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ON THE L^1-LOWER SEMICONTINUITY AND RELAXATION IN $BV(\Omega;\mathbb{R}^M)$ FOR INTEGRAL FUNCTIONALS WITH DISCONTINUOUS INTEGRAND

Tesi di Dottorato

di

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Contents

Introduction

The L^1-lower semicontinuity of the classical functional of the calculus of variations

$$F(u,\Omega) = \int_\Omega f(x,u,\nabla u)dx \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ and $u \in W^{1,1}(\Omega;\mathbb{R}^M)$ with $M, N \geq 1$, has been deeply investigated in the last years, with the aim of providing the minimal assumptions on the integrand f, that guarantee its L^1-lower semicontinuity on $W^{1,1}(\Omega;\mathbb{R}^M)$, and of finding an integral representation for its relaxed functional on the space $BV(\Omega;\mathbb{R}^M)$.

The starting point of the studies on this subject, in the scalar case ($M = 1$), is the celebrated result due to Serrin. In [45] the author proves that the functional (1) is lower semicontinuous with respect the L^1-topology by assuming that f is continuous in all its variables, convex in the last variable and by assuming on the integrand one of the following conditions: f is coercive; f is strictly convex in the gradient variable; the derivatives $f_x(x,s,\xi)$, $f_\xi(x,s,\xi)$, $f_{x\xi}(x,s,\xi)$ exist and are continuous (for further improvements see, among others, also [18, 20, 23, 33, 36, 37])

A natural extension of the functional F in (1) to the larger space $BV(\Omega)$ is given by the functional

$$\mathcal{F}(u,\Omega) = \int_\Omega f(x,u,\nabla u)dx + \int_\Omega f^\infty(x,\tilde{u},\frac{D^c u}{|D^c(u)|})d|D^c u| + \int_{J_u\cap\Omega}[\int_{u^-}^{u^+} f^\infty(x,s,\nu_u)ds]d\mathcal{H}^{N-1}$$

(for the precise definition of this functional see Section 1.4). Indeed, in [13] it is proved, by assuming continuity and coerciveness on the integrand f, that $\mathcal{F}$ coincides with the relaxed functional of F on $BV(\Omega)$. Also for this functional there exist several L^1-lower semicontinuity results in the literature. Among others we recall [6, 17, 19, 29, 32] and the reference therein. In particular, in [19], the authors prove an L^1-lower semicontinuity result for the functional $\mathcal{F}$, by assuming, in the spirit of [36], a $W^{1,1}$-dependance of the integrand in the variable x. Moreover, in that paper no continuity with respect to x is considered and this lack of regularity is compensated by assuming, on the contrary, the continuity with respect to s.

In the first part of this thesis we are interested in a similar problem. Roughly speaking, in the same hypothesis of $W^{1,1}$ dependance on x as in [19], we prove an L^1-lower semicontinuity result for the functional $\mathcal{F}$, by weakening the regularity assumptions with respect to s, but assuming continuity with respect to x.

In this direction, after Serrin's theorem, many authors extended his result by assuming weaker conditions. We recall, for instance, a classical result due to De Giorgi, Buttazzo and Dal Maso (see [23]), where they proved that for autonomous functionals the continuity of s is not necessary in order to obtain the L^1-lower semicontinuity of the functional F. A similar result for the functional $\mathcal{F}$ was proven by De Cicco in [17]. In this last paper, the lower semicontinuity is stated with respect to the weak* convergence of $BV(\Omega)$, instead of the L^1-convergence. This result was then extended to non autonomous functionals in [16].

Here, we generalize both the results of De Giorgi, Buttazzo and Dal Maso and of De Cicco, by proving the lower semicontinuity with respect to the L^1-convergence of the functional $\mathcal{F}$ on the space $BV(\Omega)$ (see Theorem 3.4). Since we admit also a dependance of the integrand f on the spatial variable (see hypothesis (3.10)), our result can be compared also with the lower semicontinuity theorem of Fonseca and Leoni (see [29]), where they assume a lower semicontinuous dependance of the integrand f in (x, s) together with a strong uniformity condition with respect to the other variables (see [33] for the consequence of this assumption). Moreover, we generalize also the lower semicontinuity result of Fusco, Giannetti and Verde (see [32]), where they assume the continuity of f in all its variables. Finally our result is an extension of the lower semicontinuity theorem of De Cicco and Leoni (see [20]), since they deal only with the space $W^{1,1}(\Omega)$.

The main tools for the proof of the lower semicontinuity theorem are a new chain rule formula (see Theorem 3.1) and an approximation result for convex functions due to De Giorgi.

Moreover, as a consequence of the lower semicontinuity theorem we obtained also a relaxation formula; more precisely, if we denote by $\overline{F}$ the relaxed functional of F in BV (i.e. the greatest lower semicontinuous functional not greater than F), we prove that $\overline{F}(u, \Omega) = \mathcal{F}(u, \Omega)$ for every $u \in BV(\Omega)$ (similar result were obtained, among others, in [4, 5, 6, 13, 29]).

This last result has been the key tool in order to prove a Γ-convergence theorem for a sequence of functionals whose integrals pointwise converge. It generalizes an analogous theorem proven in [13], since here no continuity with respect to s and no coerciveness condition are required.

