \sum_y p(x,y)\varphi(y) \\
&= \sum_y \left(\sum_{i \in C} p_i^0 \delta_1(x_1,y_1) \cdots \delta_{i-1}(x_{i-1},y_{i-1}) p_i(x_i,y_i) \delta_{i+1}(x_{i+1},y_{i+1}) \cdots \delta_n(x_n,y_n) \right. \\
&+ \left. \sum_{i \in N} p_i^0 \delta_1(x_1,y_1) \cdots \delta_{i-1}(x_{i-1},y_{i-1}) p_i(x_i,y_i) \frac{1}{m_{i+1}} \cdots \frac{1}{m_n} \right) \\
&\times \varphi_1(y_1) \cdots \varphi_{k-1}(y_{k-1}) \varphi_k(y_k) \varphi_{k+1}(y_{k+1}) \cdots \varphi_n(y_n) \\
&= \sum_{i \in C,\ i \leq k} \left(\sum_{y_i} p_i^0 p_i(x_i,y_i) \varphi_i(y_i) \right) \varphi_1(x_1) \cdots \varphi_{i-1}(x_{i-1}) \varphi_{i+1}(x_{i+1}) \cdots \varphi_n(x_n) \\
&+ \sum_{i \in C,\ i > k} \left(\sum_{y_i} p_i^0 p_i(x_i,y_i) \varphi_i(y_i) \right) \varphi_1(x_1) \cdots \varphi_{i-1}(x_{i-1}) \varphi_{i+1}(x_{i+1}) \cdots \varphi_n(x_n) \\
&+ \sum_{i \in N,\ i > k} \left(\sum_{y_i, \dots, y_n} p_i^0 p_i(x_i,y_i) \frac{1}{m_{i+1}} \cdots \frac{1}{m_n} \varphi_i(y_i) \cdots \varphi_n(y_n) \right) \varphi_1(x_1) \cdots \varphi_{i-1}(x_{i-1}) \\
&+ \chi_N(k) \sum_{y_k, \dots, y_n} p_k^0 p_k(x_k,y_k) \frac{1}{m_{k+1}} \cdots \frac{1}{m_n} \varphi_1(x_1) \cdots \varphi_{k-1}(x_{k-1}) \varphi_k(y_k) \cdots \varphi_n(y_n) \\
&= \sum_{i \in C,\ i \leq k} p_i^0 \lambda_{j_i} \varphi(x) + \sum_{i \in C,\ i > k} p_i^0 \cdot 1 \cdot \varphi(x) \\
&+ \sum_{i \in N,\ i > k} \left(\sum_{y_i} p_i^0 p_i(x_i,y_i) \varphi_i(y_i) \right) \varphi_1(x_1) \cdots \varphi_{i-1}(x_{i-1}) \varphi_{i+1}(x_{i+1}) \cdots \varphi_n(x_n) \\
&+ \chi_N(k) \sum_{y_k} p_k^0 p_k(x_k,y_k) \varphi_1(x_1) \cdots \varphi_{k-1}(x_{k-1}) \varphi_k(y_k) \varphi_{k+1}(x_{k+1}) \cdots \varphi_n(x_n) \\
&= \sum_{i \in C,\ i \leq k} p_i^0 \lambda_{j_i} \varphi(x) + \sum_{i \in C,\ i > k} p_i^0 \varphi(x) + \sum_{i \in N,\ i > k} p_i^0 \varphi(x) + \chi_N(k) p_k^0 \lambda_{j_k} \varphi(x) \\
&= \left(\sum_{i \in C,\ i < k} p_i^0 \lambda_{j_i} + p_k^0 \lambda_{j_k} + \sum_{i > k} p_i^0 \right) \varphi(x),
\end{aligned}$$

where χ_N is the characteristic function of N. Note that in this case the addends corresponding to the indices $i < k$, $i \in N$, are equal to 0 since we have supposed $j_k \neq 0$.

Consider now the function φ in the space

$$V^1_{j_1} \otimes \cdots V^{i_1-1}_{j_{i_1-1}} \otimes V^{i_1}_{j_{i_1}} \otimes V^{i_1+1}_0 \otimes \cdots \otimes V^n_0,$$

with $j_t = 0, \ldots, r_t$, for every $t = 1, \ldots, i_1$. In this case we have

$$\begin{aligned}
(P\varphi)(x) &= \sum_y p(x,y)\varphi(y) \\
&= \sum_{i\in C,\ i<i_1} \left(\sum_{y_i} p_i^0 p_i(x_i,y_i)\varphi_i(y_i)\right) \varphi_1(x_1)\cdots\varphi_{i-1}(x_{i-1})\varphi_{i+1}(x_{i+1})\cdots\varphi_n(x_n) \\
&+ \sum_{i\in C,\ i>i_1} \left(\sum_{y_i} p_i^0 p_i(x_i,y_i)\varphi_i(y_i)\right) \varphi_1(x_1)\cdots\varphi_{i-1}(x_{i-1})\varphi_{i+1}(x_{i+1})\cdots\varphi_n(x_n) \\
&+ \sum_{i\in N,\ i>i_1} \left(\sum_{y_i,\ldots,y_n} p_i^0 p_i(x_i,y_i)\frac{1}{m_{i+1}}\cdots\frac{1}{m_n}\varphi_i(y_i)\cdots\varphi_n(x_n)\right) \varphi_1(x_1)\cdots\varphi_{i-1}(x_{i-1}) \\
&+ \sum_{y_{i_1},\ldots,y_n} \left(p_{i_1}^0 p_{i_1}(x_{i_1},y_{i_1})\frac{1}{m_{i_1+1}}\cdots\frac{1}{m_n}\varphi_{i_1}(y_{i_1})\cdots\varphi_n(x_n)\right) \varphi_1(x_1)\cdots\varphi_{i_1-1}(x_{i_1-1}) \\
&= \sum_{i\in C,\ i<i_1} p_i^0\lambda_{j_i}\varphi(x) + \sum_{i\in C,\ i>i_1} p_i^0\varphi(x) + \sum_{i\in N,\ i>i_1} p_i^0\varphi(x) + p_{i_1}^0\lambda_{j_{i_1}}\varphi(x) \\
&= \left(\sum_{i=1}^{i_1} p_i^0\lambda_{j_i} + \sum_{i=i_1+1}^{n} p_i^0\right)\varphi(x).
\end{aligned}$$

Observe that, by computing the sum of the dimensions of these eigenspaces, we get

$$\sum_{k=i_1+1}^{n} m_1\cdots m_{k-1}(m_k-1) + m_1m_2\cdots m_{i_1} = m_1m_2\cdots m_n,$$

which is just the dimension of the space $X_1\times\cdots\times X_n$.

$\square$

Remark 5.4.

The expression of the eigenvalues of P given in the previous theorem tells us that if P_i is ergodic for every $i = 1, \ldots, n$, then also P is ergodic, since the eigenvalue 1 is obtained with multiplicity one and the eigenvalue -1 can never be obtained.

We can give now the matrices U, D and Δ associated with P. For every i, let U_i, D_i and Δ_i be the matrices of eigenvectors, of the coefficients of σ_i and of eigenvalues for the probability P_i, respectively. The expression of the matrix U, whose columns are an orthonormal basis of eigenvectors for P, easily follows from Theorem 5.3. In order to get the diagonal matrix D, whose entries are the coefficients of π, it suffices to consider the tensor product of the corresponding matrices associated with the probability P_i, for every $i = 1, \ldots, n$, as it follows from Theorem 5.2. Finally, to get the matrix Δ of eigenvalues of P it suffices to replace, in the expression of the matrix P, the matrix P_i by Δ_i and the matrix J_i by the corresponding diagonal matrix J_i^{diag}, which has the eigenvalue 1 with multiplicity one and the eigenvalue 0 with multiplicity $m_i - 1$. So we have the following proposition.

PROPOSITION 5.5. *Let P be the crested product of the Markov chains P_i, with $i = 1, \ldots, n$. Then we have:*

- $U = \sum_{k=i_1+1}^{n} M_1 \otimes \cdots \otimes M_{k-1} \otimes (U_k - A_k) \otimes A_{k+1} \otimes \cdots \otimes A_n$ $+ U_1 \otimes U_2 \otimes \cdots \otimes U_{i_1} \otimes A_{i_1+1} \otimes \cdots \otimes A_n$, *with*

$$M_i = \begin{cases} I_i^{\sigma_i - norm} & \text{if } i \in N \\ U_i & \text{if } i \in C \end{cases}$$

where

$$I_i^{\sigma_i - norm} = \begin{pmatrix} \frac{1}{\sqrt{\sigma_i(0)}} & & & \\ & \frac{1}{\sqrt{\sigma_i(1)}} & & \\ & & \ddots & \\ & & & \frac{1}{\sqrt{\sigma_i(m_i - 1)}} \end{pmatrix}.$$

By A_i we denote the matrix of size m_i whose entries on the first column are all 1 and the remaining ones are 0.

- $D = \bigotimes_{i=1}^{n} D_i$.
- $\Delta = \sum_{i \in C} p_i^0 \left(I_1 \otimes \cdots \otimes I_{i-1} \otimes \Delta_i \otimes I_{i+1} \otimes \cdots \otimes I_n \right)$ $+ \sum_{i \in N} p_i^0 \left(I_1 \otimes \cdots \otimes I_{i-1} \otimes \Delta_i \otimes J_{i+1}^{diag} \otimes \cdots \otimes J_n^{diag} \right)$.

Observe that another matrix U' of eigenvectors for P is given by $U' = \bigotimes_{i=1}^{n} U_i$. The matrix U that we have given above seems to be more useful whenever one wants to compute the $k-$th step transition probability $p^{(k)}(0, x)$ using the formula (11), since it contains a greater number of 0 in the first row with respect to U' and so a small number of terms in the sum are nonzero.

Suppose $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ elements in $X = X_1 \times \cdots \times X_n$. From (11) and Proposition 5.5, we have

$$\begin{aligned}
p^{(k)}(x,y) &= \pi(y)\left[\sum_{z\in X}\left(\sum_{r=i_1+1}^{n} m_1(x_1,z_1)\cdots m_{r-1}(x_{r-1},z_{r-1})(u_r-a_r)(x_r,z_r)\right.\right.\\
&\times\ a_{r+1}(x_{r+1},z_{r+1})\cdots a_n(x_n,z_n)\\
&+\ u_1(x_1,z_1)\cdots u_{i_1}(x_{i_1},z_{i_1})a_{i_1+1}(x_{i_1+1},z_{i_1+1})\cdots a_n(x_n,z_n))\,\lambda_z^k\\
&\times\ \left(\sum_{r=i_1+1}^{n} m_1(y_1,z_1)\cdots m_{r-1}(y_{r-1},z_{r-1})(u_r-a_r)(y_r,z_r)\right.\\
&\times\ a_{r+1}(y_{r+1},z_{r+1})\cdots a_n(y_n,z_n)\\
&+\ \left.u_1(y_1,z_1)\cdots u_{i_1}(y_{i_1},z_{i_1})a_{i_1+1}(y_{i_1+1},z_{i_1+1})\cdots a_n(y_n,z_n))\right],
\end{aligned}$$

where m_i, u_i, a_i are the probabilities associated with the matrices M_i, U_i, A_i occurring in Proposition 5.5.

5.2. The crossed product. The crossed product of the Markov chains P_i's can be obtained as a particular case of the crested product, by setting $C = \{1,\dots,n\}$ and it is also called *direct* product. The analogous case for product of groups has been studied in [**DSC1**].

In this case, we get the following transition probability:

$$p((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sum_{i=1}^{n} p_i^0\delta(x_1,y_1)\cdots p_i(x_i,y_i)\cdots\delta(x_n,y_n).$$

This corresponds to choose the $i-$th coordinate with probability p_i^0 and to change it according with the probability transition P_i. So we get

$$p((x_1,\dots,x_n),(y_1,\dots,y_n)) = \begin{cases} p_i^0 p_i(x_i,y_i) & \text{if } x_j = y_j \text{ for all } j\neq i\\ 0 & \text{otherwise.}\end{cases}$$

So, for $X_1 = \cdots = X_n =: X$ and $p_0^1 = \cdots = p_n^0 = \frac{1}{n}$, the probability p defines an analogous of the Ehrenfest model, where n is the number of balls and $|X| = m$ is the number of urns. In order to obtain a new configuration, we choose a ball with probability $1/n$ (let it be the $i-$th ball in the urn x_i) and with probability $p_i(x_i,y_i)$ we put it in the urn y_i.

As a consequence of Theorem 5.2, we get that if P_i is in detailed balance with π_i, then P is in detailed balance with the strict probability measure π on $X_1\times\cdots\times X_n$ defined as

$$\pi(x_1,\dots,x_n) = \pi_1(x_1)\pi_2(x_2)\cdots\pi_n(x_n).$$

The matrix P associated with the probability p is given by

$$P = \sum_{i=1}^{n} p_i^0\left(I_1\otimes\cdots\otimes P_i\otimes\cdots\otimes I_n\right). \tag{21}$$

The following proposition studies the spectral theory of the operator P and it is a straightforward consequence of Theorem 5.3.

PROPOSITION 5.6. *Let $\varphi_0^i = 1_{X_i}, \varphi_1^i, \dots, \varphi_{m_i-1}^i$ be the eigenfunctions of P_i associated with the eigenvalues $\lambda_0^i = 1, \lambda_1^i, \cdots, \lambda_{m_i-1}^i$, respectively. Then the eigenvalues of P are the $m_1 m_2 \cdots m_n$ numbers*

$$\lambda_I = \sum_{k=1}^{n} p_k^0 \lambda_{i_k}^k,$$

with $I = (i_1, \dots, i_n) \in \{0, \dots, m_1 - 1\} \times \cdots \times \{0, \dots, m_n - 1\}$. The corresponding eigenfunctions are defined as

$$\varphi_I((x_1, \dots, x_n)) = \varphi_{i_1}^1(x_1) \cdots \varphi_{i_n}^n(x_n).$$

As a consequence of Proposition 5.5, in order to get the matrices U, D and Δ associated with P, it suffices to consider the tensor product of the corresponding matrices associated with the probability P_i, for every $i = 1, \dots, n$. For every i, let U_i, D_i and Δ_i be the matrices of eigenvectors, of the coefficients of π_i and of eigenvalues for the probability P_i, respectively. We have the following corollary.

COROLLARY 5.7. *Let P be the probability defined in (21). Then we have*

$$\begin{cases} PU = U\Delta \\ U^T D U = I, \end{cases}$$

where $U = \bigotimes_{i=1}^n U_i$, $\Delta = \bigotimes_{i=1}^n \Delta_i$ and $D = \bigotimes_{i=1}^n D_i$.

In particular, we can express the $k-$th step transition probability matrix as

$$P^k = \left(\bigotimes_{i=1}^n U_i\right)\left(\bigotimes_{i=1}^n \Delta_i\right)^k \left(\bigotimes_{i=1}^n U_i\right)^T \left(\bigotimes_{i=1}^n D_i\right).$$

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Then we get

$$p^{(k)}(x, y) = \pi(y) \sum_I \varphi_I(x) \lambda_I^k \varphi_I(y) =$$

$$\pi_1(y_1) \cdots \pi_n(y_n) \sum_I \varphi_{i_1}^1(x_1) \cdots \varphi_{i_n}^n(x_n) \left(p_1^0 \lambda_{i_1}^1 + \cdots + p_n^0 \lambda_{i_n}^n\right)^k \varphi_{i_1}^1(y_1) \cdots \varphi_{i_n}^n(y_n),$$

with $I = (i_1, \dots, i_n)$.

As we said in Remark 5.4, if the matrix P_i is ergodic for every $i = 1, \dots, n$, then also the matrix P is ergodic, since the eigenvalue 1 can be obtained only by choosing $I = 0^n$ and the eigenvalue -1 can never be obtained.

REMARKS 5.8.

Put $n = 1$ and set $X = \{0, 1, \ldots, m-1\}$. Consider the action of the symmetric group S_m on X. The stabilizer of a fixed element $x_0 = 0$ is isomorphic to the symmetric group S_{m-1}. It is well known (see [**?**]) that (S_m, S_{m-1}) is a Gelfand pair and the following decomposition of $L(X)$ into irreducible representations holds:

$$L(X) = V_0 \oplus V_1,$$

where $V_0 \cong \mathbb{C}$ is the space of constant functions on X and $V_1 = \{f : X \longrightarrow \mathbb{C} : \sum_{i=0}^{m-1} f(i) = 0\}$. So we have $dimV_0 = 1$ and $dimV_1 = m-1$.

Analogously, one can consider the action of the wreath product $S_m \wr S_n$ on $X^n = X \times \cdots \times X$, defined in the natural way, and then one can study the decomposition of $L(X^n)$. We have

$$L(X^n) \cong L(X)^{\otimes^n} \cong \bigoplus_{j=0}^{n} W_j,$$

with

$$W_j = \bigoplus_{w(i_1,\ldots,i_n)=j} V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n},$$

where $w(i_1, \ldots, i_n) = \sharp\{k : i_k = 1\}$. So we have $dimW_j = \binom{n}{j}(m-1)^j$.

