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Multi-scale analysis via Γ-convergence

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Multi-scale analysis via Γ-convergence

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Introduction

In modelling a large variety of physical phenomena it happens to deal with families of variational problems involving small parameters. The notion of Γ-convergence [**29, 31, 14**] is very well suited to the variational setting and, starting by those microscopic models, is widely used to derive limiting "macro" theories not depending on any small parameter. Specifically, this notion is nearly identical to that of convergence of minimum problems. More precisely, if $\varepsilon > 0$ and (F_ε) is a given family of microscopic energies, from

$$F_\varepsilon \xrightarrow{\Gamma} F^{(0)}$$

we deduce that

(i) $m_\varepsilon := \min F_\varepsilon \longrightarrow m^{(0)} := \min F^{(0)}$ as $\varepsilon \to 0$.

Not only: if the Γ-convergence is coupled with some equi-coerciviness condition on the family (F_ε) then

(ii) if for any fixed $\varepsilon > 0$, v_ε minimizes F_ε; *i.e.,* $F_\varepsilon(v_\varepsilon) = m_\varepsilon$ then, up to an extraction, $v_\varepsilon \to v$ as $\varepsilon \to 0$ and $F^{(0)}(v) = m_0$.

The (ii) property can be sketched as

$$\{\text{limits of minimizers}\} \subseteq \operatorname{argmin}(F^{(0)}), \tag{0.1}$$

where $\operatorname{argmin}(F^{(0)}) := \{u : F^{(0)}(u) = m^{(0)}\}$ and the inclusion may well be proper, as it can be seen by very simple and natural examples. Hence, in general, the description given by $F^{(0)}$ can be too coarse and the (zero order) Γ-limit may fail to *completely* characterize the asymptotic behavior of the family (F_ε). Then, the idea is that the computation of the Γ-limit $F^{(0)}$ is only the first step in the description of the asymptotic behavior of (F_ε), as it can be necessary to refine the above limit procedure to select those minimizers of $F^{(0)}$ which are actually limits of sequences (v_ε).

The most intuitive refinement procedure of the standard Γ-convergence is the iteration of the successive Γ-limits [**9**]. Indeed, once the next meaningful scale $\lambda^{(1)}(\varepsilon)$ ($\lambda^{(1)}(\varepsilon) > 0$, $\lambda^{(1)}(\varepsilon) \to 0$ as $\varepsilon \to 0$) is conjectured, we may look at the Γ-limit of the scaled family of energies

$$F^{(1)}_\varepsilon(u) := \frac{F_\varepsilon(u) - m^{(0)}}{\lambda^{(1)}(\varepsilon)},$$

and if it exists, we denote it with $F^{(1)}$. Notice that the domain of $F^{(1)}$ is by definition a subset of the set of minimum points of $F^{(0)}$; *i.e.,*

$$\operatorname{dom}(F^{(1)}) \subseteq \operatorname{argmin}(F^{(0)}).$$

If $F^{(1)}$ is not trivial, then the iterated application of (i) leads to a better development of the minimum values

$$m_\varepsilon = m^{(0)} + \lambda^{(1)}(\varepsilon)m^{(1)} + o(\lambda^{(1)}(\varepsilon)), \quad \text{as } \varepsilon \to 0$$

with $m^{(1)} := \min F^{(1)}$.

It is also clear that the minimizers for $F^{(1)}_\varepsilon$ are exactly those for F_ε, then in view of (ii) we deduce that v not only minimizes $F^{(0)}$ but also $F^{(1)}$. Loosely speaking, we have

$$\{\text{limits of minimizers}\} \subseteq \operatorname{argmin}(F^{(1)}) \subseteq \operatorname{argmin}(F^{(0)}),$$

thus we have actually made a selection among minimum points of $F^{(0)}$.

Finally, the combined computation of the zero and of the first order Γ-limit as above is *formally* written as the Γ-*development*

$$F_\varepsilon = F^{(0)} + \lambda^{(1)}(\varepsilon)F^{(1)} + o(\lambda^{(1)}(\varepsilon)),$$

with $o(\lambda^{(1)}(\varepsilon))$ meaning that the next interesting scale is of order less than $\lambda^{(1)}(\varepsilon)$, as $\varepsilon \to 0$.

If necessary, this procedure can be iterated obtaining other scales $\lambda^{(2)}(\varepsilon), \lambda^{(3)}(\varepsilon), \dots$ and consequently other terms in the development. This may provide a considerable improvement of (0.1) and, in some cases, may give a complete characterization of the asymptotic behavior of (F_ε).

Notice that moreover, since in the applications one would like to construct theories operative at small but finite ε, a development by Γ-convergence can be also viewed as the simplest way to bring a small scale back into the problem.

A well-know example of a Γ-development is that of the *gradient theory of phase transition* [**40, 39**].

Consider the family of minimum problems

$$m_\varepsilon := \min\left\{F_\varepsilon(u) : \ u \in W^{1,2}(0,1), \int_0^1 u\,dx = d\right\}, \quad F_\varepsilon(u) := \int_0^1 \left(W(u) + \varepsilon^2 (u')^2\right)\,dx,$$

with $|d| < 1$ and W a double-well potential with wells at ± 1 (*e.g.*, $W(u) = \min\{(u-1)^2, (u+1)^2\}$). Then the Γ-limit of (F_ε) computed with respect to the weak L^2-convergence is simply

$$F^{(0)}(u) = \begin{cases} \displaystyle\int_0^1 W^{**}(u)\,dx & \text{if } u \in L^2(0,1) \text{ and } \int_0^1 u\,dx = d \\ +\infty & \text{otherwise,} \end{cases}$$

where W^{**} is the convex envelope of W.

By Jensen's Inequality $\min F^{(0)} = W^{**}(d)$, moreover $W^{**}(s) = 0 = W^{**}(d)$ for every s such that $|s| \le 1$. Then the zero order Γ-limit only provides the information that sequences of minimizers (v_ε) may develop oscillations and their weak limit can be any function $v \in L^2(0,1)$ such that $|v| \le 1$ a.e. and satisfying the volume constraint $\int_0^1 v\,dx = d$.

A simple scaling argument (see [**3, 14**]) shows that the next meaningful scale is $\lambda^{(1)}(\varepsilon) = \varepsilon$. The first-order Γ-limit is given by

$$F^{(1)}(u) = \begin{cases} C_W \# S(u) & \text{if } u \in BV((0,1);\{\pm 1\}) \text{ and } \int_0^1 u\,dx = d \\ +\infty & \text{otherwise,} \end{cases}$$

where $S(u)$ denotes the set of discontinuity points of u and $C_W := 2\int_{-1}^{1}\sqrt{W(s)}$ (Modica-Mortola's Theorem).

Now, the minimizers of $F^{(1)}$ are only the two functions $\pm\mathrm{sign}(x - \frac{1-d}{2})$ and we deduce the convergence of (v_ε) to one of this two functions. In this case, the Modica-Mortola Theorem also improves the convergence to strong L^2-convergence.

As the development of minimum values is concerned, we get

$$m_\varepsilon = \varepsilon\, C_W + o(\varepsilon), \qquad \text{as } \varepsilon \to 0.$$

On this example, it is also possible to compute the next meaningful scaling that is $\lambda^{(2)}(\varepsilon) = \varepsilon\, e^{-1/2\varepsilon}$ and thus we may further write

$$m_\varepsilon = \varepsilon\, C_W + \varepsilon\, e^{-1/2\varepsilon}\widetilde{C}_W + o(\varepsilon\, e^{-1/2\varepsilon}), \qquad \text{as } \varepsilon \to 0.$$

However, the minimizers being essentially uniquely characterized by the analysis at order ε, this last information only provides a better approximation of the minimum values m_ε.

In a general framework one does not encounter problems containing a single parameter but rather energies depending on different small parameters. In fact a physical model with a variational structure may well contain, for instance, small parameters of various nature (*e.g.,* constitutive, geometrical).

In this first chapter of this dissertation we investigate the combined effect of small-scale heterogeneities (fine microstructures) and singular gradient perturbations on the asymptotic development described above. Specifically, we focus on a *prototype* that is a special, one-dimensional variant of Modica-Mortola (or van der Waals-Cahn-Hillard) energy as we are mainly interested in a careful description of the different meaningful scales involved in the Γ-development.

The model we analyze is the following: let k be a real number such that $0 < k < 1$; for all $\varepsilon, \delta > 0$ consider the functional $F^{k(0)}_{\varepsilon,\delta} : L^2(0,1) \longrightarrow (0,+\infty]$ defined by

$$F^{k(0)}_{\varepsilon,\delta}(u) = \begin{cases} \displaystyle\int_0^1 \left(W^k\left(\frac{x}{\delta}, u\right) + \varepsilon^2 (u')^2\right) dx & \text{if } u \in W^{1,2}(0,1) \\ +\infty & \text{otherwise,} \end{cases} \tag{0.2}$$

where $W^k : \mathbb{R}\times\mathbb{R} \to [0,+\infty)$ is such that

$$W^k(y,s) := W(s - k u_0(y)) \quad \text{with} \quad u_0(y) := \frac{\sin 2\pi y}{|\sin 2\pi y|}$$

and W is the double-well potential given by

$$W(t) = \min\{(t-1)^2, (t+1)^2\}.$$

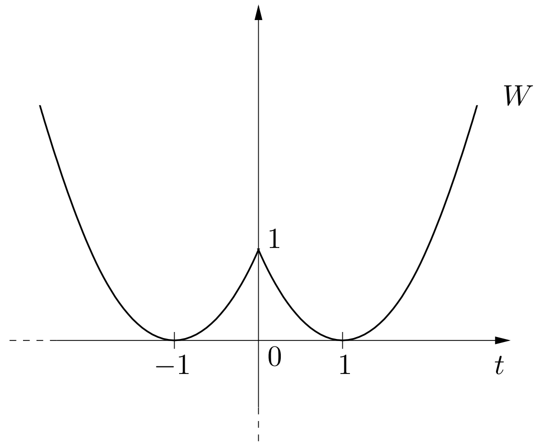

FIGURE 1. The double-well potential W.

Then we may interpret this situation as modelling the presence of spatial heterogeneities at a scale δ, which locally determine the zero set of the potential W^k. Moreover, a simple dimensional analysis shows that the pre-factor ε^2 multiplying the gradient term, introduces ε as a length scale to the problem. Finally the (fixed) parameter k, which will play an essential role in the creation of the scales occurring in the development, simply gives the width of the translation of the potential W^k with respect to W, on each period. Notice that in particular for $k = 0$, $W^k \equiv W$ and (0.2) reduces to

$$F_\varepsilon(u) = \int_0^1 (W(u) + \varepsilon^2 (u')^2)\, dx.$$

For the vectorial analogous of the investigated problem, we refer the reader to [**34**] where, among other, a complete and very general analysis of the zero order Γ-limit is given.

A similar, though in some aspects more complex, model was recently proposed in [**32**]. The authors consider a perturbation of Modica-Mortola energy by a rapidly oscillating field with zero average. More precisely they consider the functionals

$$\int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon^\gamma}\, g\Big(\frac{x}{\varepsilon^\gamma}\Big)\, u \right) dx,$$

where g is a 1-periodic function and W a general double-well potential. Then when $\gamma > 0$ both the amplitude and the frequency of g become large (for ε small) and the infimum of the energy can even tend to $-\infty$ as $\varepsilon \to 0$. Hence, to fit in the framework of Γ-convergence, the introduction of an additive renormalization is needed. So if on one hand in our model we do not encounter the difficulty arising from this renormalization (and in particular from the related fact that the functionals have nonconstant global minimizers whose energy is not uniformly bounded from below), on the other hand, our particular choice permits to detail an asymptotic expansion that is not pursued in [**32**].

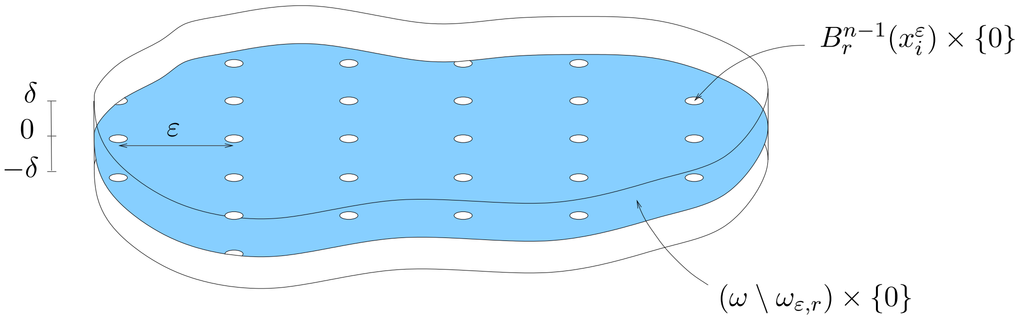

FIGURE 2. The domain $\Omega^\delta_{\varepsilon,r}$.

For an ever increasing variety of applications, another interesting (multi-scale) problem to be explored is to model the debonding of a thin film from a substrate.

If we consider a stretched film bonded to an infinite rigid substrate, the elastic energy of this film scales as its thickness. If the film debonds from the substrate, on one hand its elastic energy tends to zero, while on the other hand this creates a new surface and then an interfacial energy independent of the thickness.

In [**11**] Bhattacharya, Fonseca and Francfort examine, among other, the asymptotic behavior of a bilayer thin film allowing for the possibility of a debonding at the interface, but penalizing it postulating an interfacial energy which scales as the overall thickness of the film to some exponent. Thus the energy they consider consists of the elastic energy of the two layers and the interfacial energy with penalized debonding.

The second chapter of this dissertation deals with thin films connected by a hyperplane (sieve plane) through a periodically distributed contact zone. Thus we see the debonding as the effect of the *weak interaction* of the two thin films through this contact zone and we recover the interfacial energy term by a limit procedure.

Since we are mainly interested in describing the interaction phenomenon due to the presence of the sieve, we make a simplification choosing two thin films having the same elastic properties (for a generalization to the case of two different materials interacting, we refer the reader to [**5**]).

Consider a nonlinear elastic n-dimensional bilayer thin film of thickness 2δ with layers connected through $(n-1)$-dimensional balls $B_r^{n-1}(x_i^\varepsilon)$ centered in $x_i^\varepsilon := i\varepsilon$, $i \in \mathbb{Z}^{n-1}$ and with radius $r > 0$. Thus the investigated elastic body occupies the reference configuration parametrized as

$$\Omega^\delta_{\varepsilon,r} := \omega^{+\delta} \cup \omega^{-\delta} \cup \big(\omega_{\varepsilon,r} \times \{0\}\big)$$

where ω is a bounded open subset of $\mathbb{R}^{n-1}$, $\omega^{+\delta} := \omega \times (0,\delta)$, $\omega^{-\delta} := \omega \times (-\delta, 0)$ and $\omega_{\varepsilon,r} := \bigcup_{i\in\mathbb{Z}^{n-1}} B_r^{n-1}(x_i^\varepsilon) \cap \omega$ (see Figure 1).

In the nonlinear membrane theory setting the (scaled) elastic energy associated to the material modelled by $\Omega^\delta_{\varepsilon,r}$ is given by

$$\frac{1}{\delta}\int_{\Omega^\delta_{\varepsilon,r}} W(Du)\,dx, \tag{0.3}$$

where $u : \Omega^\delta_{\varepsilon,r} \to \mathbb{R}^m$ is the deformation field and W is the stored energy density.

The Γ-convergence approach has been used successfully in recent years to rigorously obtain limit models for various dimensional reductional problems (see for example [**12, 18, 19, 38, 47**]).

We study the multiscale asymptotic behavior of (0.3) via Γ-convergence, as ε, δ and r tend to zero, assuming that $\delta = \delta(\varepsilon)$, $r = r(\varepsilon, \delta)$ and with $W : \mathbb{R}^{m\times n} \to [0, +\infty)$, Borel function satisfying a growth condition of order p, with $1 < p < n-1$.

The case $p = n-1$ requires a further appropriate analysis and it cannot be easily derived from $p < n-1$ by slight changes. Unfortunately, three dimensional linearized elasticity falls into this framework.

Since the sieve $(\omega \setminus \omega_{\varepsilon,r}) \times \{0\}$ is not a part of the domain $\Omega^{\delta}_{\varepsilon,r}$, for any fixed $\varepsilon, \delta, r > 0$ we have no information on the admissible deformation across part of the mid-section $\omega \times \{0\}$. This possible lack of regularity might produce, in the limit, the above mentioned debonding and correspondingly an interfacial energy depending on the jump of the limit deformation. Moreover, we expect that this interfacial energy will depend on the scaling of the radius of the connecting zones with respect to the period of their distribution and the thickness of the thin film.

The cases $\delta = 1$ and $\delta = \varepsilon$ have been studied by Ansini [**5**] who proved that, to recover a non trivial limit model; *i.e.,* to obtain a limit model remembering the presence of the sieve, the meaningful radius (or critical size) of the contact zones must be of order $\varepsilon^{(n-1)/(n-p)}$ and $\varepsilon^{n/(n-p)}$, respectively. In fact a different choice should lead in the limit to two decoupled problems (if r tends to zero faster than the critical size) or to the same result that is obtained without the presence of connecting zones in the mid-section (if r tends to zero more slowly than the critical size).

The proofs of the Γ-convergence results in [**5**] (see Theorems 3.2 and 8.2 therein) are based on a technical lemma ([**5**], Lemma 3.4) that allows to modify a sequence of deformations u_ε with equi-bounded energy, on a suitable n-dimensional spherical annuli surrounding the balls $B^{n-1}_r(x^\varepsilon_i)$ without essentially changing their energies, and to study the behavior of the energies along the new modified sequence. Both in the case $\delta = 1$ and $\delta = \varepsilon$ the Γ-limits consist of three terms. The first two terms represent the contribution of the new sequence far from the balls $B^{n-1}_r(x^\varepsilon_i)$; more precisely, they are the Γ-limits of two problems defined separately on the upper and lower part (with respect to the 'sieve plane') of the considered domain. The third term describes the contribution near the balls $B^{n-1}_r(x^\varepsilon_i)$ through a nonlinear capacitary-type formula that is the same for both $\delta = 1$ and $\delta = \varepsilon$. The equality of the two formulas is due to the fact that the radii of the annuli suitably chosen to separate the two contributions are less than $c\,\varepsilon$, with c an arbitrary small positive constant. In fact as a consequence, all constructions can be performed in the interior of the domain, and the same procedure yielding the nonlinear capacitary-type formula, applies for $\delta = 1$ and for $\delta = \varepsilon$ as well. The cases $\varepsilon \sim \delta$ and $\varepsilon \ll \delta$ can be treated in the same way.

This approach follows the method introduced by Ansini-Braides in [**7, 8**] where the asymptotic behavior of periodically perforated nonlinear domains has been studied; in particular, Lemma 3.4 in [**5**] is a suitable variant, for the sieve problem, of Lemma 3.1 in [**7**].

For other problems related to this subject, we refer the reader to Attouch-Damlamian-Murat-Picard [**28**], [**41**], [**42**], Attouch-Picard [**10**], Conca [**23, 24, 25**], Del Vecchio [**30**] and Sanchez-Palencia [**45, 44, 46**], among others.

We focus our attention on the case $\delta = \delta(\varepsilon) \ll \varepsilon$. As in [**5**], we expect the existence of a meaningful radius $r = r(\varepsilon, \delta) \ll \varepsilon$ for which the limit model is nontrivial but now we expect also to find different limit regimes depending on the mutual vanishing rate of r and δ. Moreover Lemma 3.4 in [**5**] cannot be directly applied to our setting since the spherical annuli surrounding the connecting zones $B_r^{n-1}(x_i^\varepsilon)$ as above, are well contained in a strip of thickness $c\,\varepsilon$ but not in $\Omega^\delta_{\varepsilon,r}$ ($\delta \ll \varepsilon$). However, we are able to modify Lemma 3.4 in [**5**] by considering, instead of spherical annuli, suitable cylindrical annuli of thickness of order δ (see Lemma 4.2 and Lemma 4.3).

As a consequence, also in this case the asymptotic analysis of (0.3) as ε, δ and r tend to zero can be carried on studying separately the energy contributions far from and close to $B_r^{n-1}(x_i^\varepsilon)$; we get three terms in the limit. The first two terms still describe the contribution 'far' from the connecting zones; *i.e.,* they are the Γ-limits of the two dimensional reduction problems defined by

$$\frac{1}{\delta}\int_{\omega^{+\delta}} W(Du)\,dx\,, \qquad \frac{1}{\delta}\int_{\omega^{-\delta}} W(Du)\,dx\,;$$

while the third term, arising in the limit from the energy contribution close to the connecting zones, represents the asymptotic memory of the sieve: it is the above mentioned interfacial energy.

The main results of this paper are stated in Theorem 3.3 and Theorem 3.6. In Theorem 3.3 we prove a Γ-convergence result for the sequence of functionals (0.3) while in Theorem 3.6 we give an explicit characterization of the interfacial energy term occurring in the Γ-limit. More precisely, for every sequence (ε_j) converging to zero, we set $\delta_j := \delta(\varepsilon_j)$, $r_j := r(\varepsilon_j, \delta_j)$, $\Omega_j := \Omega^{\delta_j}_{\varepsilon_j, r_j}$ and

$$\mathcal{F}_j(u) := \begin{cases} \dfrac{1}{\delta_j}\displaystyle\int_{\Omega_j} W(Du)\,dx & \text{if } u \in W^{1,p}(\Omega_j;\mathbb{R}^m) \\ \\ +\infty & \text{otherwise}\,. \end{cases}$$

Up to subsequence we can define

$$\ell := \lim_{j\to+\infty} \frac{r_j}{\delta_j} \quad \text{and} \quad g(F) := \lim_{j\to+\infty} r_j^p\,\mathcal{Q}_n W(r_j^{-1}F).$$

where $\mathcal{Q}_n W$ is the n-quasiconvexification of W.

If $\ell \in (0, +\infty]$ and

$$0 < R^{(\ell)} := \lim_{j\to+\infty} \frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}} < +\infty,$$

then $(\mathcal{F}_j)$ Γ-converges to

$$\mathcal{F}^{(\ell)}(u^+, u^-) = \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^+)\,dx_\alpha + \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^-)\,dx_\alpha + R^{(\ell)}\int_\omega \varphi^{(\ell)}(u^+ - u^-)\,dx_\alpha$$

on $W^{1,p}(\omega;\mathbb{R}^m)\times W^{1,p}(\omega;\mathbb{R}^m)$ with respect to the convergence introduced in Definition 3.1, where $\overline{W}(\overline{F}) := \inf\{W(\overline{F}|z) : z\in\mathbb{R}^m\}$, $\mathcal{Q}_{n-1}\overline{W}$ is the $(n-1)$-quasiconvexification of $\overline{W}$ and $\varphi^{(\ell)} : \mathbb{R}^m \to [0,+\infty)$ is a locally Lipschitz continuous function for any $\ell\in[0,+\infty]$. Similarly, if $\ell = 0$ and

$$0 < R^{(0)} := \lim_{j\to+\infty} \frac{r_j^{n-p}}{\delta_j\, \varepsilon_j^{n-1}} < +\infty,$$

then we still have Γ-convergence, as above, to

$$\mathcal{F}^{(0)}(u^+,u^-) = \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^+)\,dx_\alpha + \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^-)\,dx_\alpha + R^{(0)}\int_\omega \varphi^{(0)}(u^+-u^-)\,dx_\alpha$$

on $W^{1,p}(\omega;\mathbb{R}^m)\times W^{1,p}(\omega;\mathbb{R}^m)$.

For any $\ell\in[0,+\infty]$, $\varphi^{(\ell)}$ is described by the following nonlinear capacitary-type formulas:

(1) if $\ell=+\infty$, then

$$\begin{aligned}\varphi^{(\infty)}(z) \;=\; \inf\Bigg\{ \int_{\mathbb{R}^{n-1}} \Big(\mathcal{Q}_{n-1}\,\bar g(D_\alpha\zeta^+) + \mathcal{Q}_{n-1}\,\bar g(D_\alpha\zeta^-)\Big)\,dx_\alpha :\ & \zeta^\pm \in W^{1,p}_{\rm loc}(\mathbb{R}^{n-1};\mathbb{R}^m),,\\ \zeta^+=\zeta^- \text{ in } B_1^{n-1}(0),\quad & D_\alpha\zeta^\pm\in L^p(\mathbb{R}^{n-1};\mathbb{R}^{m\times(n-1)}),\\ & (\zeta^+-z)\,,\zeta^-\in L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m)\Bigg\},\end{aligned}$$

where again, $\bar g(\overline{F}) := \inf\{g(\overline{F}|z) : z\in\mathbb{R}^m\}$ and $\mathcal{Q}_{n-1}\bar g$ is the $(n-1)$-quasiconvexification of $\bar g$,

(2) if $\ell=0$, then

$$\begin{aligned}\varphi^{(0)}(z) \;=\; \inf\Bigg\{ \int_{\mathbb{R}^n\setminus C_{1,\infty}} g(D\zeta)\,dx :\ & \zeta\in W^{1,p}_{\rm loc}(\mathbb{R}^n\setminus C_{1,\infty};\mathbb{R}^m),\ D\zeta\in L^p(\mathbb{R}^n\setminus C_{1,\infty};\mathbb{R}^{m\times n}),\\ & \zeta - z\in L^p(0,+\infty;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m)) \text{ and } \zeta\in L^p(-\infty,0;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))\Bigg\},\end{aligned}$$

(3) if $\ell\in(0,+\infty)$, then

$$\begin{aligned}\varphi^{(\ell)}(z) \;=\; \inf\Bigg\{ \int_{\mathbb{R}^{n-1}\times(-1,1)} g\big(D_\alpha\zeta|\ell D_n\zeta\big)\,dx :\ & \zeta\in W^{1,p}_{\rm loc}((\mathbb{R}^{n-1}\times(-1,1))\setminus C_{1,\infty};\mathbb{R}^m),\\ D\zeta\in L^p(\mathbb{R}^{n-1}\times(-1,1);\mathbb{R}^m),\qquad & \zeta - z\in L^p((0,1);L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))\\ & \zeta\in L^p((-1,0);L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))\Bigg\},\end{aligned}$$

where $C_{1,\infty} := \{(x_\alpha,0)\in\mathbb{R}^n\ :\ 1\le |x_\alpha|\}$.

Before giving a brief heuristic description of each regime, we want to point out that whatever the value of ℓ is, the interfacial energy density $\varphi^{(\ell)}$ corresponds to a cohesive interface where the surface energy increases continuously from zero with the jump in the deformation across the interface.

(1) The case $\ell = +\infty$ corresponds to $\delta_j \ll r_j \ll \varepsilon_j$, thus we expect r_j to depend only on ε_j. In this case we have a separation of scales effect. We first consider r_j and ε_j as 'fixed' and let δ_j tend to zero as if we were dealing with two pure dimensional reduction problems stated separately on the upper and lower part (with respect to the sieve plane) of Ω_j. Then this first limit procedure yields two functionals being both a copy of the functional in [**38**]. Since the two corresponding limit deformations u^+ and u^- must match inside each connecting zone, the above two terms are not completely decoupled. We are then in a situation quite similar to that of [**7, 8**], except that here both periodically 'perforated' $(n-1)$-dimensional bodies are linked each other through the 'perforations'; *i.e.,* through the holes of the sieve and not through the sieve itself. Thus it is coherent to find a critical size of order $\varepsilon^{(n-1)/(n-1-p)}$. Moreover this strong separation between the phenomena of dimension reduction and 'perforation' leads to anisotropy as it can be seen, for instance, also by an inspection of the proof of Lemma 6.2 which shows that the extra interfacial energy term appears thanks to suitable dilatations having a different scaling in the in-plane and transverse variables. Finally we note that the formula for $\varphi^{(\infty)}$ is given in terms of a 'Le Dret-Raoult type' functional involving the limit of the right capacitary scaling (that is, involving the function g).

(2) The case $\ell = 0$ corresponds to $r_j \ll \delta_j \ll \varepsilon_j$. In this case we expect that the critical size r_j depends on both δ_j and ε_j. Indeed, as already pointed out, r_j is of order $\delta_j^{1/(n-p)}\varepsilon_j^{(n-1)/(n-p)}$. Note that for $\delta_j = \varepsilon_j$ we recover $\varepsilon^{n/(n-p)}$ that is the critical size obtained in [**5**]; moreover $\varphi^{(0)}$ turns out to coincide with the function φ in [**5**] (see Remark 7.3). Contrary to the previous case, now the isotropy is preserved in fact here the dimensional reduction and 'perforation' processes are not completely decoupled: the reduction parameter δ_j is forced between both parameters r_j and ε_j. This can be seen also by noticing that now the scaling leading to the interfacial energy is the same in every direction (see for instance the proof of the Γ-limsup inequality). Moreover now in $\varphi^{(0)}$ the reduction procedure is not explicit but only witnessed by the boundary conditions expressed only on the lateral part of the boundary of the considered domain.

(3) The case $\ell \in (0, +\infty)$ corresponds to $r_j \sim \delta_j \ll \varepsilon_j$. In this case the separation of scales effect does not take place and the two previous scalings turn out to be equivalent ($R^{(0)} = \ell\, R^{(\infty)}$). Moreover we find that the interfacial energy is continuous with respect to ℓ in the extreme regimes; *i.e.,* $R^{(\ell)}\varphi^{(\ell)}(z) \to R^{(\infty)}\varphi^{(\infty)}(z)$ as $\ell \to +\infty$ and $R^{(\ell)}\varphi^{(\ell)}(z) \to R^{(0)}\varphi^{(0)}(z)$ as $\ell \to 0$. Finally, as in the previous case, the lateral boundary conditions are the only mean describing the dimensional reduction phenomenon in the procedure leading to $\varphi^{(\ell)}$.

CHAPTER 1

A model for the interaction between oscillations and surface energy

1. Zero order Γ-limit

As already observed, our energy is a particular, one-dimensional version of a more general, multidimensional energy introduced in [**34**]. Thus, with in mind the idea of a Γ-development for (0.2), in this section we adapt to our setting the Γ-convergence results of Theorem 2.1 and Theorem 2.3 in [**34**].

These two results are summarized in the following theorem.

THEOREM 1.1. *Let $\delta = \delta(\varepsilon)$ be such that $\delta \to 0$ as $\varepsilon \to 0$ and set*

$$\ell := \lim_{\varepsilon\to 0} \frac{\delta(\varepsilon)}{\varepsilon}.$$

Then the family of functionals $F_\varepsilon^{k(0)} := F_{\varepsilon,\delta(\varepsilon)}^{k(0)}$ defined as in (0.2), Γ-converges with respect to the weak L^2-convergence to the homogeneous functional defined on $L^2(0,1)$ by

$$F_\ell^{k(0)}(u) = \int_0^1 W_\ell^k(u)\, dx\,. \tag{1.1}$$

Moreover the integrand W_ℓ^k depends on ℓ in the following way:

(1) *if $\ell = +\infty$, then*

$$W_\infty^k(s) = \inf\left\{\int_0^1 W^k(x,v)\,dx \,:\, v \in L^2(0,1),\ \int_0^1 v\,dx = s\right\}; \tag{1.2}$$

(2) *if $\ell \in (0,+\infty)$, then*

$$W_\ell^k(s) = \inf_{n\in\mathbb{N}} \inf\left\{ -\!\!\!\!\!\!\int_0^n (W^k(x,v) + \frac{1}{\ell^2}(v')^2)\,dx :\ v \in W^{1,2}(0,n),\ -\!\!\!\!\!\!\int_0^n v\,dx = s\right\};$$

(3) *if $\ell = 0$, then*

$$W_0^k(s) = (\overline{W}^k)^{**}(s)$$

where

$$\overline{W}^k(s) = \int_0^1 W^k(y,s)\,dy. \tag{1.3}$$

REMARK 1.2. From the definition of W^k, a priori we only know that the family $(F_\varepsilon^{k(0)})$ is equi-coercive with respect to the weak L^2-convergence (for any choice of $\delta = \delta(\varepsilon)$), for this reason in Theorem 1.1 above, the Γ-limit is computed, in each regime, with respect to that convergence.

We only give a brief heuristic description of the result stated above while we refer the reader to [**34**], for a rigorous proof.

(1) The case $\ell = +\infty$ corresponds to $\varepsilon \ll \delta$; *i.e.,* to the case in which the scale of oscillation δ is much larger than the scale of the transition layer ε. The result is that we have a separation of scales effect, indeed we may first regard δ as fixed and let $\varepsilon \to 0$ and subsequently let $\delta \to 0$. In this way, we first obtain an inhomogeneous functional which can be explicitly computed as
$$\int_0^1 (W^k)^{**}\left(\frac{x}{\delta}, u\right)\, dx$$
where the convexification of W^k is with respect to the second argument. Then the limit as $\delta \to 0$ falls within the framework of homogenization leading to an integral functional whose density is the convex, homogenized potential given by the cell formula (1.2). Hence, we have that in this case the presence of the singular perturbation does not affect the homogenization process.

(2) The case $\ell \in (0, +\infty)$ corresponds to $\varepsilon \sim \delta$; *i.e.,* is the case in which ε and δ are comparable. Now the two effects cannot be separated and the presence of the singular perturbation contributes to the definition of W_ℓ^k.

(3) The case $\ell = 0$ corresponds to $\varepsilon \gg \delta$. In this case we again find a separation of scales phenomenon: the total effect is that the singular perturbation forces the homogenized energy to be (the convex envelope of) the average of the microscopic energy over the period.

1.1. The effective potential W_ℓ^k. Since we are interested in describing how the two different parameters ε and δ interacts in the coming up of the various scales of the Γ-development, from now on we focus only on the two regimes $\delta \gg \varepsilon$ and $\delta \ll \varepsilon$, the regime $\delta \sim \varepsilon$ being, somehow, less interesting than the extreme ones.

The starting point of our analysis consists in a complete characterization of the zero order Γ-limit. Then, recalling the definition of our given W^k, in this section we want to find the explicit expression of the effective potential W_ℓ^k for $\ell = +\infty$ and $\ell = 0$.

If $\ell = +\infty$, Theorem 1.1 asserts that W_∞^k is given in terms of the cell formula (1.2), that is equivalent to
$$W_\infty^k(s) = \min\left\{\int_0^1 (W^k)^{**}(x, v)\, dx \;:\; v \in L^2(0,1),\; \int_0^1 v\, dx = s\right\},$$
thus by using Jensen's inequality it is easy to check that
$$W_\infty^k(s) = \min\left\{\frac{1}{2}W^{**}(s_1 - k) + \frac{1}{2}W^{**}(s_2 + k) \;:\; s_1 + s_2 = 2s\right\}.$$
Finally, a straightforward calculation gives
$$W_\infty^k(s) = W^{**}(s) = \begin{cases} 0 & \text{if } \; |s| \le 1 \\ (|s| - 1)^2 & \text{if } \; |s| > 1. \end{cases} \tag{1.4}$$

If $\ell = 0$, then trivially

$$\overline{W}^k(s) = \frac{1}{2}(W^k(s-k) + W^k(s+k)) = \begin{cases} s^2 + (1-k)^2 & \text{if } |s| \le k \\ s^2 - 2|s| + k^2 + 1 & \text{if } |s| > k \end{cases}$$

hence by a direct computation we get

$$W_0^k(s) = \begin{cases} k^2 & \text{if } |s| \le 1 \\ s^2 - 2|s| + k^2 + 1 & \text{if } |s| > 1 \end{cases}$$

for $k \le \frac{1}{2}$, while

$$W_0^k(s) = \begin{cases} s^2 + (1-k)^2 & \text{if } |s| \le k - \frac{1}{2} \\ (2k-1)|s| - k + \dfrac{3}{4} & \text{if } k - \frac{1}{2} < |s| < k + \frac{1}{2} \\ s^2 - 2|s| + k^2 + 1 & \text{if } |s| > k + \frac{1}{2} \end{cases}$$

for $k > \frac{1}{2}$.

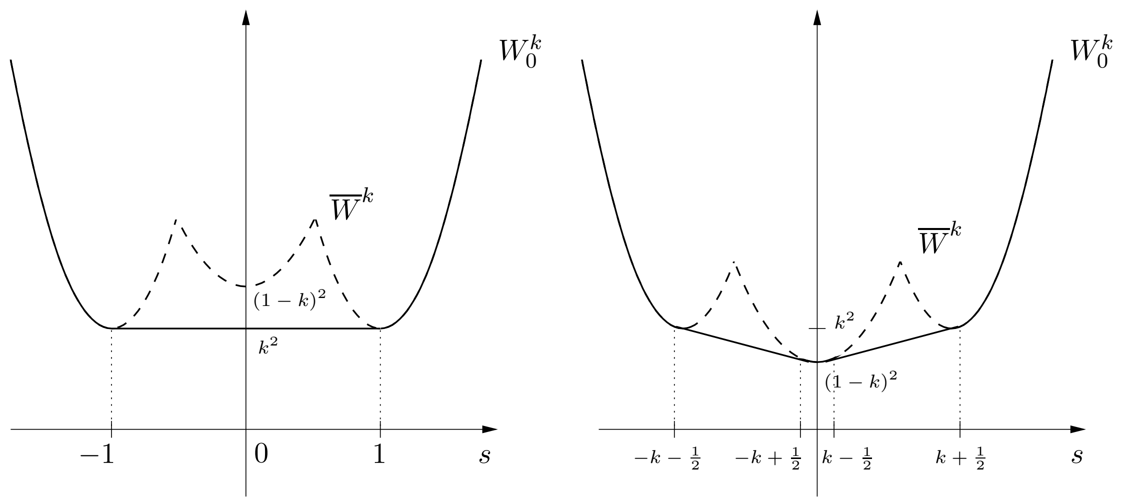

FIGURE 1. The effective potential W_0^k for $k < \frac{1}{2}$ and $k > \frac{1}{2}$.

2. Optimal scalings

In the previous section we show that the effective potential W_ℓ^k admits "many" minimizers for both $\ell = +\infty$ and $\ell = 0$, $k \le \frac{1}{2}$; more precisely, $W_\ell^k(s) = \min W_\ell^k$ for every s such that $|s| \le 1$. As a consequence, every function $u \in L^2(0,1)$ satisfying $|u| \le 1$ a.e., is a minimum point for the zero order Γ-limit $F_\ell^{k(0)}$. Hence, if for any fixed $\varepsilon > 0$, v_ε minimizes $F_\varepsilon^{k(0)}$ (notice that the existence of a minimizer for $F_\varepsilon^{k(0)}$ over $L^2(0,1)$ can be proved via standard lower semicontinuity and compactness results) then the fact that every limit point v of (v_ε) minimizes $F_\ell^{k(0)}$ actually gives few information about v.

As v_ε minimizes also

$$\frac{F_\varepsilon^{k(0)} - m_\ell^{(0)}}{\lambda^{(1)}(\varepsilon)} \tag{2.1}$$

for every $\lambda^{(1)}(\varepsilon) > 0$, with $m_\ell^{(0)} := \min F_\ell^{k(0)}$, information about the limit points of (v_ε) can be recovered also by the Γ-limit of the scaled functionals (2.1), which may be less trivial for a suitable choice of $\lambda^{(1)}(\varepsilon)$. So now the problem arises of finding the *optimal scaling*; *i.e.,* the $\lambda^{(1)}(\varepsilon)$ such that the Γ-limit of (2.1) gives the largest amount of information. Once $\lambda^{(1)}(\varepsilon)$ is determined, the Γ-limit of the scaled family of functionals (2.1) will be the *first order term* of the Γ-development.

At this point some scale analysis must be performed for both $\ell = +\infty$ and $\ell = 0$, $k \le \frac{1}{2}$, to understand what the relevant scaling $\lambda^{(1)}(\varepsilon)$ is. Moreover, we remark that we expect $\lambda^{(1)}(\varepsilon)$ to depend also on the regime ℓ; to make the dependance explicit, in the sequel we denote this scale by $\lambda_\ell^{(1)}(\varepsilon)$.

If needed, in the following we iterate this analysis to obtain more scales in the development and consequently a more accurate description of the limit points of (v_ε).

Finally, referring to the remaining case $\ell = 0$, $k > \frac{1}{2}$, we want to point out that the non strict convexity of W_0^k (see Figure 2) allows us to determine an asymptotic development for $F_\varepsilon^{k(0)}$ in this case too, as we detail in Section...

3. $\delta \gg \varepsilon$: oscillations on a larger scale than the transition layer

In this section we treat the case when the scale of oscillation δ is much larger than the scale of the transition layer ε; *i.e.,* the case $\ell = +\infty$.

In order to guess what the first meaningful scale $\lambda_\infty^{(1)}(\varepsilon)$ is, we start by performing a preliminary qualitative scale analysis.

Using the same argument proposed to examine Modica-Mortola's Model [**40, 39**] we want to estimate the order of $m_\varepsilon^{k(0)} := \min F_\varepsilon^{k(0)}$, as $\varepsilon \to 0$.

To this aim, we focus our attention on a single δ-interval: say the interval $(0, \delta)$. Then, when we come to minimize $F_\varepsilon^{k(0)}$, on one hand the term $\int_0^\delta W^k\big(\frac{x}{\delta}, u\big)\,dx$ favorites those configurations which takes values "close" to the (varying) zero set of W^k; *i.e.,* close to (at least) two different constant values one chosen in $\{1+k, -1+k\}$ when $x \in \big(0, \frac{\delta}{2}\big)$, while the other chosen in $\{1-k, -1-k\}$ when $x \in \big(\frac{\delta}{2}, \delta\big)$. In other words, the potential term in the energy favorites a phenomenon of *phase separation.* On the other hand, the gradient term $\varepsilon^2 \int_0^\delta (u')^2\,dx$ penalizes spatial inhomogeneities thus inducing a *phase transition* phenomenon as well. When ε is small the first term prevails, and the minimum of

$$\int_0^\delta \Big(W^k\Big(\frac{x}{\delta}, u\Big) + \varepsilon^2 (u')^2\Big)\,dx$$

is attained at a function which takes "mainly" values close to the set $\{1+k, -1+k\}$ in $\big(0, \frac{\delta}{2}\big)$ and close to $\{1-k, -1-k\}$ in $\big(\frac{\delta}{2}, \delta\big)$, but which also makes a transition on a "thin" layer. Then a well-known scaling argument (see *e.g.* [**3**] and [**14**], Chapter 6) proves that the transition between two different zeroes chosen as above, actually occurs in a layer of thickness of order ε (recall that $\delta \gg \varepsilon$) and gives an energy contribution of order ε too.

Clearly the previous heuristics can be repeated on each period δ thus leading to a total energy contribution of order $\frac{\varepsilon}{\delta}$. Finally what we claim is that

$$\lambda_\infty^{(1)}(\varepsilon) = \frac{\varepsilon}{\delta},$$

and the proof of this claim will be made rigorous with Theorem 3.2.

3.1. Estimate for the phase transition energy. We now move the first step towards a rigorous justification of the qualitative argument discussed in the previous section.

In what follows, we make use of some well-known facts related to the so-called *optimal profile problem* in Modica-Mortola model. For a detailed and exhaustive treatment of the one dimensional case, we refer the reader to [**3**], Section 3a or to [**14**], Remark 6.1.

We want to find an explicit formula for the phase transition energy; to this purpose we set

$$W_1^k(s) := W(s-k) \quad W_2^k(s) := W(s+k),$$

and for any fixed $\varepsilon > 0$, we let $x_1, x_2 \in \mathbb{R}$ be such that $x_1 < x_2$, $x_2 - x_1 \leq \frac{\delta}{2}$ and $\frac{\delta}{2} \in (x_1, x_2)$. We start by giving an estimate on the contribution of the integration on (x_1, x_2) in $F_\varepsilon^{k(0)}(u)$ in terms of $z_1 := u(x_1)$ and $z_2 := u(x_2)$.

We have

$$\begin{aligned}
&\int_{x_1}^{x_2} \left(W^k\left(\frac{x}{\delta}, u\right) + \varepsilon^2 (u')^2 \right) dx \\
= \;& \varepsilon \left(\int_{x_1}^{\frac{\delta}{2}} \left(\frac{1}{\varepsilon} W_1^k(u) + \varepsilon (u')^2 \right) dx + \int_{\frac{\delta}{2}}^{x_2} \left(\frac{1}{\varepsilon} W_2^k(u) + \varepsilon (u')^2 \right) dx \right) \\
= \;& \varepsilon \left(\int_{\frac{x_1}{\varepsilon}}^{\frac{\delta}{2\varepsilon}} \left(W_1^k(v) + (v')^2 \right) dx + \int_{\frac{\delta}{2\varepsilon}}^{\frac{x_2}{\varepsilon}} \left(W_2^k(v) + (v')^2 \right) dx \right), \qquad (3.1)
\end{aligned}$$

where v is defined as

$$v(x) := u(\varepsilon x).$$

By the change of variable $y = x - \frac{\delta}{2\varepsilon}$, (3.1) becomes

$$\varepsilon \left(\int_{-T_1}^{0} (W_1^k(z) + (z')^2)\, dy + \int_0^{T_2} (W_2^k(z) + (z')^2)\, dy \right),$$

with

$$T_1 := \frac{\delta - 2x_1}{2\varepsilon},\; T_2 := \frac{2x_2 - \delta}{2\varepsilon} \quad \text{and} \quad z(y) := v\Big(y + \frac{\delta}{2\varepsilon}\Big).$$

Hence we find that a lower bound for the energy of a transition between the values z_1, z_2 is given by

$$\begin{aligned}
\varepsilon \inf_{T_1, T_2 > 0} \inf \Big\{ \int_{-T_1}^{0} (W_1^k(z) + (z')^2)\, dy + \int_0^{T_2} (W_2^k(z) + (z')^2)\, dy \;: \\
z \in W^{1,2}(-T_1, T_2),\; z(-T_1) = z_1,\; z(T_2) = z_2 \Big\}. \qquad (3.2)
\end{aligned}$$

Now let Z_i^k be the set of the zeroes of W_i^k for $i = 1, 2$; *i.e.*,

$$Z_1^k = \{-1+k; 1+k\} \quad Z_2^k = \{-1-k; -1+k\},$$

if $z_i \in Z_i^k$ ($i = 1, 2$) we know that

$$\begin{aligned}\inf_{T_1>0} \inf &\left\{ \int_{-T_1}^{0} (W_1^k(z) + (z')^2)\, dy :\ z \in W^{1,2}(-T_1, 0),\ z(-T_1) = z_1,\ z(0) = z_0 \right\} \\ &= \inf \left\{ \int_{-\infty}^{0} (W_1^k(z) + (z')^2)\, dy :\ z \in W^{1,2}_{\mathrm{loc}}(-\infty, 0),\ z(-\infty) = z_1,\ z(0) = z_0 \right\}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\inf_{T_2>0} \inf &\left\{ \int_{0}^{T_2} (W_2^k(z) + (z')^2)\, dy :\ z \in W^{1,2}(0, T_2),\ z(0) = z_0,\ z(T_2) = z_2 \right\} \\ &= \inf \left\{ \int_{0}^{+\infty} (W_2^k(z) + (z')^2)\, dy :\ z \in W^{1,2}_{\mathrm{loc}}(0, +\infty),\ z(0) = z_0,\ z(+\infty) = z_2 \right\}\end{aligned} \tag{3.4}$$

where $z(-\infty)$ and $z(+\infty)$ are understood as the existence of the corresponding limits. Then, it is easy to check that (3.2) can be rewritten in terms of the two optimal profile problems (3.3) and (3.4), as

$$\begin{aligned}\varepsilon \inf_{z_0} \Big\{ &\inf \Big\{ \int_{-\infty}^{0} (W_1^k(z) + (z')^2)\, dy :\ z \in W^{1,2}_{\mathrm{loc}}(-\infty, 0),\ z(-\infty) = z_1,\ z(0) = z_0 \Big\} \\ &+ \inf \Big\{ \int_{0}^{+\infty} (W_2^k(z) + (z')^2)\, dy :\ z \in W^{1,2}_{\mathrm{loc}}(0, +\infty),\ z(0) = z_0,\ z(+\infty) = z_2, \Big\} \Big\}\end{aligned}$$

and finally as

$$\varepsilon \inf_{z_0} \left\{ 2 \left| \int_{z_1}^{z_0} \sqrt{W_1^k(s)} \right| + 2 \left| \int_{z_0}^{z_2} \sqrt{W_2^k(s)} \right| \right\}. \tag{3.5}$$

Hence, if for every $\zeta_1, \zeta_2 \in \mathbb{R}$, we set

$$C_{W^k}(\zeta_1, \zeta_2) := \inf_{z_0} \left\{ 2 \left| \int_{\zeta_1}^{z_0} \sqrt{W_1^k(s)} \right| + 2 \left| \int_{z_0}^{\zeta_2} \sqrt{W_2^k(s)} \right| \right\}, \tag{3.6}$$

we have

$$\int_{x_1}^{x_2} \left(W^k\left(\frac{x}{\delta}, u\right) + \varepsilon^2 (u')^2 \right)\, dx \geq \varepsilon\, C_{W^k}(z_1, z_2). \tag{3.7}$$

At the end, recalling the definition of the potential W^k, in order to explicitly compute $C_{W^k}(z_1, z_2)$ we have to distinguish three cases.

