Tesi di Dottorato

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Differential graded Lie algebras and deformations of holomorphic maps

Dottorato in Matematica, Roma «La Sapienza» (2006).
<http://www.bdim.eu/item?id=tesi_2006_IaconoDonatella_1>

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"La Sapienza"

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Dottorato di Ricerca in Matematica
XVIII Ciclo

Differential graded Lie algebras and deformations of holomorphic maps

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Introduction

In the last fifty years the deformations theory has played an important role in algebraic and complex geometry.

The study of small deformations of the complex structures of complex manifolds has begun with the works of K. Kodaira and D.C. Spencer [**17**] and M. Kuranishi [**19**].

Then A. Grothendieck [**12**], M. Schlessinger [**32**] and M. Artin [**1**] formalized this theory translating it into a functorial language. The idea was that to an infinitesimal deformation of a geometric object we can associate a deformation functor of Artin rings: that is a functor from the category **Art** of local artinian $\mathbb{C}$-algebras (with residue field $\mathbb{C}$) to the category **Set** of sets, that satisfies some extra conditions (see Definition I.1.10).

A modern approach to deformations theory is via differential graded Lie algebras (DGLA for short).

The guiding principle is the idea due to P. Deligne, V. Drinfeld, D. Quillen and M. Kontsevich (see [**18**]) that "in characteristic zero every deformation problem is governed by a differential graded Lie algebra".

Inspired by this principle, the aim of this thesis is to follow the modern approach to study the infinitesimal deformations of holomorphic maps of compact complex manifolds.

More precisely, a DGLA is a differential graded vector space with a structure of graded Lie algebra, plus some compatibility conditions between the differential and the bracket (of the Lie structure) (see Definition I.3.5).

Moreover, using the solutions of the Maurer-Cartan equation $dx + \frac{1}{2}[x,x] = 0$ it is well known how we can associate to a DGLA L a deformation functor Def_L. Written in details:

$$\operatorname{Def}_L : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$\operatorname{Def}_L(A) = \frac{\{x \in L^1 \otimes m_A \mid dx + \frac{1}{2}[x,x] = 0\}}{gauge}$$

where m_A is the maximal ideal of A and the gauge equivalence is induced by the gauge action $*$ of $\exp(L^0 \otimes m_A)$ on the set of solutions of the Maurer-Cartan equation (see Definition I.3.29).

Then the idea of the principle is that we can define a DGLA L (up to quasi-isomorphism) from the geometrical data of the problem, such that the deformation functor Def_L is isomorphic to the deformation functor associated to the geometric problem.

We note that it is easiest to study a deformation functor associated to a DGLA but, in general, it is not an easy task to find the right DGLA (up to quasi-isomorphism) associated to the problem.

A first example, in which the associated DGLA is well understood, is the case of deformations of complex manifolds.

Let X be a compact complex manifold. Then X is obtained gluing a finite number of polydisks in $\mathbb{C}^n$. The fundamental idea of K. Kodaira and D.C. Spencer is that "a deformation of X is considered to be the gluing of the same polydisks via different identifications" (see [**16**, pag. 182]) .

Translating it into a functorial language we define, for each $A \in$ **Art**, an infinitesimal deformation of X over $\mathrm{Spec}(A)$ as a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X_A \arrow[d, "\pi"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "a"] & \mathrm{Spec}(A) \end{tikzcd}$$

where π is a proper and flat holomorphic map and X coincides with the restriction of X_A over the closed point of $\mathrm{Spec}(A)$ (see Definition I.2.4). Moreover, we can give the notions of isomorphism and of trivial deformation ($X_A \cong X \times \mathrm{Spec}(A)$).

Then we can define the functor associated to the infinitesimal deformations of X:

$$\mathrm{Def}_X : \mathbf{Art} \to \mathbf{Set}$$

$$\mathrm{Def}_X(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations} \\ \text{of } X \text{ over } \mathrm{Spec}(A) \end{array} \right\}.$$

Therefore, following the idea of the principle, we are looking for a DGLA L such that $\mathrm{Def}_L \cong \mathrm{Def}_X$.

Let Θ_X be the holomorphic tangent bundle of X and consider the sheaf $\mathcal{A}_X^{0,*}(\Theta_X)$ of the differentials form of $(0,*)$-type, with coefficients in Θ_X.

Then we define the Kodaira-Spencer algebra $A_X^{0,*}(\Theta_X)$ of X as the graded vector space of global sections of the sheaf $\mathcal{A}_X^{0,*}(\Theta_X)$.

Considering the (opposite) Dolbeault differential and the bracket of vector fields, it is possible to endow $A_X^{0,*}(\Theta_X)$ of a natural structure of differential graded Lie algebra (see Definition II.4.1).

The DGLA $A_X^{0,*}(\Theta_X)$ governs the problem of infinitesimal deformations of X (see Theorem II.7.3):

THEOREM (A). *Let X be a complex compact manifold and $A_X^{0,*}(\Theta_X)$ its Kodaira-Spencer algebra. Then there exists an isomorphism of functors*

$$\mathrm{Def}_{A_X^{0,*}(\Theta_X)} \longrightarrow \mathrm{Def}_X .$$

In this case it is well clear the correspondence between the solutions of the Maurer-Cartan equation and the infinitesimal deformations of X, such that the gauge equivalence corresponds to the isomorphism of deformations. In particular a solution of the Maurer-Cartan equation is gauge equivalent to zero if and only if it induces a trivial deformation of X.

The next natural problem is to investigate the embedded deformations of a submanifold in a fixed manifold.

Very recently, M. Manetti in [**24**] studies this problem using the approach via DGLA.

In his work, more than to prove the existence of a DGLA that governs this geometric problem, M. Manetti develops some algebraic tools related to the DGLA.

More precisely, he describes a general construction to define a new deformation functor associated to a morphism of DGLA.

Given a morphism of differential graded Lie algebras

$$h : L \longrightarrow M$$

he defines the functor

$$\mathrm{Def}_h : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$\mathrm{Def}_h(A) = \frac{\{(l,m) \in (L^1 \otimes m_A) \times (M^0 \otimes m_A) |\ dx + \frac{1}{2}[x,x] = 0 \text{ and } e^m * h(l) = 0\}}{gauge},$$

where this gauge equivalence is a generalization of the previous one (see Remark III.1.12). This new functor is an extension of the functor associated to a single DGLA: choosing $h = M = 0$ Def_h reduces to Def_L.

Moreover, using path objects (see Example I.3.12) he shows that for every choice of L, M and h there exists a DGLA H such that $\mathrm{Def}_H \cong \mathrm{Def}_h$.

In particular this implies that if a deformation functor associated to a geometric problem is isomorphic to Def_h, for some $h : L \longrightarrow M$, then automatically we know the existence of a DGLA that governs the problem and we have an explicit description of it.

This is what M. Manetti does in [**24**]: choosing opportunely L, M and h he proves that there exists an isomorphism between the functor Def_h and the functor associated to the infinitesimal deformations of a submanifold in a fixed manifold. Actually, let X be a compact complex manifold and Z a submanifold. The infinitesimal emebedded deformations of Z can be interpreted as the deformations of the inclusion map $i : Z \hookrightarrow X$ inducing the trivial deformation on X.

Consider the Kodaira-Spencer DGLA $A_X^{0,*}(\Theta_X)$ of X and the differential graded Lie subalgebra $A_X^{0,*}(\Theta_X(-log\, Z))$ defined by the following exact sequence

$$0 \longrightarrow A_X^{0,*}(\Theta_X(-log\, Z)) \longrightarrow A_X^{0,*}(\Theta_X) \longrightarrow A_Z^{0,*}(N_{Z|X}) \longrightarrow 0,$$

where $N_{Z|X}$ is the normal bundle of Z in X (see Section II.5.1).

We have already observed that the DGLA $A_X^{0,*}(\Theta_X)$ governs the infinitesimal deformations of X, while the DGLA $A_X^{0,*}(\Theta_X(-log\, Z))$ governs the infinitesimal deformations of the couple $Z \subset X$ (each solution of the Maurer-Cartan equation in $A_X^{0,*}(\Theta_X(-log\, Z))$ define a deformation of Z and of X).

Fix $M = A_X^{0,*}(\Theta_X)$, $L = A_X^{0,*}(\Theta_X(-log\, Z))$ and h the inclusion:

$$h : A_X^{0,*}(\Theta_X(-log\, Z)) \hookrightarrow A_X^{0,*}(\Theta_X).$$

Then it is clear how we can associate to each element $(l, m) \in \mathrm{Def}_h$ a deformation of Z in X, with X fixed: the infinitesimal deformation of Z is the one corresponding to the Maurer-Cartan solution $l \in A_X^{0,*}(\Theta_X(-log\, Z))$ and it induces a trivial deformation of X, since we are requiring that $h(l)$ is gauge equivalent to zero in $A_X^{0,*}(\Theta_X)$.

These new ideas developed by M. Manetti are of fundamental importance for this thesis that, in some sense, can be considered a generalization of them. Actually, we extend these techniques to study not only the deformations of an inclusion but, in general, the deformations of holomorphic maps.

These deformations has been first studied from the classical point of view (no DGLA) by E. Horikawa [**14**] and [**15**], M. Namba [**27**] and by Z. Ran [**28**].

More precisely, let $f : X \longrightarrow Y$ be an holomorphic map of compact complex manifolds.

There are several aspects of deformations of f: we can deform just the map f fixing both X and Y, we can allow to deform f and X or Y or, more in general, we can deform everything: the map f, X and Y.

The infinitesimal deformations of f, with fixed domain and target, can be interpreted as infinitesimal deformations of the graph of f in the product $X \times Y$, with $X \times Y$ fixed (see Section V.3). Therefore we are considering the infinitesimal deformation of a submanifold in a fixed manifold and so the DGLA approach to this case is implicitly included in M. Manetti's work [**24**].

Then we turn our attention on the general case in which we deform f, X and Y.

In a functorial language, for each $A \in \mathbf{Art}$, we define an infinitesimal deformation of f over $\operatorname{Spec}(A)$ as a commutative diagram

$$\begin{tikzcd} X_A \arrow[rr, "F"] \arrow[dr, "\pi"'] & & Y_A \arrow[dl, "\mu"] \\ & S & \end{tikzcd}$$

where $S = \operatorname{Spec}(A)$, (X_A, π, S) and (Y_A, π, S) are infinitesimal deformations of X and Y respectively, and F is a holomorphic map that restricted to the fibers over the closed point of S coincides with f.

Also in this case we can give the notions of isomorphism and of trivial deformation.

Then we can define the functor of infinitesimal deformations of a holomorphic map $f : X \longrightarrow Y$:

$$\operatorname{Def}(f) : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$\operatorname{Def}(f)(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations of} \\ f \text{ over } \operatorname{Spec}(A) \end{array} \right\}.$$

Let Γ be the graph of f in $X \times Y$. An infinitesimal deformation of the map f can be interpreted as an infinitesimal deformation of Γ in $X \times Y$, such that the induced deformation of the product $X \times Y$ is a product of deformations of X and Y. In general not all the deformations of a product are product of deformations, as it was showed by K. Kodaira and D.C Spencer (see Remark II.7.5).

Consider the Kodaira-Spencer algebra $A^{0,*}_{X\times Y}(\Theta_{X\times Y})$ of the product $X \times Y$ and the differential graded Lie subalgebra $A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma))$ defined by the following exact sequence

$$0 \longrightarrow A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma)) \longrightarrow A^{0,*}_{X\times Y}(\Theta_{X\times Y}) \longrightarrow A^{0,*}_{\Gamma}(N_{\Gamma|X\times Y}) \longrightarrow 0,$$

where $N_{\Gamma|X\times Y}$ is the normal bundle of the graph Γ in $X \times Y$ (see Section II.5.1).

As before, we know that $A^{0,*}_{X\times Y}(\Theta_{X\times Y})$ governs the infinitesimal deformations of $X \times Y$ and $A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma))$ governs the infinitesimal deformations of the couple $\Gamma \subset X \times Y$ (each solution of the Maurer-Cartan equation in $A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma))$ define a deformation of Γ and of $X \times Y$).

Fix $M = A^{0,*}_{X\times Y}(\Theta_{X\times Y})$, $L = A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma))$ and h the inclusion:

$$h : A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma)) \hookrightarrow A^{0,*}_{X\times Y}(\Theta_{X\times Y}).$$

In the general case, it is not enough to consider just the DGLA L or the morphism $h : L \longrightarrow M$, since they have no control on the induced deformations on $X \times Y$.

Therefore we need to define a new functor: the deformation functor associated to a couple of morphisms.

Given morphisms of differential graded Lie algebras $h : L \longrightarrow M$ and $g : N \longrightarrow M$:

$$\begin{tikzcd} & L \arrow[d, "h"] \\ N \arrow[r, "g"] & M \end{tikzcd}$$

we define the functor

$$\mathrm{Def}_{(h,g)} : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$\mathrm{Def}_{(h,g)}(A) = \{(x, y, e^p) \in (L^1 \otimes m_A) \times (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A)|$$

$$dx + \frac{1}{2}[x, x] = 0,\ dy + \frac{1}{2}[y, y] = 0,\ g(y) = e^p * h(x)\}/gauge,$$

where this gauge equivalence is an extension of the previous ones (see Definition III.1.11).

This functor is a generalization of the previous ones: choosing $N = 0$ and $g = 0$ then $\mathrm{Def}_{(h,g)}$ reduces to Def_h; choosing $N = M = 0$ and $h = g = 0$ then $\mathrm{Def}_{(h,g)}$ reduces to Def_L.

Consider the DGLA $A^{0,*}_X(\Theta_X) \times A^{0,*}_Y(\Theta_Y)$ and the morphism $g = (p^*, q^*) : A^{0,*}_X(\Theta_X) \times A^{0,*}_Y(\Theta_Y) \longrightarrow A^{0,*}_{X\times Y}(\Theta_{X\times Y})$, where p and q are the natural projections of the product $X \times Y$ on X and Y respectively: $p : X \times Y \longrightarrow X$ and $q : X \times Y \longrightarrow Y$.

We note that the solutions $n = (n_1, n_2)$ of the Maurer-Cartan equation in $N = A^{0,*}_X(\Theta_X) \times A^{0,*}_Y(\Theta_Y)$ are in correspondences with the infinitesimal deformations of X (induced by n_1) and of Y (induced by n_2). Moreover the image $g(n)$ satisfies the Maurer-Cartan equation in $M = A^{0,*}_{X\times Y}(\Theta_{X\times Y})$ and so it is associated to an infinitesimal deformation of $X \times Y$, that is exactly the one obtained as product of the deformations of X (induced by n_1) and of Y (induced by n_2).

Therefore this g gives exactly the "control" on the deformations of the product that we are looking for.

Let $M = A^{0,*}_{X\times Y}(\Theta_{X\times Y})$, $L = A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma))$, $h : L \longrightarrow M$ the inclusion, $N = A^{0,*}_X(\Theta_X) \times A^{0,*}_Y(\Theta_Y)$ and $g = (p^*, q^*) : N \longrightarrow M$. Then we are in the following situation:

$$\begin{tikzcd} & A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma)) \arrow[d, hook, "h"] \\ A^{0,*}_X(\Theta_X)\times A^{0,*}_Y(\Theta_Y) \arrow[r, "g=(p^*,q^*)"] & A^{0,*}_{X\times Y}(\Theta_{X\times Y}) \end{tikzcd}$$

In conclusion, each deformation of the map f corresponds to a Maurer-Cartan solution $l \in A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma))$, such that $h(l)$ induces a deformation of $X \times Y$ isomorphic to a deformation induced by $g(n)$, for some Maurer-Cartan solution $n \in A^{0,*}_X(\Theta_X)\times A^{0,*}_Y(\Theta_Y)$ (that is $h(l)$ and $g(n)$ are gauge equivalent in $A^{0,*}_{X\times Y}(\Theta_{X\times Y})$).

Therefore $\mathrm{Def}_{(h,g)}$ encodes all the geometric data of the problem and the following theorem is clear (see Theorem IV.2.5).

THEOREM (B). *Let $f : X \longrightarrow Y$ be an holomorphic map of compact complex manifold. Then, in the notation above, there exists an isomorphism of functors*

$$\mathrm{Def}_{(h,g)} \cong \mathrm{Def}(f).$$

This theorem holds for the general case of infinitesimal deformations of f, the other cases are obtained as specializations of it.

For example, the deformations of f with fixed domain or fixed target, are obtained considering respectively $N = A^{0,*}_Y(\Theta_Y)$ or $N = A^{0,*}_X(\Theta_X)$.

In particular, using path objects, for each choice of $h : L \longrightarrow M$ and $g : N \longrightarrow M$, we are able to find a differential graded Lie algebra $H_{(h,g)}$ such that $\mathrm{Def}_{\mathrm{H}_{(h,g)}} \cong \mathrm{Def}_{(h,g)}$ (see Theorem III.2.36).

Therefore we give an explicit description (more than the existence) of a DGLA that governs the deformations of holomorphic maps (Theorem IV.2.6).

Finally we apply these techniques to study the obstructions to deform holomorphic maps.

The idea is the following: if we have an infinitesimal deformation of a geometric object, then we want to know if it is possible to extend it.

More precisely, let $F : \mathbf{Art} \longrightarrow \mathbf{Set}$ be a deformation functor. A (complete) obstruction space for F is a vector space V, such that for each surjection $B \longrightarrow A$ in **Art** and each element $x \in F(A)$, there exists an obstruction element $v_x \in V$, associated to x, that is zero if and only if x can be lifted to $F(B)$ (for full details see Section I.1.1).

Therefore we would like to control this obstruction space and know when the associate obstruction element is zero.

In general, we just know a vector space that contains these elements but we have no explicitly description of which elements are effectively obstructions. Among other things if W is another vector space that contains V, then also W is an obstruction space for F. Then, in some sense we are looking for the "smallest" obstruction space (see Remark I.1.26).

For example, the obstructions of the functor associated to a DGLA L are naturally contained in $H^2(L)$ (see Section I.3.5), but we don't know which classes in $H^2(L)$ are really obstructions.

In the case of a complex compact manifold X, an obstruction space for the deformation functor Def_X is the second cohomology vector space $H^2(X, \Theta_X)$ of the holomorphic tangent bundle Θ_X of X (Theorem I.2.9).

If X is also Kähler, then A. Beauville, H. Clemens [**5**] and Z. Ran [**29**] [**30**] proved that the obstructions are contained in a subspace of $H^2(X, \Theta_X)$ defined as the kernel of a well defined map. This is the so-called "Kodaira's principle" (see for example [**5**, Theorem 10.1], [**22**, Corollary 3.4] or [**7**, Corollary 12.6], [**29**, Theorem 0] or [**30**, Corollary 3.5]).

In the case of embedded deformations of a submanifold Z in a fixed manifold X, then the obstructions are naturally contained in the first choomology $H^1(Z, N_{Z|X})$ of the normal bundle $N_{Z|X}$ of Z in X. Also in this case, if X is Kähler, it is possible to define a map on $H^1(Z, N_{Z|X})$, called "semiregularity map", that contains the obstructions in the kernel. The idea of this map is due to S. Bloch [**3**] and it is also studied, using the DGLA approach, by M. Manetti [**24**, Theorem 0.1 and Section 9].

In the case of deformations of a holomorphic map $f : X \longrightarrow Y$ with fixed target, it was proved by E. Horikawa in [**14**] (see Theorem IV.1.10) that the obstructions are contained in the second hypercohomology group $\mathbb{H}^2\left(X, \mathcal{O}(\Theta_X) \xrightarrow{f_*} \mathcal{O}(f^*\Theta_Y)\right)$.

Using the approach via DGLA that we have explained, we can give an easy proof of this theorem (Proposition V.1.1) but above all we can improve it in the case of Kähler manifolds (Corollary V.1.5).

Actually, let $n = dimX$, $p = dimY - dimX$ and $\mathcal{H}$ be the space of harmonic forms on Y of type $(n+1, n-1)$. By Dolbeault theorem and Serre duality we obtain the equalities $\mathcal{H}^\nu = (H^{n-1}(Y, \Omega_Y^{n+1}))^\nu = H^{p+1}(Y, \Omega_Y^{p-1})$.

Using the contraction $\lrcorner$ of vector fields with differential forms (see Sections II.5 and V.1.2), for each $\omega \in \mathcal{H}$ we can define the following map

$$A_X^{0,*}(f^*\Theta_Y) \xrightarrow{\lrcorner\omega} A_X^{n,*+n-1}$$

$$\lrcorner\omega(\phi f^*\chi) = \phi f^*(\chi \lrcorner \omega) \in A_X^{n,p+n-1} \qquad \forall\ \phi f^*\chi \in A_X^{0,p}(f^*\Theta_Y).$$

Choosing ω such that $f^*\omega = 0$, we get the following commutative diagram (see Corollary V.1.5)

$$\begin{tikzcd} A^{0,*}_X(f^*\Theta_Y) \arrow[r, "\lrcorner\omega"] & A^{n,*+n-1}_X \\ A^{0,*}_X(\Theta_X) \arrow[u, "f_*"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

Then, for each ω we get a morphism

$$\mathbb{H}^2\left(X, \mathcal{O}(\Theta_X) \xrightarrow{f_*} \mathcal{O}(f^*\Theta_Y)\right) \longrightarrow H^n(X, \Omega^n_X).$$

that composed with the integration on X gives

$$\sigma : \mathbb{H}^2\left(X, \mathcal{O}(\Theta_X) \xrightarrow{f_*} \mathcal{O}(f^*\Theta_Y)\right) \longrightarrow H^{p+1}(Y, \Omega^{p-1}_Y).$$

Using σ we get the following theorem (see Corollary V.1.5).

THEOREM (C). *Let $f : X \longrightarrow Y$ be an holomorphic map of compact Kähler manifolds. Let $p = dimY - dimX$. Then the obstruction space to the infinitesimal deformations of f with fixed Y is contained in the kernel of the map*

$$\sigma : \mathbb{H}^2\left(X, \mathcal{O}(\Theta_X) \xrightarrow{f_*} \mathcal{O}(f^*\Theta_Y)\right) \longrightarrow H^{p+1}(Y, \Omega^{p-1}_Y).$$

The structure of the work is the following.

Chapter I contains the basic material about deformation functors. In Section I.1 we define the deformation functors of Artin rings, tangent and obstruction spaces and some related properties.

Section I.2 is devoted to study the deformation functor Def_X of the infinitesimal deformations of a compact complex manifold X.

In the last Section I.3 we introduce the differential graded Lie algebras (DGLA) and two associated functors: the Maurer-Cartan functor MC_L and the deformation functor Def_L (for each DGLA L).

In *Chapter II* we fix the notations about complex manifolds. We recall the notions of differential forms (Section II.1) of Čech and Dolbeault cohomology (Section II.3) and some properties of Kähler manifolds (Section II.2). We also study the map f_* and f^* induced by an holomorphic map f (Section II.6). In particular in Section II.4, we define the Kodaira-Spencer differential graded Lie algebra $A^{0,*}_X(\Theta_X)$ associated to a complex manifold X.

Section II.7 contains the proof of theorem A (Theorem II.7.3): we prove the existence of an isomorphism $\mathrm{Def}_{A^{0,*}_X(\Theta_X)} \cong \mathrm{Def}_X$ and so the Kodaira-Spencer algebra $A^{0,*}_X(\Theta_X)$ governs the infinitesimal deformation of X.

Chapter III is the technical bulk of this thesis. In Section III.1.2 we define the Maurer-Cartan functor $\mathrm{MC}_{(h,g)}$ and the deformation functor $\mathrm{Def}_{(h,g)}$ associated to a couple of morphisms of DGLAs $h : L \longrightarrow M$ and $g : N \longrightarrow M$. Sections III.1.3 and III.1.4 are devoted to study some properties of these functors as for example tangent and obstruction spaces.

In Section III.2 we introduce the extended deformation functors to prove the existence of a DGLA $H_{(h,g)}$ such that $\mathrm{Def}_{(h,g)} \cong \mathrm{Def}_{H_{(h,g)}}$ (Theorem III.2.36).

In *Chapter IV* we study the infinitesimal deformations of holomorphic maps.

Section IV.1 is devoted to define the deformation functor $\mathrm{Def}(f)$ of infinitesimal deformations of a holomorphic map $f : X \longrightarrow Y$ of compact complex manifolds.

In Section IV.2, we prove theorem B, i.e. the existence of a couple of morphisms of DGLA $h : A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma)) \hookrightarrow A^{0,*}_{X\times Y}(\Theta_{X\times Y})$ and $g = (p^*, q^*) : A^{0,*}_X(\Theta_X) \times A^{0,*}_Y(\Theta_Y) \longrightarrow A^{0,*}_{X\times Y}(\Theta_{X\times Y})$, such that $\mathrm{Def}_{(h,g)} \cong \mathrm{Def}(f)$ (see Theorem IV.2.5).

Chapter V contains examples and applications of the technique described before. In Section V.1 we study the infinitesimal deformations of holomorphic maps with fixed target and Section V.1.2 contains the main result about the semiregularity map (Corollary V.1.5).

Then we study infinitesimal deformations of a holomorphic map with fixed target and domain (Section V.3) and the infinitesimal deformations of an inclusion (Section V.4).

Acknowledgments. It is a pleasure for me to show my deep gratitude to the advisor of this thesis Prof. Marco Manetti, an excellent helpful professor who supports and encourages me every time. I'm indebted with him for many useful discussions, advices and suggestions. Several ideas of this work are grown under his influence. I wish also to thank him for having introduced me to the exciting subject of deformations theory.

I can not forget here my graduate study advisor Prof. Marialuisa J. de Resmini, who encouraged me to continue my studies and "sent" me to a summer school where I had my first meeting with Algebraic Geometry.

I am also indebt with all people with whom I have useful mathematical discussions or that makes my staying in Rome amusing.

In this set there are certainly Paolo Antonini, Sara Azzali, Francesco Esposito, Elena Martinengo, Silvia Montarani and Antonio Rapagnetta.

I wish also to thank three not mathematical friends of mine Arianna, Federica and Patrizia that support me in many and different ways.

This thesis is dedicated to my family Maria Teresa, Giovanni Michele, Enrico and especially to my grandmother Mina.

CHAPTER I

Functors of Artin rings

In this chapter we collect some definitions and main properties of deformation functors.

In the first section, we introduce the notion of functor of Artin rings, of deformation functors and we define the tangent and obstruction spaces.

Section I.2 is devoted to the study of the deformation functor Def_X of infinitesimal deformation of complex manifolds.

In Section I.3 we introduce the fundamental notions of differential graded Lie algebra (DGLA) L and of deformation functor associated to a DGLA Def_L.

The main references of this chapter are [**6**], [**20**], [**23**], [**31**] and [**32**].

I.1. Generalities on functors of Artin rings

Let $\mathbb{K}$ be a fixed field of characteristic zero.

Consider the following categories:

- **Set**: the category of sets in a fixed universe with $\{*\}$ a fixed set of cardinality 1;
- $\mathbf{Art} = \mathbf{Art}_{\mathbb{K}}$: the category of local Artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$ ($A/m_A = \mathbb{K}$);
- $\widehat{\mathbf{Art}} = \widehat{\mathbf{Art}}_{\mathbb{K}}$: the category of complete local noetherian $\mathbb{K}$-algebras with residue field $\mathbb{K}$ ($A/m_A = \mathbb{K}$).

  For each $S \in \widehat{\mathbf{Art}}$ we also consider:
- $\mathbf{Art}_S$: the category of local Artinian S-algebras with residue field $\mathbb{K}$ (for a such element A, the structure morphism $S \longrightarrow A$ induces a trivial extension of residue field $\mathbb{K}$);
- $\widehat{\mathbf{Art}}_S$: the category of complete local noetherian S-algebras with residue field $\mathbb{K}$.

I.1.1. Remark. We note that $\mathbf{Art}_S \subset \widehat{\mathbf{Art}}_S$. Moreover, for morphisms in a category of local object we mean local morphisms and we often use the notation $A \in \mathbf{C}$ instead of $A \in ob\mathbf{C}$), when $\mathbf{C}$ is a category.

If $\varphi : B \longrightarrow A$ and $\psi : C \longrightarrow A$ are morphisms in $\widehat{\mathbf{Art}}_S$ (respectively in $\mathbf{Art}_S$), then

$$B \times_A C = \{(b, c) \in B \times C \mid \varphi(b) = \psi(c)\}$$

is the *fiber product* of φ and ψ and $B \times_A C \in \widehat{\mathbf{Art}_S}$ (respectively in $\mathbf{Art}_S$).

I.1.2. Definition. A *small extension* in $\widehat{\mathbf{Art}_S}$ (respectively in $\mathbf{Art}_S$) is a short exact sequence

$$e: \quad 0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0,$$

where α is a morphism in $\widehat{\mathbf{Art}_S}$ (respectively in $\mathbf{Art}_S$) and the kernel J is an ideal of B annihilated by the maximal ideal m_B, $m_B \cdot J = (0)$. This implies that J is $\mathbb{K}$-vector space.

A small extension is called *principal* if J is a one dimensional vector space ($J \cong \mathbb{K}$).

We will often say that a morphism $\alpha : B \longrightarrow A$ is a small extension, meaning that $0 \longrightarrow \ker(\alpha) \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0$ is a small extension.

I.1.3. Remark. Every surjective morphism in $\mathbf{Art}_S$ can be expressed as a finite composition of small extensions.

Actually, let $B \longrightarrow A$ be a surjection with kernel J:

$$0 \longrightarrow J \longrightarrow B \longrightarrow A \longrightarrow 0.$$

Since B is a local artinian ring, its maximal ideal m is nilpotent: there exist $n_0 \in \mathbb{N}$ such that $m^n = 0$ for each $n \geq n_0$; in particular $m^n J = 0$.

Therefore it is sufficient to consider the sequence of small extensions

$$0 \longrightarrow m^n J/m^{n+1} J \longrightarrow B/m^{n+1} J \longrightarrow B/m^n J \longrightarrow 0.$$

I.1.4. Remark. In view of the previous Remark I.1.3, it will be enough to check the surjection for small extensions instead of verify the surjection for any morphism in $\mathbf{Art}_S$.

I.1.5. Definition. A *functor of Artin rings* is a covariant functor $F : \mathbf{Art}_S \longrightarrow \mathbf{Set}$, such that $F(\mathbb{K}) = \{*\}$ is the one point set.

The functors of Artin rings $F : \mathbf{Art}_S \longrightarrow \mathbf{Set}$ and their natural transformations set up a category denoted by $\mathbf{Fun}_S$. A natural transforation of functors $\gamma : F \longrightarrow G$ is an *isomorphism of functors* if and only if $\gamma(A) : F(A) \longrightarrow G(A)$ is bijective for each $A \in \mathbf{Art}_S$.

I.1.6. Example. The trivial functor $F(A) = \{*\}$, for every $A \in \mathbf{Art}_S$.

I.1.7. Example. Let $R \in \widehat{\mathbf{Art}_S}$. We define

$$h_R : \mathbf{Art}_S \longrightarrow \mathbf{Set}$$

such that

$$h_R(A) = \mathrm{Hom}_S(R, A).$$

$F \in \mathbf{Fun}_S$ is called *pro-representable* if it is isomorphic to h_R, for some $R \in \widehat{\mathbf{Art}_S}$.

If we can choose $R \in \mathbf{Art}_S$ then F is called *representable*.

Let $\mathbb{K}[\varepsilon]$, with $\varepsilon^2 = 0$, be the ring of dual number over $\mathbb{K}$. $\mathbb{K}[\varepsilon] = \mathbb{K} \oplus \mathbb{K}\varepsilon$ is a $\mathbb{K}$-vector space of dimension 2 and it has a trivial S-algebra structure (induced by $S \longrightarrow \mathbb{K} \longrightarrow \mathbb{K}[\varepsilon]$).

I.1.8. Definition. The set $t_F := F(\mathbb{K}[\varepsilon])$ is called the *tangent space* of $F \in \mathbf{Fun}_S$.

Let

$$\eta : F(B \times_A C) \longrightarrow F(B) \times_{F(A)} F(C) \tag{1}$$

be the map induced by the fiber product in $\mathbf{Art}_S$:

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & A. \end{tikzcd}$$

I.1.9. Definition. A functor F is called *homogeneous* if η is an isomorphism whenever $B \longrightarrow A$ is surjective.

I.1.10. Definition. A functor F is called a *deformation functor* if

i) η is surjective whenever $B \longrightarrow A$ is surjective;
ii) η is an isomorphism whenever $A = \mathbb{K}$.

I.1.11. Remark. The deformation functors will play an important role in this work.

In particular we will study the following four deformation functors:

1) the functor Def_X of infinitesimal deformation of an algebraic scheme, in Section I.2;
2) the functor Def_L associated to a differential graded Lie algebra L, in Section I.3;
3) the functor $\mathrm{Def}_{(h,g)}$ associated to a couple of morphisms of differential graded Lie algebras $h : L \longrightarrow M$ and $g : N \longrightarrow M$, in Section III.1;
4) the functor Def_f associated to the infinitesimal deformations of a holomorphic map f, in Section IV.1.

I.1.12. Example. Let X be an algebraic scheme over $\mathbb{K}$ (separated of finite type over $\mathbb{K}$). Define the following functor

$$\mathrm{Def}_X : \mathbf{Art}_{\mathbb{K}} \longrightarrow \mathbf{Set}$$

where $\mathrm{Def}_X(A)$ is the set of isomorphism classes of commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X_A \arrow[d, "p_A"] \\ \mathrm{Spec}(\mathbb{K}) \arrow[r] & \mathrm{Spec}(A). \end{tikzcd}$$

where i is a closed embedding and p_A a flat morphism. Def_X is a deformation functor (see [**32**, Section 3.7] or [**31**, Prop. III.3.1]).

I.1.13. Proposition. *Let F be a deformation functor. Then t_f has a natural structure of $\mathbb{K}$-vector space and every natural transformation of deformation functors $\eta : F \longrightarrow G$ induces a linear map between tangent spaces.*

Proof. Since $F(\mathbb{K})$ is just one point and η is an isomorphism for $A = \mathbb{K}$, we have $F(\mathbb{K}[\varepsilon]) \times F(\mathbb{K}[\varepsilon]) \cong F(\mathbb{K}[\varepsilon] \times_{\mathbb{K}} \mathbb{K}[\varepsilon])$.

Consider the map

$$\begin{aligned} + : \mathbb{K}[\varepsilon] \times_{\mathbb{K}} \mathbb{K}[\varepsilon] &\longrightarrow \mathbb{K}[\varepsilon] \\ (a + b\varepsilon, a + b'\varepsilon) &\longmapsto a + (b + b')\varepsilon. \end{aligned}$$

Then using the previous isomorphism, the map $+$ induces the sum on $F(\mathbb{K}[\varepsilon])$:

$$F(\mathbb{K}[\varepsilon] \times_{\mathbb{K}} \mathbb{K}[\varepsilon]) \xrightarrow{\cong} F(\mathbb{K}[\varepsilon]) \times F(\mathbb{K}[\varepsilon]) \xrightarrow{F(+)} F(\mathbb{K}[\varepsilon]).$$

Analogously for the multiplication by a scalar $k \in \mathbb{K}$ we consider the map:

$$\begin{aligned} k : \mathbb{K}[\varepsilon] &\longrightarrow \mathbb{K}[\varepsilon] \\ a + b\varepsilon &\longmapsto a + (kb)\varepsilon. \end{aligned}$$

□

I.1.14. Remark. In the previous proposition we just used the fact that $F(\mathbb{K})$ is one point and that η of (1) is an isomorphism for $A = \mathbb{K}$ and $C = \mathbb{K}[\varepsilon]$

I.1.15. Definition. A morphism $\phi : F \longrightarrow G$ in $\mathbf{Fun}_S$ is:

- *unramified* if $\phi : t_F \longrightarrow t_G$ is injective;
- *smooth* if the map

$$F(B) \longrightarrow G(B) \times_{G(A)} F(A)$$

induced by the diagram

$$\begin{tikzcd} F(B) \arrow[r] \arrow[d, "\phi"'] & F(A) \arrow[d, "\phi"] \\ G(B) \arrow[r] & G(A), \end{tikzcd}$$

is surjective for every surjection $B \longrightarrow A$ in $\mathbf{Art}_S$;
- *étale* if ϕ is both smooth and unramified.

I.1.16. Remark. If $\phi : F \longrightarrow G$ is smooth then, taking $A = \mathbb{K}$ in the previous definition, we conclude that $\phi : F(B) \longrightarrow G(B)$ is surjective.

I.1.17. Remark. If $\phi : F \longrightarrow G$ is an étale morphism, then it induces an isomorphism $\phi' : t_F \longrightarrow t_G$ on tangent spaces. Actually ϕ is unramified and so ϕ' is injective. By hypothesis ϕ is also smooth and so, applying the previous Remark I.1.16 in the special case $B = \mathbb{K}[\varepsilon]$, ϕ' is surjective.

I.1.18. Definition. A *functor F is smooth* if for every surjection $B \longrightarrow A$ of S-algebras $F(B) \longrightarrow F(A)$ is surjective (that is the morphism $F \longrightarrow *$ is smooth).

I.1.19. Proposition. *Let $\phi : F \longrightarrow G$ be an étale morphism of deformation functors. If G is homogeneous then ϕ is an isomorphism.*

Proof. Since ϕ is smooth, ϕ is surjective. Therefore it is sufficient to prove the injectivity of ϕ using that G is homogeneous and ϕ is unramified.
The proof of this fact is taken from [**20**, Lemma 2.10] for sake of completeness.
We prove it by induction on the length of A.

If $A = \mathbb{K}$ then $F(\mathbb{K}) = G(\mathbb{K}) = \{*\}$ and so it is obvious.

Let
$$0 \longrightarrow \mathbb{K}\varepsilon \longrightarrow B \longrightarrow A \longrightarrow 0.$$
be a principal small extensions ($\varepsilon \cdot m_B = 0$). By induction $\phi : F(A) \longrightarrow G(A)$ is injective.

Consider the following isomorphism of S-algebras:
$$\varphi : B \times_{\mathbb{K}} \mathbb{K}[\varepsilon] \longrightarrow B \times_A B$$
$$(b, \bar{b} + \beta\varepsilon) \longmapsto (b, b + \beta\varepsilon).$$
Since F is a deformation functor, then $F(\varphi) : F(B) \times t_F \longrightarrow F(B) \times_{F(A)} F(B)$ is surjective.

Since G is also homogeneous, then $G(\varphi) : G(B) \times t_G \longrightarrow G(B) \times_{G(A)} G(B)$ is an isomorphism. We note that $G(\varphi)(G(B) \times \{0\}) = \Delta$ diagonal.

Now, suppose that $\phi(\xi) = \phi(\eta) \in G(B)$ for ξ and $\eta \in F(B)$.

Since ϕ is injective on $F(A)$, then $(\xi, \eta) \in F(B) \times_{F(A)} F(B)$.

Moreover, the surjectivity of $F(\varphi)$ implies the existence of an element $h \in t_F$ such that $F(\varphi)(\xi, h) = (\xi, \eta)$. Then $G(\varphi)(\phi(\xi), \phi(h)) = (\phi(\xi), \phi(\eta)) \in \Delta$ and so $\phi(h) = 0$ (G is an isomorphism).

By hypothesis ϕ is unramified, therefore $h = 0$ and so $\xi = \eta$. □

I.1.20. Remark. Let F be a deformation functor and $f : B \longrightarrow A$ a surjection with $\ker f \cong \mathbb{K}$. Using the isomorphism $B \times_{\mathbb{K}} \mathbb{K}[\varepsilon] \cong B \times_A B$ of the previous Proposition I.1.19, we obtain a commutative diagram
$$\begin{tikzcd} F(B) \times t_F \arrow[r, "\pi"] \arrow[d, "\tau"'] & F(B) \arrow[d] \\ F(B) \arrow[r, "F(f)"'] & F(A). \end{tikzcd}$$
π is the projection and τ defines an action of t_F on $F(B)$ which restricts to a transitive action on each fiber of $F(f)$ (see [**6**, Lemma 2.12]).

I.1.21. Corollary. *Let F be a deformation functor, then $F = \{*\}$ if and only if $t_F = (0)$.*

PROOF. One implication is obvious. Now, let F be a deformation functor such that $t_F = (0)$. We prove that $F(A) = \{*\}$ by induction on $dim_{\mathbb{K}}(A)$. If $A = \mathbb{K}$ then it is obvious (by definition of functor of Artin rings). Now, let $\pi : B \longrightarrow A$ be a small extensions and suppose that $F(A) = \{*\}$. By the previous Remark I.1.20, $t_F = (0)$ acts transitively on the unique fiber $F(B)$ of the map $F(\pi)$. This implies $F(B) = \{*\}$. $\square$

I.1.1. Obstruction theory.

I.1.22. DEFINITION. Let F be a functor of Artin rings; an *obstruction theory* for F is a couple (V, v_e) such that:

- V is a $\mathbb{K}$-vector space, called *obstruction space*;
- for every small extension in $\mathbf{Art}_S$

$$e : 0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0$$

  $v_e : F(A) \longrightarrow V \otimes_{\mathbb{K}} J$ is an obstruction map, satisfying the following properties:
  - if $\xi \in F(A)$ can be lifted to $F(B)$ then $v_e(\xi) = 0$.
  - For every morphism $\phi : e_1 \longrightarrow e_2$ of small extension, i.e.

$$(2) \qquad \begin{array}{ll} e_1 : & \begin{tikzcd} 0 \arrow[r] & J_1 \arrow[r] \arrow[d, "\phi_J"] & B_1 \arrow[r] \arrow[d, "\phi_B"] & A_1 \arrow[r] \arrow[d, "\phi_A"] & 0 \\ 0 \arrow[r] & J_2 \arrow[r] & B_2 \arrow[r] & A_2 \arrow[r] & 0 \end{tikzcd} \\ e_2 : & \end{array}$$

    we have $v_{e_2}(\phi_A(a)) = (Id_V \otimes \phi_J)(v_{e_1}(a))$, for every $a \in F(A_1)$.

I.1.23. DEFINITION. An obstruction theory for a functor is *complete* if the lifting exists if and only if the obstruction vanishes.

I.1.24. REMARK. If F has (0) as complete obstruction space then F is smooth. In Proposition I.1.31 we will prove that the converse is also true for a deformation functor.

I.1.25. REMARK. Let $\psi : F \longrightarrow G$ be a natural transformation of functors and (V, v_e) an obstruction theory for G. Then $(V, v'_e := v_e \circ \psi)$ is an obstruction theory for F.

Actually, consider the small extension in $\mathbf{Art}$

$$e : 0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0$$

and the map

$$v'_e : F(A) \xrightarrow{F(\psi)} G(A) \xrightarrow{v_e} V \otimes_{\mathbb{K}} J.$$

Let $\xi \in F(A)$ and suppose that it can be lifted to $\tilde{\xi} \in F(B)$. Therefore $\psi(\xi) \in G(A)$ can be lifted to $\psi(\tilde{\xi}) \in G(B)$ and so $v'_e(\xi) = v_e(\psi(\xi)) = 0$.

Now, let $\phi : e_1 \longrightarrow e_2$ be a morphism of small extension as in (2). Then for each $a \in F(A_1)$ we have

$$(Id_V \otimes \phi_J)(v'_{e_1}(a)) = (Id_V \otimes \phi_J)(v_{e_1}(\psi(a))) =$$

$$v_{e_2}(\phi_A(\psi(a))) = v_{e_2}(\psi(\phi_A(a))) = v'_{e_2}(\phi_A((a))).$$

Moreover, if the morphism ψ is also smooth and (V, v_e) is complete for G, then (V, v'_e) is complete for F.