If we define on X the uniform transition probability, i.e. $p_u(x, y) = \frac{1}{m}$ for all $x, y \in X$, then the matrix P_u is the matrix J of size m.

The eigenvalues of this matrix are 1 (with multiplicity 1) and 0 (with multiplicity $m-1$). The corresponding eigenspaces in $L(X)$ are, respectively, $V_0 \cong \mathbb{C}$ and $V_1 = \{f : X \longrightarrow \mathbb{C} : \sum_{i=0}^{m-1} f(i) = 0\}$.

This means that, by choosing the uniform transition probability on X, one gets again the results obtained by considering the Gelfand pair (S_m, S_{m-1}).

Also in the case of X^n we can find again the results obtained (see [**?**]) by considering the Gelfand pair $(S_m \wr S_n, S_{m-1} \wr S_n)$. For $P_u = J$ we have $\lambda_0 = 1, \lambda_1 = \ldots = \lambda_{m-1} = 0$. Consider now the transition probability on X^n defined in (21), with $p_1^0 = \cdots = p_n^0 = \frac{1}{n}$. The eigenfunctions φ_I associated with the eigenvalue $\frac{1}{n}(n-j)$, with $j = 0, \ldots, n$, are in number of $\binom{n}{j}(m-1)^j$. Moreover

$$\sum_{j=0}^{n} \binom{n}{j}(m-1)^j = \sum_{j=0}^{n} \binom{n}{j}(m-1)^j 1^{n-j} = m^n = dimL(X^n).$$

For every $j = 0, \ldots, n$, these functions belong to W_j and they are a basis for W_j. So W_j is the eigenspace associated with the eigenvalue $\frac{1}{n}(n-j)$.

More generally, consider the case of any reversible transition probability p on X. Let $\lambda_0 = 1, \lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of P and $V_0 \cong \mathbb{C}, V_1, \ldots, V_k$ the corresponding eigenspaces. We get

$$L(X^n) \cong (V_0 \oplus V_1 \oplus \cdots \oplus V_k)^{\otimes^n}.$$

The eigenfunctions φ_I associated with the eigenvalue $\frac{1}{n}\sum_{i=0}^k r_i\lambda_i$, with $\sum_{i=0}^k r_i = n$, are

$$\binom{r_0 + r_1 + \cdots + r_k}{r_0, \ldots, r_k} \prod_{i=0}^k (dimV_i)^{r_i}$$

and the corresponding eigenspaces are the tensor products in which r_i copies of V_i, for $i = 0, 1, \ldots, k$, appear. Moreover, the number of different eigenspaces is equal to the number of integer solutions of the equation

$$r_0 + r_1 + \cdots + r_k = n, \quad r_i \geq 0,$$

so it is $\binom{k+n}{n}$.

The definition of multinomial coefficient as $\binom{r_0+r_1+\cdots+r_k}{r_0,\ldots,r_k} = \frac{(r_0+\cdots+r_k)!}{r_0!r_1!\cdots r_k!}$ guarantees that

$$\begin{aligned}\sum_{r_0+\cdots+r_k=n} \binom{n}{r_0, \ldots, r_k} (dimV_0)^{r_0} \cdots (dimV_k)^{r_k} &= (dimV_0 + \cdots + dimV_k)^n \\ &= m^n,\end{aligned}$$

as we wanted.

5.3. The nested product. The nested product of the Markov chains P_i's can be obtained as a particular case of the crested product, by setting $N = \{1, \ldots, n\}$. The term *nested* comes from the association schemes theory (see [**Bai**]).

Consider the product

$$X_1 \times \cdots \times X_n$$

and let P_i be a transition probability on X_i. We assume that p_i is in detailed balance with the strict probability measure π_i, for all $i = 1, \ldots, n$.

The formula of crested product becomes, in this case,

$$P = \sum_{i=1}^n p_i^0 \left(I_1 \otimes \cdots \otimes P_i \otimes J_{i+1} \otimes J_{i+2} \otimes \cdots \otimes J_n\right). \tag{22}$$

Theorem 5.2 tells us that P is reversible if and only if P_k is symmetric, for every $k > 1$, i.e. $\pi_i = \frac{1}{m_i}$ for every $i = 2, \ldots, n$. In this case, P is in detailed balance with the strict probability measure π on $X_1 \times \cdots \times X_n$ given by

$$\pi(x_1 \ldots, x_n) = \frac{\pi_1(x_1)}{\prod_{i=2}^n m_i}.$$

So let us assume π_i to be uniform for every $i = 2, \ldots, n$. The transition probability associated with P is

$$\begin{aligned} p((x_1, \ldots, x_n), (y_1, \ldots, y_n)) &= \frac{p_1^0 p_1(x_1, y_1)}{m_2 m_3 \cdots m_n} \\ &+ \sum_{j=2}^{n-1} \frac{\delta((x_1, \ldots, x_{j-1}), (y_1, \ldots, y_{j-1})) p_j^0 p_j(x_j, y_j)}{m_{j+1} \cdots m_n} \\ &+ \delta((x_1, \ldots, x_{n-1}), (y_1, \ldots, y_{n-1})) p_n^0 p_n(x_n, y_n). \end{aligned}$$

As we did in the case of the crossed product, we want to study the spectral theory of the operator P defined in (22).

Let

$$L(X_i) = \bigoplus_{k_i=0}^{r_i} W_{k_i}^i$$

be the spectral decomposition of $L(X_i)$, for all $i = 1, \ldots, n$ and let $\lambda_0^i = 1, \lambda_1^i, \ldots, \lambda_{r_i}^i$ the distinct eigenvalues of P_i associated with these eigenspaces. From Theorem 5.3 we get the following proposition.

Proposition 5.9. *The eigenspaces of $L(X_1 \times \cdots \times X_n)$ are*

- $L(X_1) \otimes \cdots \otimes L(X_{n-1}) \otimes W_{k_n}^n$, *of eigenvalue* $p_n^0 \lambda_{k_n}^n$, *for* $k_n = 1, \ldots, r_n$, *with multiplicity* $m_1 \cdots m_{n-1} dim(W_{k_n}^n)$;
- $L(X_1) \otimes \cdots \otimes L(X_j) \otimes W_{k_{j+1}}^{j+1} \otimes W_0^{j+2} \otimes \cdots \otimes W_0^n$, *of eigenvalue* $p_{j+1}^0 \lambda_{k_{j+1}}^{j+1} + p_{j+2}^0 + \cdots + p_n^0$, *with* $k_{j+1} = 1, \ldots, r_{j+1}$ *and for* $j = 1, \ldots, n-2$, *with multiplicity* $m_1 \cdots m_j dim(W_{k_{j+1}}^{j+1})$;
- $W_{k_1}^1 \otimes W_0^2 \otimes \cdots \otimes W_0^n$, *of eigenvalue* $p_1^0 \lambda_{k_1}^1 + p_2^0 + \cdots + p_n^0$, *for* $k_1 = 0, 1, \ldots, r_1$, *with multiplicity* $dim(W_{k_1}^1)$.

Moreover, as in the general case, one can verify that, under the hypothesis that the operators P_i are ergodic, for $i = 1, \ldots, n$, then also the operator P is ergodic.

The application of Proposition 5.5 to the case of the nested product yields the following corollary.

Corollary 5.10. *Let P be the nested product of the probabilities P_i, with $i = 1, \ldots, n$. Then we have:*

- $U = U_1 \otimes A_2 \otimes \cdots \otimes A_n$ $+ \sum_{k=2}^n I_1^{\sigma_1 - norm} \otimes \cdots \otimes I_{k-1}^{\sigma_{k-1} - norm} \otimes (U_k - A_k) \otimes A_{k+1} \otimes \cdots \otimes A_n.$
- $D = \bigotimes_{i=1}^n D_i$.

- $\Delta = \sum_{i=1}^n p_i^0 \left(I_1 \otimes \cdots \otimes I_{i-1} \otimes \Delta_i \otimes J_{i+1}^{diag} \otimes \cdots \otimes J_n^{diag} \right)$.

5.4. $k-$steps transition probability. The formula that describes the transition probability after k steps in the case of nested product can be simplified using the base of eigenvectors given in Corollary 5.10 and supposing that the starting point is $0 = (0, \dots, 0)$.

From the general formula, with the usual notations, we get

$$\begin{aligned}
p^{(k)}(0,y) &= \pi(y) \left[\sum_{z \in X} \left(\sum_{r=2}^n \delta_{\sigma_1}(0,z_1) \cdots \delta_{\sigma_{r-1}}(0,z_{r-1})(u_r - a_r)(0,z_r) \right.\right. \\
&\times \left. a_{r+1}(0,z_{r+1}) \cdots a_n(0,z_n) + u_1(0,z_1)a_2(0,z_2) \cdots a_n(0,z_n) \right) \lambda_z^k \\
&\times \left(\sum_{r=2}^n \delta_{\sigma_1}(y_1,z_1) \cdots \delta_{\sigma_{r-1}}(y_{r-1},z_{r-1})(u_r - a_r)(y_r,z_r) \right. \\
&\times a_{r+1}(y_{r+1},z_{r+1}) \cdots a_n(y_n,z_n) \\
&+ \left.\left. u_1(y_1,z_1)a_2(y_2,z_2) \cdots a_n(y_n,z_n) \right) \right] \\
&= \pi(y) \left[1 + \sum_{j=2}^n \sum_{\substack{z_j \neq 0 \\ z_i = 0,\ i \neq j}} \left(\sum_{r=j}^n \frac{1}{\sqrt{\sigma_1(0)} \cdots \sqrt{\sigma_{r-1}(0)}} (u_r - a_r)(0,z_r) \right) \right. \\
&\times \left(p_r^0 \lambda_{z_r}^r + \sum_{m > r} p_m^0 \right)^k \left(\sum_{r=j}^n \delta_{\sigma_1}(y_1,0)\delta_{\sigma_2}(y_2,z_2) \cdots \delta_{\sigma_{r-1}}(y_{r-1},z_{r-1}) \right. \\
&\times \left. (u_r - a_r)(y_r,z_r)a_{r+1}(y_{r+1},z_{r+1}) \cdots a_n(y_n,z_n) \right) \\
&+ \left. \sum_{\substack{z_1 \neq 0 \\ z_i = 0,\ i > 1}} u_1(0,z_1) \left(p_1^0 \lambda_{z_1}^1 + \sum_{m=2}^n p_m^0 \right)^k u_1(y_1,z_1) \right].
\end{aligned}$$

Observe that in this case the sum consists of no more than

$$|X_1| + \sum_{i=2}^n (|X_i| - 1) = \sum_{i=1}^n |X_i| - n + 1$$

nonzero terms.

Example 5.11.

We want to express the $k-$th step transition probability in the case $n = 2$. So consider the product $X \times Y$, with $X = \{0, 1, \dots, m\}$ and

$Y = \{0, 1, \ldots, n\}$. Let

$$L(X) = \bigoplus_{j=0}^{r} V_j \quad \text{and} \quad L(Y) = \bigoplus_{i=0}^{s} W_i$$

be the spectral decomposition of the spaces $L(X)$ and $L(Y)$, respectively. Let $\lambda_0 = 1, \lambda_1, \ldots, \lambda_r$ and $\mu_0 = 1, \mu_1, \ldots, \mu_s$ be the distinct eigenvalues of P_X and P_Y, respectively. Then the eigenspaces of $L(X \times Y)$ are $L(X) \otimes W_i$, for $i = 1, \ldots, s$, with dimension $(m+1)dimW_i$ and associated eigenvalue $p_Y^0 \mu_i$, and $V_j \otimes W_0$, for $j = 0, \ldots, r$, with dimension $dimV_j$ and associated eigenvalue $p_X^0 \lambda_j + p_Y^0$.

The expression of U becomes

$$U = I_X^{\sigma_X - norm} \otimes (U_Y - A_Y) + U_X \otimes A_Y.$$

In particular, let $\{v^0, v_1^1, \ldots, v_{dim(V_1)}^1, \ldots, v_1^r, \ldots, v_{dim(V_r)}^r\}$ and $\{w^0, w_1^1, \ldots, w_{dim(W_1)}^1, \ldots, w_1^s, \ldots, w_{dim(W_s)}^s\}$ be the eigenvectors of P_X and P_Y, respectively, i.e. they represent the columns of the matrices U_X and U_Y.

Then, the columns of the matrix U corresponding to the elements $(i, 0) \in \{0, \ldots, m\} \times \{0, \ldots, n\}$ are the eigenvectors $v^i \otimes (1, \ldots, 1)$ with eigenvalue $p_X^0 \lambda_i + p_Y^0$. On the other hand, the columns corresponding to the elements $(i, j) \in \{0, \ldots, m\} \times \{0, \ldots, n\}$, with $j = 1, \ldots, n$, are the eigenvectors $(0, \ldots, 0, \underbrace{\frac{1}{\sqrt{\sigma_X(i)}}}_{i-th \ \ place}, 0, \ldots, 0) \otimes w^j$ whose eigenvalue is $p_Y^0 \mu_j$. As a consequence, only $m + 1 + n$ of these eigenvectors can be nonzero in the first coordinate, so the probability $p^{(k)}((0,0),(x,y))$ can be expressed as a sum of $m + 1 + n$ nonzero terms: moreover, these terms become $m + 1$ if $x \neq 0$. We have

$$\begin{aligned}
p^{(k)}((0,0),(x,y)) &= \pi((x,y)) \left(\sum_{i=0}^{m} v^i(0) v^i(x) (p_X^0 \lambda_i + p_Y^0)^k \right. \\
&+ \left. \frac{1}{\sqrt{\sigma_X(0)\sigma_X(x)}} \sum_{j=1}^{n} w^j(0) \delta_0(x) w^j(y) (p_Y^0 \mu_j)^k \right) \\
&= \frac{\sigma_X(x)}{n+1} \left[\sum_{i=0}^{r} \left(\sum_{a=1}^{dim(V_i)} v_a^i(0) v_a^i(x) \right) (p_X^0 \lambda_i + p_Y^0)^k \right. \\
&+ \left. \sum_{j=1}^{s} \left(\frac{1}{\sqrt{\sigma_X(0)\sigma_X(x)}} \sum_{b=1}^{dim(W_j)} w_b^j(0) \delta_0(x) w_b^j(y) \right) (p_Y^0 \mu_j)^k \right].
\end{aligned}$$

5.5. The insect. It is clear that the product $X_1 \times \cdots \times X_n$ can be regarded as the rooted tree T of depth n, such that the root has degree m_1, each vertex of the first level has m_2 children and in general each vertex of the $i-$th level of the tree has m_{i+1} children, for every $i = 1, \dots, n-1$. We denote the $i-$th level of the tree by L_i. In this way, every vertex $x \in L_i$ can be regarded as a word $x = x_1 \cdots x_i$, where $x_j \in \{0, 1, \dots, m_j - 1\}$.

We want to show that the nested product of Markov chains is the generalization of the "insect problem" studied by A. Figà-Talamanca in [**F-T1**] and that we have described in Section 2 (in this case we generalize to the non-homogeneous case).

Let us imagine that an insect lives in a leaf $x \in L_n$ and that it performs a simple random walk on the graph T starting from x.

Then there exists a probability distribution μ_x on L_n such that, for every $y \in L_n$, $\mu_x(y)$ is the probability that y is the first point in L_n visited by the insect in the random walk. If we put $\overline{p}(x,y) = \mu_x(y)$, then we get a stochastic matrix $\overline{P} = (\overline{p}(x,y))_{x,y \in L_n}$. Since the random walk is $Aut(T)-$invariant, we can suppose that the random walk starts at the leftmost vertex, that we will call $x_0 = (0, \dots, 0)$. We recall that $Aut(T)$ is the group of all automorphisms of T, given by the iterated wreath product $S_{m_n} \wr S_{m_{n-1}} \wr \cdots \wr S_{m_1}$. We want to study this Markov chain defined on L_n.

Set $\xi_n = \emptyset$ and $\xi_i = 00 \dots 0$ ($n - i$ times). For $j \geq 0$, let α_j be the probability that the insect reaches ξ_{j+1} given that ξ_j is reached at least once. This definition implies $\alpha_0 = 1$ and $\alpha_1 = \frac{1}{m_n+1}$. In fact, with probability 1, the insect reaches the vertex ξ_1 at the first step and, starting from ξ_1, with probability $\frac{1}{m_n+1}$ it reaches ξ_2, while with probability $\frac{m_n}{m_n+1}$ it returns to L_n. Finally, we have $\alpha_n = 0$. For $1 < j < n$, there is the following recursive relation:

$$\alpha_j = \frac{1}{m_{n+1-j}+1} + \alpha_{j-1}\alpha_j \frac{m_{n+1-j}}{m_{n+1-j}+1}.$$

In fact, starting at ξ_j, with probability $\frac{1}{m_{n+1-j}+1}$ the insect reaches in one step ξ_{j+1}, otherwise with probability $\frac{m_{n+1-j}}{m_{n+1-j}+1}$ it reaches ξ_{j-1} or one of its brothers; then, with probability α_{j-1} it reaches again ξ_j and one starts the recursive argument.