Case 1:

$$z_1 = 1+k; \quad z_2 = 1-k$$

$$\begin{aligned} C_1^k := C_{W^k}(1+k, 1-k) &= \inf_{z_0} \left\{ 2 \int_{z_0}^{1+k} \sqrt{W_1^k(s)} + 2 \int_{1-k}^{z_0} \sqrt{W_2^k(s)} \right\} \\ &= 2 \int_{1}^{1+k} \sqrt{W_1^k(s)} + 2 \int_{1-k}^{1} \sqrt{W_2^k(s)} \\ &= 2k^2. \end{aligned}$$

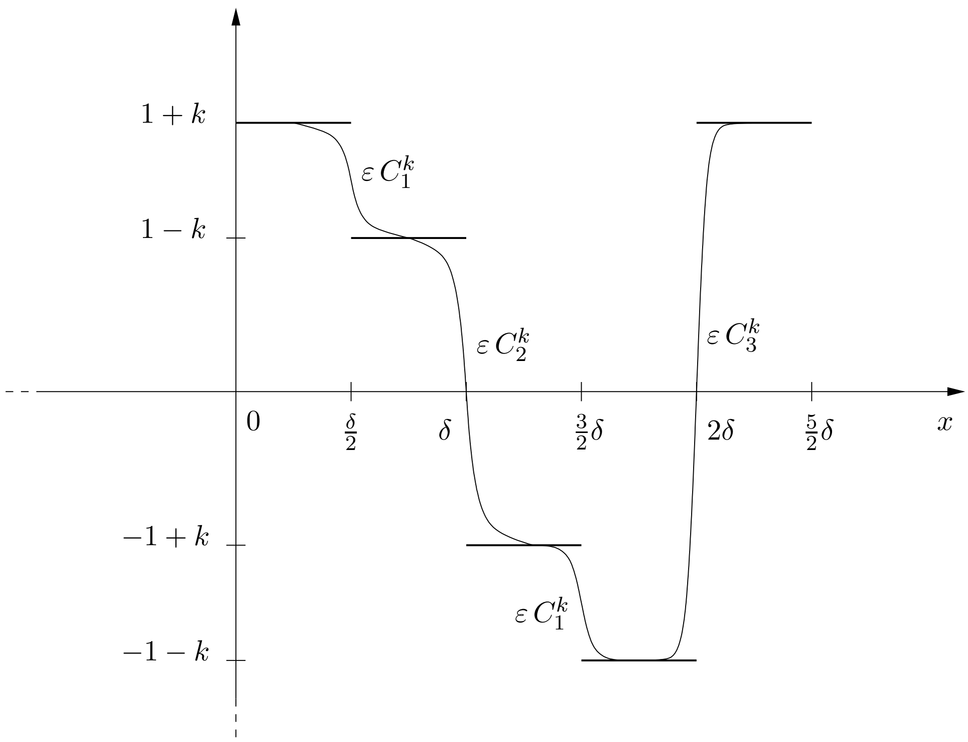

FIGURE 2. Different types of transitions with their (minimal) energy contribution, for $k < \frac{1}{2}$.

Moreover, it is immediate to prove that $C_{W^k}(-1+k, -1-k) = C_1^k$.

Case 2:

$$z_1 = -1 + k; \quad z_2 = 1 - k$$

$$\begin{aligned} C_2^k := C_{W^k}(-1+k, 1-k) &= \inf_{z_0} \left\{ 2\int_{-1+k}^{z_0} \sqrt{W_1^k(s)} + 2\int_{z_0}^{1-k} \sqrt{W_2^k(s)} \right\} \\ &= 2\int_{-1+k}^{0} \sqrt{W_1^k(s)} + 2\int_0^{1-k} \sqrt{W_2^k(s)} \\ &= 2(1-k)^2. \end{aligned}$$

Case 3:

$$z_1 = 1 + k; \quad z_2 = -1 - k$$

$$\begin{aligned} C_3^k := C_{W^k}(1+k,-1-k) &= \inf_{z_0}\left\{2\int_{z_0}^{k+1}\sqrt{W_1^k(s)}+2\int_{-1-k}^{z_0}\sqrt{W_2^k(s)}\right\} \\ &= 2\int_1^{k+1}\sqrt{W_1^k(s)}+2\int_{-k-1}^{1}\sqrt{W_2^k(s)} \\ &= 2(1+k^2). \end{aligned}$$

REMARK 3.1. The constant C_3^k is greater than both of C_1^k, C_2^k for every $k \in (0,1)$; *i.e.,* the transition between the two extreme zeroes $1+k$ and $-1-k$ is always energetically unfavorable. While

$$C_1^k < C_2^k \iff k < \frac{1}{2}, \tag{3.8}$$

or in other words, the transition from $1+k$ to $1-k$ (or equivalently from $-1+k$ to $-1-k$) is more convenient than the one from $-1+k$ to $1-k$ if and only if $k < \frac{1}{2}$.

3.2. First order Γ-limit. We are now ready to state the Γ-convergence result for the family of scaled functionals

$$F_\varepsilon^{k(1)}(u) := \frac{F_\varepsilon^{k(0)}(u)}{\lambda_\ell^{(1)}(\varepsilon)} = \begin{cases} \displaystyle\int_0^1 \left(\frac{\delta}{\varepsilon}W^k\left(\frac{x}{\delta},u\right)+\varepsilon\delta(u')^2\right)dx & \text{if } u \in W^{1,2}(0,1) \\ +\infty & \text{otherwise.}\end{cases} \tag{3.9}$$

Notice that to not overburden notation, in $F_\varepsilon^{k(1)}$ we omit its explicit dependence on ℓ.

THEOREM 3.2. *The family of functionals $F_\varepsilon^{k(1)}$ defined as in (3.9), Γ-converges with respect to the weak L^2-convergence to the integral functional defined on $L^2(0,1)$ by*

$$F^{k(1)}(u) = \begin{cases} \displaystyle\int_0^1 \psi^k(u)\,dx & \textit{if } u \in L^2(0,1) \quad \textit{and} \quad |u| \le 1 \ \textit{a.e.} \\ +\infty & \textit{otherwise}, \end{cases}$$

where

$$\psi^k(s) = \begin{cases} 2C_1^k & \textit{if } k \le \frac{1}{2} \\ 2(C_1^k - C_2^k)|s| + 2C_2^k & \textit{if } k > \frac{1}{2}\,. \end{cases}$$

REMARK 3.3. A first difference between our model and the Modica-Mortola one is that now the first order Γ-limit is again a *bulk energy* (*i.e.* an integral functional).

Before proving the Γ-convergence result for the functionals $F_\varepsilon^{k(1)}$ we need some preliminary results.

In the following proposition, η is the "small" positive parameter that we let go to zero in the Γ-limit procedure.

PROPOSITION 3.4. *i) The family of functionals G_η^k defined on $L^2(-\frac{1}{4},\frac{1}{4})$ by*

$$G_\eta^k(u) = \begin{cases} \displaystyle\int_{-\frac{1}{4}}^{\frac{1}{4}} \left(\frac{1}{\eta}W^k(x,u)+\eta(u')^2\right)dx & \textit{if } u \in W^{1,2}(-\frac{1}{4},\frac{1}{4}) \\ +\infty & \textit{otherwise} \end{cases}$$

Γ*-converges with respect to the strong* L^2*-convergence to the functional defined on* $L^2(-\frac{1}{4}, \frac{1}{4})$ *by*

$$G^k(u) = \begin{cases} C_W\left(\#(S(u)) - 1\right) + C_{W^k}(u(0^+), u(0^-)) \\ \qquad \textit{if}\, u \in BV\left(\left(-\frac{1}{4}, \frac{1}{4}\right); Z_1^k \cup Z_2^k\right) : W^k(x,u) = 0 \ \textit{a.e.} \\ +\infty \quad \textit{otherwise,} \end{cases}$$

where $u(0^+)$, $u(0^-)$ *are the values taken a.e. by* u *on* $(0, r)$ *and* $(-r, 0)$, *respectively, for* $r > 0$ *small enough.*

ii)(Compatibility with integral constraint). Let $s \in \mathbb{R}$ *and let* $G_\eta^{k,s}$ *be defined on* $L^2\left(-\frac{1}{4}, \frac{1}{4}\right)$ *by*

$$G_\eta^{k,s}(u) = \begin{cases} G_\eta^k(u) & \textit{if}\, u \in W^{1,2}\left(-\frac{1}{4}, \frac{1}{4}\right) \textit{ and } \fint_{-\frac{1}{4}}^{\frac{1}{4}} u\, dx = s \\ +\infty & \textit{otherwise.} \end{cases}$$

Then the family of functionals $G_\eta^{k,s}$ *defined as above,* Γ*-converges with respect to the strong* L^2*-convergence to the functional defined on* $L^2\left(-\frac{1}{4}, \frac{1}{4}\right)$ *by*

$$G^{k,s}(u) = \begin{cases} G^k(u) & \textit{if}\, u \in L^2\left(-\frac{1}{4}, \frac{1}{4}\right) \textit{ and } \fint_{-\frac{1}{4}}^{\frac{1}{4}} u\, dx = s \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof. The proofs of $i)$ and $ii)$ exactly follows the line of those of Theorem 6.4 and Theorem 6.6 in [**14**], with the only difference that now the set of the zeroes of the potential W^k varies with x, being equal to Z_1^k in $\left(0, \frac{1}{4}\right)$ and to Z_2^k in $\left(-\frac{1}{4}, 0\right)$, thus forcing sequences with equi-bounded energy to an additional transition in an η-neighborhood of $x = 0$. □

Corollary 3.5 (**convergence of minimum problems**). *For any fixed* $\eta > 0$ *and for every* $s \in \mathbb{R}$, *let* φ_η^k *be the function defined as*

$$\varphi_\eta^k(s) := \min\left\{ \int_{-\frac{1}{4}}^{\frac{1}{4}} \left(\frac{1}{\eta} W^k(x,u) + \eta(u')^2\right) dx : u \in W^{1,2}\left(-\frac{1}{4}, \frac{1}{4}\right), \fint_{-\frac{1}{4}}^{\frac{1}{4}} u\, dx = s \right\}. \tag{3.10}$$

Then for every $s \in \mathbb{R}$

$$\lim_{\eta \to 0} \varphi_\eta^k(s) = \varphi^k(s)$$

where

$$\varphi^k(s) = \begin{cases} C_1^k & if \quad s = -1; 1 \\ C_2^k & if \quad s = 0 \\ C_3^k & if \quad 0 < |s| < 1,\ k \le \frac{1}{2}; \quad C_2^k + C_W \quad if \quad 0 < |s| < 1,\ k > \frac{1}{2} \\ +\infty & if \quad |s| > 1. \end{cases}$$

Proof. We preliminary observe that

$$\min G^{k,\pm 1} = C_1^k, \quad \min G^{k,0} = C_2^k, \quad \min G^{k,s} = \begin{cases} C_1^k + C_W = C_3^k & \text{if } k \le \frac{1}{2} \\ C_2^k + C_W & \text{if } k > \frac{1}{2} \end{cases} \quad \text{for} \quad 0 < |s| < 1,$$

while the set of functions $u : \left(-\frac{1}{4}, \frac{1}{4}\right) \to \mathbb{R}$ such that

$$u \in BV\Big(\Big(0, \frac{1}{4}\Big); Z_1^k\Big), \quad u \in BV\Big(\Big(-\frac{1}{4}, 0\Big); Z_2^k\Big) \quad \text{and} \quad \fint_{-\frac{1}{4}}^{\frac{1}{4}} u = s, \quad \text{with} \quad |s| > 1$$

is empty. Then, since $G_\eta^{k,s} \xrightarrow{\Gamma} G^{k,s}$, the desired convergence result immediately follows from the general property of convergence of minimum values. $\square$

REMARK 3.6. By Remark 3.1 and since $C_2^k + C_W > C_1^k$, we have that $2\,(\varphi^k)^{**}(s) = \psi^k(s)$, for any s such that $|s| \le 1$, and for every $k \in (0,1)$.

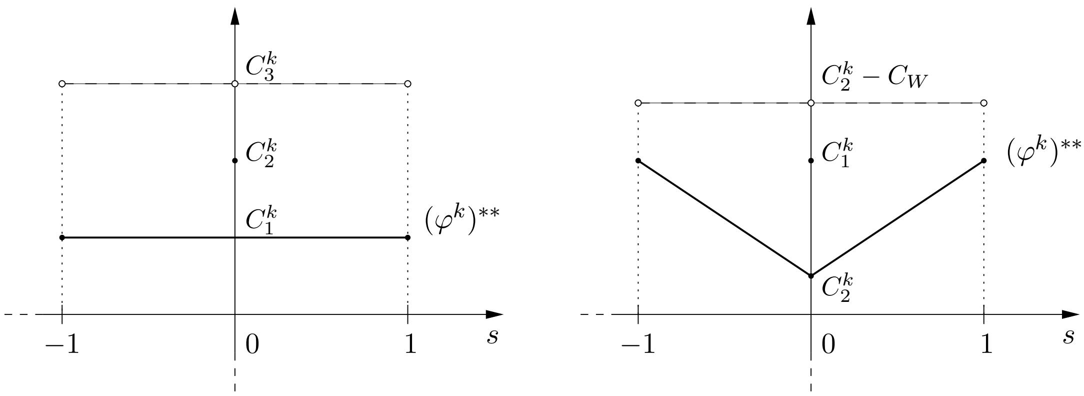

FIGURE 3. The functions φ^k and $(\varphi^k)^{**}$ for $k < \frac{1}{2}$ and $k > \frac{1}{2}$.

PROPOSITION 3.7. *Let φ_η^k be the function defined as in (3.10); then*

1. $\varphi_\eta^k(s) \le C_3^k$ *for every s such that $|s| \le 1$;*
2. *if $|s| \le 1$ and v_η^s is a test function for $\varphi_\eta^k(s)$, then there exists a constant $M > 0$ (independent of η) such that $||v_\eta^s||_\infty \le M$.*

PROOF. 1. For every s with $|s| \le 1$, we exhibit function v_η^s such that $\fint_{-\frac{1}{4}}^{\frac{1}{4}} v_\eta^s\,dx = s$ and whose energy is less then C_3^k.

Let us start by $s = 0$; then as v_η^0 we take the function defined by

$$v_\eta^0(x) := \begin{cases} v_\eta^{0,-}(x) & \text{if} \quad -\frac{1}{4} \le x \le 0 \\ v_\eta^{0,+}(x) & \text{if} \quad 0 < x \le \frac{1}{4}, \end{cases}$$

where $v_\eta^{0,-}$, $v_\eta^{0,+}$ respectively solve

$$\min_{\substack{v \in W^{1,2}(-\frac{1}{4},0) \\ v(0)=0}} \int_{-\frac{1}{4}}^{0} \Big(\frac{1}{\eta}(v - 1 + k)^2 + \eta(v')^2\Big)\,dx, \qquad \min_{\substack{v \in W^{1,2}(0,\frac{1}{4}) \\ v(0)=0}} \int_{0}^{\frac{1}{4}} \Big(\frac{1}{\eta}(v + 1 - k)^2 + \eta(v')^2\Big)\,dx;$$

or equivalently, the associated Cauchy problems

$$\begin{cases} \eta^2 v'' - v + 1 - k = 0 & \text{in} \quad \Big(-\dfrac{1}{4}, 0\Big) \\ v(0) = 0; \quad v'\Big(-\dfrac{1}{4}\Big) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \eta^2 v'' - v - 1 + k = 0 & \text{in} \quad \Big(0, \dfrac{1}{4}\Big) \\ v(0) = 0; \quad v'\Big(\dfrac{1}{4}\Big) = 0. \end{cases}$$

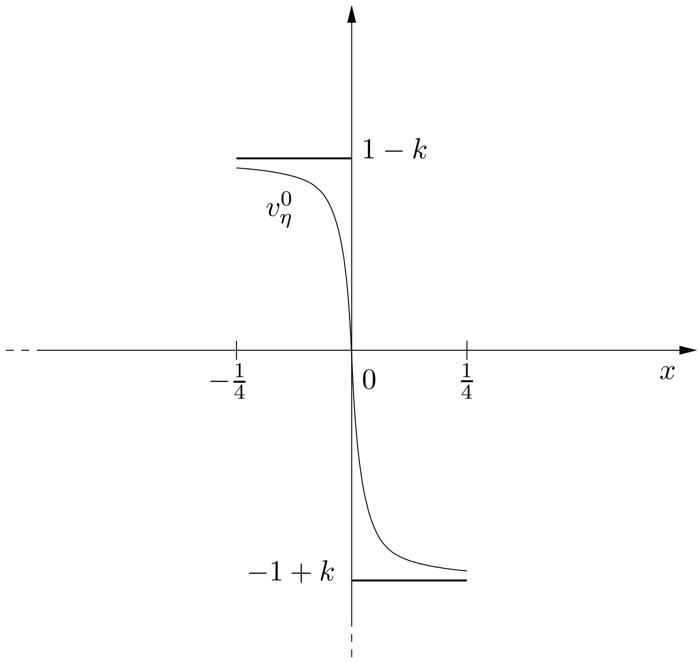

FIGURE 4. The function v_η^0.

Hence, by directly solving the above equations we get

$$v_\eta^0(x) = \begin{cases} 1-k+(k-1)\cosh\left(\dfrac{x}{\eta}\right)+(k-1)\sinh\left(\dfrac{x}{\eta}\right)\tanh\left(\dfrac{1}{4\eta}\right) & \text{if } -\frac{1}{4} \le x \le 0 \\ \\ -1+k-(k-1)\cosh\left(\dfrac{x}{\eta}\right)+(k-1)\sinh\left(\dfrac{x}{\eta}\right)\tanh\left(\dfrac{1}{4\eta}\right) & \text{if } 0 \le x \le \frac{1}{4} \end{cases} \tag{3.11}$$

thus immediately

$$\int_{-\frac{1}{4}}^{\frac{1}{4}} v_\eta^0\,dx = 0.$$

Moreover, a straightforward calculation gives

$$\int_{-\frac{1}{4}}^{\frac{1}{4}} \left(\frac{1}{\eta}W^k(x, v_\eta^0) + \eta({v_\eta^0}')^2\right)dx = C_2^k\,\tanh\left(\frac{1}{4\eta}\right),$$

and finally

$$\varphi_\eta^k(0) \le C_2^k\,\tanh\left(\frac{1}{4\eta}\right) < C_3^k.$$

If $s = 1$, we proceed as above now taking as a test function for $\varphi_\eta^k(1)$, v_η^1 defined by

$$v_\eta^1(x) := \begin{cases} v_\eta^{1,-}(x) & \text{if } \quad -\frac{1}{4} \le x \le 0 \\ v_\eta^{1,+}(x) & \text{if } \quad 0 < x \le \frac{1}{4}, \end{cases}$$

where $v_\eta^{1,-}$, $v_\eta^{1,+}$ are respectively solutions to

$$\min_{\substack{v\in W^{1,2}(-\frac14,0)\\ v(0)=1}} \int_{-\frac14}^{0} \Big(\frac{1}{\eta}(v-1+k)^2+\eta(v')^2\Big)\,dx, \qquad \min_{\substack{v\in W^{1,2}(0,\frac14)\\ v(0)=1}} \int_{0}^{\frac14} \Big(\frac{1}{\eta}(v-1-k)^2+\eta(v')^2\Big)\,dx;$$

or to

$$\begin{cases} \eta^2 v''-v+1-k=0 & \text{in } \Big(-\dfrac14,0\Big)\\ v(0)=1; \quad v'\Big(-\dfrac14\Big)=0 \end{cases} \quad \text{and} \quad \begin{cases} \eta^2 v''-v+1+k=0 & \text{in } \Big(0,\dfrac14\Big)\\ v(0)=1; \quad v'\Big(\dfrac14\Big)=0. \end{cases}$$

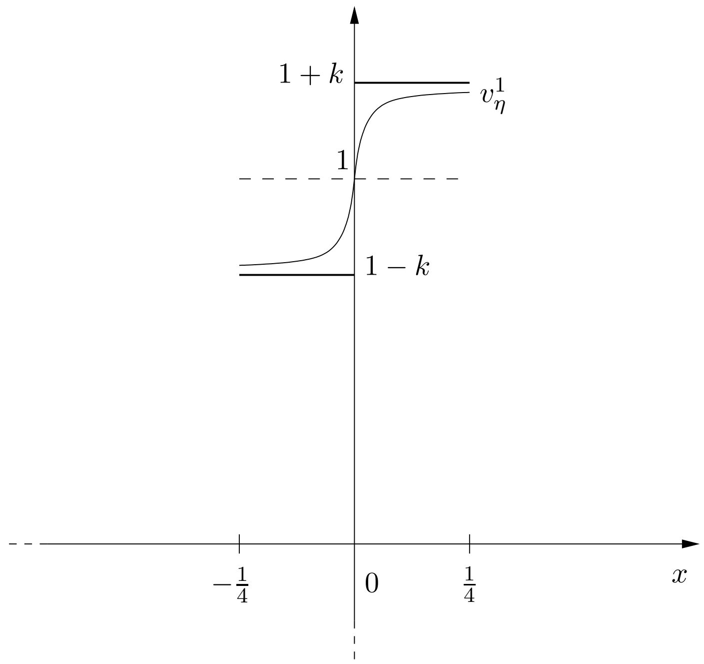

FIGURE 5. The function v_η^1.

Hence, we find

$$v_\eta^1(x)=\begin{cases} 1-k+k\cosh\Big(\dfrac{x}{\eta}\Big)+k\sinh\Big(\dfrac{x}{\eta}\Big)\tanh\Big(\dfrac{1}{4\eta}\Big) & \text{if } -\frac14\le x\le 0\\[2ex] 1+k-k\cosh\Big(\dfrac{x}{\eta}\Big)+k\sinh\Big(\dfrac{x}{\eta}\Big)\tanh\Big(\dfrac{1}{4\eta}\Big) & \text{if } 0\le x\le \frac14, \end{cases} \tag{3.12}$$

and we check that

$$⨍_{-\frac14}^{\frac14} v_\eta^1\,dx=1.$$

Then, a direct computation gives

$$\varphi_\eta^k(1)\le \int_{-\frac14}^{\frac14}\Big(\frac{1}{\eta}W^k(x,v_\eta^1)+\eta({v_\eta^1}')^2\Big)\,dx = C_1^k\,\tanh\Big(\frac{1}{4\eta}\Big)<C_3^k.$$

Notice that if $s = -1$, we simply take $v_\eta^{-1} := v_\eta^1 - 2$.

We now turn to the case $0 < |s| < 1$. Let us start by dealing with $s > 0$. We show that in this case a test function v_η^s can be obtained by suitably modifying and combining v_η^1 and an "optimal" transition between the two zeroes of W_1^k, $1 + k$ and $-1 + k$.

2. Let $|s| \le 1$ and let $v_\eta^s \in W^{1,2}(-\frac{1}{4}, \frac{1}{4})$ be a test function for $\varphi_\eta^k(s)$.

We argue by contradiction supposing the existence of a point $x' \in (-\frac{1}{4}, \frac{1}{4})$ such that, for instance

$$v_\eta^s(x') > M \ge 3(1+k). \tag{3.13}$$

To fix the ideas, and without loss of generality, we may additionally assume that $x' \in (0, \frac{1}{4})$.

Now, appealing to 1. we have

$$\varphi_\eta^k(s) = \int_{-\frac{1}{4}}^{\frac{1}{4}} \Big(\frac{1}{\eta} W^k(x, v_\eta^s) + \eta ({v_\eta^s}')^2\Big)\, dx \le C_3^k$$

and from it we deduce that (the restriction of) v_η^s converges in measure to Z_1^k in $(0, \frac{1}{4})$, as $\eta \to 0$. In fact, for any fixed $\sigma > 0$

$$\Big|\Big\{x \in \Big(0, \frac{1}{4}\Big) : \ \mathrm{dist}\big(v_\eta^s(x), Z_1^k\big) > \sigma\Big\}\Big| \min\{W(\tau) : \ ||\tau| - 1| > \eta\} \le C_3^k\, \eta \to 0 \quad \text{as} \quad \eta \to 0.$$

Then, for sufficiently small $\eta > 0$ there exists $x'' \in (0, \frac{1}{4})$ such that

$$\min\big\{|v_\eta^s(x'') - (1+k)|, |v_\eta^s(x'') - (-1+k)|\big\} \le \sigma.$$

Let us suppose that $|v_\eta^s(x'') - (1+k)| \le \sigma$, hence in particular

$$v_\eta^s(x'') \le 2(1+k), \tag{3.14}$$

having also chosen $\sigma = 1 + k$.

Finally, using the so-called "Modica-Mortola trick" together with (3.13) and (3.14), we get

$$\begin{aligned}\varphi_\eta^k(s) \quad &\ge \quad \int_0^{\frac{1}{4}} \Big(\frac{1}{\eta} W_1^k(v_\eta^s) + \eta({v_\eta^s}')^2\Big)\, dx \ge 2 \int_{v_\eta^s(x'')}^{v_\eta^s(x')} \sqrt{W_1^k} \\ &> \quad \int_{2(1+k)}^{M} (s - 1 - k) = M^2 - 2M(1+k) \ge 3\,(1+k)^2 > C_3^k\end{aligned}$$

and thus the contradiction.

Notice that if v_η^s converges in measure to the constant $-1 + k$, then since $-1 + k < 1 + k$, exactly the same argument can be again applied to get the thesis. □

In all that follows, the letter C will stand for a generic strictly-positive constant which may vary from line to line and expression to expression within the same formula.

Proof of Theorem 3.2. *Step 1: Γ-liminf inequality*
We have to prove that if $u_\varepsilon \rightharpoonup u$ in $L^2(0,1)$, then $F^{k(1)}(u) \le \liminf_{\varepsilon \to 0} F_\varepsilon^{k(1)}(u_\varepsilon)$.

By virtue of the nonnegative character of W^k, we have

$$\begin{aligned} F_\varepsilon^{k(1)}(u_\varepsilon) &= \int_0^1 \left(\frac{\delta}{\varepsilon}W^k\left(\frac{x}{\delta},u_\varepsilon\right)+\varepsilon\delta(u_\varepsilon')^2\right)dx \\ &\geq \sum_{i=1}^{\left[\frac{2}{\delta}-\frac{1}{2}\right]}\int_{(2i-1)\frac{\delta}{4}}^{(2i+1)\frac{\delta}{4}}\left(\frac{\delta}{\varepsilon}W^k\left(\frac{x}{\delta},u_\varepsilon\right)+\varepsilon\delta(u_\varepsilon')^2\right)dx\,, \end{aligned}$$

then, by the change of variable

$$x=\delta t+\frac{\delta}{2}i,$$

and setting

$$v_\varepsilon^i(t):=u_\varepsilon\Big(\delta\Big(t+\frac{i}{2}\Big)\Big),\quad i=1,\dots,\left[\frac{2}{\delta}-\frac{1}{2}\right]$$

we get

$$\begin{aligned} F_\varepsilon^{k(1)}(u_\varepsilon) &\geq \sum_{i=1}^{\left[\frac{2}{\delta}-\frac{1}{2}\right]}\delta\int_{-\frac14}^{\frac14}\left(\frac{\delta}{\varepsilon}W^k\left(t+\frac{i}{2},v_\varepsilon^i\right)+\frac{\varepsilon}{\delta}((v_\varepsilon^i)')^2\right)dt \\ &= \sum_{i\,\text{even}}\delta\int_{-\frac14}^{\frac14}\left(\frac{\delta}{\varepsilon}W^k\left(t,v_\varepsilon^i\right)+\frac{\varepsilon}{\delta}((v_\varepsilon^i)')^2\right)dt \\ &+ \sum_{i\,\text{odd}}\delta\int_{\frac14}^{\frac34}\left(\frac{\delta}{\varepsilon}W^k\left(t,w_\varepsilon^i\right)+\frac{\varepsilon}{\delta}((w_\varepsilon^i)')^2\right)dt, \end{aligned}$$

where

$$w_\varepsilon^i(t):=v_\varepsilon^i\Big(t-\frac12\Big).$$

We now remark that

$$\begin{aligned} &\min\left\{\int_{-\frac14}^{\frac14}\left(\frac{\delta}{\varepsilon}W^k(t,v)+\frac{\varepsilon}{\delta}(v')^2\right)dt:\quad \fint_{-\frac14}^{\frac14}v\,dt=s\right\} \\ &\qquad\qquad=\min\left\{\int_{\frac14}^{\frac34}\left(\frac{\delta}{\varepsilon}W^k(t,v)+\frac{\varepsilon}{\delta}(v')^2\right)dt:\quad \fint_{\frac14}^{\frac34}v\,dt=s\right\}, \end{aligned}$$

as a consequence we find

$$F_\varepsilon^{k(1)}(u_\varepsilon)\geq\sum_{i=1}^{\left[\frac{2}{\delta}-\frac{1}{2}\right]}\delta\min\Big\{\int_{-\frac14}^{\frac14}\left(\frac{\delta}{\varepsilon}W^k(t,v)+\frac{\varepsilon}{\delta}(v')^2\right)dt:\fint_{-\frac14}^{\frac14}v\,dt=\fint_{(2i-1)\frac{\delta}{4}}^{(2i+1)\frac{\delta}{4}}u_\varepsilon\,dt\Big\}. \tag{3.15}$$

Hence, by using the notation introduced in Corollary 3.5, (3.15) becomes

$$F_\varepsilon^{k(1)}(u_\varepsilon)\geq 2\sum_{i=1}^{\left[\frac{2}{\delta}-\frac{1}{2}\right]}\frac{\delta}{2}\varphi^k_{\frac{\varepsilon}{\delta}}\left(\fint_{(2i-1)\frac{\delta}{4}}^{(2i+1)\frac{\delta}{4}}u_\varepsilon\,dt\right)$$

and if we define $\tilde u_\varepsilon:(0,1)\to\mathbb{R}$ as

$$\tilde u_\varepsilon(x):=\sum_{i=1}^{\left[\frac{2}{\delta}-\frac{1}{2}\right]}\left(\fint_{(2i-1)\frac{\delta}{4}}^{(2i+1)\frac{\delta}{4}}u_\varepsilon\,dt\right)\chi_{\left((2i-1)\frac{\delta}{4},(2i+1)\frac{\delta}{4}\right)}(x),$$

we finally have

$$\liminf_{\varepsilon\to 0} F_\varepsilon^{k(1)}(u_\varepsilon) \geq 2\liminf_{\varepsilon\to 0}\int_0^1 \varphi^k_{\frac{\varepsilon}{\delta}}(\tilde{u}_\varepsilon)\,dx,$$

where in the last inequality, we have used the definition of $\tilde{u}_\varepsilon$ and the fact that $\varphi^k_{\frac{\varepsilon}{\delta}}(0) \to C_2^k$, as $\varepsilon \to 0$.

Notice that moreover, $\tilde{u}_\varepsilon \rightharpoonup u$ in $L^2(0,1)$.

Now our goal is to give an estimate from below on the function $\varphi^k_{\frac{\varepsilon}{\delta}}$. To this effect we first consider the case $|s| > 1$. On one hand (see also (1.4)), for every $s \in \mathbb{R}$ we have that

$$\begin{aligned}
\varphi^k_{\frac{\varepsilon}{\delta}}(s) &\geq \inf\left\{\frac{\delta}{\varepsilon}\int_{-\frac14}^{\frac14} W^k(t,v)\,dt : \ -\!\!\!\!\!\!\int_{-\frac14}^{\frac14} v\,dt = s\right\}\\
&= \frac{\delta}{\varepsilon}\min\left\{\frac14 W^{**}(s_1+k) + \frac14 W^{**}(s_2-k) : \ s_1+s_2 = 2s\right\}\\
&= \frac{\delta}{\varepsilon}\frac{W^{**}(s)}{2},
\end{aligned}$$

so in particular

$$\varphi^k_{\frac{\delta}{\varepsilon}}(s) \geq \frac{\varepsilon}{\delta}\frac{(|s|-1)^2}{2} \qquad \forall s \ : \ |s| > 1. \tag{3.16}$$

On the other hand, for any fixed $\eta > 0$ there exist $\sigma, \varepsilon_0 > 0$ such that

$$\varphi^k_{\frac{\varepsilon}{\delta}}(s) \geq C_1^k - \eta^2 \quad \forall s \in (1, 1+\sigma),\ \forall \varepsilon < \varepsilon_0 \tag{3.17}$$

and the above inequality can be proved by means of the following contradiction argument. If (3.17) does not hold true we can find two sequences $s_n \to 1, \varepsilon_n \to 0$ for which

$$\varphi^k_{\frac{\varepsilon_n}{\delta(\varepsilon_n)}}(s_n) < C_1^k - \eta_0^2 \tag{3.18}$$

for every $n \in \mathbb{N}$ and for some $\eta_0 > 0$. Appealing to Corollary 3.5 we can also deduce

$$C_1^k = \varphi^k(1) \leq \liminf_{n\to+\infty} \varphi^k_{\frac{\varepsilon_n}{\delta(\varepsilon_n)}}(s_n),$$

and combining it with (3.18) we find the contradiction. Note that, by symmetry, (3.17) also stands for every $s \in (-1-\sigma, -1)$.

Hence, gathering (3.16) and (3.17) we deduce that for every $\eta > 0$ and for any sufficiently small $\varepsilon > 0$,

$$\varphi^k_{\frac{\varepsilon}{\delta}}(s) \geq (C_1^k - \eta^2) \vee \left(\frac{\delta}{\varepsilon}\frac{(|s|-1)^2}{2}\right) \qquad \forall s \ : \ |s| > 1. \tag{3.19}$$

Now it remains to give an estimate on $\varphi^k_{\frac{\varepsilon}{\delta}}$ for $|s| \leq 1$. To this purpose, for any fixed $\eta > 0$, let us consider the set

$$A_\eta^\varepsilon := \left\{t \in \left(-\frac14, \frac14\right) \ : \ \mathrm{dist}(v_\varepsilon^s(t), Z^k(t)) > \eta\right\},$$

where v_ε^s is a test function for $\varphi^k_{\frac{\varepsilon}{\delta}}(s)$ and $Z^k(t)$ is defined by

$$Z^k(t) := \begin{cases} Z_2^k & \text{if} \quad t \in \left(-\frac14, 0\right) \\ Z_1^k & \text{if} \quad t \in \left(0, \frac14\right). \end{cases}$$

Then, arguing as in the proof of Proposition 3.7-2., we deduce that the measure of A^ε_η tends to zero as $\varepsilon \to 0$. In fact, we have

$$|A^\varepsilon_\eta| \min\{W(\tau) :\ ||\tau| - 1| > \eta\} \leq \frac{\varepsilon}{\delta} C_3^k \to 0\,, \quad \text{as } \varepsilon \to 0.$$

As a consequence, for any sufficiently small $\varepsilon > 0$ we can find $t^- \in (-\frac{1}{4}, 0)$, $t^+ \in (0, \frac{1}{4})$ such that $\operatorname{dist}(v^s_\varepsilon(t^\pm), Z^k(t^\pm)) \leq \eta$.

Let us suppose for a moment that one of the following inequalities holds true

$$|v^s_\varepsilon(t^-) - (-1-k)| \leq \eta, \quad |v^s_\varepsilon(t^+) - (1+k)| \leq \eta, \tag{3.20}$$

assuming for instance the first, we deduce

$$\varphi^k_{\frac{\varepsilon}{\delta}}(s) = \int_{-\frac{1}{4}}^{\frac{1}{4}} \left(\frac{\delta}{\varepsilon} W^k(t, v^s_\varepsilon) + \frac{\varepsilon}{\delta} ({v^s_\varepsilon}')^2 \right) dt \geq C_{W^k}(-1-k+\eta, -1+k-\eta),$$

with $C_{W^k}(\cdot,\cdot)$ as in (3.6); finally

$$\varphi^k_{\frac{\varepsilon}{\delta}}(s) \geq C_1^k - C\eta^2. \tag{3.21}$$

Now our plan is to prove that whenever $4\eta < |s| \leq 1$ at least one of the inequalities in (3.20) is fulfilled. Arguing by contradiction we can find a number $\eta_0 > 0$ and a sequence $\varepsilon_n \to 0$ such that for every $n \in \mathbb{N}$

$$|v^s_{\varepsilon_n}(t) - (-1-k)| > \eta_0 \quad \forall t \in \left(-\frac{1}{4}, 0\right), \quad |v^s_{\varepsilon_n}(t) - (1+k)| > \eta_0 \quad \forall t \in \left(0, \frac{1}{4}\right). \tag{3.22}$$

If we set

$$Z_0^k(t) := \begin{cases} 1-k & \text{if} \quad t \in \left(-\frac{1}{4}, 0\right) \\ -1+k & \text{if} \quad t \in \left(0, \frac{1}{4}\right), \end{cases}$$

in view of (3.22), $A^{\varepsilon_n}_{\eta_0}$ can be rewritten as

$$A^{\varepsilon_n}_{\eta_0} = \left\{ t \in \left(-\frac{1}{4}, \frac{1}{4}\right) :\ \operatorname{dist}(v^s_{\varepsilon_n}(t), Z_0^k(t)) > \eta_0 \right\}$$

and again, for the complement of $A^{\varepsilon_n}_{\eta_0}$ we have

$$\left(A^{\varepsilon_n}_{\eta_0}\right)^c = B^{\varepsilon_n,-}_{\eta_0} \cup B^{\varepsilon_n,+}_{\eta_0} \tag{3.23}$$

where

$$\begin{aligned} B^{\varepsilon_n,-}_{\eta_0} &:= \left\{ t \in \left(-\frac{1}{4}, 0\right) :\ |v^s_{\varepsilon_n}(t) - (1-k)| \leq \eta_0 \right\}, \\ B^{\varepsilon_n,+}_{\eta_0} &:= \left\{ t \in \left(0, \frac{1}{4}\right) :\ |v^s_{\varepsilon_n}(t) - (-1+k)| \leq \eta_0 \right\} \end{aligned} \tag{3.24}$$

and

$$\left|B^{\varepsilon_n,-}_{\eta_0}\right| - \left|B^{\varepsilon_n,+}_{\eta_0}\right| \to 0, \quad \text{as } \ n \to +\infty. \tag{3.25}$$

Without loss of generality, we can suppose $s > 0$, therefore

$$2\eta_0 < \int_{-\frac{1}{4}}^{\frac{1}{4}} v^s_{\varepsilon_n}\, dt = \int_{A^{\varepsilon_n}_{\eta_0}} v^s_{\varepsilon_n}\, dt + \int_{(A^{\varepsilon_n}_{\eta_0})^c} v^s_{\varepsilon_n}\, dt.$$

Now by (3.23), (3.24) and appealing to Proposition 3.7-2., we deduce

$$\begin{aligned} 2\eta_0 &< \int_{A_{\eta_0}^{\varepsilon_n}} v_{\varepsilon_n}^s \, dt + \int_{B_{\eta_0}^{\varepsilon_n,-}} v_{\varepsilon_n}^s \, dt + \int_{B_{\eta_0}^{\varepsilon_n,+}} v_{\varepsilon_n}^s \, dt \\ &\leq M|A_{\eta_0}^{\varepsilon_n}| + (\eta_0 + (1-k))|B_{\eta_0}^{\varepsilon_n,-}| + (\eta_0 + (-1+k))|B_{\eta_0}^{\varepsilon_n,+}| \\ &\leq M|A_{\eta_0}^{\varepsilon_n}| + \frac{\eta_0}{2} + (1-k)(|B_{\eta_0}^{\varepsilon_n,-}| - |B_{\eta_0}^{\varepsilon_n,+}|), \end{aligned}$$

moreover by (3.25), for any sufficiently large n, we have

$$|A_{\eta_0}^{\varepsilon_n}| > \frac{\eta_0}{M}$$

and from it, the contradiction.

Then, for $|s| \leq 4\eta$ it is easy to check that

$$\varphi^k_{\frac{\varepsilon}{\delta}(s)} \geq C_2^k - C\eta^2. \tag{3.26}$$

Finally, combining (3.19), (3.21) and (3.26) we get

$$\varphi^k_{\frac{\varepsilon}{\delta}}(s) \geq \psi^k_{\eta,\varepsilon}(s) = \begin{cases} C_2^k - C\eta^2 & \text{if } \ |s| \leq \eta \\ C_1^k - C\eta^2 & \text{if } \ \eta < |s| \leq 1 \\ (C_1^k - C\eta^2) \vee \left(\frac{\delta}{\varepsilon}\frac{(|s|-1)^2}{2}\right) & \text{if } \ |s| > 1 \end{cases}$$

for every $s \in \mathbb{R}$ and for every $0 < \eta < 1$; hence

$$\liminf_{\varepsilon \to 0} F_\varepsilon^{k(1)}(u_\varepsilon) \geq \liminf_{\varepsilon \to 0} 2 \int_0^1 \psi^k_{\eta,\varepsilon}(\tilde{u}_\varepsilon)\, dx.$$

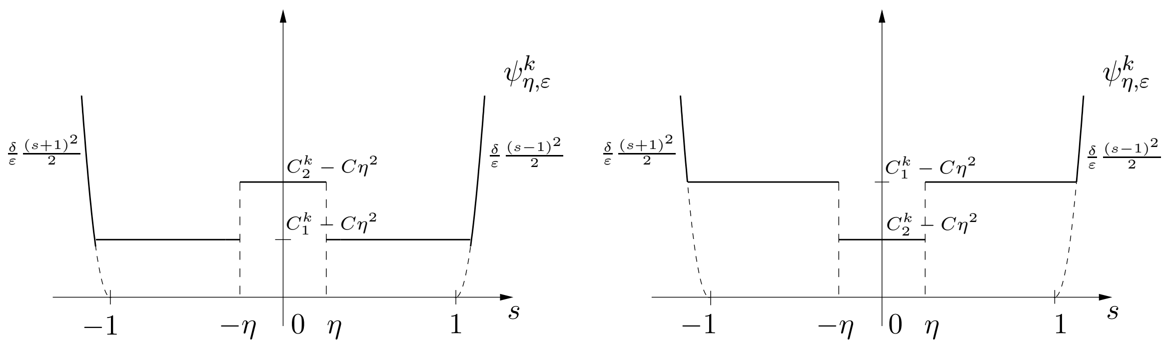

FIGURE 6. The function $\psi^k_{\eta,\varepsilon}$ for $k < \frac{1}{2}$ and $k > \frac{1}{2}$.

To conclude the proof, we note that $\left(\psi^k_{\eta,\varepsilon}\right)$ is increasing for $\varepsilon \to 0$, so in particular

$$\psi^k_{\eta,\varepsilon}(s) \geq \psi^k_{\eta,\varepsilon_0}(s), \quad \forall\, s \in \mathbb{R}\,,\ \forall\, \varepsilon \leq \varepsilon_0.$$

Then

$$\begin{aligned} \liminf_{\varepsilon \to 0} \int_0^1 \psi^k_{\eta,\varepsilon}(\tilde{u}_\varepsilon)\, dx &\geq \liminf_{\varepsilon \to 0} \int_0^1 \psi^k_{\eta,\varepsilon_0}(\tilde{u}_\varepsilon)\, dx \\ &\geq \liminf_{\varepsilon \to 0} \int_0^1 \left(\psi^k_{\eta,\varepsilon_0}\right)^{**}(\tilde{u}_\varepsilon)\, dx \geq \int_0^1 \left(\psi^k_{\eta,\varepsilon_0}\right)^{**}(u)\, dx, \end{aligned}$$

in the last inequality using the fact that $\tilde{u}_\varepsilon \rightharpoonup u$ in $L^2(0,1)$ and the L^2-weak lower semicontinuity of $u :\longrightarrow \int_0^1 \left(\psi^k_{\eta,\varepsilon_0}\right)^{**}(u)\,dx$. Moreover, by the Monotone Convergence Theorem

$$\lim_{\varepsilon_0\to 0}\int_0^1 \left(\psi^k_{\eta,\varepsilon_0}\right)^{**}(u)\,dx = \int_0^1 \lim_{\varepsilon_0\to 0}\left(\psi^k_{\eta,\varepsilon_0}\right)^{**}(u)\,dx = \int_0^1 \left(\psi^k_{\eta}\right)(u)\,dx,$$

where

$$\psi^k_\eta(s) = \begin{cases} C^k_1 - C\eta^2 & \text{if } |s|\le 1 \\ +\infty & \text{otherwise} \end{cases} \qquad \text{if} \quad k \le \frac{1}{2}$$

or

$$\psi^k_\eta(s) = \begin{cases} C^k_2 - C\eta^2 & \text{if } |s|\le \eta \\ \dfrac{C^k_1 - C^k_2}{1-\eta}|s| + C^k_2 - \dfrac{C^k_1 - C^k_2}{1-\eta}\eta - C\eta^2 & \text{if } \eta < |s| \le 1 \\ +\infty & \text{otherwise} \end{cases} \qquad \text{if} \quad k > \frac{1}{2}.$$

Collecting these inequalities we find that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F^{k(1)}_\varepsilon(u) \ge 2\int_0^1 \psi^k_\eta(u)\,dx.$$

and by the arbitrariness of η

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F^{k(1)}_\varepsilon(u) \ge 2\sup_{\eta>0}\int_0^1 \psi^k_\eta(u)\,dx.$$

Hence, again applying the Monotone Convergence Theorem we obtain the desired result for both $k\le\frac{1}{2}$ and $k>\frac{1}{2}$.

Step 2: Γ-limsup inequality
To check the limsup inequality for the Γ-limit, it will suffice to deal with the case of a constant target function $u\equiv c$ $(-1\le c\le 1)$, sice by repeating that construction we can easily deal with the case u piecewise constant and then the general case follows by density.