Actually suppose that $\xi \in F(A)$ is such that $0 = v'_e(\xi) = v_e(\psi(\xi))$. Then there exists $\eta \in G(B)$ that lifts $\psi(\xi) \in G(A)$ ((V, v_e) is complete for G). Consider the following diagram

$$\begin{tikzcd} F(B) \arrow[r] \arrow[d] & F(A) \ni \xi \arrow[d] \\ \eta \in G(B) \arrow[r] & G(A) \ni f(\xi). \end{tikzcd}$$

Since f is smooth, the map $F(B) \longrightarrow F(A) \times_{G(A)} G(B)$ is surjective and so there exists $\tilde{\xi} \in F(B)$ that lifts ξ.

I.1.26. Remark. If V is complete obstruction theory for a functor F, using embedding of vector space we can construct infinitely many complete obstruction theories. Therefore, the goal is to find a "smallest" complete obstruction theory. The main results in this context is the following Theorem I.1.28. First of all we give a definition.

I.1.27. Definition. A morphism of obstruction theories $(V, v_e) \longrightarrow (W, w_e)$ is a linear map (of vector space) $\theta : V \longrightarrow W$ such that $w_e = \theta v_e$, for every small extensions e.

An obstruction theory (O_F, ob_e) for F is called *universal* if for every obstruction theory (V, v_e) there exists a unique morphism $(O_F, ob_e) \longrightarrow (V, v_e)$.

I.1.28. Theorem. *(Fantechi, Manetti) Let F be a deformation functor. Then there exists a universal obstruction theory (O_F, ob_e) for F. Moreover the universal obstruction theory is complete and every element of the vector space O_F is of the form $ob_e(\xi)$ for some principal extension*

$$e: \ 0 \longrightarrow \mathbb{K}\varepsilon \longrightarrow B \longrightarrow A \longrightarrow 0.$$

and some $\xi \in F(A)$.

Proof. See [**6**, Th. 3.2 and Cor. 4.4] $\square$

Let $\phi : F \longrightarrow G$ be a morphism of deformation functors and (V, v_e), (W, w_e) obstructions theories for F and G respectively. A linear map $\phi' : V \longrightarrow W$ is *compatible* with ϕ if $w_e\phi = \phi' v_e$ for every small extensions e.

I.1.29. THEOREM. *Let $\phi : F \longrightarrow G$ be a morphism of deformation functors and $\phi' : (V, v_e) \longrightarrow (W, w_e)$ a compatible morphism of obstruction theories. If (V, v_e) is complete, ϕ' injective and $t_F \longrightarrow t_G$ surjective, then ϕ is smooth.*

PROOF. See [**20**, Proposition 2.17]. We have to prove that the map

$$F(B) \longrightarrow G(B) \times_{G(A)} F(A)$$

is surjective, for all small extensions

$$0 \longrightarrow \mathbb{K}\varepsilon \longrightarrow B \longrightarrow A \longrightarrow 0.$$

Let $(b', a) \in G(B) \times_{G(A)} F(A)$ and $a' \in G(A)$ their common image: that is $b' \in G(B)$ lifts $a' \in G(A)$ and so $w_e(a') = 0$.

By hypothesis ϕ' is injective and so $v_e(a) = 0$ $(0 = w_e(a') = w_e(\phi(a)) = \phi'(v_e(a)))$. Therefore there exists $b \in F(B)$ that lifts a:

$$\begin{tikzcd} b \in F(B) \arrow[r] \arrow[d] & F(A) \ni a \arrow[d] \\ b' \in G(B) \arrow[r] & G(A) \ni a'. \end{tikzcd}$$

In general b doesn't lift b'. Let $b'' = \phi(b) \in G(B)$; then $(b'', b') \in G(B) \times_{G(A)} G(B)$.

As observed in the proof of Proposition I.1.19, we have an isomorphism $B \times_{\mathbb{K}} \mathbb{K}[\varepsilon] \cong B \times_A B$; since G is a deformation functor, there exists a surjective morphism

$$\alpha : G(B) \times t_G = G(B \times_A B) \longrightarrow G(B) \times_{G(A)} G(B).$$

Therefore, there exists $h \in t_G$ such that $(b'', \overline{b''} + h)$ is a lifting of $(b'', b'' + h = b')$.

By hypothesis, $t_F \longrightarrow t_G$ is surjective and so there exists a lifting $k \in t_F$ of $h \in t_G$. Taking $k + b \in F(B)$, we produce a lifting of a that maps on b'.

□

I.1.30. REMARK. Let $\phi : F \longrightarrow G$ be a morphism of deformation functors and (O_F, v_e) and (O_G, w_e) their universal obstruction theories. Then $(O_G, w_e \circ f)$ is an obstruction theory for F. Therefore, by Theorem I.1.28, there exists a morphism $o(\phi) : (O_F, v_e) \longrightarrow (O_G, w_e)$.

In conclusion *every morphism of deformation functors induces a linear map both between tangent spaces and universal obstruction spaces.*

Now we want to consider some useful properties between these morphisms.

I.1.31. PROPOSITION. *Let $\phi : F \longrightarrow G$ be a morphism of deformation functors. Then ϕ is smooth if and only if $t_F \longrightarrow t_G$ is surjective and $o(\phi) : O_F \longrightarrow O_G$ is injective. In particular F is smooth if and only if $O_F = 0$*

PROOF. If $t_F \longrightarrow t_G$ is bijective and $o(\phi) : O_F \longrightarrow O_G$ is injective then ϕ is smooth by Theorem I.1.29.

Viceversa, suppose that ϕ is smooth, then by Remark I.1.16, $t_F = F(\mathbb{K}[\varepsilon]) \longrightarrow G(\mathbb{K}[\varepsilon]) = t_G$ is surjective. Let $B \longrightarrow A$ be a small extension and a an element of $F(A)$, with obstruction $x \in O_F$ such that $o(\phi)(x) = 0 \in O_G$. By Theorem I.1.28, O_G is complete and so $\phi(a) \in G(A)$ can be lifted to $b' \in G(B)$. Again, by Remark I.1.16, $F(B) \longrightarrow G(B)$ is surjective and so there exists $b \in F(B)$ that lifts a. Therefore the obstruction of a is zero ($x = 0$) and this proves that $o(\phi)$ is injective. □

I.1.32. COROLLARY. *A morphism of deformation functors $\phi : F \longrightarrow G$ is étale if and only if $t_F \longrightarrow t_G$ is bijective and $o(\phi) : O_F \longrightarrow O_G$ is injective.*

PROOF. If ϕ is étale, then by Remark I.1.17 $t_F \longrightarrow t_G$ is bijective. Since ϕ is also smooth $o(\phi) : O_F \longrightarrow O_G$ is injective.

Viceversa, by the Proposition I.1.31 ϕ is smooth; since by hypothesis $t_F \longrightarrow t_G$ is injective, ϕ is also unramified. □

I.1.33. COROLLARY. *Let $\phi : F \longrightarrow G$ be a morphism of deformation functor with G homogeneous. If $t_F \longrightarrow t_G$ is bijective and $o(\phi) : O_F \longrightarrow O_G$ is injective then ϕ is an isomorphism.*

PROOF. Put together Corollary I.1.32 and Proposition I.1.19. □

I.1.34. REMARK. When G is not homogeneous, ϕ could be not an injection and so we can just conclude the surjectivity of ϕ. Therefore, in these cases we prove the injectivity directly. This will happen in Theorems II.7.3 and IV.2.5.

I.1.35. COROLLARY. *If $\phi : F \longrightarrow G$ is smooth then $o(\phi) : O_F \longrightarrow O_G$ is bijective.*

PROOF. By Proposition I.1.31, we have just to prove that $o(\phi) : O_F \longrightarrow O_G$ is surjective. Let $B \longrightarrow A$ be a small extension and $y \in O_G$ the obstruction to lift $a' \in G(A)$ to $b' \in G(B)$. Since ϕ is smooth, there exists $a \in F(A)$, such that $\phi(a) = a'$, and $b \in F(B)$, such that $\phi(b) = b'$. Therefore the obstruction $x \in O_F$ to lift $a \in F(A)$ to $b \in F(B)$ is a lifting of $y \in O_G$.

□

I.2. Deformation functor of complex manifolds

In this section we study the infinitesimal deformation functor associated to a compact complex manifold. Then we will work over the complex number and so $\mathbb{K} = \mathbb{C}$ and $\mathbf{Art} = \mathbf{Art}_{\mathbb{C}}$.

The main references are [**16**, Chapter 4], [**32**], [**31**, Chapter II].

I.2.1. Definition. Let X be a compact complex manifold and $A \in \mathbf{Art}$. An *infinitesimal deformation* of X over $\mathrm{Spec}(A)$ is a commutative diagram of complex spaces

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X_A \arrow[d, "\pi"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "a"] & \mathrm{Spec}(A). \end{tikzcd}$$

where π is proper and flat holomorphic map, $a \in \mathrm{Spec}(A)$ is the closed point, i is a closed immersion and $X \cong X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathbb{C})$.

If $A = \mathbb{K}[\varepsilon]$ we call it a *first order* deformation of X.

Sometimes, for an infinitesimal deformation X_A over $\mathrm{Spec}(A)$, we also use the short notation $(X_A, \pi, \mathrm{Spec}(A))$.

I.2.2. Remark. Let X_A be an infinitesimal deformation of X. We note that, by definition it can be interpreted as a morphism of sheaves of algebras $\mathcal{O}_A \to \mathcal{O}_X$ such that $\mathcal{O}_A$ is flat over A and $\mathcal{O}_A \otimes_A \mathbb{C} \to \mathcal{O}_X$ is an isomorphism.

Given another deformation X'_A of X over $\mathrm{Spec}(A)$:

$$\begin{tikzcd} X \arrow[r, "i'"] \arrow[d] & X'_A \arrow[d, "\pi'"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "a"] & \mathrm{Spec}(A), \end{tikzcd}$$

we say that X_A and X'_A are *isomorphic* if there exists an isomorphism $\phi : X_A \longrightarrow X'_A$ over $\mathrm{Spec}(A)$ that induce the identity on X: that is the following diagram is commutative

$$\begin{tikzcd} & X \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ X_A \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & X'_A \arrow[dl, "\pi'"] \\ & \mathrm{Spec}(A). & \end{tikzcd}$$

We note that for every X we can always define the *infinitesimal product deformation*:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X \times \mathrm{Spec}(A) \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "a"] & \mathrm{Spec}(A). \end{tikzcd}$$

I.2.3. Definition. An infinitesimal deformation of X over $\mathrm{Spec}(A)$ is called *trivial* if it is isomorphic to the infinitesimal product deformation.

X is called *rigid* if every infinitesimal deformation of X over $\mathrm{Spec}(A)$ (for each $A \in \mathbf{Art}$) is trivial.

For every deformation X_A of X over $\mathrm{Spec}(A)$ and every morphism $A \longrightarrow B$ in $\mathbf{Art}$ ($\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$), there exists an associated deformation of X over $\mathrm{Spec}(B)$, called *pull-back deformation*, induced by base change:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(B). \end{tikzcd}$$

I.2.4. Definition. The *infinitesimal deformation functor* Def_X of a complex manifold X is defined as follows:

$$\mathrm{Def}_X : \mathbf{Art} \to \mathbf{Set}$$

$$A \longmapsto \mathrm{Def}_X(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations} \\ \text{of } X \text{ over } \mathrm{Spec}(A) \end{array} \right\}$$

I.2.5. Proposition. Def_X *satisfies the conditions of Definition I.1.10.*

Proof. See [**32**, Section 3.7] or [**31**, Prop. III.3.1]. □

I.2.1. Tangent and obstruction spaces of Def_X. Let X be a compact complex manifold and Θ_X its holomorphic tangent bundle.

In this section we prove that the tangent space of Def_X is $\check{H}^1(X, \Theta_X)$ and that the obstruction space is natural contained in $\check{H}^2(X, \Theta_X)$. First of all we recall an useful lemma.

I.2.6. Lemma. *Let B_0 be a $\mathbb{C}$-algebra and*

$$0 \longrightarrow J \longrightarrow B \longrightarrow A \longrightarrow 0$$

a small extension in $\mathbf{Art}$*. Then there is a 1-1 correspondence*

$$\left\{ \begin{array}{c} \textit{automorphisms of the trivial deformation } B_0 \otimes_{\mathbb{C}} B \\ \textit{inducing the identity on } B_0 \otimes_{\mathbb{C}} A \end{array} \right\} \longleftrightarrow \mathrm{Der}_{\mathbb{C}}(\mathrm{B}_0, \mathrm{B}_0) \otimes J$$

where the identity corresponds to the zero derivation, and the composition of automorphisms corresponds to the sum of derivations.

Proof. See [**31**, Lemma II.1.5] □

I.2.7. Remark. Let U be a Stein open subset of a complex manifold X. Then the previous lemma is equivalent to say that the following sequence is exact:

$$0 \longrightarrow \Gamma(U_i, \Theta_X) \otimes J \longrightarrow \mathrm{Aut}(\mathcal{O}_X(U_i) \otimes B) \longrightarrow \mathrm{Aut}(\mathcal{O}_X(U_i) \otimes A) \longrightarrow 0$$

Moreover, we note that this is a central extension.

I.2.8. THEOREM. *Let X be a complex manifold. Then there is a 1-1 correspondence:*

$$k : \frac{\{\text{first order deformation of } X\}}{\text{isomorphism}} \longrightarrow \check{H}^1(X, \Theta_X)$$

called the Kodaira-Spencer correspondence, where $\Theta_X = \mathrm{Hom}(\Omega^1_X, \mathcal{O}_X) = \mathrm{Der}_{\mathbb{K}}(\mathcal{O}_X, \mathcal{O}_X)$, *is the hololmorphic tangent bundle of X.*

Moreover $k(\xi) = 0$ if and only if ξ is the trivial deformation class.

PROOF. For completeness we take this proof from [**31**, Proposition II.1.6] where full details are available.

Let X_ε be a first order deformation of X:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_\varepsilon \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(\mathbb{C}[\varepsilon]), \end{tikzcd}$$

and $\mathcal{U} = \{U_i\}_{i \in I}$ be a Stein open cover of X such that $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$ are Stein for every i, j and $k \in I$.

For each open U_i the deformations $X_{\varepsilon|U_i}$ are trivial, then for each $i \in I$ there exist isomorphisms of deformations

$$\theta_i : U_i \times \mathrm{Spec}(\mathbb{C}[\varepsilon]) \longrightarrow X_{\varepsilon|U_i}.$$

Therefore, for each i and $j \in I$, the composition

$$\theta_{ij} = {\theta_i}^{-1}\theta_j : U_{ij} \times \mathrm{Spec}(\mathbb{C}[\varepsilon]) \longrightarrow U_{ij} \times \mathrm{Spec}(\mathbb{C}[\varepsilon])$$

is an automorphism of the trivial deformation $U_{ij} \times \mathrm{Spec}(\mathbb{C}[\varepsilon])$ of the Stein open subset U_{ij}.

Applying Lemma I.2.6, we conclude that there exists an element $d_{ij} \in \Gamma(U_{ij}, \Theta_X)$ corresponding to θ_{ij}, for each i and $j \in I$.

Moreover on each U_{ijk} the following equality holds

$$\theta_{jk}{\theta_{ik}}^{-1}\theta_{ij}^{-1} = {\theta_j}^{-1}\theta_k{\theta_k}^{-1}\theta_i{\theta_i}^{-1}\theta_j = id_{|U_{ijk} \times \mathrm{Spec}(\mathbb{K}[\varepsilon])}.$$

Therefore, applying again Lemma I.2.6, we have

$$d_{jk} - d_{ik} + d_{ij} = 0$$

that is $\{d_{ij}\}$ ia a Čech 1-cocycle and so it define an element in $\check{H}^1(X, \Theta_X)$. It can be checked that this element doesn't depend on the choice of the open cover $\mathcal{U}$.

Let X'_ε be another first order deformation

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X'_\varepsilon \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(\mathbb{C}[\varepsilon]), \end{tikzcd}$$

and ϕ an isomorphism of deformations: $\phi : X_\varepsilon \longrightarrow X'_\varepsilon$.

Then, for each $i \in I$, there exists an induced automorphism

$$\alpha_i = {\theta'_i}^{-1} \circ \phi_{|U_i} \circ \theta_i : U_i \times \mathrm{Spec}(\mathbb{C}[\varepsilon]) \xrightarrow{\theta_i} X_{\varepsilon|U_i} \xrightarrow{\phi_{|U_i}} X'_{\varepsilon|U_i} \xrightarrow{{\theta'_i}^{-1}} U_i \times \mathrm{Spec}(\mathbb{C}[\varepsilon])$$

and so a corresponding element $a_i \in \Gamma(U_i, \Theta_X)$.

Therefore we have $\theta'_i \alpha_i = \phi_{|U_i} \theta_i$ and

$$(\theta'_i \alpha_i)^{-1} (\theta'_j \alpha_j) = {\theta_i}^{-1} \phi^{-1}_{|U_{ij}} \phi_{|U_{ij}} \theta_j = {\theta_i}^{-1} \theta_j.$$

This implies

$${\alpha_i}^{-1} {\theta'_{ij}}^{-1} \alpha_j = \theta_{ij}$$

or equivalently

$$d'_{ij} + a_j - a_i = d_{ij}.$$

In conclusion the Čech cocycle $\{d_{ij}\}$ and $\{d'_{ij}\}$ are cohomologous and so they represent the same element in $\check{H}^1(X, \Theta_X)$.

Conversely, let $\theta \in \check{H}^1(X, \Theta_X)$ and $\{d_{ij}\} \in Z^1(\mathcal{U}, \Theta_X)$ be a representative of θ with respect to an open Stein cover $\{\mathcal{U}\}$. By Lemma I.2.6, we can associate to each d_{ij} an automorphism θ_{ij} of the trivial deformation $U_{ij} \times \mathrm{Spec}(\mathbb{C}[\varepsilon])$. Since the element $\{d_{ij}\}$ satisfies the cocycle condition, then θ_{ij} also satisfy this condition:

$$\theta_{jk} {\theta_{ik}}^{-1} \theta_{ij}^{-1} = id_{|U_{ijk} \times \mathrm{Spec}(\mathbb{C}[\varepsilon])}.$$

Using these automorphisms we can glue together the schemes $U_i \times \mathrm{Spec}(\mathbb{C}[\varepsilon])$ (see [**13**, pag. 69]) to obtain a scheme X_ε that is a first order deformation of X.

At this point the last assertion is clear.

□

I.2.9. Theorem. *$\check{H}^2(X, \Theta_X)$ is complete obstruction space for* Def_X.

Proof. For completeness we take the proof from [**31**, Proposition II.1.8] where full details are available.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open Stein cover of X such that U_{ij} and U_{ijk} are Stein for every i, j and $k \in I$ and

$$0 \longrightarrow J \longrightarrow B \longrightarrow A \longrightarrow 0$$

be a small extension in **Art**.

Let X_A be an infinitesimal deformation of X over $\mathrm{Spec}(A)$. Then we have isomorphisms

$$\theta_i : U_i \times \mathrm{Spec}(A) \longrightarrow X_{A|U_i}$$

such that $\theta_{ij} := {\theta_i}^{-1} \theta_j$ are automorphisms of the trivial deformations $U_{ij} \times \mathrm{Spec}(A)$ and $\theta_{jk} {\theta_{ik}}^{-1} \theta_{ij} = id_{|U_{ijk} \times \mathrm{Spec}(A)}$.

To define a deformation X_B that lifts the deformation X_A is necessary and sufficient to give automorphisms $\{\tilde{\theta}_{ij}\}$ of the trivial deformation $U_{ij} \times \mathrm{Spec}(B)$ such that

i) $\{\tilde{\theta}_{ij}\}$ glues together: $\tilde{\theta}_{jk}\tilde{\theta}_{ik}^{-1}\tilde{\theta}_{ij} = id_{|U_{ijk}\times \mathrm{Spec}(B)}$,
ii) $\{\tilde{\theta}_{ij}\}$ lifts $\{\theta_{ij}\}$: $\tilde{\theta}_{ij}$ restricts to θ_{ij} on $U_{ij} \times \mathrm{Spec}(A)$.

Let us choose automorphisms $\{\tilde{\theta}_{ij}\}$ that satisfy condition ii). Then the automorphisms

$$\tilde{\theta}_{ijk} = \tilde{\theta}_{jk}\tilde{\theta}_{ik}^{-1}\tilde{\theta}_{ij}$$

are automorphisms of the trivial deformation that restrict to the identity on $U_{ijk} \times \mathrm{Spec}(A)$. Applying Lemma I.2.6, there exists $\{\tilde{d}_{ijk}\} \in \Gamma(U_{ijk}, \Theta_X)\otimes J$ that corresponds to $\tilde{\theta}_{ijk}$. An easy calculation show that $\{\tilde{d}_{ijk}\}$ is a Čech cocylce and so $\{\tilde{d}_{ijk}\} \in Z^2(\mathcal{U}, \Theta_X) \otimes J$.

Now, let $\{\overline{\theta}_{ij}\}$ be different automorphisms of the trivial deformations $U_{ij} \times \mathrm{Spec}(B)$ that satisfy condition ii). As above, let $\overline{d}_{ijk}$ be the derivations corresponding to $\overline{\theta}_{ijk}$.

The automorphisms $\overline{\theta}_{ij}\tilde{\theta}_{ij}^{-1}$ of $U_{ij} \times \mathrm{Spec}(B)$ restrict to the identity on $U_{ij} \times \mathrm{Spec}(A)$ for each U_{ij} and so, again by Lemma I.2.6, they correspond to some $\{d_{ij}\} \in \Gamma(U_{ij}, \Theta_X) \otimes J$.

Therefore

$$\overline{d}_{ijk} = \tilde{d}_{ijk} + d_{jk} - d_{ik} + d_{ij}.$$

This implies that the Čech cocycles $\{\tilde{d}_{ijk}\}$ and $\{\overline{d}_{ijk}\}$ are cohomologous and so their cohomology classes coincide in a well defined element $v_e(X_A)$ in $\check{H}^2(X, \Theta_X) \otimes J$:

$$v_e(X_A) := [\{\tilde{d}_{ijk}\}] = [\{\overline{d}_{ijk}\}] \in \check{H}^2(X, \Theta_X) \otimes J.$$

Moreover the class $v_e(X_A)$ is zero if and only if the collection of automorphisms also satisfies condition i) and it is equivalent to the existence of a lifting X_B of the deformation X_A.

□

I.3. Differential graded Lie algebras and deformation functor

In this section we study the deformation functor associated to a differential graded Lie algebra (DGLA).

In particular, we give the fundamental definition of a DGLA (Definition I.3.5).

We also introduce the Maurer-Cartan functor MC_L (Definition I.3.16) and the deformation functor Def_L associated to a DGLA L (Definition I.3.29).

We start defining the differential graded vector spaces.

I.3.1. Differential graded vector spaces. Let $\mathbb{K}$ be a fixed field of characteristic 0. Unless otherwise specified, all vector spaces, linear maps, tensor products etc. are intended over $\mathbb{K}$.

Every *graded vector space* is a $\mathbb{Z}$-graded vector space (over $\mathbb{K}$). If $V = \oplus_i V^i$ is a graded vector space and $a \in V$ is a homogeneous element, then we denote by $\deg_V a$ the degree of a in V; we will also use the notation $\deg a = \bar{a}$, when V is clear by the context.

The morphisms of graded vector space are the degree preserving linear maps.

Given two graded vector spaces V and W, we define $\mathrm{Hom}^n_{\mathbb{K}}(V, W)$ as the vector space of $\mathbb{K}$-linear maps $f : V \longrightarrow W$, such that $f(V^i) \subset W^{i+n}$, for each $i \in \mathbb{Z}$.

Let V be a graded vector space, then $V[n]$ is the complex V with degrees shifted by n. More precisely, for $\mathbb{K}[n]$ we have

$$\mathbb{K}[n]^i = \begin{cases} \mathbb{K} & \text{if } i + n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $V[n] = \mathbb{K}[n] \otimes V$, that implies $V[n]^i = V^{i+n}$.

I.3.1. Remark. We note that there exist isomorphisms

$$\mathrm{Hom}^n_{\mathbb{K}}(V, W) = \mathrm{Hom}^0_{\mathbb{K}}(V[-n], W) = \mathrm{Hom}^0_{\mathbb{K}}(V, W[n]).$$

A *differential graded vector space* is a pair (V, d) where $V = \oplus V^i$ is a graded vector space and d is a differential of degree 1 ($d : V^i \longrightarrow V^{i+1}$ and $d \circ d = 0$).

A morphism of differential graded vector spaces is a degree preserving linear map that commutes with the differentials.

For every differential graded vector space (V, d) we use the standard notation $Z^i(V) = \ker(d\colon V^i \to V^{i+1})$, $B^i(V) = \mathrm{Im}(d\colon V^{i-1} \to V^i)$ and $H^i(V) = Z^i(V)/B^i(V)$.

A morphism is a *quasi-isomorphism* if it induces isomorphisms in cohomology.

I.3.2. Example. Given (V, d), then for each $i \in Z$, the shifted differential graded vector space $(V[i], d_{[i]})$ is defined as:

$$V[i] = \bigoplus_j V[i]^j = \bigoplus_j V^{i+j} \qquad \text{and} \qquad d_{[i]} = (-1)^i d.$$

I.3.3. Example. If (V, d_V) and (W, d_W) are differential graded vector spaces, then we can define a new differential graded vector space

$$\mathrm{Hom}^*(V, W) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}^n_{\mathbb{K}}(V, W)$$

with natural differential d' given by

$$d'(f) := d_W f - (-1)^{deg(f)} f\, d_V.$$

Moreover, for each i, there exist the following isomorphism s

$$H^i(\mathrm{Hom}^*(V, W)) \cong \mathrm{Hom}^i(H^*(V), H^*(W)).$$

I.3.4. EXAMPLE. Given (V, d_V) and (W, d_W), we can also define the following differential graded vector space

$$\mathrm{Htp}(V, W) = \mathrm{Hom}^*(V[1], W) = \bigoplus_i \mathrm{Htp}^i(V, W),$$

with

$$\mathrm{Htp}^i(V, W) = \mathrm{Hom}^i(V[1], W) = \mathrm{Hom}^{i-1}(V, W)$$

and differential δ:

$$\delta(f) = d_W(f) - (-1)^i f d_{V[1]} = d_W f + (-1)^i f d_V \qquad \forall\, f \in \mathrm{Htp}^i(V, W).$$

We will use these differential graded vector spaces in the last chapter (Section V.1.1).

I.3.2. Differential graded Lie algebras (DGLA).

I.3.5. DEFINITION. A *differential graded Lie algebra* (DGLA for short) is a triple $(L, [\,,\,], d)$, where $(L = \bigoplus_{i\in\mathbb{Z}} L^i, d)$ is a differential graded vector space and $[\,,\,] : L \times L \to L$ is a bilinear map, called bracket, satisfying the following conditions:

1. the bracket $[\,,\,]$ is homogeneous and graded skewsymmetric; i.e. $[L^i, L^j] \subset L^{i+j}$ and $[a, b] + (-1)^{\deg(a)\deg(b)}[b, a] = 0$, for every a and b homogeneous.
2. Every triple of homogeneous elements satisfies the graded Jacobi identity
$$[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a)\deg(b)}[b, [a, c]].$$
3. $d(L^i) \subseteq L^{i+1}$, $d \circ d = 0$ and
$$d[a, b] = [da, b] + (-1)^{\deg(a)}[a, db].$$

The last property is called Leibniz rule and in particular it implies that the bracket induces a structure of differential graded Lie algebra (with zero differential) on the cohomology $H^*(L) = \oplus_i H^i(L)$ of a DGLA L.

I.3.6. REMARK. If the degree of a is even then $[a, a] = 0$; if the degree is odd then $[[a, a], a] = 0$.

I.3.7. DEFINITION. A morphism of differential graded Lie algebras $\varphi : L \longrightarrow M$ is a linear map that preserves degrees and commutes with brackets and differentials; written in details we have

- $\varphi(L^i) \subseteq M^i$, for each i;
- $\varphi(d_L a) = d_M(\varphi(a))$, for each $a \in L$;
- $\varphi([a, b]) = [\varphi(a), \varphi(b)]$, for each $a, b \in L$.

A *quasi-isomorphism* of DGLA is a morphism that induces isomorphisms in cohomology. Two DGLA L and M are *quasi isomorphic* if they are equivalent under the equivalent relation $\sim$ generated by: $L \sim M$ if there exists a quasi isomorphism $\phi : L \longrightarrow M$.

A DGLA L is *formal* if it is quasi-isomorphic to its cohomology $H^*(L)$.

I.3.8. Remark. The following DGLA are isomorphic:

$$(L, [\,,\,], d) \cong (L, -[\,,\,], d) \cong (L, [\,,\,], -d) \cong (L, -[\,,\,], -d).$$

Actually, $\varphi = -id$ gives an isomorphism between $(L, [\,,\,], d)$ and $(L, -[\,,\,], d)$, while $\varphi(-) = (-1)^{deg(-)} id$ is an isomorphism between $(L, [\,,\,], d)$ and $(L, [\,,\,], -d)$.

I.3.9. Example. If $L = \oplus L^i$ is a DGLA, then L^0 is a Lie algebra in the usual sense; vice-versa, every Lie algebra is a differential graded Lie algebra concentrated in degree 0.

I.3.10. Example. Let L be a DLGA and consider the vector space decomposition $L^1 = N^1 \oplus B^1(L)$. Then the graded vector space $N = \oplus N^i$ with

$$\begin{cases} N^i = 0 & \text{for } i \leq 0 \\ N^1 = N^1 & \text{for } i = 1 \\ N^i = L^i & \text{for } i \geq 2 \end{cases}$$

is a sub-DGLA of L.

I.3.11. Example. Given a DGLA $(L = \oplus L^i, [\,,\,], d)$ we can associate a new DGLA $(L' = \oplus L'^i, [\,,\,]', d')$ where

$$\begin{cases} L'^i = L^i & \text{for } i \neq 1 \\ L'^1 = L^1 \oplus \mathbb{K}\, d & \text{for } i = 1, \end{cases}$$

$$[v + ad, w + bd]' = [v, w] + a\, d(w) - (-1)^{deg\, v} b\, d(v)$$

and

$$d'(v + a\, d) = [d, v + ad]' = dv,$$

for each $v, w \in L^i$ and $a, b \in \mathbb{K}$.

I.3.12. Example. Let M be a DGLA. Then $M[t, dt] = M \otimes \mathbb{K}[t, dt]$ is a DGLA, where $\mathbb{K}[t, dt]$ is the differential graded algebra of polynomial differential forms over the affine line. More exactly, $\mathbb{K}[t, dt] = \mathbb{K}[t] \oplus \mathbb{K}[t]dt$, where t has degree 0 and dt ha degree 1. As vector space $M[t, dt]$ is generated by elements of the form $mp(t) + nq(t)dt$, with $m, n \in M$ and $p(t), q(t) \in \mathbb{K}[t]$. The differential and the bracket on $M[t, dt]$ are defined as follows:

$$d(mp(t) + nq(t)dt) = (dm)p(t) + (-1)^{deg\, m} mp'(t)dt + (dn)q(t)dt,$$

$$[mp(t), nq(t)] = [m, n]p(t)q(t), \quad [mp(t), nq(t)dt] = [m, n]p(t)q(t)dt.$$

For every $a \in \mathbb{K}$ define the *evaluation morphism* in the following way

$$e_a : M[t,dt] \longrightarrow M \qquad e_a(\sum m_i t^i + n_i t^i dt) = \sum m_i a^i.$$

The evaluation morphism is a morphism of DGLA which is a left inverse of the inclusion and it is a surjective quasi-isomorphism for each a.

I.3.13. Example. If L is a DGLA and B is a commutative $\mathbb{K}$-algebra then $L \otimes B$ has a natural stucture of DGLA, given by

$$[l \otimes a, m \otimes b] = [l,m] \otimes ab;$$

$$d(l \otimes a) = dl \otimes a.$$

If B is also nilpotent (for example $B = m_A$ the maximal ideal of a local artinian $\mathbb{K}$-algebra A) then $L \otimes B$ is a nilpotent DGLA. Therefore, for every $a \in L^0 \otimes B$, we can define an automorphism of the DGLA $L \otimes B$:

$$e^{[a,-]} := \sum_{n=0}^{\infty} \frac{[a,-]^n}{n!} \ : L \otimes B \longrightarrow L \otimes B,$$

where

$$[a,-] : L \otimes B \longrightarrow L \otimes B$$

$$[a,-](b) := [a,b]$$

is a nilpotent derivation of degree zero (since $[a,-]([b,c]) = [[a,-](c),d] + [c,[a,-](d)]$).

I.3.14. Definition. A linear map $f : L \longrightarrow L$ is called a *derivation* of degree n if $f(L^i) \subset L^{i+n}$ and it satisfies the graded Leibniz rule:

$$f([a,b]) = [f(a),b] + (-1)^{n\bar{a}}[a,f(b)].$$

I.3.15. Example. Let (L,d) be a DGLA and $\mathrm{Der}^i(L,L)$ the space of derivations of L of degree i. Then $\mathrm{Der}^*(L,L) = \bigoplus_i \mathrm{Der}^i(L,L)$ is a DGLA with bracket

$$[f,g] = fg - (-1)^{\bar{f}\,\bar{g}} gf,$$

and differential d' given by

$$d'(f) = [d,f].$$

I.3.3. Maurer-Cartan functor associated to a DGLA.

I.3.16. Definition. The *Maurer-Cartan equation* in a DGLA L is

$$dx + \frac{1}{2}[x,x] = 0, \qquad x \in L^1.$$

The solutions of this equation are called the *Maurer-Cartan elements* of a differential graded Lie algebra L.

I.3.17. Remark. Let $x \in L^1$ be an element of degree one in the DGLA L and consider the operator of degree one

$$d(-) + [x, -] : L \longrightarrow L$$

$$(d(-) + [x, -])(a) = da + [x, a] \qquad \forall\ a \in L.$$

If x satisfies the Maurer-Cartan equation then $d(-) + [x, -]$ is a differential $((d(-) + [x, -])^2 = 0)$.

I.3.18. Remark. Let L' be the DGLA of Example I.3.11 (with $L'^1 = L^1 \oplus \mathbb{K}\, d$) then

$$dx + \frac{1}{2}[x, x] = 0 \quad \text{ if and only if } \quad [x + d, x + d]' = 0.$$

The previous definition led to the following definition of the Maurer-Cartan functor.

I.3.19. Definition. Let L be a DGLA; then the *Maurer-Cartan functor associated to* L is

$$MC_L : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$MC_L(A) = \{x \in L^1 \otimes m_A \mid dx + \frac{1}{2}[x, x] = 0\}$$

I.3.20. Remark. A morphism $\phi : L \longrightarrow M$ of DGLA preserves bracket and differential, therefore it induces a morphism of functors $\phi : \mathrm{MC}_L \longrightarrow \mathrm{MC}_M$.

I.3.21. Remark. We note that MC_L is an *homogeneous* functor, since $\mathrm{MC}_L(B \times_A C) \cong \mathrm{MC}_L(B) \times_{\mathrm{MC}_L(A)} \mathrm{MC}_L(C)$.

I.3.22. Remark. By definition the *tangent space* of MC_L is:

$$t_{\mathrm{MC}_L} := \mathrm{MC}_L(\mathbb{K}[\varepsilon]) = \{x \in L^1 \otimes \mathbb{K}\varepsilon \mid dx + \frac{1}{2}[x, x] = 0\} \cong$$

$$\{x \in L^1 \mid dx = 0\} = Z^1(L).$$

I.3.23. Lemma. *$H^2(L)$ is a complete obstruction space for* MC_L.

Proof. Let

$$e : \qquad 0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0$$

be a small extension in **Art** and $x \in \mathrm{MC}_L(A)$.

We want to define a map $v_e : \mathrm{MC}_L(A) \longrightarrow H^2(L) \otimes J$.

Let $\tilde{x} \in L^1 \otimes m_B$ be a lifting of x and define

$$h = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] \in L^2 \otimes m_B.$$

In general $\tilde{x}$ doesn't satisfy the Maurer-Cartan equation and so h is in general different from zero.

It results that $\alpha(h) = dx + \frac{1}{2}[x,x] = 0$ and so $h \in L^2 \otimes J$. Moreover,

$$dh = d^2\tilde{x} + \frac{1}{2}[d\tilde{x},\tilde{x}] - \frac{1}{2}[\tilde{x},d\tilde{x}] =^{a}$$

$$= [d\tilde{x},\tilde{x}] = [h,\tilde{x}] - \frac{1}{2}[[\tilde{x},\tilde{x}],\tilde{x}].$$

By definition $[h,\tilde{x}] \in [L^2 \otimes J, L^1 \otimes m_B] = 0$ (e is a small extension) and, by Remark I.3.6, $[[\tilde{x},\tilde{x}],\tilde{x}] = 0$.

In conclusion $dh = 0$ and so $h \in H^2(L) \otimes J$.

We note that this class doesn't depend on the choice of the lifting $\tilde{x}$. In fact, let $y \in L^1 \otimes m_B$ be another lifting of x: $\alpha(y) = \alpha(\tilde{x}) = x$. Then $y = \tilde{x} + t$ for some $t \in L^1 \otimes J$. Using $[L^1 \otimes J, L^1 \otimes m_B] = 0$, we have

$$h' = dy + \frac{1}{2}[y,y] = d\tilde{x} + dt + \frac{1}{2}[\tilde{x}+t,\tilde{x}+t] = d\tilde{x} + \frac{1}{2}[\tilde{x},\tilde{x}] + dt = h + dt$$

and so h and h' represent the same class in $H^2(L) \otimes J$.

Therefore, it is well defined the following obstruction map

$$v_e : \mathrm{MC}_L(A) \longrightarrow H^2(L) \otimes J$$

$$x \longmapsto v_e(x) = [h].$$

If $[h] = 0$ then $h = dq$ for some $q \in L^1 \otimes J$. This implies that $\overline{x} = \tilde{x} - q$ is a lifting of x that satisfies the Maurer-Cartan equation, i.e.

$$d\overline{x} + \frac{1}{2}[\overline{x},\overline{x}] = d\tilde{x} - dq + \frac{1}{2}[\tilde{x}-q,\tilde{x}-q] = d\tilde{x} - dq + \frac{1}{2}[\tilde{x},\tilde{x}] = h - dq = 0$$

Therefore v_e satisfies condition 1 of Definition I.1.22 of obstruction theory. The other property (change of base) is an easy calculation.

If $x \in \mathrm{MC}_L(A)$ can be lifted to $x' \in MC_L(B)$ then $[h] = 0$.

In conclusion $(H^2(L), v_e)$ is a complete obstruction theory for MC_L. □

I.3.24. REMARK. (**About smoothness**)

If $H^2(L) = 0$ then MC_L is smooth.

If L is abelian then MC_L is *smooth*. Actually, in this case, $\mathrm{MC}_L(A) = Z^1(L) \otimes m_A$. Moreover, if $B \twoheadrightarrow A$ then $Z^1(L) \otimes m_A \twoheadrightarrow Z^1(L) \otimes m_B$.

I.3.4. Gauge action.

I.3.25. DEFINITION. Two elements x and $y \in L^1 \otimes m_A$ are said to be *gauge equivalent* if there exists $a \in L^0 \otimes m_A$ such that

$$y = e^a * x := x + \sum_{n\geq 0} \frac{[a,-]^n}{(n+1)!}([a,x] - da)$$

[a]We have $-\frac{1}{2}[\tilde{x},d\tilde{x}] = -\frac{1}{2}(-(-1)^2[d\tilde{x},\tilde{x}]) = \frac{1}{2}[d\tilde{x},\tilde{x}]$.

The operator $*$ is called the gauge action of the group $\exp(L^0 \otimes m_A)$ on $L \otimes m_A$, in fact $e^a * e^b * x = e^{a \bullet b} * x$, where $\bullet$[b] is the Baker-Campbell-Hausdorff product in the nilpotent DGLA $L \otimes m_A$.

I.3.26. Remark. For a better understanding of the gauge action, it is convenient to consider the DGLA L' of Example I.3.11 (with $L'^1 = L^1 \oplus \mathbb{K}\, d$) and the affine embedding

$$\phi : L^1 \longrightarrow L'^1 \qquad \phi(x) = x + d \qquad \forall\, x \in L^1.$$

As already observed $dx + \dfrac{1}{2}[x, x] = 0$ if and only if $[\phi(x), \phi(x)]' = 0$.

As in Example I.3.13, for each $A \in \mathbf{Art}$ and $a \in L'^0 \otimes m_A$, we can consider the exponential of the adjoint action $e^{[a,]'} : L'^1 \otimes m_A \longrightarrow L'^1 \otimes m_A$. Using the embedding ϕ this action induce the gauge action of $L^0 \otimes m_A$ on $L^1 \otimes m_A$. Actually, for each $a \in L^0 \otimes m_A$ and $x \in L^1 \otimes m_A$

$$\phi^{-1}(e^{[a,]'}\phi(x)) = e^{[a,]'}(x + d) - d =$$

$$x + \sum_{n\geq 1} \frac{[a, -]'^n}{(n)!}(x + d) = x + \sum_{n\geq 0} \frac{[a, -]'^{n+1}}{(n + 1)!}(x + d) =$$

$$x + \sum_{n\geq 0} \frac{[a, -]'^n}{(n + 1)!}([a, x] - da) = e^a * x$$

I.3.27. Example. Let J be an ideal of $A \in \mathbf{Art}$ ($J \subset m_A$) such that $J \cdot m_A = 0$ (for example J is the kernel of a small extension).

If $x \in L^1 \otimes J$ then, for each $a \in L^0 \otimes m_A$ we have

$$e^a * x = x + \sum_{n=0}^{\infty} \frac{[a, -]^n}{(n + 1)!}([a, x] - da) = x + \sum_{n=0}^{\infty} \frac{[a, -]^n}{(n + 1)!}(-da) = x + e^a * 0.$$

or in general, if $y \in L^1 \otimes m_A$ then $e^a * (x + y) = x + e^a * y$.

If $a \in L^0 \otimes J$ then, for each $x \in L^1 \otimes m_A$:

$$e^a * x = x + \sum_{n=0}^{\infty} \frac{[a, -]^n}{(n + 1)!}([a, x] - da) = x - da.$$

or general, if $b \in L^0 \otimes m_A$, then $e^{a+b} * x = e^b * x - da$.

We note that

$$e^a * x = x \qquad \text{if and only if} \qquad [a, x] = da. \tag{3}$$

Actually $e^a * x = x$ if and only if $0 = \dfrac{e^{[a,-]} - id}{[a, -]}([a, x] - da)$. Applying the inverse of the operator $\dfrac{e^{[a,-]} - id}{[a, -]}$, we get $e^a * x = x$ if and only if $[a, x] - da = 0$.

---

[b] $a \bullet b = a + b + \frac{1}{2}[a, b] + \frac{1}{12}[a, [a, b]] - \frac{1}{12}[b, [b, a]] + \cdots$

I.3.28. Remark. The solutions of the Maurer-Cartan equation are preserved under the gauge action.