The solution, for $1 \leq j \leq n-1$, is given by

$$\begin{aligned} \alpha_j &= \frac{1 + m_n + m_n m_{n-1} + m_n m_{n-1} m_{n-2} + \cdots + m_n m_{n-1} m_{n-2} \cdots m_{n-j+2}}{1 + m_n + m_n m_{n-1} + m_n m_{n-1} m_{n-2} + \cdots + m_n m_{n-1} m_{n-2} \cdots m_{n-j+1}} \\ &= 1 - \frac{m_n m_{n-1} m_{n-2} \cdots m_{n-j+1}}{1 + m_n + m_n m_{n-1} + m_n m_{n-1} m_{n-2} + \cdots + m_n m_{n-1} m_{n-2} \cdots m_{n-j+1}}. \end{aligned}$$

Moreover, we have

$$\begin{aligned}\overline{p}(x_0,x_0) &= \frac{1}{m_n}(1-\alpha_1)+\frac{1}{m_nm_{n-1}}\alpha_1(1-\alpha_2)+\cdots\\ &+ \frac{1}{m_nm_{n-1}\cdots m_2}\alpha_1\alpha_2\cdots\alpha_{n-2}(1-\alpha_{n-1})+\frac{1}{m_n\cdots m_1}\alpha_1\cdots\alpha_{n-1}.\end{aligned}$$

Indeed the $j-$th summand is the probability of returning back to x_0 if the corresponding random walk in T reaches ξ_j but not ξ_{j+1}. It is not difficult to compute $\overline{p}(x_0,x)$, where x is a point at distance j from x_0. For $j=1$, we clearly have $\overline{p}(x_0,x_0)=\overline{p}(x_0,x)$. We observe that, for $j>1$, to reach x one is forced to first reach ξ_j, so that we have

$$\begin{aligned}\overline{p}(x_0,x) &= \frac{1}{m_n\cdots m_{n-j+1}}\alpha_1\alpha_2\cdots\alpha_{j-1}(1-\alpha_j)+\cdots\\ &+ \frac{1}{m_n\cdots m_2}\alpha_1\alpha_2\cdots\alpha_{n-2}(1-\alpha_{n-1})+\frac{1}{m_n\cdots m_1}\alpha_1\alpha_2\cdots\alpha_{n-1}.\end{aligned}$$

Since the random walk is invariant with respect to the action of $Aut(T)$, which acts isometrically on the tree, we get the same formula for any pair of vertices $x,y\in L_n$ such that $d(x,y)=j$.

Proposition 5.12. *The stochastic matrix*

$$\begin{aligned}p((x_1,\ldots,x_n),(y_1,\ldots,y_n)) &= \frac{p_1^0p_1(x_1,y_1)}{m_2m_3\cdots m_n}\\ &+ \sum_{j=2}^{n-1}\frac{\delta((x_1,\ldots,x_{j-1}),(y_1,\ldots,y_{j-1}))p_j^0p_j(x_j,y_j)}{m_{j+1}\cdots m_n}\\ &+ \delta((x_1,\ldots,x_{n-1}),(y_1,\ldots,y_{n-1}))p_n^0p_n(x_n,y_n),\end{aligned}$$

defined in (22), gives rise to the Insect Markov chain on L_n, regarded as $X_1\times\cdots\times X_n$, choosing $p_i^0=\alpha_1\alpha_2\cdots\alpha_{n-i}(1-\alpha_{n-i+1})$ for $i=1,\ldots,n-1$ and $p_n^0=1-\alpha_1$ and the transitions probabilities $p_j's$ to be uniform for all $j=1,\ldots,n$.

Proof. Set, for every $i=1,\ldots,n-1$,

$$p_i^0=\alpha_1\alpha_2\cdots\alpha_{n-i}(1-\alpha_{n-i+1})$$

and $p_n^0=1-\alpha_1$. Moreover, assume that the probability p_i on X_i is uniform, i.e.

$$P_i=J_i.$$

If $d(x_0,x)=n$, then we get

$$p(x_0,x)=\frac{\alpha_1\alpha_2\cdots\alpha_{n-1}}{m_1m_2\cdots m_n}.$$

If $d(x_0, x) = j > 1$, i.e. $x_i^0 = x_i$ for all $i = 1, \dots, n-j$, then

$$p(x_0, x) = \frac{\alpha_1\alpha_2\cdots\alpha_{n-1}}{m_1m_2\cdots m_n} + \sum_{i=1}^{n-j}\frac{\alpha_1\cdots\alpha_{n-i-1}(1-\alpha_{n-i})}{m_n\cdots m_{i+2}m_{i+1}}.$$

Finally, if $x = x_0$, we get

$$p(x_0, x_0) = \frac{\alpha_1\alpha_2\cdots\alpha_{n-1}}{m_1m_2\cdots m_n} + \sum_{i=1}^{n-2}\frac{\alpha_1\cdots\alpha_{n-i-1}(1-\alpha_{n-i})}{m_n\cdots m_{i+2}m_{i+1}} + \frac{(1-\alpha_1)}{m_n}.$$

This completes the proof. $\square$

The decomposition of the space $L(L_n) = L(X_1 \times \cdots \times X_n)$ under the action of $Aut(T)$ is known (see [**CST2**]). Denote $Z_0 \cong \mathbb{C}$ the trivial representation and, for every $j = 1, \dots, n$, define the following subspace

$$Z_j = \{f \in L(L_n) : f = f(x_1, \dots, x_j), \sum_{i=0}^{m_j-1} f(x_1, \dots, x_{j-1}, i) \equiv 0\}$$

of dimension $m_1\cdots m_{j-1}(m_j - 1)$. In virtue of the correspondence between $Aut(T)-$invariant operators and bi$-Stab_{Aut(T)}(0^n)-$invariant functions, the corresponding eigenvalues are given by the spherical Fourier transform of the convolver that represents $\overline{P}$, namely

$$\lambda_j = \sum_{x\in L_n}\overline{P}(x_0, x)\phi_j(x),$$

where ϕ_j is the $j-$th spherical function, for all $j = 0, 1, \dots, n$. It is easy verify that one get

- $\lambda_0 = 1$;
- $\lambda_j = 1 - \alpha_1\alpha_2\cdots\alpha_{n-j}$, for every $j = 1, \dots, n-1$;
- $\lambda_n = 0$.

In particular, if we set

$$p_i^0 = \alpha_1\alpha_2\cdots\alpha_{n-i}(1-\alpha_{n-i+1})$$

for every $i = 1, \dots, n-1$, with $p_n^0 = 1 - \alpha_1$ and $P_i = J_i$ for every $i = 1, \dots, n$, the eigenspaces given for $L(X_1 \times \cdots \times X_n)$ in Proposition 5.9 are exactly the Z_j's with the corresponding eigenvalues.

Let us prove that the eigenvalues that we have obtained in Proposition 5.9 coincide with the eigenvalues corresponding to the eigenspaces $Z_0, Z_1, \dots, Z_n$.

We want to get these eigenvalues by using the formulas given in Proposition 5.9 for the eigenvalues of the nested product P by setting $P_i = J_i$, then $p_i^0 = \alpha_1\alpha_2\cdots\alpha_{n-i}(1-\alpha_{n-i+1})$ for $i = 1, \dots, n-1$ and $p_n^0 = 1-\alpha_1$. First of all, we observe that the eigenvalues of the operator P_i are 1, with multiplicity one and 0, with multiplicity $m_i - 1$. So we get $L(X_i) = W_0^i \oplus W_1^i$, with $dim(W_1^i) = m_i - 1$, for all $i = 1, \ldots, n$. Following the formulas that we have given, the eigenspaces of P are:

- $L(X_1) \otimes L(X_2) \otimes \cdots \otimes L(X_{n-1}) \otimes W_1^n$;
- $L(X_1) \otimes L(X_2) \otimes \cdots \otimes L(X_{n-j-1}) \otimes W_1^{n-j} \otimes W_0^{n-j+1} \otimes \cdots \otimes W_0^n$, for every $j = 1, \ldots, n-1$;
- $W_0^1 \otimes W_0^2 \otimes \cdots \otimes W_0^n$.

The corresponding eigenvalues are:

- $p_n^0 \lambda_1^n = 0$;
- $\sum_{i=n-j+1}^n p_i^0$, for every $j = 1, \ldots, n-1$;
- $\sum_{i=1}^n p_i^0 = 1$.

We need to prove that, for every $j = 1, \ldots, n-1$, the eigenvalue $\sum_{i=n-j+1}^n p_i^0$ is equal to $1 - \alpha_1\alpha_2 \cdots \alpha_j$. We prove the assertion by induction on j.

If $j = 1$, we have $p_n^0 = 1 - \alpha_1$. Now suppose the assertion to be true for j and show that it holds also for $j+1$. We get

$$\begin{aligned} \sum_{i=n-j}^n p_i^0 &= \sum_{i=n-j+1}^n p_i^0 + p_{n-j}^0 = 1 - \alpha_1\alpha_2 \cdots \alpha_j + \alpha_1 \cdots \alpha_j(1 - \alpha_{j+1}) \\ &= 1 - \alpha_1 \cdots \alpha_j \alpha_{j+1}. \end{aligned}$$

5.6. The Second Crested Product. In this subsection we define a different kind of product of two spaces X and Y, that we will call the second crested product. In fact it contains, as particular cases, the crossed product and the nested product described in Section 5.2 and Section 5.3, respectively. We will study a Markov chain P on the set Θ_k of functions from X to Y whose domains are $k-$subsets of X, giving the spectrum and the relative eigenspaces.

Let X be a finite set of cardinality n, say $X = \{1, 2, \ldots, n\}$. For every $k = 1, \ldots, n$, denote by Ω_k the set of $k-$subsets of X, so that $|\Omega_k| = \binom{n}{k}$.

Now let Y be a finite set and let Q be a transition matrix on Y, which is in detailed balance with the strict probability τ. Let $\lambda_0 = 1, \lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of Q and denote by W_j the corresponding eigenspaces, for every $j = 0, 1, \ldots, m$, so that the following spectral decomposition holds:

$$L(Y) = \bigoplus_{j=0}^m W_j.$$

Moreover, assume that the dimension of the eigenspace associated with the eigenvalue 1 is one and set $dim(W_j) = m_j$, for every $j = 1, \ldots, m$.

Recall that the eigenspace W_0 is generated by the vector $(\underbrace{1, \ldots, 1}_{|Y| \ \ times})$ and that W_j is orthogonal to W_0 with respect to the scalar product $\langle \cdot, \cdot \rangle_\tau$, for every $j = 1, \ldots, m$.

For every $k = 1, \dots, n$, consider the space

$$\Theta_k = \{(A, \theta) : A \in \Omega_k \text{ and } \theta \in Y^A\},$$

i.e. the space of functions whose domain is a $k-$subset of X and which take values in Y.

The set $\Theta = \coprod_{k=0}^{n} \Theta_k$ is a poset with respect to the relation $\subseteq$ defined in the following way:

$$\varphi \subseteq \chi \text{ if } dom(\varphi) \subseteq dom(\chi) \text{ and } \varphi = \chi|_{dom(\varphi)}.$$

The Markov chain P on Θ_k that we are going to define can be regarded as follows. Let $0 < p_0 < 1$ a real number. Then, starting from a function $\theta \in \Theta_k$, with probability p_0 we can reach a function $\varphi \in \Theta_k$ having the same domain as θ and that can differ from θ at most in one image, according with the probability Q on Y.

On the other hand, with probability $1 - p_0$ we can reach in one step a function $\varphi \in \Theta_k$ whose domain intersects the domain of θ in $k-1$ elements (on which the functions coincide), and in such a way that the image of the $k-$th element of the domain of φ is uniformly chosen.

Note that P defines a Markovian operator on the space $L(\Theta_k)$ of all complex functions defined on Θ_k.

When Y is the ultrametric space, the Markov chain P represents the so called multi-insect, which generalizes the insect Markov chain already studied. In particular if $|X| = n$, we consider k insects living in k different subtrees and moving only one per each step in such a way that their distance is preserved, giving rise to a Markov chain on the space of all possible configurations of k insects having this property.

In fact each element in Θ_k can be ragarded as a configuration of k insects and viceversa. For example, let $\theta \in \Theta_k$ be a function such that $dom(\theta) = \{x_1, \dots, x_k\}$ and $\theta(x_i) = y_i$, with $x_i \in X$ and $y_i \in Y$ for all $i = 1, \dots, k$. Then the corresponding configuration of k insects has an insect at each leaf (x_i, y_i). They live in all different subtrees since $x_i \neq x_j$ for $i \neq j$.

We observe that the cardinality of this space is $\binom{n}{k}|Y|^k$. This space can be regarded as the variety of subtrees (see [**CST3**]) of branch indices $(k, 1)$ in the rooted tree $(n, |Y|)$.

If $\theta, \varphi \in \Theta_k$, with domains A and B respectively, then define the matrix Δ, indexed by Θ_k, whose entries are

$$\Delta_{\theta,\varphi} = \begin{cases} 1 & \text{if } |A \cap B| = k-1 \text{ and } \theta|_{A\cap B} = \varphi|_{A\cap B}, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the matrix Δ is symmetric.

The operator P can be expressed in terms of the operator associated with Δ and of another operator M as

$$P = p_0 M + (1 - p_0)\frac{\Delta}{norm(\Delta)}, \tag{23}$$

where M describes the situation in which the domain is not changed and only one of the images of the function $\theta \in \Theta_k$ is changed according with the probability Q on Y. An analytic expression for M will be presented below. On the other hand, Δ describes the situation in which we pass from a function whose domain is A to a function whose domain is $A \sqcup \{i\} \setminus \{j\}$, with $i \notin A$ and $j \in A$, and we choose uniformly the image in Y of the element i. So the action of Δ on Ω_k is an analogous of the Laplace-Bernoulli diffusion model. By $norm(\Delta)$ we indicate the number of non zero entries in each row of the matrix associated with Δ.

It is easy to check that M is in detailed balance with the strict probability measure defined as

$$\tau_M(\theta) = \frac{1}{\binom{n}{k}} \prod_{i \in A} \tau(\theta(i)),$$

where $\theta \in \Theta_k$ and $dom(\theta) = A$. On the other hand, it follows from the definition of the Markov chain Δ that the weighted graph associated with Δ is connected. From this and from the fact that the nonzero entries of Δ are all equal to 1, we can deduce that Δ is reversible and in detailed balance with a uniform probability measure. This forces τ_M to be uniform and so we have to assume that τ is uniform on Y and the matrix Q is symmetric.

In this way, P is in detailed balance with the uniform probability measure π such that $\pi(\theta) = \frac{1}{\binom{n}{k}|Y|^k}$, for every $\theta \in \Theta_k$. This choice of τ guarantees that, if f is any function in W_j, with $j = 1, \ldots, m$, then $\sum_{y \in Y} f(y) = 0$.

The spectral theory of the operator M has been studied in Section 5.2. In fact, it corresponds to choose, with probability $\frac{1}{k}$, only one element of the domain and to change the corresponding image with respect to the probability Q on Y, fixing the remaining ones. So we focus our attention to investigate the spectral theory of the operator Δ.

Let us introduce two differential operators.

DEFINITION 5.13. *(1) For every $k = 2, \dots, n$ the operator $D_k : L(\Theta_k) \longrightarrow L(\Theta_{k-1})$ is defined by*

$$(D_k F)(\varphi) = \sum_{\theta \in \Theta_k : \varphi \subseteq \theta} F(\theta),$$

for every $F \in L(\Theta_k)$ and $\varphi \in \Theta_{k-1}$.

(2) For $k = 1, \dots, n$ the operator $D_k^* : L(\Theta_{k-1}) \longrightarrow L(\Theta_k)$ is defined by

$$(D_k^* F)(\theta) = \sum_{\varphi \in \Theta_{k-1} : \varphi \subseteq \theta} F(\varphi),$$

for every $F \in L(\Theta_{k-1})$ and $\theta \in \Theta_k$.

Observe that the operator D_k^* is adjoint to D_k.
The following decomposition holds

$$L(\Theta_k) = L\left(\coprod_{A \in \Omega_k} Y^A\right) = \bigoplus_{A \in \Omega_k} L(Y^A).$$

In order to get a basis for the space $L(Y^A)$, for every $A \in \Omega_k$, we introduce some special functions that we will call fundamental functions.