We start approximating $c = 1$. Fix $\eta > 0$; by (3.2), (3.5) there exist $T_1, T_2 > 0$ and $v_1 \in W^{1,2}(-T_1, T_2)$ such that $v_1(-T_1) = 1 + k$, $v_1(T_2) = 1 - k$ and

$$\int_{-T_1}^0 \left(W^k_1(v_1) + (v_1')^2\right)\,dx + \int_0^{T_2}\left(W^k_2(v_1) + (v_1')^2\right)\,dx \le C^k_1 + \frac{\eta}{2}.$$

Note that it is not restrictive to suppose $T_1 = T_2 =: T$. Then, for instance, as a recovery sequence, we can take

$$u_\varepsilon(x) = \begin{cases} 1+k & \text{if } 0 < x \le \frac{\delta}{4} \\ \\ v^i_{\varepsilon,1}(x) & \text{if } (4i-3)\frac{\delta}{4} < x < (4i+1)\frac{\delta}{4} \quad \text{for} \qquad i = 1, \dots, \left[\frac{1}{\delta} - \frac{1}{4}\right] \\ \\ 1+k & \text{if } \left(4\left[\frac{1}{\delta} - \frac{1}{4}\right] + 1\right)\frac{\delta}{4} \le x < 1 \end{cases}$$

where

$$v^i_{\varepsilon,1}(x) = \begin{cases} 1+k & \text{if } (4i-3)\frac{\delta}{4} < x < (2i-1)\frac{\delta}{2} - \varepsilon T \\ v_1\Big(\dfrac{x-(2i-1)\frac{\delta}{2}}{\varepsilon}\Big) & \text{if } (2i-1)\frac{\delta}{2} - \varepsilon T \le x \le (2i-1)\frac{\delta}{2} + \varepsilon T \\ 1-k & \text{if } (2i-1)\frac{\delta}{2} + \varepsilon T < x < i\delta - \varepsilon T \\ v_1\Big(\dfrac{i\delta - x}{\varepsilon}\Big) & \text{if } i\delta - \varepsilon T \le x \le i\delta + \varepsilon T \\ 1+k & \text{if } i\delta + \varepsilon T < x < (4i+1)\frac{\delta}{4}. \end{cases} \qquad i \in \mathbb{N} \tag{3.27}$$

In fact, recalling that $\varepsilon \ll \delta$ it is easy to check that $u_\varepsilon \rightharpoonup 1$ in $L^2(0,1)$, while

$$\begin{aligned} \limsup_{\varepsilon\to 0} F^{k(1)}_\varepsilon(u_\varepsilon) &= \limsup_{\varepsilon\to 0} \sum_{i=1}^{\left[\frac{1}{\delta}-\frac{1}{4}\right]} \int_{(4i-3)\frac{\delta}{4}}^{(4i+1)\frac{\delta}{4}} \left(\frac{\delta}{\varepsilon} W^k\left(\frac{x}{\delta}, v^i_{\varepsilon,1}\right) + \varepsilon\delta((v^i_{\varepsilon,1})')^2 \right) dx \\ &\le \lim_{\varepsilon\to 0} \left[\frac{1}{\delta} - \frac{1}{4}\right] \delta(2C^k_1 + \eta) = 2C^k_1 + \eta, \quad \forall \eta > 0 \end{aligned}$$

permits to conclude that

$$\limsup_{\varepsilon\to 0} F^{k(1)}_\varepsilon(u_\varepsilon) \le F^{k(1)}(1).$$

Replacing $1 \pm k$ with $-1 \pm k$ and v_1 with its analogous v_{-1}, a similar construction yields $v^i_{\varepsilon,-1}$ and consequently the Γ-limsup for $c \equiv -1$.

If $-1 < c < 1$, it is necessary to make a distinction between the cases $k \le \frac{1}{2}$, $k > \frac{1}{2}$.

Let $k \le \frac{1}{2}$; writing c as a convex combination of 1 and -1, we have

$$c = \frac{c+1}{2} - \frac{1-c}{2}.$$

Now let $(n^\nu_1), (n^\nu_2)$ be two sequences of positive integers such that

$$n^\nu_1, n^\nu_2 \to +\infty \quad \text{and} \quad \frac{n^\nu_1}{n^\nu_2} \to \frac{c+1}{1-c}, \quad \text{as } \nu \to 0. \tag{3.28}$$

With fixed $\nu > 0$, we choose $\varepsilon > 0$ such that $(n_1^\nu + n_2^\nu + 1)\delta \ll 1$. With this choice we consider the $(n_1^\nu + n_2^\nu + 1)\delta$-periodic function U_ε^ν, on $\mathbb{R}^+$, which on $\left(\frac{\delta}{4}, (4(n_1^\nu + n_2^\nu) + 5)\frac{\delta}{4}\right)$ is defined as:

$$U_\varepsilon^\nu(x) = \begin{cases} v_{\varepsilon,1}^i(x) & \text{if } x \in \left((4i-3)\frac{\delta}{4}, (4i+1)\frac{\delta}{4}\right) \text{ for } i = 1, \dots, n_1^\nu \\ \\ w_\varepsilon(x) & \text{if } x \in \left((4n_1^\nu+1)\frac{\delta}{4}, (4n_1^\nu+5)\frac{\delta}{4}\right) \\ \\ v_{\varepsilon,-1}^i(x) & \text{if } x \in \left((4i-3)\frac{\delta}{4}, (4i+1)\frac{\delta}{4}\right) \text{ for } i = n_1^\nu + 2, \dots, n_1^\nu + n_2^\nu \\ \\ \tilde{w}_\varepsilon(x) & \text{if } x \in \left((4(n_1^\nu + n_2^\nu)+1)\frac{\delta}{4}, (4(n_1^\nu+n_2^\nu)+5)\frac{\delta}{4}\right) \end{cases}$$

where $v_{\varepsilon,1}^i$ is as in (3.27) and $v_{\varepsilon,-1}^i$ is its analogous. Moreover w_ε is given by

$$w_\varepsilon(x) = \begin{cases} v_{\varepsilon,1}^{n_1^\nu+1}(x) & \text{if } (4n_1^\nu+1)\frac{\delta}{4} < x \leq (2n_1^\nu+1)\frac{\delta}{2} + \varepsilon T \\ 1-k & \text{if } (2n_1^\nu+1)\frac{\delta}{2} + \varepsilon T < x < (n_1^\nu+1)\frac{\delta}{2} - \varepsilon T' \\ v_0\Big(\dfrac{x-(n_1^\nu+1)\delta}{\varepsilon}\Big) & \text{if } (n_1^\nu+1)\delta - \varepsilon T' \leq x \leq (n_1^\nu+1)\delta + \varepsilon T' \\ -1+k & \text{if } (n_1^\nu+1)\delta + \varepsilon T' < x < (4n_1^\nu+5)\frac{\delta}{4} \end{cases}$$

with $T' > 0$ and $v_0 \in W^{1,2}(-T', T')$ such that $v_0(-T') = 1-k$, $v_0(T') = -1+k$ and

$$\int_{-T'}^0 \left(W_1^k(v_0) + (v_0')^2\right) dx + \int_0^{T'} \left(W_2^k(v_0) + (v_0')^2\right) dx \leq C_2^k + \frac{\eta}{2},$$

while $\tilde{w}_\varepsilon$ is defined as

$$\tilde{w}_\varepsilon(x) = \begin{cases} -1+k & \text{if } (4(n_1^\nu+n_2^\nu)+1)\frac{\delta}{4} < x < (2(n_1^\nu+n_2^\nu)+1)\frac{\delta}{2} - \varepsilon T' \\ v_0\Big(\frac{(2(n_1^\nu+n_2^\nu)+1)\frac{\delta}{2}-x}{\varepsilon}\Big) & \text{if } (2(n_1^\nu+n_2^\nu)+1)\frac{\delta}{2} - \varepsilon T' \leq x \leq (2(n_1^\nu+n_2^\nu)+1)\frac{\delta}{2} + \varepsilon T' \\ 1-k & \text{if } (2(n_1^\nu+n_2^\nu)+1)\frac{\delta}{2} + \varepsilon T' < x < (n_1^\nu+n_2^\nu+1)\delta - \varepsilon T \\ v_{\varepsilon,1}^{n_1^\nu+n_2^\nu+1}(x) & \text{if } (n_1^\nu+n_2^\nu+1)\delta - \varepsilon T \leq x \leq (4(n_1^\nu+n_2^\nu)+5)\frac{\delta}{4}. \end{cases}$$

Taking $u_\varepsilon^\nu := U_\varepsilon^\nu\,|_{(0,1)}$, we have

$$\begin{aligned} \limsup_{\varepsilon\to 0} F_\varepsilon^{k(1)}(u_\varepsilon^\nu) &\leq \lim_{\varepsilon\to 0}((2C_1^k+\eta)(n_1^\nu+n_2^\nu)\delta + (2C_2^k+\eta)\delta)\left[\frac{1}{(n_1^\nu+n_2^\nu+1)\delta}\right] \\ &= (2C_1^k+\eta)\frac{n_1^\nu+n_2^\nu}{n_1^\nu+n_2^\nu+1} + (2C_2^k+\eta)\frac{1}{n_1^\nu+n_2^\nu+1} =: a^{k,\nu} \end{aligned}$$

Moreover,

$$\lim_{\nu\to 0} a^{k,\nu} = 2C_1^k + \eta$$

then a diagonalization argument (cf. [?], Corollary 1.18) permits to find a positive decreasing (as ε decrease) function $\nu = \nu(\varepsilon)$ such that $\nu(\varepsilon) \to 0$ as $\varepsilon \to 0$, for which

$$\limsup_{\varepsilon\to 0} F_\varepsilon^{k(1)}(u_\varepsilon^{\nu(\varepsilon)}) \leq 2C_1^k + \eta.$$

Finally, using (3.28) and the fact that $\varepsilon \ll \delta$ it is easy to check that we also have

$$u_\varepsilon^{\nu(\varepsilon)} \rightharpoonup c \quad \text{in} \;\; L^2(0,1)$$

and hence, the Γ-limsup for $-1 < c < 1$ and $k \leq \frac{1}{2}$.

Let $k > \frac{1}{2}$; now to approximate a constant c, on one hand, it is no more "optimal" to oscillate between $1+k$, $1-k$ and $-1+k$, $-1-k$, because in this case the most convenient transition is the one from $1-k$ to $-1+k$ (see Remark 3.1). While on the other hand, using convenient transitions (following the construction made for $c = 1$) only permits to approximate $c = 0$. Then, for instance, to obtain a recovery sequence for $0 < c < 1$ it is necessary to mix, in the right proportion, oscillation between $1+k$, $1-k$ with those between $1-k$, $-1+k$. In this way, following a procedure which is similar to that of the previous case, but now with

$$\frac{n_1^\nu}{n_2^\nu} \to \frac{c}{1-c} \quad \text{as} \quad \nu \to 0,$$

it is possible to construct a sequence $u_\varepsilon \rightharpoonup c$ in $L^2(0,1)$ such that

$$\begin{aligned}
\limsup_{\varepsilon\to 0} F_\varepsilon^{k(1)}(u_\varepsilon) &\leq \lim_{\varepsilon\to 0} \Big((2C_1^k+\eta)(n_1^{\nu(\varepsilon)}+1)\delta + (2C_2^k+\eta)n_2^{\nu(\varepsilon)}\delta\Big)\left[\frac{1}{(n_1^{\nu(\varepsilon)}+n_2^{\nu(\varepsilon)}+1)\delta}\right] \\
&= c(2C_1^k+\eta) + (1-c)(2C_2^k+\eta) = 2(C_1^k - C_2^k)\,c + 2C_2^k + \eta.
\end{aligned}$$

And this concludes the proof of the Γ-limsup inequality. □

3.3. Second order Γ-limit. In the spirit of studying the asymptotic behavior of the family of functionals $(F_\varepsilon^{k(0)})$, Theorem 3.2 suggests that the characterization of the limit points of sequences of minimizers, as well as the development for the minimum values m_ε^k, can be improved for $k \leq \frac{1}{2}$.

In fact, for $k \leq \frac{1}{2}$, $F^{k(1)} \equiv 2C_1^k$ so that we are again in the condition that the (first order) Γ-limit only provides the information that the weak limit of sequences of minimizers can be any function $v \in L^2(0,1)$ such that $|v| \leq 1$ a.e..

For $k > \frac{1}{2}$, the functional $F^{k(1)}$ admits the unique minimizer $u \equiv 0$. Nonetheless, as we show in Section 3.3.2, the non strict convexity of ψ^k allows as to consider a further scaling and thus another term in the Γ-development, in this case too.

Since each of the two cases $k \leq \frac{1}{2}$, $k > \frac{1}{2}$ needs a peculiar investigation, we discuss the second order asymptotic analysis for $(F_\varepsilon^{k(0)})$ in two different sections. The first one, Section 3.3.1, is devoted to the case $k \leq \frac{1}{2}$, which is also addressed to as the case of *small perturbations*; while the second one, Section 3.3.2, deals with $k > \frac{1}{2}$, the case of *large perturbations.*

3.3.1. $k < \frac{1}{2}$*: small perturbations.* In terms of the asymptotic development for the minimum value m_ε^k, the combined computation of the zero order and the first order Γ-limit gives

$$m_\varepsilon^k = \frac{\varepsilon}{\delta}\,2\,C_1^k + o\Big(\frac{\varepsilon}{\delta}\Big), \quad \text{as } \varepsilon \to 0.$$

Then to further improve the above development, we need to quantify the "small" error $o(\frac{\varepsilon}{\delta})$, and hence to identify the next meaningful scaling that we denote by $\underline{\lambda}_\infty^{(2)}(\varepsilon)$ (not to make confusion with the scaling for $k > \frac{1}{2}$ that we in the sequel denote by $\overline{\lambda}_\infty^{(2)}(\varepsilon)$).

Once $\underline{\lambda}_\infty^{(2)}(\varepsilon)$ is conjectured, we study the Γ-limit of the scaled functionals

$$F_\varepsilon^{k(2)} := \frac{F_\varepsilon^{k(0)} - \dfrac{\varepsilon}{\delta}\, 2\, C_1^k}{\underline{\lambda}_\infty^{(2)}(\varepsilon)}\,.$$

So the next step is trying to guess, by means of a heuristics, what the second meaningful scale $\underline{\lambda}_\infty^{(2)}(\varepsilon)$ is.

To this aim, we first observe that in order to keep $F_\varepsilon^{k(0)} - \frac{\varepsilon}{\delta}\, 2\, C_1^k$ bounded, a minimizing sequence must oscillate (except possibly on a finite number of δ-intervals) between $1+k$, $1-k$ or between $-1+k$, $-1-k$.

Then, we focus on a $\frac{\delta}{2}$-interval, for instance $(\frac{\delta}{4}, \frac{3}{4}\,\delta)$ and we estimate the contribution of $F_\varepsilon^{k(0)} - \frac{\varepsilon}{\delta}\, 2\, C_1^k$ over this interval. We have

$$\begin{aligned}
& \int_{\frac{\delta}{4}}^{\frac{3}{4}\,\delta} \Big(W^k\Big(\frac{x}{\delta}, u\Big) + \varepsilon^2 (u')^2 \Big)\, dx - \varepsilon\, C_1^k \\
= \;& \varepsilon \Big(\int_{\frac{\delta}{4}}^{\frac{3}{4}\,\delta} \Big(\frac{1}{\varepsilon}\, W^k\Big(\frac{x}{\delta}, u\Big) + \varepsilon (u')^2 \Big)\, dx - C_1^k \Big) \\
= \;& \varepsilon \Big(\int_{\frac{1}{4}}^{\frac{1}{2}} \Big(\frac{\delta}{\varepsilon}\, W_1^k(u) + \frac{\varepsilon}{\delta}\, (u')^2 \Big)\, dx + \int_{\frac{1}{2}}^{\frac{3}{4}} \Big(\frac{\delta}{\varepsilon}\, W_2^k(u) + \frac{\varepsilon}{\delta}\, (u')^2 \Big)\, dx - C_1^k \Big).
\end{aligned} \tag{3.29}$$

Then a direct minimization of (3.29) yields

$$\varepsilon\, C_1^k \Big(\tanh\Big(\frac{\delta}{4\varepsilon}\Big) - 1 \Big) = O\Big(\varepsilon\, e^{-\frac{\delta}{2\varepsilon}} \Big), \quad \text{as } \varepsilon \to 0,$$

and it is easy to check that the above minimum value is attained, for instance, by the function v_η^1 (defined by (3.11), Proposition 3.7) with $\eta = \frac{\varepsilon}{\delta}$. Thus, by repeating the previous argument over each $\frac{\delta}{2}$-interval (except possibly a finite number of them) we get a first energy contribution of order $\frac{\varepsilon}{\delta}\, e^{-\frac{\delta}{2\varepsilon}}$.

The energy (3.29) is minimized also by $v_{\frac{\varepsilon}{\delta}}^{-1}$ (*i.e.* by a transition with average -1), hence the total energy of a minimizing sequence may well be the result of a finite number of passages from oscillations with average 1 to oscillations with average -1 and viceversa. Since each of these passages gives an additional contribution of order ε, the total energy contribution of a minimizing sequence turns out to be of order

$$\frac{\varepsilon}{\delta}\, e^{-\frac{\delta}{2\varepsilon}} + \varepsilon.$$

If we have

$$\frac{\varepsilon}{\delta}\, e^{-\frac{\delta}{2\varepsilon}} \gg \varepsilon \quad \Longleftrightarrow \quad e^{-\frac{\delta}{2\varepsilon}} \gg \delta,$$

then $\underline{\lambda}_\infty^{(2)}(\varepsilon) = \frac{\varepsilon}{\delta}\, e^{-\frac{\delta}{2\varepsilon}}$. Loosely speaking, with this choice for the scaling we decide to pay the error that we make "cutting the tails" of the unbounded number of infinite transitions that we are gluing one each other. Thus, in this case we expect to find again a constant Γ-limit which now is given by

$$\lim_{\varepsilon \to 0} \frac{2\, C_1^k \big(\tanh\big(\frac{\delta}{4\varepsilon}\big) - 1 \big)}{e^{-\frac{\delta}{2\varepsilon}}} = -4\, C_1^k.$$

If we have

$$e^{-\frac{\delta}{2\varepsilon}} \ll \delta,$$

then $\underline{\lambda}_\infty^{(2)}(\varepsilon) = \varepsilon$ and this choice penalizes the passages from the oscillations around 1 to those around -1 and viceversa. Therefore, if $\underline{\lambda}_\infty^{(2)}(\varepsilon) = \varepsilon$ we expect that $(F_\varepsilon^{k(2)})$ Γ-converges to a surface energy which penalizes the jumps of the limit configuration, between 1 and -1.

As we are concerned not only with a better development for m_ε^k but also with an improvement in the characterization of the asymptotic behavior of sequences of minimizers, we decide to focus on the case $e^{-\frac{\delta}{2\varepsilon}} \ll \delta$ and hence, on the case

$$\underline{\lambda}_\infty^{(2)}(\varepsilon) = \varepsilon.$$

Then, we look at the scaled functionals

$$\begin{aligned} F_\varepsilon^{k(2)}(u) &= \frac{F_\varepsilon^{k(0)}(u) - \frac{\varepsilon}{\delta} 2C_1^k}{\varepsilon} \\ &= \begin{cases} \displaystyle\int_0^1 \left(\frac{1}{\varepsilon} W^k\left(\frac{x}{\delta}, u\right) + \varepsilon (u')^2\right) dx - \frac{2C_1^k}{\delta} & \text{if } u \in W^{1,2}(0,1) \\ +\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{3.30}$$

We now come to a rigorous justification of what has been only heuristically conjectured.

First we want to prove that the uniform boundedness of $F_\varepsilon^{k(2)}(u_\varepsilon)$ implies in the limit ($\varepsilon \to 0$) both the constraint $u \in \{\pm 1\}$ and that u is piecewise constant.

Lemma 3.8. *If $\sup_\varepsilon F_\varepsilon^{k(2)}(u_\varepsilon) < +\infty$ then, up to an extraction, (u_ε) converges to some function $u \in BV((0,1);\{\pm 1\})$ with respect to the weak L^2-convergence.*

Proof. Let u^-, u^+ be the 1-periodic functions on $\mathbb{R}^+$, which on $(0,1)$ are defined as

$$u^-(t) := \begin{cases} -1+k & \text{if} \quad t \in \left(0, \frac{1}{2}\right) \\ -1-k & \text{if} \quad t \in \left(\frac{1}{2}, 1\right) \end{cases} \qquad u^+(t) := \begin{cases} 1+k & \text{if} \quad t \in \left(0, \frac{1}{2}\right) \\ 1-k & \text{if} \quad t \in \left(\frac{1}{2}, 1\right). \end{cases} \tag{3.31}$$

With fixed $\varepsilon > 0$, we partition $[0,1]$ into subintervals I_i^δ, $i = 1, \dots, \left[\frac{1}{\delta}\right]$ of length δ (except possibly the last of length less than δ). Let u_ε be such that $\sup_\varepsilon F_\varepsilon^{k(2)}(u_\varepsilon) < +\infty$ and set $u_\delta^\pm(x) := u^\pm\left(\frac{x}{\delta}\right)$. The first step of the proof consists in showing that for any fixed $\eta > 0$, if $\mathcal{I}_\eta^\delta$ is the set of all the indices i in $\left\{1, \dots, \left[\frac{1}{\delta}\right]\right\}$ such that

$$\left(\fint_{I_i^\delta} \left|u_\varepsilon - u_\delta^-\right| dx\right) \wedge \left(\fint_{I_i^\delta} \left|u_\varepsilon - u_\delta^+\right| dx\right) \leq \eta\,, \tag{3.32}$$

then

$$\lim_{\varepsilon \to 0} \delta \#(\mathcal{I}_\eta^\delta) = 1. \tag{3.33}$$

In other words, we are saying that for every $\eta > 0$, (3.32) is satisfied on a "large" number of intervals I_i^δ (provided that ε is sufficiently small). In order to prove (3.33), we give an estimate on the cardinality of the family of indices i for which

$$\left(\fint_{I_i^\delta} \left|u_\varepsilon - u_\delta^-\right| dx\right) \wedge \left(\fint_{I_i^\delta} \left|u_\varepsilon - u_\delta^+\right| dx\right) > \eta.$$

Let us call $\mathcal{J}_\eta^\delta$ such a family. Before starting, we point out that the following statement

$$\text{there exists } M > 0 \text{ such that } |u_\varepsilon(x)| \le M, \ \forall\, x \in I_i^\delta \tag{3.34}$$

holds true, with the same constant M (*e.g.* $M = 2$), except for at most a bounded number of indices i. In fact, arguing as in the proof of Proposition 3.7-2., the above statement can be easily deduced by the uniform boundedness of $F_\varepsilon^{k(2)}(u_\varepsilon)$. So from now on, we focus our attention only on intervals I_i^δ in which (3.34) is satisfied.

If $i \in \mathcal{J}_\eta^\delta$ we have that

$$\begin{aligned}
\eta &< \fint_{I_i^\delta} \left|u_\varepsilon - u_\delta^+\right| dx \\
&= \frac{1}{\delta}\int_{\left\{x\in I_i^\delta:\, |u_\varepsilon - u_\delta^+| \le \frac{\eta}{2}\right\}} \left|u_\varepsilon - u_\delta^+\right| dx \\
&+ \frac{1}{\delta}\int_{\left\{x\in I_i^\delta:\, |u_\varepsilon - u_\delta^+| > \frac{\eta}{2}\right\}} \left|u_\varepsilon - u_\delta^+\right| dx \\
&\le \frac{\eta}{2} + \frac{C}{\delta}\left|\left\{x \in I_i^\delta :\ \left|u_\varepsilon - u_\delta^+\right| > \frac{\eta}{2}\right\}\right|
\end{aligned}$$

hence

$$\left|\left\{x \in I_i^\delta :\ \left|u_\varepsilon - u_\delta^+\right| > \frac{\eta}{2}\right\}\right| > C\delta.$$

Notice that the same conclusion also stands replacing u_δ^+ with u_δ^-. As a consequence

$$\int_{I_i^\delta} W^k\left(\frac{x}{\delta}, u_\varepsilon\right) dx > C\delta, \quad \text{for every } \ i \in \mathcal{J}_\eta^\delta$$

and this implies

$$F_\varepsilon^{k(0)}(u_\varepsilon) \ge \#(\mathcal{J}_\eta^\delta) C\delta. \tag{3.35}$$

By hypothesis $F_\varepsilon^{k(2)}(u_\varepsilon) \le C$, therefore

$$F_\varepsilon^{k(0)}(u_\varepsilon) \le \varepsilon\, C + \frac{\varepsilon}{\delta}\, 2C_1^k = O\left(\frac{\varepsilon}{\delta}\right) \quad \text{as } \ \varepsilon \to 0 \tag{3.36}$$

then, gathering (3.35) and (3.36) we get

$$\delta \#(\mathcal{J}_\eta^\delta) \to 0 \quad \text{as } \varepsilon \to 0$$

and hence the desired result.

Let N_ε be the overall number of transitions of u_ε between $1 + k \pm \eta$ and $-1 - k \pm \eta$; $1 + k \pm \eta$ and $-1 + k \pm \eta$; $1 - k \pm \eta$ and $-1 - k \pm \eta$; $1 - k \pm \eta$ and $-1 + k \pm \eta$. From now on we refer to these transitions as the "expensive" transitions. To conclude the proof we notice that the most convenient of such transitions is the one from $-1 + k + \eta$ to $1 - k - \eta$ and, in terms of $F_\varepsilon^{k(0)}(u_\varepsilon)$, it costs at least $\varepsilon(C_2^k - C\eta^2)$. Then, recalling that $C_2^k > C_1^k$, (for η small) from the uniform boundedness of $F_\varepsilon^{k(2)}(u_\varepsilon)$ we can deduce that $N_\varepsilon \le \overline{N}$, for some $\overline{N} \in \mathbb{N}$. As a consequence, (up to an extraction) u_ε makes a number of "expensive" transitions which is actually independent of ε; we call such a number N.

Let $S_\varepsilon = \{t_1^\varepsilon, \dots, t_{N-1}^\varepsilon\}$ (with $t_n^\varepsilon < t_{n+1}^\varepsilon$, $n = 1, \dots, N-2$) be a set of points dividing $(0,1)$ into N subintervals each containing only one expensive transition for u_ε. Up to eventual, further extractions we can suppose that

$$t_n^\varepsilon \to t_n \quad \text{as} \;\; \varepsilon \to 0, \quad \text{for } n = 1, \dots, N-1.$$

Then, for fixed $\sigma > 0$, if we consider the N intervals

$$J_\sigma^n = (t_n + \sigma, t_{n+1} - \sigma), \qquad n = 0, \dots, N-1 \quad (\text{with } t_0 = 0,\, t_N = 1)$$

we have that

$$J_\sigma^n \cap S_\varepsilon = \varnothing, \tag{3.37}$$

for sufficiently small ε and for every $n = 0, \dots, N-1$.

By virtue of (3.37), applying to J_σ^n the result established in the first part of the proof, we have that, for instance,

$$\limsup_{\varepsilon \to 0} \int_{J_\sigma^n} \left| u_\varepsilon - u_\delta^+ \right| \, dx \le C\eta. \tag{3.38}$$

On the other hand, by weak compactness we have $u_\varepsilon \rightharpoonup u$ in $L^2(J_\sigma^n)$, while from (3.31) $u_\delta^+ \rightharpoonup 1$ in $L^2(J_\sigma^n)$; thus by the weak lower semicontinuity of the L^1-norm we deduce

$$\int_{J_\sigma^n} |u - 1| \, dx \le \liminf_{\varepsilon \to 0} \int_{J_\sigma^n} \left| u_\varepsilon - u_\delta^+ \right| \, dx,$$

and combining it with (3.38) we find

$$\int_{J_\sigma^n} |u - 1| \, dx \le C\eta \quad \forall\, \eta, \sigma > 0.$$

Finally by the arbitrariness of η and σ it follows that $u \equiv 1$ on $J^n = (t_n, t_{n+1})$. Thus by repeating the above argument on all intervals of J^n $(n = 0, \dots, N-1)$, which determine a partition of $[0,1]$, we get the thesis. $\square$

We have the following Γ-convergence result.

Theorem 3.9. *Let δ be such that $\delta \gg e^{-\frac{\delta}{2\varepsilon}}$ and $\frac{1}{\delta} \in \mathbb{N}$. The family of functionals $F_\varepsilon^{k(2)}$ defined in (3.30) Γ-converges with respect to the weak L^2-convergence to the functional defined on $L^2(0,1)$ by where*

$$F^{k(2)}(u) = \begin{cases} (C_2^k - C_1^k)\#(S(u)) - C_1^k & \text{if } u \in BV((0,1); \{\pm 1\}) \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. *Step 1: Γ-liminf inequality*
We have to prove that if $u_\varepsilon \rightharpoonup u$ in $L^2(0,1)$ and $\sup_\varepsilon F_\varepsilon^{k(2)}(u_\varepsilon) < +\infty$, then $F^{k(2)}(u) \le \liminf_{\varepsilon \to 0} F_\varepsilon^{k(2)}(u_\varepsilon)$.

By Lemma 3.8 we already know that $u \in BV((0,1); \{\pm 1\})$; let us set $N := \#(S(u))$. For fixed $\varepsilon > 0$, we consider the partition of the interval $\left[\frac{\delta}{4}, 1 - \frac{\delta}{4}\right]$ into subintervals $I_i^\delta :=$

$\big[(2i-1)\frac{\delta}{4},(2i+1)\frac{\delta}{4}\big]$ with $i=1,\ldots,\frac{2}{\delta}-1$ and we rewrite $F_\varepsilon^{k(2)}(u_\varepsilon)$ as

$$F_\varepsilon^{k(2)}(u_\varepsilon)=\int_0^{\frac{\delta}{4}}\left(\frac{1}{\varepsilon}W_1^k(u_\varepsilon)+\varepsilon(u_\varepsilon')^2\right)dx+\sum_{i=1}^{\frac{2}{\delta}-1}\left(\frac{1}{\varepsilon}F_\varepsilon^{k(0)}(u_\varepsilon;I_i^\delta)-C_1^k\right)-C_1^k$$
$$+\int_{1-\frac{\delta}{4}}^1\left(\frac{1}{\varepsilon}W_2^k(u_\varepsilon)+\varepsilon(u_\varepsilon')^2\right)dx$$

where

$$F_\varepsilon^{k(0)}(u_\varepsilon;I_i^\delta):=\int_{(2i-1)\frac{\delta}{4}}^{(2i+1)\frac{\delta}{4}}\left(W^k\left(\frac{x}{\delta},u_\varepsilon\right)+\varepsilon^2(u_\varepsilon')^2\right)dx.$$

By a straightforward calculation we find that

$$\min_{v\in W^{1,2}(I_i^\delta)}\left(\frac{1}{\varepsilon}F_\varepsilon^{k(0)}(v;I_i^\delta)-C_1^k\right)=C_1^k\Big(\tanh\left(\frac{\delta}{4\varepsilon}\right)-1\Big)=O(e^{-\frac{\delta}{2\varepsilon}})\quad\text{as }\ \varepsilon\to 0,$$

for every $i=1,\ldots,\frac{2}{\delta}-1$ and the minimum is attained at

$$u_{\varepsilon,1}^i(x)=\begin{cases}v_\varepsilon^1\Big(\dfrac{i}{2}-\dfrac{x}{\delta}\Big) & \text{if } i \text{ is odd}\\[2ex] v_\varepsilon^1\Big(\dfrac{x}{\delta}-\dfrac{i}{2}\Big) & \text{if } i \text{ is even}\end{cases}\qquad i=1,\ldots,\frac{2}{\delta}-1,\tag{3.39}$$

where v_ε^1 is as in (3.12) with $\eta=\varepsilon$.

If $N=0$, since

$$F_\varepsilon^{k(2)}(u_\varepsilon)\geq\sum_{i=1}^{\frac{2}{\delta}-1}\left(\frac{1}{\varepsilon}F_\varepsilon^{k(0)}(u_\varepsilon;I_i^\delta)-C_1^k\right)-C_1^k\tag{3.40}$$

we then obtain the thesis simply taking the minimum of each term on the right hand side of (3.40) and recalling that by hypothesis

$$\lim_{\varepsilon\to 0}\frac{e^{-\frac{\delta}{2\varepsilon}}}{\delta}=0.$$

If $N>0$, let N_ε be as in Lemma 3.8, then, as already observed, N_ε is bounded and moreover

$$\liminf_{\varepsilon\to 0}N_\varepsilon\geq N.\tag{3.41}$$

To get the liminf inequality for the Γ-limit we need a lower bound for the energy of the expensive transitions. Then we first give an estimate on the measure of the set where a transition between two of the zeroes of W^k may occur. Let η be a positive number and set

$$J_i^\delta:=\left\{t\in I_i^\delta\,:\,\operatorname{dist}(u_\varepsilon,Z_i^{k,\delta}(t))>\eta\right\},$$

where

$$Z_i^{k,\delta}(t):=\begin{cases}Z_1^k & \text{if } t\in\big((2i-1)\frac{\delta}{4},i\frac{\delta}{2}\big)\\[2ex] Z_2^k & \text{if } t\in\big(i\frac{\delta}{2},(2i+1)\frac{\delta}{4}\big)\end{cases}\qquad\text{if } i \text{ is odd,}$$

while

$$Z_i^{k,\delta}(t) := \begin{cases} Z_2^k & \text{if } t \in \left((2i-1)\frac{\delta}{4}, i\frac{\delta}{2}\right) \\ \\ Z_1^k & \text{if } t \in \left(i\frac{\delta}{2}, (2i+1)\frac{\delta}{4}\right) \end{cases} \qquad \text{if } i \text{ is even.}$$

We have

$$\frac{1}{\varepsilon} F_\varepsilon^{k(0)}(u_\varepsilon; I_i^\delta) \geq \frac{1}{\varepsilon} F_\varepsilon^{k(0)}(u_\varepsilon; J_i^\delta) \geq C\eta^2 \frac{|J_i^\delta|}{\varepsilon}$$

and from $\sup_\varepsilon F_\varepsilon^{k(2)}(u_\varepsilon) < +\infty$ we deduce that, for every i, $|J_i^\delta| = O(\varepsilon)$ as ε tends to zero. Hence we can conclude that an expensive transition may only be of two different types.

Type 1: the transition entirely occurs in an interval $I_{i_0}^\delta$ for some i_0; in this case we have

$$\frac{1}{\varepsilon} F_\varepsilon^{k(0)}(u_\varepsilon; I_{i_0}^\delta) \geq C_{W^k}(1-k-\eta, -1+k+\eta) \geq C_2^k - C\eta^2. \tag{3.42}$$

Type 2: the transition occurs between two adjacent intervals $I_{i_0}^\delta, I_{i_0+1}^\delta$ for some i_0; in this case we have

$$\begin{aligned} & \frac{1}{\varepsilon} F_\varepsilon^{k(0)}(u_\varepsilon; I_{i_0}^\delta) + \frac{1}{\varepsilon} F_\varepsilon^{k(0)}(u_\varepsilon; I_{i_0+1}^\delta) \\ \geq \;& C_{W_1^k}(1+k-\eta, -1+k+\eta) \Big(= C_{W_2^k}(1-k-\eta, -1-k+\eta)\Big) \\ \geq \;& 2 - C\eta^2. \end{aligned} \tag{3.43}$$

So if we call N_ε^j $(j = 1, 2)$ the number of the expensive transitions of type j, then $N_\varepsilon = N_\varepsilon^1 + N_\varepsilon^2$. By combining (3.42) and (3.43) we find that (at least)

$$\begin{aligned} F_\varepsilon^{k(2)}(u_\varepsilon) \;&\geq\; \left(\frac{2}{\delta} - 1 - N_\varepsilon^1 - 2N_\varepsilon^2\right) C_1^k \Big(\tanh\left(\frac{\delta}{4\varepsilon}\right) - 1\Big) \\ &+\; N_\varepsilon^1 (C_2^k - C_1^k - C\eta^2) + N_\varepsilon^2 (2 - 2C_1^k - C\eta^2) - C_1^k \\ &\geq\; \frac{2}{\delta} C_1^k \Big(\tanh\left(\frac{\delta}{4\varepsilon}\right) - 1\Big) + N_\varepsilon (C_2^k - C_1^k - C\eta^2) - C_1^k \end{aligned}$$

in the last inequality using the fact that $2 \geq C_1^k + C_2^k$. Finally, passing to the liminf, in view of (3.41) we get

$$\liminf_{\varepsilon \to 0} F_\varepsilon^{k(2)}(u_\varepsilon) \geq N(C_2^k - C_1^k - C\eta^2) - C_1^k, \quad \forall\, \eta > 0$$

and thus letting η go to zero, the Γ-liminf inequality.

Step 2: Γ-limsup inequality

Let $x_0 \in (0, 1)$, to check the limsup inequality for the Γ-limit, it will suffice to deal with the case

$$u(x) = \begin{cases} -1 & \text{if } \; x < x_0 \\ 1 & \text{if } \; x \geq x_0. \end{cases}$$

Let $u^i_{\varepsilon,1}$ be as in (3.39) and set $u^i_{\varepsilon,-1} := u^i_{\varepsilon,1} - 2$ for $i = 1, \dots, \frac{2}{\delta} - 1$. As a recovery sequence we can take

$$u_\varepsilon(x) = \begin{cases} u^1_{\varepsilon,1}\Big(\frac{\delta}{4}\Big) & \text{if } x \in (0, \frac{\delta}{4}) \\[2ex] u^i_{\varepsilon,1}(x) & \text{if } x \in \big((2i-1)\frac{\delta}{4}, (2i+1)\frac{\delta}{4}\big) \ \text{ for } i = 1, \dots, 2\left[\frac{x_0}{\delta}\right] - 2 \\[2ex] \hat{w}_\varepsilon(x) & \text{if } x \in \big((4\left[\frac{x_0}{\delta}\right] - 3)\frac{\delta}{4}, (4\left[\frac{x_0}{\delta}\right] + 3)\frac{\delta}{4}\big) \\[2ex] u^i_{\varepsilon,-1}(x) & \text{if } x \in \big((2i-1)\frac{\delta}{4}, (2i+1)\frac{\delta}{4}\big) \ \text{ for } i = 2\left[\frac{x_0}{\delta}\right] + 2, \dots, \frac{2}{\delta} - 1 \\[2ex] u^{\frac{2}{\delta}-1}_{\varepsilon,-1}\Big(1 - \frac{\delta}{4}\Big) & \text{if } x \in (1 - \frac{\delta}{4}, 1) \end{cases}$$

with

$$\hat{w}_\varepsilon(x) = \begin{cases} u^{2\left[\frac{x_0}{\delta}\right]-1}_{\varepsilon,1} & \text{if } (4\left[\frac{x_0}{\delta}\right] - 3)\frac{\delta}{4} < x \leq (4\left[\frac{x_0}{\delta}\right] - 1)\frac{\delta}{4} - \varepsilon \\[2ex] l_\varepsilon(x) & \text{if } (4\left[\frac{x_0}{\delta}\right] - 1)\frac{\delta}{4} - \varepsilon < x < (4\left[\frac{x_0}{\delta}\right] - 1)\frac{\delta}{4} + \varepsilon \\[2ex] v^0_\varepsilon\Big(\frac{x}{\delta} - \left[\frac{x_0}{\delta}\right]\Big) & \text{if } (4\left[\frac{x_0}{\delta}\right] - 1)\frac{\delta}{4} + \varepsilon \leq (4\left[\frac{x_0}{\delta}\right] + 1)\frac{\delta}{4} - \varepsilon \\[2ex] l_\varepsilon\Big(x - \frac{\delta}{2}\Big) - 2 & \text{if } (4\left[\frac{x_0}{\delta}\right] + 1)\frac{\delta}{4} - \varepsilon < x \leq (4\left[\frac{x_0}{\delta}\right] + 1)\frac{\delta}{4} + \varepsilon \\[2ex] u^{2\left[\frac{x_0}{\delta}\right]+1}_{\varepsilon,-1} & \text{if } (4\left[\frac{x_0}{\delta}\right] + 1)\frac{\delta}{4} + \varepsilon < x < (4\left[\frac{x_0}{\delta}\right] + 3)\frac{\delta}{4} \end{cases}$$

where v^0_ε, v^1_ε are as in (3.11) and (3.12) respectively and l_ε is the linear function defined by

$$l_\varepsilon(x) := \frac{v^0_\varepsilon\Big(\frac{\varepsilon}{\delta} - \frac{1}{4}\Big) - v^1_\varepsilon\Big(\frac{\varepsilon}{\delta} - \frac{1}{4}\Big)}{2\varepsilon}\Big(x - \Big(4\left[\frac{x_0}{\delta}\right] - 1\Big)\frac{\delta}{4} + \varepsilon\Big) + v^0_\varepsilon\Big(\frac{\varepsilon}{\delta} - \frac{1}{4}\Big).$$

In fact it is easy to check that $u_\varepsilon \rightharpoonup u$ in $L^2(0,1)$ while

$$\begin{aligned} \limsup_{\varepsilon\to0} F^{k(2)}_\varepsilon(u_\varepsilon) &= \limsup_{\varepsilon\to0}\left(\int_{\frac{\delta}{4}}^{1-\frac{\delta}{4}}\left(\frac{1}{\varepsilon}W^k\left(\frac{x}{\delta}, u_\varepsilon\right) + \varepsilon(u'_\varepsilon)^2\right)dx - \frac{2C^k_1}{\delta}\right) \\ &\leq \limsup_{\varepsilon\to0}\left(\Big(\frac{2}{\delta} - 4\Big)C^k_1 \tanh\left(\frac{\delta}{4\varepsilon}\right) + 2C^k_1 \tanh\left(\frac{\delta}{4\varepsilon}\right) + C^k_2 \tanh\left(\frac{\delta}{4\varepsilon}\right) - \frac{2C^k_1}{\delta}\right) \\ &= (C^k_2 - C^k_1) - C^k_1 = F^{k(2)}(u) \end{aligned}$$

and this completes the proof. □

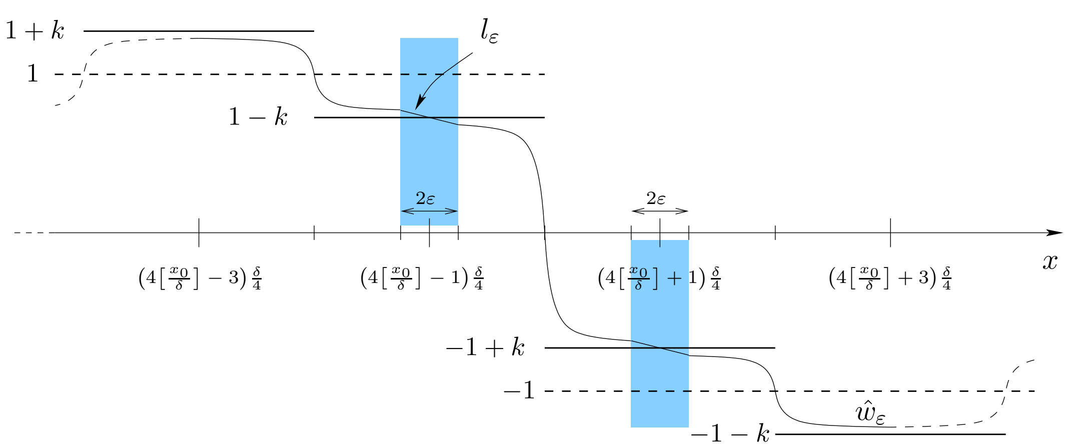

FIGURE 7. The joining transition $\hat{w}_\varepsilon$.

The Γ-convergence results stated in Theorem 1.1, Theorem 3.2 and Theorem 3.9 are (formally) summarized by the Γ-development

$$F_\varepsilon^{k(0)}(u) = \int_0^1 W^{**}(u)\,dx + \frac{\varepsilon}{\delta}2C_1^k + \varepsilon\big((C_2^k - C_1^k)\#S(u) - C_1^k\big) - \frac{\varepsilon}{\delta}e^{-\frac{\delta}{2\varepsilon}}\,4C_1^k + o\Big(\frac{\varepsilon}{\delta}e^{-\frac{\varepsilon}{2\delta}}\Big). \quad (3.44)$$

3.3.2. $k > \frac{1}{2}$*: large perturbations.* For $k > \frac{1}{2}$ Theorem 3.2 asserts that $F_\varepsilon^{k(1)} \xrightarrow{\Gamma} F^{k(1)}$ with

$$F^{k(1)} = \int_0^1 \psi^k(u)\,dx$$

where $\psi^k(s) = 2(C_1^k - C_2^k)|s| + 2C_2^k$, for every $|s| \le 1$. Then

$$\min_{|s|\le 1} \psi^k(s) = \psi^k(0) = 2C_2^k$$

and $F^{k(1)}$ admits the unique minimizer $u \equiv 0$. Nevertheless, as we will show, the nonstrict convexity of ψ^k allows as to consider a further scaling and consequently to recover some more information on sequences minimizing $F_\varepsilon^{k(0)}$ also in the case of large perturbations (*i.e.* for $k > \frac{1}{2}$).

Let us suppose that we want to study the limit behavior only of those minimizing sequences satisfying

$$\int_0^1 v_\varepsilon = d \quad (3.45)$$

with $d \neq 0$, for instance let us fix $d \in (0,1)$.

REMARK 3.10. The Γ-convergence result stated in Theorem 3.2 preserves the integral constraint (3.45).

Moreover if we consider the family of integral functionals given by

$$\mathcal{F}_\varepsilon^{k(1)}(u) := F_\varepsilon^{k(1)}(u) + \int_0^1 l(u)\,dx \quad (3.46)$$

where l is a linear function, by virtue of the stability of Γ-convergence under continuous perturbations, we have that (3.46) Γ-converges to

$$\mathcal{F}^{k(1)}(u) = F^{k(1)}(u) + \int_0^1 l(u)\, dx$$

for any $u \in L^2(0,1)$ such that $|u| \leq 1$ a.e. and satisfying the integral constraint (3.45). But actually $\mathcal{F}_\varepsilon^{k(1)}$ differs from $F_\varepsilon^{k(1)}$ by a constant so information on minimizing sequence of $F_\varepsilon^{k(1)}$ (satisfying (3.45)) can be recovered from information on those minimizing $\mathcal{F}_\varepsilon^{k(1)}$. Now because of the nonstrict convexity of ψ^k, it is possible to choose the function l in a way such that

$$\psi^k(s) + l(s)$$

attains its minimum at more than one point. Then for instance, if we set

$$l(s) = -r^k(s) := -2(C_1^k - C_2^k)s + 2C_2^k$$

we have that

$$\psi^k(s) - r^k(s) \geq 0 \quad \forall |s| \leq 1 \quad \text{and} \quad \psi^k(s) - r^k(s) = 0 \quad \forall 0 < s < 1$$

and this means that $\mathcal{F}_\varepsilon^{k(1)}$ Γ-converges to a functional having many minimizers so now it becomes natural to look for a meaningful scaling for (3.46).

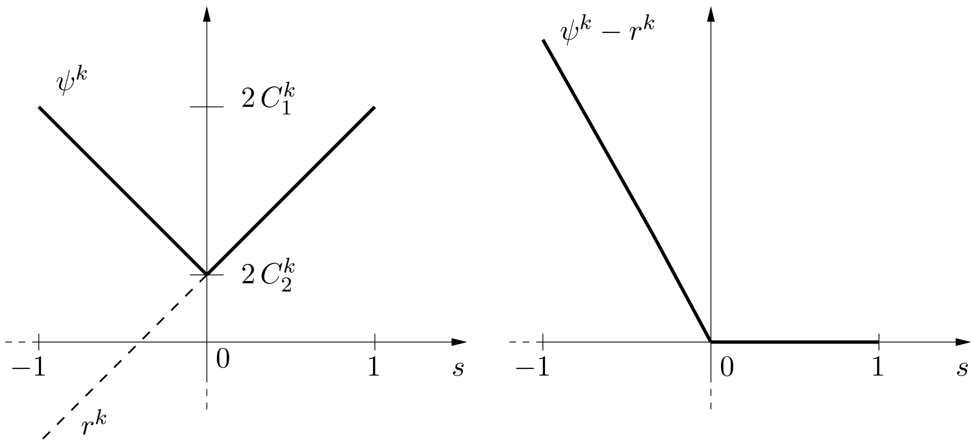

FIGURE 8. The functions ψ^k and $\psi^k - r^k$.