Actually, we have

$$d(e^a * x) = d(e^{[a,]'}(d+x) - d) = [d, e^{[a,]'}(d+x) - d]' = [d, e^{[a,]'}(d+x)]'$$

and using Remark I.3.18

$$[e^a * x, e^a * x] = [e^{[a,]'}(d+x) - d, e^{[a,]'}(d+x) - d]' =$$

$$[e^{[a,]'}(d+x), e^{[a,]'}(d+x)]' - 2[d, e^{[a,]'}(d+x)]' =$$

$$e^{[a,]'}[d+x, d+x]' - 2[d, e^{[a,]'}(d+x)]' = -2[d, e^{[a,]'}(d+x)]'.$$

Therefore

$$d(e^a * x) + \frac{1}{2}[e^a * x, e^a * x] = 0.$$

Finally, for each $x \in \mathrm{MC}_L(A)$, we define the *irrelevant stabilizer* of x:

$$Stab_A(x) = \{e^{dh+[x,h]} | \, h \in L^{-1} \otimes m_A\} \subset \exp(L^0 \otimes A).$$

The name irrelevant stabilizer is due to the fact that $e^{dh+[x,h]} * x = x$. Actually

$$[dh + [x,h], x] = [dh, x] + [[x,h], x] = d[h,x] + [h, dx] + \frac{1}{2}[h, [x,x]] =$$

$$d[h,x] + [h, dx + \frac{1}{2}[x,x]] = d[h,x] = d(dh + [x,h]).$$

Then $dh + [x,h]$ satisfies condition (3).

Moreover, we observe that $Stab_A(x)$ is a subgroup and that for each $a \in L^0 \otimes A$

$$e^a Stab_A(x) e^{-a} = Stab_A(y) \qquad \text{with} \qquad y = e^a * x.$$

I.3.5. Deformation functor associated to a DGLA.

I.3.29. Definition. The *deformation functor* associated to a differential graded Lie algebra L is:

$$\mathrm{Def}_L : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$\mathrm{Def}_L(A) = \frac{MC_L}{\exp(L^0 \otimes m_A)}.$$

Also in this case a morphism of DGLA $\phi : L \longrightarrow M$ induces a morphism of the associated functor $\phi : \mathrm{Def}_L \longrightarrow \mathrm{Def}_M$.

The name deformation functor is justified by the following proposition.

I.3.30. Proposition. Def_L *satisfies the conditions of Definition I.1.10.*

PROOF. If $A=\mathbb{K}$, then it is clear that $\mathrm{Def}_L(B\times C)=\mathrm{Def}_L(B)\times \mathrm{Def}_L(C)$ and so condition $ii)$ of Definition I.1.10 is satisfied.

Now, let $\beta: B\longrightarrow A$ and $\gamma: C\longrightarrow A$ be morphisms in **Art** with β surjective. Let $(l,m)\in \mathrm{Def}_L(B)\times_{\mathrm{Def}_L(A)}\mathrm{Def}_L(C)$ and $\tilde{l}\in MC_L(A)$ and $\tilde{m}\in \mathrm{MC}_L(C)$ be lifting of l and m respecively, such that $\beta(\tilde{l})=\gamma(\tilde{m})$. Therefore there exists $a\in L^0\otimes m_A$ such that $e^a*\beta(\tilde{l})=\gamma(\tilde{m})$. Let $b\in L^0\otimes m_B$ be a lifting of a. Replacing $\tilde{l}$ with its gauge equivalent element $l'=e^b*\tilde{l}$ we can suppose $\beta(l')=\gamma(\tilde{m})$ in $\mathrm{MC}_L(A)$. By Remark I.3.21 MC_L is homogeneous and so there exists $n\in \mathrm{MC}_L(B\times_A C)$ that lifts $(l',\tilde{m})$. This implies that

$$\mathrm{Def}_L(B\times_A C)\longrightarrow \mathrm{Def}_L(B)\times_{\mathrm{Def}_L(A)}\mathrm{Def}_L(C)$$

is surjective. Hence condition $i)$ of Definition I.1.10 also holds.

□

I.3.31. REMARK. By definition the *tangent space* of Def_L is:

$$t_{\mathrm{Def}_L}:=\mathrm{Def}_L(\mathbb{K}[\varepsilon])=\frac{\{x\in L^1\otimes\mathbb{K}\varepsilon\,|\,dx=0\}}{\{da\,|a\in L^0\otimes\mathbb{K}\varepsilon\}}\cong$$
$$H^1(L).$$

In general, if $L\otimes m_A$ is abelian then $\mathrm{Def}_L(A)=H^1(L)\otimes m_A$.

I.3.32. LEMMA. *The projection $\pi:\mathrm{MC}_L\longrightarrow\mathrm{Def}_L$ is a smooth morphism of functors.*

PROOF. Let $\alpha: B\longrightarrow A$ be a surjection in **Art** and prove that

$$\mathrm{MC}_L(B)\longrightarrow \mathrm{Def}_L(B)\times_{\mathrm{Def}_L(A)}\mathrm{MC}_L(A)$$

induced by

$$\begin{tikzcd} \mathrm{MC}_L(B) \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \mathrm{MC}_L(A) \arrow[d, "\pi"] \\ \mathrm{Def}_L(B) \arrow[r, "\alpha"'] & \mathrm{Def}_L(A), \end{tikzcd}$$

is surjective.

Let $(b,a)\in\mathrm{Def}_L(B)\times_{\mathrm{Def}_L(A)}\mathrm{MC}_L(A)$ and $\tilde{b}\in\mathrm{MC}_L(B)$ be a lifting of b. Then $\alpha(\tilde{b})$ and a have a common image in $\mathrm{Def}_L(A)$ and so $\alpha(\tilde{b})=e^t*a$, for some $t\in L^0\otimes m_A$.

Let $s\in L^0\otimes m_B$ be a lifting of t and define $b'=e^{-s}*\tilde{b}\in\mathrm{MC}_L(B)$. Then $\alpha(b')=e^{-t}*\alpha(\tilde{b})=a$ and b' lifts b. □

Therefore by Corollary I.1.35, π induces an isomorphism between universal obstruction theory.

In conclusion, Lemma I.3.23 implies that *$H^2(L)$ is a complete obstruction space of* Def_L.

I.3.33. Theorem. *Let $\phi : L \longrightarrow M$ be a morphism of DGLA and denote by*

$$H^i(\phi) : H^i(L) \longrightarrow H^i(M)$$

the induced maps in cohomology.

i) *If $H^1(\phi)$ is surjective (resp. bijective) and $H^2(\phi)$ injective. Then the morphism $\mathrm{Def}_L \longrightarrow \mathrm{Def}_M$ is smooth (resp. étale).*
ii) *If in addition to i) $H^0(\phi)$ is surjective. Then the morphism $\mathrm{Def}_L \longrightarrow \mathrm{Def}_M$ is an isomorphism.*

Proof. *i)* follows from Proposition I.1.31 (resp. Corollary I.1.32). For a proof of *ii)* see [**20**, Theorem 3.1] (it also follows from the inverse function Theorem III.2.14 of the extended case). □

I.3.34. Corollary. *Let $L \longrightarrow M$ be a quasi-isomorphism of DGLA. Then the induced morphism $\mathrm{Def}_L \longrightarrow \mathrm{Def}_M$ is an isomorphism.*

I.3.35. Corollary. *If $H^0(L) = 0$, then Def_L is homogenous.*

Proof. Let N be the DGLA introduced in Example I.3.10. Then the natural inclusion $N \longrightarrow L$ gives isomorphisms $H^i(N) \longrightarrow H^i(L)$ for each $i \geq 1$. Since $H^0(L) = 0$, then $H^0(N) \longrightarrow H^0(L)$ is surjective. Therefore Theorem I.3.33 *ii)* implies that $\mathrm{Def}_N \longrightarrow \mathrm{Def}_L$ is an isomorphism, whit $\mathrm{Def}_N \cong \mathrm{MC}_N$ homogeneous.

□

I.3.36. Remark. Let $F : \mathbf{Art} \longrightarrow \mathbf{Set}$ be the functor of the infinitesimal deformations of some algebro-geometric object defined over $\mathbb{K}$.

Then the guiding principle of Kontsevich (see [**18**]) affirms the existence of a DGLA L such that $F \cong \mathrm{Def}_L$. In spite of previous Corollary I.3.34, it is clear that this DGLA is defined only up to quasiisomorphism. In this case we say that L *governs* the deformation functor F.

In Section II.7 we will prove the existence of a DGLA that governs the infinitesimal deformation of X (Theorem II.7.3) and in Section IV.2 the existence of a DGLA that governs the infinitesimal deformations f an holomorphic map f (Theorem IV.2.6).

CHAPTER II

Deformation of complex manifolds

In the first part of this chapter we fix notations and recall some known facts about complex manifolds that will be useful in the sequel.

Therefore any book of complex varieties is a good reference for this chapter (for example [**11**], [**23**], [**33**], etc.).

In particular we decide to recall the Čech cohomology and Leray's theorem (Section II.3.1) and some properties of Kähler manifolds (Section II.2). We also study the map f_* and f^* induced by an holomorphic map f (Section II.6).

Moreover, we give the fundamental definition of the *Kodaira-Spencer differential graded Lie algebra* KS_X associated to a compact complex manifold X (Definition II.4.1), of the *contraction map* $\boldsymbol{i}$ and of the *holomorphic Lie derivative* $\boldsymbol{l}$ (Section II.5).

In the second part (Section II.7) we prove that the functors Def_X of the infinitesimal deformations of a compact complex manifold X (see Definition I.2.4) is isomorphic the deformation functor Def_{KS_X} associated the Kodaira-Spencer algebra KS_X of X (Theorem II.7.3).

THEOREM. *Let X be a complex compact manifold and KS_X its Kodaira-Spencer algebra. Then there exists an isomorphism of functors*

$$\mathrm{Def}_{KS_X} \longrightarrow \mathrm{Def}_X .$$

Therefore in spite of Remark I.3.36 we can say that the differential graded Lie algebra of Kodaira-Spencer KS_X governs the infinitesimal deformations of a complex compact manifold X.

This theorem is well known and a proof based on the theorem of Newlander-Nirenberg can be found in [**4**], [**10**] or more recently in [**23**]. Here we are interested in a simpler proof that avoid the use of this theorem.

Beware. In this chapter we will work over the complex number and so $\mathbb{K} = \mathbb{C}$.

We also assume that every variety X is smooth (complex) compact and connected.

II.1. Differential forms

Let X be a such manifold of dimension n and $T_{X,\mathbb{C}} = T^{1,0}_X \oplus T^{0,1}_X$ its complex tangent bundle, with $T^{1,0}_X =: \Theta_X$ the holomorphic tangent bundle and $T^{0,1}_X = \overline{T^{1,0}_X}$.

This decomposition induces a dual decomposition on the sheaf of differentiable forms
$$\mathcal{A}^1_X = \mathcal{A}^{1,0}_X \oplus \mathcal{A}^{0,1}_X$$
with $\mathcal{A}^{1,0}_X$ the sheaf of complex differentiable forms of type $(1,0)$. If $z_1, \dots, z_n$ are local holomorphic coordinates on X, then $\mathcal{A}^{1,0}_X$ is generated by the dz_i: each $\alpha \in \mathcal{A}^{1,0}_X$ has the form $\alpha = \sum_i \alpha_i dz_i$, with $\alpha_i \in \mathcal{A}^{0,0}_X$ for each i.

In general a (p,q)-form α can be locally written as $\alpha = \sum_{K,J} \alpha_{K,J} dz_K \wedge d\overline{z}_J$ with $K = (1 \leq k_1 < k_2 < \dots < k_p \leq n)$ a multi index of length p and $J = (1 \leq j_1 < j_2 < \dots < j_q \leq n)$ a multi index of length q.

If $\alpha = f \in \mathcal{A}^{0,0}_X$, then
$$df = \sum_{h=1}^n \frac{\partial f}{\partial z_h} dz_h + \sum_{h=1}^n \frac{\partial f}{\partial \overline{z}_h} d\overline{z}_h = \partial f + \overline{\partial} f,$$
with $\partial f \in \mathcal{A}^{1,0}_X$ and $\overline{\partial} f \in \mathcal{A}^{0,1}_X$.

In general, for $\alpha = \sum_{K,J} \alpha_{K,J} dz_K \wedge d\overline{z}_J \in \mathcal{A}^{p,q}_X$, we have
$$d\alpha = \sum_{K,J} d\alpha_{K,J} \wedge dz_K \wedge d\overline{z}_J = \partial\alpha + \overline{\partial}\alpha,$$
with
$$\partial\alpha = \sum_{I,J} \partial\alpha_{I,J} \wedge dz_K \wedge d\overline{z}_J \in \mathcal{A}^{p+1,q}$$
and
$$\overline{\partial}\alpha = \sum_{I,J} \overline{\partial}\alpha_{I,J} \wedge dz_K \wedge d\overline{z}_J \in \mathcal{A}^{p,q+1}.$$

Obviously, since $d^2 = 0$ we have $\partial^{\ 2} = \overline{\partial}^{\ 2} = \partial\overline{\partial} + \overline{\partial}\partial = 0$.

II.1.1. Proposition. *Let α be a form of type (p,q), with $q > 0$, such that $\overline{\partial}\alpha = 0$. Then there exists, locally on X, a form β of type $(p, q-1)$ such that $\overline{\partial}\beta = \alpha$.*

Proof. See [**33**, Proposition 2.31]. □

II.1.2. Definition. $(\mathcal{A}^{*,*}_X, \wedge)$ is the sheaf of graded algebras of differential forms of X, i.e. if $\mathcal{A}^{(p,q)}_X$ is the sheaf of differentiable (p,q)-forms then
$$\mathcal{A}^{*,*}_X := \bigoplus_i \mathcal{A}^i_X \qquad \text{with} \qquad \mathcal{A}^i_X = \bigoplus_{p+q=i} \mathcal{A}^{p,q}_X.$$

We use the notation $A_X^{p,q} = \Gamma(X, \mathcal{A}_X^{p,q})$ for the vector space of global sections of $\mathcal{A}_X^{p,q}$.

II.1.3. Definition. $Der^*(\mathcal{A})$ is the sheaf of $\mathbb{C}$-linear derivation on $\mathcal{A}_X^{*,*}$; more precisely if $Der^{a,b}(\mathcal{A})$ are the derivations of be-degree (a,b) then

$$Der^*(\mathcal{A}) := \bigoplus_k \bigoplus_{a+b=k} Der^{a,b}(\mathcal{A}).$$

We note that ∂ and $\overline{\partial}$ are global section of $Der^{1,0}(\mathcal{A})$ and $Der^{0,1}(\mathcal{A})$ respectively.

II.1.4. Remark. $Der^*(\mathcal{A})$ is a *sheaf of differential graded Lie algebras* with bracket and differential given by the following formulas:

$$[f,g] := f \circ g - (-1)^{deg(f)\,deg(g)} g \circ f$$

and

$$d(f) := [\partial + \overline{\partial}, f] = \partial f + \overline{\partial} f - (-1)^{deg(f)}(f\partial + f\overline{\partial}).$$

In particular, fixing $p = 0$, $(\mathcal{A}_X^{0,*}, \wedge)$ is a sheaf of graded algebras and $Der^*(\mathcal{A}^{0,*}, \mathcal{A}^{0,*}) := \bigoplus_p Der^p(\mathcal{A}^{0,*}, \mathcal{A}^{0,*})$ is a sheaf of DGLAs (in this case the differential reduces to $d(f) = [\overline{\partial}, f] = \overline{\partial} f - (-1)^{deg(f)} f\overline{\partial}$).

II.2. Kähler manifolds

This section is devoted to the compact Kähler manifolds. For definitions and properties of Kähler manifolds see for examples [**11**], [**23**] or [**33**].

We include this section just to prove an important application (Lemma II.2.2) of $\partial\overline{\partial}$-Lemma (Lemma II.2.1) that will be fundamental in the obstruction calculus of the last chapter of this thesis (Theorem V.1.4).

II.2.1. Lemma ($\partial\overline{\partial}$-Lemma). *Let X be a compact Kälher manifold and consider the operators ∂ and $\overline{\partial}$ on A_X. Then*

$$\operatorname{Im} \overline{\partial}\partial = \ker \partial \cap \operatorname{Im} \overline{\partial} = \ker \overline{\partial} \cap \operatorname{Im} \partial.$$

Proof. See for example [**23**, Theorem 6.37] and [**33**, Proposition 6.17].

□

Let $f : X \longrightarrow Y$ be an holomorphic map of compact complex manifolds. Let $\Gamma \subset X \times Y$ be the graph of f and $p : X \times Y \longrightarrow X$ and $q : X \times Y \longrightarrow Y$ be the natural projections.

II.2.2. Lemma. *If X and Y are compact Kähler, then the subcomplexes $Im(\partial) = \partial A_{X\times Y}$, ∂A_Γ, $\partial A_{X\times Y} \cap q^* A_Y$ and $\partial A_{X\times Y} \cap p^* A_X$ are acyclic.*

Proof. By hypothesis $X \times Y$ is Kähler. Then applying the $\partial\overline{\partial}$-Lemma II.2.1 to $A_{X\times Y}$ we get

$$\ker(\overline{\partial}) \cap \operatorname{Im}(\partial) = \operatorname{Im}(\overline{\partial}\partial)$$

and so $H^*_{\overline{\partial}}(\partial(A_{X\times Y})) = 0$. $\Gamma \subset X \times Y$ is also Kähler and so the same conclusion holds for ∂A_Γ: ∂A_Γ is acyclic.

Analogously, since Y is Kähler ∂A_Y and $q^*\partial A_Y$ are acyclic. Therefore to prove that $\partial A_{X\times Y} \cap q^* A_Y$ is acyclic it is sufficient to prove that $q^* A_Y \cap \partial A_{X\times Y} = q^* \partial A_Y$.

The inclusion $\supseteq$ is obvious. Let $p \in q^* A_Y \cap \partial A_{X\times Y}$, then $p = q^*\phi = \partial z$ with $\phi \in A_Y$ and $z \in A_{X\times Y}$. Therefore $\partial p = q^*\partial\phi = \partial\partial z = 0$ and so ϕ is ∂-closed ($\phi \in H_\partial(A_Y)$). Moreover $q^* : H_\partial(A_Y) \longrightarrow H_\partial(A_{X\times Y})$ is injective and $q^*[\phi] = [\partial z] = 0$. Then ϕ is ∂-exact, that is $\phi = \partial t$ with $t \in A_Y$. This implies $p = q^*\partial t \in q^*\partial A_Y$.

The case $\partial A_{X\times Y} \cap p^* A_X$ can be proved in the same way. □

II.2.3. Remark. In the previous lemma the Kähler hypothesis on X and Y can be substitute by the validity of the $\partial\overline{\partial}$-lemma in A_X,A_Y, $A_{X\times Y}$ and A_Γ.

II.3. Holomorphic fiber bundle and Dolbeault's cohomology

Let E be an holomorphic fiber bundle on X. Then the $\overline{\partial}$ operator can be extended to the *Dolbeault* operator

$$\overline{\partial}_E : \mathcal{A}^{p,q}(E) \longrightarrow \mathcal{A}^{p,q+1}(E).$$

If $e_1, \ldots, e_n$ is a local frame for E then

$$\overline{\partial}_E(\sum_i \phi_i e_i) = \sum_i \overline{\partial}(\phi) e_i.$$

Since E is an holomorphic fiber bundle this definition doesn't depend on the choice of the local frame. By definition, $\overline{\partial}_E$ satisfies the property $\overline{\partial}_E{}^2 = 0$.

Let $A^{p,q}_X(E) = \Gamma(X, \mathcal{A}^{p,q}_X(E))$ be the vector space of global sections of the sheaf $\mathcal{A}^{p,q}_X(E)$. Then we can consider, for each $p \geq 0$, the following complex:

$$0 \longrightarrow A^{p,0}_X(E) \xrightarrow{\overline{\partial}_E} A^{p,1}_X(E) \xrightarrow{\overline{\partial}_E} \cdots \xrightarrow{\overline{\partial}_E} A^{p,q}_X(E) \xrightarrow{\overline{\partial}_E} \cdots .$$

The cohomology of this complex is the *Dolbeault's cohomology* $H^{p,*}_{\overline{\partial}_E}(X, E)$ of E. We note that, for $p = 0$, $\ker(\overline{\partial}_E : A^{0,0}_X(E) \longrightarrow A^{0,1}_X(E))$ coincides with the holomorphic sections of E and

$$H^q_{\overline{\partial}_E}(X, E) = H^{0,q}_{\overline{\partial}_E}(X, E) = \frac{\ker(\overline{\partial}_E : A^{0,q}_X(E) \longrightarrow A^{0,q+1}_X(E))}{\operatorname{Im}(\overline{\partial}_E : A^{0,q-1}_X(E) \longrightarrow A^{0,q}_X(E))}.$$

Also in the case of holomorphic bundle, we have an analogous of the previous Proposition II.1.1.

II.3.1. Proposition. *Let α be a differential form with coefficient in E of type $(0,q)$ with $q>0$. If $\overline{\partial}_E\alpha=0$, then there exists locally on X a differential form β of type $(0,q-1)$, with coefficients in E, such that $\overline{\partial}_E\beta=\alpha$.*

Proof. See [**33**, Proposition 2.36]. $\square$

II.3.1. Čech cohomology and Leray's theorem. We follow [**23**, Section 1.3].

Let E be an holomorphic bundle on the complex manifold X. Let $\mathcal{U}=\{U_i\}_{i\in I}$ be a locally finite open covering of X and denote $U_{i_0\cdots i_k}=U_{i_0}\cap\cdots\cap U_{i_k}$.

Define the Čech q-chains of E:

$$\check{C}^k(\mathcal{U},E)=\{f_{i_0\cdots i_k}\mid f_{i_0\cdots i_k}:U_{i_0\cdots i_k}\longrightarrow E \text{ is an holomorphic section}\}$$

and the Čech differential

$$\check{\delta}:\check{C}^k(\mathcal{U},E)\longrightarrow\check{C}^{k+1}(\mathcal{U},E)$$

$$(\check{\delta}f)_{i_0\cdots i_{k+1}}=\sum_{j=0}^{k+1}(-1)^j\,f_{i_0\cdots\widehat{i_j}\cdots i_{k+1}}.$$

A simple calculation show that $\check{\delta}^2=0$ and so we can define the $\mathbb{C}$-vector space of Čech cohomology

$$\check{H}^k(\mathcal{U},E)=\frac{\ker(\check{\delta}:\check{C}^k(\mathcal{U},E)\longrightarrow\check{C}^{k+1}(\mathcal{U},E)}{\mathrm{Im}(\check{\delta}:\check{C}^{k-1}(\mathcal{U},E)\longrightarrow\check{C}^{k}(\mathcal{U},E)}.$$

Now, define a morphism $\theta:\check{H}^k(\mathcal{U},E)\longrightarrow H^{0,k}_{\overline{\partial}_E}(X,E)$.

Let $t_i:X\longrightarrow\mathbb{C}$, with $i\in I$, be a partition of unity subordinate to the cover $\mathcal{U}$: that is $supp(t_i)\subset U_i$, $\sum_i t_i=1$ and $\sum_i\overline{\partial}t_i=0$.

For each $f\in\check{C}^k(\mathcal{U},E)$ and $i\in I$ we define

$$\phi_i(f)=\sum_{j_1\cdots j_k}f_{ij_1\cdots j_k}\overline{\partial}t_{j_1}\wedge\cdots\wedge\overline{\partial}t_{j_k}\in\Gamma(U_i,\mathcal{A}^{0,k}(E))$$

and then

$$\phi(f)=\sum_i t_i\phi_i(f)\in\Gamma(X,\mathcal{A}^{0,k}(E)).$$

It is true that ϕ is a well defined morphism of complexes that induces a morphism θ in cohomology (for full details see [**23**, Proposition 1.22]).

II.3.2. Theorem. *Let $\mathcal{U}=\{U_i\}_{i\in I}$ be a locally finite countable open covering of a complex manifold X and E an holomorphic vector bundle. If $H^{k-q}_{\overline{\partial}_E}(U_{i_0\cdots i_q},E)=0$ for every $q<k$ and $i_0\cdots i_k$, then the morphism θ is an isomorphism*

$$\theta:\check{H}^k(\mathcal{U},E)\longrightarrow H^k_{\overline{\partial}_E}(X,E).$$

Proof. See [**23**, Theorem 1.24] or [**33**, Theorem 4.41]. □

II.3.3. Remark. If the open U_i of the cover $\mathcal{U}$ are biholomorphic to open convex subset of $\mathbb{C}^n$ then $\mathcal{U}$ satisfy hypothesis of Theorem II.3.2.

II.3.4. Remark. It is convenient to give an explicit description of the inverse map of θ, at least for $k = 2$:

$$\theta^{-1} : H^2_{\overline{\partial}_E}(X, E) \longrightarrow \check{H}^2(\mathcal{U}, E).$$

Let $h \in H^2_{\overline{\partial}_E}(X, E)$. Applying Proposition II.3.1 for each $i \in I$ there exists $\tau_i \in \Gamma(U_i, \mathcal{A}^{0,1}(E))$ such that $h_{|U_i} = \overline{\partial}\tau_i$.

Define $\sigma_{ij} = (\tau_i - \tau_j)_{|U_{ij}} \in \Gamma(U_{ij}, \mathcal{A}^{0,1}(E))$. σ_{ij} is $\overline{\partial}$-closed; actually

$$\overline{\partial}\sigma_{ij} = (\overline{\partial}\tau_i - \overline{\partial}\tau_j)_{|U_{ij}} = h_{|U_{ij}} - h_{|U_{ji}} = 0.$$

Therefore for each U_{ij} there exists $\rho_{ij} \in \Gamma(U_{ij}, \mathcal{A}^{0,0}(E))$, such that $\overline{\partial}\rho_{ij} = \sigma_{ij}$.

We observe that $(\sigma_{jk} - \sigma_{ik} + \sigma_{ij})_{|U_{ijk}} = 0$; indeed

$$(\sigma_{jk} - \sigma_{ik} + \sigma_{ij})_{|U_{ijk}} =$$

$$((\tau_j - \tau_k) - (\tau_i - \tau_k) + (\tau_i - \tau_j))_{|U_{ijk}} = 0.$$

Define $\alpha_{ijk} = (\rho_{jk} - \rho_{ik} + \rho_{ij})_{|U_{ijk}} \in \Gamma(U_{ijk}, \mathcal{A}^{0,0}(E))$. First of all we have that $\overline{\partial}\alpha_{ijk} = 0$; actually

$$\overline{\partial}\alpha_{ijk} = (\overline{\partial}\rho_{jk} - \overline{\partial}\rho_{ik} + \overline{\partial}\rho_{ij})_{|U_{ijk}} = (\sigma_{jk} - \sigma_{ik} + \sigma_{ij})_{|U_{ijk}} = 0$$

This implies that $\alpha_{ijk} \in \Gamma(U_{ijk}, E)$.

Moreover $(\check{\delta}\alpha)_{ijkl} = 0$; in fact

$$(\check{\delta}\alpha)_{ijkl} = (\alpha_{jkl} - \alpha_{ikl} + \alpha_{ijl} - \alpha_{ijk})_{|U_{ijkl}} =$$

$$((\rho_{kl}-\rho_{jl}+\rho_{jk})-(\rho_{kl}-\rho_{il}+\rho_{ik})+(\rho_{jl}-\rho_{il}+\rho_{ij})-(\rho_{jk}-\rho_{ik}+\rho_{ij}))_{|U_{ijkl}} = 0$$

This implies that $\alpha \in \check{H}^2(X, E)$.

α is independent of the choices. Actually, if we choose $\overline{\tau}_i$, such that $h_{|U_i} = \overline{\partial}\overline{\tau}_i$; then $\overline{\tau}_i = \tau_i + \overline{\partial}t_i$ and this change doesn't affect the choice of α_{ijk}.

If we choose $\overline{\rho}_{ij} \in \Gamma(U_{ij}, \mathcal{A}^{0,0}(E))$ such that $\overline{\partial}\overline{\rho}_{ij} = \sigma_{ij}$, then $\overline{\rho}_{ij} = \rho_{ij} + s_{ij}$, with $s_{ij} \in \Gamma(U_{ij}, \mathcal{A}^{0,0}(E))$ such that $\overline{\partial}s_{ij} = 0$. This implies that $s_{ij} \in \Gamma(U_{ij}, E)$. Therefore $\{\overline{\alpha}_{ijk}\} = \{\alpha_{ijk}\} + \{\check{\delta}(s_{ij})\}$ and so $\overline{\alpha}_{ijk}$ and α_{ijk} represent the same class in cohomology.

In conclusion we have defined a map

$$\vartheta : H^2_{\overline{\partial}_E}(X, E) \longrightarrow \check{H}^2(\mathcal{U}, E)$$

$$[h] \longmapsto [\alpha].$$

Finally it can be proved that this map ϑ is the inverse of θ (for full details see [**23**, Theorem 1.24] or [**33**, Theorem 4.41]).

II.4. The Kodaira-Spencer algebra KS_X

II.4.1. Definition. Let Θ_X be the holomorphic tangent bundle of a complex manifold X. The *Kodaira-Spencer* (differential graded Lie) algebra of X is

$$KS_X = \bigoplus_i \Gamma(X, \mathcal{A}_X^{0,i}(\Theta_X)) = \bigoplus_i A_X^{0,i}(\Theta_X).$$

In particular, KS_X^i is the vector space of the global sections of the sheaf of germs of the differential $(0,i)$-forms with coefficients in Θ_X.

The differential $\tilde{d}$ on KS_X is the opposite of Dolbeault differential, while the bracket is defined in local coordinates as the $\overline{\Omega}^*$-bilinear extension of the standard bracket on $\mathcal{A}_X^{0,0}(\Theta_X)$ ($\overline{\Omega}^* = \ker(\partial : \mathcal{A}_X^{0,*} \longrightarrow \mathcal{A}_X^{1,*})$ is the sheaf of antiholomorphic differential forms).

Explicitly, if $z_1, \ldots, z_n$ are local holomorphic coordinates on X, we have

$$\tilde{d}(f d\overline{z}_I \frac{\partial}{\partial z_i}) = -\overline{\partial}(f) \wedge d\overline{z}_I \frac{\partial}{\partial z_i}.$$

$$[f\frac{\partial}{\partial z_i}\, d\overline{z}_I, g\frac{\partial}{\partial z_j}\, d\overline{z}_J] = (f\frac{\partial g}{\partial z_i}\frac{\partial}{\partial z_j} - g\frac{\partial f}{\partial z_j}\frac{\partial}{\partial z_i})\, d\overline{z}_I \wedge d\overline{z}_J \qquad \forall\, f, g \in \mathcal{A}_X^{0,0}.$$

($\mathcal{A}_X^{0,*}(T_X)$ is a sheaf of DGLA).

We note that by Dolbeault theorem we have $H^i(A_X^{0,*}(\Theta_X)) \cong H^i(X, \Theta_X)$ for every i then

$$H^*(KS_X) = \frac{\ker(\overline{\partial} : A_X^{0,q}(\Theta_X) \longrightarrow A_X^{0,q+1}(\Theta_X))}{\operatorname{Im}(\overline{\partial} : A_X^{0,q-1}(\Theta_X) \longrightarrow A_X^{0,q}(\Theta_X))} \cong H^*_{\overline{\partial}}(X, \Theta_X)$$

In Theorem II.7.3, we will prove that the DGLA KS_X governs the infinitesimal deformations of X.

II.5. Contraction map and holomorphic Lie derivative

In general, for each vector space V and linear functional $\alpha : V \longrightarrow \mathbb{C}$, we can define the *contraction operator*

$$\alpha \lrcorner \bigwedge^k V \longrightarrow \bigwedge^{k-1} V$$

$$\alpha \lrcorner (v_1 \wedge \ldots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha(v_i)(v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \wedge v_k),$$

that is a derivation of degree -1 of the graded algebra $(\bigwedge^k V, \wedge)$.

Then considering the contraction $\lrcorner$ of the differential forms with vector fields we can define two injective morphisms of sheaves:

- the *contraction map*

$$\boldsymbol{i} : \mathcal{A}_X^{0,*}(\Theta_X) \longrightarrow Der^*(\mathcal{A}_X^{*,*})[-1]$$
$$a \longmapsto \boldsymbol{i}_a \quad \text{with} \quad \boldsymbol{i}_a(\omega) = a \lrcorner \omega$$

- the *holomorphic Lie derivative*

$$\boldsymbol{l} : \mathcal{A}_X^{0,*}(\Theta_X) \longrightarrow Der^*(\mathcal{A}_X^{*,*})$$
$$a \longmapsto \boldsymbol{l}_a = [\partial, \boldsymbol{i}_a] \quad \text{with} \quad \boldsymbol{l}_a(\omega) = \partial(a \lrcorner \omega) + (-1)^{deg(a)} a \lrcorner \partial\omega$$

for each $a \in \mathcal{A}_X^{0,*}(\Theta_X)$ and $\omega \in \mathcal{A}_X^{*,*}$.

II.5.1. Lemma. *In the notation above, for every $a, b \in \mathcal{A}_X^{0,*}(\Theta_X)$ we have*

$$\boldsymbol{i}_{\tilde{d}a} = -[\overline{\partial}, \boldsymbol{i}_a], \qquad \boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, [\partial, \boldsymbol{i}_b]] = [[\boldsymbol{i}_a, \partial], \boldsymbol{i}_b], \qquad [\boldsymbol{i}_a, \boldsymbol{i}_b] = 0.$$

Proof. See [**22**, Lemma2.1]. Let $z_1, z_2, \ldots, z_n$ be local holomorphic coordinates on X. By linearity, we can assume that $a = f d\overline{z}_I \dfrac{\partial}{\partial z_i}$ and $b = g d\overline{z}_J \dfrac{\partial}{\partial z_j}$ $(i \neq j)$, with $f, g \in \mathcal{A}_X^{0,0}$.

All the expressions vanish on $\mathcal{A}_X^{0,*}$ and $\mathcal{A}_X^{*,*}$ is generated as $\mathbb{C}$-algebra by $\mathcal{A}_X^{0,0} \oplus \mathcal{A}_X^{0,1} \oplus \mathcal{A}_X^{1,0}$. Therefore it is sufficient to verify the equalities on the dz_h (that generate $\mathcal{A}_X^{1,0}$).

Moreover, we note that $\overline{\partial} dz_h = \partial dz_h = \boldsymbol{i}_a \boldsymbol{i}_b dz_h = \boldsymbol{i}_b \boldsymbol{i}_a dz_h = 0$. Therefore $[\boldsymbol{i}_a, \boldsymbol{i}_b] = 0$ and the other equalities follow from the easy calculations below.

Let $\omega = dz_h$ and $\tilde{d}(a) = -\overline{\partial}(f) \wedge d\overline{z}_I \dfrac{\partial}{\partial z_i}$. Then

$$\boldsymbol{i}_{\tilde{d}a}(dz_h) = \tilde{d}a \lrcorner dz_h = \begin{cases} 0 & h \neq i, \\ -\overline{\partial}(f) \wedge d\overline{z}_I & h = i. \end{cases}$$

On the other side

$$-[\overline{\partial}, \boldsymbol{i}_a](\omega) = (-\overline{\partial}\boldsymbol{i}_a + (-1)^{\overline{a}-1} \boldsymbol{i}_a \overline{\partial})(dz_h) = -\overline{\partial}\boldsymbol{i}_a(dz_h) =$$
$$-\overline{\partial}(a \lrcorner dz_h) = \begin{cases} 0 & h \neq i, \\ -\overline{\partial}(f d\overline{z}_I) & h = i. \end{cases}$$

Then the first equalities holds.

About $\boldsymbol{i}_{[a,b]}$, we have

$$[a, b] = (f \frac{\partial g}{\partial z_i} \frac{\partial}{\partial z_j} - g \frac{\partial f}{\partial z_j} \frac{\partial}{\partial z_i}) \, d\overline{z}_I \wedge d\overline{z}_J$$

and then

$$\boldsymbol{i}_{[a,b]}(dz_h) = \begin{cases} 0 & h \neq i, j, \\ f \dfrac{\partial g}{\partial z_i} \, d\overline{z}_I \wedge d\overline{z}_J & h = j, \\ - g \, \dfrac{\partial f}{\partial z_j} \, d\overline{z}_I \wedge d\overline{z}_J & h = i. \end{cases}$$

On the other side

$$[\boldsymbol{i}_a,[\partial,\boldsymbol{i}_b]]=[\boldsymbol{i}_a,\partial\boldsymbol{i}_b-(-1)^{\overline{b}-1}\boldsymbol{i}_b\partial]=$$

$$\boldsymbol{i}_a\partial\boldsymbol{i}_b-(-1)^{\overline{b}-1}\boldsymbol{i}_a\boldsymbol{i}_b\partial-(-1)^{(\overline{a}-1)\overline{b}}(\partial\boldsymbol{i}_b\boldsymbol{i}_a-(-1)^{\overline{b}-1}\boldsymbol{i}_b\partial\boldsymbol{i}_a).$$

Then

$$[\boldsymbol{i}_a,[\partial,\boldsymbol{i}_b]](dz_h)=\boldsymbol{i}_a\partial\boldsymbol{i}_b(dz_h)-(-1)^{\overline{a}\overline{b}}\boldsymbol{i}_b\partial\boldsymbol{i}_a(dz_h)=$$

$$\begin{cases}0 & h\neq i,j,\\ \boldsymbol{i}_a\partial\boldsymbol{i}_b(dz_j)=f\dfrac{\partial g}{\partial z_i}d\overline{z}_I\wedge d\overline{z}_J & h=j,\\ -(-1)^{\overline{a}\overline{b}}\boldsymbol{i}_b\partial\boldsymbol{i}_a(dz_i)=-(-1)^{\overline{a}\overline{b}}g\dfrac{\partial f}{\partial z_j}d\overline{z}_J\wedge d\overline{z}_I & h=i.\end{cases}$$

□

The previous set of equalities is called *Cartan formulas*.

II.5.2. Definition. Let L and M be two differential graded Lie algebras and let d' be the differential on the graded vector space $\mathrm{Hom}^*(L,M)$. A linear map $i\in\mathrm{Hom}^{-1}(L,M)$ is called a *Cartan homotopy* if

$$i([a,b])=[i(a),d'i(b)]\qquad\text{and}\qquad[i(a),i(b)]=0\qquad\forall\,a,b\in L.$$

We recall that by definition (see Example I.3.3) we have

$$d'i(a)=d_M(i(a))+i(d_L(a)).$$

II.5.3. Corollary. *$\boldsymbol{i}$ is a Cartan homotopy and the Lie derivative $\boldsymbol{l}$ is a morphism of sheaves of DGLAs.*

Proof. Using Cartan formulas we get $d'(\boldsymbol{i}_b)=[d,\boldsymbol{i}_b]+\boldsymbol{i}_{\tilde{d}b}=[\partial+\overline{\partial},\boldsymbol{i}_b]-[\overline{\partial},\boldsymbol{i}_b]=[\partial,\boldsymbol{i}_b]$. Then $\boldsymbol{i}_{[a,b]}=[\boldsymbol{i}_a,[\partial,\boldsymbol{i}_b]]=[\boldsymbol{i}_a,d'(\boldsymbol{i}_b)]$. Moreover, by Lemma II.5.1 $[\boldsymbol{i}_a,\boldsymbol{i}_b]=0$ and so $\boldsymbol{i}$ is a Cartan homotopy.

As regards $\boldsymbol{l}$, we have

$$\boldsymbol{l}_{\tilde{d}a}=[\partial,\boldsymbol{i}_{\tilde{d}a}]=-[\partial,[\overline{\partial},\boldsymbol{i}_a]].$$

Moreover

$$-[\partial,[\overline{\partial},\boldsymbol{i}_a]]=-[\partial,\overline{\partial}\boldsymbol{i}_a-(-1)^{deg\,\boldsymbol{i}_a}\boldsymbol{i}_a\overline{\partial}]=$$

$$-\partial\overline{\partial}\boldsymbol{i}_a+(-1)^{deg\,\boldsymbol{i}_a}\partial\boldsymbol{i}_a\overline{\partial}-(-1)^{deg\,\boldsymbol{i}_a}\overline{\partial}\boldsymbol{i}_a\partial+\boldsymbol{i}_a\overline{\partial}\partial.$$

Therefore

$$[d,\boldsymbol{l}_a]=[\partial+\overline{\partial},\boldsymbol{l}_a]=[\partial,\boldsymbol{l}_a]+[\overline{\partial},\boldsymbol{l}_a]=$$

$$=[\partial,[\partial,\boldsymbol{i}_a]]+[\overline{\partial},[\partial,\boldsymbol{i}_a]]=-[\partial,[\overline{\partial},\boldsymbol{i}_a]]=\boldsymbol{l}_{\tilde{d}a}.$$

Using $[\boldsymbol{i}_a,\boldsymbol{i}_b]=0$ and a very boring calculation, we can also prove that

$$\boldsymbol{l}_{[a,b]}=[\boldsymbol{l}_a,\boldsymbol{l}_b].$$

□

In particular we have an injective morphism of sheaves

$$\boldsymbol{l} : \mathcal{A}_X^{0,*}(\Theta_X) \longrightarrow Der^*(\mathcal{A}_X^{0,*}, \mathcal{A}_X^{0,*})$$

$$a \longmapsto \boldsymbol{l}_a \quad \text{with} \quad \boldsymbol{l}_a(\omega) = (-1)^{dega} a \lrcorner \partial\omega. \tag{4}$$

Explicitly, in local holomorphic coordinates $z_1, z_2, \ldots, z_n$, if $a = g\, d\overline{z}_I \dfrac{\partial}{\partial z_i}$ and $\omega = f d\overline{z}_J$, then

$$\boldsymbol{l}_a(\omega) = (-1)^{dega} \frac{\partial f}{\partial z_i}\, g\, d\overline{z}_I \wedge d\overline{z}_J.$$

Using $\boldsymbol{l}$, for each $(A, m_A) \in \mathbf{Art}$, we can define the following morphism:

$$\boldsymbol{l} : \mathcal{A}_X^{0,*}(\Theta_X) \otimes A \longrightarrow Der^*(\mathcal{A}_X^{0,*} \otimes A, \mathcal{A}_X^{0,*} \otimes A).$$

In particular, for each solution of the Maurer-Cartan equation in KS_X we have the fundamental lemma below.

II.5.4. Lemma. *$x \in \mathrm{MC}_{KS_X}(A)$ if and only if*

$$\overline{\partial} + \boldsymbol{l}_x : \mathcal{A}_X^{0,*} \otimes A \longrightarrow \mathcal{A}_X^{0,*+1} \otimes A$$

is a differential of degree 1 on $\mathcal{A}_X^{0,*} \otimes A$.