DEFINITION 5.14. *Suppose that $A \in \Omega_k$ and that $F^j \in L(Y)$ for every $j \in A$. Suppose also that each F^j belongs to an eigenspace of Q and set $a_i = |\{j \in A : F^j \in W_i\}|$. Then the tensor product $\bigotimes_{j \in A} F^j$ will be called a fundamental function of type $\underline{a} = (a_0, a_1, \dots, a_m)$ in $L(Y^A)$.*

In other words, we have

$$(\bigotimes_{j \in A} F^j)(\theta) = \prod_{j \in A} F^j(\theta(j)),$$

for every $\theta \in Y^A$. We also set $\ell(\underline{a}) = a_1 + \cdots + a_m = k - a_0$.

The introduction of the fundamental functions allows to give a useful expression for the operators M and Δ.

If $F \in L(Y^A) \subseteq L(\Theta_k)$ is the fundamental function $F = \bigotimes_{j \in A} F^j$, with $|A| = k$ and $F^j : Y \longrightarrow \mathbb{C}$, then $MF = \frac{1}{k} \sum_{j \in A} \left[\left(\bigotimes_{i \in A, i \neq j} F^i \right) \otimes QF^j \right]$. So, if $\theta \in \Theta_k$ and $dom(\theta) = A$, we get

$$(MF)(\theta) = \frac{1}{k} \sum_{j \in A} \left[\prod_{i \in A, i \neq j} F^i(\theta(i)) \left(\sum_{y \in Y} q(\theta(j), y) F^j(y) \right) \right].$$

Analogously one has $(\Delta F)(\theta) = \sum_{\varphi} F(\varphi)$, where the sum is over all $\varphi \in \Theta_k$ such that $dom(\varphi) \cap dom(\theta) = k-1$ and $\varphi \equiv \theta$ on $dom(\varphi) \cap$

$dom(\theta)$. If $A = (dom(\theta) \cap A) \sqcup \{i\}$ (we denote by $\sqcup$ the disjoint union), then

$$(\Delta(\otimes_{j\in A} F^j))(\theta) = \sum_{\varphi} \bigotimes_{j\in A} F^j(\varphi) = \prod_{j\in dom(\varphi)\cap A} F^j(\theta(j)) \left(\sum_{y\in Y} F^i(y)\right).$$

Denote $P_{k,\underline{a},A}$ the subspace of $L(Y^A)$ spanned by the fundamental functions of type $\underline{a}$ and

$$P_{k,\underline{a}} = \bigoplus_{A\in\Omega_k} P_{k,\underline{a},A}.$$

LEMMA 5.15. *D_k maps $P_{k,\underline{a}}$ to $P_{k-1,\underline{a}'}$, where $\underline{a}' = (a_0-1, a_1, \ldots, a_m)$. Conversely D_k^* maps $P_{k-1,\underline{a}'}$ to $P_{k,\underline{a}}$.*

Proof. Let F be a fundamental function of type $\underline{a}$ in $L(Y^A)$ and let $B \subset A$ such that $A = B \sqcup \{i\}$. Then for every $\varphi \in Y^B$, we have

$$\begin{aligned}
(D_kF)(\varphi) &= \sum_{\theta\in Y^A:\varphi\subseteq\theta} F(\theta) \\
&= \sum_{\theta\in Y^A:\varphi\subseteq\theta} \prod_{j\in A} F^j(\theta(j)) \\
&= \left(\sum_{y\in Y} F^i(y)\right) \prod_{j\in B} F^j(\varphi(j)).
\end{aligned}$$

The value of $\sum_{y\in Y} F^i(y)$ is zero if $F^i \in W_j$ for $j = 1, \ldots, m$ and so $D_kF \equiv 0$ if $a_0 = 0$. If $F^i \in W_0$, then $D_kF \in P_{k-1,\underline{a}'}$.

Analogously, let $F \in P_{k-1,\underline{a}',B}$ with $B \in \Omega_{k-1}$. Then for every $\theta \in Y^A$, $A = B \sqcup \{i\}$, one has

$$\begin{aligned}
(D_k^*F)(\theta) &= \sum_{\varphi\in Y^B:\varphi\subseteq\theta} F(\varphi) \\
&= \prod_{j\in B} F^j(\varphi(j)) \\
&= F^i(\theta(i)) \prod_{j\in B} F^j(\theta(j)),
\end{aligned}$$

where by setting $F^i \equiv 1$ on Y (and so $F^i \in W_0$). $\square$

The restriction of D_k to $P_{k,\underline{a}}$ will be denoted by $D_{k,\underline{a}}$ and the restriction of D_k^* to $P_{k-1,\underline{a}'}$ by $D_{k,\underline{a}}^*$.

The study of the compositions of the operators $D_{k,\underline{a}}$ and $D_{k,\underline{a}}^*$ plays a central role. In fact it will be shown that the eigenspaces of these operators are also eigenspaces for Δ. Consider, for example, $D_{k+1}D_{k+1}^*$

applied to a function $F \in L(\Theta_k)$ and calculated on $\theta \in \Theta_k$. The functions $\varphi \in \Theta_{k+1}$ such that $\varphi \supseteq \theta$ are in number of $|Y|(n-k)$. Each of them covers $k+1$ functions in Θ_k, one of them is the function θ, the other ones are functions in Θ_k whose domains differ by the domain of θ of an element and coincide on their intersection. These functions are in number of $|Y|(n-k)k$ and they correspond to functions that one can reach starting from θ in the Markov chain described by Δ. From this it follows that $norm(\Delta) = |Y|(n-k)k$.

LEMMA 5.16. *Let $F \in P_{k,\underline{a},A}$, with $A \in \Omega_k$. Then*

$$D^*_{k,\underline{a}} D_{k,\underline{a}} = |Y|(k-\ell(\underline{a}))I + Q_{k,\underline{a}},$$

where $Q_{k,\underline{a}}$ is defined by setting

$$(Q_{k,\underline{a}}F)(\theta) = \begin{cases} 0 & \text{if } F^i \notin W_0, \\ |Y|F(\overline{\theta}) & \text{if } F^i \in W_0 \end{cases} \tag{24}$$

for every $\theta \in \Theta_k$ such that $|dom(\theta) \cap A| = k-1$ and $A \setminus dom(\theta) = \{i\}$. We denote by $\overline{\theta}$ the function in Θ_k whose domain is A and such that $\overline{\theta}|_{A\setminus\{i\}} = \theta$ and $\overline{\theta}(i) = \theta(i_0)$, where $dom(\theta) \setminus A = \{i_0\}$.

Proof. Take $F \in P_{k,\underline{a},A}$ and $\theta \in \Theta_k$. We have

$$\begin{aligned} (D^*_{k,\underline{a}} D_{k,\underline{a}} F)(\theta) &= \sum_{\varphi \in \Theta_{k-1} : \varphi \subseteq \theta} (D_{k,\underline{a}}F)(\varphi) \\ &= \sum_{\varphi \in \Theta_{k-1} : \varphi \subseteq \theta} \; \sum_{\substack{\omega \in \Theta_k : \omega \supseteq \varphi, \\ dom(\omega) = A}} F(\omega). \end{aligned}$$

If $dom(\theta) = A$, then we get

$$\begin{aligned} (D^*_{k,\underline{a}} D_{k,\underline{a}} F)(\theta) &= \sum_{j \in A} \left(\sum_{y \in Y} F^j(y) \right) \prod_{t \in A \setminus \{j\}} F^t(\theta(t)) \\ &= |Y|(k-\ell(\underline{a})) \prod_{t \in A} F^t(\theta(t)) \\ &= |Y|(k-\ell(\underline{a}))F(\theta), \end{aligned}$$

where the second equality follows from the fact that $\sum_{y \in Y} F^j(y) = |Y|$ if $F^j \in W_0$ and $\sum_{y \in Y} F^j(y) = 0$ whenever $F^j \notin W_0$.

On the other hand, if $|dom(\theta) \cap A| = k-1$, with $A \setminus dom(\theta) = \{i\}$, then

$$\begin{aligned} (D^*_{k,\underline{a}} D_{k,\underline{a}} F)(\theta) &= \left(\sum_{y \in Y} F^i(y) \right) \prod_{j \in A \setminus \{i\}} F^j(\theta(j)) \\ &= \begin{cases} 0 & \text{if } F^i \notin W_0, \\ |Y|F(\overline{\theta}) & \text{if } F^i \in W_0 \end{cases} \end{aligned}$$

which is just the definition of $Q_{k,\underline{a}}$. $\square$

LEMMA 5.17. *Let $F \in P_{k,\underline{a}',A}$, with $A \in \Omega_k$. Then*

$$D_{k+1,\underline{a}} D^*_{k+1,\underline{a}} = |Y|(n-k)I + Q_{k,\underline{a}},$$

where $Q_{k,\underline{a}}$ is defined as in (24).

Proof. Take $F \in P_{k,\underline{a}',A}$ and $\theta \in \Theta_k$. We have

$$\begin{aligned}
(D_{k+1,\underline{a}} D^*_{k+1,\underline{a}} F)(\theta) &= \sum_{\varphi \in \Theta_{k+1} : \theta \subseteq \varphi} (D^*_{k+1,\underline{a}} F)(\varphi) \\
&= \sum_{\varphi \in \Theta_{k+1} : \theta \subseteq \varphi} \sum_{\substack{\omega \in \Theta_k : \omega \supseteq \varphi, \\ dom(\omega) = A}} F(\omega).
\end{aligned}$$

If $dom(\theta) = A$, then we get

$$\begin{aligned}
(D_{k+1,\underline{a}} D^*_{k+1,\underline{a}} F)(\theta) &= \sum_{j \in A^C} \sum_{y \in Y} F(\theta) \\
&= |Y|(n-k)F(\theta).
\end{aligned}$$

On the other hand, if $|dom(\theta) \cap A| = k-1$, with $A \setminus dom(\theta) = \{i\}$, then

$$\begin{aligned}
(D_{k+1,\underline{a}} D^*_{k+1,\underline{a}} F)(\theta) &= \left(\sum_{y \in Y} F^i(y) \right) \prod_{j \in A \setminus \{i\}} F^j(\theta(j)) \\
&= \begin{cases} 0 & \text{if } F^i \notin W_0, \\ |Y| F(\overline{\theta}) & \text{if } F^i \in W_0 \end{cases} \\
&= (Q_{k,\underline{a}} F)(\theta).
\end{aligned}$$

This completes the proof. $\square$

The following corollary easily follows.

COROLLARY 5.18. *Let $F \in P_{k,\underline{a}',A}$, with $A \in \Omega_k$. Then*

$$D_{k+1,\underline{a}} D^*_{k+1,\underline{a}} - D^*_{k,\underline{a}'} D_{k,\underline{a}'} = |Y|(n + \ell(\underline{a}) - 2k)I.$$

Consider now the operator $D_{k,\underline{a}} : P_{k,\underline{a}} \longrightarrow P_{k-1,\underline{a}'}$.

DEFINITION 5.19. *For $0 \leq \ell(\underline{a}) \leq k \leq n$, set*

$$P_{k,\underline{a},k} = Ker(D_{k,\underline{a}})$$

and inductively, for $k \leq h \leq n$, set

$$P_{h,\underline{a},k} = D^*_{h,\underline{a}} P_{h-1,\underline{a}',k}.$$

These spaces have a fundamental importance because they exactly constitute the eigenspaces of the operator Δ. This will be a consequence of the following proposition.

Proposition 5.20. *$P_{h,\underline{a}',k}$ is an eigenspace for the operator $D_{h+1,\underline{a}}D^*_{h+1,\underline{a}}$ and the corresponding eigenvalue is $|Y|(n+\ell(\underline{a})-k-h)(h-k+1)$.*

Proof. We prove the assertion by induction on h. If $h=k$, from the last corollary we get $D_{k+1,\underline{a}}D^*_{k+1,\underline{a}}|_{P_{k,\underline{a}',k}} = |Y|(n+\ell(\underline{a})-2k)I$, since $D_{k,\underline{a}'}P_{k,\underline{a}',k}=0$ by definition of $P_{k,\underline{a}',k}$.

Now suppose the lemma to be true for $k \le t \le h$ and recall that, by definition, we have $P_{h+1,\underline{a}',k} = D^*_{h+1,\underline{a}'}P_{h,\underline{a}'',k}$. Moreover, Corollary 5.18 gives

$$D_{h+2,\underline{a}}D^*_{h+2,\underline{a}} - D^*_{h+1,\underline{a}'}D_{h+1,\underline{a}'} = |Y|(n+\ell(\underline{a})-2(h+1))I.$$

So we get

$$\begin{aligned} D_{h+2,\underline{a}}D^*_{h+2,\underline{a}}|_{P_{h+1,\underline{a}',k}} &= D^*_{h+1,\underline{a}'}|_{D_{h+1,\underline{a}'}D^*_{h+1,\underline{a}'}P_{h,\underline{a}'',k}} \\ &+ |Y|(n+\ell(\underline{a})-2(h+1))P_{h+1,\underline{a}',k} \\ &= |Y|(n+\ell(\underline{a})-k-h)(h-k+1)D^*_{h+1,\underline{a}'}P_{h,\underline{a}'',k} \\ &+ |Y|(n+\ell(\underline{a})-2(h+1))P_{h+1,\underline{a}',k} \\ &= |Y|(n+\ell(\underline{a})-k-h-1)(h-k+2)P_{h+1,\underline{a}',k}, \end{aligned}$$

where the second equality follows from the inductive hypothesis and the third one from an easy computation. This completes the proof. □

Corollary 5.21. *$P_{h,\underline{a}',k}$ is an eigenspace for Δ of eigenvalue $|Y|(n+\ell(\underline{a})-k-h)(h-k+1)-|Y|(n-h)$.*

Proof. It suffices to observe that the operator $Q_{h,\underline{a}}$ defined in (24) coincides with the operator Δ on the space $P_{h,\underline{a}}$ and then the assertion follows from Lemma 5.17 and Proposition 5.20. □

In particular, after normalizing the matrix Δ we obtain $\frac{\Delta}{norm(\Delta)}$ and the corresponding eigenvalue is $\frac{1}{|Y|(n-h)h}(|Y|(n+\ell(\underline{a})-k-h)(h-k+1)-|Y|(n-h))$.

The following lemma holds.

Lemma 5.22. *Given $\ell(\underline{a})$ and h then, for $\ell(\underline{a}) \le k \le \min\left\{h, \frac{n+\ell(\underline{a})}{2}\right\}$, the spaces $P_{h,\underline{a}',k}$ are mutually orthogonal.*

Proof. Each $P_{h,\underline{a}',k}$ is an eigenspace for the self-adjoint operator $D_{h+1,\underline{a}}D^*_{h+1,\underline{a}}$. Since the eigenvalue $|Y|(n+\ell(\underline{a})-k-h)(h-k+1)$ is a strictly decreasing function of k for $k \le \frac{n+\ell(\underline{a})}{2}$, then to different values of k correspond different eigenvalues. This proves the assertion. □

Recall that, if $\underline{a}=(a_0,a_1,\dots,a_m)$, we set $\underline{a}'=(a_0-1,a_1,\dots,a_m)$ and, inductively, $\underline{a}^{h+1}=\underline{a}^h-(1,0,\dots,0)$.

PROPOSITION 5.23. *Let F be a function in $P_{k,\underline{a}^{h-k},k}$. Then, for $\ell(\underline{a}) \leq k \leq \frac{n+\ell(\underline{a})}{2}$ and $k \leq h \leq n+\ell(\underline{a})-k$, we have*

$$\|D^*_{h,\underline{a}}D^*_{h-1,\underline{a}'}\cdots D^*_{k+1,\underline{a}^{h-k-1}}F\|^2 = \frac{(n+\ell(\underline{a})-2k)!(h-k)!}{(n+\ell(\underline{a})-k-h)!}|Y|^{h-k}\|F\|^2.$$

In particular, $D^*_{h,\underline{a}}D^*_{h-1,\underline{a}'}\cdots D^*_{k+1,\underline{a}^{h-k-1}}$ is an isomorphism of $P_{k,\underline{a}^{h-k},k}$ onto $P_{h,\underline{a},k}$.