LEMMA 3.11. *Let $L > 0$, $u \in W^{1,2}(0, L)$ and set $C_\varepsilon := \int_0^L (u^2 + \varepsilon^2 (u')^2)\, dx$, then*

$$||u||_\infty \leq \sqrt{\frac{C_\varepsilon}{L} + \frac{C_\varepsilon}{\varepsilon}}.$$

PROOF. We start noticing that

$$|u(x)| \leq \sqrt{\frac{C_\varepsilon}{L}} \qquad \text{for some } x \in (0, L). \tag{3.47}$$

In fact if $|u(x)| > \sqrt{\frac{C_\varepsilon}{L}}$ for every $x \in (0, L)$, from

$$C_\varepsilon \geq \int_0^L u^2\,dx > \frac{C_\varepsilon}{L}\,L = C_\varepsilon$$

we find a contradiction. If moreover $|u(x)| < \sqrt{\frac{C_\varepsilon}{L}}$ for any $x \in (0, L)$, then $||u||_\infty \leq \sqrt{\frac{C_\varepsilon}{L}}$ and by the positivity of $C_\varepsilon/\varepsilon$ we get the thesis.

Then to complete the proof, we now assume that $|u(x)| \geq \sqrt{\frac{C_\varepsilon}{L}}$ for some $x \in (0, L)$. This combined with (3.47) and in view of the continuity of u, implies the existence of a point $x' \in (0, L)$ for which

$$|u(x')| = \sqrt{\frac{C_\varepsilon}{L}}.$$

Starting by the existence of such a point we want to prove the thesis.

We argue by contradiction supposing that $||u||_\infty > \sqrt{\frac{C_\varepsilon}{L} + \frac{C_\varepsilon}{\varepsilon}}$. Then for instance, we suppose that there exists $x'' \in (0, L)$ such that

$$u(x'') > \sqrt{\frac{C_\varepsilon}{L} + \frac{C_\varepsilon}{\varepsilon}}.$$

Hence a direct application of the Modica-Mortola trick gives

$$\begin{aligned} C_\varepsilon &\geq \left|\int_{x'}^{x''} (u^2 + \varepsilon^2 (u')^2)\,dx\right| \geq \varepsilon \int_{(\pm)\sqrt{\frac{C_\varepsilon}{L}}}^{u(x'')} 2|s| > \varepsilon \int_{(\pm)\sqrt{\frac{C_\varepsilon}{L}}}^{\sqrt{\frac{C_\varepsilon}{L}+\frac{C_\varepsilon}{\varepsilon}}} 2|s| \\ &\geq \varepsilon \left(\frac{C_\varepsilon}{L} + \frac{C_\varepsilon}{\varepsilon} - \frac{C_\varepsilon}{L}\right) = C_\varepsilon \end{aligned}$$

and thus the contradiction.

It is immediate to check that the case $u(x'') < -\sqrt{\frac{C_\varepsilon}{L} + \frac{C_\varepsilon}{\varepsilon}}$ can be treated exactly as above.

□

To determine the next meaningful scale we write

$$\begin{aligned} &\mathcal{F}_\varepsilon^{k(1)}(u_\varepsilon) = \frac{\delta}{\varepsilon} F^{k(0)}(u_\varepsilon) - \int_0^1 r^k(u_\varepsilon)\,dx \\ \geq &\sum_{i=1}^{\left[\frac{2}{\delta}-\frac{1}{2}\right]} \left[\int_{(2i-1)\frac{\delta}{4}}^{(2i+1)\frac{\delta}{4}} \left(\frac{\delta}{\varepsilon} W^k\left(\frac{x}{\delta}, u_\varepsilon\right) + \varepsilon\delta (u_\varepsilon')^2\right)\,dx - \int_{(2i-1)\frac{\delta}{4}}^{(2i+1)\frac{\delta}{4}} r^k(u_\varepsilon)\,dx\right] \\ = &\sum_{i=1}^{\left[\frac{2}{\delta}-\frac{1}{2}\right]} \left[\frac{\delta}{2} \int_{(2i-1)\frac{\delta}{4}}^{(2i+1)\frac{\delta}{4}} 2\left(\frac{1}{\varepsilon} W^k\left(\frac{x}{\delta}, u_\varepsilon\right) + \varepsilon (u_\varepsilon')^2\right)\,dx - \fint_{(2i-1)\frac{\delta}{4}}^{(2i+1)\frac{\delta}{4}} r^k(u_\varepsilon)\,dx\right] \end{aligned}$$

then arguing as in Theorem 3.2 *Step 1* we find

$$\begin{aligned} \mathcal{F}_\varepsilon^{k(1)}(u_\varepsilon) &\geq \sum_{i=1}^{\left[\frac{2}{\delta}-\frac{1}{2}\right]} \frac{\delta}{2}\left(2\varphi^k_{\frac{\varepsilon}{\delta}}(\tilde{u}_\varepsilon) - r^k(\tilde{u}_\varepsilon)\right) \\ &\simeq \int_0^1 (2\varphi^k_{\frac{\varepsilon}{\delta}}(\tilde{u}_\varepsilon) - r^k(\tilde{u}_\varepsilon))\,dx, \end{aligned}$$

where $\varphi^k_{\frac{\varepsilon}{\delta}}$ and $\tilde{u}_\varepsilon$ are defined as before. So now we want to estimate from below the function

$$g^k_\varepsilon(s) := \varphi^k_{\frac{\varepsilon}{\delta}}(s) - r^k(s).$$

LEMMA 3.12. *Let φ^k_η be defined as in Corollary 3.5; then*

$$\varphi^k_\eta(s) = \begin{cases} \dfrac{(s+1)^2}{2\,\eta} + C^k_1 \tanh\left(\dfrac{1}{4\eta}\right) & \textit{if} \quad |s+1| \le c\sqrt{\eta} \\ \dfrac{s^2}{2\,\eta} + C^k_2 \tanh\left(\dfrac{1}{4\eta}\right) & \textit{if} \quad |s| \le c\sqrt{\eta} \\ \dfrac{(s-1)^2}{2\,\eta} + C^k_1 \tanh\left(\dfrac{1}{4\eta}\right) & \textit{if} \quad |s-1| \le c\sqrt{\eta} \end{cases} \tag{3.48}$$

for some positive constant c.

PROOF. We prove the equality (3.48) only for $|s| \le c\sqrt{\eta}$ (with c suitably chosen) the proof of the other two cases being analogous.

Let $|s| \le c\sqrt{\eta}$, with $c > 0$ to be determined. We start giving an estimate on above on φ^k_η.

By definition, we trivially have

$$\begin{aligned} \varphi^k_\eta(s) &\le \min\left\{ \int_{-\frac{1}{4}}^{\frac{1}{4}} \left(\frac{1}{\eta} W^k(x,u) + \eta(u')^2\right) dx : u \in W^{1,2}\left(-\frac{1}{4}, \frac{1}{4}\right), \fint_{-\frac{1}{4}}^{\frac{1}{4}} u\,dx = s,\ ||u||_\infty \le k \right\} \\ &= \min\left\{ \int_{-\frac{1}{4}}^{\frac{1}{4}} \left(\frac{1}{\eta} \mathcal{W}^k(x,u) + \eta(u')^2\right) dx : u \in W^{1,2}\left(-\frac{1}{4}, \frac{1}{4}\right), \fint_{-\frac{1}{4}}^{\frac{1}{4}} u\,dx = s \right\}, \end{aligned} \tag{3.49}$$

where

$$\mathcal{W}^k(x,u) := \begin{cases} (u-1+k)^2 & \text{if} \quad -\frac{1}{4} \le x \le 0 \\ (u+1-k)^2 & \text{if} \quad 0 \le x \le \frac{1}{4}\,. \end{cases} \tag{3.50}$$

Following the Lagrange Multipliers Method we explicitly determine the minimum value (3.49) by means of the auxiliary minimum problem

$$M^k_\eta(\lambda) := \min\left\{ \int_{-\frac{1}{4}}^{\frac{1}{4}} \left(\frac{1}{\eta} \mathcal{W}^k(x,u) + \eta(u')^2 + \lambda\,u\right) dx : u \in W^{1,2}\left(-\frac{1}{4}, \frac{1}{4}\right) \right\}, \tag{3.51}$$

with $\lambda \in \mathbb{R}$.

Also taking into account the definition of $\mathcal{W}^k$ (3.50), it is easy to check that $M^k_\eta(\lambda)$ can be equivalently expressed as

$$\begin{aligned} M^k_\eta(\lambda) = \min_{u_0} \Bigg\{ & \min_{\substack{u \in W^{1,2}\left(-\frac{1}{4},0\right) \\ u(0)=u_0}} \int_{-\frac{1}{4}}^{0} \left(\frac{1}{\eta}(u-1+k)^2 + \eta(u')^2 + \lambda\,u\right) dx \\ & + \min_{\substack{u \in W^{1,2}\left(0,\frac{1}{4}\right) \\ u(0)=u_0}} \int_{0}^{\frac{1}{4}} \left(\frac{1}{\eta}(u+1-k)^2 + \eta(u')^2 + \lambda\,u\right) dx \Bigg\}. \end{aligned}$$

Then by a straightforward computation we find that the minimum (3.51) is attained at

$$u_\eta^\lambda(x) = \begin{cases} 1 - k - \dfrac{\lambda\eta}{2} + (k-1)\cosh\Big(\dfrac{1+4x}{4\eta}\Big)\Big(\cosh\Big(\dfrac{1}{4\eta}\Big)\Big)^{-1} & \text{if} \quad -\dfrac{1}{4} \le x \le 0 \\ \\ -1 + k - \dfrac{\lambda\eta}{2} - (k-1)\cosh\Big(\dfrac{x}{\eta}\Big) + (k-1)\sinh\Big(\dfrac{x}{\eta}\Big)\tanh\Big(\dfrac{1}{4\eta}\Big) & \text{if} \quad 0 \le x \le \dfrac{1}{4}\,. \end{cases} \tag{3.52}$$

Moreover, in (3.52) the dependence on λ can be rephrased in terms of s by imposing the integral constraint

$$\int_{-\frac{1}{4}}^{\frac{1}{4}} u_\eta^\lambda(x)\,dx = \frac{s}{2},$$

which gives $\lambda = -\frac{2s}{\eta}$.

Finally, evaluating the energy in (3.49) at $u_\eta^{-\frac{2s}{\eta}}$, by a direct computation we get

$$\varphi_\eta^k(s) \le \frac{s^2}{2\eta} + C_2^k \tanh\Big(\frac{1}{4\eta}\Big). \tag{3.53}$$

Now we want to prove that (3.53) is actually an equality. We show that in particular if v_η^s is a test function for $\varphi_\eta^k(s)$, then $||v_\eta^s||_\infty < k$. To this effect, we additionally assume that $s > 0$ (the case $s < 0$ being symmetric).

To start we claim that supposing $v_\eta^s(0) = k$, yields to a contradiction. In fact, on one hand we have

$$\begin{aligned} \varphi_\eta^k(s) \;&\ge\; \min\Big\{\int_{-\frac{1}{4}}^{0}\Big(\frac{1}{\eta}(u-1+k)^2 + \eta(u')^2\Big)\,dx : \quad u \in W^{1,2}\Big(-\frac{1}{4},0\Big),\, u(0)=k\Big\} \\ &+\; \min\Big\{\int_{0}^{\frac{1}{4}}\Big(\frac{1}{\eta}(u+1-k)^2 + \eta(u')^2\Big)\,dx : \quad u \in W^{1,2}\Big(0,\frac{1}{4}\Big),\, u(0)=k\Big\} \\ &=\; \tanh\Big(\frac{1}{4\eta}\Big) + (2k-1)^2\tanh\Big(\frac{1}{4\eta}\Big) \\ &=\; 1 + (2k-1)^2 + (1+(2k-1)^2)\Big(1 - \tanh\Big(\frac{1}{4\eta}\Big)\Big) \qquad (3.54) \\ &=\; 2k^2 + C_2^k + o(1), \qquad \text{as } \eta \to 0. \qquad (3.55) \end{aligned}$$

While on the other hand, from (3.53) and since $0 < s < c\sqrt{\eta}$, we also find

$$\varphi_\eta^k(s) < \frac{c}{2} + C_2^k + o(1). \tag{3.56}$$

As a consequence if we choose $c \le 4k^2$, gathering (3.55) and (3.56) we get the contradiction and thus the claim.

Then it is easy to check that the case $v_\eta^s(0) = k$ is actually the most energetically convenient one among those for which the function v_η^s does not satisfy $||v_\eta^s|| < k$. So in particular this permits to exclude the existence of a point $x_\eta \in \big(-\frac{1}{4}, \frac{1}{4}\big)$ such that $v_\eta^s(x_\eta) \ge k$.

Moreover, we notice that the additional hypothesis $s > 0$ combined with the previous argument also excludes the possibility $v^s_\eta(x_\eta) \le -k$ for some $x_\eta \in \left(-\frac{1}{4}, \frac{1}{4}\right)$ which would clearly be even more unfavorable. This concludes the proof of the lemma for $s > 0$. □

THEOREM 3.13. *Let δ be such that $\delta \ll \varepsilon^2$ and $\frac{1}{\delta} \in \mathbb{N}$. The family of functionals $\mathcal{F}^{k(2)}_\varepsilon$ Γ-converges with respect to the weak L^2-convergence to the functional defined on $L^2(0,1)$ by*

$$\mathcal{F}^{k(2)}(u) = \begin{cases} -(C^k_1 - C^k_2)^2 & \text{if } u \in L^2(0,1),\ 0 \le u \le 1 \text{ a.e., and } \int_0^1 u = d \\ +\infty & \text{otherwise.} \end{cases}$$

On one hand, a straightforward calculation gives

$$\varphi^k_{\frac{\varepsilon}{\delta}}(s) = \frac{\delta}{2\varepsilon}s^2 + C^k_2 \tanh\left(\frac{\delta}{4\varepsilon}\right) \quad \text{for } s \le 0$$

and

$$\varphi^k_{\frac{\varepsilon}{\delta}}(s) = \frac{\delta}{2\varepsilon}(s-1)^2 + C^k_1 \tanh\left(\frac{\delta}{4\varepsilon}\right) \quad \text{for } s \ge 1;$$

on the other hand from Corollary 3.5 we have

$$\varphi^k_{\frac{\varepsilon}{\delta}} \to C^k_3 \quad \text{uniformly in } (0,1).$$

Then we have that for any sufficiently small ε

$$g^k_\varepsilon(s) \ge f^k_\varepsilon(s) := \begin{cases} \frac{\delta}{\varepsilon}\, s^2 - 2(C^k_1 - C^k_2)s + 2C^k_2\left(\tanh\left(\frac{\delta}{4\varepsilon}\right) - 1\right) & \text{if } s \le 0 \\ 2(C^k_3 - C^k_1) & \text{if } 0 < s < 1 \\ \frac{\delta}{\varepsilon}\,(s-1)^2 - 2(C^k_1 - C^k_2)(s-1) + 2C^k_1\left(\tanh\left(\frac{\delta}{4\varepsilon}\right) - 1\right) & \text{if } s \ge 1. \end{cases}$$

Recalling Remark 3.1 we notice that $C^k_3 - C^k_1 > 0$, while it is immediate to check that the two parabolas defining f^k_ε for $s \le 0$ and $s \ge 1$ have their vertexes respectively in

$$\left(\frac{\varepsilon}{\delta}(C^k_1 - C^k_2);\ -\frac{\varepsilon}{\delta}(C^k_1 - C^k_2)^2 + 2C^k_2\left(\tanh\left(\frac{\delta}{4\varepsilon}\right) - 1\right)\right)$$

and

$$\left(\frac{\varepsilon}{\delta}(C^k_1 - C^k_2) + 1;\ -\frac{\varepsilon}{\delta}(C^k_1 - C^k_2)^2 + 2C^k_1\left(\tanh\left(\frac{\delta}{4\varepsilon}\right) - 1\right)\right).$$

Then, for instance, from

$$-\frac{\varepsilon}{\delta}(C^k_1 - C^k_2)^2 + 2C^k_2\left(\tanh\left(\frac{\delta}{4\varepsilon}\right) - 1\right) = O\left(\frac{\varepsilon}{\delta}\right) + O\left(e^{-\frac{\delta}{2\varepsilon}}\right) = O\left(\frac{\varepsilon}{\delta}\right), \quad \text{for } \varepsilon \to 0$$

we deduce that the next meaningful scaling is $\dfrac{\varepsilon}{\delta}$.

4. $\delta \ll \varepsilon$: oscillations on a finer scale than the transition layer

THEOREM 4.1. *Let $k \le \frac{1}{2}$ and let δ be such that*

$$\lim_{\varepsilon\to 0} \frac{\delta}{\varepsilon\sqrt{\varepsilon}} = 0. \tag{4.1}$$

Then the functionals I_ε^k defined on $L^2(0,1)$ by

$$I_\varepsilon^k(u) := \begin{cases} \displaystyle\int_0^1 \left(\frac{1}{\varepsilon}\left(W^k\left(\frac{x}{\delta}, u\right) - k^2\right) + \varepsilon (u')^2 \right) dx & \text{if } u \in W^{1,2}(0,1) \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converge with respect to the strong L^2-convergence to the functional

$$I^k(u) = \begin{cases} \left(2\displaystyle\int_{-1}^1 \sqrt{\overline{W}^k(s) - k^2}\right) \#(S(u)) & \text{if } u \in BV((0,1);\{\pm 1\}) \\ +\infty & \text{otherwise} \end{cases}$$

with $\overline{W}^k$ as in (1.3).

PROOF. *Step 1: Γ-liminf inequality*
Let $u_\varepsilon \to u$ in $L^2(0,1)$ be such that $\sup_\varepsilon I_\varepsilon^k(u_\varepsilon) < +\infty$; with fixed $\varepsilon > 0$ let us define the set I^δ and, on I^δ, the function v_ε respectively as

$$I^\delta := \bigcup_{i=1}^{[\frac{1}{\delta}]} ((i-1)\delta, i\delta) \qquad v_\varepsilon(x) := \sum_{i=1}^{[\frac{1}{\delta}]} u_\varepsilon^i \chi_{((i-1)\delta, i\delta)}(x)$$

with

$$u_\varepsilon^i := \fint_{(i-1)\delta}^{i\delta} u_\varepsilon \, dt \quad \text{for } i = 1, \dots, \left[\frac{1}{\delta}\right].$$

By using Jensen's Inequality it is immediate to check that

$$||v_\varepsilon||_{L^2(I^\delta)} \le ||u_\varepsilon||_{L^2(I^\delta)} \tag{4.2}$$

while from the Poincaré Inequality and its scaling properties we have

$$||u_\varepsilon - v_\varepsilon||_{L^2(I^\delta)} \le \delta ||u_\varepsilon'||_{L^2(I^\delta)}. \tag{4.3}$$

A first estimate gives

$$I_\varepsilon^k(u_\varepsilon) \ge \int_{I^\delta} \left(\frac{1}{\varepsilon}\left(W^k\left(\frac{x}{\delta}, u_\varepsilon\right) - k^2\right) + \varepsilon (u_\varepsilon')^2 \right) dx - \frac{k^2}{\varepsilon}\int_{\delta[\frac{1}{\delta}]}^1 dx$$

hence

$$\liminf_{\varepsilon\to 0} I_\varepsilon^k(u_\varepsilon) \ge \liminf_{\varepsilon\to 0} \int_{I^\delta} \left(\frac{1}{\varepsilon}\left(W^k\left(\frac{x}{\delta}, u_\varepsilon\right) - k^2\right) + \varepsilon (u_\varepsilon')^2 \right) dx.$$

What we want to prove now is that the quantity

$$\frac{1}{\varepsilon}\int_{I^\delta} \left(W^k\left(\frac{x}{\delta}, u_\varepsilon\right) - \overline{W}^k(u_\varepsilon)\right) dx \tag{4.4}$$

tends to 0 as $\varepsilon \to 0$. To this purpose we first ramark that $W^k(y,\cdot)$ satisfies the following local Lipschitz property

$$|W^k(y,s_1) - W^k(y,s_2)| \le \alpha(1+|s_1|+|s_2|)|s_1-s_2| \quad \text{for a.e. } \ y\in\mathbb{R},\ \forall\, s_1,s_2\in\mathbb{R} \tag{4.5}$$

for some positive α. A simple averaging over $(0,1)$ demonstrates that (4.5) is satisfied also by $\overline{W}^k$. Moreover by the definition of v_ε and the 1-periodicity of $W^k(\cdot,s)$ the following string of equalities holds true

$$\begin{aligned}
\int_{I^\delta} W^k\left(\frac{x}{\delta},u_\varepsilon\right)\,dx &= \sum_{i=1}^{[\frac{1}{\delta}]}\int_{(i-1)\delta}^{i\delta} W^k\left(\frac{x}{\delta},u_\varepsilon^i\right)\,dx\\
&= \sum_{i=1}^{[\frac{1}{\delta}]}\int_{0}^{\delta} W^k\left(\frac{x}{\delta},u_\varepsilon^i\right)\,dx\\
&= \sum_{i=1}^{[\frac{1}{\delta}]}\delta\int_{0}^{1} W^k\left(x,u_\varepsilon^i\right)\,dx\\
&= \sum_{i=1}^{[\frac{1}{\delta}]}\delta\overline{W}^k(u_\varepsilon^i) = \int_{I^\delta}\overline{W}^k(v_\varepsilon)\,dx.
\end{aligned}$$

Then by adding and subtracting $\frac{1}{\varepsilon}\int_{I^\delta} W^k\left(\frac{x}{\delta},v_\varepsilon\right)\,dx$ in (4.4) and by virtue of (4.5) and the local Lipschitz continuity of $\overline{W}^k$ we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\left|\int_{I^\delta}\left(W^k\left(\frac{x}{\delta},u_\varepsilon\right)-\overline{W}^k(u_\varepsilon)\right)\,dx\right|\\
\le\ & \frac{1}{\varepsilon}\int_{I^\delta}\left|W^k\left(\frac{x}{\delta},u_\varepsilon\right)-W^k\left(\frac{x}{\varepsilon},v_\varepsilon\right)\right|\,dx+\frac{1}{\varepsilon}\int_{I^\delta}\left|\overline{W}^k\left(u_\varepsilon\right)-\overline{W}^k\left(v_\varepsilon\right)\right|\,dx\\
\le\ & \frac{1}{\varepsilon}\int_{I^\delta}2\alpha(1+|u_\varepsilon|+|v_\varepsilon|)|u_\varepsilon-v_\varepsilon|\,dx\\
\le\ & \frac{1}{\varepsilon}C(1+||u_\varepsilon||_{L^2(I^\delta)}+||v_\varepsilon||_{L^2(I^\delta)})||u_\varepsilon-v_\varepsilon||_{L^2(I^\delta)}\\
\le\ & C\frac{\delta}{\varepsilon}||u'_\varepsilon||_{L^2(0,1)}
\end{aligned}\tag{4.6}$$

in the last inequality having used (4.2) and (4.3).

Recalling that $\sup_\varepsilon I^k_{\varepsilon,\delta}(u_\varepsilon)<+\infty$, in particular implies

$$||u'_\varepsilon||_{L^2(0,1)}\le\frac{C}{\sqrt{\varepsilon}}, \tag{4.7}$$

by combining (4.6), (4.7) and invoking hypothesis (4.1) we obtain the desired result. At the end we find that

$$\liminf_{\varepsilon\to0} I^k_\varepsilon(u_\varepsilon)\ge\liminf_{\varepsilon\to0}\int_0^{\delta[\frac{1}{\delta}]}\left(\frac{1}{\varepsilon}(\overline{W}^k(u_\varepsilon)-k^2)+\varepsilon(u'_\varepsilon)^2\right)\,dx$$

and invoking the Modica-Mortola Theorem

$$\begin{aligned}\liminf_{\varepsilon\to 0} I^k_\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon\to 0}\int_0^a \left(\frac{1}{\varepsilon}(\overline{W}^k(u_\varepsilon)-k^2)+\varepsilon(u'_\varepsilon)^2\right)dx\\ &\geq \left(2\int_{-1}^1\sqrt{\overline{W}^k(s)-k^2}\right)\#(S(u)\cap(0,a)),\end{aligned}$$

for any fixed $a \in (0,1)$. Then, passing to the sup on $a \in (0,1)$ in (4.8), we get the Γ-liminf inequality.

Finally, notice that by $\sup_\varepsilon I^k_\varepsilon(u_\varepsilon) < +\infty$ and the Γ-liminf inequality, we immediately deduce that $u \in BV((0,1);\{\pm 1\})$.

Step 2: Γ*-limsup inequality*
We have to construct a recovery sequence for $u \in PC(0,1)$ with $u \in \{\pm 1\}$ a.e.; it will suffice to approximate

$$u(x) = \begin{cases} -1 & \text{if } x < x_0 \\ 1 & \text{if } x \geq x_0, \end{cases} \tag{4.8}$$

with $x_0 \in (0,1)$.

We want to show that the limsup inequality can be easily obtained acting as if we were studying the convergence of the functionals

$$\int_0^1 \left(\frac{1}{\varepsilon}(\overline{W}^k(u)-k^2)+\varepsilon(u')^2\right)dx. \tag{4.9}$$

To this effect, arguing as in Modica-Mortola construction, for any fixed $\eta > 0$ we can find a number $T > 0$ and a function $v \in W^{1,2}(-T,T)$ such that $v(-T) = -1$, $v(T) = 1$ and

$$\int_{-T}^T (\overline{W}^k(v)-k^2+(v')^2)\,dx \leq 2\int_{-1}^1\sqrt{\overline{W}^k(s)-k^2}+\eta \tag{4.10}$$

then, for instance, a recovery sequence for (4.8)-(4.9) is given by

$$u_\varepsilon(x) = \begin{cases} -1 & \text{if } x < x_0^\delta - \varepsilon T \\ v\left(\frac{x-x_0^\delta}{\varepsilon}\right) & \text{if } x_0^\delta - \varepsilon T \leq x \leq x_0^\delta + \varepsilon T \\ 1 & \text{if } x > x_0^\delta + \varepsilon T \end{cases}$$

with $x_0^\delta = \left[\frac{x_0}{\delta}\right]\delta$. We next claim that u_ε is a recovery sequence also for $I^k_{\varepsilon,\delta}$. In order to prove it, testing $I^k_{\varepsilon,\delta}$ on u_ε, we find

$$\begin{aligned}I^k_\varepsilon(u_\varepsilon) &= \int_{x_0^\delta-\varepsilon T}^{x_0^\delta+\varepsilon T}\left(\frac{1}{\varepsilon}\left(W^k\left(\frac{x}{\delta},u_\varepsilon\right)-k^2\right)+\varepsilon(u'_\varepsilon)^2\right)dx\\ &= \int_{-T}^T\left(W^k\left(\frac{\varepsilon}{\delta}x,v\right)-k^2+(v')^2\right)dx.\end{aligned}$$

Then the next step consists in proving that

$$\lim_{\varepsilon\to 0}\int_{-T}^T W^k\left(\frac{\varepsilon}{\delta}x,v\right)dx = \int_{-T}^T \overline{W}^k(v)\,dx. \tag{4.11}$$

Setting

$$W^k_{\frac{\delta}{\varepsilon}}(x) := W^k\left(\frac{\varepsilon}{\delta}x, v\right) \quad \text{for a.e. } x \in (-T,T),$$

we have

$$0 \leq W^k_{\frac{\delta}{\varepsilon}} \leq \beta(1+|v|^2) \quad \text{a.e. in } (-T,T) \quad \text{for some positive } \beta,$$

and from it we deduce:

i) $\exists C > 0$ such that $||W^k_{\frac{\delta}{\varepsilon}}||_{L^1(-T,T)} \leq C$;

ii) $(W^k_{\frac{\delta}{\varepsilon}})$ is equi-integrable on $(-T,T)$;

then by applying Dunford-Pettis criterion, upon passing to a subsequence (not relabelled)

$$W^k_{\frac{\delta}{\varepsilon}} \rightharpoonup f \quad \text{in } L^1(-T,T). \tag{4.12}$$

and by Lebesgue Theorem

$$f(x) = \lim_{r\to 0^+} \fint_{x-r}^{x+r} f(y)\,dy \quad \text{for a.e. } x \in (-T,T).$$

Moreover from (4.12) we have that in particular, for $x \in (-T,T)$ and for sufficiently small $r > 0$,

$$\lim_{\varepsilon\to 0} \fint_{x-r}^{x+r} W^k_{\frac{\delta}{\varepsilon}}(y)\,dy = \fint_{x-r}^{x+r} f(y)\,dy$$

and consequently

$$\lim_{r\to 0^+}\lim_{\varepsilon\to 0} \fint_{x-r}^{x+r} W^k_{\frac{\delta}{\varepsilon}}(y)\,dy = f(x) \quad \text{for a.e. } x \in (-T,T).$$

On the other hand, from

$$\fint_{x-r}^{x+r} W^k_{\frac{\delta}{\varepsilon}}(y)\,dy = \fint_{x-r}^{x+r} W^k\left(\frac{\varepsilon}{\delta}y, v\right)\,dy - \fint_{x-r}^{x+r} W^k\left(\frac{\varepsilon}{\delta}y, v(x)\right)\,dy$$
$$+ \fint_{x-r}^{x+r} W^k\left(\frac{\varepsilon}{\delta}y, v(x)\right)\,dy \tag{4.13}$$

with

$$\left|\fint_{x-r}^{x+r} \left(W^k\left(\frac{\varepsilon}{\delta}y, v\right) - W^k\left(\frac{\varepsilon}{\delta}y, v(x)\right)\right)\,dy\right| \leq \alpha \fint_{x-r}^{x+r} (1+|v(x)|+|v|)|v-v(x)|\,dy$$

and

$$\lim_{\varepsilon\to 0} \fint_{x-r}^{x+r} W^k\left(\frac{\varepsilon}{\delta}y, v(x)\right)\,dy = \fint_{x-r}^{x+r} \overline{W}^k\left(v(x)\right)\,dy = \overline{W}^k(v(x)).$$

Passing to the limit in (4.13) first letting ε, then r go to zero, we obtain

$$f(x) = \overline{W}^k(v(x)) \quad \text{for a.e. } x \in (-T,T)$$

hence, from (4.12), (4.11). Finally by combining (4.11) and (4.10) we get

$$\begin{aligned} \limsup_{\varepsilon\to 0} I^k_\varepsilon(u_\varepsilon) &\leq 2\int_{-1}^{1} \sqrt{\overline{W}^k(s) - k^2} + \eta \\ &= I^k(u) + \eta \end{aligned}$$

and by the arbitrariness of η, the thesis. □

REMARK 4.2. Since as for the Modica-Mortola functionals, the equi-coercivity at scale ε improves to strong-L^2 equi-coercivity, then we may (a posteriori) compute also the zero order Γ-limit with respect to the strong L^2-convergence, obtaining

$$\widetilde{F}_0^{k(0)}(u) = \int_0^1 \overline{W}^k(u)\,dx.$$

Thus, for $\delta \ll \varepsilon$, $k > \frac{1}{2}$ we have that a Γ-development for $F_\varepsilon^{k(0)}$ with respect to the weak L^2-convergence is given by

$$F_\varepsilon^{k(0)}(u) = \int_0^1 (\overline{W}^k)^{**}(u)\,dx + \varepsilon\left(2\int_{-1}^1 \sqrt{\overline{W}^k(s) - k^2}\right)\#(S(u)) + o(\varepsilon), \tag{4.14}$$

while a Γ-development with respect to the strong L^2-convergence is

$$F_\varepsilon^{k(0)}(u) = \int_0^1 \overline{W}^k(u)\,dx + \varepsilon\left(2\int_{-1}^1 \sqrt{\overline{W}^k(s) - k^2}\right)\#(S(u)) + o(\varepsilon). \tag{4.15}$$

Set

$$\tau^k(s) := (2k-1)s - k + \frac{3}{4}$$

THEOREM 4.3. *Let $k > \frac{1}{2}$ and choose δ satisfying (4.1). Then the functionals $\mathcal{I}_\varepsilon^k$ defined on $L^2(0,1)$ by*

$$\mathcal{I}_\varepsilon^k(u) := \begin{cases} \displaystyle\int_0^1 \left(\frac{1}{\varepsilon}\left(W^k\left(\frac{x}{\delta}, u\right) - \tau^k(u)\right) + \varepsilon(u')^2\right)dx & \text{if } u \in W^{1,2}(0,1) \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converge with respect to the strong L^2-convergence to the functional

$$\mathcal{I}^k(u) = \begin{cases} \left(2\displaystyle\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \sqrt{\widetilde{W}^k(s)}\right)\#(S(u)) & \text{if } u \in PC((0,1);\{k \pm \frac{1}{2}\}) \\ +\infty & \text{otherwise} \end{cases}$$

where $\widetilde{W}^k(s) := \overline{W}^k(s) - \tau^k(s)$.

CHAPTER 2

The Neumann sieve problem and dimensional reduction

1. Plan of the chapter

This chapter is organized as follows: after recalling some useful notation in Section 2, we state the main results, Theorem 3.3 and Theorem 3.6, in Section 3. Then, in Section 4 we list some auxiliary results as rescaled Poincaré type inequalities and joining lemmas. Section 5 is devoted to give a preliminary definition of the interfacial energy density which is in terms of limit of minimum problems. In Section 6 we prove the Γ-convergence result (Theorem 3.3). It is only in Section 7 that we compute the explicit expression of the interfacial energy density of each regime (Theorem 3.6).

2. Notation

Given $x \in \mathbb{R}^n$, we set $x = (x_\alpha, x_n)$ where $x_\alpha := (x_1, \ldots, x_{n-1})$ is the in-plane variable and $D_\alpha := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\right)$ (resp. D_n) the derivative with respect to x_α (resp. x_n).

The notation $\mathbb{R}^{m\times n}$ stands for the set of $m \times n$ real matrices. Given a matrix $F \in \mathbb{R}^{m\times n}$, we write $F = (\overline{F}|F_n)$ where $\overline{F} = (F_1, \ldots, F_{n-1})$ and F_i denotes the i-th column of F, $1 \le i \le n$ and $\overline{F} \in \mathbb{R}^{m\times(n-1)}$.

The Lebesgue measure in $\mathbb{R}^n$ will be denoted by $\mathcal{L}^n$ and the Hausdorff $(n-1)$-dimensional measure by $\mathcal{H}^{n-1}$. Let A be an open subset of $\mathbb{R}^d$ ($d = n-1$, $d = n$). If $s \in [1, +\infty]$, we use standard notation for Lebesgue and Sobolev spaces $L^s(A; \mathbb{R}^m)$ and $W^{1,s}(A; \mathbb{R}^m)$.

Let ω be a bounded open subset of $\mathbb{R}^{n-1}$ and $I = (-1, 1)$, we define $\Omega := \omega \times I$. In the sequel, we will identify $L^s(\omega; \mathbb{R}^m)$ (resp. $W^{1,s}(\omega; \mathbb{R}^m)$) with the space of functions $v \in L^s(\Omega; \mathbb{R}^m)$ (resp. $W^{1,s}(\Omega; \mathbb{R}^m)$) such that $D_n v = 0$ in the sense of distribution.

For every $(a, b) \subset \mathbb{R}$ with $a < b$ and $q_1, q_2 \ge 1$, $L^{q_1}(a, b; L^{q_2}(\mathbb{R}^{(n-1)}; \mathbb{R}^m))$ is the space of measurable m-vectorial functions ζ such that

$$\int_b^a \left(\int_{\mathbb{R}^{n-1}} |\zeta(x_\alpha, x_n)|^{q_2}\, dx_\alpha\right)^{\frac{q_1}{q_2}} dx_n < +\infty.$$

Let $a \in \mathbb{R}^{n-1}$ and $\rho > 0$, we denote by $B^{n-1}_\rho(a)$ the open ball of $\mathbb{R}^{n-1}$ of center a and radius ρ and by $Q^{n-1}_\rho(a)$ the open cube of $\mathbb{R}^{n-1}$ with center a and length side ρ. We write B^{n-1}_ρ instead of $B^{n-1}_\rho(0)$ not to overburden notation. Let $x^\varepsilon_i = i\varepsilon$ with $i \in \mathbb{Z}^{n-1}$, we set $Q^{n-1}_{i,\varepsilon} := Q^{n-1}_\varepsilon(x^\varepsilon_i)$.

We define $U^{+a} := U \times (0, a)$ and $U^{-a} := U \times (-a, 0)$ with $U \subseteq \mathbb{R}^{n-1}$ and $a > 0$, while if $a = 1$, then $U^+ = U^{+1}$ and $U^- = U^{-1}$.

We set $C_{1,\infty} := \{(x_\alpha, 0) \in \mathbb{R}^n : 1 \le |x_\alpha|\}$ and $C_{1,N} := \{(x_\alpha, 0) \in \mathbb{R}^n : 1 \le |x_\alpha| < N\}$ for every $N > 1$.

Let $p \geq 1$, we denote by $\mathrm{Cap}_p(B_1^{n-1}; A)$ the p-capacity of B_1^{n-1} with respect to $A \subseteq \mathbb{R}^d$:

$$\mathrm{Cap}_p(B_1^{n-1}; A) = \inf \left\{ \int_A |D\psi|^p \, dx : \ \psi \in W_0^{1,p}(A) \text{ and } \psi = 1 \text{ on } B_1^{n-1} \right\}.$$

The letter c will stand for a generic strictly-positive constant which may vary from line to line and expression to expression within the same formula.

3. Statements of the main results

Since we are going to work with varying domains, we have to precise the meaning of 'converging sequences'.

DEFINITION 3.1. *Let $\Omega_j = \omega^{+\delta_j} \cup \omega^{-\delta_j} \cup \big(\omega_{r_j,\varepsilon_j} \times \{0\}\big)$. Given a sequence $(u_j) \subset W^{1,p}(\Omega_j; \mathbb{R}^m)$, we define $\hat{u}_j(x_\alpha, x_n) := u_j(x_\alpha, \delta_j\, x_n)$. We say that (u_j) converges (resp. converges weakly) to $(u^+, u^-) \in W^{1,p}(\omega; \mathbb{R}^m) \times W^{1,p}(\omega; \mathbb{R}^m)$ if we have*

$$\begin{aligned} \hat{u}_j^+ \ &:= \ \hat{u}_j|_{\omega^+} \to u^+ \ \textit{in} \quad L^p(\omega^+; \mathbb{R}^m) \quad (\textit{resp. weakly in } W^{1,p}(\omega^+; \mathbb{R}^m)),\\ \hat{u}_j^- \ &:= \ \hat{u}_j|_{\omega^-} \to u^- \ \textit{in} \quad L^p(\omega^-; \mathbb{R}^m) \quad (\textit{resp. weakly in } W^{1,p}(\omega^-; \mathbb{R}^m)). \end{aligned}$$

Similarly if we replace Ω_j by $\omega^{\pm\delta_j}$.

We say that the sequence $(|Du_j|^p/\delta_j)$ is equi-integrable on $\omega^{\pm\delta_j}$ if $\big(\big|\big(D_\alpha \hat{u}_j \big| \frac{1}{\delta_j} D_n \hat{u}_j\big)\big|^p\big)$ is equi-integrable on $\omega^\pm$.

REMARK 3.2. By virtue of Definition 3.1, a sequence $(u_j) \subset W^{1,p}(\Omega_j; \mathbb{R}^m)$ converges to $(u^+, u^-) \in W^{1,p}(\omega; \mathbb{R}^m) \times W^{1,p}(\omega; \mathbb{R}^m)$ if and only if

$$\lim_{j \to +\infty} \frac{1}{\delta_j} \int_{\omega^{\pm\delta_j}} |u_j - u^\pm|^p \, dx = 0, \tag{3.1}$$

while (3.1) and

$$\sup_{j \in \mathbb{N}} \frac{1}{\delta_j} \int_{\omega^{\pm\delta_j}} |Du_j|^p \, dx = \sup_{j \in \mathbb{N}} \int_{\omega^\pm} \left| \left(D_\alpha \hat{u}_j \middle| \frac{1}{\delta_j} D_n \hat{u}_j \right) \right|^p dx < +\infty \tag{3.2}$$

imply the weak convergence.

Note that Remark 3.2 is still valid if we consider the domain $\omega^{+\delta_j} \cup \omega^{-\delta_j}$ in place of Ω_j.

The main results of this chapter are the following:

THEOREM 3.3 (Γ-convergence). *Let $1 < p < n-1$. Let ω be a bounded open subset of $\mathbb{R}^{n-1}$ satisfying $\mathcal{H}^{n-1}(\partial\omega) = 0$ and $W : \mathbb{R}^{m \times n} \to [0, +\infty)$ be a Borel function such that $W(0) = 0$ and satisfying a growth condition of order p : there exists a constant $\beta > 0$ such that*

$$|F|^p - 1 \leq W(F) \leq \beta(|F|^p + 1), \quad \textit{for every } F \in \mathbb{R}^{m \times n}. \tag{3.3}$$

Let (ε_j), (δ_j) and (r_j) be sequences of strictly positive numbers converging to zero such that

$$\lim_{j \to +\infty} \frac{\delta_j}{\varepsilon_j} = 0$$

and set

$$\ell := \lim_{j\to+\infty} \frac{r_j}{\delta_j}\,.$$

If

$$\ell \in (0,+\infty], \qquad \textit{and} \qquad 0 < R^{(\ell)} := \lim_{j\to+\infty} \frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}} < +\infty$$

or

$$\ell = 0, \qquad \textit{and} \qquad 0 < R^{(0)} := \lim_{j\to+\infty} \frac{r_j^{n-p}}{\delta_j\,\varepsilon_j^{n-1}} < +\infty\,,$$

then, up to an extraction, the sequence of functionals $\mathcal{F}_j : L^p(\Omega_j;\mathbb{R}^m) \to [0,+\infty]$ *defined by*

$$\mathcal{F}_j(u) := \begin{cases} \dfrac{1}{\delta_j}\displaystyle\int_{\Omega_j} W(Du)\,dx & \textit{if } u \in W^{1,p}(\Omega_j;\mathbb{R}^m), \\ \\ +\infty & \textit{otherwise} \end{cases}$$

Γ*-converges to*

$$\mathcal{F}^{(\ell)}(u^+,u^-) = \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^+)\,dx_\alpha + \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^-)\,dx_\alpha + R^{(\ell)}\int_\omega \varphi^{(\ell)}(u^+-u^-)\,dx_\alpha$$

on $W^{1,p}(\omega;\mathbb{R}^m)\times W^{1,p}(\omega;\mathbb{R}^m)$ *with respect to the convergence introduced in Definition* 3.1*, where* $\overline{W}(\overline{F}) := \inf\{W(\overline{F}|z) : z \in \mathbb{R}^m\}$*,* $\mathcal{Q}_{n-1}\overline{W}$ *is the* $(n-1)$*-quasiconvexification of* $\overline{W}$ *and* $\varphi^{(\ell)} : \mathbb{R}^m \to [0,+\infty)$ *is a locally Lipschitz continuous function for any* $\ell \in [0,+\infty]$*.*

REMARK 3.4. Note that if $\ell \in (0,+\infty]$ the only meaningful scaling for r_j is that of order $\varepsilon_j^{(n-1)/(n-1-p)}$; *i.e.*, for both $R^{(\ell)} = 0$ and $R^{(\ell)} = +\infty$ we loose the asymptotic memory of the sieve. In fact, if $R^{(\ell)} = 0$, we obtain two uncoupled problems in the limit, while if $R^{(\ell)} = +\infty$, limit deformations (u^+,u^-) with finite energy are continuous across the mid-section ($u^+ = u^-$ in ω) as in Le Dret-Raoult [**38**]. Similarly, for $\ell = 0$.

REMARK 3.5. If $\ell \in (0,+\infty)$ then

$$0 < R^{(\ell)} = \lim_{j\to+\infty} \frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}} < +\infty \qquad \text{if and only if} \qquad 0 < R^{(0)} = \lim_{j\to+\infty} \frac{r_j^{n-p}}{\delta_j\,\varepsilon_j^{n-1}} < +\infty\,;$$

hence, in this case the two meaningful scalings are equivalent.

The following result provides a characterization of the interfacial energy density $\varphi^{(\ell)}$ for each $\ell \in [0,+\infty]$.

THEOREM 3.6 (Representation formulas). *Let* $p^* = (n-1)p/(n-1-p)$ *be the Sobolev exponent in dimension* $(n-1)$*. Then, upon extracting a subsequence, there exists the limit*

$$g(F) := \lim_{j\to+\infty} r_j^p \mathcal{Q}_n W(r_j^{-1}F),$$

for all $F \in \mathbb{R}^{m\times n}$*, where* $\mathcal{Q}_n W$ *denotes the* n*-quasiconvexification of* W*, so that:*
if $\ell \in (0,+\infty)$*,*

$$\begin{aligned}\varphi^{(\ell)}(z) \;&:=\; \inf\Bigg\{\int_{(\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty}} g\big(D_\alpha\zeta|\ell D_n\zeta\big)\,dx : \zeta \in W^{1,p}_{\mathrm{loc}}((\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty};\mathbb{R}^m),\\ &\qquad D\zeta \in L^p((\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty};\mathbb{R}^{m\times n}), \quad \zeta - z \in L^p(0,1;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))\\ &\qquad \zeta \in L^p(-1,0;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))\Bigg\};\end{aligned}$$

if $\ell = +\infty$

$$\begin{aligned}\varphi^{(\infty)}(z) \;&:=\; \inf\Bigg\{\int_{\mathbb{R}^{n-1}}\Big(\mathcal{Q}_{n-1}\,\overline{g}(D_\alpha\zeta^+) + \mathcal{Q}_{n-1}\,\overline{g}(D_\alpha\zeta^-)\Big)\,dx_\alpha :\ \zeta^\pm \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^{n-1};\mathbb{R}^m),\\ &\qquad \zeta^+ = \zeta^- \ \text{in } B_1^{n-1}, \quad D_\alpha\zeta^\pm \in L^p(\mathbb{R}^{n-1};\mathbb{R}^{m\times(n-1)}),\\ &\qquad (\zeta^+ - z)\,,\zeta^- \in L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m)\Bigg\},\end{aligned}$$

where $\overline{g}(\overline{F}) := \inf\{g(\overline{F}|z) : z \in \mathbb{R}^m\}$ *and* $\mathcal{Q}_{n-1}\overline{g}$ *is the* $(n-1)$*-quasiconvexification of* $\overline{g}$*;*
if $\ell = 0$

$$\begin{aligned}\varphi^{(0)}(z) \;&=\; \inf\Bigg\{\int_{\mathbb{R}^n\setminus C_{1,\infty}} g(D\zeta)\,dx :\ \zeta \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n\setminus C_{1,\infty};\mathbb{R}^m),\ D\zeta \in L^p(\mathbb{R}^n\setminus C_{1,\infty};\mathbb{R}^{m\times n}),\\ &\qquad \zeta - z \in L^p(0,+\infty;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m)),\ \zeta \in L^p(-\infty,0;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))\Bigg\},\end{aligned}$$

for all $z \in \mathbb{R}^m$*.*

REMARK 3.7. Without loss of generality we may assume that W is quasiconvex (upon first relaxing the energy); hence, by (3.3), W satisfies the following p-Lipschitz condition (see e.g. [**26**]):

$$|W(F_1) - W(F_2)| \le c\,(1 + |F_1|^{p-1} + |F_2|^{p-1})|F_1 - F_2|, \qquad \text{for all } F_1,\ F_2 \in \mathbb{R}^{m\times n}\,. \tag{3.4}$$

4. Preliminary results

4.1. Some rescaled Poincaré Inequalities. Since we deal with varying domains, depending on different parameters, it is useful to note how the constant in Poincaré type inequalities rescale with respect to such parameters.

LEMMA 4.1. *Let* A *be an open bounded and connected subset of* $\mathbb{R}^{n-1}$ *with Lipschitz boundary and let* $A_\rho := \rho A$ *for* $\rho > 0$*.*

(i) *There exists a constant $c > 0$ (depending only on (A, n, p)) such that for every $\rho, \delta > 0$*

$$\int_{A_\rho^{\pm\delta}} |u - \overline{u}_{A_\rho^{\pm\delta}}|^p \, dx \leq c \int_{A_\rho^{\pm\delta}} (\rho^p |D_\alpha u|^p + \delta^p |D_n u|^p) \, dx,$$

for every $u \in W^{1,p}(A_\rho^{\pm\delta}; \mathbb{R}^m)$ where $\overline{u}_{A_\rho^{\pm\delta}} = ⨍_{A_\rho^{\pm\delta}} u \, dx$.