Proof. Since $\boldsymbol{l}$ is a morphism of DGLAs we have

$$(\overline{\partial} + \boldsymbol{l}_x)^2 = \overline{\partial}\boldsymbol{l}_x + \boldsymbol{l}_x\overline{\partial} + \boldsymbol{l}_x^2 = [\overline{\partial}, \boldsymbol{l}_x] + \frac{1}{2}[\boldsymbol{l}_x, \boldsymbol{l}_x] = \boldsymbol{l}(\tilde{d}x + \frac{1}{2}[x, x]).$$

□

Moreover using $\boldsymbol{l}$, we can also define, for each $(A, m_A) \in \mathbf{Art}$ and $a \in \mathcal{A}_X^{0,0}(\Theta_X) \otimes m_A$, an automorphism e^a of $\mathcal{A}_X^{0,*} \otimes A$:

$$e^a : \mathcal{A}_X^{0,*} \otimes A \longrightarrow \mathcal{A}_X^{0,*} \otimes A, \qquad e^a(f) = \sum_{n=0}^{\infty} \frac{\boldsymbol{l}_a^n}{n!}(f). \tag{5}$$

II.5.5. Lemma. *For every local Artinian $\mathbb{C}$-algebra (A, m_A), $a \in \mathcal{A}_X^{0,0}(\Theta_X) \otimes m_A$ and $x \in \mathrm{MC}_{KS_X}(A)$ we have*

$$e^a \circ (\overline{\partial} + \boldsymbol{l}_x) \circ e^{-a} = \overline{\partial} + e^a * \boldsymbol{l}_x \;:\; \mathcal{A}_X^{0,0} \otimes A \longrightarrow \mathcal{A}_X^{0,1} \otimes A. \tag{6}$$

where $*$ is the gauge action (and $e^a * \boldsymbol{l}_x \in \mathcal{A}_X^{0,1}(\Theta_X) \otimes m_A$ acts on $\mathcal{A}_X^{0,0} \otimes A$ as defined in (4)). In particular

$$\ker(\overline{\partial} + e^a * \boldsymbol{l}_x : \mathcal{A}_X^{0,0} \otimes A \longrightarrow \mathcal{A}_X^{0,1} \otimes A) = e^a(\ker(\overline{\partial} + \boldsymbol{l}_x : \mathcal{A}_X^{0,0} \otimes A \longrightarrow \mathcal{A}_X^{0,1} \otimes A)).$$

Proof. It follows from definition of gauge action. More precisely, since $e^a \circ e^b \circ e^{-a} = e^{[a,\,]}(b)$, we have

$$e^a \circ (\overline{\partial} + l_x) \circ e^{-a} = e^{[a,\,]'}(\overline{\partial} + l_x) = \sum_{n=0}^{\infty} \frac{[a,\,]'}{n!}(\overline{\partial} + l_x) = \overline{\partial} + l_x + \sum_{n=1}^{\infty} \frac{[a,\,]'}{n!}(\overline{\partial} + l_x) =$$

$$= \overline{\partial} + l_x + \sum_{n=0}^{\infty} \frac{[a,\ ]'^{n+1}}{(n+1)!}(\overline{\partial} + l_x) = \overline{\partial} + l_x + \sum_{n=0}^{\infty} \frac{[a,\ ]'^{n}}{(n+1)!}([a,\overline{\partial}]' + [a, l_x]) =$$
$$\overline{\partial} + l_x + \sum_{n=0}^{\infty} \frac{[a,\ ]'^{n}}{(n+1)!}([a, l_x] - \overline{\partial} a) = \overline{\partial} + e^a * l_x$$

□

II.5.6. REMARK. Let ϕ_i be automorphism of the A-module $\mathcal{A}_X^{0,*} \otimes A$ whose specialization to the residue field $\mathbb{C}$ is the identity. Let $\phi = \sum_i \phi = id + \eta$ with $\eta \in \mathrm{Hom}^0(\mathcal{A}_X^{0,*}, \mathcal{A}_X^{0,*}) \otimes m_A$. Since we are in characteristic zero, we can take the logarithm so that $\phi = e^a$ with $a \in \mathrm{Hom}^0(\mathcal{A}_X^{0,*}, \mathcal{A}_X^{0,*}) \otimes m_A$

II.5.1. The DGLA of a submanifolds. Let X be a complex manifold and $i : X \hookrightarrow Y$ be the inclusion of a submanifold X. Let $i^* : \mathcal{A}_X^{0,*} \longrightarrow \mathcal{A}_Y^{0,*}$ be the restriction morphism (of sheaves of DGLAs). Finally, denote by Θ_Y the holomorphic tangent bundle of Y and by $N_{X|Y}$ the normal bundle of X in Y. Define the sheaf $\mathcal{L}' = \oplus_i \mathcal{L}'^i$ such that

$$0 \longrightarrow \mathcal{L}' \longrightarrow \mathcal{A}_Y^{0,*}(\Theta_Y) \longrightarrow \mathcal{A}_X^{0,*}(N_{X|Y}) \longrightarrow 0.$$

Let $z_1, \ldots, z_n$ be holomorphic coordinates on Y such that $Y \supset X = \{z_{t+1} = \cdots = z_n = 0\}$. Then $\eta \in \mathcal{L}'^i$ if and only if $\eta = \sum_{j=1}^{n} \omega_j \frac{\partial}{\partial z_j}$, with $\omega_j \in \mathcal{A}_Y^{0,i}$ such that $\omega_j \in \ker i^*$ for $j \geq t$. In particular $\mathcal{L}'^0$ is the sheaf of differentiable vector field on Y that are tangent to X.

II.5.7. LEMMA. *$\mathcal{L}'$ is a sheaf of differential graded Lie subalgebras of $\mathcal{A}_Y^{0,*}(\Theta_Y)$ such that $\boldsymbol{l}_a(\ker i^*) \subset \ker i^*$ if and only if $a \in \mathcal{L}' \subset \mathcal{A}_Y^{0,*}(\Theta_Y)$.*

PROOF. See [**24**, Section 5]. It is an easy calculation in local holomorphic coordinates.

□

Moreover, consider the automorphism e^a of $\mathcal{A}_X^{0,*} \otimes A$ defined in (5): if $a \in \mathcal{L}'^0 \otimes m_A$ then $e^a(\ker(i^*) \otimes A) = \ker(i^*) \otimes A$.

Let L' be the DGLA of the global section of $\mathcal{L}'$:

$$0 \longrightarrow L' \longrightarrow A_Y^{0,*}(\Theta_Y) \xrightarrow{\pi'} A_X^{0,*}(N_{X|Y}) \longrightarrow 0.$$

In the literature, there also exists the notation $L' = A_Y^{0,*}(\Theta_Y(-log\ X))$.

In Section V.4 we will prove that L' governs the embedded deformations of the inclusion $i : X \hookrightarrow Y$ (Corollary V.4.1).

II.6. Induced map f_* and f^* by a holomorphic map f

This section is devoted to study the maps f_* and f^* induced by an holomorphic map f. In particular we prove a property of these map (Lemma II.6.1) that will be used in the last chapter (Section V.1.2).

Let $f : X \longrightarrow Y$ be an holomorphic map of compact complex manifolds.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_i\}_{i\in I}$ be finite Stein open covers of X and Y, respectively, such that $f(U_i) \subset V_i$ (U_i is allowed to be empty). Then f induces morphisms

$$f^* : \check{C}^p(\mathcal{V}, \Theta_Y) \longrightarrow \check{C}^p(\mathcal{U}, f^*\Theta_Y).$$

and

$$f_* : \check{C}^p(\mathcal{U}, \Theta_X) \longrightarrow \check{C}^p(\mathcal{U}, f^*\Theta_Y)$$

Explicitly, for each $i \in I$ and local holomorphic coordinate systems $z = (z_1, z_2, \dots, z_n)$ on U_i and $w = (w_1, w_2, \dots w_m)$ on V_i such that $f(z_1, z_2, \dots, z_n) = (f_1(z), \dots, f_m(z))$, we have

$$f^* : \Gamma(V_i, \Theta_Y) \longrightarrow \Gamma(U_i, f^*\Theta_Y)$$

$$f^*(\sum_j g_j(w)\frac{\partial}{\partial w_j}) = \sum_j g_j(f(z))\frac{\partial}{\partial w_j}$$

and

$$f_* : \Gamma(U_i, \Theta_X) \longrightarrow \Gamma(U_i, f^*\Theta_Y)$$

$$f_*(\sum_k h_k(z)\frac{\partial}{\partial z_k}) = \sum_{k,j} h_k(z)\frac{\partial f_j(z)}{\partial z_k}\frac{\partial}{\partial w_j}$$

Moreover f_* and f^* commute with the Čech differential and they don't depend on the choice of the cover. Therefore we get linear maps in cohomology :

$$f^* : \check{H}^p(Y, \Theta_Y) \longrightarrow \check{H}^p(X, f^*\Theta_Y).$$

and

$$f_* : \check{H}^p(X, \Theta_X) \longrightarrow \check{H}^p(X, f^*\Theta_Y)$$

Analogously f induces morphisms

$$f^* : A_Y^{p,q}(\Theta_Y) \longrightarrow A_X^{p,q}(f^*\Theta_Y)$$

and

$$f_* : A_X^{p,q}(\Theta_X) \longrightarrow A_X^{p,q}(f^*\Theta_Y).$$

Let $\mathcal{U}$ and $\mathcal{V}$ be Stein covers and $z = (z_1, z_2, \dots, z_n)$ on U_i and $w = (w_1, w_2, \dots w_m)$ on V_i local holomorphic coordinate systems as above. Let $K = (1 \leq k_1 < k_2 < \cdots < k_p \leq n)$ be a multi index of length p and $J = (1 \leq j_1 < j_2 < \cdots < j_q \leq n)$ a multi index of length q. Then

$$f_* : A^{p,q}(U_i, \Theta_X) \longrightarrow A_X^{p,q}(U_i, f^*\Theta_Y)$$

$$f_*\left(h(z)dz_K \wedge d\overline{z}_J\frac{\partial}{\partial z_i}\right) = h(z)dz_K \wedge d\overline{z}_J \sum_j \frac{\partial f_j(z)}{\partial z_i}\frac{\partial}{\partial w_j}.$$

and

$$f^* : A^{p,q}(V_i, \Theta_Y) \longrightarrow A_X^{p,q}(U_i, f^*\Theta_Y)$$

$$f^*\left(g(w)dw_K \wedge d\overline{w}_J \frac{\partial}{\partial w_i}\right) = g(f(z))\partial f_K \wedge \overline{\partial} f_J \frac{\partial}{\partial w_i}$$

where

$$\partial f_K = \sum_{h=1}^{n} \frac{\partial f_{k_1}}{\partial z_h} dz_h \wedge \cdots \wedge \sum_{h=1}^{n} \frac{\partial f_{k_p}}{\partial z_h} dz_h$$

and

$$\overline{\partial} f_J = \sum_{h=1}^{n} \frac{\partial f_{j_1}}{\partial \overline{z}_h} d\overline{z}_h \wedge \cdots \wedge \sum_{h=1}^{n} \frac{\partial f_{j_q}}{\partial \overline{z}_h} d\overline{z}_h;$$

We note that f^* and f_* commutes with ∂ and $\overline{\partial}$.

Moreover, for each k, there exists the following commutative diagrams

$$\begin{tikzcd}
\check{C}^q(\mathcal{V}, \Theta_Y) \arrow[r, "\phi"] \arrow[d, "f^*"] & A_Y^{0,q}(\Theta_Y) \arrow[d, "f^*"] \\
\check{C}^q(\mathcal{U}, f^*\Theta_Y) \arrow[r, "\phi"] & A_X^{0,q}(f^*\Theta_Y)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\check{C}^q(\mathcal{U}, \Theta_X) \arrow[r, "\phi"] \arrow[d, "f_*"] & A_X^{0,q}(\Theta_X) \arrow[d, "f_*"] \\
\check{C}^q(\mathcal{U}, f^*\Theta_Y) \arrow[r, "\phi"] & A_X^{0,q}(f^*\Theta_Y),
\end{tikzcd}$$

where ϕ is the map defined in Section II.3.1. Therefore $f_*\phi = \phi f_*$ and $f^*\phi = \phi f^*$.

II.6.1. LEMMA. *Let $f : X \longrightarrow Y$ be an holomorphic map of complex manifolds. Let $\chi \in \mathcal{A}_Y^{0,*}(\Theta_Y)$ and $\eta \in \mathcal{A}_X^{0,*}(\Theta_X)$ such that $f^*\chi = f_*\eta \in \mathcal{A}_X^{0,*}(f^*\Theta_Y)$. Then for each $\omega \in \mathcal{A}_Y^{*,*}$*

$$f^*(\chi \lrcorner \omega) = \eta \lrcorner f^*\omega.$$

PROOF. Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_i\}_{i\in I}$ be finite open Stein covers of X and Y, respectively, as above. For each $i \in I$, let z be local holomorphic coordinates system on U_i and w on V_i such that $f(z) = (f_1(z), \dots, f_m(z))$.

Let

$$\mathcal{A}_X^{0,r}(\Theta_X) \ni \eta = \sum_{i=1}^{n} h_i(z) d\overline{z}_I \frac{\partial}{\partial z_i}$$

and

$$\mathcal{A}_Y^{0,r}(\Theta_Y) \ni \chi = \sum_{h=1}^{m} \varphi_h(w) d\overline{w}_H \frac{\partial}{\partial w_h},$$

with $I = (1 \leq i_1 < i_2 < \cdots < i_r \leq n)$ and $H = (1 \leq h_1 < h_2 < \cdots < h_r \leq n)$ multi indexes of length r.

Therefore

$$f_*\eta = \sum_{i=1}^{n} h_i(z) d\overline{z}_I \sum_{h=1}^{m} \frac{\partial f_h}{\partial z_i} \frac{\partial}{\partial w_h} = \sum_{h=1}^{m} \left(\sum_{i=1}^{n} h_i(z) \frac{\partial f_h}{\partial z_i} d\overline{z}_I \right) \frac{\partial}{\partial w_h}$$

and

$$f^*\chi = \sum_{h=1}^{m} \varphi_h(f(z)) \overline{\partial} f_H \frac{\partial}{\partial w_h}.$$

By hypothesis $f^*\chi = f_*\eta \in \mathcal{A}_X^{0,*}(f^*\Theta_Y)$, then

$$\varphi_h(f(z)) \overline{\partial} f_H = \sum_{i=1}^{n} h_i(z) \frac{\partial f_h}{\partial z_i} d\overline{z}_I \qquad \forall\ h = 1, \ldots, m. \tag{7}$$

Now, let

$$\mathcal{A}^{p,q}(V_i) \ni \omega = g(w)\, dw_K \wedge d\overline{w}_J$$

with $K = (1 \leq k_1 < k_2 < \cdots < k_p \leq n)$ a multi index of length p and $J = (1 \leq j_1 < j_2 < \cdots < j_q \leq n)$ a multi index of length q. Then

$$f^*\omega = g(f(z))\, \partial f_K \wedge \overline{\partial} f_J.$$

and

$$\chi \lrcorner \omega = \sum_{h=1}^{m} \varphi_h(w) g(w)\, d\overline{w}_H \wedge \left(\frac{\partial}{\partial w_h} \lrcorner dw_K \right) \wedge d\overline{w}_J =$$

$$\sum_{h=1}^{p} (-1)^{p-1} \varphi_{k_h}(w) g(w)\, d\overline{w}_H \wedge dw_{K-\{k_h\}} \wedge d\overline{w}_J,$$

with $dw_{K-\{k_h\}} = dw_{k_1} \wedge \ldots \wedge \widehat{dw_{k_h}} \wedge \ldots \wedge dw_{k_p}$.

Therefore

$$f^*(\chi \lrcorner \omega) = \sum_{h=1}^{p} (-1)^{h-1} \varphi_{k_h}(f(z)) g(f(z))\, \overline{\partial} f_H \wedge \partial f_{K-\{k_h\}} \wedge \overline{\partial} f_J$$

and using (7) we get

$$f^*(\chi \lrcorner \omega) = \sum_{h=1}^{p} (-1)^{h-1} g(f(z)) \left(\sum_{i=1}^{n} h_i(z) \frac{\partial f_{k_h}}{\partial z_i} d\overline{z}_I \right) \wedge \partial f_{K-\{k_h\}} \wedge \overline{\partial} f_J.$$

On the other and

$$\eta \lrcorner f^*\omega =$$

$$\sum_{i=1}^{n} h_i(z) g(f(z))\, d\overline{z}_I \left(\frac{\partial}{\partial z_i} \lrcorner \partial f_K \right) \wedge \overline{\partial} f_J =$$

$$\sum_{i=1}^{n} h_i(z) g(f(z))\, d\overline{z}_I \wedge \left(\sum_{h=1}^{p} (-1)^{h-1} \frac{\partial f_{k_h}}{\partial z_i} \partial f_{K-\{k_h\}} \right) \wedge \overline{\partial} f_J =$$

$$f^*(\chi \lrcorner \omega).$$

$\square$

II.7. Deformations of complex manifolds

In this section we prove that the infinitesimal deformations of a complex compact manifold X are governed by

the differential graded Lie algebra of Kodaira-Spencer KS_X: that is $\mathrm{Def}_{KS_X} \cong \mathrm{Def}_X$.

We start with some lemmas and we postpone the proof in subSection II.7.1 where we also give an explicit description of the isomorphism (see Theorem II.7.3).

II.7.1. Lemma. *Let $A \in \mathbf{Art}$ and $x \in MC_{KS_X}(A)$, then there exists a cover $\mathcal{U} = \{U_i\}$ of X, such that $x_{|U_i} \sim 0$ for each i.*

Proof. By Proposition II.3.1, there exists a cover $\mathcal{U} = \{U_i\}$ such that $H^1(U_i, \Theta_X) = 0$ for each i. Moreover, by Remark I.3.31, $H^1(X, \Theta_X)$ is the tangent space of the deformation functor Def_{KS_X}. Therefore by Corollary I.1.21, Def_{KS_X} is locally trivial and so each $x \in MC_{KS_X}(A)$ is locally gauge equivalent to zero.

$\square$

In Section II.4 we have defined a morphism of sheaves

$$\boldsymbol{l} : \mathcal{A}_X^{0,*}(\Theta_X) \otimes A \longrightarrow Der^*(\mathcal{A}_X^{0,*} \otimes A, \mathcal{A}_X^{0,*} \otimes A)$$

$$a \longmapsto \boldsymbol{l}_a \quad \text{with} \quad \boldsymbol{l}_a(\omega) = (-1)^{dega} a \lrcorner \partial\omega.$$

Let $x \in MC_{KS_X}(A)$. Explicitly, in local holomorphic coordinates $z_1, z_2, \ldots, z_n$ if $x = \displaystyle\sum_{i,j} x_{ij} d\bar{z}_i \frac{\partial}{\partial z_j}$ and $\omega = f d\overline{z}_J$, then

$$\boldsymbol{l}_x(f) = -\sum_{i,j} x_{ij} \frac{\partial f}{\partial z_j} d\bar{z}_i \wedge d\overline{z}_J.$$

We also proved that for $x \in MC_{KS_X}(A)$

$$\overline{\partial} + \boldsymbol{l}_x : \mathcal{A}_X^{0,*} \otimes A \longrightarrow \mathcal{A}_X^{0,*+1} \otimes A$$

is a differential (Lemma II.5.4).

Define $\mathcal{O}_A(x)$ as the kernel of $\overline{\partial} + \boldsymbol{l}_x : \mathcal{A}_X^{0,0} \otimes A \longrightarrow \mathcal{A}_X^{0,1} \otimes A$. Then we have

$$0 \longrightarrow \mathcal{O}_A(x) \longrightarrow \mathcal{A}_X^{0,0} \otimes A \xrightarrow{\overline{\partial}+\boldsymbol{l}_x} \mathcal{A}_X^{0,1} \otimes A \xrightarrow{\overline{\partial}+\boldsymbol{l}_x} \cdots \xrightarrow{\overline{\partial}+\boldsymbol{l}_x} \mathcal{A}_X^{0,n} \otimes A \longrightarrow 0.$$

In Section II.4 we have also defined for each $s \in \mathcal{A}_X^{0,0}(\Theta_X) \otimes m_A$ an automorphism e^s of $\mathcal{A}_X^{0,*} \otimes A$.

II.7.2. Lemma. *Let $F, G : \mathbf{Art} \longrightarrow \mathbf{Set}$ be the following functors*

$$F(A) := \{\textit{isomorphisms of complexes } e^s : (\mathcal{A}_X^{0,*} \otimes A, \overline{\partial} + \boldsymbol{l}_x) \longrightarrow (\mathcal{A}_X^{0,*} \otimes A, \overline{\partial} + \boldsymbol{l}_y)$$
$$\textit{with } s \in \mathcal{A}_X^{0,0}(\Theta_X) \otimes m_A \textit{ that specialize to identity}\}$$

$$G(A) = \{\textit{isomorphisms of sheaves of } A\textit{-module } \psi : \mathcal{O}_A(x) \longrightarrow \mathcal{O}_A(y)$$
$$\textit{that specialize to identity}\}.$$

Then the restriction morphism $\phi : F \longrightarrow G$ is surjective.

PROOF. We proceed by induction on $d = dim_{\mathbb{C}} A$.

If $A = \mathbb{C}$, then $G(\mathbb{C}) = \{\text{ identity}\}$ and so it can be lifted.

Assume that $d \geq 2$ and let

$$0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0$$

be a small extension; by induction each element in $G(A)$ can be lifted to $F(A)$.
Let ψ be an isomorphism between $\mathcal{O}_B(x)$ and $\mathcal{O}_B(y)$ ($\psi \in G(B)$); we want to lift it to an isomorphism e^s.

$\alpha(x)$ and $\alpha(y)$ are in $MC_{KS_X}(A)$ and ψ induces an isomorphism of sheaves of A module $\overline{\psi} : \mathcal{O}_A(\alpha(x)) \longrightarrow \mathcal{O}_A(\alpha(y))$. Therefore, by induction hypothesis we can lift $\overline{\psi}$ to an isomorphism of complexes $e^{\overline{s}}$: i.e $e^{\overline{s}-1} \circ (\overline{\partial} + \alpha(x)) \circ e^{\overline{s}} = \overline{\partial} + \alpha(y)$ with $\overline{s} \in \mathcal{A}^{0,0}_X(\Theta_X) \otimes m_A$.

Then we can suppose that $\alpha(x) = \alpha(y) \in A^{0,1}(\Theta_X) \otimes m_A$ and that $e^{\overline{s}}$ is the identity.

This implies the existence of an element $p \in \mathcal{A}^{0,1}_X(\Theta_X) \otimes J$ such that $x = y + p$. Since x and y satisfy the Maurer-Cartan equation, then $\overline{\partial} p = 0$. In fact

$$0 = dx + \frac{1}{2}[x, x] = d(y + p) + \frac{1}{2}[y + p, y + p] = dy + dp + \frac{1}{2}[y, y] = dp.$$

Therefore, by Proposition II.3.1, there exists a Stein cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that p is locally $\overline{\partial}$-exact: i.e. for each $i \in I$ there exists $t_i \in \mathcal{A}^{0,0}_X(\Theta_X) \otimes J$ such that $\overline{\partial} t_i = p_{|U_i}$. Then

$$y_{|U_i} = (x - p)_{|U_i} = x_{|U_i} - \overline{\partial} t_i = e^{t_i} * x_{|U_i},$$

where we use the fact that $J \cdot m_B = 0$ as in Example I.3.27. In particular by Lemma II.5.5, $e^{t_i} : (\mathcal{A}^{0,*}(U_i) \otimes B, \overline{\partial} + \boldsymbol{l}_x) \longrightarrow (\mathcal{A}^{0,*}(U_i) \otimes B, \overline{\partial} + \boldsymbol{l}_y)$ is an isomorphism of complexes, that lifts the isomorphism $e^{t_i} : \mathcal{O}_B(x)(U_i) \longrightarrow \mathcal{O}_B(y)(U_i)$. We note that e^{t_i} restricts to identity on $\mathcal{O}_A(x)(U_i)$.

On the other side, by previous Lemma II.7.1, the Maurer-Cartan element x is locally gauge equivalent to zero. Then for each $i \in I$ there exists $a_i \in \mathcal{A}^{0,0}(U_i, \Theta_X) \otimes m_B$ such that $e^{a_i} * x_{|U_i} = 0$. As before, Lemma II.5.5 implies that $e^{a_i} : (\mathcal{A}^{0,*}(U_i) \otimes B, \overline{\partial} + \boldsymbol{l}_x) \longrightarrow (\mathcal{A}^{0,*}(U_i) \otimes B, \overline{\partial})$ is an isomorphism of complexes, that lifts the isomorphism $e^{a_i} : \mathcal{O}_B(x)(U_i) \longrightarrow \mathcal{O}_X(U_i) \otimes B$.

Now consider the following isomorphism

$$\varphi_{|U_i} : \mathcal{O}_X(U_i) \otimes B \longrightarrow \mathcal{O}_X(U_i) \otimes B$$

defined as follows:

$$\varphi_{|U_i} = e^{a_i} \circ e^{-t_i} \circ \psi_{|U_i} \circ e^{-a_i} :$$

$$\mathcal{O}_X(U_i) \otimes B \xrightarrow{e^{-a_i}} \mathcal{O}_B(x)(U_i) \xrightarrow{\psi_{|U_i}} \mathcal{O}_B(y)(U_i) \xrightarrow{e^{-t_i}} \mathcal{O}_B(x)(U_i) \xrightarrow{e^{a_i}} \mathcal{O}_X(U_i) \otimes B.$$

Then $\varphi_{|U_i}$ is an automorphism of $\mathcal{O}_X(U_i)\otimes B$ that restricts to identity on $\mathcal{O}_X(U_i)\otimes A$. Therefore Lemma I.2.6 implies the existence of $q_i \in \Gamma(U_i,\Theta_X)\otimes J$ such that $\varphi_{|U_i} = e^{q_i}$. In particular $e^{a_i}\circ e^{-t_i}\circ\psi_{|U_i}\circ e^{-a_i} = e^{q_i}$; by Remark I.2.7, the automorphism e^{q_i} commutes with the other automorphisms and so

$$\psi_{|U_i} = e^{t_i+q_i}.$$

Let $s_i = t_i + q_i \in \mathcal{A}^{0,0}(U_i,\Theta_X)\otimes J$, then $e^{s_i} = \psi_{|U_i}$ and so we have locally lifted the isomorphism ψ.

Now, we prove that the automorphisms e^{s_i} can be glued together to obtain an automorphism e^s of $\mathcal{A}_X^{0,*}\otimes A$ that lifts ψ. Consider the intersection U_{ij}, then the isomorphisms coincide on $\mathcal{O}_B(x)(U_{ij})$, i.e $e^{s_i}_{|U_{ij}} = \psi_{|U_{ij}} = e^{s_j}_{|U_{ij}} : \mathcal{O}_B(x)(U_{ij}) \longrightarrow \mathcal{O}_B(y)(U_{ij})$. Therefore the isomorphism $e^{s_i-s_j}$ is the identity on $\mathcal{O}_B(x)(U_{ij})$. Since the action of $\mathcal{A}_X^{0,0}(\Theta_X)\otimes m_B$ on $A_X^{0,0}(U_{ij})\otimes B$ is faithful on $\mathcal{O}_X(U_{ij})\otimes B$, it follows that $(s_i - s_j)_{|U_{ij}} = 0$.

□

II.7.1. KS_X governs the infinitesimal deformations of X. This section is devoted to prove that the Kodaira-Spencer algebra of a complex manifold X governs the infinitesimal deformations of X.

II.7.3. Theorem. *Let X be a complex compact manifold and KS_X its Kodaira-Spencer algebra. Then there exists an isomorphism of functors*

$$\gamma' : \mathrm{Def}_{KS_X} \longrightarrow \mathrm{Def}_X\,.$$

defined in the following way: given a local Artinian $\mathbb{C}$-algebra (A, m_A) and a solution of the Maurer-Cartan equation $x \in A_X^{0,1}(\Theta_X)\otimes m_A$ we set

$$\mathcal{O}_A(x) = \ker(\mathcal{A}_X^{0,0}\otimes A \xrightarrow{\overline{\partial}+\boldsymbol{\imath}_x} \mathcal{A}_X^{0,1}\otimes A),$$

and the map $\mathcal{O}_A(x)\longrightarrow\mathcal{O}_X$ is induced by the projection $\mathcal{A}_X^{0,0}\otimes A \longrightarrow \mathcal{A}_X^{0,0}\otimes\mathbb{C} = \mathcal{A}_X^{0,0}$.

II.7.4. Remark. As observed in subSection I.2.1, a deformation of X can be interpreted as a morphism of sheaves of algebras $\mathcal{O}_A \longrightarrow \mathcal{O}_X$ such that $\mathcal{O}_A$ is flat over A and $\mathcal{O}_A\otimes_A\mathbb{C}\longrightarrow\mathcal{O}_X$ is an isomorphism.

For this, the first part of the following proof consists of showing the A-flatness of $\mathcal{O}_A(x)$ and the isomorphism $\mathcal{O}_A(x)\otimes_A\mathbb{C}\cong\mathcal{O}_X$.

Proof. For each $(A, m_A) \in$ **Art** and $x \in MC_{KS_X}(A)$, we have defined

$$\mathcal{O}_A(x) = \ker(\mathcal{A}_X^{0,0}\otimes A \xrightarrow{\overline{\partial}+\boldsymbol{\imath}_x} \mathcal{A}_X^{0,1}\otimes A).$$

First of all, we observe that the projection π on the residue field $A/m_A \cong \mathbb{C}$ gives the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_A(x) \arrow[r] \arrow[d, "\pi"] & \mathcal{A}_X^{0,0}\otimes A \arrow[r, "\overline{\partial}"] \arrow[d, "\pi"] & \cdots \arrow[r, "\overline{\partial}"] & \mathcal{A}_X^{0,n}\otimes A \arrow[r] \arrow[d, "\pi"] & 0 \\
& \mathcal{O}_A(x)\otimes_A \mathbb{C} \arrow[r] \arrow[d] & \mathcal{A}_X^{0,0} \arrow[r, "\overline{\partial}"] \arrow[d, "id"] & \cdots \arrow[r, "\overline{\partial}"] & \mathcal{A}_X^{0,n} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{A}_X^{0,0} \arrow[r, "\overline{\partial}"] & \cdots \arrow[r, "\overline{\partial}"] & \mathcal{A}_X^{0,n} \arrow[r] & 0
\end{tikzcd}$$

Then π induces the morphism $\mathcal{O}_A(x) \longrightarrow \mathcal{O}_X$.

Using Lemma II.7.1, the Maurer-Cartan solution x is locally gauge equivalent to zero, therefore there exist a cover $\mathcal{U} = \{U_i\}$ and elements $a_i \in A^{0,0}(U_i, \Theta_X) \otimes m_A$ such that $e^{a_i} * x_{|U_i} = 0$, for each i. Therefore, by Lemma II.5.5, $e^{a_i} \circ (\overline{\partial} + \boldsymbol{l}_{x_{|U_i}}) \circ e^{-a_i} = e^{a_i} * (\overline{\partial} + \boldsymbol{l}_{x_{U_i}}) = \overline{\partial} + e^{a_i} * x = \overline{\partial}$ and so we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_A(x)(U_i) \arrow[r] \arrow[d, "e^{a_i}"] & \mathcal{A}_X^{0,0}(U_i)\otimes A \arrow[r, "\overline{\partial}+\boldsymbol{l}_{x_{|U_i}}"] \arrow[d, "e^{a_i}"] & \cdots \arrow[r, "\overline{\partial}+\boldsymbol{l}_{x_{|U_i}}"] & \mathcal{A}_X^{0,n}(U_i)\otimes A \arrow[r] \arrow[d, "e^{a_i}"] & 0 \\
0 \arrow[r] & \mathcal{O}_X(U_i)\otimes A \arrow[r] & \mathcal{A}_X^{0,0}(U_i)\otimes A \arrow[r, "\overline{\partial}"] & \cdots \arrow[r, "\overline{\partial}"] & \mathcal{A}_X^{0,n}(U_i)\otimes A \arrow[r] & 0,
\end{tikzcd}$$

where the vertical arrow are isomorphisms.

This implies that the deformation $\mathcal{O}_A(x)$ is locally trivial, i.e. $\mathcal{O}_A(x)(U_i) \cong \mathcal{O}_X(U_i) \otimes A$. Since $\mathcal{O}_X(U_i) \otimes A$ is flat over A, then $\mathcal{O}_A(x)(U_i)$ is also flat. Since flatness is a local property, $\mathcal{O}_A(x)$ is A-flat.

Using the isomorphism $\mathcal{O}_A(x)(U_i) \cong \mathcal{O}_X(U_i) \otimes A$ we can also conclude that $\mathcal{O}_A(x)(U_i) \otimes_A \mathbb{C} \cong \mathcal{O}_X(U_i)$ and so $\mathcal{O}_A(x) \otimes_A \mathbb{C} \longrightarrow \mathcal{O}_X$ is an isomorphism.

Then it is well defined the following morphism of functors of Artin rings

$$\gamma : \mathrm{MC}_{KS_X} \longrightarrow \mathrm{Def}_X$$

such that

$$\gamma(A) : \mathrm{MC}_{KS_X}(A) \longrightarrow \mathrm{Def}_X(A)$$

$$x \longmapsto \mathcal{O}_A(x).$$

Now, we prove that the deformations $\mathcal{O}_A(x)$ and $\mathcal{O}_A(y)$ are isomorphic if and only if $x, y \in \mathrm{MC}_{KS_X}(A)$ are gauge equivalent.

Actually, if $\mathcal{O}_A(x) \cong \mathcal{O}_A(y)$, applying Proposition II.7.2, we can lift the isomorphism

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_A(x) \arrow[r] \arrow[d, "\cong"] & \mathcal{A}_X^{0,0} \otimes A \arrow[r, "\overline{\partial}+\boldsymbol{l}_x"] \arrow[d, "e^s"] & \cdots \arrow[r, "\overline{\partial}+\boldsymbol{l}_x"] & \mathcal{A}_X^{0,n} \otimes A \arrow[r] \arrow[d, "e^s"] & 0 \\
0 \arrow[r] & \mathcal{O}_A(y) \arrow[r] & \mathcal{A}_X^{0,0} \otimes A \arrow[r, "\overline{\partial}+\boldsymbol{l}_y"] & \cdots \arrow[r, "\overline{\partial}+\boldsymbol{l}_y"] & \mathcal{A}_X^{0,n} \otimes A \arrow[r] & 0.
\end{tikzcd}$$

The commutativity of the diagram and Lemma II.5.5 imply that $\overline{\partial} + \boldsymbol{l}_y = e^{-s} \circ (\overline{\partial} + \boldsymbol{l}_x) \circ e^s = \overline{\partial} + e^s * \boldsymbol{l}_x$. Therefore $e^s * x = y$.

In conclusion, the map γ', induced by γ on $\mathrm{Def}_{KS_X} = MC_{KS_X}/gauge$, is a well defined injective morphism:

$$\gamma' : \mathrm{Def}_{KS_X} \hookrightarrow \mathrm{Def}_X\, .$$

To conclude that γ' is an isomorphism we prove that γ' is étale (and so γ' is surjective).

Using Corollary I.1.32, we need to prove that:

1) γ' induces a bijective map on the tangent spaces;

2) γ' induces an injective map on the obstruction spaces.

As regards Def_{KS_X}, by Remark I.3.31 the tangent space is isomorphic to $H^1_{\overline{\partial}}(X, \Theta_X)$ and Lemma I.3.23 implies that the obstructions are naturally contained in $H^2_{\overline{\partial}}(X, \Theta_X)$. As regards Def_X, Theorems I.2.8 and I.2.9 show that the tangent space is isomorphic to $\check{H}^1(X, \Theta_X)$ and the obstructions are naturally contained in $\check{H}^2(X, \Theta_X)$.

Then we will prove that the maps induced by γ' coincide with the Leray isomorphisms (see Theorem II.3.2 and Remark II.3.4).

1) Tangent Spaces. Let us prove that the map γ'_ε induced by γ' on the tangent space

$$\gamma'_\varepsilon : \mathrm{Def}_{KS_X}(\mathbb{C}[\varepsilon]) \longrightarrow \mathrm{Def}_X(\mathbb{C}[\varepsilon])$$

is the Leray isomorphism.

By Remark I.3.31, we have $\mathrm{Def}_{KS_X}(\mathbb{C}[\varepsilon]) = H^1(X, \Theta_X)$. Proceeding as in Remark II.3.4, there exists a Stein cover $\mathcal{U} = \{U_i\}$ so that we can associate to each $x \in \mathrm{Def}_{KS_X}(\mathbb{C}[\varepsilon])$ an element $[\sigma] = [\{\sigma_{ij} = (a_i - a_j)_{|U_{ij}}\}] \in \check{H}^1(X, \Theta_X) \otimes \mathbb{C}\varepsilon$, with $x_{|U_i} = \overline{\partial} a_i$, that doesn't depend on the choice of a_i.

Now, let $\gamma(x) = \mathcal{O}_{\mathbb{C}[\varepsilon]}(x)$ be the deformation associated to x, i.e.

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}[\varepsilon]}(x) \longrightarrow \mathcal{A}_X^{0,0} \otimes \mathbb{C}[\varepsilon] \xrightarrow{\overline{\partial}+\boldsymbol{l}_x} \mathcal{A}_X^{0,1} \otimes \mathbb{C}[\varepsilon] \longrightarrow \cdots\, .$$

As before, the deformation $\mathcal{O}_{\mathbb{C}[\varepsilon]}(x)$ is locally trivial; then there exists $b_i \in \mathcal{A}^{0,0}(U_i, \Theta_X) \otimes \mathbb{C}\varepsilon$ such that such that $e^{b_i} * x_{|U_i} = 0$ and so

$$\mathcal{O}_{\mathbb{C}[\varepsilon]}(x)(U_i) \xrightarrow[\cong]{e^{b_i}} \mathcal{O}_X(U_i) \otimes \mathbb{C}[\varepsilon].$$

Proceeding as in the proof of Theorem I.2.8, for each i and j

$$\varphi_{ij} := e^{b_i - b_j} : \mathcal{O}_X(U_{ij}) \otimes \mathbb{C}[\varepsilon] \longrightarrow \mathcal{O}_X(U_{ij}) \otimes \mathbb{C}[\varepsilon]$$

is an automorphism of the trivial deformation $\mathcal{A}_X^{0,0}(U_{ij}) \otimes \mathbb{C}[\varepsilon]$ that restricts to the identity.

Applying Lemma I.2.6, the class $[\{\tau_{ij}\}] = [\{(b_i - b_j)_{|U_{ij}}\}] \in \check{H}^1(X, \Theta_X) \otimes \mathbb{C}\varepsilon$ is the Check 1-cocycle associated to the deformation $\gamma'_\varepsilon(x)$.

Since $e^{b_i} * x_{|U_i} = 0$, then $\overline{\partial} b_i = x_{|U_i} = \overline{\partial} a_i$ and so $b_i = a_i + c_i$ with $c_i \in \Gamma(U_i, \Theta_X) \otimes \mathbb{C}\varepsilon$.

Therefore $[\tau_{ij}] = [\{(b_i - b_j)_{|U_{ij}}\}] = [\sigma]$. This shows that γ'_ε coincides with the Leray isomorphism.

2) Obstruction

Let

$$0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0$$

be a small extension.

First we consider the obstruction class $[h]$ of $x \in \mathrm{Def}_{KS_X}$.

Let $x \in \mathrm{Def}_{KS_X}(A)$, and $\tilde{x} \in {KS_X}^1 \otimes m_B$ be a lifting of x. The obstruction class associated to x is $[h] \in H^2(KS_X) \otimes J$ with

$$h = \overline{\partial}\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}]$$

and this class doesn't depend on the choice of the lifting $\tilde{x}$ as we show in Lemma I.3.23.

Proceeding as in Remark II.3.4, there exists a Stein cover $\mathcal{U} = \{U_i\}$ such that the class $[\{\alpha_{ijk}\}] = [\{\rho_{jk} - \rho_{ik} + \rho_{ij}\}] \in \check{H}^2(X, \Theta_X) \otimes J$ is the class associated to h by the Leray isomorphism, where $h_{|U_i} = \overline{\partial}\tau_i$ and $\overline{\partial}\rho_{ij} = (\tau_i - \tau_j)_{|U_{ij}}$. In particular we note that $\rho_{ij} \in \Gamma(U_{ij}, \mathcal{A}^{0,0}(\Theta_X)) \otimes J$.

Since $h_{|U_i} = \overline{\partial}\tau_i$, x can be locally lifted to a solution of the Maurer-Cartan equation

$$\overline{x}_i = \tilde{x}_{|U_i} - \tau_i \in A^{0,1}(U_i, \Theta_X) \otimes m_B.$$

In fact on U_i, we have

$$\alpha(\overline{x}) = \alpha(\tilde{x}) = x \quad \text{and} \quad \overline{\partial}\overline{x} + \frac{1}{2}[\overline{x}, \overline{x}] = \overline{\partial}\tilde{x} - \overline{\partial}\tau_i + \frac{1}{2}[\tilde{x}, \tilde{x}] = h_{|U_i} - \overline{\partial}\tau_i = 0.$$

Moreover, $e^{\rho_{ij}} * \overline{x}_{j|U_{ij}} = \overline{x}_{i|U_{ij}}$, in fact (see Example I.3.27) we have

$$e^{\rho_{ij}} * \overline{x}_{j|U_{ij}} = e^{\rho_{ij}} * (\tilde{x} - \tau_j)_{|U_{ij}} = (-\overline{\partial}\rho_{ij} + \tilde{x} - \tau_j)_{|U_{ij}} = (\tilde{x} - \tau_i)_{|U_{ij}} = \overline{x}_{i|U_{ij}}.$$

As above, x is locally equivalent to zero therefore for each i there exists $a_i \in A^{0,0}(U_i, \Theta_X) \otimes m_A$, such that $e^{a_i} * x_{|U_i} = 0$.

Analogously, for each i, there exists $b_i \in A^{0,0}(U_i, \Theta_X) \otimes m_B$ that is a lifting of a_i such that

$$e^{b_i} * \overline{x}_i = 0.$$

Now, let $\gamma'(x) = \mathcal{O}_A(x)$ be the deformation of X induced by x. As above the deformation is locally trivial and so there exist a cover $\mathcal{U} = \{U_i\}$ and $a_i \in \mathcal{A}^{0,0}(U_i, T_X) \otimes m_A$ such that

$$\mathcal{O}_A(x)(U_i) \overset{e^{a_i}}{\cong} \mathcal{O}_X(U_i) \otimes A$$

Let φ_{ij} the following isomorphism

$$\varphi_{ij} : \mathcal{O}_X(U_{ij}) \otimes A \xrightarrow{e^{-a_j}} \mathcal{O}_A(x)(U_{ij}) \xrightarrow{e^{a_i}} \mathcal{O}_X(U_{ij}) \otimes A.$$

Now, proceeding as in the proof of Theorem I.2.9, since $b_i \in \mathcal{A}^{0,0}(U_i, \Theta_X) \otimes m_B$ are liftings of a_i, then $\tilde{\varphi_{ij}} = e^{-b_j} e^{\rho_{ij}} e^{b_i} \in \mathrm{Aut}(\mathcal{O}_X(U_{ij}) \otimes B)$ defined as

$$\tilde{\varphi}_{ij} : \mathcal{O}_X(U_{ij}) \otimes B \xrightarrow{e^{-b_j}} \mathcal{O}_B(\overline{x}_j)(U_{ij}) \xrightarrow{e^{\rho_{ij}}} \mathcal{O}_B(\overline{x}_i)(U_{ij}) \xrightarrow{e^{b_i}} \mathcal{O}_X(U_{ij}) \otimes B,$$

is a lifting of φ_{ij}.

By remarkI.2.7 the automorphisms $e^{\rho_{ij}}$, $e^{\rho_{ik}}$ and $e^{\rho_{jk}}$ commutes with the other automorphisms. Then $\Phi_{ijk} = \tilde{\varphi_{jk}} \tilde{\varphi_{ik}}^{-1} \tilde{\varphi_{ij}} = e^{\rho_{jk} - \rho_{ik} + \rho_{ij}}$ is an automorphism of the trivial deformation that restricts to identity ($\Phi_{ijk|\mathcal{O}_X(U_{ijk}) \otimes A} = id$). Therefore by Lemma I.2.6 the element $[\{\alpha_{ijk}\}] = [\{\rho_{jk} - \rho_{ik} + \rho_{ij}\}] \in \check{H}^2(\mathcal{U}, \Theta_X) \otimes J$ is the obstruction class associated to $\gamma'(x)$. In conclusion also in the obstruction case, the map induced by γ' coincides with the Leray isomorphism.

□

II.7.2. Deformations of a product. As an application of the previous Theorem II.7.3 we study the deformation of a product of compact complex manifolds X and Y. The following remark will be used in Section IV.2.

II.7.5. Remark. In general not all the deformations of the product $X \times Y$ are products of deformations of X and of Y.