Proof. We prove the assertion by induction on h. For $h = k+1$ and $F \in P_{k,\underline{a}',k}$, we have

$$\begin{aligned}
\|D^*_{k+1,\underline{a}}F\|^2 &= < D^*_{k+1,\underline{a}}F,\ D^*_{k+1,\underline{a}}F > \\
&= < D_{k+1,\underline{a}}D^*_{k+1,\underline{a}}F,\ F > \\
&= |Y|(n+\ell(\underline{a})-2k)\|F\|^2
\end{aligned}$$

by Proposition 5.20, so the assertion is true. For $h > k+1$, applying Proposition 5.20 to $D_{h,\underline{a}}D^*_{h,\underline{a}}$, we get

$$\begin{aligned}
&\|D^*_{h,\underline{a}}D^*_{h-1,\underline{a}'}\cdots D^*_{k+1,\underline{a}^{h-k-1}}F\|^2 \\
=&< D_{h,\underline{a}}D^*_{h,\underline{a}}D^*_{h-1,\underline{a}'}\cdots D^*_{k+1,\underline{a}^{h-k-1}}F,\ D^*_{h-1,\underline{a}'}\cdots D^*_{k+1,\underline{a}^{h-k-1}}F > \\
=& |Y|(n+\ell(\underline{a})-k-h+1)(h-k)\|D^*_{h-1,\underline{a}'}\cdots D^*_{k+1,\underline{a}^{h-k-1}}F\|^2.
\end{aligned}$$

Now the proposition follows by induction. $\square$

PROPOSITION 5.24. *Assume $\ell(\underline{a}) \leq h \leq \frac{n+\ell(\underline{a})}{2}$. Then*

(1) $P_{h,\underline{a}} = \bigoplus_{k=\ell(\underline{a})}^{\min\{h,n+\ell(\underline{a})-h\}} P_{h,\underline{a},k}$;

(2) $D^*_{h+1,\underline{a}} : P_{h,\underline{a}'} \longrightarrow P_{h+1,\underline{a}}$ is an injective map.

Proof. We prove the assertion by induction on h.

Assume that (1) and (2) are true for $\ell(\underline{a})-1 \leq h \leq t \leq \frac{n+\ell(\underline{a})-1}{2}$. For $h = \ell(\underline{a})-1$ we have $P_{\ell(\underline{a})-1,\underline{a}} = 0$ and so the proposition trivially holds.

Since the operator $D^*_{h,\underline{a}}$ is the adjoint of $D_{h,\underline{a}}$ we have the following decomposition:

$$\begin{aligned}
P_{h,\underline{a}} &= Ker(D_{h,\underline{a}}) \oplus D^*_{h,\underline{a}}P_{h-1,\underline{a}'} \\
&= P_{h,\underline{a},h} \oplus D^*_{h,\underline{a}}P_{h-1,\underline{a}'}.
\end{aligned}$$

In particular

$$P_{t+1,\underline{a}} = P_{t+1,\underline{a},t+1} \oplus D^*_{t+1,\underline{a}}P_{t,\underline{a}'}.$$

By induction

$$P_{t,\underline{a}'} = \bigoplus_{k=\ell(\underline{a})}^{t} P_{t,\underline{a}',k}$$

and so

$$\begin{aligned} P_{t+1,\underline{a}} &= P_{t+1,\underline{a},t+1} \oplus D^*_{t+1,\underline{a}} \left(\bigoplus_{k=\ell(\underline{a})}^{t} P_{t,\underline{a}',k} \right) \\ &= \bigoplus_{k=\ell(\underline{a})}^{t+1} P_{t+1,\underline{a},k}. \end{aligned}$$

This proves (1), while (2) follows from (1) and Proposition 5.23 . $\square$

Corollary 5.25. *The dimension of the spaces $P_{h,\underline{a},k}$ that appear in decomposition of $P_{h,\underline{a}}$ is*

$$\frac{n+\ell(\underline{a})+1-2k}{n-k+1}\binom{n}{k}\binom{k}{\ell(\underline{a})}\binom{\ell(\underline{a})}{a_1,\ldots,a_m}\prod_{j=1}^{m}(\dim(W_j))^{a_j}.$$

Proof. From the previous proposition it follows

$$\dim(P_{t+1,\underline{a},t+1}) = \dim(P_{t+1,\underline{a}}) - \dim(P_{t,\underline{a}'}).$$

Now

$$\dim(P_{t+1,\underline{a}}) = \binom{n}{t+1}\binom{t+1}{a_0,a_1,\ldots,a_m}\prod_{j=1}^{m}(\dim(W_j))^{a_j}.$$

In fact, $\binom{n}{t+1}$ represents the number of $(t+1)-$subsets in X and $\binom{t+1}{a_0,a_1,\ldots,a_m}\prod_{j=1}^{m}(\dim(W_j))^{a_j}$ represents the number of possible choices in the fundamental function $F=\prod_{r\in A}F^r$ of a_i functions belonging to the eigenspace W_i of $L(Y)$. Thus

$$\begin{aligned} \dim(P_{t+1,\underline{a},t+1}) &= \binom{n}{t+1}\binom{t+1}{a_0,a_1,\ldots,a_m}\prod_{j=1}^{m}(\dim(W_j))^{a_j} \\ &- \binom{n}{t}\binom{t}{a_0-1,a_1,\ldots,a_m}\prod_{j=1}^{m}(\dim(W_j))^{a_j} \\ &= \frac{n-t-a_0}{n-t}\binom{n}{t+1}\binom{t+1}{a_0,a_1,\ldots,a_m}\prod_{j=1}^{m}(\dim(W_j))^{a_j}. \end{aligned}$$

Since, by Proposition 5.23, $\dim(P_{h,\underline{a},k}) = \dim(P_{k,\underline{a}^{h-k},k})$ one can obtain the result replacing t by $k-1$ and $\underline{a}$ by $\underline{a}^{h-k}$. $\square$

We want to find now the eigenvector of $\frac{\Delta}{norm(\Delta)}$ associated with the eigenvalue 1. Consider in $P_{1,(1,0,\ldots,0)}$ the function

$$f=\sum_{i=1}^{n} f_i,$$

where f_i is the fundamental function of type $(1,0,\dots,0)$ whose domain is $\{i\}$. Set

$$< f >= P_{1,(1,0,\dots,0),0} =: D^*_{1,(1,0,\dots,0)} P_{0,(0,\dots,0),0}.$$

So the element $F_0 = D^*_{h,(h,0,\dots,0)} \cdots D^*_{3,(3,0,\dots,0)} D^*_{2,(2,0,\dots,0)} f$ is the generator of the space $P_{h,(h,0,\dots,0),0}$, which has dimension 1. Corollary 5.21 implies that $P_{h,(h,0,\dots,0),0}$ is an eigenspace for $\frac{\Delta}{norm(\Delta)}$ and the corresponding eigenvalue is 1. Moreover, the connectedness of the graph associated with Δ implies that this is the unique (up to constant) eigenvector of eigenvalue 1. We denote by $P_{1,(1,0,\dots,0),1}$ the orthogonal subspace to $P_{1,(1,0,\dots,0),0}$ in $P_{1,(1,0,\dots,0)}$. It has dimension $n-1$.

Observe that the definition of fundamental functions is strictly linked to the spectral theory of the operator Q and so of the operator M restricted to each domain. In fact, if F is a fundamental function in $P_{h,\underline{a},A}$, with $\underline{a} = (a_0, a_1, \dots, a_m)$ and $A \in \Omega_h$, then it is an eigenvector for the operator M and the corresponding eigenvalue is $\frac{1}{h}\sum_{j=0}^m a_j\lambda_j$. So the set of the eigenvalues of M is given by $\binom{n}{h}$ copies of these values. In particular, the eigenspace $P_{h,\underline{a},k}$ of $\frac{\Delta}{norm(\Delta)}$ is also an eigenspace for M and an eigenvector in this space has eigenvalue $\frac{1}{h}\sum_{j=0}^m a_j\lambda_j$. So, by Corollary 5.21 and definition (23) of P, we get the following theorem.

THEOREM 5.26. *$P_{h,\underline{a},k}$ is an eigenspace for P with eigenvalue*

$$p_0 \cdot \frac{1}{h}\sum_{j=0}^{m} a_j\lambda_j + (1-p_0)\frac{(n+\ell(\underline{a})-k-h)(h-k+1)-(n-h)}{h(n-h)}.$$

REMARK 5.27.

It is easy to check that the operator M is not ergodic. In fact its associated graph contains $\binom{n}{h}$ connected components and so the multiplicity of the eigenvalue 1 for M is $\binom{n}{h}$.

On the other hand we already observed that the operator $\frac{\Delta}{norm(\Delta)}$ has the eigenvalue 1 with multiplicity one. To conclude that it is ergodic it suffices to show that -1 is not an eigenvalue, i.e. the associated graph is not bipartite. In fact consider $\theta \in \Theta_h$ with domain $\{i_1, \dots, i_h\}$ and $\theta(i_j) = y_j$, for every $j = 1, \dots, h$. By definition of Δ we can connect θ with φ, whose domain is $\{i_1, \dots, i_{h-1}, i_t\}$, $i_h \neq i_t$ and such that $\varphi(i_j) = y_j = \theta(i_j)$ for all $j = 1, \dots, h-1$ and $\varphi(i_t) = y_t$. Moreover θ can also be connected with ϱ whose domain is $\{i_1, \dots, i_{h-2}, i_h, i_t\}$ and such that $\varrho(i_j) = y_j = \theta(i_j)$ for all $j = 1, \dots, h-2, h$ and $\varrho(i_t) = y_t = \varphi(i_t)$. On the other hand φ and ϱ are connected as well and this proves that the graph is not bipartite.

From Theorem 5.26 we can deduce the ergodicity for the operator P, since the multiplicity of the eigenvalue 1 is one and the eigenvalue -1 does not appear in the spectrum of P.

REMARK 5.28.

The second crested product reduces to the crossed product if $k = n$ and to the nested product if $k = 1$.

In fact, if $k = n$, the domain of a function $\theta \in \Theta_n$ cannot be changed and θ can be identified with the $n-$tuple $(y_1, \dots, y_n) \in Y^n$ of its images. The operator P becomes

$$P = \frac{1}{n}\sum_{i=1}^{n} I_1 \otimes \cdots \otimes I_{i-1} \otimes Q \otimes I_{i+1} \otimes \cdots \otimes I_n,$$

which is the crossed product on the space Y^n.

If $k = 1$, then Δ has the following expression:

$$\Delta = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & 1 & & \vdots \\ \vdots & 1 & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 0 \end{pmatrix}$$

and $norm(\Delta) = n - 1$. So we get

$$P = p_0(I_X \otimes Q) + (1 - p_0)\left(\frac{\Delta}{norm(\Delta)} \otimes J_Y\right),$$

which is just the nested product of X and Y, with $P_X = \frac{\Delta}{norm(\Delta)}$ and $P_Y = Q$.

5.7. Bi-insect. In what follows, we take Y as a homogeneous rooted tree of degree q and depth $m-1$ and we give an explicit description of the spectrum of the operator $P = p_0 M + (1-p_0)\frac{\Delta}{norm(\Delta)}$ acting on the space $L(\Theta_2)$. Therefore we are considering functions in Θ_2 such that the image of each element of the domain is an insect. Suppose X to be a set of cardinality n and let $m \geq 3$. Recall that we have the decomposition

$$L(Y) = \bigoplus_{j=0}^{m-1} W_j,$$

where $W_0 \cong \mathbb{C}$ and

$$W_j = \{f \in L(L_{m-1}) : f = f(x_1, \dots, x_j), \sum_{i=0}^{q-1} f(x_1, \dots, x_{j-1}, i) \equiv 0\},$$

for every $j = 1, \dots, m-1$. Observe that $dim(W_j) = q^{j-1}(q-1)$.

The eigenspaces relative to the operator $\Delta/norm(\Delta)$ are the subspaces of the form $P_{2,(a_0,a_1,\dots,a_{m-1}),k}$, with $k = 0, 1, 2$. The corresponding eigenvalue is

$$\frac{1}{q^{m-1}(n-2)2}\left[q^{m-1}(n+\ell(\underline{a})-k-2)(2-k+1)-q^{m-1}(n-2)\right].$$

So, by dependence of $\ell(\underline{a})$, we get the following eigenspaces:

- $P_{2,(a_0,a_1,\dots,a_{m-1}),2}\begin{cases} a_0 = 0 & \text{with eigenvalue } \lambda = 0, \\ a_0 = 1 & \text{with eigenvalue } \lambda = -\frac{1}{2(n-2)}, \\ a_0 = 2 & \text{with eigenvalue } \lambda = -\frac{1}{n-2}. \end{cases}$

- $P_{2,(a_0,a_1,\dots,a_{m-1}),1}\begin{cases} a_0 = 1 & \text{with eigenvalue } \lambda = \frac{1}{2}, \\ a_0 = 2 & \text{with eigenvalue } \lambda = \frac{n-4}{2(n-2)}. \end{cases}$

- $P_{2,(2,0,\dots,0),0} \Rightarrow\ a_0 = 2$ with eigenvalue $\lambda = 1$.

Now we describe the eigenvalues of these eigenspaces with respect to the operator M and to join the results.

If F is a fundamental function of type $(a_0, a_1, \dots, a_{m-1})$, then it has eigenvalue $\frac{1}{2}\sum_{j=0}^{m-1} a_j\lambda_j$, where $\lambda_j = 1 - \frac{q-1}{q^{m-j}-1}$ is the eigenvalue of the eigenspace W_j, of dimension $q^{j-1}(q-1)$, occurring in the spectral decomposition of $L(Y)$. From this we can fill the following tabular in which we give the eigenspaces, together with the corresponding eigenvalue and dimension.

- $P_{2,(a_0,a_1,\dots,a_{m-1}),2}$. We have three different cases:

  (1) if $a_0 = 0$, the corresponding eigenspace is

  $$P_{2,(0,\dots,0,\underbrace{1}_{i-th\ place},0,\dots,0,\underbrace{1}_{j-th\ place},0,\dots,0),2}$$

  of dimension $n(n-1)(q-1)^2q^{i-1}q^{j-1}$, with eigenvalue $\frac{p_0}{2}(\lambda_i + \lambda_j)$;

  (2) if $a_0 = 1$, the corresponding eigenspace is

  $$P_{2,(1,\dots,0,\underbrace{1}_{i-th\ place},0,\dots,0),2}$$

  of dimension $n(n-2)(q-1)q^{i-1}$, with eigenvalue $p_0\frac{1+\lambda_i}{2} + (1-p_0)\frac{-1}{2(n-2)}$;

  (3) if $a_0 = 2$, the corresponding eigenspace is $P_{2,(2,0,\dots,0),2}$ of dimension $\frac{n(n-3)}{2}$ with eigenvalue $p_0 + (1-p_0)\frac{-1}{n-2}$.

- $P_{2,(a_0,a_1,\dots,a_{m-1}),1}$. We have two different cases:

(1) if $a_0 = 1$, the corresponding eigenspace is

$$P_{2,(1,\dots,0,\underbrace{1}_{i-th\ place},0,\dots,0),1}$$

of dimension $n(q-1)q^{i-1}$, with eigenvalue $p_0\frac{1+\lambda_i}{2} + \frac{1-p_0}{2}$;

(2) if $a_0 = 2$, the corresponding eigenspace is $P_{2,(2,0,\dots,0),1}$ of dimension $n-1$, with eigenvalue $p_0 + (1-p_0)\frac{n-4}{2(n-2)}$

- $P_{2,(2,0,\dots,0),0}$. In this case, the dimension of the eigenspace is 1 with eigenvalue 1.

Appendix: Association schemes

The theory of the Association Schemes is strictly linked to the theory of Gelfand pairs. It is a combinatorial tool that gives an equivalent description of the theory developed for groups and for Markov chains.

Association schemes are about relations between pairs of elements of a set Ω, that we suppose to be finite. Three equivalent definitions of association scheme can be given: in terms of partitions, graphs and matrices, respectively. A complete theory is developed in [**Bai**].

6. First definition

DEFINITION 6.1. *An* ***association scheme*** *with s associate classes on a finite set Ω is a partition of $\Omega\times\Omega$ into nonempty sets $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_s$, called the* ***associate classes****, such that*

(1) $\mathcal{C}_0 = Diag(\Omega) = \{(\omega,\omega) : \omega \in \Omega\}$.

(2) $\mathcal{C}_i$ *is symmetric for every* $i = 1, \ldots, s$, *i.e.* $\mathcal{C}_i = \mathcal{C}_i'$, *where* $\mathcal{C}_i'$ *denotes the dual of* $\mathcal{C}_i$ *defined as* $\mathcal{C}_i' = \{(\beta,\alpha) : (\alpha,\beta) \in \mathcal{C}_i\}$.

(3) For all $i, j, k \in \{0, 1, \ldots, s\}$ *there exists an integer* p_{ij}^k *such that, for all* $(\alpha,\beta) \in \mathcal{C}_k$,

$$|\{\gamma \in \Omega : (\alpha,\gamma) \in \mathcal{C}_i \text{ and } (\gamma,\beta) \in \mathcal{C}_j\}| = p_{ij}^k.$$

We will say that the rank of this association scheme is $s+1$. Observe that the conditions (2) and (3) imply $p_{ij}^k = p_{ji}^k$. The elements α and β are called $i-$th associates if $(\alpha,\beta) \in \mathcal{C}_i$. In particular, the set of $i-$th associates of α is denoted by

$$\mathcal{C}_i(\alpha) = \{\beta \in \Omega : (\alpha,\beta) \in \mathcal{C}_i\}.$$

Condition (2) implies $p_{ij}^0 = 0$ if $i \neq j$. Similarly, $p_{0j}^k = 0$ if $j \neq k$ and $p_{i0}^k = 0$ if $i \neq k$, while $p_{0j}^j = p_{i0}^i = 1$. Moreover, the condition (3) implies that each element of Ω has $p_{ii}^0 = a_i$ $i-$th associates.