(ii) *If B is an open and connected subset of A with Lipschitz boundary and $B_\rho := \rho B$ then there exists a constant $c > 0$ (depending only on (A, B, n, p)) such that for every $\rho, \delta > 0$*

$$\int_{A_\rho^{\pm\delta}} |u - \overline{u}_{B_\rho^{\pm\delta}}|^p \, dx \leq c \int_{A_\rho^{\pm\delta}} (\rho^p |D_\alpha u|^p + \delta^p |D_n u|^p) \, dx,$$

for every $u \in W^{1,p}(A_\rho^{\pm\delta}; \mathbb{R}^m)$ where $\overline{u}_{B_\rho^{\pm\delta}} = ⨍_{B_\rho^{\pm\delta}} u \, dx$.

Proof. Let us define $v(x_\alpha, x_n) := u(\rho x_\alpha, \delta x_n)$ then $v \in W^{1,p}(A^\pm; \mathbb{R}^m)$. By a change of variable, we get that $\overline{u}_{A_\rho^{\pm\delta}} = \overline{v}_{A^\pm}$. Moreover, by the Poincaré Inequality, there exists a constant $c = c(A, n, p) > 0$ such that

$$\begin{aligned}
\int_{A_\rho^{\pm\delta}} |u - \overline{u}_{A_\rho^{\pm\delta}}|^p \, dx &= \delta \rho^{n-1} \int_{A^\pm} |v - \overline{v}_{A^\pm}|^p \, dy \\
&\leq c \delta \rho^{n-1} \int_{A^\pm} |Dv|^p \, dy \\
&= c \int_{A_\rho^{\pm\delta}} (\rho^p |D_\alpha u|^p + \delta^p |D_n u|^p) \, dx
\end{aligned}$$

and it completes the proof of (i). Now, if $B_\rho \subset A_\rho$, we get that

$$\begin{aligned}
&\int_{A_\rho^{\pm\delta}} |u - \overline{u}_{B_\rho^{\pm\delta}}|^p \, dx \\
\leq\ & c \Big(\int_{A_\rho^{\pm\delta}} |u - \overline{u}_{A_\rho^{\pm\delta}}|^p \, dx + \delta \rho^{n-1} \mathcal{H}^{n-1}(A) |\overline{u}_{A_\rho^{\pm\delta}} - \overline{u}_{B_\rho^{\pm\delta}}|^p \Big) \\
\leq\ & c \int_{A_\rho^{\pm\delta}} |u - \overline{u}_{A_\rho^{\pm\delta}}|^p \, dx + c \frac{\mathcal{H}^{n-1}(A)}{\mathcal{H}^{n-1}(B)} \left(\int_{B_\rho^{\pm\delta}} |u - \overline{u}_{A_\rho^{\pm\delta}}|^p \, dx + \int_{B_\rho^{\pm\delta}} |u - \overline{u}_{B_\rho^{\pm\delta}}|^p \, dx \right) \\
\leq\ & c \int_{A_\rho^{\pm\delta}} (\rho^p |D_\alpha u|^p + \delta^p |D_n u|^p) \, dx.
\end{aligned}$$

□

4.2. A joining lemma on varying domains. If not otherwise specified, in all that follows the convergence of a sequence of functions has to be intended in the sense of Definition 3.1.

The following lemma, is the key tool in the proof of Theorem 3.3. It is a technical result which allows to modify sequences of functions 'near' the sets $B_{r_j}^{(n-1)}(x_i^{\varepsilon_j})$. It is very close in spirit to Lemma 3.4 in [**5**] although now the geometry of the problem yields a different construction involving suitable cylindrical (instead of spherical) annuli to surround the connecting zones.

LEMMA 4.2. *Let* (ε_j), (δ_j) *be sequences of strictly positive numbers converging to* 0 *and such that* $\delta_j \ll \varepsilon_j$. *Let* $(u_j) \subset W^{1,p}(\omega^{+\delta_j} \cup \omega^{-\delta_j}; \mathbb{R}^m)$ *be a sequence converging to* $(u^+, u^-) \in W^{1,p}(\omega; \mathbb{R}^m) \times W^{1,p}(\omega; \mathbb{R}^m)$ *satisfying* $\sup_j \mathcal{F}_j(u_j) < +\infty$; *let* $k \in \mathbb{N}$. *Set* $\rho_j = \gamma\varepsilon_j$ *with* $\gamma < 1/2$ *and*

$$Z_j := \{i \in \mathbb{Z}^{n-1} : \ \mathrm{dist}(x_i^{\varepsilon_j}, \mathbb{R}^{n-1} \setminus \omega) > \varepsilon_j\}.$$

For every $i \in Z_j$, *there exists* $k_i \in \{0, \dots, k-1\}$ *such that having set*

$$C_j^i := \left\{x_\alpha \in \omega : \ 2^{-k_i-1}\rho_j < |x_\alpha - x_i^{\varepsilon_j}| < 2^{-k_i}\rho_j\right\},$$

$$u_j^{i\pm} := \fint_{(C_j^i)^{\pm\delta_j}} u_j \, dx \tag{4.1}$$

and

$$\rho_j^i := \frac{3}{4} 2^{-k_i} \rho_j,$$

there exists a sequence $(w_j) \subset W^{1,p}(\omega^{+\delta_j} \cup \omega^{-\delta_j}; \mathbb{R}^m)$ *weakly converging to* (u^+, u^-) *such that*

$$w_j = u_j \quad in \quad \Big(\omega \setminus \bigcup_{i \in Z_j} C_j^i\Big)^{\pm\delta_j}, \tag{4.2}$$

$$w_j = u_j^{i\pm} \quad on \quad \big(\partial B^{n-1}_{\rho_j^i}(x_i^{\varepsilon_j})\big)^{\pm\delta_j} \tag{4.3}$$

and satisfying

$$\limsup_{j\to+\infty} \frac{1}{\delta_j} \int_{\omega^{\pm\delta_j}} \big|W(Dw_j) - W(Du_j)\big| \, dx \le \frac{c}{k}. \tag{4.4}$$

Proof. For every $j \in \mathbb{N}$, $i \in Z_j$, $k \in \mathbb{N}$ and $h \in \{0, \dots, k-1\}$, we define

$$C_j^{i,h} := \left\{x_\alpha \in \omega : \ 2^{-h-1}\rho_j < |x_\alpha - x_i^{\varepsilon_j}| < 2^{-h}\rho_j\right\},$$

$$(u_j^{i,h})^\pm := \fint_{(C_j^{i,h})^{\pm\delta_j}} u_j \, dx$$

and

$$\rho_j^{i,h} := \frac{3}{4} 2^{-h} \rho_j. \tag{4.5}$$

Let $\phi \equiv \phi_j^{i,h} \in \mathcal{C}_c^\infty(C_j^{i,h}; [0,1])$ be a cut-off function such that $\phi = 1$ on $\partial B^{n-1}_{\rho_j^{i,h}}(x_i^{\varepsilon_j})$ and $|D_\alpha \phi| \le c/\rho_j^{i,h}$. In $(C_j^{i,h})^{\pm\delta_j}$, we set

$$w_j^{i,h}(x) := \phi(x_\alpha)(u_j^{i,h})^\pm + (1 - \phi(x_\alpha))u_j,$$

then

$$\begin{aligned}
\int_{(C_j^{i,h})^{\pm\delta_j}} |Dw_j^{i,h}|^p \, dx \quad &\le \quad c\int_{(C_j^{i,h})^{\pm\delta_j}} \Big(|D_\alpha\phi|^p |u_j - (u_j^{i,h})^\pm|^p + |Du_j|^p\Big) \, dx \\
&\le \quad c\int_{(C_j^{i,h})^{\pm\delta_j}} \left(\frac{|u_j - (u_j^{i,h})^\pm|^p}{(\rho_j^{i,h})^p} + |Du_j|^p\right) dx.
\end{aligned}$$

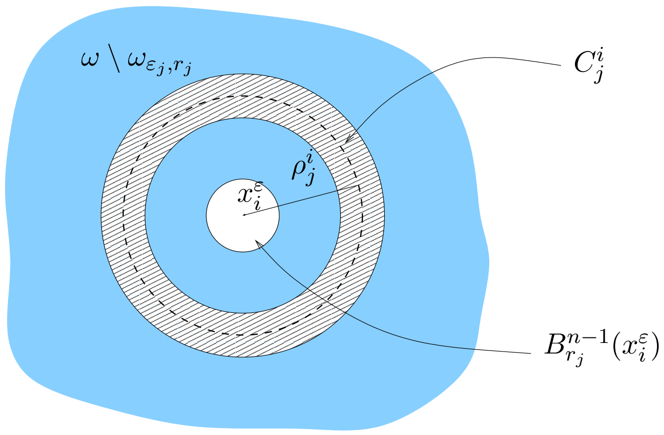

FIGURE 1. The $(n-1)$-dimensional annuli C_j^i.

Applying Lemma 4.1 (i), with $\rho = \rho_j^{i,h}$ and $A_\rho = C_j^{i,h}$, we have that

$$
\begin{aligned}
&\int_{(C_j^{i,h})^{\pm\delta_j}} |Dw_j^{i,h}|^p \, dx \\
\leq\ & c \int_{(C_j^{i,h})^{\pm\delta_j}} \left(|D_\alpha u_j|^p + \Big(\frac{\delta_j}{\rho_j^{i,h}}\Big)^p |D_n u_j|^p \right) dx + c \int_{(C_j^{i,h})^{\pm\delta_j}} |Du_j|^p \, dx \\
\leq\ & m_j(k,\gamma)\, c \int_{(C_j^{i,h})^{\pm\delta_j}} |Du_j|^p \, dx, \qquad (4.6)
\end{aligned}
$$

where by (4.5)

$$
m_j(k,\gamma) := \max\Big\{1, \Big(\frac{2^{k+1}}{3\gamma}\Big)^p \Big(\frac{\delta_j}{\varepsilon_j}\Big)^p\Big\}
$$

and since $\delta_j \ll \varepsilon_j$, $m_j(k,\gamma) \to 1$ as $j \to +\infty$. As

$$
\sum_{h=0}^{k-1} \int_{(C_j^{i,h})^{\pm\delta_j}} (1 + |Du_j|^p) \, dx \leq \int_{B_{\rho_j}^{n-1}(x_i^{\varepsilon_j})^{\pm\delta_j}} (1 + |Du_j|)^p \; dx,
$$

there exists $k_i \in \{0, \dots, k-1\}$ such that, having set $C_j^i := C_j^{i,k_i}$, we get

$$
\int_{(C_j^i)^{\pm\delta_j}} (1 + |Du_j|^p) \, dx \leq \frac{1}{k} \int_{B_{\rho_j}^{n-1}(x_i^{\varepsilon_j})^{\pm\delta_j}} (1 + |Du_j|^p) \, dx. \qquad (4.7)
$$

Hence, if we define the sequence

$$
w_j := \begin{cases} w_j^{i,k_i} & \text{in } (C_j^i)^{\pm\delta_j} \ \text{ for } \ i \in Z_j \\[2ex] u_j & \text{otherwise}\,, \end{cases}
$$

by the p-growth condition (3.3), (4.6), (4.7) and Remark 3.2 we have

$$\begin{aligned}\frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}}\big|W(Dw_j)-W(Du_j)\big|\,dx &= \sum_{i\in Z_j}\frac{1}{\delta_j}\int_{(C^i_j)^{\pm\delta_j}}\big|W(Dw_j^{i,k_i})-W(Du_j)\big|\,dx\\ &\leq \frac{c}{k}\,m_j(k,\gamma)\sum_{i\in Z_j}\frac{1}{\delta_j}\int_{B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{\pm\delta_j}}(1+|Du_j|^p)\,dx\\ &\leq \frac{c}{k}\,m_j(k,\gamma)\left(1+\sup_{j\in\mathbb{N}}\frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}}|Du_j|^p\,dx\right)\\ &\leq \frac{c}{k}\,m_j(k,\gamma)\,,\end{aligned}$$

which concludes the proof of (4.4). Note that, by construction, (w_j) satisfies (4.2) and (4.3) and it converges weakly to (u^+,u^-). In fact,

$$\begin{aligned}\frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}}|w_j-u^\pm|^p\,dx &= \frac{1}{\delta_j}\sum_{i\in Z_j}\int_{(C^i_j)^{\pm\delta_j}}|\phi u_j^{i\pm}+(1-\phi)u_j-u^\pm|^p\,dx\\ &\quad+\frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}\setminus\bigcup_{i\in Z_j}(C^i_j)^{\pm\delta_j}}|u_j-u^\pm|^p\,dx\\ &\leq \frac{c}{\delta_j}\int_{\omega^{\pm\delta_j}}|u_j-u^\pm|^p\,dx+\frac{c}{\delta_j}\sum_{i\in Z_j}\int_{(C^i_j)^{\pm\delta_j}}|u_j-u_j^{i\pm}|^p\,dx,\end{aligned}$$

while by Lemma 4.1 (i) applied with $\rho=\rho^i_j$ and since $\delta_j\ll\varepsilon_j$, $\rho^i_j\leq\varepsilon_j$, we get

$$\frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}}|w_j-u^\pm|^p\,dx\leq\frac{c}{\delta_j}\int_{\omega^{\pm\delta_j}}|u_j-u^\pm|^p\,dx+c\varepsilon_j^p\frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}}|Du_j|^p\,dx\,. \tag{4.8}$$

Moreover by (4.6) we have

$$\frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}}|Dw_j|^p\,dx\leq\frac{c}{\delta_j}\int_{\omega^{\pm\delta_j}}|Du_j|^p\,dx. \tag{4.9}$$

Hence (4.8), (4.9), the convergence of (u_j) towards (u^+,u^-), $\sup_j\frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}}|Du_j|^p\,dx<+\infty$ together with Remark 3.2 imply the weak convergence of (w_j) towards (u^+,u^-). ☐

REMARK 4.1. Note that to prove Lemma 4.2 we essentially use that $\rho_j<\varepsilon_j/2$ (but not necessarily equal to $\gamma\varepsilon_j$) and $\lim_{j\to+\infty}(\delta_j/\rho_j)=0$. Hence, Lemma 4.2 is still true if we replace the assumptions $\delta_j\ll\varepsilon_j$ and $\rho_j=\gamma\varepsilon_j$ by $\rho_j<\varepsilon_j/2$ and $\lim_{j\to+\infty}(\delta_j/\rho_j)=0$.

Since we will apply Lemma 4.2 when $\rho_j=\gamma\varepsilon_j$ ($\gamma<1/2$) and $\delta_j\ll\varepsilon_j$, we prefer to prove it directly under these assumptions.

If the sequence $(|Du_j|^p/\delta_j)$ is equi-integrable on $\omega^{\pm\delta_j}$ (see Definition 3.1), then we do not have to choose for every $i\in Z_j$ a suitable annulus C^i_j but we may consider the same radius independently of i as the following lemma shows.

LEMMA 4.3. *Let* (u_j), (ε_j), (δ_j), (ρ_j) *and* Z_j *be as in Lemma* 4.2 *and suppose that* $(|Du_j|^p/\delta_j)$ *is equi-integrable on* $\omega^{\pm\delta_j}$. *Set*

$$C_j^i := \left\{x_\alpha \in \omega : \frac{2}{3}\rho_j < |x_\alpha - x_i^{\varepsilon_j}| < \frac{4}{3}\rho_j\right\} \quad \text{and} \quad u_j^{i\pm} := \fint_{(C_j^i)^{\pm\delta_j}} u_j \, dx$$

for every $i \in Z_j$. *Then, there exists a sequence* $(w_j) \subset W^{1,p}(\omega^{+\delta_j} \cup \omega^{-\delta_j}; \mathbb{R}^m)$ *weakly converging to* (u^+, u^-) *such that*

$$w_j = u_j \quad \text{in} \quad \Big(\omega \setminus \bigcup_{i \in Z_j} C_j^i\Big)^{\pm\delta_j}, \tag{4.10}$$

$$w_j = u_j^{i\pm} \quad \text{on} \quad \big(\partial B_{\rho_j}^{n-1}(x_i^{\varepsilon_j})\big)^{\pm\delta_j} \tag{4.11}$$

and

$$\limsup_{j\to+\infty} \frac{1}{\delta_j} \int_{\omega^{\pm\delta_j}} \big|W(Dw_j) - W(Du_j)\big| \, dx \le o(1) \quad \text{as} \quad \gamma \to 0^+ . \tag{4.12}$$

Moreover, the sequence $(|Dw_j|^p/\delta_j)$ *is equi-integrable on* $\omega^{\pm\delta_j}$.

Proof. Let $\phi \equiv \phi_j^i \in \mathcal{C}_c^\infty(C_j^i; [0,1])$ be a cut-off function such that $\phi = 1$ on $\partial B_{\rho_j}^{n-1}(x_i^{\varepsilon_j})$ and $|D_\alpha \phi| \le c/\rho_j$. In $(C_j^i)^{\pm\delta_j}$, we define

$$w_j^i := \phi(x_\alpha) u_j^{i\pm} + (1 - \phi(x_\alpha)) u_j.$$

Then, reasoning as in the proof of Lemma 4.2, we have that

$$\int_{(C_j^i)^{\pm\delta_j}} W(Dw_j^i) \, dx \le c \int_{(C_j^i)^{\pm\delta_j}} (1 + |Du_j|^p) \, dx.$$

Hence, if we define

$$w_j := \begin{cases} w_j^i & \text{in } (C_j^i)^{\pm\delta_j} \text{ for } i \in Z_j, \\ u_j & \text{otherwise,} \end{cases}$$

w_j satisfies (4.10) and (4.11). Moreover,

$$\begin{aligned} \frac{1}{\delta_j} \int_{\omega^{\pm\delta_j}} \big|W(Dw_j) - W(Du_j)\big| \, dx &\le \sum_{i \in Z_j} \frac{1}{\delta_j} \int_{(C_j^i)^{\pm\delta_j}} \big|W(Dw_j^i) - W(Du_j)\big| \, dx \\ &\le c \sum_{i \in Z_j} \frac{1}{\delta_j} \int_{(B_{4\rho_j/3}^{n-1}(x_i^{\varepsilon_j}) \cap \omega)^{\pm\delta_j}} (1 + |Du_j|^p) \, dx. \end{aligned}$$

Since $\#(Z_j) \le c/\varepsilon_j^{n-1}$, we get that

$$\mathcal{H}^{n-1}\bigg(\bigcup_{i \in Z_j} (B_{4\rho_j/3}^{n-1}(x_i^{\varepsilon_j}) \cap \omega)\bigg) \le c\gamma^{n-1}$$

and by the equi-integrability of $(|Du_j|^p/\delta_j)$ we obtain (4.12). Finally, the weak convergence of (w_j) can be proved as in Lemma 4.2 while the equi-integrability of $(|Dw_j|^p/\delta_j)$ is just a consequence of the definition of (w_j). □

5. A preliminary analysis of the energy contribution 'close' to the connecting zones

For later references, in the following section we study the asymptotic behavior of a sequence of functions which will turn out to represent the energy contribution 'close' to the connecting zones. The results listed in this section will be applied in Section 6 to prove the Γ-convergence of $(\mathcal{F}_j)$ as well as in Section 7 to compute the explicit formula for $\varphi^{(\ell)}$.

Before starting, let us recall that we consider the domain $\Omega_j = \omega^{+\delta_j} \cup \omega^{-\delta_j} \cup \big(\omega_{r_j,\varepsilon_j} \times \{0\}\big)$ where $\omega_{r_j,\varepsilon_j} := \bigcup_{i\in\mathbb{Z}^{n-1}} B^{n-1}_{r_j}(x_i^{\varepsilon_j}) \cap \omega$. Our Γ-convergence analysis deals with the case where the thickness δ_j of Ω_j is much smaller than the period of distribution of the connecting zones ε_j; *i.e.*,

$$\lim_{j\to+\infty} \frac{\delta_j}{\varepsilon_j} = 0\,.$$

Moreover, we can exclude that $r_j \geq \varepsilon_j/2$ otherwise the zones may overlap. More precisely, we assume that $r_j \ll \varepsilon_j$; *i.e.*,

$$\lim_{j\to+\infty} \frac{r_j}{\varepsilon_j} = 0\,. \tag{5.1}$$

This choice will be justify a posteriori since (5.1) will be the only admissible assumption to get a non trivial Γ-convergence result (see Remark 3.4).

Finally, it remains to fix the behavior of r_j with respect to δ_j. Let us define

$$\ell := \lim_{j\to+\infty} \frac{r_j}{\delta_j}.$$

This yields to consider all the possible scenario, namely to distinguish between the cases: ℓ finite, infinite or zero.

For any fixed $\ell \in [0,+\infty]$, we consider the sequence of functions $(\varphi^{(\ell)}_{\gamma,j})$ defined in (5.2) and (5.13). Propositions 5.1 and 5.2 establish the existence of the function $\varphi^{(\ell)}$ as the (locally uniform) limit of $(\varphi^{(\ell)}_{\gamma,j})$ as $j \to +\infty$ and $\gamma \to 0^+$ while Proposition 5.3 will allow us to prove that $\varphi^{(\ell)}$ is actually the interfacial energy density in $\mathcal{F}^{(\ell)}$ (see e.g. Proposition 6.2).

5.1. The case $\ell \in (0,+\infty]$.

Setting $N_j = \varepsilon_j/r_j$, we define the space

$$X^\gamma_j(z) := \Big\{\zeta \in W^{1,p}((B^{n-1}_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j};\mathbb{R}^m) :\ \zeta = z \text{ on } (\partial B^{n-1}_{\gamma N_j})^+,\ \zeta = 0 \text{ on } (\partial B^{n-1}_{\gamma N_j})^-\Big\}\,,$$

where $I = (-1,1)$ and we consider the following minimum problem

$$\varphi^{(\ell)}_{\gamma,j}(z) := \inf\left\{ \int_{(B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}} r_j^p\, W\left(r_j^{-1} D_\alpha\zeta | \delta_j^{-1} D_n\zeta\right) dx :\quad \zeta \in X^\gamma_j(z) \right\}. \tag{5.2}$$

In the next proposition we study the behavior of $(\varphi^{(\ell)}_{\gamma,j})$ as $j \to +\infty$ and $\gamma \to 0^+$.

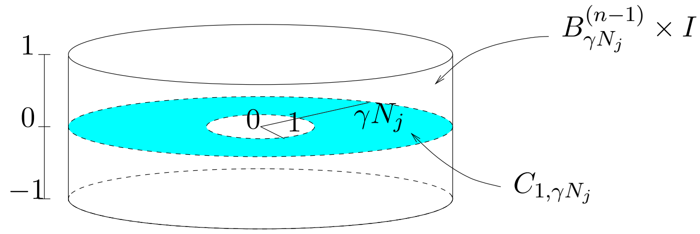

FIGURE 2. The domain $(B^{(n-1)}_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}$.

PROPOSITION 5.1. *Let* $\ell \in (0, +\infty]$. *If*

$$0 < R^{(\ell)} := \lim_{j \to +\infty} \frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}} < +\infty \tag{5.3}$$

then,

(i) there exists a constant $c > 0$ *(independent of* j *and* γ*) such that*

$$0 \leq \varphi^{(\ell)}_{\gamma,j}(z) \leq c\left(|z|^p + \gamma^{n-1}\right)$$

for all $z \in \mathbb{R}^m$, $j \in \mathbb{N}$ *and* $\gamma > 0$*;*

(ii) there exists a constant $c > 0$ *(independent of* j *and* γ*) such that*

$$|\varphi^{(\ell)}_{\gamma,j}(z) - \varphi^{(\ell)}_{\gamma,j}(w)| \leq c\,|z - w|\left(\gamma^{(n-1)(p-1)/p} + r_j^{p-1} + |z|^{p-1} + |w|^{p-1}\right) \tag{5.4}$$

for every $z, w \in \mathbb{R}^m$, $j \in \mathbb{N}$ *and* $\gamma > 0$*;*

(iii) for every fixed $\gamma > 0$*, up to subsequences,* $\varphi^{(\ell)}_{\gamma,j}$ *converges locally uniformly on* $\mathbb{R}^m$ *to* $\varphi^{(\ell)}_{\gamma}$ *as* $j \to +\infty$ *and*

$$|\varphi^{(\ell)}_{\gamma}(z) - \varphi^{(\ell)}_{\gamma}(w)| \leq c\,|z - w|\left(\gamma^{(n-1)(p-1)/p} + |z|^{p-1} + |w|^{p-1}\right) \tag{5.5}$$

for every $z, w \in \mathbb{R}^m$ *;*

(iv) up to subsequences, $\varphi^{(\ell)}_{\gamma}$ *converges locally uniformly on* $\mathbb{R}^m$*, as* $\gamma \to 0^+$*, to a continuous function* $\varphi^{(\ell)} : \mathbb{R}^m \to [0, +\infty)$ *satisfying*

$$0 \leq \varphi^{(\ell)}(z) \leq c|z|^p\,, \qquad |\varphi^{(\ell)}(z) - \varphi^{(\ell)}(w)| \leq c\,|z - w|\left(|z|^{p-1} + |w|^{p-1}\right) \tag{5.6}$$

for every $z, w \in \mathbb{R}^m$.

Proof. Fix $\gamma > 0$, then $\gamma N_j > 2$ for j large enough.

(i) According to the p-growth condition (3.3),

$$0 \leq \varphi^{(\ell)}_{\gamma,j}(z) \leq \beta\left(\mathcal{C}_{\gamma,j}(z) + \mathcal{H}^{n-1}(B_1^{n-1})\gamma^{n-1}\frac{\varepsilon_j^{n-1}}{r_j^{n-1-p}}\right), \tag{5.7}$$

where

$$\mathcal{C}_{\gamma,j}(z) := \inf\left\{\int_{(B^{n-1}_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} \left|\left(D_\alpha\zeta\Big|\frac{r_j}{\delta_j}D_n\zeta\right)\right|^p dx : \quad \zeta \in X_j^\gamma(z)\right\}.$$

Since $\mathcal{C}_{\gamma,j}(z)$ is invariant by rotations, reasoning as in [**5**] Section 4.1, we can consider the minimization problem with respect to a particular class of scalar test functions as follows

$$\begin{aligned}\frac{\mathcal{C}_{\gamma,j}(z)}{|z|^p} &= \inf\left\{ \int_{(B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}} \left|\left(D_\alpha\psi\Big|\frac{r_j}{\delta_j}D_n\psi\right)\right|^p dx \,:\, \psi\in W^{1,p}((B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}),\right.\\ &\qquad\qquad \left.\psi=1 \text{ on } \left(\partial B^{n-1}_{\gamma N_j}\right)^+ \text{ and } \psi=0 \text{ on } \left(\partial B^{n-1}_{\gamma N_j}\right)^-\right\}\\ &\le \inf\left\{\int_{B^{n-1}_{\gamma N_j}} \left(|D_\alpha\psi^+|^p+|D_\alpha\psi^-|^p\right)dx :\quad (\psi^+-1)\,,\,\psi^-\in W^{1,p}_0(B^{n-1}_{\gamma N_j})\right.\\ &\qquad\qquad \left.\text{and } \psi^+=\psi^- \text{ in } B^{n-1}_1\right\}. \end{aligned} \tag{5.8}$$

Let $\psi_1^\pm$ be the unique minimizer of the strictly convex minimization problem (5.8). It turns out that $\psi_2^\pm := 1-\psi_1^\mp$ is also a minimizer. Thus by uniqueness, $\psi_1^\pm=\psi_2^\pm$ and in particular, $\psi_1^\pm=1/2$ in B_1^{n-1}. Hence,

$$\begin{aligned}\mathcal{C}_{\gamma,j}(z) &\le |z|^p\inf\left\{\int_{B^{n-1}_{\gamma N_j}}\left(|D_\alpha\psi^+|^p+|D_\alpha\psi^-|^p\right)dx_\alpha \,:\, (\psi^+-1)\,,\,\psi^-\in W^{1,p}_0(B^{n-1}_{\gamma N_j}),\right.\\ &\qquad\qquad\left.\text{and } \psi^+=\psi^-=\frac{1}{2} \text{ in } B^{n-1}_1\right\}\\ &= 2|z|^p\inf\left\{\int_{B^{n-1}_{\gamma N_j}}|D_\alpha\psi|^p\,dx_\alpha :\quad \psi\in W^{1,p}_0(B^{n-1}_{\gamma N_j}) \text{ and } \psi=\frac{1}{2} \text{ in } B^{n-1}_1\right\}\\ &= \frac{|z|^p}{2^{p-1}}\inf\left\{\int_{B^{n-1}_{\gamma N_j}}|D_\alpha\psi|^p\,dx_\alpha :\quad \psi\in W^{1,p}_0(B^{n-1}_{\gamma N_j}) \text{ and } \psi=1 \text{ in } B^{n-1}_1\right\}\\ &= \frac{|z|^p}{2^{p-1}}\mathrm{Cap}_p\big(B^{n-1}_1;B^{n-1}_{\gamma N_j}\big). \end{aligned} \tag{5.9}$$

Since

$$\lim_{j\to+\infty}\mathrm{Cap}_p\big(B^{n-1}_1;B^{n-1}_{\gamma N_j}\big)=\mathrm{Cap}_p\big(B^{n-1}_1;\mathbb{R}^{n-1}\big)<+\infty\,;$$

hence, by (5.3), (5.7) and (5.9) we conclude the proof of (i).

(ii) For every $\eta>0$, there exists $\zeta_{\gamma,j}\in X^\gamma_j(z)$ such that

$$\int_{(B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}} r_j^p\,W\left(r_j^{-1}D_\alpha\zeta_{\gamma,j}|\delta_j^{-1}D_n\zeta_{\gamma,j}\right)dx\le\varphi^{(\ell)}_{\gamma,j}(z)+\eta. \tag{5.10}$$

We want to modify $\zeta_{\gamma,j}$ in order to get an admissible test function for $\varphi^{(\ell)}_{\gamma,j}(w)$. More precisely, we just have to modify $\zeta_{\gamma,j}$ on a neighborhood of $(\partial B^{n-1}_{\gamma N_j})^+$ to change the boundary condition z into w. To this aim we introduce a cut-off function $\theta\in\mathcal{C}^\infty_c(\mathbb{R}^{n-1};[0,1])$, independent of x_n, such that

$$\theta(x_\alpha) = \begin{cases} 1 & \text{if} \quad x_\alpha \in B_1^{n-1}, \\ \\ 0 & \text{if} \quad x_\alpha \notin B_2^{n-1} \end{cases} \qquad \text{and} \qquad |D_\alpha \theta| \le c\,.$$

Hence, we define $\tilde{\zeta}_{\gamma,j} \in X_j^\gamma(w)$ as follows

$$\tilde{\zeta}_{\gamma,j} = \begin{cases} \zeta_{\gamma,j} + (1-\theta(x_\alpha))(w-z) & \text{in} \quad (B_{\gamma N_j}^{n-1})^+ \\ \\ \zeta_{\gamma,j} & \text{in} \quad (B_{\gamma N_j}^{n-1})^- \cup \big(B_1^{n-1} \times \{0\}\big)\,. \end{cases}$$

By (5.10), since $\zeta_{\gamma,j} = \tilde{\zeta}_{\gamma,j}$ in $(B_{\gamma N_j}^{n-1})^-$, we have that

$$\begin{aligned} & \varphi_{\gamma,j}^{(\ell)}(w) - \varphi_{\gamma,j}^{(\ell)}(z) \\ \le\ & r_j^p \int_{(B_{\gamma N_j}^{n-1} \times I) \setminus C_{1,\gamma N_j}} \Big(W\big(r_j^{-1} D_\alpha \tilde{\zeta}_{\gamma,j} | \delta_j^{-1} D_n \tilde{\zeta}_{\gamma,j}\big) - W\big(r_j^{-1} D_\alpha \zeta_{\gamma,j} | \delta_j^{-1} D_n \zeta_{\gamma,j}\big)\Big)\, dx \,+ \eta \\ =\ & r_j^p \int_{(B_{\gamma N_j}^{n-1})^+} \Big(W\big(r_j^{-1} D_\alpha \tilde{\zeta}_{\gamma,j} | \delta_j^{-1} D_n \tilde{\zeta}_{\gamma,j}\big) - W\big(r_j^{-1} D_\alpha \zeta_{\gamma,j} | \delta_j^{-1} D_n \zeta_{\gamma,j}\big)\Big)\, dx \,+ \eta\,. \end{aligned}$$

By (3.4) and Hölder's Inequality, we obtain that

$$\begin{aligned} & \varphi_{\gamma,j}^{(\ell)}(w) - \varphi_{\gamma,j}^{(\ell)}(z) \,-\eta \\ \le\ & c \int_{(B_{\gamma N_j}^{n-1})^+} \left(r_j^{p-1} + \left| \left(D_\alpha \zeta_{\gamma,j} \Big| \frac{r_j}{\delta_j} D_n \zeta_{\gamma,j} \right) \right|^{p-1} + \left| \left(D_\alpha \tilde{\zeta}_{\gamma,j} \Big| \frac{r_j}{\delta_j} D_n \tilde{\zeta}_{\gamma,j} \right) \right|^{p-1} \right) \\ & \qquad\qquad \times \left| \left(D_\alpha \tilde{\zeta}_{\gamma,j} - D_\alpha \zeta_{\gamma,j} \Big| \frac{r_j}{\delta_j} (D_n \tilde{\zeta}_{\gamma,j} - D_n \zeta_{\gamma,j}) \right) \right| \, dx \\ \le\ & c \int_{(B_{\gamma N_j}^{n-1})^+} \left(r_j^{p-1} + 2 \left| \left(D_\alpha \zeta_{\gamma,j} \Big| \frac{r_j}{\delta_j} D_n \zeta_{\gamma,j} \right) \right|^{p-1} + |D_\alpha \theta|^{p-1} \, |w-z|^{p-1} \right) |D_\alpha \theta| \, |w-z| \, dx \\ \le\ & c\,|z-w|^p \int_{B_{\gamma N_j}^{n-1}} |D_\alpha \theta|^p \, dx_\alpha + c\, r_j^{p-1} |z-w| \int_{B_{\gamma N_j}^{n-1}} |D_\alpha \theta| \, dx_\alpha \\ & +2c\,|z-w| \, \|D_\alpha \theta\|_{L^p(B_{\gamma N_j}^{n-1}; \mathbb{R}^{n-1})} \left\| \left(D_\alpha \zeta_{\gamma,j} \Big| \frac{r_j}{\delta_j} D_n \zeta_{\gamma,j} \right) \right\|_{L^p\big((B_{\gamma N_j}^{n-1})^+; \mathbb{R}^{m \times n}\big)}^{p-1}\,. \end{aligned}$$

Since $\gamma N_j > 2$ and $\mathrm{Supp}(\theta) \subset B_2^{n-1}$, we obtain that

$$\begin{aligned} & \varphi_{\gamma,j}^{(\ell)}(w) - \varphi_{\gamma,j}^{(\ell)}(z) \\ & \quad \le c|z-w| \left(|z-w|^{p-1} + r_j^{p-1} + \left\| \left(D_\alpha \zeta_{\gamma,j} \Big| \frac{r_j}{\delta_j} D_n \zeta_{\gamma,j} \right) \right\|_{L^p\big((B_{\gamma N_j}^{n-1})^+; \mathbb{R}^{m \times n}\big)}^{p-1} \right) + \eta. \qquad (5.11) \end{aligned}$$

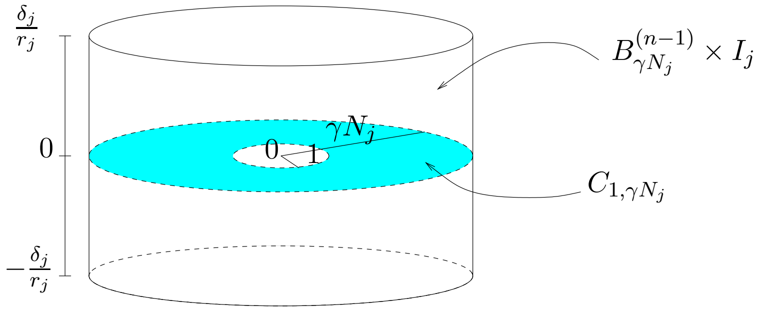

FIGURE 3. The domain $(B^{(n-1)}_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}$.

By the p-growth condition (3.3), (5.10) and (i), we have that

$$
\begin{aligned}
&\int_{(B^{n-1}_{\gamma N_j})^+} \left| \left(D_\alpha \zeta_{\gamma,j} \Big| \frac{r_j}{\delta_j} D_n \zeta_{\gamma,j} \right) \right|^p dx \\
\leq\ & \int_{(B^{n-1}_{\gamma N_j})^+} r_j^p \, W\left(r_j^{-1} D_\alpha \zeta_{\gamma,j} | \delta_j^{-1} D_n \zeta_{\gamma,j} \right) dx + r_j^p \, \mathcal{H}^{n-1}(B^{n-1}_{\gamma N_j}) \\
\leq\ & \varphi^{(\ell)}_{\gamma,j}(z) + \eta + c\gamma^{n-1} \frac{\varepsilon_j^{n-1}}{r_j^{n-1-p}} \\
\leq\ & c(|z|^p + \gamma^{n-1}) + \eta + c\gamma^{n-1} \frac{\varepsilon_j^{n-1}}{r_j^{n-1-p}}. \qquad (5.12)
\end{aligned}
$$

Hence, by (5.11), (5.12) and (5.3) we have that

$$
\varphi^{(\ell)}_{\gamma,j}(w) - \varphi^{(\ell)}_{\gamma,j}(z) \leq c\,|z-w| \Big(|z|^{p-1} + |w|^{p-1} + r_j^{p-1} + \gamma^{(n-1)(p-1)/p} + \eta^{(p-1)/p} \Big) + \eta
$$

and (5.4) follows by the arbitrariness of η.

By (ii) and Ascoli-Arzela's Theorem we have that, up to subsequences, $\varphi^{(\ell)}_{\gamma,j}$ converges uniformly on compact sets of $\mathbb{R}^m$ to $\varphi^{(\ell)}_\gamma$ as $j \to +\infty$. Moreover, passing to the limit in (5.4) as $j \to +\infty$ we get

$$
|\varphi^{(\ell)}_\gamma(w) - \varphi^{(\ell)}_\gamma(z)| \leq c\,|z-w| \Big(|z|^{p-1} + |w|^{p-1} + \gamma^{(n-1)(p-1)/p} \Big).
$$

Hence, we can apply again Ascoli-Arzela's Theorem to conclude that, up to subsequences, $\varphi^{(\ell)}_\gamma$ converges uniformly on compact sets of $\mathbb{R}^m$ to $\varphi^{(\ell)}$ as $\gamma \to 0^+$. In particular, $\varphi^{(\ell)} : \mathbb{R}^m \to [0, +\infty)$ is a continuous function and

$$
0 \leq \varphi^{(\ell)}(z) \leq c|z|^p, \qquad |\varphi^{(\ell)}(z) - \varphi^{(\ell)}(w)| \leq c\left(|z|^{p-1} + |w|^{p-1}\right)|z-w|
$$

for every $z, w \in \mathbb{R}^m$. $\square$

5.2. The case $\ell = 0$. In this case we expect that the energy contribution due to the presence of the sieve is obtained studying the behavior, as $j \to +\infty$ and $\gamma \to 0^+$, of the sequence $(\varphi^{(0)}_{\gamma,j})$ defined as follows

$$\begin{aligned}\varphi^{(0)}_{\gamma,j}(z) \quad &:= \quad \frac{\delta_j}{r_j} \inf\left\{ \int_{(B^{n-1}_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} r_j^p \, W\left(r_j^{-1} D_\alpha \zeta | \delta_j^{-1} D_n \zeta\right) dx : \quad \zeta \in X_j^\gamma(z) \right\} \\ &= \quad \inf\left\{ \int_{(B^{n-1}_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} r_j^p \, W(r_j^{-1} D\zeta)\, dx : \quad \zeta \in Y_j^\gamma(z) \right\} \end{aligned} \tag{5.13}$$

where $I_j := (-\delta_j/r_j, \delta_j/r_j)$ and

$$\begin{aligned} Y_j^\gamma(z) = \Big\{ \zeta \in W^{1,p}((B^{n-1}_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}; \mathbb{R}^m) \quad : \quad &\zeta = z \text{ on } (\partial B^{n-1}_{\gamma N_j})^{+(\delta_j/r_j)}, \\ &\zeta = 0 \text{ on } (\partial B^{n-1}_{\gamma N_j})^{-(\delta_j/r_j)} \Big\}. \end{aligned}$$

Note that in this case we are interested in the limit behavior of a sequence that is obtained from the one corresponding to $\ell \in (0, +\infty]$ multiplying it by δ_j/r_j (see (5.13) and recall (5.2)). Let us try to motivate this choice.

Let $\ell \in (0, +\infty)$, then starting from (5.2) by a change of variable it is immediate to check that

$$\varphi^{(\ell)}_{\gamma,j}(z) = \frac{r_j}{\delta_j} \inf\left\{ \int_{(B^{n-1}_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} r_j^p \, W(r_j^{-1} D\zeta)\, dx : \quad \zeta \in Y_j^\gamma(z) \right\}. \tag{5.14}$$

Now assuming that $\lim_{j\to+\infty} r_j^{n-p}/(\delta_j\, \varepsilon_j^{n-1}) < +\infty$ (or equivalently that $\lim_{j\to+\infty} r_j^{n-1-p}/\varepsilon_j^{n-1} < +\infty$; see Remark 3.5) we know that the sequence $(\varphi^{(\ell)}_{\gamma,j})$ converges to $\ell\, \tilde{\varphi}^{(\ell)}$, for some $\tilde{\varphi}^{(\ell)}$, locally uniformly in $\mathbb{R}^m$, as $j \to +\infty$ and $\gamma \to 0^+$ (Proposition 5.1). Then if $\ell \in (0, +\infty)$, studying the limit behavior of (5.13) is perfectly equivalent to study the limit behavior of (5.2). While if $\ell = \lim_{j\to+\infty} r_j/\delta_j = 0$, (5.14) suggests that, to recover nontrivial information in the limit, we have to study the asymptotic behavior of the sequence obtained from (5.14) dividing it by r_j/δ_j, that is to study the asymptotic behavior of the sequence given by (5.13).

Following the line of the proof of Proposition 5.1, we want to establish an analogous result for the sequence $(\varphi^{(0)}_{\gamma,j})$.

PROPOSITION 5.2. *Let $\ell = 0$. If*

$$0 < R^{(0)} = \lim_{j\to+\infty} \frac{r_j^{n-p}}{\varepsilon_j^{n-1}\delta_j} < +\infty \tag{5.15}$$

then,

(i) there exists a constant $c > 0$ (independent of j and γ) such that

$$0 \le \varphi^{(0)}_{\gamma,j}(z) \le c\left(|z|^p + \gamma^{n-1}\right)$$

for all $z \in \mathbb{R}^m$, $j \in \mathbb{N}$ and $\gamma > 0$;

(ii) there exists a constant $c > 0$ (independent of j and γ) such that

$$|\varphi^{(0)}_{\gamma,j}(z) - \varphi^{(0)}_{\gamma,j}(w)| \leq c\,|z-w|\left(\gamma^{(n-1)(p-1)/p} + r_j^{n-1} + |z|^{p-1} + |w|^{p-1}\right) \tag{5.16}$$

for every $z, w \in \mathbb{R}^m$, $j \in \mathbb{N}$ and $\gamma > 0$;

(iii) for every fixed $\gamma > 0$, up to subsequences, $\varphi^{(0)}_{\gamma,j}$ converges locally uniformly in $\mathbb{R}^m$ to $\varphi^{(0)}_{\gamma}$ as $j \to +\infty$, and

$$|\varphi^{(0)}_{\gamma}(z) - \varphi^{(0)}_{\gamma}(w)| \leq c\,|z-w|\Big(\gamma^{(n-1)(p-1)/p} + |z|^{p-1} + |w|^{p-1}\Big) \tag{5.17}$$

for every $z, w \in \mathbb{R}^m$;

(iv) up to subsequences, $\varphi^{(0)}_{\gamma}$ converges locally uniformly in $\mathbb{R}^m$, as $\gamma \to 0^+$, to a continuous function $\varphi^{(0)} : \mathbb{R}^m \to [0, +\infty)$ satisfying

$$0 \leq \varphi^{(0)}(z) \leq c|z|^p, \qquad |\varphi^{(0)}(z) - \varphi^{(0)}(w)| \leq c\,|z-w|\left(|z|^{p-1} + |w|^{p-1}\right) \tag{5.18}$$

for every $z, w \in \mathbb{R}^m$.

Proof. Fix $\gamma > 0$, then $\gamma N_j > 2$ and $\delta_j / r_j > 2$ for j large enough.

(i) According to the p-growth condition (3.3),

$$0 \leq \varphi^{(0)}_{\gamma,j}(z) \leq \beta\left(\mathcal{C}_{\gamma,j}(z) + 2\mathcal{H}^{n-1}(B_1^{n-1})\,\gamma^{n-1}\,\frac{\delta_j\,\varepsilon_j^{n-1}}{r_j^{n-p}}\right), \tag{5.19}$$

where

$$\mathcal{C}_{\gamma,j}(z) = \inf\left\{\int_{(B^{n-1}_{\gamma N_j} \times I_j)\setminus C_{1,\gamma N_j}} |D\zeta|^p\,dx : \quad \zeta \in Y_j^{\gamma}(z)\right\}.$$

Arguing similarly than in the proof of Proposition 5.1, we can rewrite

$$\begin{aligned}\frac{\mathcal{C}_{\gamma,j}(z)}{|z|^p} \;=\; \inf\Bigg\{ & \int_{(B^{n-1}_{\gamma N_j} \times I_j)\setminus C_{1,\gamma N_j}} |D\psi|^p\,dx \;:\; \psi \in W^{1,p}((B^{n-1}_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}), \\ & \psi = 1 \text{ on } (\partial B^{n-1}_{\gamma N_j})^{+(\delta_j/r_j)}, \quad \psi = 0 \text{ on } (\partial B^{n-1}_{\gamma N_j})^{-(\delta_j/r_j)}\Bigg\}.\end{aligned} \tag{5.20}$$

Let ψ_1 be the unique minimizer of the strictly convex minimization problem (5.20). It turns out that $\psi_2(x_\alpha, x_n) := 1 - \psi_1(x_\alpha, -x_n)$ is also a minimizer. Thus by uniqueness, $\psi_1 = \psi_2$ and in particular, $\psi_1 = \psi_2 = 1/2$ on $B_1^{n-1} \times \{0\}$. Thus

$$\begin{aligned}
\mathcal{C}_{\gamma,j}(z) &= 2|z|^p \inf \Bigg\{ \int_{(B_{\gamma N_j}^{n-1})^{+(\delta_j/r_j)}} |D\psi|^p \, dx : \quad \psi \in W^{1,p}((B_{\gamma N_j}^{n-1})^{+(\delta_j/r_j)}), \\
&\qquad \psi = 0 \text{ on } (\partial B_{\gamma N_j}^{n-1})^{+(\delta_j/r_j)} \text{ and } \psi = \frac{1}{2} \text{ on } B_1^{n-1} \times \{0\} \Bigg\} \\
&= \frac{|z|^p}{2^{p-1}} \inf \Bigg\{ \int_{(B_{\gamma N_j}^{n-1})^{+(\delta_j/r_j)}} |D\psi|^p \, dx : \quad \psi \in W^{1,p}((B_{\gamma N_j}^{n-1})^{+(\delta_j/r_j)}), \\
&\qquad \psi = 0 \text{ on } (\partial B_{\gamma N_j}^{n-1})^{+(\delta_j/r_j)} \text{ and } \psi = 1 \text{ on } B_1^{n-1} \times \{0\} \Bigg\} \\
&\leq \frac{|z|^p}{2^p} \mathrm{Cap}_p\big(B_1^{n-1}; B_{\gamma N_j}^{n-1} \times I_j\big) \,.
\end{aligned} \tag{5.21}$$

Since

$$\lim_{j \to +\infty} \mathrm{Cap}_p\big(B_1^{n-1}; B_{\gamma N_j}^{n-1} \times I_j\big) = \mathrm{Cap}_p\big(B_1^{n-1}; \mathbb{R}^n\big) < +\infty \,;$$

hence, by (5.15), (5.19) and (5.21) we conclude the proof of (i).