The first example in this way was given by Kodaira and Spencer in their work ([**17**, pag. 436]) when they showed one of the first example of obstructed varieties. More precisely, they considered the product of the projective line and the complex tori of dimension $q \geq 2$ $\mathbb{P}^1 \times \mathbb{C}^q/G$ and they proved that it is obstructed thought the two manifolds are unobstructed.

A sufficient and necessary condition to have an isomorphism between products of deformations and deformations of the product is given by the lemma below.

II.7.6. Lemma. *The morphism*

$$F : \mathrm{Def}_{KS_X} \times \mathrm{Def}_{KS_Y} \longrightarrow \mathrm{Def}_{KS_{X \times Y}}$$

is an isomorphism if and only if $H^1(\mathcal{O}_X) \otimes H^0(\Theta_Y) = H^1(\mathcal{O}_Y) \otimes H^0(\Theta_X) = 0$.

Proof. Let p and q the natural projections of the product $X \times Y$ respectively on X and Y. Consider the morphism of DGLA

$$F : KS_X \times KS_Y \longrightarrow KS_{X\times Y}$$
$$(n_1, n_2) \longmapsto p^*n_1 + q^*n_2$$

and denote by $H^i(F)$ the induced map on cohomology.

By Theorem I.3.33 if $H^0(F)$ is surjective, $H^1(F)$ is bijective and $H^2(F)$ is injective then F induces an isomorphism of deformation functors.

$H^0(KS_{X\times Y}) = H^0(\Theta_{X\times Y})$ and so by the Kunneth formula $H^0(KS_{X\times Y}) = H^0(\Theta_X) \oplus H^0(\Theta_Y) = H^0(KS_X \times KS_Y)$. This implies that $H^0(F)$ is surjective.

Again by Kunneth formula we get

$$H^1(KS_{X\times Y}) \cong H^1(\Theta_{X\times Y}) \cong$$
$$H^1(\Theta_X) \oplus H^1(\Theta_Y) \oplus H^1(\mathcal{O}_X) \otimes H^0(\Theta_Y) \oplus H^1(\mathcal{O}_Y) \otimes H^0(\Theta_X),$$

and

$$H^1(KS_X \times KS_Y) = H^1(\Theta_X) \oplus H^1(\Theta_Y).$$

Then the hypothesis imply that $H^1(F)$ is bijective.

Finally, reasoning as above we get that

$$H^2(F) : H^2(\Theta_X) \oplus H^2(\Theta_Y) \longrightarrow H^2(\Theta_{X\times Y})$$

is injective. This implies that F induce an isomorphism of deformation functors.

On the other hand if F is an isomorphism of deformation functors, then $H^1(F)$ is a bijection on the tangent spaces and so $H^1(\mathcal{O}_X) \otimes H^0(\Theta_Y) = H^1(\mathcal{O}_Y) \otimes H^0(\Theta_X) = 0$. □

CHAPTER III

Deformation functor of a couple of morphisms of DGLAs

In this chapter we give the key definition of deformation functor associated to a couple of morphisms of differential graded Lie algebras.

In the first section we define the (non extended) functors of Artin rings $\mathrm{MC}_{(h,g)}$ (Definition III.1.7) and $\mathrm{Def}_{(h,g)}$ (Definition III.1.11) associated to a couple $h : L \longrightarrow M$ and $g : N \longrightarrow M$. These functor will play an important role in the infinitesimal deformations of holomorphic maps of next chapter.

Then Section III.2 is devoted to introduce the extended deformation functors (Definition III.2.4). In particular we define the functors $\widetilde{\mathrm{MC}}_{(h,g)}$ and $\widetilde{\mathrm{Def}}_{(h,g)}$ that are a generalization of the previous $\mathrm{MC}_{(h,g)}$ and $\mathrm{Def}_{(h,g)}$.

We introduce the extended functors, since using their properties, we can show the existence of a DGLA $\mathrm{H}_{(h,g)}$ such that $\mathrm{Def}_{\mathrm{H}_{(h,g)}} \cong \mathrm{Def}_{(h,g)}$ (Theorem III.2.36).

III.1. Functors $\mathrm{MC}_{(h,g)}$ and $\mathrm{Def}_{(h,g)}$

In this section we introduce the functors $\mathrm{MC}_{(h,g)}$ and $\mathrm{Def}_{(h,g)}$ (Section III.1.2) associated to a couple $h : L \longrightarrow M$ and $g : N \longrightarrow M$ and we study some properties (Section III.1.3). First of all we recall the definition of the mapping cone associate to morphisms of complexes.

Beware. In this section, we suppose that M is concentrated in non negative degree. Since in the main application M will be the Kodaira-Spencer algebra of a manifold, this extra hypothesis is not restrictive. Anyway, in Section III.2 we will remove this hypothesis.

III.1.1. The mapping cone of a couple of morphisms. The *suspension of the mapping cone of a morphism* of complexes $h : (L, d) \longrightarrow (M, d)$ is the differential graded vector space $(C_h^{\cdot}, \delta)$, where

$$C_h^i = L^i \oplus M^{i-1}$$

and the differential δ is

$$\delta(l, m) = (dl, -dm + h(l)) \in L^{i+1} \oplus M^i \qquad \forall\, (l, m) \in L^i \oplus M^{i-1}.$$

Actually $\delta^2(l, m) = \delta(dl, -dm + h(l)) = (d^2 l, d^2 m - d(h(l)) + h(d(l))) = (0, 0)$.

III.1.1. REMARK. Consider the projection $\pi : M \longrightarrow \operatorname{coker}(h)$. Then there exists a morphism of complexes

$$\varphi : (C_h^{\cdot}, \delta) \longrightarrow (\operatorname{coker}(h)[-1], d[-1])$$

with

$$\varphi(l,m) = \pi(m) \qquad \forall (l,m) \in C_h^i.$$

Actually

$$\begin{tikzcd} (l,m) \arrow[r, "\varphi"] \arrow[d, "\delta"'] & \pi(m) \arrow[d, "d[-1]"] \\ (dl, -dm + h(l)) \arrow[r, "\varphi"] & -d\pi(m). \end{tikzcd}$$

If h is injective then $(C_h^{\cdot}, \delta)$ and $(\operatorname{coker}(h)[-1], d[-1])$ are quasi isomorphic.

Now suppose that $h : (L,d) \longrightarrow (M,d)$ and $g : (N,d) \longrightarrow (M,d)$ are morphism of complexes:

$$\begin{tikzcd} & L \arrow[d, "h"] \\ N \arrow[r, "g"] & M. \end{tikzcd}$$

III.1.2. DEFINITION. The *suspension of the mapping cone of a couple of morphisms* (h,g) is the differential graded vector space $(\mathrm{C}^{\cdot}_{(h,g)}, D)$, where

$$\mathrm{C}^i_{(h,g)} = L^i \oplus N^i \oplus M^{i-1}$$

and the differential D is

$$L^i \oplus N^i \oplus M^{i-1} \ni (l,n,m) \xrightarrow{D} (dl, dn, -dm - g(n) + h(l)) \in L^{i+1} \oplus N^{i+1} \oplus M^i.$$

Actually, $D^2(l,n,m) = D(dl, dn, -dm - g(n) + h(l)) = (d^2 l, d^2 n, d^2 m + dg(n) - dh(l) - g(dn) + h(dl)) = (0,0,0)$.

III.1.3. REMARK. By definition, $(\mathrm{C}^{\cdot}_{(h,g)}, D)$ coincides with the suspended mapping cone associated to the morphism of complexes $h - g : L \oplus N \longrightarrow M$.

Moreover, the projection $\mathrm{C}^{\cdot}_{(h,g)} \longrightarrow L^{\cdot} \oplus N^{\cdot}$ is a morphism of complexes and so there exists the following exact sequence

$$0 \longrightarrow (M^{\cdot -1}, -d) \longrightarrow (C^{\cdot}_{(h,g)}, D) \longrightarrow (L^{\cdot} \oplus N^{\cdot}, d) \longrightarrow 0$$

that induces

(8)
$$\cdots \longrightarrow H^i(\mathrm{C}^{\cdot}_{(h,g)}) \longrightarrow H^i(L^{\cdot} \oplus N^{\cdot}) \longrightarrow H^i(M^{\cdot}) \longrightarrow H^{i+1}(\mathrm{C}^{\cdot}_{(h,g)}) \longrightarrow \cdots.$$

III.1.4. REMARK. The complexes $\mathrm{C}^{\cdot}_{(h,g)}$ and $\mathrm{C}^{\cdot}_{(g,h)}$ are isomorphic. Actually, let $\gamma : \mathrm{C}^{\cdot}_{(h,g)} \longrightarrow \mathrm{C}^{\cdot}_{(g,h)}$ defined as

$$\mathrm{C}^i_{(h,g)} \ni (l,n,m) \overset{\gamma}{\longmapsto} (-l, -n, m) \in \mathrm{C}^i_{(g,h)}.$$

Then

$$\begin{tikzcd}
(l,n,m) \arrow[r, "\gamma"] \arrow[d, "D"'] & (-l,-n,m) \arrow[d, "\delta"] \\
(dl, dn, -dm - g(n) + h(l)) \arrow[r, "\gamma"] & (-dl, -dn, -dm - g(n) + h(l)).
\end{tikzcd}$$

and so γ is a well defined morphism of complexes that is a quasi-isomorphism ($\gamma^2 = id$).

III.1.5. Lemma. *If $h : L \longrightarrow M$ is injective: i.e. there exists the exact sequence of complexes*

$$0 \longrightarrow L \xrightarrow{h} M \xrightarrow{\pi} \operatorname{coker}(h) \longrightarrow 0.$$

Then $(\mathrm{C}^{\cdot}_{(h,g)}, D)$ is quasi isomorphic to $(C^{\cdot}_{\pi\circ g}, \delta)$.

Proof. Let $\gamma : \mathrm{C}^{\cdot}_{(h,g)} \longrightarrow C^{\cdot}_{\pi\circ g}$ defined as

$$\mathrm{C}^i_{(h,g)} \ni (l,n,m) \overset{\gamma}{\longmapsto} (-n, \pi(m)) \in C^i_{\pi\circ g},$$

then

$$\begin{tikzcd}
(l,n,m) \arrow[r, "\gamma"] \arrow[d, "D"'] & (-n, \pi(m)) \arrow[d, "\delta"] \\
(dl, dn, -dm - g(n) + h(l)) \arrow[r, "\gamma"] & (-dn, -d\pi(m) - \pi(g(n))).
\end{tikzcd}$$

Therefore γ is a well defined morphism of complexes and we denote by the same γ the map induced in cohomology.

The fact that the induced morphism γ is an isomorphism in cohomology is an easy calculation but we state it for completeness.

γ is injective. Suppose that $\gamma[(l,n,m)] = [(-n, \pi(m))]$ is zero in $H^i(C^{\cdot}_{\pi\circ g})$. Then $dl = dn = -dm - g(n) + l = 0$ and there exists $(b,c) \in N^{i-1} \times \operatorname{coker}(h)^{i-2}$ such that $-n = db$ and $\pi(m) = -dc + \pi\circ g(b)$. Let $m' \in M$ be a lifting of c, i.e. $\pi(m') = c$, $n' = -b$ and $l' = m + dm' + g(n')$. Then $l' \in L$, actually $\pi(l') = \pi(m) + \pi(dm') + \pi\circ g(n') = -dc + \pi\circ g(b) + dc - \pi\circ g(b) = 0$. Therefore $D(l', n', m') = (dl', dn', -dm' - g(n') + l') = (dm - g(db), -db, -dm' + g(b) + (m + dm' + g(n'))) = (l,n,m)$.

γ is surjective. Let $[(n,t)] \in H^i(C^{\cdot}_{\pi\circ g})$; then $dn = 0$ and $-dt + \pi\circ g(n) = 0$. Let $m \in M^{i-1}$ be a lifting of t, i.e. $\pi(m) = t$ and $l = -g(n) + dm \in M^i$. Then $l \in L^i$, in fact $\pi(l) = -\pi\circ g(n) + dt = 0$. Moreover $(l, -n, m) \in H^i(\mathrm{C}^{\cdot}_{(h,g)})$ ($dl = dn = 0$ and $-dm + g(n) + l = 0$) and $\gamma[(l,-n,m)] = [n, \pi(m)] = [(n,t)]$.

$\square$

III.1.6. Remark. If L, M and N are DGLA and $h : L \longrightarrow M$ and $g : N \longrightarrow M$ are morphisms of DGLA. Also in this case we can't define a canonical DGLA structure on $C^{\cdot}_{(h,g)}$ such that the projection $C^{\cdot}_{(h,g)} \longrightarrow L^{\cdot} \oplus N^{\cdot}$ is a morphism of DGLA.

III.1.2. Definition of $\mathrm{MC}_{(h,g)}$ and $\mathrm{Def}_{(h,g)}$.

III.1.7. Definition. Let $h : L \longrightarrow M$ and $g : N \longrightarrow M$ be morphisms of differential graded Lie algebras:

$$\begin{tikzcd} & L \arrow[d, "h"] \\ N \arrow[r, "g"] & M. \end{tikzcd}$$

For each $(A, m_A) \in \mathbf{Art}$ the *Maurer-Cartan functor associated to the couple* (h, g) is defined as follows

$$\mathrm{MC}_{(h,g)} : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$\mathrm{MC}_{(h,g)}(A) = \{(x, y, e^p) \in (L^1 \otimes m_A) \times (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A)|$$
$$dx + \frac{1}{2}[x, x] = 0,\ dy + \frac{1}{2}[y, y] = 0,\ g(y) = e^p * h(x)\}.$$

III.1.8. Remark. In [**24**, Section 2],M. Manetti defined the functor MC_h associated to a morphism $h : L \longrightarrow M$ of DGLAs:

$$\mathrm{MC}_h : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$\mathrm{MC}_h(A) =$$

$$\{(x, e^p) \in (L^1 \otimes m_A) \times \exp(M^0 \otimes m_A)|\ dx + \frac{1}{2}[x, x] = 0,\ e^p * h(x) = 0\}.$$

Therefore if we take $N = 0$ and $g = 0$, the new functor $\mathrm{MC}_{(h,g)}$ reduce to the old one MC_h.

Choosing $N = 0$ and $h = g = 0$, $\mathrm{MC}_{(h,g)}$ reduces to the Maurer-Cartan functor MC_L associated to the DGLA L (Definition I.3.16).

III.1.9. Remark. As in the case of a differential graded Lie algebra (see Remark I.3.21), $\mathrm{MC}\,(h, g)$ is an *homogeneous* functor, since $\mathrm{MC}_{(h,g)}(B \times_A C) \cong \mathrm{MC}_{(h,g)}(B) \times_{\mathrm{MC}_{(h,g)}(A)} \mathrm{MC}_{(h,g)}(C)$.

As in the case of a differential graded Lie algebra (see Definition I.3.25), we can define for each $(A, m_A) \in \mathbf{Art}$ a gauge action over $\mathrm{MC}_{(h,g)}(A)$.

III.1.10. Definition. The gauge action of $\exp(L^0 \otimes m_A) \times \exp(N^0 \otimes m_A)$ over $\mathrm{MC}_{(h,g)}(A)$ is given by:

$$(e^a, e^b) * (x, y, e^p) = (e^a * x, e^b * y, e^{g(b)} e^p e^{-h(a)}).$$

This is well defined since

$$e^{g(b)} e^p e^{-h(a)} * h(e^a * x) = e^{g(b)} e^p * h(x) = e^{g(b)} * g(y) = g(e^b * y).$$

and so $(e^a, e^b) * (x, y, e^p) \in \mathrm{MC}_{(h,g)}(A)$.

In conclusion, it makes sense to consider the following functor.

III.1.11. DEFINITION. The *deformation functor associated to a couple* (h,g) of morphisms of differential graded Lie algebras is:

$$\mathrm{Def}_{(h,g)} : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$\mathrm{Def}_{(h,g)}(A) = \frac{\mathrm{MC}_{(h,g)}(A)}{\exp(L^0 \otimes m_A) \times \exp(N^0 \otimes m_A)}$$

III.1.12. REMARK. In [**24**, Section 2],M. Manetti defined the functor Def_h associated to a morphism $h : L \longrightarrow M$ of DGLAs:

$$\mathrm{Def}_h : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$\mathrm{Def}_h(A) = \frac{\mathrm{MC}_h(A)}{\exp(L^0 \otimes m_A) \times \exp(dM^{-1} \otimes m_A)},$$

with the gauge action of $\exp(L^0 \otimes m_A) \times \exp(dM^{-1} \otimes m_A)$ given by the formula

$$(e^a, e^{dm}) * (x, e^p) = (e^a * x, e^{dm} e^p e^{-h(a)}) \qquad \forall\, a \in L^0 \otimes m_A, m \in M^{-1} \otimes m_A.$$

Therefore if we take $N = 0$ and $g = 0$, the new functor $\mathrm{Def}_{(h,g)}$ reduce to the old one Def_h (M is concentrated in non negative degree).

Choosing $N = M = 0$ and $h = g = 0$, $\mathrm{Def}_{(h,g)}$ reduces to the Maurer-Cartan functor Def_L associated to the DGLA L.

The name deformation functor is justified by the theorem below.

III.1.13. THEOREM. $\mathrm{Def}_{(h,g)}$ *satisfies the conditions of Definition I.1.10.*

PROOF. If $C = \mathbb{K}$, then $\mathrm{Def}_{(h,g)}(\mathbb{K}) = \{\text{one element}\}$; therefore $ii)$ of Definition I.1.10 holds, i.e. $\mathrm{Def}_{(h,g)}(A \times B) = \mathrm{Def}_{(h,g)}(A) \times \mathrm{Def}_{(h,g)}(B)$.

Let $\beta : B \longrightarrow A$ and $\gamma : C \longrightarrow A$ be morphisms in **Art** and $(v, w) \in \mathrm{Def}_{(h,g)}(B) \times_{\mathrm{Def}_{(h,g)}(A)} \mathrm{Def}_{(h,g)}(C)$. Then we are looking for a lifting $z \in \mathrm{Def}_{(h,g)}(B \times_A C)$, whenever $\beta : B \longrightarrow A$ is surjective.

Let $(x, y, e^p) \in \mathrm{MC}_{(h,g)}(B)$ and $(s, t, e^r) \in \mathrm{MC}_{(h,g)}(C)$ liftings for v and w respectively.

By hypothesis $\beta(v) = \gamma(w) \in \mathrm{Def}_{(h,g)}(A)$; therefore $\beta(x, y, e^p)$ and $\gamma(s, t, e^r)$ are gauge equivalent in $\mathrm{MC}_{(h,g)}(A)$: i.e. there exist $a \in L^0 \otimes m_A$ and $b \in N^0 \otimes m_A$ such that

$$e^a * \beta(x) = \gamma(s) \qquad e^b * \beta(y) = \gamma(t) \qquad e^{g(b)} e^{\beta(p)} e^{-h(a)} = e^{\gamma(r)}.$$

Let $c \in L^0 \otimes m_B$ such that $\beta(c) = a$ and $d \in N^0 \otimes m_B$ such that $\beta(d) = b$.

Up to substitute (x, y, e^p) with the gauge equivalent element $(e^c * x, e^d * y, e^{g(d)} e^p e^{-h(c)})$ (they both lift v), we can assume that[a] $\beta(x, y, e^p) = \gamma(s, t, e^r) \in \mathrm{MC}_{(h,g)}(A)$.

[a] $\beta(e^c * x, e^d * y, e^{g(d)} e^p e^{-h(c)}) = (e^{\beta(c)} * \beta(x), e^{\beta(d)} * \beta(y), e^{g(\beta(d))} e^p e^{-h(\beta(c))}) = (e^a * \beta(x), e^b * \beta(y), e^{g(b)} e^{\beta(p)} e^{-h(a)}) = \gamma(s, t, e^r)$

Since $\mathrm{MC}_{(h,g)}(B \times_A C) = \mathrm{MC}_{(h,g)}(B) \times_{\mathrm{MC}_{(h,g)}(A)} \mathrm{MC}_{(h,g)}(C)$, it is well defined a lifting $z \in \mathrm{MC}_{(h,g)}(B \times_A C)$ and so it is sufficient to take its class $[z] \in \mathrm{Def}_{(h,g)}(B \times_A C)$.

□

III.1.14. Remark. Consider the functor $\mathrm{Def}_{(h,g)}$. Then the projection ϱ on the second factor:

$$\varrho : \mathrm{Def}_{(h,g)} \longrightarrow \mathrm{Def}_N$$

$$\mathrm{Def}_{(h,g)}(A) \ni (x, y, e^p) \xrightarrow{\varrho} y \in \mathrm{Def}_N(A)$$

is a morphism of deformation functors.

III.1.15. Remark. If the morphism h *is injective*, then for each $(A, m_A) \in \mathbf{Art}$ the functor $\mathrm{MC}_{(h,g)}$ has the following form:

$$\mathrm{MC}_{(h,g)}(A) = \{(x, e^p) \in (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A)|$$

$$dx + \frac{1}{2}[x, x] = 0,\ e^{-p} * g(x) \in L^1 \otimes m_A\}.$$

In this case the gauge equivalence $\sim$ is given by

$$(x, e^p) \sim (e^b * x, e^{g(b)} e^p e^a), \qquad \text{with } a \in L^0 \otimes m_A \text{ and } b \in N^0 \otimes m_A.$$

III.1.16. Lemma. *The projection* $\pi : \mathrm{MC}_{(h,g)} \longrightarrow \mathrm{Def}_{(h,g)}$ *is a smooth morphism of functors.*

Proof. Let $\alpha : B \longrightarrow A$ be a surjection in $\mathbf{Art}$ and prove that

$$\mathrm{MC}_{(h,g)}(B) \longrightarrow \mathrm{Def}_{(h,g)}(B) \times_{\mathrm{Def}_{(h,g)}(A)} \mathrm{MC}_{(h,g)}(A)$$

is surjective.

Let $((x, y, e^p), (l, n, e^m)) \in \mathrm{Def}_{(h,g)}(B) \times_{\mathrm{Def}_{(h,g)}(A)} \mathrm{MC}_{(h,g)}(A)$, that is the class of (l, n, m) and $\alpha(x, y, e^p)$ are the same element in $\mathrm{Def}_{(h,g)}(A)$.

Then there exists $(a, b) \in \exp(L^0 \otimes m_A) \times \exp(N^0 \otimes m_A)$ such that

$$(l, n, e^m) = (e^a, e^b) * (\alpha(x, y, e^p)) = (e^a * \alpha(x), e^b * \alpha(y), e^{g(b)} e^{\alpha(p)} e^{-h(a)}).$$

Let $c \in L^0 \otimes m_B$ be a lifting of a and $d \in N^0 \otimes m_B$ be a lifting of b. Then

$$t = (e^c * x, e^d * y, e^{g(d)} e^p e^{-h(c)})$$

lies in $\mathrm{MC}_{(h,g)}(B)$ and it is a lifting of $((x, y, e^p), (l, n, e^m))$.

Actually, t is gauge equivalent to (x, y, e^p) and

$$\alpha(t) = (e^{\alpha(c)} * \alpha(x), e^{\alpha(d)} * \alpha(y), e^{g(\alpha(d))} e^{\alpha(p)} e^{-h(\alpha(c))}) =$$

$$(e^a * \alpha(x), e^b * \alpha(y), e^{g(b)} e^{\alpha(p)} e^{-h(a)}) = (l, n, e^m).$$

□

III.1.3. Tangent and obstructions spaces of $\mathrm{MC}_{(h,g)}$ and $\mathrm{Def}_{(h,g)}$. By Definition III.1.7, *the tangent space of* $\mathrm{MC}_{(h,g)}$ is

$$\mathrm{MC}_{(h,g)}(\mathbb{K}[\varepsilon]) =$$
$$= \{(x,y,e^p) \in (L^1 \otimes \mathbb{K}\varepsilon) \times (N^1 \otimes \mathbb{K}\varepsilon) \times \exp(M^0 \otimes \mathbb{K}\varepsilon)|$$
$$dx = dy = 0, h(x) - g(y) - dp = 0\}$$
$$\cong \{(x,y,p) \in L^1 \times N^1 \times M^0 |\ dx = dy = 0, g(y) = h(x) - dp\} =$$
$$\ker(D : \mathrm{C}^1_{(h,g)} \longrightarrow \mathrm{C}^2_{(h,g)}).$$

By Definition III.1.11, *the tangent space of* $\mathrm{Def}_{(h,g)}$ is

$$\mathrm{Def}_{(h,g)}(\mathbb{K}[\varepsilon]) =$$
$$\{(x,y,e^p) \in (L^1 \otimes \mathbb{K}\varepsilon) \times (N^1 \otimes \mathbb{K}\varepsilon) \times (\exp M^0 \otimes \mathbb{K}\varepsilon)|$$
$$dx = dy = 0, g(y) = h(x) - dp\}\ /$$
$$\{(-da, -db, g(b) - h(a))|\ a \in (L^0 \otimes \mathbb{K}[\varepsilon]), b \in (N^0 \otimes \mathbb{K}[\varepsilon])\}$$
$$\cong H^1(\mathrm{C}^{\cdot}_{(h,g)})$$

III.1.17. Remark. In the last equality we use the extra hypothesis that $M^{-1} = 0$.

About the obstruction space of $\mathrm{MC}_{(h,g)}$, we prove below that it is naturally contained in $H^2(\mathrm{C}^{\cdot}_{(h,g)})$. Since $\pi : \mathrm{MC}_{(h,g)} \longrightarrow \mathrm{Def}_{(h,g)}$ is a smooth, then Corollary I.1.35 implies that the obstructions space of $\mathrm{Def}_{(h,g)}$ is also contained in $H^2(\mathrm{C}^{\cdot}_{(h,g)})$.

III.1.18. Lemma. *$H^2(\mathrm{C}^{\cdot}_{(h,g)})$ is a complete obstruction space for $\mathrm{MC}_{(h,g)}$.*

Proof. Let
$$0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0$$
be a small extension and $(x,y,e^p) \in \mathrm{MC}_{(h,g)}(A)$.

Since α is surjective there exist $\tilde{x} \in L^1 \otimes m_B$ that lifts x, $\tilde{y} \in N^1 \otimes m_B$ that lifts y, and $q \in M^0 \otimes m_B$ that lifts p. Let
$$l = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] \in L^2 \otimes m_B$$
and
$$k = d\tilde{y} + \frac{1}{2}[\tilde{y}, \tilde{y}] \in N^2 \otimes m_B.$$
As in Lemma I.3.23, we can easily prove that $\alpha(l) = \alpha(k) = dl = dk = 0$; then $l \in H^2(L) \otimes J$ and $k \in H^2(N) \otimes J$.

Let $r = -g(\tilde{y}) + e^q * h(\tilde{x}) \in M^1 \otimes m_B$; in particular $\alpha(r) = 0$ and so $r \in M^1 \otimes J$.

Now we prove that $(l,k,r) \in Z^2(\mathrm{C}^{\cdot}_{(h,g)}) \otimes J$.

Since $dl = dk = 0$, it remains to prove that $-dr - g(k) + h(l) = 0$.

By definition, $h(\tilde{x}) = e^{-q} * (g(\tilde{y}) + r) = r + e^{-q} * g(\tilde{y})$ (in the last equalities we use Example I.3.27) and so

$$h(l) = d(h(\tilde{x}))+\frac{1}{2}[h(\tilde{x}), h(\tilde{x})] = dr+d(e^{-q}*g(\tilde{y}))+\frac{1}{2}[e^{-q}*g(\tilde{y}), e^{-q}*g(\tilde{y})].$$

Let $A = d(e^{-q} * g(\tilde{y}))$ and $B = [e^{-q} * g(\tilde{y}), e^{-q} * g(\tilde{y})]$. Therefore it is sufficient to prove $A + \frac{1}{2}B = g(k)$.

We have

$$B = [e^{-q}*g(\tilde{y}), e^{-q}*g(\tilde{y})] = [e^{[-q,]'}(d+g(\tilde{y}))-d, e^{[-q,]'}(d+g(\tilde{y}))-d]' =$$
$$[e^{[-q,]'}(d+g(\tilde{y})), e^{[-q,]'}(d+g(\tilde{y}))]'+[e^{[-q,]'}(d+g(\tilde{y})), -d]'+[-d, e^{[-q,]'}(d+g(\tilde{y}))]' =$$
$$e^{[-q,]'}[d + g(\tilde{y}), d + g(\tilde{y})]' - 2[d, e^{[-q,]'}(d + g(\tilde{y}))]' =$$
$$e^{[-q,]'}(2dg(\tilde{y})+[g(\tilde{y}), g(\tilde{y})])-2[d, e^{[-q,]'}(d+g(\tilde{y}))-d]' = 2e^{[-q,]}(g(k))-2A.$$

By assumption $g(k) \in M^2 \otimes J$ and so $e^{[-q,]}(g(k)) = g(k)$.

Therefore

$$A + \frac{1}{2}B = A + \frac{1}{2}(2g(k) - 2A) = g(k)$$

and so $[(l, k, r)] \in H^2(\mathrm{C}^{\cdot}_{(h,g)}) \otimes J$.

This class doesn't depend on the choice of the liftings.

Actually, let $\tilde{x}' \in L^1 \otimes m_B$ be another lifting of x. Then $\tilde{x}' = \tilde{x} + j_x$, for some $j_x \in L^1 \otimes J$ and so

$$l' = d\tilde{x}' + \frac{1}{2}[\tilde{x}', \tilde{x}'] = d\tilde{x} + dj_x + \frac{1}{2}[\tilde{x}, \tilde{x}] = l + dj_x.$$

Analogously, if $\tilde{y}'$ is another lifting of y then there exists $j_y \in N^1 \otimes J$ such that

$$k' = k + dj_y.$$

Moreover

$$r' = -g(\tilde{y}')+e^q*h(\tilde{x}') = -g(\tilde{y})-g(j_y)+e^q*h(\tilde{x})+h(j_x) = r-g(j_y)+h(j_x).$$

Therefore $[(l', k', r')] = [(l+dj_x, k+dj_y, r-g(j_y)+h(j_x))] = [(l, k, r)] \in H^2(\mathrm{C}^{\cdot}_{(h,g)}) \otimes J$.

In conclusion, $[(l, k, r)] \in H^2(\mathrm{C}^{\cdot}_{(h,g)}) \otimes J$ is the obstruction class associated to the element $(x, y, e^p) \in \mathrm{MC}_{(h,g)}(A)$.

If this class vanishes, then there exists $(u, v, z) \in \mathrm{C}^1_{(h,g)} \otimes J$ such that $(du, dv, -dz - g(v) + h(u)) = (l, k, r)$. In this case, define $\overline{x} = \tilde{x} - u$, $\overline{y} = \tilde{y} - v$ and $\overline{z} = q - z$. Then $(\overline{x}, \overline{y}, e^{\overline{z}}) \in \mathrm{MC}_{(h,g)}(B)$ and it lifts (x, y, e^p). Actually

$$d\overline{x} + [\overline{x}, \overline{x}] = d\tilde{x} - du + [\tilde{x}, \tilde{x}] = l - du = 0,$$
$$d\overline{y} + [\overline{y}, \overline{y}] = d\tilde{y} - dv + [\tilde{y}, \tilde{y}] = k - dv = 0$$

and

$$g(\overline{y}) - e^{\overline{z}} * h(\overline{x}) = g(\tilde{y}) - g(v) - e^{q-z} * (h(\tilde{x}) - h(u)) = {}^{\mathrm{b}}$$

[b] See Example I.3.27

$$g(\tilde{y}) - g(v) - e^{q-z} * h(\tilde{x}) + h(u) =$$

$$(g(\tilde{y}) - e^{q} * h(\tilde{x})) - dz - g(v) + h(u) = -r + r = 0.$$

□

III.1.4. Properties.

III.1.19. Lemma. *Let $h : L \longrightarrow M$ and $g : N \longrightarrow M$ be morphisms of abelian DGLAs. Then the functor $\mathrm{MC}_{(h,g)}$ is smooth.*

Proof. We have to prove that for every surjection $\varphi : B \longrightarrow A \in$ **Art** the map

$$\mathrm{MC}_{(h,g)}(B) \longrightarrow \mathrm{MC}_{(h,g)}(A)$$

is surjective. By hypothesis we have

$$\mathrm{MC}_{(h,g)}(A) = \{(x, y, e^{p}) \in (L^1 \otimes m_A) \times (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A) |$$

$$dx = dy = 0 \qquad g(y) = e^{p} * h(x) = h(x) - dp\}.$$

This implies that the Maurer-Cartan equation reduces to the linear equations $dx = dy = 0, g(y) + dp - h(x) = 0$; then $\mathrm{MC}_{(h,g)}(A) = Z^1(\mathrm{C}^{\cdot}_{(h,g)} \otimes m_A) = Z^1(C) \otimes m_A$ and so the lifting exists ($Z^1(\mathrm{C}^{\cdot}_{(h,g)} \otimes m_A) \twoheadrightarrow Z^1(\mathrm{C}^{\cdot}_{(h,g)} \otimes m_B)$).

□

III.1.20. Remark. Every commutative diagram of morphisms of DGLA

$$\begin{tikzcd}
 & L \arrow[r, "\alpha'"] \arrow[d, "h"'] & P \arrow[d, "\eta"] \\
 & M \arrow[r, "\alpha"] & Q \\
N \arrow[ur, "g"] \arrow[r, "\alpha''"] & R \arrow[ur, "\mu"'] &
\end{tikzcd}$$

induces a *morphism* $\varphi^{\cdot}$ *of complexes*

$$\mathrm{C}^{i}_{(h,g)} \ni (l, n, m) \xmapsto{\varphi^i} (\alpha'(l), \alpha''(n), \alpha(m)) \in \mathrm{C}^{i}_{(\eta,\mu)}$$

and a natural *transformation* F of the associated deformation functors:

$$F : \mathrm{Def}_{(h,g)} \longrightarrow \mathrm{Def}_{(\eta,\mu)} .$$

Then we obtain the following proposition that is a generalization of [**24**, Prop. 2.3].

III.1.21. Proposition. *Let*

$$\begin{tikzcd} & L \arrow[r, "\alpha'"] \arrow[d, "h"'] & P \arrow[d, "\eta"] \\ & M \arrow[r, "\alpha"] & Q \\ N \arrow[ur, "g"] \arrow[r, "\alpha''"] & R \arrow[ur, "\mu"'] & \end{tikzcd}$$

be a commutative diagram of differential graded Lie algebras. If the functor $\mathrm{Def}_{(\eta,\mu)}$ *is smooth, then the obstruction space of* $\mathrm{Def}_{(h,g)}$ *is contained in the kernel of the map*

$$H^2(\mathrm{C}^{\cdot}_{(h,g)}) \longrightarrow H^2(\mathrm{C}^{\cdot}_{(\eta,\mu)}).$$

PROOF. The morphism $F : \mathrm{Def}_{(h,g)} \longrightarrow \mathrm{Def}_{(\eta,\mu)}$ induces a linear map between obstruction spaces. If $\mathrm{Def}_{(\eta,\mu)}$ is smooth, then its obstruction space is zero (Proposition I.1.31).

$\square$

III.1.22. THEOREM. *If* $\varphi^{\cdot} : \mathrm{C}^{\cdot}_{(h,g)} \longrightarrow \mathrm{C}^{\cdot}_{(\eta,\mu)}$ *is a quasi isomorphism of complexes then* $F : \mathrm{Def}_{(h,g)} \longrightarrow \mathrm{Def}_{(\eta,\mu)}$ *is an isomorphism of functors.*

The proof of this theorem is postponed at the end of Section III.2.4.

Now, let

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[d, "\beta"'] & L \arrow[d, "h"] \\ N \arrow[r, "g"] & M \end{tikzcd}$$

be a commutative diagram of morphism of DGLAs. Then it induces a morphism of complexes

$$H \xrightarrow{\varphi^{\cdot}} \mathrm{C}^{\cdot}_{(h,g)} \qquad \text{with} \qquad \varphi(x) = (\alpha(x), \beta(x), 0),$$

and a morphism of functors

$$\mathrm{Def}_H \xrightarrow{F} \mathrm{Def}_{(h,g)} \qquad \text{with} \qquad F(x) = (\alpha(x), \beta(x), 0).$$

III.1.23. THEOREM. *In the notation above, if* $\varphi^{\cdot}$ *is a quasi isomorphism then* F *is an isomorphism of functors.*

Also the proof of this theorem is postponed at the end of Section III.2.4.

III.2. Extended Deformation Functors

In this section we study the extended deformation functors. In particular we are interested in the functors $\widetilde{\mathrm{MC}}_{(h,g)}$ and $\widetilde{\mathrm{Def}}_{(h,g)}$ that are a generalization of the functors $\mathrm{MC}_{(h,g)}$ and $\mathrm{Def}_{(h,g)}$ introduced in

Section III.1. Here, we remove the restrictive hypothesis of M concentrated in non negative degree.

The main references of this chapter are [**21**], [**23**, Sections 5.7 and 5.8] and [**24**, Sections 6 and 7].

III.2.1. Notations. We denote by:

$\mathbf{C}$ the category of nilpotent (associative and commutative) differential graded algebras which are finite dimensional as $\mathbb{K}$-vector spaces.

$\mathbf{C}_0$ the full subcategory of $\mathbf{C}$ of algebras with trivial multiplication.

III.2.1. Example. Define the complex $(\Omega = \Omega_0 \oplus \Omega_1, d)$, where $\Omega_0 = \mathbb{K}$, $\Omega_1 = \mathbb{K}[-1]$ and $d : \Omega_0 \longrightarrow \Omega_1$ the canonical linear isomorphism $d(1[0]) = 1[-1]$. $\Omega \in \mathbf{C}_0$ and the projection $p : \Omega \longrightarrow \Omega_0 = \mathbb{K}$ and the inclusion $\Omega_1 \longrightarrow \Omega$ are morphism in $\mathbf{C}_0$.

Moreover $\Omega[n] = \mathbb{K}[n] \otimes \Omega$ is an acyclic complex in $\mathbf{C}_0$, for each $n \in \mathbb{N}$.

III.2.2. Definition of extended functors. Let $A \in \mathbf{C}$ and $J \subset A$ a differential ideal; then $J \in \mathbf{C}$ and the inclusion $J \longrightarrow A$ is a morphism of differential graded algebras.

III.2.2. Definition. A *small extension* in $\mathbf{C}$ is a short exact sequence

$$0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0$$

such that α is a morphism in $\mathbf{C}$ and J is an ideal of B such that $BJ = 0$; in addition it is called *acyclic* if J is an acyclic complex, or equivalently α is a quasi-isomorphism.

III.2.3. Definition. A covariant functor $F : \mathbf{C} \longrightarrow \mathbf{Set}$ is called a *predeformation functor* if the following conditions are satisfied:

III.2.3.1) $F(0) = \{*\}$ is the one point set.

III.2.3.2) For every $A, B \in \mathbf{C}$, the natural map

$$F(A \times B) \longrightarrow F(A) \times F(B)$$

is bijective.

III.2.3.3) For every surjective morphism $\alpha : A \longrightarrow C$ in $\mathbf{C}$, with C an acyclic complex in $\mathbf{C}_0$, the natural morphism

$$F(\ker(\alpha)) = F(A \times_C 0) \longrightarrow F(A) \times_{F(C)} F(0) = F(A)$$

is bijective.

III.2.3.4) For every pair of morphisms $\beta : B \longrightarrow A$ and $\gamma : C \longrightarrow A$ in $\mathbf{C}$, with β surjective, the natural map

$$F(B \times_A C) \longrightarrow F(B) \times_{F(A)} F(C)$$

is surjective.

III.2.3.5) For every small acyclic extension

$$0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0$$

the induced map $F(B) \longrightarrow F(A)$ is surjective.

III.2.4. DEFINITION. A covariant functor $F : \mathbf{C} \longrightarrow \mathbf{Set}$ is called *deformation functor* if it is a predeformation functor and $F(J) = 0$ for every acyclic complex $J \in \mathbf{C}_0$.

III.2.2.1. *Examples.* Let L be a differential graded Lie algebra and $A \in \mathbf{C}$. Then $L \otimes A$ has a natural structure of (nilpotent) DGLA given by:

$$(L \otimes A)^i = \bigoplus_{j \in \mathbb{Z}} L^j \otimes A^{i-j};$$

$$d(x \otimes a) = dx \otimes a + (-1)^{deg(x)} x \otimes da;$$

$$[x \otimes a, y \otimes b] = (-1)^{deg(a)deg(y)} [x, y] \otimes ab.$$

III.2.5. DEFINITION. The *extended Maurer-Cartan* functor associated to a DGLA L is

$$\widetilde{\mathrm{MC}}_L(A) = \{(x \in (L \otimes A)^1 |\, dx + \frac{1}{2}[x, x] = 0\}$$

III.2.6. LEMMA. *$\widetilde{\mathrm{MC}}_L$ is a predeformation functor.*

PROOF. See [**21**, Lemma 2.15]. We also give a proof in Section III.2.4, since it is a particular case of Theorem III.2.20. □

III.2.7. REMARK. We note that, for each $C \in \mathbf{C}_0$, we have:

$$\widetilde{\mathrm{MC}}_L(C) = \{x \in (L \otimes C)^1 |\, dx = 0\} = Z^1(L \otimes C).$$

III.2.8. DEFINITION. The *extended deformation* functor associated to a DGLA L is

$$\widetilde{\mathrm{Def}}_L(A) = \frac{\{x \in (L \otimes A)^1 |\, dx + \frac{1}{2}[x, x] = 0\}}{\text{gauge action of } \exp{(L \otimes A)^0}},$$

where the gauge action of $a \in (L \otimes A)^0$ is the analogous of the non extended case: i.e.

$$e^a * x := x + \sum_{n=0}^{\infty} \frac{[a, -]^n}{(n+1)!}([a, x] - da)$$

III.2.9. LEMMA. *$\widetilde{\mathrm{Def}}_L$ is a deformation functor.*

PROOF. See [**21**, Theorem 2.16]. We also give a proof in Section III.2.4, since it is a particular case of Theorem III.2.26.

□

III.2.10. REMARK. For each $C \in \mathbf{C}_0$ we note that:

$$\widetilde{\mathrm{Def}}_L(C) = \frac{\{x \in (L \otimes C)^1 |\, dx = 0\}}{\{da |\, a \in (L \otimes C)^0\}} = H^1(L \otimes C).$$

III.2.3. Properties. As in the not extended case, for every predeformation functor F and every $C \in \mathbf{C}_0$, there exists a natural structure of vector space on $F(A)$ with:

- sum given by the map $F(C \times C) = F(C) \times F(C) \xrightarrow{+} F(C)$ induced by $C \times C \xrightarrow{+} C$;
- scalar multiplication by s given by the map $F(C) \xrightarrow{\cdot s} F(C)$ induced by $C \xrightarrow{\cdot s} C$.

III.2.11. Remark. 1) For each morphism $B \longrightarrow A$ in $\mathbf{C}_0$, the induced map $F(B) \longrightarrow F(A)$ is $\mathbb{C}$-linear.

2) For each natural transformation of predeformation functors $F \longrightarrow G$, the induced map $F(C) \longrightarrow G(C)$ is $\mathbb{C}$-linear for each $C \in \mathbf{C}_0$.

III.2.12. Definition. Let F be a predeformation functor, the *tangent space* of F is the graded vector space $TF[1]$, where

$$TF = \bigoplus_{n \in \mathbb{Z}} T^n F, \quad T^{n+1}F = TF[1]^n = \operatorname{coker}(F(\Omega[n]) \xrightarrow{p} F(\mathbb{K}[n])), n \in \mathbb{Z}$$

and p is the linear map induced by the projection $\Omega[n] \longrightarrow \mathbb{K}[n]$ (see Example III.2.1).