EXAMPLE 6.2.

Let Ω be a finite set, with $|\Omega| = n$. Let $\mathcal{C}_0$ be the diagonal subset and set

$$\mathcal{C}_1 = \{(\alpha,\beta) \in \Omega\times\Omega : \alpha \neq \beta\} = (\Omega\times\Omega) \setminus \mathcal{C}_0.$$

This is the **trivial** association scheme, the only scheme on Ω having only one associate class. It has $a_1 = n - 1$ and it is denoted by $\underline{\underline{n}}$.

EXAMPLE 6.3.

Let Ω an $m \times n$ rectangular array, with $m, n \geq 2$. Set

- $\mathcal{C}_1 = \{(\alpha, \beta) : \alpha, \beta$ are in the same row but $\alpha \neq \beta\}$;
- $\mathcal{C}_2 = \{(\alpha, \beta) : \alpha, \beta$ are in the same column but $\alpha \neq \beta\}$;
- $\mathcal{C}_3 = \{(\alpha, \beta) : \alpha, \beta$ are in different rows and columns$\}$.

It is clear that $\mathcal{C}_3 = (\Omega \times \Omega) \setminus \mathcal{C}_0 \setminus \mathcal{C}_1 \setminus \mathcal{C}_2$. This is an association scheme with three associate classes and $a_1 = n - 1$, $a_2 = m - 1$, $a_3 = (m-1)(n-1)$. It is called the **rectangular** association scheme $R(m, n)$ and is also denoted by $\underline{\underline{m}} \times \underline{\underline{n}}$.

EXAMPLE 6.4.

Consider the partition $\Omega = \Delta_1 \sqcup \ldots \sqcup \Delta_m$ of the set Ω into m subsets of size n. These subsets are traditionally called *groups*. We declare α and β to be:

- first associates if they are in the same groups but $\alpha \neq \beta$;
- second associates if they are in different groups.

It is easy to verify that, if $\omega \in \Omega$, then it has $n - 1$ first associates and $(m-1)n$ second associates. So this is an association scheme with $s = 2$ and $a_1 = n - 1$, $a_2 = (m-1)n$. It is called the **group-divisible** association scheme, denoted by $GD(m, n)$ or also $\underline{\underline{m}}/\underline{\underline{n}}$.

7. Second definition

DEFINITION 7.1. *An association scheme with s associate classes on a finite set Ω is a colouring of the edges of the complete undirected graph, whose vertices are indexed by Ω, by s colours such that:*

(1) for all $i, j, k \in \{1, \ldots, s\}$ there exists an integer p_{ij}^k such that, if $\{\alpha, \beta\}$ is an edge of colour k, then

$$|\{\gamma \in \Omega : \{\alpha, \gamma\} \text{ has colour } i \text{ and } \{\gamma, \beta\} \text{ has colour } j\}| = p_{ij}^k;$$

(2) every colour is used at least once;

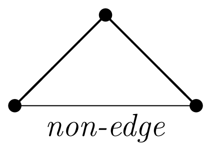
(3) there exist integers a_i, for $i = 1, \ldots, s$, such that each vertex is contained in exactly a_i edges of colour i.

We do not need an analogous of the conditions (1) and (2) of the first definition. In fact, every edge consists of two distinct vertices and the graph is supposed to be undirected. The new condition (1) says that if we consider any two different vertices α and β and fix two colours i and j, then the number of triangles consisting of the edge $\{\alpha, \beta\}$ and an $i-$coloured edge through α and a $j-$coloured edge through β is exactly p_{ij}^k, where k is the colour of $\{\alpha, \beta\}$. The new condition (2) has not an analogous in the partition definition, since we specified that the subsets in the partition are nonempty. Finally, since the condition (1) of the graph definition does not deal with the analogue of the diagonal subset, this is explicitly given in condition (3).

If an association scheme has two associate classes, the two colours can be regarded as "visible" and "invisible". The corresponding graph is *strongly regular*, according with the following definition.

DEFINITION 7.2. *A finite graph is strongly regular if:*

- *it is regular, i.e. each vertex is contained in the same number of edges;*
- *every edge is contained in the same number of triangles;*
- *every non-edge is contained in the same number of configurations like*



- *it is neither complete (all pairs are edges) nor null (no pairs are edges).*

8. Third definition

Given an association scheme with associate classes $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_s$, we can associate to each class $\mathcal{C}_i$ its **adjacency matrix** A_i, i.e. the matrix of size $|\Omega|$ defined as

$$(A_i)_{\alpha\beta} = \begin{cases} 1 & \text{if } (\alpha, \beta) \in \mathcal{C}_i \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma holds.

LEMMA 8.1. *Given an association scheme with associate classes $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_s$, let A_i be the corresponding adjacency matrices. Then*

$$A_i A_j = \sum_{k=0}^{s} p_{ij}^k A_k. \tag{25}$$

Proof. Suppose $(\alpha, \beta) \in \mathcal{C}_k$. Then the $(\alpha, \beta)-$entry of the right-hand side of (25) is equal to p_{ij}^k, while the $(\alpha, \beta)-$entry of the left-hand side is equal to

$$\begin{aligned} (A_i A_j) &= \sum_{\gamma \in \Omega} A_i(\alpha, \gamma) A_j(\gamma, \beta) \\ &= |\{\gamma : (\alpha, \gamma) \in \mathcal{C}_i \text{ and } (\gamma, \beta) \in \mathcal{C}_j\}| \\ &= p_{ij}^k, \end{aligned}$$

because the product $A_i(\alpha, \gamma) A_j(\gamma, \beta)$ is zero unless $(\alpha, \gamma) \in \mathcal{C}_i$ and $(\gamma, \beta) \in \mathcal{C}_j$, in which case it is 1. $\square$

This lemma leads us to the third definition of association schemes, in terms of adjacency matrices.

DEFINITION 8.2. *An association scheme with s associate classes on a finite set* Ω *is a set of nonzero matrices* $A_0, A_1, \ldots, A_s$*, with rows and columns indexed by* Ω*, whose entries are equal to 0 or 1 and such that:*

(1) $A_0 = I_\Omega$*, where* I_Ω *denotes the identity matrix of size* $|\Omega|$*;*
(2) A_i *is symmetric for every* $i = 1, \ldots, s$*;*
(3) for all $i, j \in \{1, \ldots, s\}$*, the product* A_iA_j *is a linear combination of* $A_0, A_1, \ldots, A_s$*;*
(4) $\sum_{i=0}^{s} A_i = J_\Omega$*, where* J_Ω *denotes the all–1 matrix of size* $|\Omega|$*.*

Observe that the condition (4) of this definition gives an analogue of the fact that the subsets $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_s$ constitute a partition of $\Omega \times \Omega$.

PROPOSITION 8.3. *If* $A_0, A_1, \ldots, A_s$ *are the adjacency matrices of an association scheme, then* $A_iA_j = A_jA_i$ *for all* $i, j \in \{0, 1, \ldots, s\}$*.*

Proof. We have

$$\begin{aligned}
A_jA_i &= A_j^T A_i^T, \quad \text{because the adjacency matrices are symmetric,}\\
&= (A_iA_j)^T\\
&= \left(\sum_k p_{ij}^k A_k\right)^T, \quad \text{by Equation (25),}\\
&= \sum_k p_{ij}^k A_k^T\\
&= \sum_k p_{ij}^k A_k, \quad \text{because the adjacency matrices are symmetric,}\\
&= A_iA_j. \qquad \square
\end{aligned}$$

EXAMPLE 8.4.

Let $\prod$ be a **Latin square** of size n, i.e. an $n \times n$ array filled with n letters in such a way that each letter occurs once in each row and once in each column.

a	d	b	c
c	a	d	b
b	c	a	d
d	b	c	a

Fig.13. A Latin square of size 4.

Let Ω be the set of n^2 cells of the array. Consider $\alpha, \beta \in \Omega$, with $\alpha \neq \beta$. We declare α and β to be first associates if they are in the same row or in the same column or have the same letter. Otherwise, they are second associates. It is easy to check that so we get an association scheme on Ω, with two associate classes.

9. The Bose-Mesner algebra

Consider an association scheme with adjacency matrices $A_0, \ldots, A_s$. Let $\mathcal{A}$ be the space of all real linear combinations of $A_0, A_1, \ldots, A_s$. This is a real vector space of dimension $s+1$. In fact, the matrices $A_0, A_1, \ldots, A_s$ are linearly independent because, given α and β in Ω, there exists only one index i such that $A_i(\alpha, \beta) \neq 0$. It follows from Lemma 8.1 that $\mathcal{A}$ is closed under multiplication and so it is an algebra. Proposition 8.3 tells us that $\mathcal{A}$ is a commutative algebra, called the **Bose-Mesner algebra**.

Since every adjacency matrix is symmetric, a matrix $M \in \mathcal{A}$ is symmetric and so it is diagonalizable on $\mathbb{R}$, i.e. it has distinct real eigenvalues $\lambda_1, \ldots, \lambda_r$ such that:

- $L(\Omega) = \bigoplus_{i=1}^{r} V_i$, where V_i is the eigenspace associated with the eigenvalue λ_i;
- the eigenspaces V_i and V_j are orthogonal, for $i \neq j$.

The orthogonality of eigenspaces is with respect to the inner product on $L(\Omega)$ defined as

$$\langle f, g \rangle = \sum_{\omega \in \Omega} f(\omega) g(\omega), \qquad \text{for all } f, g \in L(\Omega).$$

Definition 9.1. *The orthogonal projector P on a subspace W is the map $P : L(\Omega) \longrightarrow L(\Omega)$ defined by*

$$Pv \in W \qquad \textit{and} \qquad v - Pv \in W^{\perp}.$$

Now put

$$P_1 = \frac{(M - \lambda_2 I) \cdots (M - \lambda_r I)}{(\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_r)}.$$

It is easy to check that, if $v \in V_1$, then $P_1 v = v$, while if $Mv = \lambda_i v$ for $i > 1$, then $P_1 v = 0$. So P_1 is the orthogonal projector onto V_1. Analogously for V_i, with $i > 1$.

Now let M_1 and M_2 be two matrices in $\mathcal{A}$ and let $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_m$ be the respective eigenprojectors. They commute with each other, since they are polynomials in M_1 and M_2, respectively. The following properties of $P_i Q_j$'s hold:

- they are orthogonal, in fact $P_i Q_j P_{i'} Q_{j'} = P_i P_{i'} Q_j Q_{j'}$, which is zero unless $i = i'$ and $j = j'$;
- they are idempotents, in fact $P_i Q_j P_i Q_j = P_i P_i Q_j Q_j = P_i Q_j$;
- $\sum_i \sum_j P_i Q_j = (\sum_i P_i)(\sum_j Q_j) = I^2 = I$;
- the subspaces which they project onto are contained in eigenspaces of both M_1 and M_2.

If we apply this argument to $A_0, A_1, \ldots, A_s$, we deduce that there exist mutually orthogonal subspaces $W_0, W_1, \ldots, W_r$, with orthogonal projectors $S_0, S_1, \ldots, S_r$, such that

- $L(\Omega) = W_0 \oplus W_1 \oplus \cdots \oplus W_r$;

- each W_i is contained in an eigenspace of every A_j;
- each S_i is a polynomial in $A_1, \ldots, A_s$ and so in $\mathcal{A}$.

Thus there are unique constant $D(e,i)$ such that

$$S_e = \sum_i D(e,i) A_i.$$

On the other hand, if $C(i,e)$ is the eigenvalue of A_i on W_e, then

$$A_i = \sum_{e=0}^{r} C(i,e) S_e.$$

Moreover, the projectors $S_0, \ldots, S_r$ are linearly independent because $S_e S_f = \delta_{ef} S_e$ and so they constitute another basis for $\mathcal{A}$. Therefore we have $r = s$ and $D = C^{-1}$.

The subspaces W_e are called **strata**, while the matrices S_e are called **stratum projectors**. The matrix C is the **character table** of the association scheme.

10. Crossed and nested product of association schemes

Definition 10.1. *Let $\mathcal{Q}_1$ be an association scheme on Ω_1 with classes $\mathcal{C}_i$, for $i \in \mathcal{K}_1$ and let $\mathcal{Q}_2$ be an association scheme on Ω_2 with classes $\mathcal{D}_j$, for $j \in \mathcal{K}_2$. Then $\mathcal{Q}_1$ is isomorphic to $\mathcal{Q}_2$ if there exist bijections*

$$\phi : \Omega_1 \longrightarrow \Omega_2 \qquad \text{and} \qquad \pi : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$$

such that

$$(\alpha, \beta) \in \mathcal{C}_i \;\Leftrightarrow\; (\phi(\alpha), \phi(\beta)) \in \mathcal{D}_{\pi(i)}.$$

In this case, we say that the pair (ϕ, π) is an **isomorphism** between association schemes and write $\mathcal{Q}_1 \cong \mathcal{Q}_2$.

We can now introduce two special product of association schemes, called the crossed product and the nested product, respectively.

So let $\mathcal{Q}_1$ be an association scheme on the finite set Ω_1 with adjacency matrices $A_0, A_1, \ldots, A_m$, and let $\mathcal{Q}_2$ be an association scheme on the finite set Ω_2 with adjacency matrices $B_0, B_1, \ldots, B_r$.

Definition 10.2. *The **crossed product** of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is the association scheme $\mathcal{Q}_1 \times \mathcal{Q}_2$ on $\Omega_1 \times \Omega_2$ whose adjacency matrices are*

$$A_i \otimes B_j,$$

for $i = 0, \ldots, m$ and $j = 0, \ldots, r$.

The crossed product of two association schemes is also called *direct product*. For example, one can easily verify that the rectangular association scheme $R(m,n)$ can be obtained as the crossed product of the schemes $\underline{\underline{m}}$ and $\underline{\underline{n}}$.

Definition 10.3. *The* ***nested product*** *of* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *is the association scheme* $\mathcal{Q}_1/\mathcal{Q}_2$ *on* $\Omega_1 \times \Omega_2$ *whose adjacency matrices are*

- $A_i \otimes J_{\Omega_2}$, *with* $i \neq 0$;
- $I_{\Omega_1} \otimes B_j$, *for every* $j = 0, 1, \ldots, r$.

The nested product of two association schemes is also called *wreath product*. For example, one can easily verify that the group-divisible association scheme $GD(m,n)$ can be obtained as the nested product of the schemes $\underline{\underline{m}}$ and $\underline{\underline{n}}$.

Proposition 10.4. *The following properties of crossed and nested product hold:*

(1) *crossing is commutative, in the sense that* $\mathcal{Q}_1 \times \mathcal{Q}_2 \cong \mathcal{Q}_2 \times \mathcal{Q}_1$;
(2) *crossing is associative, in the sense that* $\mathcal{Q}_1 \times (\mathcal{Q}_2 \times \mathcal{Q}_3) \cong (\mathcal{Q}_1 \times \mathcal{Q}_2) \times \mathcal{Q}_3$;
(3) *nesting is associative, in the sense that* $\mathcal{Q}_1/(\mathcal{Q}_2/\mathcal{Q}_3) \cong (\mathcal{Q}_1/\mathcal{Q}_2)/\mathcal{Q}_3$.

11. Crested product of association schemes

In this section we introduce the *crested product* of two association schemes $\mathcal{Q}_1$ and $\mathcal{Q}_2$, giving a new association scheme on the space $\Omega_1 \times \Omega_2$ that has both crossed and nested products as special cases. Our main source is [**BaCa**].

11.1. Orthogonal block structures. Given a partition F of a finite set Ω, let R_F be the $|\Omega| \times |\Omega|$ relation matrix of F, i.e.

$$R_F(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same part of } F \\ 0 & \text{otherwise.} \end{cases}$$

Definition 11.1. *A partition of* Ω *is* ***uniform*** *if all its parts have the same size.*

The trivial partitions of Ω are the **universal** partition U, which has a single part and whose relation matrix is J_Ω, and the **equality partition** E, all of whose parts are singletons and whose relation matrix is I_Ω.

The partitions of Ω constitute a poset with respect to the relation $\preccurlyeq$, where $F \preccurlyeq G$ if every part of F is contained in a part of G. Given any two partitions F and G, their *infimum* is denoted $F \wedge G$ and is the partition whose parts are intersections of $F-$parts with $G-$parts; their *supremum* is denoted $F \vee G$ and is the partition whose parts are minimal subject to being unions of $F-$parts and $G-$parts.