(ii) We can proceed as in the proof of Proposition 5.1 (ii) using a different cut-off function also depending on x_n. Namely, let $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^n; [0,1])$ be such that

$$\theta(x_\alpha, x_n) = \begin{cases} 1 & \text{if} \quad (x_\alpha, x_n) \in B_1^{n-1} \times (-1,1), \\ \\ 0 & \text{if} \quad (x_\alpha, x_n) \notin B_2^{n-1} \times (-2,2) \end{cases} \qquad \text{and} \qquad |D\theta| \leq c.$$

Hence, if $\zeta_{\gamma,j} \in Y_j^\gamma(z)$ is a sequence which 'almost attains' the infimum value $\varphi_{\gamma,j}^{(0)}$, we define $\tilde{\zeta}_{\gamma,j} \in Y_j^\gamma(w)$ as follows

$$\tilde{\zeta}_{\gamma,j} = \begin{cases} \zeta_{\gamma,j} + (1 - \theta(x))(w - z) & \text{in} \quad (B_{\gamma N_j}^{n-1})^{+(\delta_j/r_j)}, \\ \\ \zeta_{\gamma,j} & \text{in} \quad \big((B_{\gamma N_j}^{n-1})^{-(\delta_j/r_j)}\big) \cup \big(B_1^{n-1} \times \{0\}\big). \end{cases}$$

By (5.15) we conclude the proof of (ii) reasoning as in the proof of Proposition 5.1 (ii).

The proof of (iii) and (iv) follows the line of the proof of (iii) and (iv) in Proposition 5.1. □

Now we are able to describe the energy contribution close to the connecting zones as $j \to +\infty$ and $\gamma \to 0^+$.

Proposition 5.3 (Discrete approximation of the interfacial energy). *Let $(u_j) \subset W^{1,p}(\Omega_j; \mathbb{R}^m) \cap L^\infty(\Omega_j; \mathbb{R}^m)$ be a sequence converging to $(u^+, u^-) \in W^{1,p}(\omega; \mathbb{R}^m) \times W^{1,p}(\omega; \mathbb{R}^m)$ such that $\sup_j \mathcal{F}_j(u_j) < +\infty$ and satisfying $\sup_{j \in \mathbb{N}} \|u_j\|_{L^\infty(\Omega_j; \mathbb{R}^m)} < +\infty$. Let $(u_j^{i\pm})$ be as in* (4.1). *If*

$$\ell \in (0, +\infty] \qquad \textit{and} \qquad 0 < R^{(\ell)} = \lim_{j \to +\infty} \frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}} < +\infty$$

or

$$\ell = 0 \quad \textit{and} \quad 0 < R^{(0)} = \lim_{j\to+\infty} \frac{r_j^{n-p}}{\delta_j \varepsilon_j^{n-1}} < +\infty$$

then

$$\lim_{\gamma\to0^+} \limsup_{j\to+\infty} \int_\omega \Big| \sum_{i\in Z_j} \varphi_{\gamma,j}^{(\ell)}(u_j^{i+} - u_j^{i-}) \chi_{Q_{i,\varepsilon_j}^{n-1}} - \varphi^{(\ell)}(u^+ - u^-) \Big| \, dx_\alpha = 0 \,, \tag{5.22}$$

for every $\ell \in [0, +\infty]$.

Proof. Since $\sup_{j\in\mathbb{N}} \|u_j\|_{L^\infty(\Omega_j;\mathbb{R}^m)} < +\infty$ by Propositions 5.1 or 5.2 we have that

$$\begin{aligned}
&\limsup_{j\to+\infty} \int_\omega \Big| \sum_{i\in Z_j} \varphi_{\gamma,j}^{(\ell)}(u_j^{i+} - u_j^{i-}) \chi_{Q_{i,\varepsilon_j}^{n-1}} - \varphi^{(\ell)}(u^+ - u^-) \Big| \, dx_\alpha \\
\le\ & \limsup_{j\to+\infty} \int_\omega \sum_{i\in Z_j} \Big| \varphi_{\gamma,j}^{(\ell)}(u_j^{i+} - u_j^{i-}) - \varphi^{(\ell)}(u_j^{i+} - u_j^{i-}) \Big| \chi_{Q_{i,\varepsilon_j}^{n-1}} \, dx_\alpha \\
&+ \limsup_{j\to+\infty} \int_\omega \Big| \sum_{i\in Z_j} \varphi^{(\ell)}(u_j^{i+} - u_j^{i-}) \chi_{Q_{i,\varepsilon_j}^{n-1}} - \varphi^{(\ell)}(u^+ - u^-) \Big| \, dx_\alpha \\
\le\ & o(1) + \limsup_{j\to+\infty} \int_\omega \Big| \sum_{i\in Z_j} \varphi^{(\ell)}(u_j^{i+} - u_j^{i-}) \chi_{Q_{i,\varepsilon_j}^{n-1}} - \varphi^{(\ell)}(u^+ - u^-) \Big| \, dx_\alpha,
\end{aligned}$$

as $\gamma \to 0^+$. By (5.6) or (5.18) and Hölder's Inequality we have that

$$\begin{aligned}
&\limsup_{j\to+\infty} \int_\omega \Big| \sum_{i\in Z_j} \varphi^{(\ell)}(u_j^{i+} - u_j^{i-}) \chi_{Q_{i,\varepsilon_j}^{n-1}} - \varphi^{(\ell)}(u^+ - u^-) \Big| \, dx_\alpha \\
=\ & \limsup_{j\to+\infty} \sum_{i\in Z_j} \int_{Q_{i,\varepsilon_j}^{n-1}} |\varphi^{(\ell)}(u_j^{i+} - u_j^{i-}) - \varphi^{(\ell)}(u^+ - u^-)| \, dx_\alpha \\
\le\ & c \limsup_{j\to+\infty} \Big(\sum_{i\in Z_j} \int_{Q_{i,\varepsilon_j}^{n-1}} |u_j^{i+} - u^+|^p + |u_j^{i-} - u^-|^p \, dx_\alpha \Big)^{1/p} .
\end{aligned}$$

Hence, it remains to prove that

$$\limsup_{j\to+\infty} \sum_{i\in Z_j} \int_{Q_{i,\varepsilon_j}^{n-1}} |u^\pm - u_j^{i\pm}|^p \, dx_\alpha = 0 \,. \tag{5.23}$$

By Lemma 4.1 (ii) applied with $\rho = \varepsilon_j$, $B_\rho = C_j^i$ and $A_\rho = Q_{i,\varepsilon_j}^{n-1}$ and since $\delta_j \ll \varepsilon_j$, we have

$$\begin{aligned}
\int_{Q_{i,\varepsilon_j}^{n-1}} |u^\pm - u_j^{i\pm}|^p \, dx_\alpha \ &\le\ \frac{c}{\delta_j} \Big(\int_{(Q_{i,\varepsilon_j}^{n-1})^{\pm\delta_j}} |u_j - u^\pm|^p \, dx + \int_{(Q_{i,\varepsilon_j}^{n-1})^{\pm\delta_j}} |u_j - u_j^{i\pm}|^p \, dx \Big) \\
&\le\ \frac{c}{\delta_j} \int_{(Q_{i,\varepsilon_j}^{n-1})^{\pm\delta_j}} |u_j - u^\pm|^p \, dx + \frac{c\,\varepsilon_j^p}{\delta_j} \int_{(Q_{i,\varepsilon_j}^{n-1})^{\pm\delta_j}} |Du_j|^p \, dx \,, \quad (5.24)
\end{aligned}$$

for all $i \in Z_j$; hence, summing up on $i \in Z_j$, we find

$$\sum_{i\in Z_j} \int_{Q_{i,\varepsilon_j}^{n-1}} |u_j - u_j^{i\pm}|^p \, dx_\alpha \le \frac{c}{\delta_j} \int_{\omega^{\pm\delta_j}} |u_j - u^\pm|^p \, dx + \frac{c\,\varepsilon_j^p}{\delta_j} \int_{\omega^{\pm\delta_j}} |Du_j|^p \, dx \,,$$

then passing to the limit as $j \to +\infty$ by the convergence of (u_j) towards (u^+, u^-) and $\sup_j \mathcal{F}_j(u_j) < +\infty$ we get (5.23) and then (5.22). $\square$

6. Γ-convergence result

6.1. The liminf inequality. Let $(u_j) \subset W^{1,p}(\Omega_j;\mathbb{R}^m) \cap L^\infty(\Omega_j;\mathbb{R}^m)$ be a sequence converging to $(u^+, u^-) \in W^{1,p}(\omega,\mathbb{R}^m) \times W^{1,p}(\omega,\mathbb{R}^m)$ such that $\sup_{j\in\mathbb{N}} \|u_j\|_{L^\infty(\Omega_j;\mathbb{R}^m)} < +\infty$ and

$$\liminf_{j\to+\infty} \mathcal{F}_j(u_j) < +\infty\,.$$

By Lemma 4.2, for every fixed $k \in \mathbb{N}$, there exists a sequence $(w_j) \subset W^{1,p}(\Omega_j;\mathbb{R}^m) \cap L^\infty(\Omega_j;\mathbb{R}^m)$ weakly converging to (u^+, u^-) satisfying (4.2), (4.3) and such that

$$\begin{aligned}
&\liminf_{j\to+\infty} \frac{1}{\delta_j}\left(\int_{\omega^{+\delta_j}} W(Du_j)\,dx + \int_{\omega^{-\delta_j}} W(Du_j)\,dx\right)\\
\geq\ &\liminf_{j\to+\infty} \frac{1}{\delta_j}\left(\int_{\omega^{+\delta_j}} W(Dw_j)\,dx + \int_{\omega^{-\delta_j}} W(Dw_j)\,dx\right) - \frac{c}{k}\\
\geq\ &\liminf_{j\to+\infty} \frac{1}{\delta_j}\left(\int_{(\omega\setminus E_j)^{+\delta_j}} W(Dw_j)\,dx + \int_{(\omega\setminus E_j)^{-\delta_j}} W(Dw_j)\,dx\right)\\
&+\liminf_{j\to+\infty} \frac{1}{\delta_j}\left(\int_{E_j^{+\delta_j}} W(Dw_j)\,dx + \int_{E_j^{-\delta_j}} W(Dw_j)\,dx\right) - \frac{c}{k},
\end{aligned} \tag{6.1}$$

where $E_j := \bigcup_{i\in Z_j} B^{n-1}_{\rho^i_j}(x_i^{\varepsilon_j})$.

We first consider the energy contribution 'far' from the connecting zones. In this case, we suitably modify the sequence (w_j) in order to get a constant inside each half cylinder $B^{(n-1)}_{\rho^i_j}(x_i^{\varepsilon_j})^{\pm\delta_j}$. Then, we apply the classical result of dimensional reduction proved in [**38**] to $\omega^{+\delta_j}$ and $\omega^{-\delta_j}$, separately.

Proposition 6.1. *We have*

$$\begin{aligned}
&\liminf_{j\to+\infty} \frac{1}{\delta_j}\left(\int_{(\omega\setminus E_j)^{+\delta_j}} W(Dw_j)\,dx + \int_{(\omega\setminus E_j)^{-\delta_j}} W(Dw_j)\,dx\right)\\
&\geq \int_\omega \left(\mathcal{Q}_{n-1}\overline{W}(D_\alpha u^+) + \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^-)\right)\,dx_\alpha.
\end{aligned}$$

Proof. We define

$$v_j := \begin{cases} w_j & \text{in} \quad (\omega\setminus E_j)^{\pm\delta_j},\\ u_j^{i\pm} & \text{in} \quad B^{n-1}_{\rho^i_j}(x_i^{\varepsilon_j})^{\pm\delta_j} \text{ if } i \in Z_j. \end{cases} \tag{6.2}$$

Then $(v_j) \subset W^{1,p}(\Omega_j;\mathbb{R}^m)$ converges weakly to (u^+, u^-). In fact,

$$\sup_{j\in\mathbb{N}} \frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}} |Dv_j|^p\,dx \leq \sup_{j\in\mathbb{N}} \frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}} |Du_j|^p\,dx < +\infty. \tag{6.3}$$

Moreover, since $\rho^i_j < \rho_j < \varepsilon_j/2$, then $B^{n-1}_{\rho^i_j}(x_i^{\varepsilon_j}) \subset Q^{n-1}_{i,\varepsilon_j}$; hence,

$$\int_{\omega^{\pm\delta_j}} |v_j - u^\pm|^p\,dx \leq \int_{(\omega\setminus E_j)^{\pm\delta_j}} |w_j - u^\pm|^p\,dx + \sum_{i\in Z_j}\int_{(Q^{n-1}_{i,\varepsilon_j})^{\pm\delta_j}} |u^\pm - u_j^{i\pm}|^p\,dx$$

and, by (5.24), we obtain that

$$\begin{aligned}\frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}}|v_j-u^\pm|^p\,dx \;\leq\;& \frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}}|w_j-u^\pm|^p\,dx+\frac{c}{\delta_j}\int_{\omega^{\pm\delta_j}}|u_j-u^\pm|^p\,dx\\ &+c\,\varepsilon_j^p\sup_{j\in\mathbb{N}}\frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}}|Du_j|^p\,dx. \end{aligned}\tag{6.4}$$

Passing to the limit as $j\to+\infty$ in (6.4), by (6.3) and Remark 3.2 we get that (v_j) converges weakly to (u^+,u^-).

Since $W(0)=0$, by (6.2) and [**38**] Theorem 2, we have

$$\begin{aligned}&\liminf_{j\to+\infty}\frac{1}{\delta_j}\left(\int_{(\omega\setminus E_j)^{+\delta_j}}W(Dw_j)\,dx+\int_{(\omega\setminus E_j)^{-\delta_j}}W(Dw_j)\,dx\right)\\
=\;&\liminf_{j\to+\infty}\frac{1}{\delta_j}\left(\int_{(\omega\setminus E_j)^{+\delta_j}}W(Dv_j)\,dx+\int_{(\omega\setminus E_j)^{-\delta_j}}W(Dv_j)\,dx\right)\\
=\;&\liminf_{j\to+\infty}\frac{1}{\delta_j}\left(\int_{\omega^{+\delta_j}}W(Dv_j)\,dx+\int_{\omega^{-\delta_j}}W(Dv_j)\,dx\right)\\
\geq\;&\int_\omega\mathcal{Q}_{n-1}\overline{W}(D_\alpha u^+)\,dx_\alpha+\int_\omega\mathcal{Q}_{n-1}\overline{W}(D_\alpha u^-)\,dx_\alpha\,.\end{aligned}$$

□

Now let us deal with the contribution 'near' the connecting zones. We always work under the assumption

$$\ell\in(0,+\infty]\quad\text{ and }\quad 0<R^{(\ell)}=\lim_{j\to+\infty}\frac{r_j^{(n-1-p)}}{\varepsilon_j^{n-1}}<+\infty,$$

or

$$\ell=0\quad\text{ and }\quad 0<R^{(0)}=\lim_{j\to+\infty}\frac{r_j^{(n-p)}}{\delta_j\varepsilon_j^{n-1}}<+\infty.$$

In the following proposition we suitably modify (w_j) in each surrounding cylinder in order to get an admissible test function for the minimum problem (5.2) or (5.13).

Proposition 6.2. *Let $\ell\in[0,+\infty]$. Then*

$$\liminf_{j\to+\infty}\frac{1}{\delta_j}\left(\int_{E_j^{+\delta_j}}W(Dw_j)\,dx+\int_{E_j^{-\delta_j}}W(Dw_j)\,dx\right)\geq R^{(\ell)}\int_\omega\varphi^{(\ell)}(u^+-u^-)\,dx_\alpha+o(1)\,,$$

as $\gamma\to0^+$.

Proof. Let $\ell \in (0,+\infty]$, the case $\ell = 0$ can be treated similarly. Let $i \in Z_j$ and $N_j = \frac{\varepsilon_j}{r_j}$. Since $\rho_j^i < \gamma\varepsilon_j$, we can define

$$\zeta_j^i := \begin{cases} w_j(x_i^{\varepsilon_j} + r_j\, y_\alpha, \delta_j\, y_n) - u_j^{i-} & \text{in } \big(B^{n-1}_{\rho_j^i/r_j} \times I\big) \setminus C_{1,\rho_j^i/r_j} \\ \\ (u_j^{i+} - u_j^{i-}) & \text{in } \big(B^{n-1}_{\gamma N_j} \setminus B^{n-1}_{\rho_j^i/r_j}\big)^+ \\ \\ 0 & \text{in } \big(B^{n-1}_{\gamma N_j} \setminus B^{n-1}_{\rho_j^i/r_j}\big)^-, \end{cases}$$

where $N_j = \varepsilon_j/r_j$. Then $\zeta_j^i \in W^{1,p}((B^{n-1}_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}; \mathbb{R}^m)$, $\zeta_j^i = (u_j^{i+} - u_j^{i-})$ on $\big(\partial B^{n-1}_{\gamma N_j}\big)^+$ and $\zeta_j^i = 0$ on $\big(\partial B^{n-1}_{\gamma N_j}\big)^-$. Since $W(0) = 0$, changing variable, by (5.2) we get

$$\begin{aligned}
&\frac{1}{\delta_j}\left(\int_{B^{n-1}_{\rho_j^i}(x_i^{\varepsilon_j})^{+\delta_j}} W(Dw_j)\,dx + \int_{B^{n-1}_{\rho_j^i}(x_i^{\varepsilon_j})^{-\delta_j}} W(Dw_j)\,dx\right) \\
=\ & r_j^{n-1}\left(\int_{\big(B^{n-1}_{\rho_j^i/r_j}\big)^+} W\Big(r_j^{-1}D_\alpha\zeta_j^i | \delta_j^{-1}D_n\zeta_j^i\Big)\,dy + \int_{\big(B^{n-1}_{\rho_j^i/r_j}\big)^-} W\Big(r_j^{-1}D_\alpha\zeta_j^i | \delta_j^{-1}D_n\zeta_j^i\Big)\,dy\right) \\
=\ & r_j^{n-1}\int_{(B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}} W\Big(r_j^{-1}D_\alpha\zeta_j^i | \delta_j^{-1}D_n\zeta_j^i\Big)\,dy \\
\geq\ & r_j^{n-1-p}\varphi^{(\ell)}_{\gamma,j}(u_j^{i+} - u_j^{i-})\,.
\end{aligned} \tag{6.5}$$

Summing up in (6.5), for $i \in Z_j$, we get that

$$\begin{aligned}
&\frac{1}{\delta_j}\left(\int_{E_j^{+\delta_j}} W(Dw_j)\,dx + \int_{E_j^{-\delta_j}} W(Dw_j)\,dx\right) \\
=\ & \sum_{i\in Z_j}\frac{1}{\delta_j}\left(\int_{B^{n-1}_{\rho_j^i}(x_i^{\varepsilon_j})^{+\delta_j}} W(Dw_j)\,dx + \int_{B^{n-1}_{\rho_j^i}(x_i^{\varepsilon_j})^{-\delta_j}} W(Dw_j)\,dx\right) \\
\geq\ & r_j^{n-1-p}\sum_{i\in Z_j}\varphi^{(\ell)}_{\gamma,j}(u_j^{i+} - u_j^{i-}) = \frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}}\sum_{i\in Z_j}\varepsilon_j^{n-1}\varphi^{(\ell)}_{\gamma,j}(u_j^{i+} - u_j^{i-})\,.
\end{aligned} \tag{6.6}$$

Passing to the limit as $j \to +\infty$ we get, by (5.3) and Proposition 5.3, that

$$\begin{aligned}
&\liminf_{j\to+\infty}\frac{1}{\delta_j}\left(\int_{E_j^{+\delta_j}} W(Dw_j)\,dx + \int_{E_j^{-\delta_j}} W(Dw_j)\,dx\right) \\
\geq\ & R^{(\ell)}\int_\omega \varphi^{(\ell)}(u^+ - u^-)\,dx_\alpha \\
& + R^{(\ell)}\liminf_{j\to+\infty}\int_\omega\Big(\sum_{i\in Z_j}\varphi^{(\ell)}_{\gamma,j}(u_j^{i+} - u_j^{i-})\chi_{Q^{n-1}_{i,\varepsilon_j}} - \varphi^{(\ell)}(u^+ - u^-)\Big)\,dx_\alpha \\
=\ & R^{(\ell)}\int_\omega \varphi^{(\ell)}(u^+ - u^-)\,dx_\alpha + o(1)\,,
\end{aligned}$$

as $\gamma \to 0^+$, which completes the proof. □

We now prove the liminf inequality for any arbitrary converging sequence.

LEMMA 6.3. *Let $\ell \in [0, +\infty]$. For every sequence (u_j) converging to (u^+, u^-) we have*

$$\begin{aligned}\liminf_{j\to+\infty} \mathcal{F}_j(u_j) \quad &\geq \quad \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^+)\,dx_\alpha + \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^-)\,dx_\alpha \\ &\quad + R^{(\ell)} \int_\omega \varphi^{(\ell)}(u^+ - u^-)\,dx_\alpha\,.\end{aligned}$$

Proof. Let $(u_j) \to (u^+, u^-)$ be such that $\liminf_{j\to+\infty} \mathcal{F}_j(u_j) < +\infty$. Reasoning as in [**5**] Proposition 5.2, by [**17**] Lemma 3.5, upon passing to a subsequence, for every $M > 0$ and $\eta > 0$, we have the existence of $R_M > M$ and of a Lipschitz function $\Phi_M \in \mathcal{C}^1_c(\mathbb{R}^m; \mathbb{R}^m)$ with $\mathrm{Lip}(\Phi_M) = 1$ such that

$$\Phi_M(z) = \begin{cases} z & \text{if} \quad |z| < R_M, \\ \\ 0 & \text{if} \quad |z| > 2R_M \end{cases}$$

and

$$\liminf_{j\to+\infty} \mathcal{F}_j(u_j) \geq \liminf_{j\to+\infty} \mathcal{F}_j(\Phi_M(u_j)) - \eta\,. \tag{6.7}$$

Note that $(\Phi_M(u_j)) \subset W^{1,p}(\Omega_j; \mathbb{R}^m) \cap L^\infty(\Omega_j; \mathbb{R}^m)$, $\sup_{j\in\mathbb{N}} \|\Phi_M(u_j)\|_{L^\infty(\Omega_j;\mathbb{R}^m)} < R_M$ and it converges to $(\Phi_M(u^+), \Phi_M(u^-))$ as $j \to +\infty$. Hence, if we apply (6.1), Propositions 6.1 and 6.2 to $(\Phi_M(u_j))$ in place of (u_j), letting $\gamma \to 0$ and $k \to +\infty$, we get that

$$\begin{aligned}\liminf_{j\to+\infty} \mathcal{F}_j(\Phi_M(u_j)) \quad &\geq \quad \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha \Phi_M(u^+))\,dx_\alpha + \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha \Phi_M(u^-))\,dx_\alpha \\ &\quad + R^{(\ell)} \int_\omega \varphi^{(\ell)}(\Phi_M(u^+) - \Phi_M(u^-))\,dx_\alpha.\end{aligned} \tag{6.8}$$

Moreover $\Phi_M(u^\pm) \rightharpoonup u^\pm$ weakly in $W^{1,p}(\omega; \mathbb{R}^m)$ as $M \to +\infty$; hence, by (6.7), (6.8), the lower semicontinuity of $\int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u)\,dx_\alpha$ with respect to the weak $W^{1,p}(\omega; \mathbb{R}^m)$-convergence, and (5.6) we have that

$$\begin{aligned}&\liminf_{j\to+\infty} \mathcal{F}_j(u_j) \\ \geq \quad &\int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^+)\,dx_\alpha + \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^-)\,dx_\alpha + R^{(\ell)} \int_\omega \varphi^{(\ell)}(u^+ - u^-)\,dx_\alpha - \eta\,,\end{aligned} \tag{6.9}$$

and by the arbitrariness of η, the thesis. □

6.2. The limsup inequality. For every $(u^+, u^-) \in W^{1,p}(\omega, \mathbb{R}^m) \times W^{1,p}(\omega, \mathbb{R}^m)$ the limsup inequality is obtained by suitably modifying the recovery sequences $(u_j^\pm)$ for the Γ-limits of

$$\frac{1}{\delta_j} \int_{\omega^{+\delta_j}} W(Du)\,dx \quad \text{and} \quad \frac{1}{\delta_j} \int_{\omega^{-\delta_j}} W(Du)\,dx.$$

LEMMA 6.4. *Let $\ell \in [0,+\infty]$ and let ω be an open bounded subset of $\mathbb{R}^{n-1}$ such that $\mathcal{H}^{n-1}(\partial\omega) = 0$. Then, for all $(u^+,u^-) \in W^{1,p}(\omega,\mathbb{R}^m) \times W^{1,p}(\omega,\mathbb{R}^m)$ and for all $\eta > 0$ there exists a sequence $(\bar{u}_j) \subset W^{1,p}(\Omega_j;\mathbb{R}^m)$ converging to (u^+,u^-) such that*

$$\begin{aligned}\limsup_{j\to+\infty} \mathcal{F}_j(\bar{u}_j) \quad &\leq \quad \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^+)\,dx_\alpha + \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^-)\,dx_\alpha \\ &\qquad + R^{(\ell)} \int_\omega \varphi^{(\ell)}(u^+ - u^-)\,dx_\alpha + \eta R^{(\ell)}\mathcal{H}^{n-1}(\omega)\,.\end{aligned}$$

Proof. The proof of the limsup is divided into three steps. We first construct a sequence $(\bar{u}_j) \subset W^{1,p}(\Omega_j;\mathbb{R}^m)$ that we expect to be a recovery sequence. In the second step we prove that $(\bar{u}_j)$ converges to (u^+,u^-). Finally, we prove that it satisfies the limsup inequality. We first deal with the case $\ell \in (0,+\infty]$.

Step 1: Definition of a recovery sequence. Let $u^\pm \in W^{1,p}(\omega;\mathbb{R}^m)\cap L^\infty(\omega;\mathbb{R}^m)$. According to [**38**] Theorem 2 and [**13**] Theorem 1.1, there exist two sequences $(u_j^\pm) \subset W^{1,p}(\omega^{\pm\delta_j};\mathbb{R}^m)$ such that $u_j^\pm \to u^\pm$, the sequences of gradients $(|Du_j^\pm|^p/\delta_j)$ are equi-integrable on $\omega^{\pm\delta_j}$, respectively, and

$$\lim_{j\to+\infty} \frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}} W(Du_j^\pm)\,dx = \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^\pm)\,dx_\alpha\,. \tag{6.10}$$

Moreover, using a truncation argument (as in [**7**] Lemma 6.1, Step 2) we may assume without loss of generality that

$$\sup_{j\in\mathbb{N}} \|u_j^\pm\|_{L^\infty(\omega^{\pm\delta_j};\mathbb{R}^m)} < +\infty\,.$$

Let $u_j := u_j^+\chi_{\omega^{+\delta_j}} + u_j^-\chi_{\omega^{-\delta_j}} \in W^{1,p}(\omega^{+\delta_j}\cup\omega^{-\delta_j};\mathbb{R}^m)$ and let (w_j) be the sequence obtained from (u_j) as in Lemma 4.3, then $\sup_{j\in\mathbb{N}}\|w_j\|_{L^\infty(\omega^{\pm\delta_j};\mathbb{R}^m)} < +\infty$.

We first define $(\bar{u}_j)$ 'far' from the connecting zones; *i.e.*,

$$\bar{u}_j := w_j \;\text{ in }\; \Big(\omega \setminus \bigcup_{i\in\mathbb{Z}^{n-1}} B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\Big)^{\pm\delta_j}\,. \tag{6.11}$$

Then we pass to define $(\bar{u}_j)$ on each $B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{\pm\delta_j}$ making a distinction between the indices $i \in Z_j$ and $i \in \mathbb{Z}^{n-1}\setminus Z_j$.

If $i \in Z_j$, by (5.2), for every $\eta > 0$ there exists $\zeta^i_{\gamma,j} \in X_j^\gamma(u_j^{i+} - u_j^{i-})$ such that

$$\int_{(B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}} r_j^p\, W\left(r_j^{-1}D_\alpha\zeta^i_{\gamma,j}|\delta_j^{-1}D_n\zeta^i_{\gamma,j}\right)dx \leq \varphi^{(\ell)}_{\gamma,j}(u_j^{i+} - u_j^{i-}) + \eta. \tag{6.12}$$

Then, we define

$$\bar{u}_j := \zeta^i_{\gamma,j}\left(\frac{x_\alpha - x_i^{\varepsilon_j}}{r_j}, \frac{x_n}{\delta_j}\right) + u_j^{i-} \;\text{ in }\; B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{\pm\delta_j}\,,\quad i \in Z_j\,. \tag{6.13}$$

In particular, $\bar{u}_j = u_j^{i\pm} = w_j$ on $\left(\partial B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\right)^{\pm\delta_j}$.

Let us now deal with the contact zones not well contained in ω; *i.e.,* with the indices $i \notin Z_j$. For fixed $\gamma > 0$ and j large enough we have that $\gamma N_j > 2$. Let $\psi \in W^{1,p}(B_2^{n-1};[0,1])$ be such that $\psi = 1$ on ∂B_2^{n-1} and $\psi = 0$ in B_1^{n-1} and define

$$\psi_{\gamma,j}(x) := \begin{cases} 0 & \text{in} \quad (B_{\gamma N_j}^{n-1})^- \\ \psi(x_\alpha) & \text{in} \quad (B_2^{n-1})^+ \\ 1 & \text{in} \quad (B_{\gamma N_j}^{n-1} \setminus B_2^{n-1})^+ . \end{cases}$$

Then $\psi_{\gamma,j} \in W^{1,p}((B_{\gamma N_j}^{n-1} \times I) \setminus C_{1,\gamma N_j};[0,1])$, $\psi_{\gamma,j} = 1$ on $\left(\partial B_{\gamma N_j}^{n-1}\right)^+$ and $\psi_{\gamma,j} = 0$ on $\left(\partial B_{\gamma N_j}^{n-1}\right)^-$. Let $w_j^\pm = w_j\,\chi_{\omega^{\pm\delta_j}}$, we extend both of them to the whole $\omega \times (-\delta_j,\delta_j)$ by reflection; *i.e.,* we define $\tilde{w}_j^\pm(x_\alpha,x_n) = w_j^\pm(x_\alpha,-x_n)$ for $x \in \omega^{\mp\delta_j}$ and $\tilde{w}_j^\pm(x) = w_j^\pm(x)$ for $x \in \omega^{\pm\delta_j}$. Hence, we define

$$\bar{u}_j := \psi_{\gamma,j}\left(\frac{x_\alpha - x_i^{\varepsilon_j}}{r_j}, \frac{x_n}{\delta_j}\right)\tilde{w}_j^+ + \left(1 - \psi_{\gamma,j}\left(\frac{x_\alpha - x_i^{\varepsilon_j}}{r_j}, \frac{x_n}{\delta_j}\right)\right)\tilde{w}_j^- \tag{6.14}$$

in $\left(B_{\rho_j}^{n-1}(x_i^{\varepsilon_j}) \times (-\delta_j,\delta_j)\right) \cap \Omega_j$ and for $i \in \mathbb{Z}^{n-1} \setminus Z_j$. In particular, we have that $\bar{u}_j = w_j$ on $\left(\partial B_{\rho_j}^{n-1}(x_i^{\varepsilon_j}) \times (-\delta_j,\delta_j)\right) \cap \Omega_j$; thus $(\bar{u}_j) \subset W^{1,p}(\Omega_j;\mathbb{R}^m)$.

Step 2: The sequence $(\bar{u}_j)$ weakly converges to (u^+,u^-). Let us check (3.1) and (3.2). We will only treat the upper cylinder $\omega^{+\delta_j}$, the lower part being analogous. First

$$\begin{aligned}
&\frac{1}{\delta_j}\int_{\omega^{+\delta_j}} |\bar{u}_j - u^+|^p\,dx \\
= \ &\frac{1}{\delta_j}\int_{\left(\omega \setminus \bigcup_{i\in\mathbb{Z}^{n-1}} B_{\rho_j}^{n-1}(x_i^{\varepsilon_j})\right)^{+\delta_j}} |w_j^+ - u^+|^p\,dx \\
&+\frac{1}{\delta_j}\sum_{i\in Z_j}\int_{B_{\rho_j}^{n-1}(x_i^{\varepsilon_j})^{+\delta_j}} \left|\zeta_{\gamma,j}^i\left(\frac{x_\alpha - x_i^{\varepsilon_j}}{r_j}, \frac{x_n}{\delta_j}\right) + u_j^{i-} - u^+\right|^p dx \\
&+\frac{1}{\delta_j}\sum_{i\in\mathbb{Z}^{n-1}\setminus Z_j}\int_{\left(\omega\cap B_{\rho_j}^{n-1}(x_i^{\varepsilon_j})\right)^{+\delta_j}} \left|\psi_{\gamma,j}\left(\frac{x_\alpha - x_i^{\varepsilon_j}}{r_j}, \frac{x_n}{\delta_j}\right)(w_j^+ - \tilde{w}_j^-) + \tilde{w}_j^- - u^+\right|^p dx \\
\le \ &\frac{1}{\delta_j}\int_{\omega^{+\delta_j}} |w_j - u^+|^p\,dx + c\sum_{i\in Z_j}\int_{B_{\rho_j}^{n-1}(x_i^{\varepsilon_j})} |u^+ - u_j^{i+}|^p\,dx_\alpha \\
&+\frac{c}{\delta_j}\sum_{i\in Z_j}\int_{B_{\rho_j}^{n-1}(x_i^{\varepsilon_j})^{+\delta_j}} \left|\zeta_{\gamma,j}^i\left(\frac{x_\alpha - x_i^{\varepsilon_j}}{r_j}, \frac{x_n}{\delta_j}\right) - (u_j^{i+} - u_j^{i-})\right|^p dx \\
&+\frac{c}{\delta_j}\int_{\left(\omega\cap\bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j} B_{\rho_j}^{n-1}(x_i^{\varepsilon_j})\right)^{+\delta_j}} \left(|w_j^+|^p + |\tilde{w}_j^-|^p + |u^+|^p\right)dx\,.
\end{aligned} \tag{6.15}$$

Since $\lim_{j\to+\infty}\mathcal{H}^{n-1}\left(\omega\cap\bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j} B_{\rho_j}^{n-1}(x_i^{\varepsilon_j})\right) = 0$ and $\sup_{j\in\mathbb{N}}\|w_j^\pm\|_{L^\infty(\omega^{\pm\delta_j};\mathbb{R}^m)} < +\infty$, we have that

$$\lim_{j\to+\infty}\frac{c}{\delta_j}\int_{\left(\omega\cap\bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j} B_{\rho_j}^{n-1}(x_i^{\varepsilon_j})\right)^{+\delta_j}} \left(|w_j^+|^p + |\tilde{w}_j^-|^p + |u^+|^p\right)dx = 0\,. \tag{6.16}$$

Moreover, reasoning as in the proof of Proposition 5.3 (see inequality (5.24)), we have that

$$\lim_{j\to+\infty}\sum_{i\in Z_j}\int_{B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})}|u^+-u_j^{i+}|^p\,dx_\alpha=0\,, \tag{6.17}$$

and, by the convergence $w_j\to(u^+,u^-)$, it remains only to prove that

$$\lim_{j\to+\infty}\frac{1}{\delta_j}\sum_{i\in Z_j}\int_{B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{+\delta_j}}\left|\zeta^i_{\gamma,j}\left(\frac{x_\alpha-x_i^{\varepsilon_j}}{r_j},\frac{x_n}{\delta_j}\right)-(u_j^{i+}-u_j^{i-})\right|^p dx=0\,. \tag{6.18}$$

In fact, changing variable, we get that

$$\begin{aligned}&\frac{1}{\delta_j}\sum_{i\in Z_j}\int_{B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{+\delta_j}}\left|\zeta^i_{\gamma,j}\left(\frac{x_\alpha-x_i^{\varepsilon_j}}{r_j},\frac{x_n}{\delta_j}\right)-(u_j^{i+}-u_j^{i-})\right|^p dx\\ =\;& r_j^{n-1}\sum_{i\in Z_j}\int_{(B^{n-1}_{\gamma N_j})^+}\left|\zeta^i_{\gamma,j}(x)-(u_j^{i+}-u_j^{i-})\right|^p dx\,,\end{aligned}$$

and by, Poincaré's Inequality

$$\int_{B^{n-1}_{\gamma N_j}}\left|\zeta^i_{\gamma,j}(x_\alpha,x_n)-(u_j^{i+}-u_j^{i-})\right|^p dx_\alpha\le c\,(\gamma N_j)^p\int_{B^{n-1}_{\gamma N_j}}|D_\alpha\zeta^i_{\gamma,j}(x_\alpha,x_n)|^p\,dx_\alpha$$

for a.e. $x_n\in(0,1)$. Hence, by the p-growth condition (3.3) and (6.12) if we integrate with respect to x_n and sum up in $i\in Z_j$, we get that

$$\begin{aligned}&\frac{1}{\delta_j}\sum_{i\in Z_j}\int_{B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{+\delta_j}}\left|\zeta^i_{\gamma,j}\left(\frac{x_\alpha-x_i^{\varepsilon_j}}{r_j},\frac{x_n}{\delta_j}\right)-(u_j^{i+}-u_j^{i-})\right|^p dx\\ \le\;& c\,r_j^{n-1}\gamma^pN_j^p\sum_{i\in Z_j}\int_{(B^{n-1}_{\gamma N_j})^+}|D_\alpha\zeta^i_{\gamma,j}|^p\,dx\\ \le\;& c\,r_j^{n-1}\gamma^pN_j^p\sum_{i\in Z_j}\int_{(B^{n-1}_{\gamma N_j})^+}\left|\left(D_\alpha\zeta^i_{\gamma,j}\Big|\frac{r_j}{\delta_j}D_n\zeta^i_{\gamma,j}\right)\right|^p dx\\ \le\;& c\,r_j^{n-1}\gamma^pN_j^p\sum_{i\in Z_j}\left(\varphi^{(\ell)}_{\gamma,j}(u_j^{i+}-u_j^{i-})+\eta+r_j^p\,\mathcal{H}^{n-1}(B^{n-1}_{\gamma N_j})\right)\\ \le\;& c\,\gamma^p\,\varepsilon_j^p\,\frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}}\left(\sum_{i\in Z_j}\varepsilon_j^{n-1}\varphi^{(\ell)}_{\gamma,j}(u_j^{i+}-u_j^{i-})+\left(\eta+c\,\gamma^{n-1}\frac{\varepsilon_j^{n-1}}{r_j^{n-1-p}}\right)\mathcal{H}^{n-1}(\omega)\right).\end{aligned} \tag{6.19}$$

By Proposition 5.3 and (5.3), passing to the limit as $j\to+\infty$ in (6.19), we get (6.18).

It remains to prove that (3.2) holds. In fact,

$$\begin{aligned}
&\frac{1}{\delta_j}\int_{\omega^{+\delta_j}} |D\bar{u}_j|^p\,dx \\
= \;&\frac{1}{\delta_j}\int_{\left(\omega\setminus\bigcup_{i\in\mathbb{Z}^{n-1}} B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\right)^{+\delta_j}} |Dw_j^{\pm}|^p\,dx \\
&+\frac{1}{\delta_j}\int_{\bigcup_{i\in Z_j} B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{+\delta_j}} \left|\left(r_j^{-1}D_\alpha\zeta^i_{\gamma,j}\Big(\frac{x_\alpha-x_i^{\varepsilon_j}}{r_j},\frac{x_n}{\delta_j}\Big)\Big|\delta_j^{-1}D_n\zeta^i_{\gamma,j}\Big(\frac{x_\alpha-x_i^{\varepsilon_j}}{r_j},\frac{x_n}{\delta_j}\Big)\right)\right|^p dx \\
&+\frac{1}{\delta_j}\int_{\left(\bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j} B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega\right)^{+\delta_j}} |D\bar{u}_j|^p\,dx\,. \qquad (6.20)
\end{aligned}$$

It can be easily shown that

$$\begin{aligned}
&\frac{1}{\delta_j}\int_{\bigcup_{i\in Z_j} B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{+\delta_j}} \left|\left(r_j^{-1}D_\alpha\zeta^i_{\gamma,j}\Big(\frac{x_\alpha-x_i^{\varepsilon_j}}{r_j},\frac{x_n}{\delta_j}\Big)\Big|\delta_j^{-1}D_n\zeta^i_{\gamma,j}\Big(\frac{x_\alpha-x_i^{\varepsilon_j}}{r_j},\frac{x_n}{\delta_j}\Big)\right)\right|^p dx \\
\le\;& \frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}}\Big(\sum_{i\in Z_j}\varepsilon_j^{n-1}\,\varphi^{(\ell)}_{\gamma,j}(u_j^{i+}-u_j^{i-})\Big)+\mathcal{H}^{n-1}(\omega)\Big(\eta\frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}}+\gamma^{n-1}\Big)\,; \qquad (6.21)
\end{aligned}$$

while,

$$\begin{aligned}
&\frac{1}{\delta_j}\int_{\left(\bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j} B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega\right)^{+\delta_j}} |D\bar{u}_j|^p\,dx \\
\le\;& c\sum_{i\in\mathbb{Z}^{n-1}\setminus Z_j}\left(\frac{1}{r_j^p\delta_j}\int_{(B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega)^{+\delta_j}}\Big|D_\alpha\psi_{\gamma,j}\Big(\frac{x_\alpha-x_i^{\varepsilon_j}}{r_j},\frac{x_n}{\delta_j}\Big)\Big|^p\left(|w_j^+|^p+|\tilde{w}_j^-|^p\right)dx\right. \\
&\qquad\left.+\frac{1}{\delta_j}\int_{(B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega)^{+\delta_j}}(|Dw_j^+|^p+|D\tilde{w}_j^-|^p)\,dx\right) \\
\le\;& c\sum_{i\in\mathbb{Z}^{n-1}\setminus Z_j}\left(r_j^{n-1-p}\int_{B_2^{n-1}}|D_\alpha\psi|^p\,dx_\alpha+\frac{1}{\delta_j}\int_{(B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega)^{+\delta_j}}|Dw_j^+|^p\,dx\right. \\
&\qquad\left.+\frac{1}{\delta_j}\int_{(B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega)^{-\delta_j}}|Dw_j^-|^p\,dx\right) \\
\le\;& c\sum_{i\in\mathbb{Z}^{n-1}\setminus Z_j}\left(\frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}}\mathcal{H}^{n-1}(Q^{n-1}_{i,\varepsilon_j})+\frac{1}{\delta_j}\int_{(B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega)^{\pm\delta_j}}|Dw_j^{\pm}|^p\,dx\right. \\
&\qquad\left.+\frac{1}{\delta_j}\int_{(B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega)^{-\delta_j}}|Dw_j^-|^p\,dx\right)\,. \qquad (6.22)
\end{aligned}$$

Note that the previous sum can be computed over all $i\in\mathbb{Z}^{n-1}\setminus Z_j$ such that $Q^{n-1}_{i,\varepsilon_j}\cap\omega\neq\emptyset$. Let

$$\omega'_j := \bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j,\,Q^{n-1}_{i,\varepsilon_j}\cap\omega\neq\emptyset} Q^{n-1}_{i,\varepsilon_j}\,,$$

then

$$\sum_{i\in\mathbb{Z}^{n-1}\setminus Z_j,\, Q^{n-1}_{i,\varepsilon_j}\cap\omega\neq\emptyset} \mathcal{H}^{n-1}(Q^{n-1}_{i,\varepsilon_j}) = \mathcal{H}^{n-1}(\omega'_j) \to \mathcal{H}^{n-1}(\partial\omega) = 0. \tag{6.23}$$

Moreover, by Lemma 4.3 we have that $\sup_j \frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}} |Dw_j^{\pm}|^p\,dx < +\infty$; hence, by Proposition 5.3, (5.3), (6.20), (6.21) and (6.22) we get (3.2).

Step 3: The sequence $(\bar u_j)$ is a recovery sequence. We now prove the limsup inequality.

$$\begin{aligned}
&\limsup_{j\to+\infty}\int_{\omega^{\pm\delta_j}} W(D\bar u_j)\,dx\\
&= \limsup_{j\to+\infty}\frac{1}{\delta_j}\Bigg(\int_{\left(\omega\setminus\bigcup_{i\in\mathbb{Z}^{n-1}} B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\right)^{\pm\delta_j}} W(D\bar u_j)\,dx + \int_{\bigcup_{i\in Z_j} B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{\pm\delta_j}} W(D\bar u_j)\,dx\\
&\qquad + \int_{\left(\omega\cap\bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j} B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\right)^{\pm\delta_j}} W(D\bar u_j)\,dx\Bigg).
\end{aligned}\tag{6.24}$$

We deal with the first term in (6.24). The definition of $\bar u_j$ (6.11), Lemma 4.3 and (6.10), yield

$$\begin{aligned}
&\limsup_{j\to+\infty}\frac{1}{\delta_j}\int_{\left(\omega\setminus\bigcup_{i\in\mathbb{Z}^{n-1}} B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\right)^{\pm\delta_j}} W(D\bar u_j)\,dx\\
&= \limsup_{j\to+\infty}\frac{1}{\delta_j}\int_{\left(\omega\setminus\bigcup_{i\in\mathbb{Z}^{n-1}} B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\right)^{\pm\delta_j}} W(Dw_j)\,dx\\
&\le \limsup_{j\to+\infty}\frac{1}{\delta_j}\int_{\omega^{\pm\delta_j}} W(Du_j^{\pm})\,dx + o(1)\\
&= \int_\omega \mathcal{Q}_{n-1}\overline{W}(D_\alpha u^{\pm})\,dx_\alpha + o(1)\,,
\end{aligned}\tag{6.25}$$

as $\gamma\to 0^+$. For every $i\in Z_j$, by (6.13) and (6.12) we get that

$$\begin{aligned}
&\frac{1}{\delta_j}\left(\int_{B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{+\delta_j}} W(D\bar u_j)\,dx + \int_{B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{-\delta_j}} W(D\bar u_j)\,dx\right)\\
&= r_j^{n-1}\int_{(B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}} W\left(r_j^{-1}D_\alpha\zeta^i_{\gamma,j}|\delta_j^{-1}D_n\zeta^i_{\gamma,j}\right)dx\\
&\le r_j^{n-1-p}\left(\varphi^{(\ell)}_{\gamma,j}(u_j^{i+}-u_j^{i-})+\eta\right);
\end{aligned}$$

hence, by (5.3) and Proposition 5.3 we get

$$\begin{aligned}
&\limsup_{j\to+\infty}\frac{1}{\delta_j}\left(\int_{\bigcup_{i\in Z_j}B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{+\delta_j}}W(D\bar u_j)\,dx+\int_{\bigcup_{i\in Z_j}B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{-\delta_j}}W(D\bar u_j)\,dx\right)\\
\le\;& R^{(\ell)}\int_\omega\varphi^{(\ell)}(u^+-u^-)\,dx_\alpha+R^{(\ell)}\,\mathcal{H}^{n-1}(\omega)\,\eta\\
&+\limsup_{j\to+\infty}\int_\omega\Big|\sum_{i\in Z_j}\varphi^{(\ell)}_{\gamma,j}(u_j^{i+}-u_j^{i-})\chi_{Q^{n-1}_{i,\varepsilon_j}}-\varphi^{(\ell)}(u^+-u^-)\Big|\,dx_\alpha\\
=\;& R^{(\ell)}\int_\omega\varphi^{(\ell)}(u^+-u^-)\,dx_\alpha+R^{(\ell)}\,\mathcal{H}^{n-1}(\omega)\,\eta+o(1)\,,
\end{aligned}\tag{6.26}$$

as $\gamma\to0^+$. Finally, for $i\notin Z_j$, by the p-growth condition (3.3) and (6.22), we obtain

$$\begin{aligned}
&\frac{1}{\delta_j}\left(\int_{\left(\bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j}B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega\right)^{\pm\delta_j}}W(D\bar u_j)\,dx\right)\\
\le\;&\sum_{i\in\mathbb{Z}^{n-1}\setminus Z_j}\frac{\beta}{\delta_j}\left(\int_{(B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega)^{\pm\delta_j}}(1+|D\bar u_j|^p)\,dx\right)\\
\le\;& c\,\mathcal{H}^{n-1}\Big(\bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j}B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega\Big)\\
&+c\sum_{i\in\mathbb{Z}^{n-1}\setminus Z_j}\left(\frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}}\mathcal{H}^{n-1}(Q^{n-1}_{i,\varepsilon_j})+\frac{1}{\delta_j}\int_{(B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega)^{+\delta_j}}|Dw_j^+|^p\,dx\right.\\
&\qquad\left.+\frac{1}{\delta_j}\int_{(B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega)^{-\delta_j}}|Dw_j^-|^p\,dx\right).
\end{aligned}$$

Since

$$\lim_{j\to+\infty}\mathcal{H}^{n-1}\Big(\bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j}B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\cap\omega\Big)=0\,,$$

by (5.3), the equi-integrability of $(|Dw_j^\pm|^p/\delta_j)$ on $\omega^{\pm\delta_j}$ and (6.23), we deduce

$$\limsup_{j\to+\infty}\frac{1}{\delta_j}\int_{\left(\omega\cap\bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j}B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})\right)^{\pm\delta_j}}W(D\bar u_j)\,dx=0\,.\tag{6.27}$$

Gathering (6.24)-(6.27) and passing to the limit as $\gamma\to0^+$ we get the limsup inequality for every $u^\pm\in W^{1,p}(\omega;\mathbb{R}^m)\cap L^\infty(\omega;\mathbb{R}^m)$.