In particular, if F is a deformation functor then $F(\Omega[n]) = 0$ for every n. Therefore $T^{n+1}F = TF[1]^n = F(\mathbb{K}[\varepsilon])$, where ε is an indeterminate of degree $-n \in \mathbb{Z}$, such that $\varepsilon^2 = 0$.

III.2.13. Definition. A natural transformation of predeformation functors $F \longrightarrow G$ is called a *quasi-isomorphism* if it induces isomorphisms on tangent spaces: i.e. $T^n F \cong T^n G$.

III.2.14. Theorem. *(Inverse function theorem) A natural transformation of deformation functors is an isomorphism if and only if it is a quasi-isomorphism.*

Proof. See [**21**, Corollary 3.2] or [**23**, Corollary 5.72]. □

III.2.15. Theorem. *(Manetti) Let F be a predeformation functor, then there exists a deformation functor F^+ and a natural transformation $\nu : F \longrightarrow F^+$, that is a quasi isomorphism, such that for every deformation functor G and every natural transformation $\phi : F \longrightarrow G$ there exists a unique natural transformation $\psi : F^+ \longrightarrow G$ such that $\phi = \psi\nu$.*

Proof. See [**21**, Th. 2.8]. □

III.2.16. Remark. Given a natural transformation of predeformation functors $\alpha : F \longrightarrow G$, there is a natural transformation of associated deformation functors $\alpha^+ : F^+ \longrightarrow G^+$. Actually, let $\eta : G \longrightarrow G^+$. Then by composition we get $\beta = \eta \circ \alpha : F \longrightarrow G^+$ and so, by previous Theorem III.2.15, there exists $\alpha^+ : F^+ \longrightarrow G^+$.

III.2.17. THEOREM. *(Manetti) Let S be a complex of vector spaces and assume that the functor*

$$\mathbf{C}_0 \longrightarrow \mathbf{Set}$$

$$C \longmapsto Z^1(S \otimes C)$$

is the restriction of a predeformation functor F. Then for every complex $C \in \mathbf{C}_0$ holds the equality $F^+(C) = H^1(S \otimes C)$; in particular $T^iF^+ = H^i(S)$.

PROOF. See [**21**, Lemma 2.10]. □

III.2.18. COROLLARY. *(Manetti) For every differential graded Lie algebra L there exists a natural isomorphism $\widetilde{\mathrm{MC}}_L^+ \cong \widetilde{\mathrm{Def}}_L$.*

PROOF. By Remark III.2.7 and Theorem III.2.17, we have that $T^i\widetilde{\mathrm{MC}}_L^+ = H^i(L)$. Therefore the natural projection $\widetilde{\mathrm{MC}}_L \longrightarrow \widetilde{\mathrm{Def}}_L$ induces (by Theorem III.2.15) a natural transformation $\widetilde{\mathrm{MC}}_L^+ \longrightarrow \widetilde{\mathrm{Def}}_L$ which is an isomorphism on tangent spaces. □

III.2.4. Extended deformation functor of a couple of morphisms. Let L, M, N be DGLA, and $h : L \longrightarrow M$ and $g : N \longrightarrow M$ be morphisms of DGLA:

$$\begin{tikzcd} & L \arrow[d, "h"] \\ N \arrow[r, "g"'] & M. \end{tikzcd}$$

III.2.19. DEFINITION. The *extended Maurer-Cartan functor associated to the couple* (h, g) is

$$\widetilde{\mathrm{MC}}_{(h,g)} : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$\widetilde{\mathrm{MC}}_{(h,g)}(A) = \{(x, y, e^p) \in (L \otimes A)^1 \times (N \otimes A)^1 \times \exp(M \otimes A)^0 |$$

$$dx + \frac{1}{2}[x, x] = 0,\ dy + \frac{1}{2}[y, y] = 0,\ g(y) = e^p * h(x)\}.$$

III.2.20. THEOREM. *$\widetilde{\mathrm{MC}}_{(h,g)}$ is a predeformation functor.*

PROOF. $\widetilde{\mathrm{MC}}_{(h,g)}(0) = 0$ and so (III.2.3.1) is satisfied.
For each pair of morphisms $\beta : B \longrightarrow A$ and $\gamma : C \longrightarrow A$ in $\mathbf{C}$, we have

$$\widetilde{\mathrm{MC}}_{(h,g)}(B \times_A C) = \widetilde{\mathrm{MC}}_{(h,g)}(B) \times_{\widetilde{\mathrm{MC}}_{(h,g)}(A)} \widetilde{\mathrm{MC}}_{(h,g)}(C)$$

and so $\widetilde{\mathrm{MC}}_{(h,g)}$ satisfies properties (III.2.3.2),(III.2.3.3) and (III.2.3.4).

Let $0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0$ be an acyclic small extension and $(x, y, e^p) \in \widetilde{\mathrm{MC}}_{(h,g)}(A)$. We want to prove the existence of a lifting $(\overline{x}, \overline{y}, e^{\overline{p}}) \in \widetilde{\mathrm{MC}}_{(h,g)}(B)$ so that the induced map $\widetilde{\mathrm{MC}}_{(h,g)}(B) \longrightarrow \widetilde{\mathrm{MC}}_{(h,g)}(A)$ is surjective.
Since α is surjective, there exists $(r, s, e^t) \in (L \otimes B)^1 \times (N \otimes B)^1 \times \exp(M \otimes B)^0$ such that $\alpha(r) = x, \alpha(s) = y$ and $\alpha(t) = p$.

Let $l \in (L \otimes J)^2$ and $k \in (N \otimes J)^2$ defined as follows

$$l = dr + \frac{1}{2}[r, r] \qquad k = ds + \frac{1}{2}[s, s].$$

Then

$$dl = \frac{1}{2}d[r, r] = [dr, r] = [l, r] - \frac{1}{2}[[r, r], r]$$

and the same holds for k, i.e.

$$dk = [k, s] - \frac{1}{2}[[s, s], s].$$

Since $BJ = 0$, we have $[l, r] = [k, s] = 0$; moreover, using Jacobi identity (see Remark I.3.6), $[[r, r], r] = [[s, s], s] = 0$. This implies that $dl = dk = 0$.

By hypothesis, J is acyclic and so, by the Künneth formula, the complexes $L \otimes J$ and $N \otimes J$ are acyclic. Therefore there exist $w \in (L \otimes J)^1$ and $z \in (N \otimes J)^1$, such that $dw = l$ and $dz = k$.

Let

$$\overline{x} = r - w \in (L \otimes B)^1 \qquad \text{and} \qquad \overline{y} = s - z \in (N \otimes B)^1;$$

We have

$$\alpha(\overline{x}) = \alpha(r) - \alpha(w) = \alpha(r) = x, \qquad \alpha(\overline{y}) = \alpha(s) - \alpha(z) = \alpha(s) = y,$$

$$d\overline{x} + \frac{1}{2}[\overline{x}, \overline{x}] = dr - l + \frac{1}{2}[r, r] = 0 \qquad \text{and} \qquad d\overline{y} + \frac{1}{2}[\overline{y}, \overline{y}] = 0.$$

Therefore $\overline{x}$ and $\overline{y}$ lift, respectively, x and y and they satisfy the Maurer-Cartan equation.

Let $z = e^t * h(\overline{x}) - g(\overline{y}) \in (M \otimes B)^1$. Since $\alpha(z) = e^{\alpha(t)} * h(\alpha(\overline{x})) - g(\alpha(\overline{y})) = e^p * h(x) - g(y) = 0$, then $z \in (M \otimes J)^1$. Moreover $dz = 0$; in fact

$$2dz = 2d(e^t * h(\overline{x})) - 2d(g(\overline{y})) = -[e^t * h(\overline{x}), e^t * h(\overline{x})] + [g(\overline{y}), g(\overline{y})] =$$
$$-[e^t * h(\overline{x}), e^t * h(\overline{x})] + [g(\overline{y}), e^t * h(\overline{x})] - [g(\overline{y}), e^t * h(\overline{x})] + [g(\overline{y}), g(\overline{y})] =$$
$$-[e^t * h(\overline{x}) - g(\overline{y}), e^t * h(\overline{x})] - [g(\overline{y}), e^t * h(\overline{x}) - g(\overline{y})] = 0.$$

$(z = e^t * h(\overline{x}) - g(\overline{y}) \in (M \otimes J)$ and so $[e^t * h(\overline{x}) - g(\overline{y}), -] = 0.)$

Since $M \otimes J$ is acyclic, there exist $v \in (M \otimes J)^0$ such that $z = dv$. Therefore

$$e^t * h(\overline{x}) = z + g(\overline{y}) = dv + g(\overline{y}) = e^{-v} * g(\overline{y})$$

and so

$$e^v e^t * h(\overline{x}) = g(\overline{y}).$$

This implies that $e^{\overline{p}} = e^v e^t$ lifts e^p.

Then the triple $(\overline{x}, \overline{y}, e^{\overline{p}}) \in \widetilde{\mathrm{MC}}_{(h,g)}(B)$ lifts $(x, y, e^p) \in \widetilde{\mathrm{MC}}_{(h,g)}(A)$ and so (III.2.3.5) holds.

$\square$

Proof of Lemma III.2.6. It is sufficient to apply the previous theorem with $M = N = 0$. $\square$

III.2.21. Remark. If the DGLA M is concentrated in non negative degree, then for every $(A, m_A) \in \mathbf{Art}$, $\widetilde{\mathrm{MC}}_{(h,g)}(m_A) = \mathrm{MC}_{(h,g)}(A)$.

Applying Theorem III.2.15, we can conclude the existence of a deformation functor $\widetilde{\mathrm{MC}}^{+}_{(h,g)}$ associated to $\widetilde{\mathrm{MC}}_{(h,g)}$.

III.2.22. Proposition. $T^i\widetilde{\mathrm{MC}}^{+}_{(h,g)} \cong H^i(\mathrm{C}_{(h,g)})$.

Proof. For each $C \in \mathbf{C}_0$ we have

$$\widetilde{\mathrm{MC}}_{(h,g)}(C) = \{(l, n, e^m) \in (L \otimes C)^1 \times (N \otimes C)^1 \times \exp(M \otimes C)^0 | \\ dl = dn = 0,\ g(n) = e^m * h(l) = h(l) - dm\}.$$

and

$$Z^1(\mathrm{C}_{(h,g)} \otimes C) = \{(l, n, m) \in (L \otimes C)^1 \times (N \otimes C)^1 \times (M \otimes C)^0 | \\ dl = dn = -dm - g(n) + h(l) = 0\}.$$

Therefore $\widetilde{\mathrm{MC}}_{(h,g)}(-)_{|\mathbf{C}_0} = Z^1(\mathrm{C}_{(h,g)} \otimes -)_{|\mathbf{C}_0}$. Then we can apply Theorem III.2.17 to conclude the proof. $\square$

Now, we consider on $\widetilde{\mathrm{MC}}_{(h,g)}(A)$ the following equivalence relation $\approx$:

$$(x_1, y_1, e^{p_1}) \ \approx\ (x_2, y_2, e^{p_2})$$

if and only if there exist $a \in (L \otimes A)^0$, $b \in (N \otimes A)^0$ and $c \in (M \otimes A)^{-1}$ such that

$$x_2 = e^a * x_1, \qquad y_2 = e^b * y_1$$

and

$$e^{p_2} = e^{g(b)} e^T e^{p_1} e^{-h(a)} \quad \text{with} \quad T = dc + [g(y_1), c] \in Stab_A(g(y_1)).$$

III.2.23. Lemma. *The relation $\approx$ is an equivalence relation.*

Proof. The reflexivity is obvious. As regards the symmetry and transitivity, we use the following property of the irrelevant stabilizers $Stab_A(-)$(see Section I.3.4): for each $x \in \widetilde{MC}_M(A)$, $a \in (M \otimes A)^0$ and $T = dc + [x, c] \in Stab_A(x)$, there exist $f \in (M \otimes A)^{-1}$ such that

$$e^a e^T = e^{T'} e^a$$

where $T' = df + [y, f] \in Stab_A(y)$ and $y = e^a * x$.

Symmetry. Let $(x_1, y_1, e^{p_1}) \approx (x_2, y_2, e^{p_2})$. Therefore there exist $a \in (L \otimes A)^0$, $b \in (N \otimes A)^0$ and $c \in (M \otimes A)^{-1}$ such that

$$x_2 = e^a * x_1, \qquad y_2 = e^b * y_1$$

and

$$e^{p_2} = e^{g(b)} e^T e^{p_1} e^{-h(a)} \quad \text{with} \quad T = dc + [g(y_1), c].$$

This implies the existence of $f \in (M \otimes A)^{-1}$ such that

$$e^{p_2} = e^{T'} e^{g(b)} e^{p_1} e^{-h(a)} \quad \text{with} \quad T' = df + [g(y_2), f].$$

Therefore, choosing $\alpha = -a \in (L \otimes A)^0$, $\beta = -b \in (N \otimes A)^0$ and $\gamma = -f \in (M \otimes A)^{-1}$ we get

$$x_1 = e^\alpha * x_2, \qquad y_1 = e^\beta * y_2$$

and

$$e^{p_1} = e^{-g(b)} e^{-T'} e^{p_2} e^{h(a)} = e^{g(\beta)} e^{T'} e^{p_2} e^{-h(\alpha)} \quad \text{with} \quad T' = d\gamma + [g(y_1), \gamma].$$

Then $(x_2, y_2, e^{p_2}) \approx (x_1, y_1, e^{p_1})$.

Transitivity. Suppose

$$(x_1, y_1, e^{p_1}) \approx (x_2, y_2, e^{p_2}) \quad \text{and} \quad (x_2, y_2, e^{p_2}) \approx (x_3, y_3, e^{p_3}).$$

Therefore there exist $a_1, a_2 \in (L \otimes A)^0$, $b_1, b_2 \in (N \otimes A)^0$ and $c_1, c_2 \in (M \otimes A)^{-1}$ such that

$$x_2 = e^{a_1} * x_1, \qquad y_2 = e^{b_1} * y_1 \qquad e^{p_2} = e^{g(b_1)} e^{T_1} e^{p_1} e^{-h(a_1)}$$

and

$$x_3 = e^{a_2} * x_2, \qquad y_3 = e^{b_2} * y_2 \qquad e^{p_3} = e^{g(b_2)} e^{T_2} e^{p_2} e^{-h(a_2)}$$

with $T_1 = dc_1 + [g(y_1), c_1]$ and $T_2 = dc_2 + [g(y_2), c_2]$. Then

$$e^{p_3} = e^{g(b_2)} e^{T_2} e^{g(b_1)} e^{T_1} e^{p_1} e^{-h(a_1)} e^{-h(a_2)} = e^{g(b_2)} e^{g(b_1)} e^{T_2'} e^{T_1} e^{p_1} e^{-h(a_1)} e^{-h(a_2)}$$

for some $c' \in (M \otimes A)^{-1}$ and $T_2' = dc' + [g(y_1), c']$.

Since $Stab_A(g(y_1))$ is a subgroup, there exists $c \in (M \otimes A)^{-1}$ such that $e^{T_2'} e^{T_1} = e^T$ with $T = dc + [g(y_1), c]$.

Let $a = a_2 \bullet a_1 \in (L \otimes A)^0$, $b = b_2 \bullet b_1 \in (N \otimes A)^0$ and $c \in (M \otimes A)^{-1}$ as above. Then

$$x_3 = e^a * x_1, \qquad y_3 = e^b * y_3 \qquad e^{p_3} = e^{g(b)} e^T e^{p_1} e^{-h(a)}$$

and so $(x_1, y_1, e^{p_1}) \approx (x_3, y_3, e^{p_3})$.

□

III.2.24. Remark. We note that this equivalence relation generalizes the equivalence relation induced by the gauge action given in Definition III.1.10, when M is concentrated in non negative degree.

III.2.25. Definition. Define the functor

$$\widetilde{\mathrm{Def}}_{(h,g)} : \mathbf{C} \longrightarrow \mathbf{Set}$$

$$\widetilde{\mathrm{Def}}_{(h,g)}(A) = \widetilde{\mathrm{MC}}_{(h,g)}(A) / \approx .$$

III.2.26. THEOREM. *$\widetilde{\mathrm{Def}}_{(h,g)} : \mathbf{C} \longrightarrow \mathbf{Set}$ is a deformation functor with $T^i\widetilde{\mathrm{Def}}_{(h,g)} = H^i(C^{\cdot}_{(h,g)})$.*

PROOF. We first prove that $\widetilde{\mathrm{Def}}_{(h,g)}$ is a predeformation functor.

Since $\widetilde{\mathrm{Def}}_{(h,g)}$ is the quotient of the predeformation functor $\widetilde{\mathrm{MC}}_{(h,g)}$, then the conditions (III.2.3.1) and (III.2.3.5) are verified.

An easy calculation shows that $\widetilde{\mathrm{Def}}_{(h,g)}$ satisfies (III.2.3.2).

Now we verify condition (III.2.3.4). Let $\beta : B \longrightarrow A$ and $\gamma : C \longrightarrow A$ be morphisms in $\mathbf{C}$, with β surjective. We prove that the natural map $\widetilde{\mathrm{Def}}_{(h,g)}(B \times_A C) \longrightarrow \widetilde{\mathrm{Def}}_{(h,g)}(B) \times_{\widetilde{\mathrm{Def}}_{(h,g)}(A)} \widetilde{\mathrm{Def}}_{(h,g)}(C)$ is surjective.

Let $(x_1, y_1, e^{p_1}) \in \widetilde{\mathrm{MC}}_{(h,g)}(B)$ and $(x_2, y_2, e^{p_2}) \in \widetilde{\mathrm{MC}}_{(h,g)}(C)$, such that $\beta(x_1, y_1, e^{p_1})$ and $\gamma(x_2, y_2, e^{p_2})$ are the same element in $\widetilde{\mathrm{Def}}_{(h,g)}(A)$.

Then, there exist $a \in (L \otimes A)^0$, $b \in (N \otimes A)^0$ and $c \in (M \otimes A)^{-1}$ such that

$$\gamma(x_2) = e^a * \beta(x_1), \qquad \gamma(y_2) = e^b * \beta(y_1)$$

and

$$e^{\gamma(p_2)} = e^{g(b)}e^T e^{\beta(p_1)}e^{-h(a)} \quad \text{with } T = dc + [g(\beta(y_1)), c].$$

Let $\tilde{a} \in (L \otimes B)^0$ be a lifting of a, $\tilde{b} \in (N \otimes B)^0$ a lifting of b and $\tilde{c} \in (M \otimes B)^{-1}$ a lifting of c, so that $\tilde{T} = d\tilde{c} + [g((\beta(y_1)), \tilde{c}]$ lifts T.

Up to substitute (x_1, y_1, e^{p_1}) with its equivalent $(e^{\tilde{a}}*x_1, e^{\tilde{b}}*y_1, e^{g(\tilde{b})}e^{\tilde{T}}e^{p_1}e^{h(\tilde{a})})$, we can suppose that[c] $\beta(x_1, y_1, e^{p_1}) = \gamma(x_2, y_2, e^{p_2}) \in \widetilde{\mathrm{MC}}_{(h,g)}(A)$.

Then $((x_1, y_1, e^{p_1}), (x_2, y_2, e^{p_2})) \in \widetilde{\mathrm{MC}}_{(h,g)}(B) \times_{\widetilde{\mathrm{MC}}_{(h,g)}(A)} \widetilde{\mathrm{MC}}_{(h,g)}(C)$ and so, since $\widetilde{\mathrm{MC}}_{(h,g)}$ is a predeformation functor, there exists a lifting in $\widetilde{\mathrm{MC}}_{(h,g)}(B \times_A C)$. Now, it is sufficient to take its equivalence class in $\widetilde{\mathrm{Def}}_{(h,g)}$.

Finally, we prove that condition (III.2.3.3) is satisfied.

Let $\alpha : A \longrightarrow C$ be a surjection with $C \in \mathbf{C}_0$ an acyclic complex. Let (x_1, y_1, e^{p_1}) and $(x_2, y_2, e^{p_2}) \in \widetilde{\mathrm{Def}}_{(h,g)}(\ker\alpha)$ (in particular $g(y_1) = e^{p_1} * h(x_1)$ and $g(y_2) = e^{p_2} * h(x_2)$) be such that there exist $a \in (L \otimes A)^0$, $b \in (N \otimes A)^0$ and $c \in (M \otimes A)^{-1}$ with

$$x_2 = e^a * x_1 \qquad y_2 = e^b * y_1$$

$$e^{p_2} = e^{g(b)}e^T e^{p_1}e^{-h(a)} \quad \text{with} \qquad T = dc + [g(y_1), c].$$

We are looking for $\overline{a} \in (L \otimes \ker\alpha)^0$, $\overline{b} \in (N \otimes \ker\alpha)^0$ and $\overline{c} \in (M \otimes \ker\alpha)^{-1}$ such that $\overline{T} = d\overline{c} + [g(y_1), \overline{c}]$ and

$$e^{\overline{a}} * x_1 = x_2 \qquad e^{\overline{b}} * y_1 = y_2 \qquad e^{g(\overline{b})}e^{\overline{T}}e^{p_1}e^{-h(\overline{a})} = e^{p_2}.$$

---

[c] $\beta(e^{\tilde{a}} * x_1, e^{\tilde{b}} * y_1, e^{g(b)}e^{\tilde{T}}e^{p_1}e^{h(\tilde{a})}) = (e^{\beta(\tilde{a})} * \beta(x_1), e^{\beta(\tilde{b})} * \beta(y_1), e^{g(\beta(\tilde{b}))}e^{\beta(\tilde{T})}e^{\beta(p_1)}e^{h(\beta(\tilde{a}))}) = (e^a * \beta(x_1), e^b * \beta(y_1), e^{g(b)}e^T e^{\beta(p_1)}e^{-h(a)}) = (\gamma(x_2), \gamma(y_2), e^{\gamma(p_2)}) = \gamma(x_2, y_2, e^{p_2})$

Since $L\otimes C$ and $N\otimes C$ are abelian DGLAs and $\alpha(x_i)=\alpha(y_i)=0$, for $i=1,2$, we have

$$0=\alpha(x_2)=e^{\alpha(a)}*\alpha(x_1)=-d\alpha(a)$$

and

$$0=\alpha(y_2)=e^{\alpha(b)}*\alpha(y_1)=-d\alpha(y).$$

Moreover, $L\otimes C$ and $N\otimes C$ are acyclic; therefore there exist $l\in(L\otimes A)^{-1}$ and $k\in(N\otimes A)^{-1}$ such that $d\alpha(l)=-\alpha(a)$ and $d\alpha(k)=-\alpha(b)$. This implies $dl+a\in(L\otimes\ker\alpha)^0$ and $dk+b\in(N\otimes\ker\alpha)^0$.

Set $w_1=dl+[x_1,l]\in Stab_A(x_1)$, $w_2=dk+[y_1,k]\in Stab_A(y_1)$ and define $\overline{a}=a\bullet w_1$ and $\overline{b}=b\bullet w_2$.

We claim that $\overline{a}\in(L\otimes\ker\alpha)^0$ and $\overline{b}\in(N\otimes\ker\alpha)^0$. Actually

$$\overline{a}=a\bullet w_1\equiv a+w_1\equiv a+dl\ (mod\,[L\otimes A,L\otimes A]);$$

since $A\cdot A\subset\ker\alpha$, we conclude $a\equiv a+dl\equiv 0\ (mod\,L\otimes\ker\alpha)$. An analogous calculation implies that $\overline{b}\in(N\otimes\ker\alpha)^0$.

Moreover, we note that

$$e^{\overline{a}}*x_1=e^ae^{w_1}*x_1=x_2\qquad\text{and}\qquad e^{\overline{b}}*y_1=e^be^{w_2}*y_1=y_2.$$

As regard e^{p_1}, since $e^aStab_A(x)e^{-a}=Stab_A(y)$ with $y=e^a*x$, we have

$$e^{-g(w_2)}e^Te^{p_1}e^{h(w_1)}=e^Se^{p_1}$$

for some $S=df+[g(y_1),f]$ with $f\in(M\otimes A)^{-1}$.

Therefore

$$e^{p_2}=e^{g(b)}e^Te^{p_1}e^{-h(a)}=$$
$$e^{g(b)}e^{g(w_2)}e^{-g(w_2)}e^Te^{p_1}e^{h(w_1)}e^{-h(w_1)}e^{-h(a)}=e^{g(\overline{b})}e^Se^{p_1}e^{-h(\overline{a})}$$

This implies that $e^S=e^{-g(\overline{b})}e^{p_2}e^{h(\overline{a})}e^{-p_1}$ lies in the subgroup $\exp((M\otimes\ker\alpha)^0)$ or equivalently $S=df+[g(y_1),f]\in(M\otimes\ker\alpha)^0$.

Since C is acyclic, the inclusion $M\otimes\ker\alpha\longrightarrow M\otimes A$ is a quasi isomorphism and it remains a quasi isomorphism if we consider the deformed differentials $d(-)+[g(y_1),-]$ on both $M\otimes\ker\alpha$ and $M\otimes A$ ($g(y_1)$ satisfies the Maurer-Cartan equation and so by Remark I.3.17 $d(-)+[g(y_1),-]$ is a differential).

Therefore, since the class of S is trivial in $M\otimes A$, it is also trivial in $M\otimes\ker\alpha$: i.e. there exist $\overline{c}\in(M\otimes\ker\alpha)^{-1}$ such that $S=\overline{T}=d\overline{c}+[g(y_1),\overline{c}]$.

In conclusion $e^{p_2}=e^{g(\overline{b})}e^{\overline{T}}e^{p_1}e^{-h(\overline{a})}$ and so $\widetilde{\mathrm{Def}}_{(h,g)}$ is a predeformation functor.

Now, we prove that $\widetilde{\mathrm{Def}}_{(h,g)}$ is a deformation functor.

Actually, if $C\in\mathbf{C}_0$ then

$$\widetilde{\mathrm{MC}}_{(h,g)}(C)=\{(l,n,e^m)\in(L\otimes C)^1\times(N\otimes C)^1\times\exp(M\otimes C)^0|$$
$$dl=dn=0,\ g(n)=e^m*h(l)=h(l)-dm\}=Z^1(\mathrm{C}_{(h,g)}\otimes C).$$

and

$$\widetilde{\mathrm{Def}}_{(h,g)}(C) = \widetilde{\mathrm{MC}}_{(h,g)}(C) \;/ \; \{(-da, -db, dc+g(b)-h(a)) | \, a \in (L\otimes C)^0, \; b \in (N\otimes C)^0, \; c \in (M\otimes C)^{-1}\}. \tag{9}$$

Therefore $\widetilde{\mathrm{Def}}_{(h,g)}(C)$ is isomorphic to the first cohomology group of the suspended cone of the couple of morphism $h : L \otimes C \longrightarrow M \otimes C$ and $g : N \otimes C \longrightarrow M \otimes C$.

If C is also acyclic, then $\widetilde{\mathrm{Def}}_{(h,g)}(C) = 0$. This implies that $\widetilde{\mathrm{Def}}_{(h,g)}$ satisfies the condition of Definition III.2.4 and so it is a deformation functor.

Finally equation (9) implies also that $T^i\widetilde{\mathrm{Def}}_{(h,g)} \cong H^i(C^{\cdot}_{(h,g)})$.

□

Proof of Lemma III.2.9. It is sufficient to apply the previous theorem with $M = N = 0$. □

III.2.27. Theorem. $\widetilde{\mathrm{Def}}_{(h,g)} \cong \widetilde{\mathrm{MC}}^{+}_{(h,g)}$.

Proof. The projection to the quotient $\widetilde{\mathrm{MC}}_{(h,g)} \longrightarrow \widetilde{\mathrm{Def}}_{(h,g)}$ induces, by Theorem III.2.15, a map $\widetilde{\mathrm{MC}}^{+}_{(h,g)} \longrightarrow \widetilde{\mathrm{Def}}_{(h,g)}$ that is a quasi isomorphism by Proposition III.2.22 and Theorem III.2.26. □

III.2.28. Corollary. *Let M be concentrated in non negative degree. Then for every $(A, m_A) \in$ **Art** we have $\widetilde{\mathrm{Def}}_{(h,g)}(m_A) = \mathrm{Def}_{(h,g)}(A)$.*

Proof. Evident. □

III.2.29. Remark. Every commutative diagram of differential graded Lie algebras

$$\begin{array}{ccccc}
 & & L & \xrightarrow{\alpha'} & P \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \eta} \\
 & & M & \xrightarrow{\alpha} & Q \\
 & \nearrow{\scriptstyle g} & & \nearrow{\scriptstyle \mu} & \\
N & \xrightarrow{\alpha''} & R & &
\end{array}$$

induces a natural transformation $\widetilde{F}$ of the associated deformation functors:

$$\widetilde{F} : \widetilde{\mathrm{Def}}_{(h,g)} \longrightarrow \widetilde{\mathrm{Def}}_{(\eta,\mu)}$$

Moreover the inverse function Theorem III.2.14 implies that $\widetilde{F}$ is an isomorphism if and only if the maps $(\alpha', \alpha, \alpha'')$ induces a quasi isomorphism of complexes $\varphi^{\cdot} : \mathrm{C}^{\cdot}_{(h,g)} \longrightarrow \mathrm{C}^{\cdot}_{(\eta,\mu)}$.

Proof of Theorem III.1.22. It is sufficient to apply the previous remark and Corollary III.2.28.

□

Proof of Theorem III.1.23. It is sufficient to apply the inverse function Theorem III.2.14 and Corollary III.2.28.

$\square$

III.2.5. Fibred product. In Example I.3.12, we have defined a DGLA structure on $M[t,dt] = M \otimes \mathbb{C}[t,dt]$ and evaluation morphisms e_a, for each $a \in \mathbb{C}$: i.e.

$$e_a : M[t,dt] \longrightarrow M \qquad e_a(\sum m_i t^i + n_i t^i dt) = \sum m_i a^i.$$

Define $K \subset L \times N \times M[t,dt] \times M[s,ds]$ as follows

$$K = \{(l,n,m_1(t,dt),m_2(s,ds)) |\, h(l) = e_1(m_2(s,ds)), g(n) = e_0(m_1(t,dt))\}.$$

K is a DGLA with bracket and differential δ defined as the natural ones on each component.

Define the following morphisms of DGLAs:

$$e_0 : K \longrightarrow M \qquad (l,n,m_1(t,dt),m_2(s,ds)) \longmapsto e_0(m_1(t,dt))$$

and

$$e_1 : K \longrightarrow M \qquad (l,n,m_1(t,dt),m_2(s,ds)) \longmapsto e_1(m_2(s,ds)).$$

Then we can construct the following simplicial diagram of DGLA

$$\begin{tikzcd}
L \times N \arrow[r, "G"] \arrow[d, shift right, "h"'] \arrow[d, shift left, "g"] & K \subset L \times N \times M[t,dt] \times M[s,ds] \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_0"] \\
M \arrow[r, "id"] & M
\end{tikzcd} \tag{10}$$

with:

$$\begin{tikzcd}
(l,n) \arrow[r, "G"] \arrow[d, "g"] \arrow[dd, bend right, "h"'] & (l,n,g(n),h(l)) \arrow[d, "e_0"] \arrow[dd, bend right, "e_1"'] \\
g(n) \arrow[r, "id"] & g(n) \\
h(l) \arrow[r, "id"] & h(l).
\end{tikzcd}$$

The diagram is commutative in a simplicial meaning and G is a quasi-isomorphism.

III.2.30. Lemma. *The complexes $C^{\cdot}_{(e_1-e_0)}$ and $C^{\cdot}_{(h-g)}$ are quasi-isomorphic.*

Proof. Consider the following commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & (M^{\cdot -1}, -d) \arrow[r] \arrow[d, "id"] & C^{\cdot}_{(h-g)} \arrow[r] \arrow[d, "\alpha"] & (L^{\cdot} \oplus N^{\cdot}, d) \arrow[r] \arrow[d, "G"] & 0 \\
0 \arrow[r] & (M^{\cdot -1}, -d) \arrow[r] & C^{\cdot}_{(e_1-e_0)} \arrow[r] & (K^{\cdot}, \delta) \arrow[r] & 0,
\end{tikzcd}$$

with $\alpha(l,n,m) = (l,n,g(n),h(l),m)$, for $(l,n,m) \in C^{\cdot}_{(h-g)}$. Since id and G are quasi-isomorphism, then α is a quasi-isomorphism. $\square$

III.2.31. Proposition. $\widetilde{\mathrm{Def}}_{(h,g)} \cong \widetilde{\mathrm{Def}}_{(e_1,e_0)}$.

Proof. Since G is a morphism of DGLAs, using diagram (10), we can define a morphism of Maurer-Cartan functors:

$$G : \widetilde{\mathrm{MC}}_{(h,g)} \longrightarrow \widetilde{\mathrm{MC}}_{(e_1,e_0)}$$
$$(l,n,m) \longmapsto (G(l,n),m), \quad \text{with } G(l,n) = (l,n,g(n),h(l)).$$

It is well-defined since $G(l,n)$ satisfies the Maurer-Cartan equation and

$$e^m * e_1(G(l,n)) = e^m * e_1(h(l)) = e^m * h(l) = g(n) = e_0(g(n)) = e_0(G(l,n)).$$

Therefore G induces a morphism between the deformation functors $\widetilde{\mathrm{Def}}_{(h,g)}$ and $\widetilde{\mathrm{Def}}_{(e_1,e_0)}$ that is a quasi-isomorphism by Lemma III.2.30 (and so an isomorphism by Theorem III.2.14). $\square$

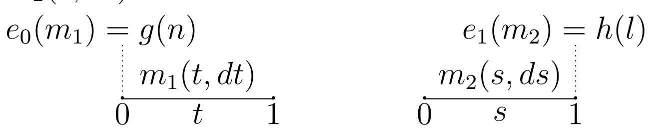
III.2.5.1. *Definition of* $\mathrm{H}_{(h,g)}$*, properties and barycenter subdivision.* Let $H \subset K$ defined as follow

$$H = \{(l,n,m_1(t,dt),m_2(s,ds)) \in K | \; e_1(m_1(t,dt)) = e_0(m_2(s,ds))\},$$

or written in more details

$$H = \{(l,n,m_1(t,dt),m_2(s,ds)) \in L \times N \times M[t,dt] \times M[s,ds] \mid$$
$$h(l) = e_1(m_2(s,ds)), \; g(n) = e_0(m_1(t,dt)), \; e_1(m_1(t,dt)) = e_0(m_2(s,ds))\}.$$

Let $k = (l,n,m_1(t,dt),m_2(s,ds)) \in K$. Then the couple $m_1(t,dt)$ and $m_2(s,ds)$ have fixed values in one of the extreme of the unit interval. More precisely the value of $m_1(t,dt)$ is fixed at the origin and $m_2(s,ds)$ is fixed in 1: i.e.



If k also lies in H, then there are conditions on the other extremes: the value of $m_1(t,dt)$ in 1 has to coincide with the value of $m_2(s,ds)$ in 0.

Let

$$\mathrm{H}_{(h,g)} = \tag{11}$$
$$\{(l,n,m(t,dt)) \in L \times N \times M[t,dt] \mid h(l) = e_1(m(t,dt)), \; g(n) = e_0(m(t,dt))\}.$$

Since e_i are morphisms of DGLA it is clear that $\mathrm{H}_{(h,g)}$ is a DGLA.

Moreover, considering the barycenter subdivision we get an injective quasi isomorphism

$$\mathrm{H}_{(h,g)} \hookrightarrow H.$$

$$(l,n,m(t,dt)) \longmapsto (l,n,m\,(\frac{1}{2}\,t,dt),m\,(\frac{s+1}{2},ds))$$

III.2.32. Definition. $\mathrm{H}_{(h,g)}$ is the *differential graded Lie algebra associated to the couple* (h,g).

III.2.33. Proposition. $\mathrm{H}_{(h,g)}$ *is a quasi isomorphic to the complex* $C^{\cdot}_{e_1-e_0}$.

Proof. It is sufficient to consider the following commutative diagram of complexes

$$\begin{tikzcd} H \arrow[r, hook] \arrow[d] & K \arrow[d, "e_1-e_0"] \\ 0 \arrow[r] & M. \end{tikzcd}$$

where $e_1 - e_0$ is surjective. $\square$

III.2.34. Proposition. *Let* $h : L \longrightarrow M$ *and* $g : N \longrightarrow M$ *be morphism of DGLAs. If the morphism* $g - h : N \times L \longrightarrow M$ *is surjective, then* $\widetilde{\mathrm{Def}}_{L\times_N M}$ *is isomorphic to* $\widetilde{\mathrm{Def}}_{(h,g)}$.

Proof. We recall that by definition

$$\widetilde{\mathrm{MC}}_{L\times_M N}(A) =$$

$$\{(l,n) \in (L\otimes A)^1 \times (N\otimes A)^1 | dl + \frac{1}{2}[l,l] = 0,\ dn + \frac{1}{2}[n,n] = 0,\ h(l) = g(n)\},$$

and

$$\widetilde{\mathrm{MC}}_{(h,g)}(A) = \{(l,n,m) \in (L\otimes A)^1 \times (N\otimes A)^1 \times (M\otimes A)^0 |$$

$$dl + \frac{1}{2}[l,l] = 0,\ dn + \frac{1}{2}[n,n] = 0,\ g(n) = e^m * h(l)\}.$$

Moreover $T^i\widetilde{\mathrm{Def}}_{L\times_M N} \cong H^i(L\times_M N)$ and $T^i\widetilde{\mathrm{Def}}_{(h,g)} \cong H^i(C^{\cdot}_{(h,g)})$.

Let

$$\psi : \widetilde{\mathrm{MC}}_{L\times_M N}(A) \longrightarrow \widetilde{\mathrm{MC}}_{(h,g)}(A)$$

$$(l,n) \longmapsto (l,n,0).$$

and $\varphi : \widetilde{\mathrm{Def}}_{L\times_N M} \longrightarrow \widetilde{\mathrm{Def}}_{(h,g)}$ the induced map between the associated extended deformation functors. Then φ is a quasi isomorphism.

For completeness we state all details, denoting by the same φ the map induced on cohomology.

φ is injective. Suppose that $\varphi([(l,n)]) = [(l,n,0)] = 0$ in $H^i(C^{\cdot}_{(h,g)})$. Then $[(l,n,0)] = (dr, ds, -dt - g(s) + h(r))$ with $r \in (L)^{i-1}$, $s \in (N)^{i-1}$ and $t \in (M)^{i-2}$.

Since $g-h$ is surjective, there exists $p \in L^{i-2} \otimes A$ and $q \in N^{i-2} \otimes A$ such that $g(q) - h(p) = -t$ and so $g(dq) - h(dp) = -dt$. Let $(l', n') = (r-dp, s-dq)$; then $h(l') = h(r) - h(dp) = g(s) + dt - dt - g(dq) = g(n')$.

Therefore $(l', n') \in L \times_M N$ and $d(l', n') = (dr, ds) = (l, n)$ and so $[(l, n)] = 0 \in H^i(L \times_M N)$.

φ is surjective. Let $[(l, n, m)] \in H^i(C^{\cdot}_{(h,g)})$, i.e. $dl = dn = 0$ and $-dm - g(n) + h(l) = 0$. Since the class $[(l, n, m)]$ coincides with the class $[l + dr, n + ds, m - dt - g(s) + h(r)] \in H^i(C^{\cdot}_{(h,g)})$, we are looking for $r \in L^{i-1}$, $s \in N^{i-1}$ and $t \in M^{i-2}$ such that $(l+dr, n+ds) \in H^i(L \times_M N)$ and $m - dt - g(s) + h(r) = 0$ (thus $[(l, n, m)] \in \mathrm{Im}(\varphi)$).

Since $g - h$ is surjective, there exists $r \in L^{i-1}$ and $s \in N^{i-1}$, such that $g(s) - h(r) = m - dt$ and so $g(ds) - h(dr) = dm$. Therefore $h(l + dr) = h(l) + g(ds) - dm = h(l) + g(ds) - h(l) + g(n) = g(n + ds)$, that is $(l+dr, n+ds) \in L \times_M N$ and $\varphi([(l+dr, n+ds)]) = [(l, n, m)]$. $\square$

III.2.35. Remark. Let $\alpha_0, \alpha_1 : L \longrightarrow M$ be morphisms of DGLAs with $\alpha_0 - \alpha_1$ surjective and $T = \{t \in L |\, \alpha_0(t) = \alpha_1(t)\}$ (T is called the equalizer of α_0 and α_1). In this particular case the previous proposition implies $\widetilde{\mathrm{Def}}_T \cong \widetilde{\mathrm{Def}}_{(\alpha_1, \alpha_0)}$.

In conclusion we have the following theorem.

III.2.36. Theorem. $\widetilde{\mathrm{Def}}_{\mathrm{H}_{(h,g)}} \cong \widetilde{\mathrm{Def}}_{(h,g)}$.

Proof. It is sufficient to apply Proposition III.2.34 to $e_0, e_1 : K \longrightarrow M$ (with $e_0 - e_1$ surjective) to conclude that $\widetilde{\mathrm{Def}}_{\mathrm{H}_{(h,g)}} \cong \widetilde{\mathrm{Def}}_{(e_1, e_0)}$ and then Proposition III.2.31. $\square$

CHAPTER IV

Deformations of holomorphic maps

This chapter is devoted to the main topic of this thesis: infinitesimal deformations of holomorphic maps of complex compact manifolds.

These deformations were first studied dring the 70s by E. Horikawa in his works ([**14**] and [**15**]) and then by Namba [**27**] and Ran [**28**].

Our purpose is to study these deformations using a technique based on differential graded Lie algebra.

Beware. Unless otherwise specified X *and* Y *are compact complex connected smooth varieties.*

IV.1. Deformations functor Def(f) of holomorphic maps

The main references of this section are [**14**] and [**27**, Section 3.6].

IV.1.1. Definition. Let $f : X \longrightarrow Y$ be an holomorphic map and $A \in$ **Art**. An *infinitesimal deformation of* f *with fixed domain and target over* Spec(A) is a commutative diagram

$$\begin{tikzcd} X \times S \arrow[rr, "\mathcal{F}"] \arrow[dr] & & Y \times S \arrow[dl] \\ & S & \end{tikzcd}$$

where $S = \operatorname{Spec}(A)$, the morphism to S are the projections, $\mathcal{F}$ is a holomorphic map and f coincides with the restriction of $\mathcal{F}$ to the fibers over the closed point of S.

If $A = \mathbb{K}[\varepsilon]$ we have a *first order deformation of* f *with fixed domain and target*.

Two infinitesimal deformations of f with fixed domain and target

$$\begin{tikzcd} X \times S \arrow[rr, "\mathcal{F}"] \arrow[dr] & & Y \times S \arrow[dl] \\ & S & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \times S \arrow[rr, "\mathcal{F}'"] \arrow[dr] & & Y \times S \arrow[dl] \\ & S & \end{tikzcd}$$

are equivalent if there exist automorphisms $\phi : X \longrightarrow X$ and $\psi : Y \longrightarrow Y$ such that the following diagram is commutative:

$$\begin{tikzcd} X \times S \arrow[r, "\mathcal{F}"] \arrow[d, "(\phi,id)"'] & Y \times S \arrow[d, "(\psi,id)"] \\ X \times S \arrow[r, "\mathcal{F}'"] & Y \times S. \end{tikzcd}$$

IV.1.2. Definition. The *infinitesimal deformation functor* $\mathrm{Def}(X \xrightarrow{f} Y)$ of infinitesimal deformation of an holomorphic map f with fixed domain and target is defined as follows:

$$\mathrm{Def}(X \xrightarrow{f} Y) : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$A \longmapsto \mathrm{Def}(X \xrightarrow{f} Y)(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations of } f \\ \text{with fixed domain and target} \\ \text{over } \mathrm{Spec}(A) \end{array} \right\}$$

IV.1.3. Remark. When the domain and target are fixed, an infinitesimal deformation of an holomorphic map f can be interpreted as an infinitesimal deformation of the graph of the map f in the product $X \times Y$, with $X \times Y$ fixed.