Definition 11.2. *A set* $\mathcal{F}$ *of uniform partitions of* Ω *is an* ***orthogonal block structure*** *if:*

(1) $\mathcal{F}$ *contains* U *and* E;
(2) *for all* F *and* $G \in \mathcal{F}$, $\mathcal{F}$ *contains* $F \wedge G$ *and* $F \vee G$;

(3) for all F and $G \in \mathcal{F}$, the matrices R_F and R_G commute with each other.

Given a partition F belonging to an orthogonal block structure $\mathcal{F}$ on Ω, we define the adjacency matrix A_F as

$$A_F(\alpha, \beta) = \begin{cases} 1 & \text{if } F = \bigwedge\{G \in \mathcal{F} : R_G(\alpha, \beta) = 1\} \\ 0 & \text{otherwise.} \end{cases}$$

One can verify that the set $\{A_F : F \in \mathcal{F}, A_F \neq 0\}$ is an association scheme on Ω.

Given two partitions F and G of two sets Ω_1 and Ω_2, respectively, denote $F \times G$ the partition of $\Omega_1 \times \Omega_2$ whose relation matrix is $R_F \otimes R_G$.

Now let $\mathcal{F}$ and $\mathcal{G}$ be two orthogonal block structures on Ω_1 and Ω_2, respectively. Then their *crossed product* is given by

$$\mathcal{F} \times \mathcal{G} = \{F \times G : F \in \mathcal{F}, G \in \mathcal{G}\}$$

and their *nested product* is given by

$$\mathcal{F}/\mathcal{G} = \{F \times U_2 : F \in \mathcal{F}\} \cup \{E_1 \times G : G \in \mathcal{G}\},$$

where E_i and U_i are the trivial partitions of Ω_i. One can show that the operation of deriving the association scheme from the orthogonal block structure commutes with both crossing and nesting.

Definition 11.3. *For $i = 1, 2$, let $\mathcal{F}_i$ be an orthogonal block structure on a set Ω_i and choose $F_i \in \mathcal{F}_i$. The **crested product** of $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to F_1 and F_2 is the set $\mathcal{G}$ of partitions of $\Omega_1 \times \Omega_2$ given by*

$$\mathcal{G} = \{G_1 \times G_2 : G_1 \in \mathcal{F}_1, G_2 \in \mathcal{F}_2, G_1 \preccurlyeq F_1 \text{ or } G_2 \succcurlyeq F_2\}. \tag{26}$$

The following theorem holds (see [**BaCa**] for the proof).

Theorem 11.4. *The crested product defined in (26) is an orthogonal block structure on $\Omega_1 \times \Omega_2$.*

Observe that:

- if $F_1 = U_1$ or $F_2 = E_2$, then $\mathcal{G}$ is the crossed product $\mathcal{F}_1 \times \mathcal{F}_2$;
- if $F_1 = E_1$ and $F_2 = U_2$, then $\mathcal{G}$ is the nested product $\mathcal{F}_1/\mathcal{F}_2$.

11.2. Partitions in association schemes.

Definition 11.5. *Let $\mathcal{Q}$ be an association scheme on Ω with adjacency matrices A_i, for $i \in \mathcal{K}$. Then a partition F of Ω is **inherent** in $\mathcal{Q}$ if its relation matrix R_F is in the Bose-Mesner algebra of $\mathcal{Q}$, i.e. if there exists a subset $\mathcal{L}$ of $\mathcal{K}$ such that $R_F = \sum_{i \in \mathcal{L}} A_i$.*

It is easy to check that the trivial partitions E and U are inherent in every association scheme.

Example 11.6.

Consider the 12 edges of the cube and define an association scheme on the set Ω of these edges in the following way:

- two edges α and β are 1−st associates if they meet at a vertex;
- two edges α and β are 2−nd associates if they are diagonally opposite;
- two edges α and β are 3−rd associates if they are parallel but not opposite;
- two edges α and β are 4−th associates if they are skew.

The partitions inherent in this scheme have relation matrices $A_0 = I_\Omega$, $A_0 + A_2$, $A_0 + A_2 + A_3$ and $A_0 + A_1 + A_2 + A_3 + A_4 = J_\Omega$.

Theorem 11.7. *If $\mathcal{Q}$ is an association scheme on Ω, then the set $\mathcal{F}$ of partitions of Ω which are inherent in $\mathcal{Q}$ is an orthogonal block structure on Ω.*

See [**BaCa**] for the proof.

Now let $\mathcal{P}$ be a partition of $\Omega \times \Omega$ and let $V(\mathcal{P})$ be the real span of the adjacency matrices of its classes. It is clear that

$$\mathcal{Q} \preccurlyeq \mathcal{P} \qquad \Longleftrightarrow \qquad V(\mathcal{P}) \leq \mathcal{A},$$

where $\mathcal{A}$ is the Bose-Mesner algebra of $\mathcal{Q}$.

Definition 11.8. *Let $\mathcal{Q}$ be an association scheme on Ω. A partition $\mathcal{P}$ of $\Omega \times \Omega$ is **ideal** for $\mathcal{Q}$ if $V(\mathcal{P})$ is an ideal of $\mathcal{A}$, i.e. $V(\mathcal{P}) \leq \mathcal{A}$ and $AD \in V(\mathcal{P})$ whenever $A \in \mathcal{A}$ and $D \in V(\mathcal{P})$.*

Theorem 11.9. *Let $\mathcal{Q}$ be an association scheme with adjacency matrices A_i, for $i \in \mathcal{K}$. If $\mathcal{Q}$ has an inherent partition F with relation matrix R_F, then there exists an ideal partition $\vartheta(F)$ of $\mathcal{Q}$ whose adjacency matrices are scalar multiples of A_iR, for $i \in \mathcal{K}$.*

Proof. (Sketch) Let $\mathcal{L}$ be the subset of $\mathcal{K}$ such that $R_F = \sum_{i\in\mathcal{L}} A_i$. So there exist positive integers m_{ij} such that

$$R_F A_i = A_i R_F = \sum_{j\in\mathcal{K}} m_{ij} A_j.$$

It follows from the definition that

$$m_{ij} = (A_i R_F)(\alpha, \beta) = |\mathcal{C}_i(\alpha) \cap F(\beta)|,$$

where $F(\beta)$ denotes the $F-$class containing β. Put $i \sim j$ if $m_{ij} \neq 0$. One can check that $\sim$ is an equivalence relation. Define $[i] = \{j \in \mathcal{K} : j \sim i\}$ and $B_{[i]} = \sum_{j\sim i} A_j$. Then the distinct $B_{[i]}$ are the adjacency matrices of a partition $\mathcal{P}$ of $\Omega \times \Omega$ such that $\mathcal{Q} \preccurlyeq \mathcal{P}$. Moreover, it is easy to verify that $A_j B_{[i]} \in V(\mathcal{P})$. $\square$

Indeed, the inverse construction can be done, as the following theorem shows (see [**BaCa**]).

THEOREM 11.10. *Let $\mathcal{P}$ be an ideal partition for $\mathcal{Q}$. Let A_i be the adjacency matrices of $\mathcal{Q}$, for $i \in \mathcal{K}$, and let D_m be the adjacency matrices of $\mathcal{P}$, for $m \in \mathcal{M}$. Denote by σ the surjection from $\mathcal{K}$ to $\mathcal{M}$ such that class i of $\mathcal{Q}$ is contained in class $\sigma(i)$ of $\mathcal{P}$. Put $R = D_{\sigma(0)}$. Then R is the relation matrix of an inherent partition in $\mathcal{Q}$. Moreover, for all $i \in \mathcal{K}$, the matrix A_iR is an integer multiple of $D_{\sigma(i)}$.*

11.3. Crested product of association schemes. Let F be a partition in an orthogonal block structure $\mathcal{F}$, so that $R_F = \sum_{G\in\mathcal{L}} A_G$, where $\mathcal{L} = \{G \in \mathcal{F} : G \preccurlyeq F\}$. This implies that F is inherent in the association scheme derived from $\mathcal{F}$. Then $\{A_G : G \in \mathcal{L}\}$ and $\{R_G : G \in \mathcal{L}\}$ span the same subspace $\mathcal{A}|_F$ of $\mathcal{A}$, which is closed under matrix multiplication.

Let $\mathcal{P}$ be the ideal partition $\vartheta(F)$. For $G \in \mathcal{F}$, R_G is in the ideal of $\mathcal{A}$ generated by R_F if and only if $F \preccurlyeq G$, so $V(\mathcal{P})$ is the span of $\{R_G : G \in \mathcal{F},\ G \succcurlyeq F\}$. We denote $V(\vartheta(F))$ by $A|^F$.

Consider now the crested product $\mathcal{G}$ of the orthogonal block structures $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to the partitions F_1 and F_2. The span of the relation matrices of the partitions in $\mathcal{G}$ is

$$(\mathcal{A}_1|_{F_1} \otimes \mathcal{A}_2) + (\mathcal{A}_1 \otimes \mathcal{A}_2|^{F_2}),$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are the Bose-Mesner algebra of the association schemes derived by $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. The adjacency matrices of the association scheme derived by $\mathcal{G}$ are:

- $A_G \otimes A_H$, for $G \in \mathcal{L}$ and $H \in \mathcal{F}_2$;
- $A_G \otimes D$, for $G \in \mathcal{F}_1 \setminus \mathcal{L}$ and D an adjacency matrix of $\mathcal{P}$,

where $\mathcal{L} = \{G \in \mathcal{F}_1 : G \preccurlyeq F_1\}$ and $\mathcal{P} = \vartheta(F_2)$. This leads to the following definition.

DEFINITION 11.11. *For $r = 1, 2$, let $\mathcal{Q}_r$ be an association scheme on a set Ω_r and let F_r be an inherent partition in $\mathcal{Q}_r$. Put $\mathcal{P} = \vartheta(F_2)$ and $\Omega = \Omega_1 \times \Omega_2$. Let the adjacency matrices of $\mathcal{Q}_1, \mathcal{Q}_2$ and $\mathcal{P}$ be A_i, for $i \in \mathcal{K}_1$, B_j, for $j \in \mathcal{K}_2$ and D_m, for $m \in \mathcal{M}$, respectively. Let $\mathcal{L}$ be the subset of $\mathcal{K}_1$ such that $R_{F_1} = \sum_{i\in\mathcal{L}} A_i$. The **crested product** of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with respect to F_1 and F_2 is the association scheme $\mathcal{Q}$ on Ω whose adjacency matrices are*

- *$A_i \otimes B_j$, for $i \in \mathcal{L}$ and $j \in \mathcal{K}_2$;*
- *$A_i \otimes D_m$, for $i \in \mathcal{K}_1 \setminus \mathcal{L}$ and $m \in \mathcal{M}$.*

Observe that the crested product reduces to the crossed product if $F_1 = U_1$ or $F_2 = E_2$ (in which case $\mathcal{P} = \mathcal{Q}_2$) and it reduces to the nested product if $F_1 = E_1$ and $F_2 = U_2$ (in which case $\mathcal{P} = U_{\Omega_2\times\Omega_2}$).

Moreover, the interesting fact is that the character table of the crested product $\mathcal{Q}$ can be described using the character table of $\mathcal{Q}_1$ and $\mathcal{Q}_2$. See [**BaCa**] for more details.

12. Examples

Let Q be an association scheme on a finite set Ω and let $A_0 = I_\Omega, A_1, \ldots, A_m$ the adjacency matrices associated with Q. Consider also an association scheme Q' on a second finite set Ω', whose adjacency matrices are $A'_0 = I_{\Omega'}, A'_1, \ldots, A'_m$.

The *nested* product Q/Q' of the schemes Q and Q' is well defined: it is the association scheme on the set $\Omega \times \Omega'$ whose adjacency matrices are

- $A_i \otimes J_{\Omega'}$, for $i \neq 0$;
- $I_\Omega \otimes A'_j$, for $j = 0, 1, \ldots, m'$.

Consider now the inherent partition F of $\Omega \times \Omega'$ whose relation matrix is

$$R_F = \sum_{j=0}^{m'} (I_\Omega \otimes A'_j) = I_\Omega \otimes J_{\Omega'},$$

i.e. the partition $\Omega \times \Omega' = \bigsqcup_{\alpha \in \Omega} \{(\alpha, \alpha') : \alpha' \in \Omega'\}$. We can ask which is the ideal partition associated with F.

In general, if Q is an association scheme on the set X with matrices $A_0, A_1, \ldots, A_m$ and F is an inherent partition of X with relation matrix $R_F = \sum_{i \in \mathcal{L}} A_i$, then the adjacency matrices of the ideal partition $\mathcal{P}$ of $X \times X$ associated with F are $D_i = \sum_{i \sim j} A_j$, where $\sim$ is the equivalence relation defined by $i \sim j$ if $m_{ij} \neq 0$ and the m_{ij}'s are defined by

$$m_{ij} = |\mathcal{C}_i(\alpha) \cap F(\beta)|, \quad \text{for all } (\alpha, \beta) \in \mathcal{C}_j.$$

So, if $m_{ij} \neq 0$ and $(\alpha, \beta) \in \mathcal{C}_j$, then there exists some $\gamma \in \mathcal{C}_i(\alpha)$ such that $F(\beta) = F(\gamma)$. One can easily check that $[0] = \mathcal{L}$. We will use also the notation $A_i \sim A_j$ to indicate $i \sim j$.

In our case we have $I_\Omega \otimes A'_j \sim I_\Omega \otimes A'_k$ for every $j, k = 0, 1, \ldots, m'$. Moreover, it is easy to verify that, for $i, j \neq 0$, one has $A_i \otimes J_{\Omega'} \not\sim A_j \otimes J_{\Omega'}$ for $i \neq j$. So the adjacency matrices of the ideal partition $\mathcal{P}$ associated with F are

$$A_i \otimes J_{\Omega'}, \qquad \text{for } i = 0, 1, \ldots, m.$$

Consider now an association scheme S on a finite set Θ with adjacency matrices $B_0 = I_\Theta, B_1, \ldots, B_n$ and an association scheme S' on a finite set Θ' whose adjacency matrices are $B'_0 = I_{\Theta'}, B'_1, \ldots, B'_n$. Take now the nested product S/S' defined on the product $\Theta \times \Theta'$, i.e. the association scheme on $\Theta \times \Theta'$ whose adjacency matrices are

- $B_i \otimes J_{\Theta'}$, for $i \neq 0$;
- $I_\Theta \otimes B'_j$, for $j = 0, 1, \ldots, n'$.

We can consider the inherent partition G of $\Theta \times \Theta'$ defined as in the previous case, so that its relation matrix is

$$R_G = \sum_{j=0}^{n'} I_\Theta \otimes B'_j = I_\Theta \otimes J_{\Theta'},$$

i.e. we have the partition $\Theta \times \Theta' = \coprod_{\theta \in \Theta} \{(\theta, \theta') : \theta' \in \Theta'\}$.

We can now consider the *crested* product of the schemes S/S' and Q/Q' with respect to the inherent partition G and F defined above. So we get a new association scheme on the set

$$\Theta \times \Theta' \times \Omega \times \Omega'$$

whose adjacency matrices are

- $(I_\Theta \otimes B'_j) \otimes (A_i \otimes J_{\Omega'})$, with $j = 0, 1, \ldots, n'$ and $i \neq 0$;
- $(I_\Theta \otimes B'_j) \otimes (I_\Omega \otimes A'_k)$, with $j = 0, 1, \ldots, n'$ and $k = 0, 1, \ldots, m'$;
- $(B_i \otimes J_{\Theta'}) \otimes (A_j \otimes J_{\Omega'})$, with $i \neq 0$ and $j = 0, 1, \ldots, m$.

Moreover, by choosing the inherent partition G for $\Theta \times \Theta'$ and the universal partition $U_{\Omega \times \Omega'}$ for $\Omega \times \Omega'$, i.e. the partition whose relation matrix is $R_{U_{\Omega \times \Omega'}} = J_\Omega \otimes J_{\Omega'}$, we can get a different crested product of the schemes S/S' and Q/Q'. Observe that the only adjacency matrix of the ideal partition $\mathcal{P}$ associated with $U_{\Omega \times \Omega'}$ is $J_\Omega \otimes J_{\Omega'}$. So the adjacency matrices of the crested product of the schemes S/S' and Q/Q' are

- $(I_\Theta \otimes B'_j) \otimes (A_i \otimes J_{\Omega'})$, with $j = 0, 1, \ldots, n'$ and $i \neq 0$;
- $(I_\Theta \otimes B'_j) \otimes (I_\Omega \otimes A'_k)$, with $j = 0, 1, \ldots, n'$ and $k = 0, 1, \ldots, m'$;
- $(B_i \otimes J_{\Theta'}) \otimes (J_\Omega \otimes J_{\Omega'})$, with $i \neq 0$.