We remove the boundedness assumption simply noting that any arbitrary $W^{1,p}(\omega;\mathbb{R}^m)$ function can approximated by a sequence of functions belonging to $W^{1,p}(\omega;\mathbb{R}^m)\cap L^\infty(\omega;\mathbb{R}^m)$, with respect to the strong $W^{1,p}(\omega;\mathbb{R}^m)$-convergence. Then, by the lower semicontinuity of the Γ-limsup and the continuity of

$$(v^+,v^-)\mapsto\int_\omega\mathcal{Q}_{n-1}\overline{W}(D_\alpha v^+)\,dx_\alpha+\int_\omega\mathcal{Q}_{n-1}\overline{W}(D_\alpha v^-)\,dx_\alpha+R^{(\ell)}\int_\omega\varphi^{(\ell)}(v^+-v^-)\,dx_\alpha$$

with respect to the strong $W^{1,p}(\omega;\mathbb{R}^m)$-convergence we get the thesis for $\ell\in(0,+\infty]$.

If $\ell = 0$, we can follow the line of the previous case with slight changes. Let us start by dealing with Step 1. First, we have to notice that for the definition of $(\bar{u}_j)$ in $B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{\pm\delta_j}$, for $i \in Z_j$, we have to consider, for any $\eta > 0$, a function $\zeta_{\gamma,j} \in Y_j^\gamma(z)$ such that

$$\int_{(B^{n-1}_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} r_j^p\, W\left(r_j^{-1} D\zeta_{\gamma,j}\right) dx \leq \varphi^{(0)}_{\gamma,j}(z) + \eta\,;$$

hence,

$$\bar{u}_j(x_\alpha, x_n) := \zeta^i_{\gamma,j}\left(\frac{x_\alpha - x_i^{\varepsilon_j}}{r_j}, \frac{x_n}{r_j}\right) + u_j^{i-} \ \text{ in } \ B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{\pm\delta_j}\,, \qquad \text{for } \ i \in Z_j\,.$$

While for the definition of $(\bar{u}_j)$ in $B^{n-1}_{\rho_j}(x_i^{\varepsilon_j})^{\pm\delta_j}$, for $i \in \mathbb{Z}^{n-1} \setminus Z_j$, we have to introduce a suitable function $\psi_{\gamma,j}$ different from the one used in (6.14). In fact, for a fixed $\gamma > 0$ and j large enough we can always assume that $\gamma N_j > 2$ and $\delta_j / r_j > 2$. Let $\psi \in W^{1,p}(B_2^{n-1} \times (0,2); [0,1])$ such that $\psi = 0$ on $B_1^{n-1} \times \{0\}$ and $\psi = 1$ on $\partial B_2^{n-1} \times (0,2)$. We then define

$$\psi_{\gamma,j}(x) := \begin{cases} 0 & \text{in} \quad (B^{n-1}_{\gamma N_j})^{-(\delta_j/r_j)}, \\ \psi(x) & \text{in} \quad (B_2^{n-1})^{+2}, \\ 1 & \text{in} \quad (B^{n-1}_{\gamma N_j})^{+(\delta_j/r_j)} \setminus (B_2^{n-1})^{+2}. \end{cases}$$

The functions $\psi_{\gamma,j}$ belong to $W^{1,p}((B^{n-1}_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}; [0,1])$ and satisfy $\psi_{\gamma,j} = 1$ on $(\partial B^{n-1}_{\gamma N_j})^{+(\delta_j/r_j)}$ and $\psi_{\gamma,j} = 0$ in $(B^{n-1}_{\gamma N_j})^{-(\delta_j/r_j)}$. Hence, we define

$$\bar{u}_j := \psi_{\gamma,j}\left(\frac{x_\alpha - x_i^{\varepsilon_j}}{r_j}, \frac{x_n}{r_j}\right)\tilde{w}_j^+ + \left(1 - \psi_{\gamma,j}\left(\frac{x_\alpha - x_i^{\varepsilon_j}}{r_j}, \frac{x_n}{r_j}\right)\right)\tilde{w}_j^-$$

in $\left(B^{n-1}_{\rho_j}(x_i^{\varepsilon_j}) \times (-\delta_j, \delta_j)\right) \cap \Omega_j$ and for $i \in \mathbb{Z}^{n-1} \setminus Z_j$. In particular, we have that $\bar{u}_j = w_j$ on $\left(\partial B^{n-1}_{\rho_j}(x_i^{\varepsilon_j}) \times (-\delta_j, \delta_j)\right) \cap \Omega_j$.

Taking into account the definition of $(\bar{u}_j)$ we can proceed as in Steps 2 and 3 also for $\ell = 0$. □

7. Representation formula for the interfacial energy density

This section is devoted to describe explicitly the interfacial energy density $\varphi^{(\ell)}$ for $\ell \in [0, +\infty]$. As in [**5**], we expect to find a capacitary type formula for each regime $\ell \in (0, +\infty)$, $\ell = +\infty$ and $\ell = 0$.

We recall that $\varphi^{(\ell)}$ is the pointwise limit of the sequence $(\varphi^{(\ell)}_{\gamma,j})$, as $j \to +\infty$ and $\gamma \to 0^+$ where for $\ell \in (0, +\infty]$

$$\varphi^{(\ell)}_{\gamma,j}(z) = \inf\left\{\int_{(B^{n-1}_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} r_j^p\, W\left(r_j^{-1}\Big(D_\alpha\zeta\Big|\frac{r_j}{\delta_j}D_n\zeta\Big)\right) dx : \quad \zeta \in X_j^\gamma(z)\right\},$$

while for $\ell = 0$,

$$\varphi^{(0)}_{\gamma,j}(z) = \inf\left\{\int_{(B^{n-1}_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} r_j^p\, W(r_j^{-1} D\zeta)\, dx : \quad \zeta \in Y_j^\gamma(z)\right\}$$

(see Section 5). The main difficulty occurring in the description of $\varphi^{(\ell)}$ is due to the fact that the above minimum problems are stated on (increasingly) varying domains. This do not permit, for example, to deal with a direct Γ-convergence approach in order to apply the classical result on the convergence of associated minimum problems. Thus the proof of the representation formula will be performed in three main steps: we first prove an auxiliary Γ-convergence result for a suitable sequence of energies stated on a fixed domain, then we describe the functional space occurring in the limit capacitary formula, finally, we prove that $\varphi^{(\ell)}$ is described by a representation formula of capacitary-type.

We introduce some convenient notation for the sequel. Let $g_j : \mathbb{R}^{m\times n} \to [0,+\infty)$ be the sequence of functions given by

$$g_j(F) := r_j^p\, W(r_j^{-1}F)$$

for every $F \in \mathbb{R}^{m\times n}$. By (3.3) and (3.4) it follows that

$$|F|^p - r_j^p \le g_j(F) \le \beta(r_j^p + |F|^p), \quad \text{for all } F \in \mathbb{R}^{m\times n} \tag{7.1}$$

and the following p-Lipschitz condition holds:

$$|g_j(F_1) - g_j(F_2)| \le c(r_j^{p-1} + |F_1|^{p-1} + |F_2|^{p-1})|F_1 - F_2|, \qquad \text{for all } F_1,\, F_2 \in \mathbb{R}^{m\times n}.$$

Then, according to Ascoli-Arzela's Theorem, up to subsequences, g_j converges locally uniformly in $\mathbb{R}^{m\times n}$ to a function g satisfying:

$$|F|^p \le g(F) \le \beta|F|^p, \quad \text{for all } F \in \mathbb{R}^{m\times n} \tag{7.2}$$

and

$$|g(F_1) - g(F_2)| \le c(|F_1|^{p-1} + |F_2|^{p-1})|F_1 - F_2|, \qquad \text{for all } F_1,\, F_2 \in \mathbb{R}^{m\times n}. \tag{7.3}$$

7.1. The case $\ell \in (0,+\infty)$. We define

$$X_N(z) \;:=\; \Big\{\zeta \in W^{1,p}((B_N^{n-1}\times I)\setminus C_{1,N};\mathbb{R}^m) : \quad \zeta = z \text{ on } (\partial B_N^{n-1})^+ \text{ and } \zeta = 0 \text{ on } (\partial B_N^{n-1})^- \Big\}$$

for $N > 1$ and $I = (-1,1)$. We recall the following Γ-convergence result.

PROPOSITION 7.1. *Let*

$$\ell = \lim_{j\to+\infty} \frac{r_j}{\delta_j} \in (0,+\infty)\,,$$

then the sequence of functionals $G_j^{(\ell)} : L^p((B_N^{n-1}\times I)\setminus C_{1,N};\mathbb{R}^m) \to [0,+\infty]$, *defined by*

$$G_j^{(\ell)}(\zeta) := \begin{cases} \displaystyle\int_{(B_N^{n-1}\times I)\setminus C_{1,N}} g_j\left(D_\alpha\zeta\Big|\frac{r_j}{\delta_j}D_n\zeta\right)dx & \text{if } \zeta \in X_N(z) \\[2ex] +\infty & \text{otherwise}\,, \end{cases}$$

Γ*-converges, with respect to the* L^p*-convergence, to*

$$G^{(\ell)}(\zeta) := \begin{cases} \displaystyle\int_{(B_N^{n-1}\times I)\setminus C_{1,N}} g(D_\alpha\zeta|\ell D_n\zeta)\,dx & \textit{if } \zeta\in X_N(z) \\ \\ +\infty & \textit{otherwise}\,. \end{cases}$$

Proof. Since $\ell = \lim_{j\to+\infty}(r_j/\delta_j)\in(0,+\infty)$, by the locally uniform convergence of g_j to g we have that the sequence of quasiconvex functions $F\mapsto g_j(\overline{F}|(r_j/\delta_j)F_n)$ pointwise converges to $F\mapsto g(\overline{F}|\ell F_n)$. Hence the conclusion comes from [**16**] Propositions 12.8 and 11.7. $\square$

REMARK 7.1. We denote by p^* the Sobolev exponent in dimension $(n-1)$ i.e.

$$p^* := \frac{(n-1)p}{n-1-p}.$$

We recall that if $(a,b)\subset\mathbb{R}$, the space $L^p(a,b;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))$ is a reflexive and separable Banach space (see e.g. [**4**] or [**49**]). Hence, by the Banach-Alaoglu-Bourbaki Theorem, any bounded sequence admits a weakly converging subsequence.

PROPOSITION 7.2 (Limit space). *Let*

$$\ell = \lim_{j\to+\infty}\frac{r_j}{\delta_j}\in(0,+\infty)\,,\qquad 0<R^{(\ell)}=\lim_{j\to+\infty}\frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}}<+\infty \tag{7.4}$$

and let $(\zeta_{\gamma,j})\in X_j^\gamma(z)$ *such that, for every fixed* $\gamma>0$,

$$\sup_{j\in\mathbb{N}}\int_{(B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}} g_j\Big(D_\alpha\zeta_{\gamma,j}\Big|\frac{r_j}{\delta_j}D_n\zeta_{\gamma,j}\Big)\,dx\le c\,. \tag{7.5}$$

Then, there exists a sequence $\tilde\zeta_j\in W^{1,p}_{\mathrm{loc}}((\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty};\mathbb{R}^m)$ *such that*

$$\tilde\zeta_j=\zeta_{\gamma,j}\quad \textit{in}\quad (B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}$$

and such that, up to subsequences, it converges weakly to ζ *in* $W^{1,p}_{\mathrm{loc}}((\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty};\mathbb{R}^m)$. *Moreover, the function* ζ *satisfies the following properties*

$$\begin{cases} D\zeta\in L^p((\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty};\mathbb{R}^{m\times n}), \\ \\ \zeta-z\in L^p(0,1;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m)), \\ \\ \zeta\in L^p(-1,0;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))\,. \end{cases} \tag{7.6}$$

Proof. By (7.1), (7.4) and (7.5) we deduce that, for every fixed $\gamma>0$,

$$\sup_{j\in\mathbb{N}}\int_{(B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}}\Big|\Big(D_\alpha\zeta_{\gamma,j}\Big|\frac{r_j}{\delta_j}D_n\zeta_{\gamma,j}\Big)\Big|^p\,dx\le c\,. \tag{7.7}$$

We define

$$\tilde{\zeta}_j := \begin{cases} z & \text{in} \quad \big(\mathbb{R}^{n-1} \setminus B^{n-1}_{\gamma N_j}\big)^+ , \\ \\ \zeta_{\gamma,j} & \text{in} \quad (B^{n-1}_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j} , \\ \\ 0 & \text{in} \quad \big(\mathbb{R}^{n-1} \setminus B^{n-1}_{\gamma N_j}\big)^- ; \end{cases}$$

hence,

$$\tilde{\zeta}_j(\cdot, x_n) - z \in W^{1,p}(\mathbb{R}^{n-1}; \mathbb{R}^m) \quad \text{for} \quad \text{a.e.} \quad x_n \in (0,1)$$

and

$$\tilde{\zeta}_j(\cdot, x_n) \in W^{1,p}(\mathbb{R}^{n-1}; \mathbb{R}^m) \quad \text{for} \quad \text{a.e.} \quad x_n \in (-1,0) .$$

Moreover by (7.7) we get that

$$\int_{(\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}} \left| \left(D_\alpha \tilde{\zeta}_j \Big| \frac{r_j}{\delta_j} D_n \tilde{\zeta}_j \right) \right|^p dx = \int_{(B^{n-1}_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} \left| \left(D_\alpha \zeta_{\gamma,j} \Big| \frac{r_j}{\delta_j} D_n \zeta_{\gamma,j} \right) \right|^p dx \leq c . \tag{7.8}$$

Since $p < n-1$, according to the Sobolev Inequality (see e.g. [**4**]), there exists a constant $c = c(n,p) > 0$ (independent of x_n) such that

$$\left(\int_{\mathbb{R}^{n-1}} |\tilde{\zeta}_j(x_\alpha, x_n) - z|^{p^*} \, dx_\alpha \right)^{p/p^*} \leq c \int_{\mathbb{R}^{n-1}} |D_\alpha \tilde{\zeta}_j(x_\alpha, x_n)|^p \, dx_\alpha \tag{7.9}$$

for a.e. $x_n \in (0,1)$, and

$$\left(\int_{\mathbb{R}^{n-1}} |\tilde{\zeta}_j(x_\alpha, x_n)|^{p^*} \, dx_\alpha \right)^{p/p^*} \leq c \int_{\mathbb{R}^{n-1}} |D_\alpha \tilde{\zeta}_j(x_\alpha, x_n)|^p \, dx_\alpha \tag{7.10}$$

for a.e. $x_n \in (-1,0)$. If we integrate (7.9) and (7.10) with respect to x_n, by (7.8) and Remark 7.1, we get that there exist $\zeta_1 \in L^p(0,1; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m))$ and $\zeta_2 \in L^p(-1,0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m))$ such that, up to subsequences,

$$\begin{cases} \tilde{\zeta}_j - z \rightharpoonup \zeta_1 & \text{in} \quad L^p(0,1; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)), \\ \tilde{\zeta}_j \rightharpoonup \zeta_2 & \text{in} \quad L^p(-1,0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)), \\ \\ D\tilde{\zeta}_j \rightharpoonup D\zeta_1 & \text{in} \quad L^p((\mathbb{R}^{n-1})^+; \mathbb{R}^{m \times n}), \\ D\tilde{\zeta}_j \rightharpoonup D\zeta_2 & \text{in} \quad L^p((\mathbb{R}^{n-1})^-; \mathbb{R}^{m \times n}). \end{cases}$$

In particular, we have that

$$\begin{cases} \tilde{\zeta}_j \rightharpoonup \zeta_1 + z & \text{in} \quad W^{1,p}_{\mathrm{loc}}((\mathbb{R}^{n-1})^+; \mathbb{R}^m), \\ \\ \tilde{\zeta}_j \rightharpoonup \zeta_2 & \text{in} \quad W^{1,p}_{\mathrm{loc}}((\mathbb{R}^{n-1})^-; \mathbb{R}^m) . \end{cases}$$

Then, since $\zeta_1 + z = \zeta_2$ on B^{n-1}_1 in the sense of traces, we can define

$$\zeta := \begin{cases} \zeta_1 + z & \text{in} \quad (\mathbb{R}^{n-1})^+ \\ \zeta_2 & \text{in} \quad (\mathbb{R}^{n-1})^- \cup \big(B^{n-1}_1 \times \{0\}\big), \end{cases}$$

and it satisfies (7.6). $\square$

Now we are able to describe the interfacial energy density $\varphi^{(\ell)}$ as the following nonlinear capacitary formula.

PROPOSITION 7.3 (Representation formula). *We have*

$$\begin{aligned}\varphi^{(\ell)}(z) \;=\; \inf\Bigg\{ \int_{(\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty}} & g\big(D_\alpha\zeta|\ell D_n\zeta\big)\,dx : \zeta \in W^{1,p}_{\mathrm{loc}}((\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty};\mathbb{R}^m), \\ & D\zeta \in L^p((\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty};\mathbb{R}^{m\times n}),\ \zeta - z \in L^p(0,1;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m)) \\ & \textit{and } \zeta \in L^p(-1,0;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))\Bigg\}\end{aligned}$$

for every $z \in \mathbb{R}^m$.

Proof. We define

$$\begin{aligned}\psi^{(\ell)}(z) \;:=\; \inf\Bigg\{ \int_{(\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty}} & g\big(D_\alpha\zeta|\ell D_n\zeta\big)\,dx : \zeta \in W^{1,p}_{\mathrm{loc}}((\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty};\mathbb{R}^m), \\ & D\zeta \in L^p((\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty};\mathbb{R}^{m\times n}),\ \zeta - z \in L^p(0,1;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m)) \\ & \text{and } \zeta \in L^p(-1,0;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))\Bigg\},\end{aligned}$$

we want to prove that $\varphi^{(\ell)}(z) = \psi^{(\ell)}(z)$ for every $z \in \mathbb{R}^m$. For every fixed $\eta > 0$, by definition of $\varphi^{(\ell)}_{\gamma,j}(z)$ (see (5.2)), there exists $\zeta_{\gamma,j} \in X^\gamma_j(z)$ such that

$$\int_{(B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}} g_j\left(D_\alpha\zeta_{\gamma,j}\Big|\frac{r_j}{\delta_j}D_n\zeta_{\gamma,j}\right)dx \le \varphi^{(\ell)}_{\gamma,j}(z) + \eta.$$

By Proposition 5.1(i) we have that (7.5) is fulfilled, then by Propositions 7.2 and 7.1 we get

$$\begin{aligned}\lim_{j\to+\infty}\varphi^{(\ell)}_{\gamma,j}(z) + \eta \;&\ge\; \liminf_{j\to+\infty}\int_{(B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}} g_j\left(D_\alpha\tilde\zeta_j\Big|\frac{r_j}{\delta_j}D_n\tilde\zeta_j\right)dx \\ &\ge\; \liminf_{j\to+\infty}\int_{(B^{n-1}_{N}\times I)\setminus C_{1,N}} g_j\left(D_\alpha\tilde\zeta_j\Big|\frac{r_j}{\delta_j}D_n\tilde\zeta_j\right)dx \\ &\ge\; \int_{(B^{n-1}_{N}\times I)\setminus C_{1,N}} g(D_\alpha\zeta|\ell D_n\zeta)\,dx\end{aligned}$$

with $\zeta \in W^{1,p}_{\mathrm{loc}}((\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty};\mathbb{R}^m)$ satisfying (7.6). Note that for every fixed $\gamma > 0$ and j large enough we can always assume that $\gamma N_j > N$ for some fixed $N > 2$. Hence, passing to the limit as $N \to +\infty$ and $\gamma \to 0^+$, we obtain

$$\varphi^{(\ell)}(z) + \eta \ge \int_{(\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty}} g(D_\alpha\zeta|\ell D_n\zeta)\,dx \ge \psi^{(\ell)}(z) \tag{7.11}$$

and by the arbitrariness of η we get the first inequality.

We now prove the converse inequality. For every fixed $\eta > 0$ there exists $\zeta \in W^{1,p}_{\mathrm{loc}}((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^m)$ satisfying (7.6) such that

$$\int_{(\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty}} g(D_\alpha\zeta|\ell D_n\zeta)\,dx \le \psi^{(\ell)}(z) + \eta\,. \tag{7.12}$$

Let $N > 2$, for every fixed $\gamma > 0$ and j large enough we have that $\gamma N_j > N$. We consider a cut-off function $\theta_N \in \mathcal{C}^\infty_c(B^{n-1}_N; [0,1])$ such that $\theta_N = 1$ in $B^{n-1}_{N/2}$, $|D_\alpha\theta_N| \le c/N$ and we define

$$\zeta_N := \begin{cases} \theta_N(x_\alpha)\zeta + (1-\theta_N(x_\alpha))z & \text{in } (B^{n-1}_N)^+, \\ \theta_N(x_\alpha)\zeta & \text{in } (B^{n-1}_N)^- \cup (B^{n-1}_1 \times \{0\}) \end{cases}$$

so that $\zeta_N \in X_N(z)$. By Proposition 7.1, there exists a sequence $(\zeta^N_j) \subset X_N(z)$ strongly converging to ζ_N in $L^p((B^{n-1}_N \times I) \setminus C_{1,N}; \mathbb{R}^m)$ such that

$$\int_{(B^{n-1}_N\times I)\setminus C_{1,N}} g(D_\alpha\zeta_N|\ell D_n\zeta_N)\,dx = \lim_{j\to+\infty}\int_{(B^{n-1}_N\times I)\setminus C_{1,N}} g_j\left(D_\alpha\zeta^N_j\Big|\frac{r_j}{\delta_j}D_n\zeta^N_j\right)dx \tag{7.13}$$

Let us define $\zeta_{\gamma,j} \in X^\gamma_j(z)$ as

$$\zeta_{\gamma,j} := \begin{cases} z & \text{in } (B^{n-1}_{\gamma N_j} \setminus B^{n-1}_N)^+, \\ \zeta^N_j & \text{in } (B^{n-1}_N \times I) \setminus C_{1,N}, \\ 0 & \text{in } (B^{n-1}_{\gamma N_j} \setminus B^{n-1}_N)^-. \end{cases}$$

Consequently, $\zeta_{\gamma,j}$ is an admissible test function for (5.2) and since $g_j(0) = 0$ we get that

$$\begin{aligned} \varphi^{(\ell)}_{\gamma,j}(z) &\le \int_{(B^{n-1}_{\gamma N_j}\times I)\setminus C_{1,\gamma N_j}} g_j\left(D_\alpha\zeta_{\gamma,j}\Big|\frac{r_j}{\delta_j}D_n\zeta_{\gamma,j}\right)dx \\ &= \int_{(B^{n-1}_N\times I)\setminus C_{1,N}} g_j\left(D_\alpha\zeta^j_N\Big|\frac{r_j}{\delta_j}D_n\zeta^j_N\right)dx. \end{aligned}$$

Passing to the limit as $j \to +\infty$, using (7.13) and the p-growth condition (7.2) satisfied by g, we obtain

$$\begin{aligned} \lim_{j\to+\infty}\varphi^{(\ell)}_{\gamma,j}(z) &\le \int_{(B^{n-1}_N\times I)\setminus C_{1,N}} g(D_\alpha\zeta_N|\ell D_n\zeta_N)\,dx \\ &\le \int_{(B^{n-1}_{N/2}\times I)\setminus C_{1,N/2}} g(D_\alpha\zeta|\ell D_n\zeta)\,dx + c\int_{(B^{n-1}_N\setminus B^{n-1}_{N/2})^+}|D\zeta_N|^p\,dx \\ &\qquad + c\int_{(B^{n-1}_N\setminus B^{n-1}_{N/2})^-}|D\zeta_N|^p\,dx\,. \end{aligned} \tag{7.14}$$

Let us examine the contribution of the gradient in (7.14),

$$\begin{aligned}
&\int_{(B_N^{n-1}\setminus B_{N/2}^{n-1})^+} |D\zeta_N|^p\,dx + \int_{(B_N^{n-1}\setminus B_{N/2}^{n-1})^-} |D\zeta_N|^p\,dx \\
&\quad\le c\int_{(B_N^{n-1}\setminus B_{N/2}^{n-1})^+} (|D_\alpha\theta_N|^p|\zeta-z|^p + |D\zeta|^p)\,dx \\
&\qquad + c\int_{(B_N^{n-1}\setminus B_{N/2}^{n-1})^-} (|D_\alpha\theta_N|^p|\zeta|^p + |D\zeta|^p)\,dx \\
&\quad\le c\left(\int_{(\mathbb{R}^{n-1}\setminus B_{N/2}^{n-1})^+} |D\zeta|^p\,dx + \int_{(\mathbb{R}^{n-1}\setminus B_{N/2}^{n-1})^-} |D\zeta|^p\,dx\right) \\
&\qquad + \frac{c}{N^p}\left(\int_{(B_N^{n-1}\setminus B_{N/2}^{n-1})^+} |\zeta-z|^p\,dx + \int_{(B_N^{n-1}\setminus B_{N/2}^{n-1})^-} |\zeta|^p\,dx\right). \qquad (7.15)
\end{aligned}$$

Since $p^* > p$ we can apply Hölder Inequality with $q = p^*/p$ obtaining

$$\begin{aligned}
&\frac{c}{N^p}\left(\int_{(B_N^{n-1}\setminus B_{N/2}^{n-1})^+} |\zeta-z|^p\,dx + \int_{(B_N^{n-1}\setminus B_{N/2}^{n-1})^-} |\zeta|^p\right) \\
&\quad\le c\int_0^1 \Big(\int_{B_N^{n-1}\setminus B_{N/2}^{n-1}} |\zeta-z|^{p^*}\,dx_\alpha\Big)^{p/p^*} dx_n \\
&\qquad\qquad + c\int_{-1}^0 \Big(\int_{B_N^{n-1}\setminus B_{N/2}^{n-1}} |\zeta|^{p^*}\,dx_\alpha\Big)^{p/p^*} dx_n \\
&\quad\le c\int_0^1 \Big(\int_{\mathbb{R}^{n-1}\setminus B_{N/2}^{n-1}} |\zeta-z|^{p^*}\,dx_\alpha\Big)^{p/p^*} dx_n \\
&\qquad\qquad + c\int_{-1}^0 \Big(\int_{\mathbb{R}^{n-1}\setminus B_{N/2}^{n-1}} |\zeta|^{p^*}\,dx_\alpha\Big)^{p/p^*} dx_n. \qquad (7.16)
\end{aligned}$$

Hence by (7.6), (7.15) and (7.16) we have that, for every fixed $\gamma > 0$,

$$\lim_{N\to+\infty}\int_{(B_N^{n-1}\setminus B_{N/2}^{n-1})^\pm} |D\zeta_N|^p\,dx = 0$$

which thanks to (7.12) and (7.14) implies that

$$\lim_{j\to+\infty}\varphi^{(\ell)}_{\gamma,j}(z) \le \psi^{(\ell)}(z) + \eta.$$

Then we get the converse inequality by letting $\gamma \to 0^+$ and by the arbitrariness of η. $\square$

7.2. The case $\ell = +\infty$. In this case the study leading to the representation formula for $\varphi^{(\infty)}$ involves a dimensional reduction problem stated on a varying domain. As before, we start proving some Γ-convergence results (see Propositions 7.4 and 7.5) for suitable sequences of functionals stated on fixed domains. This will allow as to apply some well-known Γ-convergence and integral representation theorems due to Le Dret-Raoult [**38**] and Braides-Fonseca-Francfort [**19**] respectively.

Let $G_j^{\pm} : L^p((B_N^{n-1})^{\pm};\mathbb{R}^m) \to [0,+\infty]$ be defined by

$$G_j^+(\zeta) := \begin{cases} \displaystyle\int_{(B_N^{n-1})^+} g_j\left(D_\alpha\zeta\Big|\frac{r_j}{\delta_j}D_n\zeta\right)dx & \text{if } \begin{cases}\zeta\in W^{1,p}((B_N^{n-1})^+;\mathbb{R}^m)\\ \zeta = z \text{ on } (\partial B_N^{n-1})^+\end{cases} \\ +\infty & \text{otherwise}\end{cases}$$

and

$$G_j^-(\zeta) := \begin{cases} \displaystyle\int_{(B_N^{n-1})^-} g_j\left(D_\alpha\zeta\Big|\frac{r_j}{\delta_j}D_n\zeta\right)dx & \text{if } \begin{cases}\zeta\in W^{1,p}((B_N^{n-1})^-;\mathbb{R}^m)\\ \zeta = 0 \text{ on } (\partial B_N^{n-1})^-\end{cases} \\ +\infty & \text{otherwise.}\end{cases}$$

PROPOSITION 7.4. *Let*

$$\ell = \lim_{j\to+\infty}\frac{r_j}{\delta_j} = +\infty\,,$$

then, the sequences of functionals $(G_j^{\pm})$ Γ*-converge, with respect to the* L^p*-convergence, to*

$$G^+(\zeta) := \begin{cases} \displaystyle\int_{B_N^{n-1}} \mathcal{Q}_{n-1}\,\overline{g}(D_\alpha\zeta)\,dx_\alpha & \textit{if } \zeta - z\in W_0^{1,p}(B_N^{n-1};\mathbb{R}^m) \\ +\infty & \textit{otherwise}\end{cases}$$

and

$$G^-(\zeta) := \begin{cases} \displaystyle\int_{B_N^{n-1}} \mathcal{Q}_{n-1}\,\overline{g}(D_\alpha\zeta)\,dx_\alpha & \textit{if } \zeta\in W_0^{1,p}(B_N^{n-1};\mathbb{R}^m) \\ +\infty & \textit{otherwise}\,,\end{cases}$$

respectively, where $\overline{g}(\overline{F}) = \inf\{g(\overline{F}|F_n) :\ F_n\in\mathbb{R}^m\}$ *for every* $\overline{F}\in\mathbb{R}^{m\times(n-1)}$.

Proof. We prove the Γ-convergence result only for (G_j^+), the other one being analogous. According to [**19**] Theorem 2.5 and Lemma 2.6 there exists a continuous function $\hat{g} : \mathbb{R}^{m\times(n-1)} \to [0,+\infty)$ such that, up to subsequence, (G_j^+) Γ-converges to

$$G^+(\zeta) := \begin{cases} \displaystyle\int_{B_N^{n-1}} \hat{g}(D_\alpha\zeta)\,dx_\alpha & \text{if } \zeta - z\in W_0^{1,p}(B_N^{n-1};\mathbb{R}^m) \\ +\infty & \text{otherwise}\,.\end{cases}$$

Hence, it remains to show that $\hat{g} = \mathcal{Q}_{n-1}\,\overline{g}$. By [**19**] Lemma 2.6, it is enough to consider $W^{1,p}$-functions without boundary condition; hence, it will suffice to deal with affine functions. Let $\zeta(x_\alpha) := \overline{F}\cdot x_\alpha$, by [**19**] Theorem 2.5, there exists a sequence $(\zeta_j)\subset W^{1,p}((B_N^{n-1})^+;\mathbb{R}^m)$ (the so-called recovery sequence) converging to ζ in $L^p((B_N^{n-1})^+;\mathbb{R}^m)$, such that

$$\hat{g}(\overline{F})\,c_N = G^+(\zeta) = \lim_{j\to+\infty}\int_{(B_N^{n-1})^+} g_j\left(D_\alpha\zeta_j\Big|\frac{r_j}{\delta_j}D_n\zeta_j\right)dx \tag{7.17}$$

where $c_N = \mathcal{H}^{n-1}(B_N^{n-1})$. Moreover, by [**13**] Theorem 1.1, we can assume, without loss of generality, that the sequence $\left(\left|\left(D_\alpha\zeta_j\middle|\frac{r_j}{\delta_j}D_n\zeta_j\right)\right|^p\right)$ is equi-integrable. By (7.17) and (7.1), we have that

$$\sup_{j\in\mathbb{N}}\int_{(B_N^{n-1})^+}\left|\left(D_\alpha\zeta_j\middle|\frac{r_j}{\delta_j}D_n\zeta_j\right)\right|^p dx\le c\,;$$

hence, for every fixed $M>0$, if we define

$$A_j^M:=\left\{x\in(B_N^{n-1})^+:\quad\left|\left(D_\alpha\zeta_j(x)\middle|\frac{r_j}{\delta_j}D_n\zeta_j(x)\right)\right|\le M\right\},$$

we get that $\mathcal{L}^n((B_N^{n-1})^+\setminus A_j^M)\le c/M^p$ for some constant $c>0$ independent of j and M. Fix $M>0$, by (7.17), we have

$$\hat{g}(\overline{F})\,c_N\ge\limsup_{j\to+\infty}\int_{A_j^M}g_j\left(D_\alpha\zeta_j\middle|\frac{r_j}{\delta_j}D_n\zeta_j\right)dx. \tag{7.18}$$

Moreover, for all $x\in A_j^M$,

$$\left|g_j\left(D_\alpha\zeta_j(x)\middle|\frac{r_j}{\delta_j}D_n\zeta_j(x)\right)-g\left(D_\alpha\zeta_j(x)\middle|\frac{r_j}{\delta_j}D_n\zeta_j(x)\right)\right|\le\sup_{|F|\le M}|g_j(F)-g(F)|,$$

and then,

$$\begin{aligned}&\int_{A_j^M}\left|g_j\left(D_\alpha\zeta_j\middle|\frac{r_j}{\delta_j}D_n\zeta_j\right)-g\left(D_\alpha\zeta_j\middle|\frac{r_j}{\delta_j}D_n\zeta_j\right)\right|dx\\ \le\ &c_N\sup_{|F|\le M}|g_j(F)-g(F)|.\end{aligned}$$

Hence, by the local uniform convergence of g_j to g, we have that

$$\lim_{j\to+\infty}\int_{A_j^M}\left(g_j\left(D_\alpha\zeta_j\middle|\frac{r_j}{\delta_j}D_n\zeta_j\right)-g\left(D_\alpha\zeta_j\middle|\frac{r_j}{\delta_j}D_n\zeta_j\right)\right)dx=0.$$

By (7.18), we get

$$\hat{g}(\overline{F})\,c_N\ge\limsup_{j\to+\infty}\int_{A_j^M}g\left(D_\alpha\zeta_j\middle|\frac{r_j}{\delta_j}D_n\zeta_j\right)dx. \tag{7.19}$$

Note that, since $\mathcal{L}^n((B_N^{n-1})^+\setminus A_j^M)\to 0$ as $M\to+\infty$, by the p-growth condition (7.2) and the equi-integrability assumption, we find

$$\limsup_{j\to+\infty}\int_{(B_N^{n-1})^+\setminus A_j^M}g\left(D_\alpha\zeta_j\middle|\frac{r_j}{\delta_j}D_n\zeta_j\right)dx=o(1)\,,\qquad\text{as}\quad M\to+\infty\,. \tag{7.20}$$

Consequently, (7.19) and (7.20) imply that

$$\hat{g}(\overline{F})\,c_N\ge\limsup_{j\to+\infty}\int_{(B_N^{n-1})^+}g\left(D_\alpha\zeta_j\middle|\frac{r_j}{\delta_j}D_n\zeta_j\right)dx. \tag{7.21}$$

Finally, from [**38**] Theorem 2, we know that

$$\liminf_{j\to+\infty}\int_{(B_N^{n-1})^+}g\left(D_\alpha\zeta_j\middle|\frac{r_j}{\delta_j}D_n\zeta_j\right)dx\ge\mathcal{Q}_{n-1}\,\overline{g}(\overline{F})\,c_N\,;$$

hence, by (7.21) we obtain that $\hat{g}(\overline{F})\ge\mathcal{Q}_{n-1}\,\overline{g}(\overline{F})$.

We now prove the converse inequality. By [**38**] Theorem 2, there exists a sequence (ζ_j) belonging to $W^{1,p}((B_N^{n-1})^+;\mathbb{R}^m)$ and converging to ζ in $L^p((B_N^{n-1})^+;\mathbb{R}^m)$ such that

$$\mathcal{Q}_{n-1}\overline{g}(\overline{F})\, c_N = \lim_{j\to+\infty}\int_{(B_N^{n-1})^+} g\left(D_\alpha\zeta_j\Big|\frac{r_j}{\delta_j}D_n\zeta_j\right)\,dx\,. \tag{7.22}$$

Without loss of generality, we can still assume that the sequence $\left(\left|\left(D_\alpha\zeta_j|\frac{r_j}{\delta_j}D_n\zeta_j\right)\right|^p\right)$ is equi-integrable. Thus arguing as above, from (7.22) we deduce

$$\mathcal{Q}_{n-1}\,\overline{g}(\overline{F})\, c_N \geq \limsup_{j\to+\infty}\int_{(B_N^{n-1})^+} g_j\left(D_\alpha\zeta_j\Big|\frac{r_j}{\delta_j}D_n\zeta_j\right)\,dx\,. \tag{7.23}$$

Now, by [**19**] Theorem 2.5, we have that

$$\liminf_{j\to+\infty}\int_{(B_N^{n-1})^+} g_j\left(D_\alpha\zeta_j\Big|\frac{r_j}{\delta_j}D_n\zeta_j\right)\,dx \geq \hat{g}(\overline{F})\, c_N\,;$$

hence, $\mathcal{Q}_{n-1}\,\overline{g}(\overline{F}) \geq \hat{g}(\overline{F})$, which concludes the proof. □

Remark 7.2. By [**38**] Theorem 2, for every $\zeta \in W^{1,p}(B_N^{n-1};\mathbb{R}^m)$ the recovery sequence is given by $\zeta_j(x_\alpha,x_n) := \zeta(x_\alpha) + (\delta_j/r_j)\,x_n\,b_j(x_\alpha)$ for a suitable sequence of functions $(b_j)\subset \mathcal{C}_c^\infty(B_N^{n-1};\mathbb{R}^m)$. Note that by definition (ζ_j) keeps the boundary conditions of ζ. Reasoning as in the proof of Proposition 7.4 we can observed that (ζ_j) is also a recovery sequence for (G_j^+) (see e.g. (7.23)). The same remark holds for (G_j^-).

Proposition 7.5. *Let*

$$\ell = \lim_{j\to+\infty}\frac{r_j}{\delta_j} = +\infty\,,$$

then the sequence of functionals $G_j^{(\infty)} : L^p((B_N^{n-1}\times I)\setminus C_{1,N};\mathbb{R}^m)\to[0,+\infty]$ *defined by*

$$G_j^{(\infty)}(\zeta) := \begin{cases} \displaystyle\int_{(B_N^{n-1}\times I)\setminus C_{1,N}} g_j\left(D_\alpha\zeta\Big|\frac{r_j}{\delta_j}D_n\zeta\right)\,dx & \text{if } \zeta\in X_N(z)\\[2ex] +\infty & \text{otherwise}\end{cases}$$

Γ*-converges, with respect to the* L^p*-convergence, to*

$$G^{(\infty)}(\zeta) := \begin{cases} \displaystyle\int_{(B_N^{n-1}\times I)\setminus C_{1,N}} \mathcal{Q}_{n-1}\,\overline{g}(D_\alpha\zeta)\,dx & \text{if } \zeta\in X_N(z) \text{ and } D_n\zeta = 0\\[2ex] +\infty & \text{otherwise}\,.\end{cases}$$

Proof. The lim inf inequality is a straightforward consequence of Proposition 7.4.

Dealing with the lim sup inequality, let us consider $\zeta\in X_N(z)$ with $D_n\zeta = 0$. We denote by $\zeta^\pm \in W^{1,p}(B_N^{n-1}(0);\mathbb{R}^m)$ the restriction of ζ to $(B_N^{n-1})^+$ and $(B_N^{n-1})^-$, respectively. By

Proposition 7.4 and Remark 7.2, there exist two sequences $(\zeta_j^{\pm}) \subset W^{1,p}((B_N^{n-1})^{\pm};\mathbb{R}^m)$ such that

$$\begin{aligned} &\zeta_j^+ \to \zeta^+ \text{ in } L^p((B_N^{n-1})^+;\mathbb{R}^m)\,, \quad \zeta_j^+ = z \text{ on } (\partial B_N^{n-1})^+ \\ &\zeta_j^- \to \zeta^- \text{ in } L^p((B_N^{n-1})^-;\mathbb{R}^m)\,, \quad \zeta_j^- = 0 \text{ on } (\partial B_N^{n-1})^- \end{aligned} \tag{7.24}$$

and

$$\begin{aligned} \lim_{j\to+\infty} \int_{(B_N^{n-1})^+} g_j\left(D_\alpha \zeta_j^+ \Big| \frac{r_j}{\delta_j} D_n \zeta_j^+\right) dx &= \int_{B_N^{n-1}} \mathcal{Q}_{n-1}\,\overline{g}(D_\alpha \zeta^+)\, dx_\alpha \\ \lim_{j\to+\infty} \int_{(B_N^{n-1})^-} g_j\left(D_\alpha \zeta_j^- \Big| \frac{r_j}{\delta_j} D_n \zeta_j^-\right) dx &= \int_{B_N^{n-1}} \mathcal{Q}_{n-1}\,\overline{g}(D_\alpha \zeta^-)\, dx_\alpha\,. \end{aligned} \tag{7.25}$$

Moreover, since $\zeta \in W^{1,p}((B_N^{n-1}\times I)\setminus C_{1,N};\mathbb{R}^m)$, by Remark 7.2, (ζ_j^+) and (ζ_j^-) have the same trace on $B_1^{n-1}\times\{0\}$; hence, $\zeta_j^+ = \zeta_j^- = \zeta$ on $B_1^{n-1}\times\{0\}$. Then we can define

$$\bar\zeta_j := \begin{cases} \zeta_j^+ & \text{in} \quad (B_N^{n-1})^+, \\ \zeta & \text{on} \quad B_1^{n-1}\times\{0\}, \\ \zeta_j^- & \text{in} \quad (B_N^{n-1})^-, \end{cases}$$

with $\bar\zeta_j \in W^{1,p}((B_N^{n-1}\times I)\setminus C_{1,N};\mathbb{R}^m)$. In particular, by (7.24) we have that $\bar\zeta_j \in X_N(z)$ and $\bar\zeta_j \to \zeta$ in $L^p((B_N^{n-1}\times I)\setminus C_{1,N};\mathbb{R}^m)$. Finally, by (7.25) , we have

$$\begin{aligned} \lim_{j\to+\infty} G_j^{(\infty)}(\bar\zeta_j) &= \lim_{j\to+\infty} \int_{(B_N^{n-1}\times I)\setminus C_{1,N}} g_j\left(D_\alpha \bar\zeta_j \Big| \frac{r_j}{\delta_j} D_n \bar\zeta_j\right) dx \\ &= \int_{B_N^{n-1}} \mathcal{Q}_{n-1}\,\overline{g}(D_\alpha\zeta^+)\,dx_\alpha + \int_{B_N^{n-1}} \mathcal{Q}_{n-1}\,\overline{g}(D_\alpha\zeta^-)\,dx_\alpha \\ &= \int_{(B_N^{n-1}\times I)\setminus C_{1,N}} \mathcal{Q}_{n-1}\,\overline{g}(D_\alpha\zeta)\,dx \end{aligned}$$

which completes the proof of the lim sup inequality. □

Proposition 7.6 (Limit space). *Let*

$$\ell = \lim_{j\to+\infty} \frac{r_j}{\delta_j} = +\infty\,, \qquad 0 < R^{(\infty)} = \lim_{j\to+\infty} \frac{r_j^{n-1-p}}{\varepsilon_j^{n-1}} < +\infty$$

and let $\zeta_{\gamma,j} \in X_j^\gamma(z)$ such that, for every fixed $\gamma > 0$,

$$\sup_{j\in\mathbb{N}} \int_{(B_{\gamma N_j}^{n-1}\times I)\setminus C_{1,\gamma N_j}} g_j\Big(D_\alpha\zeta_{\gamma,j}\Big|\frac{r_j}{\delta_j}D_n\zeta_{\gamma,j}\Big)\,dx \le c\,. \tag{7.26}$$

Then, there exists a sequence $\tilde\zeta_j \in W^{1,p}_{\mathrm{loc}}((\mathbb{R}^{n-1}\times I)\setminus C_{1,\infty};\mathbb{R}^m)$ such that

$$\tilde\zeta_j = \zeta_{\gamma,j} \quad \textit{in} \quad (B_{\gamma N_j}^{n-1}\times I)\setminus C_{1,\gamma N_j}$$

and such that, up to subsequences, it converges weakly to ζ^+ in $W^{1,p}_{\rm loc}((\mathbb{R}^{n-1})^+;\mathbb{R}^m)$ and to ζ^- in $W^{1,p}_{\rm loc}((\mathbb{R}^{n-1})^-;\mathbb{R}^m)$. Moreover, the functions $\zeta^\pm$ satisfy the following properties

$$\begin{cases} \zeta^\pm \in W^{1,p}_{\rm loc}(\mathbb{R}^{(n-1)};\mathbb{R}^m), \\[2ex] \zeta^+ = \zeta^- \quad \text{in } B_1^{n-1}, \\[2ex] D_\alpha \zeta^\pm \in L^p(\mathbb{R}^{n-1};\mathbb{R}^{m\times(n-1)}), \\[2ex] (\zeta^+ - z) \text{ and } \zeta^- \in L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m). \end{cases}$$

Proof. We can reason as in Proposition 7.2 using the fact that, by (7.26),

$$\int_{(\mathbb{R}^{n-1})^\pm} |D_n\tilde{\zeta}_j|^p\,dx \le c\left(\frac{\delta_j}{r_j}\right)^p;$$

hence, in the limit we have that $D_n\zeta = 0$ a.e. in $(\mathbb{R}^{n-1})^\pm$. □

Proposition 7.7 (Representation formula). *We have*

$$\begin{aligned}\varphi^{(\infty)}(z) \;=\; \inf\Big\{ \int_{\mathbb{R}^{n-1}} \big(\mathcal{Q}_{n-1}\,\overline{g}(D_\alpha\zeta^+) + \mathcal{Q}_{n-1}\,\overline{g}(D_\alpha\zeta^-)\big)\,dx_\alpha :\ & \zeta^\pm \in W^{1,p}_{\rm loc}(\mathbb{R}^{n-1};\mathbb{R}^m), \\ \zeta^+ = \zeta^- \text{ in } B_1^{n-1}, \quad & D_\alpha\zeta^\pm \in L^p(\mathbb{R}^{n-1};\mathbb{R}^{m\times(n-1)}), \\ & (\zeta^+ - z) \text{ and } \zeta^- \in L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m)\Big\}\end{aligned}$$

for every $z \in \mathbb{R}^m$.