In the previous case, we were just deforming the map f. In general we can also deform both the domain and the target.

IV.1.4. Definition. Let $f : X \longrightarrow Y$ be an holomorphic map and $A \in \mathbf{Art}$. An *infinitesimal deformation of* f *over* $\mathrm{Spec}(A)$ is a commutative diagram of complex spaces

$$\begin{tikzcd} X_A \arrow[rr, "\mathcal{F}"] \arrow[dr, "\pi"'] & & Y_A \arrow[dl, "\mu"] \\ & S & \end{tikzcd}$$

where $S = \mathrm{Spec}(A)$, (X_A, π, S) and (Y_A, π, S) are infinitesimal deformations of X and Y respectively (Definition I.2.1), $\mathcal{F}$ is an holomorphic map that restricted to the fibers over the closed point of S coincides with f.

If $A = \mathbb{K}[\varepsilon]$ we have a *first order deformation* of f.

IV.1.5. Definition. Let

$$\begin{tikzcd} X_A \arrow[rr, "\mathcal{F}"] \arrow[dr, "\pi"'] & & Y_A \arrow[dl, "\mu"] \\ & S & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X'_A \arrow[rr, "\mathcal{F}'"] \arrow[dr, "\pi'"'] & & Y'_A \arrow[dl, "\mu'"] \\ & S & \end{tikzcd}$$

two infinitesimal deformations of f. They are *equivalent* if there exist bi-holomorphic maps $\phi : X_A \longrightarrow X'_A$ and $\psi : Y_A \longrightarrow Y'_A$ (that are equivalence of infinitesimal deformations of X and Y respectively) such that the following diagram is commutative:

$$\begin{tikzcd} X_A \arrow[r, "\mathcal{F}"] \arrow[d, "\phi"'] & Y_A \arrow[d, "\psi"] \\ X'_A \arrow[r, "\mathcal{F}'"] & Y'_A. \end{tikzcd}$$

IV.1.6. Definition. The *functor of infinitesimal deformations* of an holomorphic map $f : X \longrightarrow Y$ is

$$\mathrm{Def}(f) : \mathbf{Art} \longrightarrow \mathbf{Set}$$

$$A \longmapsto \mathrm{Def}(f)(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations of} \\ f \text{ over } \mathrm{Spec}(A) \end{array} \right\}$$

IV.1.7. Proposition. Def(f) *satisfies the conditions of Definition I.1.10.*

Proof. It follows from the fact that the functors Def_X and Def_Y of infinitesimal deformations of X and Y are deformation functors. □

IV.1.8. Remark. In this general case, the infinitesimal deformations of f can be interpreted as infinitesimal deformations $\widetilde{\Gamma}$ of the graph Γ of the map f in the product $X \times Y$, such that the induced deformations $\widetilde{X \times Y}$ of $X \times Y$ are products of infinitesimal deformations of X and of Y. Since not all the deformations of a product are product of deformations (see Remark II.7.5), we are not simply considering the deformations of the graph in the product.

Moreover, with this interpretation, two infinitesimal deformations $\widetilde{\Gamma} \subset \widetilde{X \times Y}$ and $\widetilde{\Gamma}' \subset \widetilde{X \times Y}'$ are equivalent if there exists an isomorphism $\phi : \widetilde{X \times Y} \longrightarrow \widetilde{X \times Y}'$ of infinitesimal deformations of $X \times Y$ such that $\phi(\widetilde{\Gamma}) = \widetilde{\Gamma}'$.

IV.1.1. Tangent and obstruction spaces of Def(f)**.** Let $\mathcal{U} = \{U_i\}$ and $\mathcal{W} = \{W_j\}$ be Stein coves of X and Y respectively, such that $f(U_i) \subset V_i$ for each i. For any integer $p \geq 0$, let

$$C^p(\mathcal{U}, \mathcal{W}, \Theta_X, \Theta_Y, f^*\Theta_Y) = \check{C}^p(\mathcal{U}, \Theta_X) \oplus \check{C}^p(\mathcal{W}, \Theta_Y) \oplus \check{C}^{p-1}(\mathcal{U}, f^*\Theta_Y)$$

($\check{C}^{-1}(\mathcal{U}, f^*\Theta_Y) = 0$). Define a linear map

$$\check{D} : C^p(\mathcal{U}, \mathcal{W}, \Theta_X, \Theta_Y, f^*\Theta_Y) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{W}, \Theta_X, \Theta_Y, f^*\Theta_Y)$$

$$(x, y, z) \longmapsto (\check{\delta}x, \check{\delta}y, \check{\delta}z + (-1)^p(f_*x - f^*y)).$$

Using the equalities $f_*\check{\delta} = \check{\delta}f_*$ and $f^*\check{\delta} = \check{\delta}f^*$, we conclude that $\check{D}$ is a differential ($\check{D} \circ \check{D} = 0$). Therefore the $\mathbb{C}$-vector spaces of cohomology $\check{H}^p(\mathcal{U}, \mathcal{W}, \Theta_X, \Theta_Y, f^*\Theta_Y)$ are well defined.

IV.1.9. Lemma. $\check{H}^{\cdot}(\mathcal{U}, \mathcal{W}, \Theta_X, \Theta_Y, f^*\Theta_Y)$ *doesn't depend on the choice of the covers and so we denote it* $\check{H}^{\cdot}(\Theta_X, \Theta_Y, f^*\Theta_Y)$.

Proof. The following linear maps are well defined:

$$\check{H}^p(\mathcal{U},\Theta_X)\oplus\check{H}^p(\mathcal{V},\Theta_Y)\longrightarrow\check{H}^p(\mathcal{U},f^*\Theta_Y)$$

$$(\{n_1\},\{n_2\})\longmapsto\{f_*n_1-f^*n_2\};$$

$$\check{H}^p(\mathcal{U},f^*\Theta_Y)\longrightarrow\check{H}^{p+1}(\mathcal{U},\mathcal{W},\Theta_X,\Theta_Y,f^*\Theta_Y)$$

$$\{a\}\longmapsto\{(0,0,a)\};$$

$$H^p(\mathcal{U},\mathcal{W},\Theta_X,\Theta_Y,f^*\Theta_Y)\longrightarrow\check{H}^p(\mathcal{U},\Theta_X)\oplus\check{H}^p(\mathcal{V},\Theta_Y)$$

$$\{(n_1,n_2,a)\}\longmapsto(\{n_1\},\{n_2\}).$$

Then the sequence below is exact:

$$\cdots\longrightarrow\check{H}^p(\mathcal{U},\Theta_X)\oplus\check{H}^p(\mathcal{V},\Theta_Y)\longrightarrow\check{H}^p(\mathcal{U},f^*\Theta_Y)\longrightarrow$$

$$\longrightarrow\check{H}^{p+1}(\mathcal{U},\mathcal{W},\Theta_X,\Theta_Y,f^*\Theta_Y)\longrightarrow$$

$$\longrightarrow\check{H}^{p+1}(\mathcal{U},\Theta_X)\oplus\check{H}^{p+1}(\mathcal{V},\Theta_Y)\longrightarrow\check{H}^{p+1}(\mathcal{U},f^*\Theta_Y)\longrightarrow\cdots.$$

Moreover, $\check{H}^{\cdot}(\mathcal{U},\Theta_X)$, $\check{H}^{\cdot}(\mathcal{U},\Theta_X)$ and $\check{H}^{\cdot}(\mathcal{U},f^*\Theta_Y)$ doesn't depend on the choice of the Stein covers $\mathcal{U}$ and $\mathcal{V}$. Hence applying the five lemma , $\check{H}^{\cdot}(\mathcal{U},\mathcal{W},\Theta_X,\Theta_Y,f^*\Theta_Y)$ doesn't depend on the choice of the covers. □

In [**14**] E. Horikawa used the vector spaces $\check{H}^{\cdot}(\Theta_X,\Theta_Y,f^*\Theta_Y)$ to describe the tangent and obstruction spaces of the deformation functor $\mathrm{Def}(f)$.

IV.1.10. Theorem. *$\check{H}^1(\Theta_X,\Theta_Y,f^*\Theta_Y)$ is in 1-1 correspondence with the first order deformations of $f:X\longrightarrow Y$.*

The obstructions space is naturally contained in $\check{H}^2(\Theta_X,\Theta_Y,f^*\Theta_Y)$.

Proof. See [**27**, Section 3.6] □

IV.1.11. Remark. Consider a first order deformation f_ε of f: in particular we are considering a first order deformations X_ε and Y_ε of X and of Y, respectively. Using Theorem I.2.8, we associate to X_ε a class $x\in\check{H}^1(X,\Theta_X)$ and to Y_ε a class $y\in\check{H}^1(Y,\Theta_Y)$.

Therefore the class in $\check{H}^1(\Theta_X,\Theta_Y,f^*\Theta_Y)$ associated to f_ε is $[(x,y,z)]$ with $z\in\check{C}^0(\mathcal{U},f^*\Theta_Y)$ such that $\check{\delta}z=f_*x-f^*y$.

Analogously, let $0\longrightarrow J\longrightarrow B\longrightarrow A\longrightarrow 0$ be a small extension and f_A an infinitesimal deformation of f over $\mathrm{Spec}(A)$. If $h\in\check{H}^2(X,\Theta_X)$ and $k\in\check{H}^2(Y,\Theta_Y)$ are the obstruction class associated to X_A and Y_A respectively, then the obstruction class in $\check{H}^2(\Theta_X,\Theta_Y,f^*\Theta_Y)$ associated to f_A is $[(h,k,r)]$, with $r\in\check{C}^1(\mathcal{U},f^*\Theta_Y)$ such that $\check{\delta}r=-(f_*x-f^*y)$.

IV.1.12. Remark. We have defined the $\mathbb{C}$-vector space $\check{H}^*(\Theta_X, \Theta_Y, f^*\Theta_Y)$ using the Čech cohomology. For convenience we reinterpret it, using the Dolbeault cohomology.

Let $(B^\cdot, D_{\overline{\partial}})$ be the complex below

$$B^p = A_X^{(0,p)}(\Theta_X) \oplus A_Y^{(0,p)}(\Theta_Y) \oplus A_X^{(0,p-1)}(f^*\Theta_Y)$$

and

$$D_{\overline{\partial}} : B^p \longrightarrow B^{p+1} \qquad (x, y, z) \longmapsto (\overline{\partial}x, \overline{\partial}y, \overline{\partial}z + (-1)^p(f_*x - f^*y))$$

IV.1.13. Lemma. *The complexes* $(B^\cdot, D_{\overline{\partial}})$ *and* $(C^\cdot(\mathcal{U}, \mathcal{W}, \Theta_X, \Theta_Y, f^*\Theta_Y), \check{D})$ *are quasi isomorphic.*

Proof. Let $\mathcal{U} = \{U_i\}$ and $\mathcal{W} = \{W_j\}$ as above and denote by $\phi_1 : \check{C}^*(\mathcal{U}, \Theta_X) \longrightarrow A_X^{0,*}(\Theta_X)$, $\phi_2 : \check{C}^*(\mathcal{V}, \Theta_Y) \longrightarrow A_Y^{0,*}(\Theta_Y)$ and $\phi_3 : \check{C}^*(\mathcal{U}, f^*\Theta_X) \longrightarrow A_X^{0,*}(f^*\Theta_X)$ the quasi isomorphism of complexes of Leray's theorem, defined in Section II.3.1. We recall that $\phi_i\check{\delta} = \overline{\partial}\phi_i$, $f_*\phi_i = \phi_i f_*$ and $f^*\phi_i = \phi_i f^*$, for each $i = 1, 2, 3$.

Now, define the following morphism

$$\gamma : (C^\cdot(\mathcal{U}, \mathcal{W}, \Theta_X, \Theta_Y, f^*\Theta_Y), \check{D}) \longrightarrow (B^\cdot, D_{\overline{\partial}})$$
$$\gamma(x, y, z) = (\phi_1(x), \phi_2(y), \phi_3(z)).$$

Then γ is a morphism of complexes: for each $(x, y, z) \in C^p(\mathcal{U}, \mathcal{W}, \Theta_X, \Theta_Y, f^*\Theta_Y)$

$$\begin{tikzcd}
(x,y,z) \arrow[r, "\gamma"] \arrow[d, "\check{D}"'] & (\phi_1(x), \phi_2(y), \phi_3(z)) \arrow[d, "D_{\overline{\partial}}"] \\
(\check{\delta}x, \check{\delta}y, \check{\delta}z + (-1)^p(f_*x - f^*y)) \arrow[r] & b
\end{tikzcd}$$

where $b = (\overline{\partial}\phi_1(x), \overline{\partial}\phi_2(y), \overline{\partial}\phi_3(z) + (-1)^p(f_*\phi_1(x) - f^*\phi_2(y)))$.

Moreover, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \check{C}^{\cdot-1}(\mathcal{U}, f^*\Theta_Y) \arrow[r] \arrow[d, "\phi_3"] & \mathcal{C}^\cdot \arrow[r] \arrow[d, "\gamma"] & \check{C}^\cdot(\mathcal{U}, \Theta_X) \oplus \check{C}^\cdot(\mathcal{W}, \Theta_Y) \arrow[r] \arrow[d, "(\phi_1,\phi_2)"] & 0 \\
0 \arrow[r] & A_X^{(0,\cdot-1)}(f^*\Theta_Y) \arrow[r] & B^\cdot \arrow[r] & A_X^{(0,\cdot)}(\Theta_X) \oplus A_Y^{(0,\cdot)}(\Theta_Y) \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{C}^\cdot$ stands for $C^\cdot(\mathcal{U}, \mathcal{W}, \Theta_X, \Theta_Y, f^*\Theta_Y)$.

Since ϕ_3 and (ϕ_1, ϕ_2) are quasi isomorphism, γ is a quasi isomorphism.

□

IV.2. Infinitesimal deformations of holomorphic maps

Let $f : X \longrightarrow Y$ be an holomorphic map and Γ its graph in $Z := X \times Y$.

Let

$$F : X \longrightarrow \Gamma \subseteq Z := X \times Y$$

$$x \longmapsto (x, f(x)).$$

and $p : Z \longrightarrow X$ and $q : Z \longrightarrow Y$ the natural projections

Then we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "F"] \arrow[dr, "id"'] \arrow[drr, "f"] & & Z \arrow[dl, "p"'] \arrow[d, "q"] \\
& X & Y.
\end{tikzcd}$$

In particular, $F^* \circ p^* = id$ and $F^* \circ q^* = f^*$.

Since $\Theta_Z = p^*\Theta_X \oplus q^*\Theta_Y$, it follows that $F^*(\Theta_Z) = \Theta_X \oplus f^*\Theta_Y$.

Define the morphism $\gamma : \Theta_Z \longrightarrow f^*\Theta_Y$ as the composition

$$\Theta_Z \xrightarrow{F^*} \Theta_X \oplus f^*\Theta_Y \xrightarrow{(f^*,-id)} f^*\Theta_Y.$$

and let π be the following surjective morphism:

$$\mathcal{A}^{0,j}_Z(\Theta_Z) \xrightarrow{\pi} \mathcal{A}^{0,j}_X(f^*\Theta_Y) \longrightarrow 0$$

$$\pi(\omega\, u) = F^*(\omega)\gamma(u) \qquad \forall\ \omega \in \mathcal{A}^{0,j}_Z,\ u \in \Theta_Z.$$

Now, since each $u \in \Theta_Z$ can be written as $u = p^*v_1 + q^*v_2$, for some $v \in \Theta_X$ and $w \in \Theta_Y$, we have also

$$\pi(\omega u) = F^*(\omega)(f_*(v_1) - f^*(v_2)).$$

Since $F^*\overline{\partial} = \overline{\partial}F^*$, π is a morphism of complexes.

For convenience we give an explicit description of the map π.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_i\}_{i\in I}$ be finite Stein open covers of X and Y, respectively, such that $f(U_i) \subset V_i$ (U_i is allowed to be empty). Moreover let $z = (z_1, z_2, \dots, z_n)$ on U_i and $w = (w_1, w_2, \dots w_m)$ on V_i be local holomorphic coordinate systems for each $i \in I$. As in Section II.6, if $v_1 = \displaystyle\sum_{i=1}^{n} \varphi_i(z)\frac{\partial}{\partial z_i}$ and $v_2 = \displaystyle\sum_{h=1}^{m} \psi_h(w)\frac{\partial}{\partial w_h}$ then $f_*v_1 = \displaystyle\sum_{i=1}^{n} \varphi_i(z) \sum_{h=1}^{m} \frac{\partial f_h}{\partial z_i}\frac{\partial}{\partial w_h}$ and $f^*v_2 = \displaystyle\sum_{h=1}^{m} \psi_h(f(z))\frac{\partial}{\partial w_h}$. Let K and J be multi-indexes of length respectively k and j, and fix $\omega = \Phi(z,w)d\overline{z}_K \wedge d\overline{z}_J \in \mathcal{A}^{0,k+j}_Z$. Then

$$\pi(\omega u) = F^*(\omega)(f_*(v_1) - f^*(v_2)) =$$

$$\Phi(z, f(z))d\overline{z}_K \wedge \overline{\partial} f_J \sum_{h=1}^{m} \left(\sum_{i=1}^{n} \varphi_i(z)\frac{\partial f_h}{\partial z_i} - \psi_h(f(z)) \right) \frac{\partial}{\partial w_h}.$$

Let $\mathcal{L}$ be the kernel of π:

$$0 \longrightarrow \mathcal{L} \xrightarrow{h} \mathcal{A}^{0,*}_Z(\Theta_Z) \xrightarrow{\pi} \mathcal{A}^{0,*}_X(f^*\Theta_Y) \longrightarrow 0$$

and h the inclusion.

IV.2.1. Lemma. *$\mathcal{L}$ is a sheaf of differential graded subalgebra of $\mathcal{A}^{0,*}_Z(\Theta_Z)$ and h is a morphism of differential graded Lie algebras.*

Proof. There is a canonical isomorphism between the normal bundle $N_{\Gamma|Z}$ of Γ in Z and the pull-back f^*T_Y.

Therefore there exists the following exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{h} \mathcal{A}^{0,*}_Z(\Theta_Z) \xrightarrow{\pi} \mathcal{A}^{0,*}_\Gamma(N_{\Gamma|Z}) \longrightarrow 0$$

Then by Lemma II.5.7, $\mathcal{L}$ is a sheaf of differential graded subalgebras of KS_Z.

□

Let L be the differential graded Lie algebra of global section of $\mathcal{L}$.

Let M be the Kodaira-Spencer algebra of the product $X \times Y$: $M = KS_{X\times Y} = KS_Z$ and $h : L \longrightarrow M$ be the inclusion.

Let N be the product of the Kodaira-Spencer algebra: $N = KS_X \times KS_Y$ and $g : KS_X \times KS_Y \longrightarrow KS_{X\times Y}$ be given by $g = p^* + q^*$ (for $n = (n_1, n_2)$, we use both the notation $g(n)$ and $p^*n_1 + q^*n_2$).

Therefore we get a diagram

(12)
$$\begin{tikzcd} & L \arrow[d, hook, "h"] \\ N = KS_X \times KS_Y \arrow[r, "g=(p^*,q^*)"] & M = KS_{X\times Y}. \end{tikzcd}$$

IV.2.2. Remark. Given the morphisms of DGLAs $h : L \longrightarrow KS_{X\times Y}$ and $g : KS_X \times KS_Y \longrightarrow KX_{X\times Y}$ we can consider the complex $(\mathrm{C}^{\cdot}_{(h,g)}, D)$ with $C^i_{(h,g)} = L^i \oplus KS^i_X \oplus KS^i_Y \oplus KS^{i-1}_{X\times Y}$ and differential $D(l, n_1, n_2, m) = (\overline{\partial}l, \overline{\partial}n_1, \overline{\partial}n_2, -\overline{\partial}m - p^*n_1 - q^*n_2 + h(l))$.

Using the morphism $\pi : KS_{X\times Y} \longrightarrow A^{0,*}_X(f^*\Theta_Y)$ we can define a morphism

$$\beta : (\mathrm{C}^{\cdot}_{(h,g)}, D) \longrightarrow (B^{\cdot}, D_{\overline{\partial}})$$

$$\beta(l, n_1, n_2, m) = (n_1, n_2, (-1)^i\pi(m)) \qquad \forall\ (l, n_1, n_2, m) \in \mathrm{C}^i_{(h,g)}\,.$$

IV.2.3. Proposition. *$\beta : (\mathrm{C}^{\cdot}_{(h,g)}, D) \longrightarrow (B^{\cdot}, D_{\overline{\partial}})$ is a morphism of complexes that is a quasi isomorphism.*

Proof. β commute with the differentials, i.e $\beta \circ D = D_{\overline{\partial}} \circ \beta$, in fact for each $(l, n_1, n_2, m) \in C^i_{(h,g)}$ we have the following commutative diagram

$$\begin{tikzcd} (l, n_1, n_2, m) \arrow[r, "\beta"] \arrow[d, "D"'] & (n_1, n_2, (-1)^i\pi(m)) \arrow[d, "D_{\overline{\partial}}"] \\ c \arrow[r, "\beta"] & b. \end{tikzcd}$$

where c stands for $(\overline{\partial}l, \overline{\partial}n_1, \overline{\partial}n_2, -\overline{\partial}m - p^*n_1 - q^*n_2 + h(l))$ and $b = \beta(c) = (\overline{\partial}n_1, \overline{\partial}n_2, (-1)^i(\overline{\partial}\pi(m) + f_*n_1 - f^*n_2))$.

Therefore β induces a map in cohomology that we again call β that is a quasi isomorphism. The proof is standard but we state it.

β is injective. Let $[(l, n_1, n_2, m)] \in H^i(C^{\cdot}_{(h,g)})$[a] be such that $\beta([(l, n_1, n_2, m)]) = [(n_1, n_2, (-1)^i\pi(m))]$ is zero in $H^i(B)$.

Then there exists $(a_1, a_2, b) \in B^{i-1}$ such that $n_1 = \overline{\partial}a_1$, $n_2 = \overline{\partial}a_2$, and $(-1)^i\pi(m) = \overline{\partial}b + (-1)^{i-1}(f_*a_1 - f^*a_2)$, so that $\pi(m) = (-1)^i\overline{\partial}b - f_*a_1 + f^*a_2$.

Let $n'_1 = a_1 \in KS_X^{i-1}$ and $n'_2 = a_2 \in KS_Y^{i-1}$; then $\overline{\partial}n'_1 = n_1$ and $\overline{\partial}n'_2 = n_2$.

Let $z \in KS_{X\times Y}$ be a lifting of $-b$ (i.e. $\pi(z) = -b$) and $l' = m + (-1)^i\overline{\partial}z + p^*n'_1 + q^*n'_2 \in KS_{X\times Y}$. Then $l' \in (L)^{i-1}$, in fact $\pi(l') = \pi(m) - (-1)^i\overline{\partial}b + f_*a_1 - f^*a_2 = 0$; moreover $\overline{\partial}l' = \overline{\partial}m + p^*n_1 + q^*n_2 = l$. Therefore $[(l, n_1, n_2, m)] = [\overline{\partial}(l', n'_1, n'_2, (-1)^i z)]$ is zero in $H^i(C^{\cdot}_{(h,g)})$.

β is surjective. Let $[(a_1, a_2, b)] \in H^i(B)$. Then $\overline{\partial}a_1 = \overline{\partial}a_2 = 0$ and $\overline{\partial}b + (-1)^i(f_*a_1 - f^*a_2) = 0$. Let $m \in KS^{i-1}_{X\times Y}$ be a lift of $(-1)^i b$, i.e. $\pi(m) = (-1)^i b$. Let $n_1 = a_1 \in KS^i_X$ and $n_2 = a_2 \in KS^i_Y$. Finally let $l = \overline{\partial}m + p^*a_1 + q^*a_2 \in KS^i_{X\times Y}$. Then $l \in L^i$, in fact $\pi(l) = \pi(\overline{\partial}m) + f_*a_1 - f^*a_2 = (-1)^i\overline{\partial}b + f_*a_1 - f^*a_2 = 0$.

Since $\overline{\partial}l = \overline{\partial}n_1 = \overline{\partial}n_2 = 0$ and $-\overline{\partial}m - p^*n_1 - q^*n_2 + l = 0$, $[(l, n_1, n_2, m)] \in H^i(C^{\cdot}_{(h,g)})$ and $\beta[(l, n_1, n_2, m)] = [(n_1, n_2, (-1)^i\pi(m))] = [(a_1, a_2, b)] \in H^i(B)$. □

IV.2.1. $\mathrm{Def}_{(h,g)}$ is isomorphic to $\mathrm{Def}(f)$. Using the notation above and diagram (12), consider the functor $\mathrm{Def}_{(h,g)}$. Since h is injective, by Remark III.1.15 for each $(A, m_A) \in \mathbf{Art}$ we have

$$\mathrm{Def}_{(h,g)}(A) = \{(n, e^m) \in (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A) |$$

$$dn + \frac{1}{2}[n, n] = 0,\ e^{-m} * g(n) \in L^1 \otimes m_A\}/gauge.$$

IV.2.4. Remark. Let $(n, e^m) \in \mathrm{Def}_{(h,g)}$. In particular $n = (n_1, n_2)$ satisfies the Maurer-Cartan equation and so $n_1 \in KS_X$ and $n_2 \in KS_Y$. Therefore, there are associated to n infinitesimals deformations X_A of X (induced by n_1) and Y_A of Y (induced by n_2). Moreover, since $g(n)$ satisfies Maurer-Cartan equation in $M = KS_{X\times Y}$, it defines an infinitesimal deformation $(X \times Y)_A$ of $X \times Y$. By construction, the deformation $(X \times Y)_A$ is the product of the deformations X_A and Y_A.

Let $i^* : \mathcal{A}_Z^{0,*} \longrightarrow \mathcal{A}_\Gamma^{0,*}$ be the restriction morphism and let $I = \ker i^* \cap \mathcal{O}_Z$ be the holomorphic ideal sheaf of the graph $\Gamma$of f in Z.

[a]In particular this implies that $-\overline{\partial}m - p^*n_1 - q^*n_2 + l = 0$

Consider an infinitesimal deformation of the holomorphic map f over $\operatorname{Spec}(A)$ as an infinitesimal deformation $\widetilde{\Gamma}$ of Γ over $\operatorname{Spec}(A)$ and $\widetilde{Z}$ of Z over $\operatorname{Spec}(A)$, with $\widetilde{Z}$ product of deformations of X and of Y over $\operatorname{Spec}(A)$.

Applying the previous Remark IV.2.4 and Theorem II.7.3, the condition on the deformation $\widetilde{Z}$ is equivalent to require $\mathcal{O}_{\widetilde{Z}} = \mathcal{O}_A(g(n))$, for some Maurer-Cartan element $n \in KS_X \times KS_Y$.

The deformation $\widetilde{\Gamma}$ of the graph corresponds to an infinitesimal deformation $I_A \subset \mathcal{O}_{\widetilde{Z}}$ of the holomorphic ideal sheaf I over $\operatorname{Spec}(A)$, that is I_A is a sheaf flat over A such that $I_A \otimes_A \mathbb{C} \cong I$.

In conclusion, to give an infinitesimal deformation of f over $\operatorname{Spec}(A)$ (an element in $\operatorname{Def}(f)(A)$) is sufficient to give an ideal sheaf $I_A \subset \mathcal{O}_A(g(n))$ (for some $n \in \operatorname{MC}_{KS_X \times KS_Y}$) with I_A A-flat and $I_A \otimes_A \mathbb{C} \cong I$.

IV.2.5. THEOREM. *Let h, g and i^* be as above. Then there exists an isomorphism of functors*

$$\gamma : \operatorname{Def}_{(h,g)} \longrightarrow \operatorname{Def}(f).$$

Given a local Artinian $\mathbb{C}$-algebra A and an element $(n, e^m) \in \operatorname{MC}_{(h,g)}(A)$, we define a deformation of f over $\operatorname{Spec}(A)$ as a deformation $I_A(n, e^m)$ of the holomorphic ideal sheaf of the graph in the following way

$$\gamma(n, e^m) = I_A(n, e^m) := (\ker(\mathcal{A}_Z^{0,0} \otimes A \overset{\overline{\partial} + l_{g(n)}}{\longrightarrow} \mathcal{A}_Z^{0,1} \otimes A)) \cap e^m (\ker i^* \otimes A) =$$
$$= \mathcal{O}_A(g(n)) \cap e^m (\ker i^* \otimes A),$$

where $\mathcal{O}_A(g(n))$ is the infinitesimal deformation of Z that corresponds to $g(n) \in MC_{KS_{X \times Y}}$ (see Theorem II.7.3).

PROOF. For each $(n, e^m) \in \operatorname{MC}_{(h,g)}(A)$ we have defined

$$I_A(n, e^m) = \mathcal{O}_A(g(n)) \cap e^m (\ker i^* \otimes A).$$

First of all we verify that this sheaf $I_A(n, e^m) \subset \mathcal{O}_A(g(n))$ define an infinitesimal deformation of f: therefore we need to prove that $I_A(n, e^m)$ is flat over A and $I_A \otimes_A \mathbb{C} \cong I$. It is equivalent to verify these properties for $e^{-m} I_A(n, e^m)$.

Applying Lemma II.5.5, we obtain

$$e^{-m}(\mathcal{O}_A(g(n))) = \ker(\overline{\partial} + e^{-m} * g(n) : \mathcal{A}_Z^{0,0} \otimes A \longrightarrow \mathcal{A}_Z^{0,1} \otimes A)$$

and also

$$e^{-m} I_A(n, e^m) = e^{-m}(\mathcal{O}_A(g(n))) \cap (\ker i^* \otimes A) =$$
$$= \ker(\overline{\partial} + e^{-m} * g(n)) \cap (\ker i^* \otimes A).$$

Since flatness is a local property, we can assume that Z is a Stein manifold. Then $H^1(Z, \Theta_Z) = 0$ and $H^0(Z, \Theta_Z) \longrightarrow H^0(Z, N_{\Gamma|Z})$ is surjective. Since the following sequence is exact

$$\cdots \longrightarrow H^0(Z, \Theta_Z) \longrightarrow H^0(Z, N_{\Gamma|Z}) \longrightarrow H^1(Z, L) \longrightarrow H^1(Z, \Theta_Z) \longrightarrow \cdots,$$

we conclude that $H^1(L) = 0$ or equivalently that the tangent space of the functor Def_L is trivial. Therefore, by Corollary I.1.21, Def_L is the trivial functor.

This implies the existence of $\nu \in L^0 \otimes m_A$ such that $e^{-m} * g(n) = e^{\nu} * 0$ (by hypothesis $e^{-m} * g(n)$ is a solution of Maurer-Cartan in L). Moreover, we recall that if $a \in \mathcal{L}^0 \otimes m_A$ then $e^a(\ker i^* \otimes A) = \ker i^* \otimes A$ (see Section II.5.1).

Therefore

$$e^{-m} I_A(n, e^m) = \ker(\overline{\partial} + e^{\nu} * 0) \cap (\ker i^* \otimes A) = \mathcal{O}_A(e^{\nu} * 0) \cap (\ker i^* \otimes A)$$
$$= e^{\nu}(\mathcal{O}_A(0)) \cap e^{\nu}(\ker i^* \otimes A) = e^{\nu}(I \otimes A).$$

Then $I_A(n, e^m)$ defines a deformation of f and so it is well defined the morphism

$$\gamma : \mathrm{MC}_{(h,g)} \longrightarrow \mathrm{Def}(f)$$

such that

$$\gamma(A) : \mathrm{MC}_{(h,g)}(A) \longrightarrow \mathrm{Def}(f)(A)$$
$$(n, e^m) \longmapsto \gamma(n, e^m) = I_A(n, e^m)$$

Moreover γ is also well defined on $\mathrm{Def}_{(h,g)}(A) = \mathrm{MC}_{(h,g)}(A)/gauge$. Actually, for each $a \in L^0 \otimes m_A$ and $b \in N^0 \otimes m_A$, we have

$$\gamma(e^b * n, e^{g(b)} e^m e^a) = \mathcal{O}_A(e^{g(b)} * g(n)) \cap e^{g(b)} e^m e^a (\ker i^* \otimes A) =$$
$$e^{g(b)} \mathcal{O}_A(g(n)) \cap e^{g(b)} e^m (\ker i^* \otimes A) = e^{g(b)} \gamma(n, e^m).$$

This implies that the deformations $\gamma(n, e^m)$ and $\gamma(e^b * n, e^{g(b)} e^m e^a)$ are isomorphic (see Remark IV.1.8).

In conclusion $\gamma : \mathrm{Def}_{(f,g)} \longrightarrow \mathrm{Def}(f)$ is well defined.

In order to prove that γ is an isomorphism of functors it is sufficient to prove that

i) γ is injective;
ii) γ induces a bijective map on the tangent spaces;
iii) γ induces an injective map on the obstruction spaces.

Actually by Corollary I.1.32, conditions *ii)* and *iii)* imply that γ is étale and so surjective.

i) *γ is injective.* Suppose that $\gamma(n, e^m) = \gamma(r, e^s)$, i.e. the deformations induced respectively by (n, e^m) and (r, e^s) are isomorphic. We want to conclude that (n, e^m) is gauge equivalent to (r, e^s): i.e there exist $a \in L^0 \otimes m_A$ and $b \in N^0 \otimes m_A$ such that $e^b * r = n$ and $e^{g(b)} e^s e^a = e^m$.

By hypothesis $\gamma(n, e^m)$ and $\gamma(r, e^s)$ are isomorphic deformations, then, in particular, the deformations induced on Z are isomorphic. This implies that there exists $b \in N^0 \otimes m_A$ such that $e^b * r = n$ and so $e^{g(b)}(\mathcal{O}_A(g(r))) = \mathcal{O}_A(g(n))$. Up to substitute (r, e^s) with its equivalent $(e^b * r, e^{g(b)} e^s)$ we can assume to be in the following situation

$$\mathcal{O}_A(g(n)) \cap e^m(\ker i^* \otimes A) = \mathcal{O}_A(g(n)) \cap e^{m'}(\ker i^* \otimes A).$$

Let $e^a = e^{-m'}e^m$, then

$$e^a(e^{-m}(\mathcal{O}_A(g(n))) \cap (\ker i^* \otimes A)) = e^{-m'}(\mathcal{O}_A(g(n))) \cap (\ker i^* \otimes A).$$

In particular, $e^a(e^{-m}(\mathcal{O}_A(g(n))) \cap (\ker i^* \otimes A)) \subseteq \ker i^* \otimes A$.

Now we prove by induction that $a \in L^0 \otimes m_A$ (thus $e^m = e^{m'}e^a = e^{g(b)}e^s e^a$).

Let $z_1, \ldots, z_n$ be holomorphic coordinates on Z such that $Z \supset \Gamma = \{z_{t+1} = \cdots = z_n = 0\}$. Consider the projection on the residue field

$$e^{-m}(\mathcal{O}_A(g(n))) \cap (\ker i^* \otimes A) \longrightarrow \mathcal{O}_Z \cap \ker i^*.$$

Then $z_i \in \ker i^* \cap \mathcal{O}_Z$, for $i > t$. Since $e^{-m}(\mathcal{O}_A(g(n))) \cap (\ker i^* \otimes A)$ is flat over A we can lift z_i to $\tilde{z}_i = z_i + \varphi_i \in e^{-m}(\mathcal{O}_A(g(n))) \cap (\ker i^* \otimes A)$. By hypothesis

$$e^a(\tilde{z}_i) = e^a(z_i) + e^a(\varphi_i) \in \ker i^* \otimes A. \tag{13}$$

By Lemma II.5.7, to prove that $a \in L^0 \otimes m_A$ it is sufficient to verify that $e^a(z_i) \in \ker i^* \otimes A$ and so by (13) that $e^a(\varphi_i) \in \ker i^* \otimes A$.

If $A = \mathbb{C}[\varepsilon]$, then $\varphi_i \in \ker i^* \otimes \mathbb{C}\varepsilon$, $a \in \mathcal{A}^{0,0}_Z \otimes \mathbb{C}\varepsilon$, this implies $e^a(\varphi_i) = \varphi_i \in \ker i^* \otimes \mathbb{C}\varepsilon$.

Now, let $0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0$ be a small extension. By hypothesis $\alpha(a) \in L^0 \otimes m_A$, that is $\alpha(a) = \sum_{j=1}^n \overline{a}_j \frac{\partial}{\partial z_j}$ with $\overline{a}_j \in \ker i^* \otimes m_A$ for $j > t$.

Let a'_j be liftings of $\overline{a}_j$. Then $a'_j \in \ker i^* \otimes m_B$ for $j > t$, $a' = \sum_{j=1}^n a'_j \frac{\partial}{\partial z_j} \in L^0 \otimes m_B$ and $e^{a'}(\varphi_i) \in \ker i^* \otimes m_B$. Since $\alpha(a) = \alpha(a')$, then $a = a' + j$ with $j \in M^0 \otimes J$. This implies that $e^a(\varphi_i) = e^{a'+j}(\varphi_i) = e^{a'}(\varphi_i) \ker i^* \otimes m_B$.

As regards, tangent and obstruction spaces, by Theorem IV.1.10 and Lemma IV.1.13, the tangent space of $\mathrm{Def}(f)$ is isomorphic to $H^1(B^\cdot)$ and the obstruction space is naturally contained in $H^2(B^\cdot)$, where $(B^\cdot, D_{\overline{\partial}})$ is the complex with $B^p = A_X^{(0,p)}(\Theta_X) \oplus A_Y^{(0,p)}(\Theta_Y) \oplus A_X^{(0,p-1)}(f^*\Theta_Y)$ and $D_{\overline{\partial}}(x, y, z) = (\overline{\partial}x, \overline{\partial}y, \overline{\partial}z + (-1)^p(f_*x - f^*y))$, for each $(x, y, z) \in B^p$. As regard the functor $\mathrm{Def}_{(h,g)}$, in Section III.1.3, we have proved that the tangent space is $H^1(\mathrm{C}^\cdot_{(h,g)})$ and the obstruction space is naturally contained in $H^2(\mathrm{C}^\cdot_{(h,g)})$, where $\mathrm{C}^\cdot_{(h,g)}$ is the suspended come associated to the couple (h, g) (Section III.1.1). Moreover, Proposition IV.2.3 shows the existence of a quasi isomorphism β between the previous complexes $B^\cdot$ and $\mathrm{C}^\cdot_{(h,g)}$.

ii) γ induces a bijection on the tangent spaces: we prove that γ coincides with the isomorphism β of Proposition IV.2.3.

Let $(n_1, n_2, m) \in \mathrm{Def}_{(h,g)}(\mathbb{C}[\varepsilon])$. Then $\overline{\partial}n_1 = \overline{\partial}n_2 = 0$, $\overline{\partial}m + g(n) = \overline{\partial}m + p^*n_1 + q^*n_2 \in H^1(L)$ and so (n_1, n_2, m) determines the class $[(g(n) + \overline{\partial}m, n_1, n_2, m)] \in H^1(C_{(h,g)})$. Moreover, we note that $\overline{\partial}\pi(m) + f_*n_1 - f^*n_2 = 0$ and $\beta[(g(n) + \overline{\partial}m, n_1, n_2, m)] = [(n_1, n_2, -\pi(m))]$.

Now, consider the image $\gamma(n_1, n_2, m) = \mathcal{O}_A(g(n)) \cap e^m(\ker i^* \otimes A)$. Using Theorem IV.1.10 and Remark IV.1.11, we associate to $\gamma(n_1, n_2, m)$ the class [b] $[(n_1, n_2, a)] \in H^1(B^{\cdot})$, such that $\overline{\partial} a = f_* n_1 - f^* n_2$. Then $\overline{\partial} a = -\overline{\partial}\pi(m)$ and so $[(n_1, n_2, a)] = [(n_1, n_2, -\pi(m))] = \beta([(g(n) + \overline{\partial} m, n_1, n_2, m)]) \in H^1(B^{\cdot})$.

This implies that the map induced by γ on the tangent spaces coincides with the isomorphism β.

iii) γ' induces an injective map on the obstruction spaces. Also in this case, we prove that the map induced by γ on the obstruction space coincides with β and so it is injective.

Actually, let

$$0 \longrightarrow J \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0$$

a small extension. The obstruction class associated to $(n_1, n_2, m) \in \mathrm{Def}_{(h,g)}(A)$ is the class $[(k, h_1, h_2, r)] \in H^2(\mathrm{C}^{\cdot}_{(h,g)}) \otimes J$ defined in III.1.18. We note that $\overline{\partial} r + p^* h_1 + q^* h_2 \in H^2(L))$ and so $\pi(\overline{\partial} r) = -f_* h_1 + f^* h_2$.

Now, again by Theorem IV.1.10 and Remark IV.1.11, the obstruction class associated to the image $\gamma(n_1, n_2, m)$ is $[(h_1, h_2, a)] \in H^2(B^{\cdot}) \otimes J$ with $\overline{\partial} a = -(f_* h_1 - f^* h_2)$ and as above $[(h_1, h_2, a)] = \beta([(k, h_1, h_2, r)])$. $\square$

In conclusion, choosing opportunely L, M, and h, g, Theorem IV.2.5 shows that the infinitesimal deformation functor $\mathrm{Def}(f)$ of an holomorphic map f is isomorphic to the functor $\mathrm{Def}_{(h,g)}$.

IV.2.6. THEOREM. *Let $f : X \longrightarrow Y$ be an holomorphic map. Then the DGLA $\mathrm{H}_{(h,g)}$ associated to the above morphisms $h : L \hookrightarrow KS_{X \times Y}$ and $g = (p^*, q^*) : KS_X \times KS_Y \longrightarrow KS_{X \times Y}$ (see Definition III.2.32) governs the infinitesimal deformation of f:*

$$\mathrm{Def}_{\mathrm{H}_{(h,g)}} \cong \mathrm{Def}(f).$$

PROOF. It is sufficient to apply the previous Theorem IV.2.5, Corollary III.2.28 and Theorem III.2.36. $\square$

This theorem is very interesting from the point of view of "guiding principle", since it shows the existence of a DGLA that governs the geometric problem of infinitesimal deformation of holomorphic maps.

Anyway, in Chapter V we will see that in the applications it is more convenient to use the functor $\mathrm{Def}_{(h,g)}$ than $\mathrm{Def}_{\mathrm{H}_{(h,g)}}$.

IV.2.7. REMARK. Consider the diagram

[b] By Theorem II.7.3 the class associated to the first order deformation of X induced by n_1 is n_1 itself and the class associated to the first order deformation of Y induced by n_2 is n_2 itself.

$$\begin{tikzcd}
& L \arrow[d, "h"] & \\
KS_X \times KS_Y \arrow[r, "g"] \arrow[rr, bend right, "\pi\circ g"'] & KS_{X\times Y} \arrow[r, "\pi"] & A_X^{0,*}(f^*T_Y).
\end{tikzcd}$$

Since h is injective, Lemma III.1.5 implies the existence of a quasi-isomorphism of complexes $(\mathrm{C}^{\cdot}_{(h,g)}, D)$ and $(C^{\cdot}_{\pi\circ g}, \check{\delta})$.

Therefore we get the following exact sequence

(14)

$$\cdots \longrightarrow H^1(C^{\cdot}_{\pi\circ g}) \xrightarrow{\varrho^1} H^1(X,\Theta_X)\oplus H^1(Y,\Theta_Y) \longrightarrow H^1(X, f^*\Theta_Y) \longrightarrow$$

$$\longrightarrow H^2(C^{\cdot}_{\pi\circ g}) \xrightarrow{\varrho^2} H^2(X,\Theta_X)\oplus H^2(Y,\Theta_Y) \longrightarrow H^2(X, f^*\Theta_Y) \longrightarrow \cdots$$

where ϱ^1 and ϱ^2 are the projections on the first factors and are induced by the projection morphism $\varrho : \mathrm{Def}_{(h,g)} \longrightarrow \mathrm{Def}_N$ (see Remark III.1.14).