Finally, by choosing the identity partition $E_{\Theta \times \Theta'}$ for $\Theta \times \Theta'$ and the inherent partition F for $\Omega \times \Omega'$, we can get again a different crested product of the schemes S/S' and Q/Q', whose adjacency matrices are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_i \otimes J_{\Omega'})$, with $i \neq 0$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_k)$, with $k = 0, 1, \ldots, m'$;
- $(I_\Theta \otimes B'_k) \otimes (A_i \otimes J_{\Omega'})$, with $i = 0, 1, \ldots, m$ and $k \neq 0$;
- $(B_j \otimes J_{\Theta'}) \otimes (A_i \otimes J_{\Omega'})$, with $j \neq 0$ and $i = 0, 1, \ldots, m$.

This completes the description of the nontrivial crested products that we can get from the schemes S/S' and Q/Q'. By choosing the identity partition $E_{\Theta \times \Theta'}$ as inherent partition of $\Theta \times \Theta'$ and the universal partition $U_{\Omega \times \Omega'}$ as inherent partition of $\Omega \times \Omega'$, we get the nested product

$$S/S'/Q/Q'.$$

This notation is correct because of the associativity of iterating the nested product of association schemes. The adjacency matrices of the scheme $S/S'/Q/Q'$ are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_i \otimes J_{\Omega'})$, with $i \neq 0$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_k)$, with $k = 0, 1, \ldots, m'$;
- $(I_\Theta \otimes B'_k) \otimes (J_\Omega \otimes J_{\Omega'})$, with $k \neq 0$;

- $(B_j \otimes J_{\Theta'}) \otimes (J_\Omega \otimes J_{\Omega'})$, with $j \neq 0$.

The remaining choices for the inherent partitions of $\Theta \times \Theta'$ and $\Omega \times \Omega'$ give rise to the crossed product

$$(S/S') \times (Q/Q'),$$

i.e. the association scheme on $\Theta \times \Theta' \times \Omega \times \Omega'$ whose adjacency matrices are

- $(I_\Theta \otimes B'_j) \otimes (I_\Omega \otimes A'_k)$, with $j = 0, 1, \ldots, n'$ and $k = 0, 1, \ldots, m'$;
- $(I_\Theta \otimes B'_j) \otimes (A_i \otimes J_{\Omega'})$, with $j = 0, 1, \ldots, n'$ and $i \neq 0$;
- $(B_i \otimes J_{\Theta'}) \otimes (I_\Omega \otimes A'_k)$, with $i \neq 0$ and $k = 0, 1, \ldots, m'$;
- $(B_i \otimes J_{\Theta'}) \otimes (A_k \otimes J_{\Omega'})$, with $i, k \neq 0$.

As an easy example, we can consider the case when $\Theta = \Theta' = \Omega = \Omega' = \{1, 2\}$ and $S = S' = Q = Q' = \underline{\underline{2}}$. We recall that $\underline{\underline{2}}$ denotes the trivial association scheme on two elements, whose adjacency matrices are

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let us call these matrices B_0 and B_1 in the case of S, B'_0 and B'_1 in the case of S', A_0 and A_1 in the case of Q, A'_0 and A'_1 in the case of Q', respectively.

So the adjacency matrices of the nested product Q/Q' are

- $A_1 \otimes J_{\Omega'}$;
- $I_\Omega \otimes I_{\Omega'}$;
- $I_\Omega \otimes A'_1$.

Consider now the inherent partition F of $\Omega \times \Omega'$ whose relation matrix is

$$R_F = I_\Omega \otimes I_{\Omega'} + I_\Omega \otimes A'_1 = I_\Omega \otimes J_{\Omega'},$$

i.e. the partition $\Omega \times \Omega' = \{(1,1), (1,2)\} \coprod \{(2,1), (2,2)\}$.

The adjacency matrices of the ideal partition $\mathcal{P}$ associated with F are

- $I_\Omega \otimes J_{\Omega'}$;
- $A_1 \otimes J_{\Omega'}$.

Analogously, the adjacency matrices associated with the nested product S/S' defined on the product $\Theta \times \Theta'$ are

- $B_1 \otimes J_{\Theta'}$;
- $I_\Theta \otimes I_{\Theta'}$;
- $I_\Theta \otimes B'_1$.

We can consider the inherent partition G of $\Theta \times \Theta'$ defined as in the previous case, so that its relation matrix is

$$R_G = I_\Theta \otimes I_{\Theta'} + I_\Theta \otimes B'_1 = I_\Theta \otimes J_{\Theta'},$$

corresponding to the partition $\Theta \times \Theta' = \{(1,1), (1,2)\} \coprod \{(2,1), (2,2)\}$.

We can now consider the crested product of the schemes $\underline{\underline{2}}/\underline{\underline{2}}$ and $\underline{\underline{2}}/\underline{\underline{2}}$ with respect to the inherent partition G and F defined above. So we get the association scheme on the set

$$\Theta \times \Theta' \times \Omega \times \Omega'$$

whose adjacency matrices are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_1)$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes A'_1)$;
- $(B_1 \otimes J_{\Theta'}) \otimes (I_\Omega \otimes J_{\Omega'})$;
- $(B_1 \otimes J_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$.

By choosing the inherent partition G for $\Theta \times \Theta'$ and the universal partition $U_{\Omega\times\Omega'}$ for $\Omega \times \Omega'$, i.e. the partition whose relation matrix is $R_{U_{\Omega\times\Omega'}} = J_\Omega \otimes J_{\Omega'}$, we get a different crested product of the schemes $\underline{\underline{2}}/\underline{\underline{2}}$ and $\underline{\underline{2}}/\underline{\underline{2}}$. The only adjacency matrix of the ideal partition $\mathcal{P}$ associated with $U_{\Omega\times\Omega'}$ is $J_\Omega \otimes J_{\Omega'}$. So the adjacency matrices of the crested product of the schemes $\underline{\underline{2}}/\underline{\underline{2}}$ and $\underline{\underline{2}}/\underline{\underline{2}}$ are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_1)$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes A'_1)$;
- $(B_1 \otimes J_{\Theta'}) \otimes (J_\Omega \otimes J_{\Omega'})$.

Finally, by choosing the identity partition $E_{\Theta\times\Theta'}$ for $\Theta \times \Theta'$ and the inherent partition F for $\Omega \times \Omega'$, we get again a different crested product of the schemes $\underline{\underline{2}}/\underline{\underline{2}}$ and $\underline{\underline{2}}/\underline{\underline{2}}$, whose adjacency matrices are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_1)$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes J_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (A_1 \otimes J_{\Omega'})$;
- $(B_1 \otimes J_{\Theta'}) \otimes (I_\Omega \otimes J_{\Omega'})$;
- $(B_1 \otimes J_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$.

This completes the description of the nontrivial crested products that we can get from the schemes $\underline{\underline{2}}/\underline{\underline{2}}$ and $\underline{\underline{2}}/\underline{\underline{2}}$. By choosing the identity partition $E_{\Theta\times\Theta'}$ as inherent partition of $\Theta \times \Theta'$ and the universal partition $U_{\Omega\times\Omega'}$ as inherent partition of $\Omega \times \Omega'$, we get the nested product

$$\underline{\underline{2}}/\underline{\underline{2}}/\underline{\underline{2}}/\underline{\underline{2}}.$$

The adjacency matrices of this scheme are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_1)$;
- $(I_\Theta \otimes B'_1) \otimes (J_\Omega \otimes J_{\Omega'})$;
- $(B_1 \otimes J_{\Theta'}) \otimes (J_\Omega \otimes J_{\Omega'})$.

The remaining choices of inherent partitions of $\Theta \times \Theta'$ and $\Omega \times \Omega'$ give rise to the crossed product

$$(\underline{\underline{2}}/\underline{\underline{2}}) \times (\underline{\underline{2}}/\underline{\underline{2}}),$$

whose adjacency matrices are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_1)$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes A'_1)$;
- $(I_\Theta \otimes B'_1) \otimes (A_1 \otimes J_{\Omega'})$;
- $(B_1 \otimes J_{\Theta'}) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(B_1 \otimes J_{\Theta'}) \otimes (I_\Omega \otimes A'_1)$;
- $(B_1 \otimes J_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$.

These products have also another interpretation from the orthogonal block structures point of view.

Remark 12.1.

A ultrametric space has in a natural way an orthogonal block structure: if we fix a level of the tree, this level induces a partition in spheres. Considering this partition in spheres for each level, we get an orthogonal block structure.

Take now two rooted trees of depth 2 with branch indices (m, n) and (p, q), respectively. Consider the corresponding orthogonal block structures: each block consists of three partitions with sizes $1, n, mn$ and $1, q, pq$, respectively. We denote these partitions by F_0, F_1, F_2 for the first tree and by G_0, G_1, G_2 for the second tree. So the relation matrices in the case of the first tree are

- $R_0 = I_m \otimes I_n$;
- $R_1 = I_m \otimes J_n$;
- $R_2 = J_m \otimes J_n$

and in the case of the second tree are

- $S_0 = I_p \otimes I_q$;
- $S_1 = I_p \otimes J_q$;
- $S_2 = J_p \otimes J_q$.

The corresponding association schemes that we can get considering the matrices A_F defined above are Q, with adjacency matrices

- $A_0 = I_m \otimes I_n$;
- $A_1 = I_m \otimes (J_n - I_n)$;
- $A_2 = (J_m - I_m) \otimes J_n$

and Q', with adjacency matrices

- $A'_0 = I_p \otimes I_q$;
- $A'_1 = I_p \otimes (J_q - I_q)$;
- $A'_2 = (J_p - I_p) \otimes J_q$.

So we can observe that the association scheme Q is just the scheme $\underline{\underline{m}}/\underline{\underline{n}}$ and the association scheme Q' is just the scheme $\underline{\underline{p}}/\underline{\underline{q}}$. We can do the crested product of these schemes with respect to the possible inherent partitions, whose relation matrices are R_0 or S_0 in the case of the equality partition, then R_1 or S_1 and finally R_2 or S_2 in the case of the universal partition.

We can also do the crested product of orthogonal block structures and then we can associate to the block obtained a new association scheme by using the matrices A_F. Actually, we can show that the operation of deriving the association scheme from the orthogonal block structure commutes with cresting. Let us verify it in all cases.

The relation matrices of the block obtained by the crest product with respect to the partition F_1 and G_1 are

- $R_0 \otimes S_0$, with associated adjacency matrix $A_{0,0} = I_m \otimes I_n \otimes I_p \otimes I_q$;
- $R_0 \otimes S_1$, with $A_{0,1} = I_m \otimes I_n \otimes I_p \otimes (J_q - I_q)$;
- $R_0 \otimes S_2$, with $A_{0,2} = I_m \otimes I_n \otimes (J_p - I_p) \otimes J_q$;
- $R_1 \otimes S_0$, with $A_{1,0} = I_m \otimes (J_n - I_n) \otimes I_p \otimes I_q$;
- $R_1 \otimes S_1$, with $A_{1,1} = I_m \otimes (J_n - I_n) \otimes I_p \otimes (J_q - I_q)$;
- $R_1 \otimes S_2$, with $A_{1,2} = I_m \otimes (J_n - I_n) \otimes (J_p - I_p) \otimes J_q$;
- $R_2 \otimes S_1$, with $A_{2,1} = (J_m - I_m) \otimes J_n \otimes I_p \otimes J_q$;
- $R_2 \otimes S_2$, with $A_{2,2} = (J_m - I_m) \otimes J_n \otimes (J_p - I_p) \otimes J_q$

and these matrices $A_{i,j}$'s are just the adjacency matrices of the association scheme obtained by the crested product of the association schemes Q and Q' by choosing the partitions F_1 and G_1 as inherent partitions, respectively.

The relation matrices of the block obtained with the crest product with respect to the partition F_1 and G_2 are

- $R_0 \otimes S_0$, with associated adjacency matrix $A_{0,0} = I_m \otimes I_n \otimes I_p \otimes I_q$;
- $R_0 \otimes S_1$, with $A_{0,1} = I_m \otimes I_n \otimes I_p \otimes (J_q - I_q)$;
- $R_0 \otimes S_2$, with $A_{0,2} = I_m \otimes I_n \otimes (J_p - I_p) \otimes J_q$;
- $R_1 \otimes S_0$, with $A_{1,0} = I_m \otimes (J_n - I_n) \otimes I_p \otimes I_q$;
- $R_1 \otimes S_1$, with $A_{1,1} = I_m \otimes (J_n - I_n) \otimes I_p \otimes (J_q - I_q)$;
- $R_1 \otimes S_2$, with $A_{1,2} = I_m \otimes (J_n - I_n) \otimes (J_p - I_p) \otimes J_q$;
- $R_2 \otimes S_2$, with $A_{2,2} = (J_m - I_m) \otimes J_n \otimes J_p \otimes J_q$

and these matrices $A_{i,j}$'s are just the adjacency matrices of the association scheme obtained by the crested product of the association schemes Q and Q' by choosing the partitions F_1 and G_2 as inherent partitions, respectively.

The relation matrices of the block obtained with the crest product with respect to the partition F_0 and G_1 are

- $R_0 \otimes S_0$, with associated adjacency matrix $A_{0,0} = I_m \otimes I_n \otimes I_p \otimes I_q$;
- $R_0 \otimes S_1$, with $A_{0,1} = I_m \otimes I_n \otimes I_p \otimes (J_q - I_q)$;
- $R_0 \otimes S_2$, with $A_{0,2} = I_m \otimes I_n \otimes (J_p - I_p) \otimes J_q$;
- $R_1 \otimes S_1$, with $A_{1,1} = I_m \otimes (J_n - I_n) \otimes I_p \otimes J_q$;
- $R_2 \otimes S_1$, with $A_{2,1} = (J_m - I_m) \otimes J_n \otimes I_p \otimes J_q$
- $R_1 \otimes S_2$, with $A_{1,2} = I_m \otimes (J_n - I_n) \otimes (J_p - I_p) \otimes J_q$;
- $R_2 \otimes S_2$, with $A_{2,2} = (J_m - I_m) \otimes J_n \otimes (J_p - I_p) \otimes J_q$

and these matrices $A_{i,j}$'s are just the adjacency matrices of the association scheme obtained by the crested product of the association schemes Q and Q' by choosing the partitions F_0 and G_1 as inherent partitions, respectively.

The same result can be obtained by considering the crossed product and the nested product.

In fact, the relation matrices of the block obtained with the crest product with respect to the partition F_0 and G_2 are

- $R_0 \otimes S_0$, with associated adjacency matrix $A_{0,0} = I_m \otimes I_n \otimes I_p \otimes I_q$;
- $R_0 \otimes S_1$, with $A_{0,1} = I_m \otimes I_n \otimes I_p \otimes (J_q - I_q)$;
- $R_0 \otimes S_2$, with $A_{0,2} = I_m \otimes I_n \otimes (J_p - I_p) \otimes J_q$;
- $R_1 \otimes S_2$, with $A_{1,2} = I_m \otimes (J_n - I_n) \otimes J_p \otimes J_q$;
- $R_2 \otimes S_2$, with $A_{2,2} = (J_m - I_m) \otimes J_n \otimes J_p \otimes J_q$

and these matrices $A_{i,j}$'s are just the adjacency matrices of the association scheme obtained by the crested product of the association schemes Q and Q' by choosing the partitions F_0 and G_2 as inherent partitions, respectively. The remaining choices for the partitions give rise to the crossed product. The relation matrices of the block obtained with the crossed product are

- $R_0 \otimes S_0$, with associated adjacency matrix $A_{0,0} = I_m \otimes I_n \otimes I_p \otimes I_q$;
- $R_0 \otimes S_1$, with $A_{0,1} = I_m \otimes I_n \otimes I_p \otimes (J_q - I_q)$;
- $R_0 \otimes S_2$, with $A_{0,2} = I_m \otimes I_n \otimes (J_p - I_p) \otimes J_q$;
- $R_1 \otimes S_0$, with $A_{1,0} = I_m \otimes (J_n - I_n) \otimes I_p \otimes I_q$;
- $R_1 \otimes S_1$, with $A_{1,1} = I_m \otimes (J_n - I_n) \otimes I_p \otimes (J_q - I_q)$;
- $R_1 \otimes S_2$, with $A_{1,2} = I_m \otimes (J_n - I_n) \otimes (J_p - I_p) \otimes J_q$;
- $R_2 \otimes S_0$, with $A_{2,0} = (J_m - I_m) \otimes J_n \otimes I_p \otimes I_q$;
- $R_2 \otimes S_1$, with $A_{2,1} = (J_m - I_m) \otimes J_n \otimes I_p \otimes (J_q - I_q)$;
- $R_2 \otimes S_2$, with $A_{2,2} = (J_m - I_m) \otimes J_n \otimes (J_p - I_p) \otimes J_q$.

The interesting fact is that the nested product of the two original blocks gives an orthogonal block structure on a set with $mnpq$ elements, which is exactly the block of spherical partitions of the fourth level of the rooted tree of depth 4 and branch indices (m, n, p, q). The remaining crested product give other orthogonal block structures corresponding to different partitions which are not induced by the spheres of the trees.

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