Proof. Reasoning as in the proof of Proposition 7.3, by Propositions 7.5 and 7.6 we get the representation formula for $\varphi^{(\infty)}$. □

7.3. The case $\ell = 0$. We first recall the following Γ-convergence result.

Proposition 7.8. *The sequence of functionals $G_j^{(0)} : L^p((B_N^{n-1}\times(-N,N))\setminus C_{1,N};\mathbb{R}^m) \to [0,+\infty]$, defined by*

$$G_j^{(0)}(\zeta) := \begin{cases} \displaystyle\int_{(B_N^{n-1}\times(-N,N))\setminus C_{1,N}} g_j(D\zeta)\,dx & \text{if } \zeta \in W^{1,p}((B_N^{n-1}\times(-N,N))\setminus C_{1,N};\mathbb{R}^m), \\[2ex] +\infty & \text{otherwise}, \end{cases}$$

Γ-converges, with respect to the L^p-convergence, to

$$G^{(0)}(\zeta) := \begin{cases} \displaystyle\int_{(B_N^{n-1}\times(-N,N))\setminus C_{1,N}} g(D\zeta)\,dx & \text{if } \zeta \in W^{1,p}((B_N^{n-1}\times(-N,N))\setminus C_{1,N};\mathbb{R}^m), \\[2ex] +\infty & \text{otherwise}. \end{cases}$$

Proof. The result is an immediate consequence of the pointwise convergence of the sequence of quasiconvex functions g_j towards g together with Proposition 12.8 in [**16**]. $\square$

PROPOSITION 7.9 (Limit space). *Let*

$$\ell = \lim_{j\to+\infty} \frac{r_j}{\delta_j} = 0\,, \qquad 0 < R^{(0)} = \lim_{j\to+\infty} \frac{r_j^{n-p}}{\varepsilon_j^{n-1}\delta_j} < +\infty \tag{7.27}$$

and let $\zeta_{\gamma,j} \in Y_j^\gamma(z)$ such that, for every fixed $\gamma > 0$,

$$\sup_{j\in\mathbb{N}} \int_{(B^{n-1}_{\gamma N_j}\times I_j)\setminus C_{1,\gamma N_j}} g_j(D\zeta_{\gamma,j})\,dx \le c\,. \tag{7.28}$$

Then, there exists a sequence $\tilde\zeta_j \in W^{1,p}_{\rm loc}(\mathbb{R}^n\setminus C_{1,\infty};\mathbb{R}^m)$ such that

$$\tilde\zeta_j = \zeta_{\gamma,j} \quad \text{in} \quad (B^{n-1}_{\gamma N_j}\times I_j)\setminus C_{1,\gamma N_j}$$

and such that, up to subsequences, it converges weakly to ζ in $W^{1,p}_{\rm loc}(\mathbb{R}^n\setminus C_{1,\infty};\mathbb{R}^m)$. Moreover, the function ζ satisfies the following properties

$$\begin{cases} D\zeta \in L^p(\mathbb{R}^n\setminus C_{1,\infty};\mathbb{R}^{m\times n}), \\[2ex] \zeta - z \in L^p(0,+\infty; L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m)), \\[2ex] \zeta \in L^p(-\infty,0; L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))\,. \end{cases} \tag{7.29}$$

Proof. By (7.28), (7.1) and (7.27), we deduce that, for every fixed $\gamma > 0$,

$$\sup_{j\in\mathbb{N}} \int_{(B^{n-1}_{\gamma N_j}\times I_j)\setminus C_{1,\gamma N_j}} |D\zeta_{\gamma,j}|^p\,dx \le c\,. \tag{7.30}$$

Let us first extend $\zeta_{\gamma,j}$ by reflection

$$\bar\zeta_{\gamma,j}(x) = \begin{cases} \zeta_{\gamma,j}\Big(x_\alpha, 2\frac{\delta_j}{r_j} - x_n\Big) & \text{if} \quad x_\alpha \in B^{n-1}_{\gamma N_j} \text{ and } x_n \in (\delta_j/r_j, 2\delta_j/r_j), \\[2ex] \zeta_{\gamma,j}(x) & \text{if} \quad x \in (B^{n-1}_{\gamma N_j}\times I_j)\setminus C_{1,\gamma N_j}, \\[2ex] \zeta_{\gamma,j}\Big(x_\alpha, -2\frac{\delta_j}{r_j} - x_n\Big) & \text{if} \quad x_\alpha \in B^{n-1}_{\gamma N_j} \text{ and } x_n \in (-2\delta_j/r_j, -\delta_j/r_j) \end{cases} \tag{7.31}$$

and then, we extend it by $(2\delta_j/r_j)$-periodicity in the x_n direction. The resulting sequence, still denoted by $\bar\zeta_{\gamma,j}$, is defined in $\big(B^{n-1}_{\gamma N_j}\times\mathbb{R}\big)\setminus C_{1,\gamma N_j}$. Hence, we define on $\mathbb{R}^n\setminus C_{1,\infty}$,

$$\bar\zeta_j(x) := \begin{cases} z & \text{in} \quad \big(\mathbb{R}^{n-1}\setminus B^{n-1}_{\gamma N_j}\big)\times(0,+\infty), \\[2ex] \bar\zeta_{\gamma,j}(x) & \text{in} \quad \big(B^{n-1}_{\gamma N_j}\times\mathbb{R}\big)\setminus C_{1,\gamma N_j}, \\[2ex] 0 & \text{in} \quad \big(\mathbb{R}^{n-1}\setminus B^{n-1}_{\gamma N_j}\big)\times(-\infty,0)\,. \end{cases} \tag{7.32}$$

Let us now introduce the cut-off functions $\phi_j \in \mathcal{C}_c^\infty((-2\delta_j/r_j, 2\delta_j/r_j); [0,1])$ such that $\phi_j(x_n) = 1$ if $|x_n| \leq \delta_j/r_j$, $\phi_j(x_n) = 0$ if $|x_n| \geq 2\delta_j/r_j$ and $|D_n\phi_j| \leq c(r_j/\delta_j)$. Then, we introduce our last sequence,

$$\tilde{\zeta}_j(x_\alpha, x_n) := \begin{cases} \phi_j(x_n)\bar{\zeta}_j(x_\alpha, x_n) + (1-\phi_j(x_n))z & \text{if} \quad (x_\alpha, x_n) \in \mathbb{R}^{n-1} \times (0, +\infty), \\ \\ \phi_j(x_n)\bar{\zeta}_j(x_\alpha, x_n) & \text{if} \quad (x_\alpha, x_n) \in \mathbb{R}^{n-1} \times (-\infty, 0). \end{cases}$$

Note that

$$\tilde{\zeta}_j = \zeta_{\gamma,j} \quad \text{in} \quad (B^{n-1}_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}\,. \tag{7.33}$$

Moreover, by (7.30)-(7.33) we have that

$$\sup_{j\in\mathbb{N}} \int_{\mathbb{R}^n \setminus C_{1,\infty}} |D_\alpha \tilde{\zeta}_j|^p \, dx \leq c\,, \tag{7.34}$$

while, for every $(a,b) \subset \mathbb{R}$, with $a < b$, we have

$$\int_{\left(\mathbb{R}^{n-1}\times(a,b)\right)\setminus C_{1,\infty}} |D_n \tilde{\zeta}_j|^p \, dx \leq c\,, \tag{7.35}$$

for j large enough and c independent of (a,b). Reasoning as in Proposition 7.2, with $(0,+\infty)$ and $(-\infty, 0)$ in place of $(0,1)$ and $(-1,0)$, respectively, we can conclude that there exist $\zeta_1 \in L^p(0,+\infty; L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))$ and $\zeta_2 \in L^p(-\infty,0; L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))$ such that, up to subsequences,

$$\tilde{\zeta}_j - z \rightharpoonup \zeta_1 \quad \text{in} \quad L^p(0,+\infty; L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))$$

and

$$\tilde{\zeta}_j \rightharpoonup \zeta_2 \quad \text{in} \quad L^p(-\infty,0; L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))\,.$$

Moreover, by (7.34) and (7.35), we have that, up to subsequences, $\tilde{\zeta}_j$ converges weakly to ζ in $W^{1,p}_{\text{loc}}(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^m)$ where

$$\zeta = \begin{cases} \zeta_1 + z & \text{in} \quad \mathbb{R}^{n-1} \times (0,+\infty) \\ \zeta_2 & \text{in} \quad (\mathbb{R}^{n-1} \times (-\infty,0)) \cup (B_1^{n-1} \times \{0\})\,. \end{cases}$$

In particular, for any compact set $K \subset \mathbb{R}^n \setminus C_{1,\infty}$, we have that

$$\int_K |D\zeta|^p \, dx \leq \liminf_{j\to+\infty} \int_K |D\tilde{\zeta}_j|^p \, dx \leq c$$

for some constant c independent of K; hence, we get that $D\zeta \in L^p(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^{m\times n})$ which concludes the description of the limit function ζ. □

Proposition 7.10 (Representation formula). *We have*

$$\varphi^{(0)}(z) \;=\; \inf\Bigg\{ \int_{\mathbb{R}^n\setminus C_{1,\infty}} g(D\zeta)\,dx : \ \zeta \in W^{1,p}_{\text{loc}}(\mathbb{R}^n \setminus C_{1,\infty};\mathbb{R}^m),\ D\zeta \in L^p(\mathbb{R}^n \setminus C_{1,\infty};\mathbb{R}^{m\times n}),$$
$$\zeta - z \in L^p(0,+\infty; L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m)) \ \text{and}\ \zeta \in L^p(-\infty,0; L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m)) \Bigg\}$$

for every $z \in \mathbb{R}^m$.

Proof. We define

$$\psi^{(0)}(z) := \inf\Big\{ \int_{\mathbb{R}^n\setminus C_{1,\infty}} g(D\zeta)\,dx : \zeta \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus C_{1,\infty};\mathbb{R}^m),\ D\zeta \in L^p(\mathbb{R}^n \setminus C_{1,\infty};\mathbb{R}^{m\times n}),$$
$$\zeta - z \in L^p(0,+\infty;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m)) \text{ and } \zeta \in L^p(-\infty,0;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))\Big\}$$

and let us prove that $\varphi^{(0)}(z) = \psi^{(0)}(z)$ for every $z \in \mathbb{R}^m$.

By definition of $\varphi^{(0)}_{\gamma,j}$ (see (5.13)), for every fixed $\eta > 0$, there exists $\zeta_{\gamma,j} \in Y^\gamma_j(z)$ such that

$$\int_{(B^{n-1}_{\gamma N_j}\times I_j)\setminus C_{1,\gamma N_j}} g_j(D\zeta_{\gamma,j})\,dx \le \varphi^{(0)}_{\gamma,j}(z) + \eta\,; \tag{7.36}$$

hence, by Proposition 5.2 (i), (7.28) is satisfied. Then by Propositions 7.8 and 7.9 we get that

$$\begin{aligned}
\lim_{j\to+\infty} \varphi^{(0)}_{\gamma,j}(z) + \eta &\ge \liminf_{j\to+\infty} \int_{(B^{n-1}_{\gamma N_j}\times I_j)\setminus C_{1,\gamma N_j}} g_j(D\tilde{\zeta}_j)\,dx \\
&\ge \liminf_{j\to+\infty} \int_{\left(B^{n-1}_N\times(-N,N)\right)\setminus C_{1,N}} g_j(D\tilde{\zeta}_j)\,dx \\
&\ge \int_{\left(B^{n-1}_N\times(-N,N)\right)\setminus C_{1,N}} g(D\zeta)\,dx
\end{aligned} \tag{7.37}$$

for some fixed $N > 1$, where ζ satisfies (7.29). Thus, passing to the limit in (7.37) as $N \to +\infty$ and $\gamma \to 0^+$, it follows that

$$\varphi^{(0)}(z) \ge \int_{\mathbb{R}^n\setminus C_{1,\infty}} g(D\zeta)\,dx \ge \psi^{(0)}(z)\,.$$

Let us prove the converse inequality. For any fixed $\eta > 0$, let $\zeta \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus C_{1,\infty};\mathbb{R}^m)$ be as in (7.29) and satisfying

$$\int_{\mathbb{R}^n\setminus C_{1,\infty}} g(D\zeta)\,dx \le \psi^{(0)}(z) + \eta. \tag{7.38}$$

For every $j \in \mathbb{N}$ and $\gamma > 0$, we consider a cut-off function $\theta_{\gamma,j} \in \mathcal{C}^\infty_c(B^{n-1}_{\gamma N_j};[0,1])$ such that $\theta_{\gamma,j} = 1$ in $B^{n-1}_{(\gamma N_j)/2}$, $|D_\alpha\theta_{\gamma,j}| \le c/\gamma N_j$ and we define $\zeta_{\gamma,j} \in Y^\gamma_j(z)$ by

$$\zeta_{\gamma,j} := \begin{cases} \theta_{\gamma,j}(x_\alpha)\zeta + (1-\theta_{\gamma,j}(x_\alpha))z & \text{in } \ (B^{n-1}_{\gamma N_j})^{+(\delta_j/r_j)} \\ \\ \theta_{\gamma,j}(x_\alpha)\zeta & \text{in } \ (B^{n-1}_{\gamma N_j})^{-(\delta_j/r_j)} \cup (B^{n-1}_1 \times \{0\})\,. \end{cases}$$

Consequently, $\zeta_{\gamma,j}$ is an admissible test function for (5.13) and we get that

$$\varphi^{(0)}_{\gamma,j}(z) \le \int_{(B^{n-1}_{\gamma N_j}\times I_j)\setminus C_{1,\gamma N_j}} g_j(D\zeta_{\gamma,j})\,dx.$$

The same kind of computations as those already employed in the proof of Lemma 7.3 now with g_j in place of g and with other obvious replacements (see (7.14)-(7.16)) gives

$$\lim_{j\to+\infty} \varphi^{(0)}_{\gamma,j}(z) \le \limsup_{j\to+\infty} \int_{(B^{n-1}_{\gamma N_j}\times I_j)\setminus C_{1,\gamma N_j}} g_j(D\zeta)\,dx + o(1)\,, \qquad \text{as} \quad \gamma \to 0^+\,.$$

On the other hand, Fatou's Lemma and (7.1) imply

$$\limsup_{j\to+\infty}\int_{(B^{n-1}_{\gamma N_j}\times I_j)\setminus C_{1,\gamma N_j}} g_j(D\zeta)\,dx \le \int_{\mathbb{R}^n\setminus C_{1,\infty}} g(D\zeta)\,dx + o(1)\,, \qquad \text{as} \quad \gamma\to 0^+.$$

Hence by (7.38), passing to the limit as $\gamma\to 0^+$, we get that

$$\varphi^{(0)}(z) \le \psi^{(0)}(z) + \eta$$

and by the arbitrariness of η, the thesis. □

REMARK 7.3. As already recalled, in [**5**] it is proved that if $\delta_j = 1$ or $\delta_j = \varepsilon_j$ then the critical size r_j of the contact zones is of order $\varepsilon_j^{(n-1)/(n-p)}$ or $\varepsilon_j^{n/(n-p)}$, respectively; moreover, the interfacial energy density is described by the following formula

$$\begin{aligned}\varphi(z) \;=\; \inf\Big\{\int_{\mathbb{R}^n\setminus C_{1,\infty}} g(D\zeta)\,dx :\ & \zeta\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n\setminus C_{1,\infty};\mathbb{R}^m)\\ & \zeta - z\in W^{1,p}(\mathbb{R}^n_+;\mathbb{R}^m),\ \zeta\in W^{1,p}(\mathbb{R}^n_-;\mathbb{R}^m)\Big\}\end{aligned}$$

where $\mathbb{R}^n_+ = \mathbb{R}^{n-1}\times(0,+\infty)$, $\mathbb{R}^n_- = \mathbb{R}^{n-1}\times(-\infty,0)$ (see [**5**] Section 7, the case $p=q$, with $\rho_{\varepsilon_j} = r_j$, $W_p = U_p = W$, $\widehat{W}_p = \widehat{U}_p = g$ and $\mathbb{R}^n_{+,-}\cup B_1^{n-1}(0) = \mathbb{R}^n\setminus C_{1,\infty}$).

We want to point out that from the analysis we carried on in the case $\ell = 0$ and in particular from

$$0 < R^{(0)} = \lim_{j\to+\infty}\frac{r_j^{n-p}}{\delta_j\varepsilon_j^{n-1}}$$

we recovered both the critical sizes founded in [**5**] and correspondent to the two cases $\delta_j = 1$ and $\delta_j = \varepsilon_j$.

Moreover we want to show that $\varphi = \varphi^{(0)}$. We have to check only the inequality $\varphi\le\varphi^{(0)}$, the other one being obvious.

For any fixed $\eta>0$ let $\zeta\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n\setminus C_{1,\infty};\mathbb{R}^m)$ be such that $\zeta - z\in L^p(0,+\infty;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))$, $\zeta\in L^p(-\infty,0;L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))$, $D\zeta\in L^p(\mathbb{R}^n\setminus C_{1,\infty};\mathbb{R}^{m\times n})$ and

$$\int_{\mathbb{R}^n\setminus C_{1,\infty}} g(D\zeta)\,dx \le \varphi^{(0)}(z) + \eta\,. \tag{7.39}$$

For every $N>2$ we denote by B_N the n-dimensional ball of radius N centered in zero and by $B_N^{\pm}$ the set of the points $x\in B_N$ such that $\pm x_n > 0$; we consider a cut-off function $\theta_N\in\mathcal{C}^\infty_c(B_N;[0,1])$ such that $\theta_N = 1$ in $B_{N/2}$, $|D\theta_N|\le c/N$ and we define

$$\bar\zeta := \begin{cases} \theta_N(\zeta - z) + z & \text{in} \quad B_N^+,\\[2ex] \theta_N\zeta & \text{in} \quad B_N^-\cup(B_1^{n-1}\times\{0\})\end{cases}$$

so that $\bar\zeta\in W^{1,p}(B_N\setminus C_{1,N};\mathbb{R}^m)$, $\bar\zeta = z$ on ∂B_N^+ and $\bar\zeta = 0$ on ∂B_N^-. Hence,

$$\int_{B_N\setminus C_{1,N}} g(D\bar\zeta)\,dx = \int_{B_{N/2}\setminus C_{1,N/2}} g(D\zeta)\,dx + \int_{(B_N\setminus B_{N/2})\setminus C_{1,N}} g(D\bar\zeta)\,dx\,;$$

in particular, by (7.2), we have

$$\begin{aligned}\int_{(B_N\setminus B_{N/2})\setminus C_{1,N}} g(D\bar{\zeta})\,dx &\leq \beta\Big(\int_{B_N^+\setminus B_{N/2}^+} |D\theta_N|^p|\zeta - z|^p\,dx + \int_{B_N^-\setminus B_{N/2}^-} |D\theta_N|^p|\zeta|^p\,dx \\ &\quad + \int_{(B_N\setminus B_{N/2})\setminus C_{1,N}} |D\zeta|^p\,dx\Big) \\ &\leq \frac{c}{N^p}\Big(\int_{B_N^+\setminus B_{N/2}^+} |\zeta - z|^p\,dx + \int_{B_N^-\setminus B_{N/2}^-} |\zeta|^p\,dx\Big) \\ &\quad + \int_{(\mathbb{R}^n\setminus B_{N/2})\setminus C_{1,\infty}} |D\zeta|^p\,dx\,.\end{aligned}$$

Since $\zeta - z \in L^p(0,+\infty; L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))$, $\zeta \in L^p(-\infty,0; L^{p^*}(\mathbb{R}^{n-1};\mathbb{R}^m))$ and $D\zeta \in L^p(\mathbb{R}^n \setminus C_{1,\infty};\mathbb{R}^{m\times n})$, we can easily conclude that

$$\lim_{N\to+\infty}\int_{(B_N\setminus B_{N/2})\setminus C_{1,N}} g(D\bar{\zeta})\,dx = 0\,. \tag{7.40}$$

Hence, by (7.40), we deduce

$$\begin{aligned}\varphi^{(0)}(z) + \eta &\geq \int_{\mathbb{R}^n\setminus C_{1,\infty}} g(D\zeta)\,dx \geq \int_{B_{N/2}\setminus C_{1,N/2}} g(D\zeta)\,dx \\ &= \int_{B_N\setminus C_{1,N}} g(D\bar{\zeta})\,dx + o(1) \\ &\geq \inf\Big\{\int_{B_N\setminus C_{1,N}} g(D\zeta)\,dx :\ \zeta \in W^{1,p}(B_N\setminus C_{1,N};\mathbb{R}^m) \\ &\qquad\qquad \zeta = z \text{ on } \partial B_N^+,\ \zeta = 0 \text{ on } \partial B_N^-\Big\} + o(1)\end{aligned}$$

as $N \to +\infty$. Finally, passing to the limit as $N \to +\infty$, by the arbitrariness of η, we get $\varphi^{(0)} \geq \varphi$.

Note that the proof of the explicit formula for φ in [**5**] relies on the fact that δ_j is of order ε_j or bigger than it, while in Proposition 7.9 and Proposition 7.12 we have to take into account that $\delta_j \ll \varepsilon_j$. This is the reason why our proof is different from the one of [**5**] even if, at the end, the two representation formulas turn out to coincide.

APPENDIX A

Equi-integrability in dimension reduction problems

1. A brief overview

A very handy tool in the study of the asymptotic behavior of variational problems defined on Sobolev spaces is Fonseca, Müller and Pedregal's *equi-integrability Lemma* [**33**] (see Theorem 2.1 below; see also earlier work by Acerbi and Fusco [**2**] and by Kristensen [**36**]), which allows to substitute a sequence (w_j) with (∇w_j) bounded in L^p by a sequence (z_j) with $(|\nabla z_j|^p)$ equi-integrable, such that the two sequences are equal except on a set of vanishing measure. In this way the asymptotic behavior of integral energies of p-growth involving ∇w_j can be computed using ∇z_j and thus avoiding to consider concentration effects. This method is very helpful for example in the computation of lower bounds for Γ-limits (see, *e.g.*, [**14**]).

In the framework of dimensional reduction, we encounter sequences of functions (w_ε) defined on cylindrical sets with some 'thin dimension' ε; *e.g.*, in the physical three-dimensional case either *thin films* defined on some set of the type $\omega \times (0,\varepsilon)$ (see, *e.g.*, [**38, 19**]), or *thin wires* defined on $\varepsilon\omega \times (0,1)$ (see, *e.g.*, [**1, 37**]), where ω is some two-dimensional bounded open set. In order to carry on some asymptotic analysis such functions are usually rescaled to an ε-independent reference configuration Ω (see Fig. 1), so that a new sequence (u_ε) is constructed, satisfying some 'degenerate' bounds of the form

$$\int_\Omega \Big(|\nabla_\alpha u_\varepsilon|^p + \frac{1}{\varepsilon^p}|\nabla_\beta u_\varepsilon|^p\Big)dx \le C < +\infty \tag{1.1}$$

whenever the sequence of the gradients (∇w_ε) satisfied some corresponding L^p bound on the unscaled domain. Here, ∇_α represents the gradient with respect to the unscaled coordinates (denoted by x_α) and ∇_β represents the gradient with respect to the 'thin' coordinate directions (denoted by x_β). In the case described above of thin films $x_\beta = x_3$; for thin wires, $x_\beta = (x_1, x_2)$.

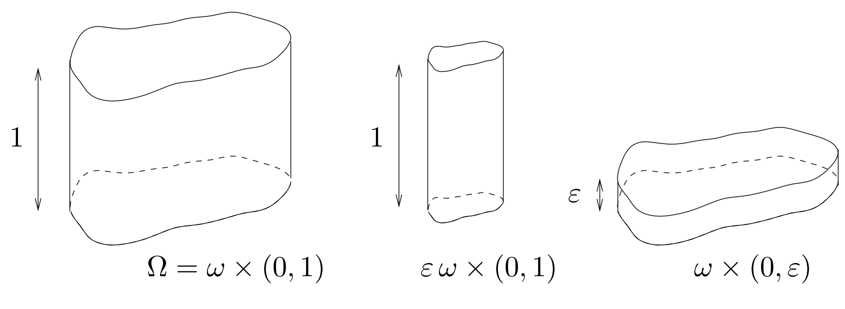

FIGURE 1. Scaled domain, a wire and a thin film.

A theorem by Bocea and Fonseca [**13**] states that an analog of Fonseca, Müller and Pedregal's result still holds in this framework, and an 'equivalent sequence' v_ε can be constructed such that the sequence $(|\nabla_\alpha v_\varepsilon|^p + \frac{1}{\varepsilon^p}|\nabla_\beta v_\varepsilon|^p)$ is equi-integrable on Ω. In their result they deal specifically with the case of thin films; *i.e.*, when the space of the x_β is one-dimensional in the notation above. An earlier mention of the equi-integrability result in this form can be found without proof in a paper by Shu [**47**], where it is suggested that the same argument of [**33**] could be followed. This path is not pursued by Bocea and Fonseca's as it would necessitate re-proving a number of fine results for maximal functions in a periodic context; their proof instead relies on a direct argument.

This appendix provides an alternative proof to that of Bocea and Fonseca, that we think worth pointing out since its method could be applied to other types of problems involving thin structures and extends to a general nD-to-$(n-k)$D dimensional-reduction framework. Its argument is essentially the following: we consider the unscaled functions w_ε defined on some Ω_ε (*e.g.*, $\omega \times (0,\varepsilon)$) on which we have an L^p bound of the gradient and extend them to 2ε-periodic functions in the x_β directions. These extended functions still satisfy an L^p bound, now on each fixed Ω (*e.g.*, a cube), so that we may apply Fonseca, Müller and Pedregal's result to find z_ε with the equi-integrability property. This property is quantified by de la Vallée Poussin's Criterion, which ensures the existence of a positive Borel function φ with superlinear growth such that $\int_\Omega \varphi(|\nabla z_\varepsilon|^p)\,dx \le C < +\infty$. By this remark and a simple but careful counting argument we can choose a set differing from the original Ω_ε by a 2ε-periodic translation in the x_β directions (and hence it is not restrictive to suppose that this set is precisely Ω_ε) such that

$$\frac{1}{\varepsilon^k}\int_{\Omega_\varepsilon} \varphi(|\nabla z_\varepsilon|^p)\,dx \le C < +\infty, \tag{1.2}$$

(k denotes the dimension of the space of the x_β) and still z_ε equals w_ε except for a set with relative measure tending to zero in Ω_ε. By scaling such z_ε we conclude the proof since (1.2) exactly states the desired equi-integrability property.

Since our method does not rely on space dimensions, we state and proof our result in a general n-dimensional setting. In particular it also comprises the physical case of thin wires not covered in [**13**]. Thin wires are generally dealt with by more direct arguments exploiting their one-dimensional limit nature, but our general equi-integrability result may nevertheless be useful in the case of thin wires with an unprescribed heterogeneous nature, in order to obtain general compactness results as for thin films (see [**19**]).

2. Preliminaries

In this section we recall two results which will be the key tools in the proof of Theorem 3.1. The first one is Fonseca-Müller-Pedregal's decomposition Theorem for 'unscaled gradients' while the second is a classical equi-integrability criterion.

In what follows m, n will be two positive integers, Ω a bounded open subset of $\mathbb{R}^n$ and p a real number such that $1 < p < +\infty$.

THEOREM 2.1 ([**33**] Lemma 1.2). *Let (w_j) be a bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$. Then there exists a subsequence of (w_j) (not relabelled) and a sequence (z_j) in $W^{1,p}(\Omega;\mathbb{R}^m)$ such that*

$$\mathcal{L}^n(\{z_j \neq w_j\} \cup \{\nabla z_j \neq \nabla w_j\}) \to 0,$$

as $j \to +\infty$, and $(|\nabla z_j|^p)$ is equi-integrable on Ω. If Ω is Lipschitz, then each z_j can be chosen to be a Lipschitz function.

PROPOSITION 2.2 (de la Vallée Poussin's Criterion). *Let (w_j) be in $L^1(\Omega;\mathbb{R}^m)$; then (w_j) is equi-integrable on Ω if and only if there exists a positive Borel function $\varphi : [0,+\infty) \to [0,+\infty]$ such that*

$$\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty \qquad \text{and} \qquad \sup_j \int_\Omega \varphi(|w_j|)\,dx < +\infty.$$

A proof of de la Vallée Poussin's Criterion can be found in Dellacherie-Meyer [**31**].

3. Statement and proof of the main result

Let k be a positive integer such that $k < n$. Given $x \in \mathbb{R}^n$, we set $x = (x_\alpha, x_\beta)$ where $x_\alpha = (x_1, \dots, x_{n-k})$ and $x_\beta = (x_{n-k+1}, \dots, x_n)$ is the 'thin variable'; then $\nabla_\alpha = \big(\partial_{x_1}, \dots, \partial_{x_{n-k}}\big)$ is the gradient with respect to x_α and $\nabla_\beta = \big(\partial_{x_{n-k+1}}, \dots, \partial_{x_n}\big)$ the gradient with respect to x_β.

THEOREM 3.1. *Let $\omega_\alpha \subset \mathbb{R}^{n-k}$, $\omega_\beta \subset \mathbb{R}^k$ be open bounded sets and assume that ω_β is connected and with Lipschitz boundary. Let (ε_j) be a sequence of positive real numbers converging to zero and let (u_j) be a bounded sequence in $W^{1,p}(\omega_\alpha \times \omega_\beta;\mathbb{R}^m)$ satisfying*

$$\sup_j \int_{\omega_\alpha\times\omega_\beta} \Big(|\nabla_\alpha u_j|^p + \frac{1}{\varepsilon_j^p}|\nabla_\beta u_j|^p\Big)\,dx < +\infty. \tag{3.1}$$

Then there exists a subsequence of (u_j) (not relabelled) and a sequence (v_j) in $W^{1,p}(\omega_\alpha\times\omega_\beta;\mathbb{R}^m)$ such that

$$\mathcal{L}^n(\{v_j \neq u_j\} \cup \{\nabla v_j \neq \nabla u_j\}) \to 0, \tag{3.2}$$

as $j \to +\infty$, and $\left(|\nabla_\alpha v_j|^p + \frac{1}{\varepsilon_j^p}|\nabla_\beta v_j|^p\right)$ is equi-integrable on $\omega_\alpha \times \omega_\beta$. If ω_α is Lipschitz then each v_j can be chosen to be a Lipschitz function.

PROOF. Let (u_j) be a bounded sequence in $W^{1,p}(\omega_\alpha \times \omega_\beta;\mathbb{R}^m)$ satisfying (3.1). Since ω_β is connected and with Lipschitz boundary, by applying a standard extension technique (see for instance Adams [**4**], Theorems 4.26 and 4.28, and Section 4.29 for details) we may assume to deal with a $W^{1,p}(\omega_\alpha \times Q^k;\mathbb{R}^m)$-sequence, for $Q^k \subset \mathbb{R}^k$ open cube containing ω_β, still preserving the same boundedness properties of (u_j). Moreover, up to possible scalings and translations, we can always suppose that $Q^k = (0,1)^k$.

Set $\hat u_j(x) := u_j(x_\alpha, \frac{x_\beta}{\varepsilon_j})$; then $(\hat u_j) \subset W^{1,p}(\omega_\alpha \times (0,\varepsilon_j)^k;\mathbb{R}^m)$ and by hypothesis

$$\sup_j \frac{1}{\varepsilon_j^k} \int_{\omega_\alpha\times(0,\varepsilon_j)^k} |\hat u_j|^p\,dx = \sup_j \int_{\omega_\alpha\times(0,1)^k} |u_j|^p\,dx < +\infty, \tag{3.3}$$

while

$$\sup_j \frac{1}{\varepsilon_j^k} \int_{\omega_\alpha \times (0,\varepsilon_j)^k} (|\nabla_\alpha \hat{u}_j|^p + |\nabla_\beta \hat{u}_j|^p)\, dx$$
$$= \sup_j \int_{\omega_\alpha \times (0,1)^k} \Big(|\nabla_\alpha u_j|^p + \frac{1}{\varepsilon_j^p} |\nabla_\beta u_j|^p \Big)\, dx < +\infty, \quad (3.4)$$

and from (3.4) in particular

$$\sup_j \frac{1}{\varepsilon_j^k} \int_{\omega_\alpha \times (0,\varepsilon_j)^k} |\nabla \hat{u}_j|^p \, dx < +\infty. \quad (3.5)$$

We extend $\hat{u}_j$ to $\omega_\alpha \times (-\varepsilon_j, \varepsilon_j)^k$ by reflection in the k variables $x_{n-k+1}, \dots, x_n$ by defining

$$\tilde{u}_j(x) := \hat{u}_j(x_\alpha, |x_{n-k+1}|, \dots, |x_n|) \quad \text{in } \omega_\alpha \times (-\varepsilon_j, \varepsilon_j)^k.$$

Note that $(\tilde{u}_j) \subset W^{1,p}(\omega_\alpha \times (-\varepsilon_j, \varepsilon_j)^k; \mathbb{R}^m)$ and $\tilde{u}_j(x_\alpha, \cdot)$ has the same trace on the opposite faces of $(-\varepsilon_j, \varepsilon_j)^k$ for a.e. $x_\alpha \in \omega_\alpha$. Thus $\tilde{u}_j$ can be extended by $(-\varepsilon_j, \varepsilon_j)^k$-periodicity in x_β, to the whole $\omega_\alpha \times \mathbb{R}^k$ obtaining the $W^{1,p}_{\rm loc}(\omega_\alpha \times \mathbb{R}^k; \mathbb{R}^m)$-sequence defined as follows

$$\bar{u}_j(x) := \tilde{u}_j(x_\alpha, x_\beta - 2\varepsilon_j\, i) \quad \text{in} \quad \omega_\alpha \times (2\varepsilon_j\, i + (-\varepsilon_j, \varepsilon_j)^k), \text{ for } i = (i_1, \dots, i_k) \in \mathbb{Z}^k.$$

We want to prove that $(\bar{u}_j)$ is bounded in $W^{1,p}(\omega_\alpha \times (0,1)^k; \mathbb{R}^m)$. By the periodicity and symmetry properties of $\bar{u}_j$, denoting by $[t]$ the integer part of $t \in \mathbb{R}$, we have

$$\begin{aligned}
\int_{\omega_\alpha \times (0,1)^k} |\bar{u}_j|^p \, dx &\le \sum_{i_1,\dots,i_k=0}^{[1/2\varepsilon_j]+1} \int_{\omega_\alpha \times (2\varepsilon_j\, i + (-\varepsilon_j,\varepsilon_j)^k)} |\bar{u}_j|^p \, dx \\
&= \sum_{i_1,\dots,i_k} \int_{\omega_\alpha \times (-\varepsilon_j,\varepsilon_j)^k} |\tilde{u}_j|^p \, dx = 2^k \sum_{i_1,\dots,i_k} \int_{\omega_\alpha \times (0,\varepsilon_j)^k} |\hat{u}_j|^p \, dx \\
&= 2^k \left(\left[\frac{1}{2\varepsilon_j} \right] + 2 \right)^k \int_{\omega_\alpha \times (0,\varepsilon_j)^k} |\hat{u}_j|^p \, dx \\
&\le \frac{2^k}{\varepsilon_j^k} \int_{\omega_\alpha \times (0,\varepsilon_j)^k} |\hat{u}_j|^p \, dx
\end{aligned} \quad (3.6)$$

for j sufficiently large.

Gathering (3.6) and (3.3) we deduce

$$\sup_j \int_{\omega_\alpha \times (0,1)^k} |\bar{u}_j|^p \, dx < +\infty;$$

an analogous argument combined with (3.5) yields

$$\sup_j \int_{\omega_\alpha \times (0,1)^k} |\nabla \bar{u}_j|^p \, dx < +\infty.$$

By these estimates $(\bar{u}_j)$ fulfills the hypothesis of Theorem 2.1, which ensures (up to an extraction) the existence of a sequence $(z_j) \subset W^{1,p}(\omega_\alpha \times (0,1)^k; \mathbb{R}^m)$ satisfying

$$\mathcal{L}^n((\{z_j \neq \bar{u}_j\} \cup \{\nabla z_j \neq \nabla \bar{u}_j\}) \cap (\omega_\alpha \times (0,1)^k)) \to 0, \quad \text{as } j \to +\infty$$

and such that $(|\nabla z_j|^p)$ (or equivalently $(|\nabla_\alpha z_j|^p+|\nabla_\beta z_j|^p)$) is equi-integrable on $\omega_\alpha\times(0,1)^k$. As a consequence, in view of Proposition 2.2, there exists a positive Borel function $\varphi:\,[0,+\infty)\to[0,+\infty]$ such that

$$\lim_{t\to+\infty}\frac{\varphi(t)}{t}=+\infty \quad\text{and}\quad \sup_j\int_{\omega_\alpha\times(0,1)^k}\varphi(|\nabla_\alpha z_j|^p+|\nabla_\beta z_j|^p)\,dx<+\infty.$$

Hence, $(0,[1/\varepsilon_j]\varepsilon_j)^k\subset(0,1)^k$ and the nonnegative character of φ yield

$$\int_{\omega_\alpha\times(0,[1/\varepsilon_j]\varepsilon_j)^k}\varphi(|\nabla_\alpha z_j|^p+|\nabla_\beta z_j|^p)\,dx\le\int_{\omega_\alpha\times(0,1)^k}\varphi(|\nabla_\alpha z_j|^p+|\nabla_\beta z_j|^p)\,dx \tag{3.7}$$

while the monotonicity of the Lebesgue measure implies

$$\begin{aligned}\mathcal{L}^n((\{z_j\neq\bar u_j\}\cup\{\nabla z_j\neq\nabla\bar u_j\})\cap(\omega_\alpha\times(0,[1/\varepsilon_j]\varepsilon_j)^k))&\\ \le\mathcal{L}^n((\{z_j\neq\bar u_j\ \}\cup\{\nabla z_j\neq\nabla\bar u_j\})\cap(\omega_\alpha\times(0,1)^k)).&\end{aligned} \tag{3.8}$$

To shorten notation, set

$$\begin{aligned}M_j&:=\int_{\omega_\alpha\times(0,1)^k}\varphi\left(|\nabla_\alpha z_j|^p+|\nabla_\beta z_j|^p\right)\,dx,\\ m_j&:=\mathcal{L}^n((\{z_j\neq\bar u_j\ \}\cup\{\nabla z_j\neq\nabla\bar u_j\})\cap(\omega_\alpha\times(0,1)^k))\end{aligned} \tag{3.9}$$

and recall that

$$\text{(i)}\quad\sup_j M_j<+\infty,\qquad\text{(ii)}\quad m_j\to0. \tag{3.10}$$

From (3.9) and $(0,[1/\varepsilon_j]\varepsilon_j)^k=\bigcup_{i_1,\dots,i_k=0}^{[1/\varepsilon_j]-1}(\varepsilon_j\,i+(0,\varepsilon_j)^k)$, (3.7)-(3.8) can be rewritten respectively as

$$\sum_{i_1,\dots,i_k=0}^{[1/\varepsilon_j]-1}\int_{\omega_\alpha\times(\varepsilon_j i+(0,\varepsilon_j)^k)}\varphi\left(|\nabla_\alpha z_j|^p+|\nabla_\beta z_j|^p\right)dx\le M_j, \tag{3.11}$$

and

$$\sum_{i_1,\dots,i_k=0}^{[1/\varepsilon_j]-1}\mathcal{L}^n((\{z_j\neq\bar u_j\}\cup\{\nabla z_j\neq\nabla\bar u_j\})\cap(\omega_\alpha\times(\varepsilon_j\,i+(0,\varepsilon_j)^k)))\le m_j. \tag{3.12}$$

For fixed j, we now consider only those cubes $\varepsilon_j\,i+(0,\varepsilon_j)^k$ with $i=2h$ for h in $\mathcal{I}_j:=\{h\in\mathbb{Z}^k:\ 0\le h_1,\dots,h_k\le\frac12(\,[1/\varepsilon_j]-1)\}$. Note that for $h\in\mathcal{I}_j$, $\bar u_j|_{\omega_\alpha\times2\varepsilon_jh+(0,\varepsilon_j)^k}$ coincide with the $2\varepsilon_jh$-translation of $\hat u_j$ in the x_β variable.

By (3.11) and (3.12) we have that in particular

$$\sum_{h\in\mathcal{I}_j}\int_{\omega_\alpha\times(2\varepsilon_jh+(0,\varepsilon_j)^k)}\varphi\left(|\nabla_\alpha z_j|^p+|\nabla_\beta z_j|^p\right)dx\le M_j \tag{3.13}$$

$$\sum_{h\in\mathcal{I}_j}\mathcal{L}^n((\{z_j\neq\bar u_j\}\cup\{\nabla z_j\neq\nabla\bar u_j\})\cap(\omega_\alpha\times(2\varepsilon_j\,h+(0,\varepsilon_j)^k)))\le m_j. \tag{3.14}$$

Then from (3.13), for at least half of the indices $h\in\mathcal{I}_j$ (*i.e.*, for $[1/2\,\#(\mathcal{I}_j)]$ indices) we must have

$$\int_{\omega_\alpha\times(2\varepsilon_jh+(0,\varepsilon_j)^k)}\varphi\left(|\nabla_\alpha z_j|^p+|\nabla_\beta z_j|^p\right)dx\le(\#(\mathcal{I}_j)-[1/2\,\#(\mathcal{I}_j)]+1)^{-1}M_j. \tag{3.15}$$

In fact, let otherwise $\mathcal{I}_j^{'} := \{h \in \mathcal{I}_j : \text{ (3.15) does not hold}\}$ be such that

$$\#(\mathcal{I}_j^{'}) \geq \#(\mathcal{I}_j) - [1/2\,\#(\mathcal{I}_j)] + 1 \tag{3.16}$$

then

$$\begin{aligned}
&\sum_{h\in\mathcal{I}_j} \int_{\omega_\alpha\times(2\varepsilon_j h+(0,\varepsilon_j)^k)} \varphi\left(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p\right) dx \\
\geq\ &\sum_{h\in\mathcal{I}_j^{'}} \int_{\omega_\alpha\times(2\varepsilon_j h+(0,\varepsilon_j)^k)} \varphi\left(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p\right) dx \\
>\ &\#(\mathcal{I}_j^{'})(\#(\mathcal{I}_j) - [1/2\,\#(\mathcal{I}_j)] + 1)^{-1} M_j
\end{aligned}$$

and combining it with (3.16), by (3.13) we find a contradiction.

Since $\#(\mathcal{I}_j) = \left(\left[\frac{1}{2}\left([1/\varepsilon_j] - 1\right)\right] + 1\right)^k$ it can be easily checked that, for j large enough

$$\#(\mathcal{I}_j) - [1/2\,\#(\mathcal{I}_j)] + 1 > \frac{1}{2^{2k+1}\varepsilon_j^k};$$

therefore from (3.15) we get that for at least $[1/2\,\#(\mathcal{I}_j)]$ indices $h \in \mathcal{I}_j$

$$\int_{\omega_\alpha\times(2\varepsilon_j h+(0,\varepsilon_j)^k)} \varphi\left(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p\right) < 2^{2k+1}\varepsilon_j^k M_j, \tag{3.17}$$

for any sufficiently large j. Moreover, in view of (3.14) we can again use an averaging procedure to find among those $[1/2\,\#(\mathcal{I}_j)]$ indices h satisfying (3.17), an index such that

$$\begin{aligned}
&\mathcal{L}^n((\{z_j \neq \bar{u}_j\} \cup \{\nabla z_j \neq \nabla \bar{u}_j\}) \cap (\omega_\alpha \times (2\varepsilon_j\, h + (0,\varepsilon_j)^k))) \\
&\qquad\qquad \leq [1/2\,\#(\mathcal{I}_j)]^{-1} m_j \leq 2^{3k+1}\varepsilon_j^k\, m_j,
\end{aligned} \tag{3.18}$$

for j large enough.

Finally, we have selected an index in $\mathcal{I}_j$ for which both (3.17) and (3.18) (definitively) hold true. Let us call this index $h^\star$. Then by the $(-\varepsilon_j,\varepsilon_j)^k$-periodicity of $\bar{u}_j$ in the x_β variable, up to at most k translations in the $x_{n-k+1},\dots,x_n$-directions, we can always suppose that $h^\star = (0,\dots,0)$.

Abusing notation we denote by z_j the restriction of z_j to $\omega_\alpha \times (0,\varepsilon_j)^k$; we show that our (v_j) can be obtained from (z_j) just by unscaling. In fact, having set

$$v_j(x) := z_j(x_\alpha, \varepsilon_j x_\beta),$$

then $(v_j) \subset W^{1,p}(\omega_\alpha \times (0,1)^k; \mathbb{R}^m)$ and by (3.17) with $h = h^\star = (0,\dots,0)$ we have that

$$\begin{aligned}
&\int_{\omega_\alpha\times(0,1)^k} \varphi\Big(|\nabla_\alpha v_j|^p + \frac{1}{\varepsilon_j^p}|\nabla_\beta v_j|^p\Big)\, dx \\
&\qquad\qquad = \frac{1}{\varepsilon_j^k}\int_{\omega_\alpha\times(0,\varepsilon_j)^k} \varphi(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p)\, dx < 2^{2k+1}\, M_j.
\end{aligned}$$

Thus, by virtue of (3.10)(i), again applying de la Vallée Poussin's Criterion we get that $\big(|\nabla_\alpha v_j|^p + \frac{1}{\varepsilon_j^p}|\nabla_\beta v_j|^p\big)$ is equi-integrable on $\omega_\alpha \times (0,1)^k$. Moreover by (3.18) we deduce

$$\mathcal{L}^n(\{v_j \neq u_j\} \cup \{\nabla v_j \neq \nabla u_j\}) \\ = \frac{1}{\varepsilon_j^k}\,\mathcal{L}^n((\{z_j \neq \bar{u}_j\} \cup \{\nabla z_j \neq \nabla \bar{u}_j\}) \cap (\omega_\alpha \times (0,\varepsilon_j)^k)) \leq 2^{3k+1} m_j$$

and by (3.10)(ii) we find (3.2). Clearly these two conditions can be restricted to $\omega_\alpha \times \omega_\beta$ if such was the domain of the starting sequence.

Finally, note that if ω_α is Lipschitz, by appealing to Theorem 2.1 we can choose any z_j to be a Lipschitz function, then for every $x, y \in \omega_\alpha \times (0,1)^k$

$$|v_j(x) - v_j(y)| = |z_j(x_\alpha, \varepsilon_j x_\beta) - z_j(y_\alpha, \varepsilon_j y_\beta)| \leq \mathrm{Lip}_{z_j}|x-y|,$$

thus v_j is still a Lipschitz function and $\mathrm{Lip}_{v_j} \leq \mathrm{Lip}_{z_j}$. □

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