In particular, $\varrho : \mathrm{Def}(f) \longrightarrow \mathrm{Def}_{KS_X\times KS_Y}$ associates to an infinitesimal deformation of f the induced infinitesimal deformation of X and of Y.

Then ϱ^1 associates to a first order deformations of f the induced first order deformations of X and Y and ϱ^2 is a morphism of obstruction theory: the obstruction to deform f is mapped in the induced obstruction to deform X and Y (see also Remark IV.2.4).

In [**28**], Z. Ran studied the infinitesimal deformations of holomorphic map $f : X \longrightarrow Y$ of singular compact complex spaces. He introduced some algebraic objects T^i_f, $i = 1, 2$, that classify the deformations of a map f, getting the following exact sequence

(15)

$$T^1_f \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)\oplus \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega_Y, \mathcal{O}_Y) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X) \longrightarrow$$

$$\longrightarrow T^2_f \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)\oplus \mathrm{Ext}^2_{\mathcal{O}_Y}(\Omega_Y, \mathcal{O}_Y) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X).$$

IV.2.8. Lemma. *If X and Y are compact complex manifold, then the exact sequence (15) reduces to (14).*

Proof. If X and Y are smooth, then Ω_X and Ω_Y are locally free. Then applying the spectral sequence associated to Ext (see [**9**, Lemme 7.4.1]) we get $\mathrm{Ext}^i_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X) \cong H^i(X,\Theta_X)$, $\mathrm{Ext}^i_{\mathcal{O}_Y}(\Omega_Y, \mathcal{O}_Y) \cong H^i(Y,\Theta_Y)$ and $\mathrm{Ext}^i_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X) \cong H^i(X, f^*\Theta_Y)$.

$\square$

CHAPTER V

Semiregularity maps

In the previous chapter we have studied the infinitesimal deformations of holomorphic maps.

More precisely, let $f : X \longrightarrow Y$ be an holomorphic map of compact complex manifolds and $\Gamma \subset X \times Y$ its graph. Let $M = KS_{X\times Y}$, $N = KS_X \times KS_Y$ and $g = (p^*, q^*) : KS_X \times KS_Y \longrightarrow KS_{X\times Y}$, where p and q are the projections of $X\times Y$ respectively on X and Y. Moreover, let $L = A^{0,*}_{X\times Y}(-log\,\Gamma)$ be the DGLA defined by the following exact sequence (see Section IV.2):

$$0 \longrightarrow L \xrightarrow{h} KS_{X\times Y} \xrightarrow{\pi} A^{0,*}_X(f^*\Theta_Y) \longrightarrow 0.$$

Then we get the following diagram

$$\begin{tikzcd} & L \arrow[d, "h"] \\ KS_X \times KS_Y \arrow[r, "(p^*,q^*)"] & KS_{X\times Y}. \end{tikzcd} \tag{16}$$

Theorem IV.2.5 of the previous chapter shows the existence of an isomorphism between the functor $\mathrm{Def}(f)$ of infinitesimal deformations of f and the functor $\mathrm{Def}_{(h,g)}$ associated to the couple of morphism (h, g):

$$\mathrm{Def}(f) \cong \mathrm{Def}_{(h,g)} \qquad \text{(Theorem IV.2.5)}.$$

Moreover, Theorem IV.2.6 gives an explicit description of the DGLA $H_{(h,g)}$ (Definition III.2.32) that governs the infinitesimal deformations of f:

$$\mathrm{Def}(f) \cong \mathrm{Def}_{H(h,g)} \qquad \text{(Theorem IV.2.6)}.$$

In particular, if $i : X \hookrightarrow Y$ is an inclusion, we can find an easy description of the DGLA associated to $\mathrm{Def}(i)$ (see Section V.4). Actually let L' be the the DGLA L' introduced in Section II.5.1 (see also [**24**, Sect. 5])

$$0 \longrightarrow L' \longrightarrow A^{0,*}_Y(\Theta_Y) \xrightarrow{\pi'} A^{0,*}_X(N_{X|Y}) \longrightarrow 0.$$

Then L' governs the deformation of i (Corollary V.4.1).

In general, without an easy description of $H_{(h,g)}$ it is convenient to use the deformations functor associated to the previous diagram (16).

For example, if we want to study the infinitesimal deformations of f with fixed domain or fixed target, it is sufficient to consider diagram (16) after erasing respectively KS_X (Section V.2) and KS_Y (Section V.1).

Anyway, the main application of the techniques developed in the previous chapters concerns the study of the obstructions to deforms an holomorphic map f and the "semiregularity" maps.

In general, we can find a vector space V (most of time a cohomology vector space) that contains the obstruction space, but we don't know an explicit description of the elements that are obstructions.

Then the idea is to restrict the vector space V as for example defining a map on V, the so called "semiregularity" map, that contains the obstructions in the kernel.

In particular, let $f : X \longrightarrow Y$ be an holomorphic mapand consider the infinitesimal deformations of f with fixed target Y. Then in [**14**] Horikawa proved the following theorem.

V.0.9. THEOREM. *(Horikawa) Let $f : X \longrightarrow Y$ be an holomorphic map and consider the functor of infinitesimal deformations of f with fixed target Y. Then the tangent space is isomorphic to the hypercohomology vector space $\mathbb{H}^1\left(X, \mathcal{O}(\Theta_X) \xrightarrow{f_*} \mathcal{O}(f^*\Theta_Y)\right)$ and the obstruction space is contained in $\mathbb{H}^2\left(X, \mathcal{O}(\Theta_X) \xrightarrow{f_*} \mathcal{O}(f^*\Theta_Y)\right)$.*

Using the techniques introduced in the previous section, we can improve this result in the case of Kähler manifolds, defining a map that contains the obstruction space in the kernel.

THEOREM. *Let $f : X \longrightarrow Y$ be an holomorphic map of compact Kähler manifolds. Let $p = dimY - dimX$. Then the obstruction space of deformations of f (with fixed Y) is contained in the kernel of a map*

$$\sigma : \mathbb{H}^2\left(X, \mathcal{O}(\Theta_X) \xrightarrow{f_*} \mathcal{O}(f^*\Theta_Y)\right) \longrightarrow H^{p+1}(Y, \Omega_Y^{p-1}).$$

The previous map is the generalization of the semiregularity map defined by Bloch (see [**3**] or [**24**, Sec. 9]) obtained when f is the inclusion map $X \hookrightarrow Y$, i.e.

$$\sigma : H^1(X, N_{X|Y}) \longrightarrow H^{p+1}(Y, \Omega_Y^{p+1}).$$

The proof of this theorem is postponed in the next Section V.1.2 (Corollary V.1.5), where we also give an explicit description of the map σ.

V.1. Semiregularity for deformations with fixed target

This section is devoted to study infinitesimal deformations of a holomorphic map $f : X \longrightarrow Y$, with fixed target Y.

In this case the DGLA N reduces to KS_X and so diagram (16) reduces to

$$\begin{tikzcd} & L \arrow[d, "h"] & \\ A_X^{0,*}(\Theta_X) \arrow[r, "p^*"] \arrow[rrd, bend right=10, "f_*"'] & A_{X\times Y}^{0,*}(\Theta_{X\times Y}) \arrow[rd, "\pi"] & \\ & & A_X^{0,*}(f^*\Theta_Y). \end{tikzcd}$$

where f_* is the composition $\pi \circ p^*$.

Using this diagram and Theorem IV.2.5, we can easily prove the Theorem V.0.9 of Horikawa.

V.1.1. Proposition. *The tangent space of the infinitesimal deformation functor of holomorphic map $f : X \longrightarrow Y$, with Y fixed, is $H^1(C^{\cdot}_{f_*})$ and the obstruction space is naturally contained in $H^2(C^{\cdot}_{f_*})$.*

Proof. Theorem IV.2.5 implies that the infinitesimal deformations functor of f, with Y fixed, is isomorphic to $\mathrm{Def}_{(h,p^*)}$. Therefore the tangent space is $H^1(C^{\cdot}_{(h,p^*)})$ and the obstruction space is naturally contained in $H^2(C^{\cdot}_{(h,p^*)})$.

Since h is injective, Lemma III.1.5 implies that, for each i, $H^i(C^{\cdot}_{(h,p^*)}) \cong H^i(C^{\cdot}_{\pi\circ p^*}) = H^i(C^{\cdot}_{f_*})$. $\square$

As we already announced in Section V.1.2 we improve this theorem in the case of Kähler manifolds.

The next section is devoted to some preliminary lemmas.

V.1.1. Preliminaries. Let Z be a complex manifold.

Then $KS_Z = A_Z^{0,*}(\Theta_Z)$ is the Kodaira-Spencer algebra of Z and in Section II.5, we have defined the contraction map $\boldsymbol{i}$:

$$\boldsymbol{i} : KS_Z \longrightarrow \mathrm{Hom}^*(A_Z, A_Z)$$

$$\boldsymbol{i}_a(\omega) = a \lrcorner \omega$$

for each $a \in KS_Z$ and $\omega \in A_Z^{*,*}$.

Therefore $\boldsymbol{i}(A_Z^{0,j}(\Theta_Z)) \subset \oplus_{h,l} \mathrm{Hom}^0(A_Z^{h,l}, A_Z^{h-1,l+j}) \subset \mathrm{Hom}^{j-1}(A_Z, A_Z)$.

To interpret $\boldsymbol{i}$ as a morphism of DGLAs, the key idea due to M. Manetti [**24**, Section 8] is to substitute $\mathrm{Hom}^*(A_Z, A_Z)$ with the graded vector space $\mathrm{Htp}\left(\ker(\partial), \dfrac{A_Z}{\partial A_Z}\right) = \bigoplus_i \mathrm{Hom}^{i-1}\left(\ker(\partial), \dfrac{A_Z}{\partial A_Z}\right)$ (see Example I.3.4). Consider on $\mathrm{Htp}\left(\ker(\partial), \dfrac{A_Z}{\partial A_Z}\right)$ the following differential δ and bracket $\{\ ,\ \}$:

$$\delta(f) = -\overline{\partial} f - (-1)^{deg(f)} f \overline{\partial},$$

$$\{f,g\} = f\partial g - (-1)^{deg(f)deg(g)} g\partial f.$$

V.1.2. Proposition. Htp $\left(\ker(\partial), \dfrac{A_Z}{\partial A_Z}\right)$ *is a DGLA and the linear map*

$$\boldsymbol{i} : A_Z^{0,*}(T_Z) \longrightarrow \text{Htp}\left(\ker(\partial), \frac{A_Z}{\partial A_Z}\right)$$

is a morphism of DGLA.

Proof. See [**24**, Prop. 8.1]. An easy calculation show that the graded vector space Htp $\left(\ker(\partial), \dfrac{A_Z}{\partial A_Z}\right)$ is a DGLA.

Moreover $\boldsymbol{i}$ is a linear map that preserves degree and commutes with differential and bracket. Actually, using Cartan fomulas [a](see Lemma II.5.1), we get

$$\boldsymbol{i}_{\tilde{d}a} = -[\overline{\partial}, \boldsymbol{i}_a] = -\overline{\partial}\boldsymbol{i}_a + (-1)^{\overline{a}-1}\boldsymbol{i}_a\overline{\partial}$$

and by definition of δ

$$\delta(\boldsymbol{i}_a) = -\overline{\partial}\boldsymbol{i}_a - (-1)^{\overline{a}}\boldsymbol{i}_a\overline{\partial}.$$

As regard the bracket, again by Cartan fomulas, we have

$$\boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, [\partial, \boldsymbol{i}_b]] = [\boldsymbol{i}_a, \partial\boldsymbol{i}_b - (-1)^{\overline{b}-1}\boldsymbol{i}_b\partial] =$$

$$\boldsymbol{i}_a\partial\boldsymbol{i}_b - (-1)^{\overline{b}-1}\boldsymbol{i}_a\boldsymbol{i}_b\partial - (-1)^{(\overline{a}-1)\overline{b}}(\partial\boldsymbol{i}_b\boldsymbol{i}_a - (-1)^{\overline{b}-1}\boldsymbol{i}_b\partial\boldsymbol{i}_a) = {}^{\mathrm{b}}$$

$$\boldsymbol{i}_a\partial\boldsymbol{i}_b + (-1)^{\overline{a}\overline{b}-1}\boldsymbol{i}_b\partial\boldsymbol{i}_a$$

and by definition of $\{\ ,\ \}$

$$\{\boldsymbol{i}_a, \boldsymbol{i}_b\} = \boldsymbol{i}_a\partial\boldsymbol{i}_b - (-1)^{\overline{a}\,\overline{b}}\boldsymbol{i}_b\partial\boldsymbol{i}_a.$$

□

Now, let $f : X \longrightarrow Y$ be an holomorphic map, fix $Z = X \times Y$ and Γ the graph of f in Z. Let $I_\Gamma \subset A_Z$ be the space of the differential forms vanishing on Γ and $L \subset KS_Z$ be the DGLA defined as in Lemma IV.2.1:

$$0 \longrightarrow L \longrightarrow KS_Z \longrightarrow A_X^{0,*}(f^*\Theta_Y) \longrightarrow 0.$$

We recall that

$$L \subset \{a \in A_Z^{0,*}(\Theta_Z) |\ \boldsymbol{i}_a(I_\Gamma) \subset I_\Gamma\}$$

and moreover

$$p^*A_X^{0,*}(\Theta_X) \subset \{a \in A_Z^{0,*}(\Theta_Z) |\ \boldsymbol{i}_a(q^*A_Y) = 0\},$$

where p and q are the projection of Z on X and Y respectively.

In conclusion, we can define the following commutative diagram of morphisms of DGLA:

---

[a] $\boldsymbol{i}_{\tilde{d}a} = -[\overline{\partial}, \boldsymbol{i}_a],\ \boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, [\partial, \boldsymbol{i}_b]] = [[\boldsymbol{i}_a, \partial], \boldsymbol{i}_b],\ [\boldsymbol{i}_a, \boldsymbol{i}_b] = 0.$

[b] $\boldsymbol{i}_a\boldsymbol{i}_b\partial$ and $\partial\boldsymbol{i}_b\boldsymbol{i}_a$ are zero in Htp $\left(\ker(\partial), \dfrac{A_Z}{\partial A_Z}\right)$

(17)

$$\begin{tikzcd}
L \arrow[r] \arrow[d, hook, "h"] & K=\left\{f\in \operatorname{Htp}\left(\ker(\partial),\dfrac{A_Z}{\partial A_Z}\right)\mid f(I_\Gamma\cap\ker(\partial))\subset\dfrac{I_\Gamma}{I_\Gamma\cap\partial A_Z}\right\} \arrow[d, hook, "\eta"] \\
A_Z^{0,*}(\Theta_Z) \arrow[r] & \operatorname{Htp}\left(\ker(\partial),\dfrac{A_Z}{\partial A_Z}\right) \\
A_X^{0,*}(\Theta_X) \arrow[r] \arrow[u, "p^*"] & J=\left\{f\in \operatorname{Htp}\left(\ker(\partial),\dfrac{A_Z}{\partial A_Z}\right)\mid f(\ker(\partial)\cap q^*A_Y)=0\right\} \arrow[u, hook, "\mu"']
\end{tikzcd}$$

where the horizontal morphisms are all given by $\boldsymbol{i}$.

Therefore diagram (17) induces a morphism of deformation functors:

$$\mathcal{I}:\operatorname{Def}_{(h,p^*)}\longrightarrow \operatorname{Def}_{(\eta,\mu)}\,.$$

V.1.3. Lemma. *If the differential graded vector spaces* $(\partial A_Z,\overline{\partial})$, $(\partial A_\Gamma,\overline{\partial})$ *and* $(\partial A_Z\cap q^*A_Y,\overline{\partial})$ *are acyclic, then the functor* $\operatorname{Def}_{(\eta,\mu)}$ *is unobstructed. In particular the obstruction space of* $\operatorname{Def}_{(h,p^*)}$ *is naturally contained in the kernel of the map*

$$H^2(C^\cdot_{(h,p^*)})\xrightarrow{\mathcal{I}} H^2(C^\cdot_{(\eta,\mu)}).$$

Proof. This proof is an extension of the proof of [**24**, Lemma 8.2].

The projection $\ker(\partial)\to\ker(\partial)/\partial A_Z$ induces a commutative diagram

(18)

$$\begin{tikzcd}
K \arrow[d, "\eta"] & \{f\in K\mid f(\partial A_Z)=0\} \arrow[l, "\alpha"] \arrow[d, "\eta'"'] \\
\operatorname{Htp}\left(\ker(\partial),\dfrac{A_Z}{\partial A_Z}\right) & \operatorname{Htp}\left(\dfrac{\ker(\partial)}{\partial A_Z},\dfrac{A_Z}{\partial A_Z}\right) \arrow[l, "\beta"] \\
J \arrow[u, "\mu"] & \{f\in J\mid f(\partial A_Z)=0\} \arrow[l, "\gamma"] \arrow[u, "\mu'"']
\end{tikzcd}$$

Since ∂A_Z is acyclic, β is a quasi isomorphism of DGLA. Since

$$\operatorname{coker}(\alpha)=\{f\in \operatorname{Htp}\left(\partial A_Z,\frac{A_Z}{\partial A_Z}\right)|f(I_\Gamma\cap\partial A_Z)\subset\frac{I_\Gamma}{I_\Gamma\cap\partial A_Z}\},$$

there exists an exact sequence

$$0\to \operatorname{Htp}\left(\frac{\partial A_Z}{I_\Gamma\cap\partial A_Z},\frac{A_Z}{\partial A_Z}\right)\to\operatorname{coker}(\alpha)\to \operatorname{Htp}\left(I_\Gamma\cap\partial A_Z,\frac{I_\Gamma}{I_\Gamma\cap\partial A_Z}\right)\to 0.$$

Moreover, the exact sequence

$$0 \longrightarrow I_\Gamma \cap A_Z \longrightarrow \partial A_Z \longrightarrow \partial A_\Gamma \longrightarrow 0$$

implies that $I_\Gamma \cap A_Z$ and $\dfrac{\partial A_Z}{I_\Gamma \cap \partial A_Z} = \partial A_\Gamma$ are acyclic. By Example I.3.3, this implies that $\operatorname{Htp}\left(\dfrac{\partial A_Z}{I_\Gamma \cap \partial A_Z}, \dfrac{A_Z}{\partial A_Z}\right)$ and $\operatorname{Htp}\left(I_\Gamma \cap \partial A_Z, \dfrac{I_\Gamma}{I_\Gamma \cap \partial A_Z}\right)$ are acyclic and so the same holds for $\operatorname{coker}(\alpha)$: i.e. α is a quasiisomorphism.

As regard γ, we have

$$\operatorname{coker}(\gamma) =$$

$$\{f \in \operatorname{Htp}\left(\partial A_Z, \frac{A_Z}{\partial A_Z}\right) \mid f(\partial A_Z \cap q^* A_Y) = 0\} = \operatorname{Htp}\left(\frac{\partial A_Z}{\partial A_Z \cap q^* A_Y}, \frac{A_Z}{\partial A_Z}\right).$$

By hypothesis $\partial A_Z \cap q^* A_Y$ and ∂A_Z are acyclic and so the same holds for $\dfrac{\partial A_Z}{\partial A_Z \cap q^* A_Y}$. Then $\operatorname{coker}(\gamma)$ is acyclic: i.e. γ is also a quasiisomorphism.

Therefore, Theorem III.1.22 implies the existence of an isomorphism of functors $\operatorname{Def}_{(\eta,\mu)} \cong \operatorname{Def}_{(\eta',\mu')}$.

We note that the elements of the three algebras of the second column of (18) vanish on ∂A_Z. Then by definition of the bracket $\{\ ,\ \}$, these algebras are abelian. Therefore Lemma III.1.19 implies that the functor $\operatorname{Def}_{(\eta,\mu)} \cong \operatorname{Def}_{(\eta',\mu')}$ is smooth. Finally, Proposition III.1.21 guarantees that the obstruction space lies in the kernel of $H^2(C^\cdot_{(h,p^*)}) \xrightarrow{\mathcal{I}} H^2(C^\cdot_{(\eta,\mu)})$. □

V.1.2. Semiregularity map for deformations with fixed target.

In the notation of the previous section we have the following theorem.

V.1.4. Theorem. *Let $f : X \longrightarrow Y$ be an holomorphic map of compact Kähler manifolds. Then the obstruction space to the infinitesimal deformations of f with fixed target is contained in the kernel of the following map*

$$H^2(C^\cdot_{f_*}) \xrightarrow{\mathcal{I}'} H^1\left(\operatorname{Htp}(I_\Gamma \cap \ker(\partial) \cap q^* A_Y, A_\Gamma)\right).$$

Proof. Lemma II.2.2 implies that the complexes $(\partial A_Z, \overline{\partial})$, $(\partial A_\Gamma, \overline{\partial})$ and $(\partial A_Z \cap q^* A_Y, \overline{\partial})$ are acyclic. Then we can apply Lemma V.1.3 to conclude that the obstruction space lies in the kernel of the following map

$$H^2(C^\cdot_{(h,p^*)}) \xrightarrow{\mathcal{I}} H^2(C^\cdot_{(\eta,\mu)}).$$

Since h is injective, by Lemma III.1.5, $H^2(C^\cdot_{(h,p^*)}) \cong H^2(C^\cdot_{\pi\circ p^*}) \cong H^2(C^\cdot_{f_*})$. Then the obstructions lies in the kernel of $\mathcal{I} : H^2(C^\cdot_{f_*}) \longrightarrow H^2(C^\cdot_{(\eta,\mu)})$.

As regard $H^2(C^{\cdot}_{(\eta,\mu)})$, consider

$$K=\left\{f\in \mathrm{Htp}\left(\ker(\partial),\frac{A_Z}{\partial A_Z}\right)\mid f(I_\Gamma\cap\ker(\partial))\subset\frac{I_\Gamma}{I_\Gamma\cap\partial A_Z}\right\}$$

and the exact sequence

$$0\longrightarrow K\xrightarrow{\eta}\mathrm{Htp}\left(\partial A_Z,\frac{A_Z}{\partial A_Z}\right)\xrightarrow{\pi'}\mathrm{coker}(\eta)\longrightarrow 0,$$

with $\mathrm{coker}(\eta)=\mathrm{Htp}\left(I_\Gamma\cap\ker(\partial),\dfrac{A_\Gamma}{\partial A_\Gamma}\right)$.

Applying again Lemma III.1.5, there exists an isomorphism $H^2(C^{\cdot}_{(\eta,\mu)})\cong H^2(C^{\cdot}_{\pi'\circ\mu})$. By Remark III.1.1 there exists also a map $\mathcal{I}'':H^2(C^{\cdot}_{\pi'\circ\mu})\longrightarrow H^1(\mathrm{coker}(\pi'\circ\mu))$.

Moreover, we note that

$$J=\left\{f\in\mathrm{Htp}\left(\ker(\partial),\frac{A_Z}{\partial A_Z}\right)\mid f(\ker(\partial)\cap q^*A_Y)=0\right\},$$

$$\pi'\circ\mu:J\longrightarrow\mathrm{Htp}\left(I_\Gamma\cap\ker(\partial),\frac{A_\Gamma}{\partial A_\Gamma}\right)$$

and

$$\mathrm{coker}(\pi'\circ\mu)=\mathrm{Htp}(I_\Gamma\cap\ker(\partial)\cap q^*A_Y,\frac{A_\Gamma}{\partial A_\Gamma}).$$

Finally, since the complex ∂A_Γ is acyclic, the projection

$$H^1(\mathrm{Htp}(I_\Gamma\cap\ker(\partial)\cap q^*A_Y,\frac{A_\Gamma}{\partial A_\Gamma})\longrightarrow H^1(\mathrm{Htp}(I_\Gamma\cap\ker(\partial)\cap q^*A_Y,A_\Gamma))$$

is an isomorphism.

Therefore, the obstruction space is contained in the kernel of $\mathcal{I}':H^2(C^{\cdot}_{f_*})\longrightarrow H^1(\mathrm{Htp}(I_\Gamma\cap\ker(\partial)\cap q^*A_Y,A_\Gamma))$: i.e

$$\begin{tikzcd}
H^2(C^{\cdot}_{(\eta,\mu)}) \arrow[r, "\mathcal{I}"] \arrow[drr, "\mathcal{I}'"] & H^2(C^{\cdot}_{\pi'\circ\mu}) \arrow[r, "\mathcal{I}''"] & H^1\left(\mathrm{Htp}(I_\Gamma\cap\ker(\partial)\cap q^*A_Y,\frac{A_\Gamma}{\partial A_\Gamma})\right) \arrow[d, "\cong"] \\
 & & \oplus_i\mathrm{Hom}(H^i(I_\Gamma\cap\ker(\partial)\cap q^*A_Y),H^i(A_\Gamma))
\end{tikzcd}$$

□

V.1.5. Corollary. *Let $f:X\longrightarrow Y$ be an holomorphic map of compact Kähler manifolds. Let $p=dimY-dimX$. Then the obstruction space to the infinitesimal deformations of f with fixed Y is contained in the kernel of the map*

$$\sigma:\mathbb{H}^2\left(X,\mathcal{O}(\Theta_X)\xrightarrow{f_*}\mathcal{O}(f^*\Theta_Y)\right)\longrightarrow H^{p+1}(Y,\Omega_Y^{p-1}).$$

PROOF. Let $n = dimX$, $p = dimY - dimX$ and $\mathcal{H}$ be the space of harmonic forms on Y of type $(n+1, n-1)$. By Dolbeault theorem and Serre duality we obtain the equalities $\mathcal{H}^\nu = (H^{n-1}(Y, \Omega_Y^{n+1}))^\nu = H^{p+1}(Y, \Omega_Y^{p-1})$.

Let $\omega \in \mathcal{H}$ such that $f^*\omega = 0$. Considering the contraction with ω we define a morphism of complexes

$$(A_X^{0,*}(f^*\Theta_Y), \overline{\partial}) \xrightarrow{\lrcorner\omega} (A_X^{n,*+n-1}, \overline{\partial})$$

$$\lrcorner\omega(\phi f^*\chi) = \phi f^*(\chi\lrcorner\omega) \in A_X^{n,p+n-1} \qquad \forall\ \phi f^*\chi \in A_X^{0,p}(f^*\Theta_Y).$$

Actually, since $\overline{\partial}\omega = 0$, then $\overline{\partial}(\phi f^*(\chi\lrcorner\omega)) = (\overline{\partial}\phi) f^*(\chi\lrcorner\omega) = \lrcorner\omega(\overline{\partial}\phi f^*\chi)$.

In particular, since $f^*\omega = 0$, using the identity of Lemma II.6.1[c] we have the following commutative diagram

$$\begin{tikzcd} A_X^{0,*}(f^*\Theta_Y) \arrow[r, "\lrcorner\omega"] & A_X^{n,*+n-1} \\ A_X^{0,*}(\Theta_X) \arrow[u, "f_*"] \arrow[r] & 0. \arrow[u, "\alpha"'] \end{tikzcd}$$

Then we get a morphism between the second cohomology group of the cone associated to the morphisms f_* and α:

$$H^2(C^\cdot_{f_*}) \longrightarrow H^2(C^\cdot_\alpha) \cong H^n(X, \Omega_X^n).$$

Composing the previous morphism with the integration on X we get

$$\sigma : \mathbb{H}^2\left(X, \mathcal{O}(\Theta_X) \xrightarrow{f_*} \mathcal{O}(f^*\Theta_Y)\right) \longrightarrow H^{p+1}(Y, \Omega_Y^{p-1}).$$

Since $q^*\mathcal{H}$ is contained in $I_\Gamma \cap \ker \overline{\partial} \cap \ker \partial \cap q^*A_Y$, we conclude the proof applying Theorem V.1.4.

□

V.1.6. REMARK. We recall that as already observed in Remark II.2.3, the Kähler hypothesis is just used to have the $\partial\overline{\partial}$-lemma on A_X,A_Y, $A_{X\times Y}$ and A_Γ.

V.2. Deformations of a map with fixed domain

In this section we study infinitesimal deformations of a holomorphic map $f : X \longrightarrow Y$, with fixed domain X.

In this case the DGLA N reduces to KS_Y and so diagram (16) reduces to

[c] $f^*(\chi\lrcorner\omega) = \eta\lrcorner f^*\omega$, for each $\omega \in A_Y^{*,*}$, $\chi \in A_Y^{0,*}(\Theta_Y)$ and $\eta \in A_X^{0,*}(\Theta_X)$ such that $f^*\chi = f_*\eta$.

$$\begin{tikzcd}
 & L \arrow[d, "h"] & \\
A^{0,*}_{Y}(\Theta_Y) \arrow[r, "q^*"] \arrow[rrd, bend right=10, "-f^*"'] & A^{0,*}_{X\times Y}(\Theta_{X\times Y}) \arrow[rd, "\pi"] & \\
 & & A^{0,*}_{X}(f^*\Theta_Y).
\end{tikzcd}$$

where $-f^*$ is the composition $\pi \circ q^*$.

Using this diagram and Theorem IV.2.5, analogously to the case of fixed target we can prove the following proposition

V.2.1. Proposition. *The tangent space of the infinitesimal deformation functor of holomorphic map $f : X \longrightarrow Y$, with X fixed, is $H^1(C^{\cdot}_{f^*})$ and the obstruction space is naturally contained in $H^2(C^{\cdot}_{f^*})$.*

Proof. Analogous of the proof of Proposition V.1.1. □

V.3. Semiregularity for deformations with fixed target and domain

Let $f : X \longrightarrow Y$ be an holomorphic map and consider the infinitesimal deformations of f with fixed target and domain (see Definition IV.1.1).

As we have already observed in Remark IV.1.3, these deformations can be interpreted as infinitesimal deformations of the graph Γ in $X \times Y$, with $X \times Y$ fixed.

Therefore, in diagram (16), we don't need to consider the DGLA $N = KS_X \times KS_Y$ and g. Then the functor $\mathrm{Def}_{(h,g)}$ reduces to Def_h with

$$h : L = A^{0,*}_{X\times Y}(-log\,\Gamma) \hookrightarrow M = KS_{X\times Y}.$$

This implies that the previous Theorem IV.2.5 reduces to:

V.3.1. Corollary. *Let $f : X \longrightarrow Y$ be an holomorphic map. Then the functor $\mathrm{Def}(X \xrightarrow{f} Y)$ of infinitesimal deformations of f with fixed target and domain is isomorphic to Def_h:*

$$Def_h \cong \mathrm{Def}(X \xrightarrow{f} Y).$$

Proof. Apply Theorem IV.2.5 with $N = g = 0$. □

V.3.2. Remark. This corollary is equivalent to [**24**, Theorem 5.2].

V.3.1. Semiregularity map. As regards the obstructions to deform a map f fixing both X and Y, Lemma V.1.3 has the form below.

V.3.3. LEMMA. *If the differential graded vector spaces* $(\partial A_Z, \overline{\partial})$ *and* $(\partial A_\Gamma, \overline{\partial})$ *are acyclic, then the functor* Def_η *is unobstructed. In particular the obstruction space of* Def_h *is naturally contained in the kernel of the map*

$$H^2(C_h^{\cdot}) \xrightarrow{\mathcal{I}} H^2(C_\eta^{\cdot}) \cong \oplus_i \operatorname{Hom}\left(H^i(I_\Gamma \cap \ker(\partial)), H^i(\frac{A_Z}{\partial A_Z}) \right).$$

PROOF. See [**24**, Lemma 8.2]. It follows from Lemma V.1.3, with $N = J = 0$.

□

We have also an analogue of Theorem V.1.4 and then Corollary V.1.5 becomes:

V.3.4. COROLLARY. *Let* $f : X \longrightarrow Y$ *be an holomorphic map of compact Kähler manifolds. Let* $p = dimY - dimX$*. Then the obstruction space to the infinitesimal deformations of* f*, with fixed* X *and* Y*, is contained in the kernel of the map*

$$\sigma : H^1(X, f^*\Theta_Y) \longrightarrow H^{p+1}(Y, \Omega_Y^{p-1}).$$

PROOF. See [**24**, Corollary 9.2]. It follows from previous Lemma V.3.3 and Corollary V.1.5.

□

V.4. Semiregularity for the inclusion map

In this section, we focus our attention on the infinitesimal deformations of an inclusion $i : X \hookrightarrow Y$ of compact complex manifolds. The DGLA L, M, N and the morphisms g, h and π are as before.

Consider the DGLA L' introduced in Section II.5.1 (see also [**24**, Sect. 5])

$$0 \longrightarrow L' \longrightarrow A_Y^{0,*}(\Theta_Y) \xrightarrow{\pi'} A_X^{0,*}(N_{X|Y}) \longrightarrow 0.$$

V.4.1. COROLLARY. L' *governs the infinitesimal deformations of the inclusion* $i : X \hookrightarrow Y$*.*

The proof is postponed at the end of this section after two preliminary lemmas.

V.4.2. LEMMA. *If* $i : X \hookrightarrow Y$ *is the inclusion, then the morphism* $g - h : L \times N \longrightarrow M$ *is surjective.*

PROOF. We want to prove that for each $\phi \in M = A^{0,*}_{X\times Y}(\Theta_{X\times Y})$ there exist $n_1 \in A_X^{0,*}(\Theta_X)$ and $n_2 \in A_Y^{0,*}(\Theta_Y)$ such that $g(n_1, n_2) - \phi = p^*n_1 + q^*n_2 - \phi \in L = \ker \pi$, that is $\pi(\phi) = i_*n_1 - i^*n_2 \in A_X^{0,*}(i^*\Theta_Y) = A_X^{0,*}(\Theta_{Y|X})$. Then the proof immediately follows from the fact that the restriction morphism $i^* : A_Y^{0,*}(\Theta_Y) \longrightarrow A_X^{0,*}(\Theta_{Y|X})$ is surjective. □

V.4.3. Lemma. *In the notation above, $L \times_M N \cong L'$.*

Proof. By definition, $L \times_M N = \{(l, n_1, n_2) \in L \times KS_X \times KS_Y \mid h(l) = p^* n_1 + q^* n_2\}$ and so $0 = \pi h(l) = i_* n_1 - i^* n_2$.

Define $\gamma : L \times_M N \longrightarrow KS_Y$ as the projection on KS_Y.

Then γ is an injective morphism of DGLA with L' as image.

Actually, suppose that $\gamma(l, n_1, n_2) = n_2 = 0$; then $i_* n_1 = 0$ and so $n_1 = n_2 = l = 0$.

About the image, consider the following exact sequences

$$\begin{tikzcd}
0 \arrow[r] & L' \arrow[r] \arrow[d] & A_Y^{0,*}(\Theta_Y) \arrow[r, "\pi'"] \arrow[d, "i^*"] & A_X^{0,*}(N_{X|Y}) \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & A_X^{0,*}(\Theta_X) \arrow[r, "i_*"] & A_Y^{0,*}(\Theta_{Y|X}) \arrow[r, "\beta"] & A_X^{0,*}(N_{X|Y}) \arrow[r] & 0.
\end{tikzcd}$$

Let $\gamma(l, n_1, n_2) = n_2 \in KS_Y$; then $\beta(i^* n_2) = \beta(i_* n_1) = 0$ and so $\pi'(n_2) = 0$. This implies that $\gamma(l, n_1, n_2) \in L'$. □

Proof of Corollary V.4.1. By Theorem IV.2.5, the infinitesimal deformation functor $\mathrm{Def}(i)$ is isomorphic to the functor $\mathrm{Def}_{(h,g)}$. By Lemma V.4.2 and Proposition III.2.34, $\mathrm{Def}_{L \times_M N} \cong \mathrm{Def}_{(h,g)}$. Finally the previous Lemma V.4.3 implies that $\mathrm{Def}_{L'} \cong \mathrm{Def}_{(h,g)}$. Then $\mathrm{Def}(i) \cong \mathrm{Def}_{L'}$. □

V.4.1. Example. We can generalize the case of one inclusion $i : X \hookrightarrow Y$ considering more subvarieties.

For example, let X be a manifold of dimension n and $D_1, \ldots, D_m$ smooth hypersurfaces, with $0 < m \leq n-2$. Moreover, assume that $D_1, \ldots, D_m$ intersect transversally in a smooth subvariety S.

Define

$$\mathrm{Def}_{X; D_1, \ldots, D_m} : \mathbf{Art} \longrightarrow \mathbf{Set}$$

as the functor of infinitesimal deformations of the holomorphic map

$$f : \overset{\circ}{\bigcup} D_i \longrightarrow X,$$

where $f_{|D_i}$ is the inclusion. Equivalently, for each $A \in \mathbf{Art}$, $\mathrm{Def}_{X; D_1, \ldots, D_m}(A)$ is the data of an infinitesimal deformation X_A of X over $\mathrm{Spec}(A)$ and of infinitesimal deformations $\mathcal{D}_i \subset X_A$ of the D_i.

Let $\Theta_X(-log\, D) \subset \Theta_X$ be the subsheaf of vector fields that are tangent to D_i, for every i, and $N_{D_i|X}$ be the normal bundle of D_i in X.

Define $L' := A_X^{0,*}(\Theta_X(-log\, D))$ as in Section II.5.1. Then L' is a DGLA and we have the following exact sequence

$$0 \longrightarrow L' \longrightarrow A_X^{0,*}(\Theta_X) \xrightarrow{\pi'} A_S^{0,*}(\oplus_i N_{D_i|X}) \longrightarrow 0.$$

Denoting for convenience $D^\circ = \overset{\circ}{\bigcup} D_i$, we define $M = KS_{D^\circ \times X}$, $N = KS_{D^\circ} \times KS_X$ and the morphism

$$g = (p^*, q^*) : N = KS_{D^\circ} \longrightarrow M = KS_{D^\circ \times X},$$

where p and q are the projections of $D^\circ \times X$ on D° and X respectively.

Finally, let L be the DGLA defined, as in Section IV.2, by the following exact sequence:

$$0 \longrightarrow L \xrightarrow{h} M \xrightarrow{\pi} A^{0,*}_{D^\circ}(f^*\Theta_X) \longrightarrow 0.$$

V.4.4. Corollary. *L' governs the functor* $\mathrm{Def}_{X;D_1,\ldots,D_m}$. *In particular, the tangent space of* $\mathrm{Def}_{X;D_1,\ldots,D_m}$ *is* $H^1(X, \Theta_X(-log\, D))$ *and the obstruction space is naturally contained in* $H^2(X, \Theta_X(-log\, D))$.

Proof. In the notation above, applying Theorem IV.2.5, there is an isomorphism of functors $\mathrm{Def}_{X;D_1,\ldots,D_m} \cong \mathrm{Def}_{(h,g)}$. Then proceeding as in the case of inclusion it is sufficient to prove the two steps below.

Step 1. The morphism $g - h : N \times L \longrightarrow M$ is surjective (analogous of Lemma V.4.2).

Step 2 . $L' \cong L \times_M N$ (analogous of Lemma V.4.3).

Actually, *Step* 1 and Proposition III.2.34 imply that $\mathrm{Def}_{X;D_1,\ldots,D_m} \cong \mathrm{Def}_{(h,g)} \cong \mathrm{Def}_{L \times_M N}$. Finally *Step* 2 implies $\mathrm{Def}_{X;D_1,\ldots,D_m} \cong \mathrm{Def}_{L'}$.

Proof of Step 1. We want to prove that for each $\phi \in M = A^{0,*}_{D^\circ \times X}(\Theta_{D^\circ \times X})$ there exist $n_1 \in KS_{D^\circ} = A^{0,*}_{D^\circ}(\Theta_{D^\circ})$ and $n_2 \in A^{0,*}_X(\Theta_X)$ such that $g(n_1, n_2) - \phi = p^*n_1 + q^*n_2 - \phi \in L = \ker \pi$, or equivalently $\pi(\phi) = f_*n_1 - f^*n_2 \in A^{0,*}_{D^\circ}(f^*\Theta_X)$. Then we have to prove that $(f^*, -f_*) : A^{0,*}_X(\Theta_X) \times A^{0,*}_{D^\circ}(\Theta_{D^\circ}) \longrightarrow A^{0,*}_{D^\circ}(f^*\Theta_X)$ is surjective and it follows by the hypothesis on D_i.

Proof of Step 2. By definition, $L \times_M N = \{(l, n_1, n_2) \in L \times KS_{D^\circ} \times KS_X \mid h(l) = p^*n_1 + q^*n_2\}$ and so $0 = \pi h(l) = f_*n_1 - f^*n_2$.

Define $\gamma : L \times_M N \longrightarrow KS_X$ as the projection on KS_X.

Then γ is an injective morphism of DGLAs and its image is $L' = A^{0,*}_X(\Theta_X(-log\, D))$.

Actually, suppose that $\gamma(l, n_1, n_2) = n_2 = 0$; then $f_*n_1 = 0$ and so $n_1 = n_2 = l = 0$.

About the image, consider the following exact sequences

$$\begin{tikzcd}
0 \arrow[r] & L' \arrow[r] \arrow[d] & A^{0,*}_X(\Theta_X) \arrow[r, "\pi'"] \arrow[d, "f^*"] & A_S(\oplus_i N_{D_i|X}) \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & A^{0,*}_{D^\circ}(\Theta_{D^\circ}) \arrow[r, "f_*"] & A^{0,*}_{D^\circ}(f^*\Theta_X) \arrow[r, "\beta"] & A_S(\oplus_i N_{D_i|X}) \arrow[r] & 0.
\end{tikzcd}$$

Let $\gamma(l, n_1, n_2) = n_2 \in KS_X$, then $\beta(f^*n_2) = \beta(f_*n_1) = 0$ and so $\pi'(n_2) = 0$. This implies that $\gamma(l, n_1, n_2) \in L'$.

□

V.4.2. Semiregularity map for the inclusion. Let $i : X \hookrightarrow Y$ be the inclusion of a submanifold X in Y and L' the DGLA (defined in Section II.5.1):

$$0 \longrightarrow L' \longrightarrow A_Y^{0,*}(\Theta_Y) \xrightarrow{\pi'} A_X^{0,*}(N_{X|Y}) \longrightarrow 0.$$

In Corollary V.4.1, we proved that L' govern the infinitesimal deformations of the inclusion i. In particular this implies that the obstructions are naturally contained in $H^2(L')$.

Moreover as in Section V.1.1 we can consider the following morphism of DGLAs

$$\boldsymbol{i} : L' \longrightarrow K' = \left\{ f \in \operatorname{Htp}\left(\ker(\partial), \frac{A_Y}{\partial A_Y}\right) \mid f(I_X \cap \ker(\partial)) \subset \frac{I_X}{I_X \cap \partial A_Y} \right\}.$$

Then we get the following corollary, whose proof is essentially contained in Manetti [**24**, Corollary 9.2].

V.4.5. Corollary. *Let $i : X \longrightarrow Y$ be the inclusion of a submanifold X in compact Kähler manifold Y. Let $p = dimY - dimX$. Then the obstruction space to the infinitesimal deformations of i is contained in the kernel of the map*

$$\sigma : H^2(L') \longrightarrow H^{p-1,p+2}(Y, I_X)$$

where $I_X = \ker i^* \subset A_Y^{*,*}$ is the subcomplex of differential forms vanishing on X.

Proof. It follows from previous Corollary V.3.4, reminding that $L' \subset \{a \in A_Y^{0,*}(\Theta_Y) |\ \boldsymbol{i}_a(I_X) \subset I_X\}$.

□

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