In the second part of the thesis we focus our attention on the vectorial case ($M > 1$), where the situation is much more delicate. In [29], when the integrand f does not depend on s, Fonseca and Leoni, by assuming the continuity (uniform with respect to ξ) with respect to x and the quasiconvexity with respect to the variable ξ of the integrand f proved the following relaxation formula

$$(2)\quad \overline{F}(u,\Omega) = \int_\Omega f(x,\nabla u)dx + \int_\Omega f^\infty(x, \frac{D^c u}{|D^c(u)|})d|D^c u| + \int_{J_u \cap \Omega} f^\infty(x,(u^+ - u^-)\otimes \nu_u)d\mathcal{H}^{N-1}$$

(for the precise definition of this functional see Section 1.4). When f depends also on s, the integral representation formula on the jump set $J_u \cap \Omega$ is rather complicated. (see Theorem 1.10 of [29]). Therefore we have chosen to restrict ourselves to integrands of the form $f = f(x,\xi)$. In the same spirit of [4, 5], we assume that, with respect to x, f is an H^{N-1}-a.e. approximately continuous function, belonging to $W^{1,1}(\Omega)$ and it is convex with respect to the variable ξ. Under these conditions we state our two L^1-lower semicontinuity results: the first one holds in the space $BV(\Omega;\mathbb{R}^M)$ along sequences equibounded with respect to L^∞- norm; the second one holds along sequences equibounded with respect to $L^{\frac{N}{N-1}}$-norm, but by assuming an extra summability condition on the weak gradient of f (see hypothesis (4.25) of Theorem 3.2).

We remark that, even if the convexity assumption is not natural in the vectorial setting, it could be the first step in order to attack the general case of the quasiconvex integrand or at least the polyconvex case. Moreover the convexity permits to use again the techniques of the scalar case. Indeed the main tools of the proof of the lower semicontinuity theorem still are a chain rule formula due to De Cicco, Fusco and Verde (see Theorem 2.1 of [19]) and the approximation result for convex functions due to De Giorgi.

The thesis is organized as follows: Chapter 1 is devoted to notations and preliminaries. In Chapter 2 we present the main problems which will be discussed in the following of the thesis. In Chapter 3 we consider the scalar case and state the lower semicontinuity theorem, the relaxation formula and the Γ-convergence result. Finally, in Chapter 4 we address L^1-lower semicontinuity and relaxation results in the vectorial case.

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Notation

Throughout the thesis $d, M, N \geq 1$ are fixed integers. Ω will be an open bounded set of $\mathbb{R}^N$. If x, y belong to the space $\mathbb{R}^d$ we denote by $\langle x, y\rangle_{\mathbb{R}^d}$ or simply $\langle x, y\rangle$ the canonical scalar product between x and y, and by $\|x\|_{\mathbb{R}^d}$ or $|x|$ the norm of $\mathbb{R}^d$. The absolute value of a real number r is denoted by $|r|$. If $\rho > 0$ and $x \in \mathbb{R}^N$, we set $B_\rho(x) = \{y \in \mathbb{R}^N : \|y - x\|_{\mathbb{R}^N} < \rho\}$, and $\mathcal{S}^{N-1} = \{y \in \mathbb{R}^N : \|y\|_{\mathbb{R}^N} = 1\}$. For any set $F \subset \mathbb{R}^N$, we indicate by χ_F its characteristic function.

We denote by $\mathcal{A}(\Omega)$ the family of all open bounded subsets A of Ω and by $\mathbb{B}(\Omega)$ the σ-algebra of all Borel subsets of Ω. We denote by $\mathcal{M}(X; \mathbb{R}^N)$ the space of the all $\mathbb{R}^N$-valued finite Radon measures.

Let $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ be the Lebesgue and the Hausdorff measure of dimension $(N-1)$ on $\mathbb{R}^N$, respectively.

If $1 \leq p \leq +\infty$, we denote by $L^p(\Omega; \mathbb{R}^M)$ the vectorial space of all M-tuples $f_1, ..., f_M$ of real function in $L^p(\Omega)$. The space $L^p(\Omega; \mathbb{R}^M)$ becomes a Banach space if it is endowed with the norm

$$\|u\|_p := \Big(\int_\Omega |u(x)|^p dx\Big)^{\frac{1}{p}}, \quad \text{when } 1 \leq p < +\infty$$

and

$$\|u\|_\infty := \text{esssup}|u(x)|, \quad \text{when } p = +\infty.$$

Analogously, we say that $u \in W^{1,p}(\Omega;\mathbb{R}^M)$ if u belongs to $L^p(\Omega;\mathbb{R}^M)$ together with its distributional derivatives $\frac{\partial u^i}{\partial x_j}$, $1 \le i \le M$, $1 \le j \le N$. The $M \times N$ matrix of these derivatives will be denoted by ∇u.

We indicate by $C_0^k(\Omega)$, $k = 0, ..., \infty$ the space of all C^k-functions with compact support in Ω. Finally we denote by $\mathcal{D}'(\Omega)$ the space of distributions defined on Ω.

Chapter 1

Preliminaries

This chapter is devoted to preliminary results which will be useful in the sequel.

1.1 Measure Theory

We start by classical definitions and theorems of measure theory. For a general survey on measures we refer, among others, to [9].

Definition 1.1 *Let (X, Σ) be a measure space and $\mu : \Sigma \to [0, \infty)$. We say that μ is a positive measure if $\mu(\emptyset) = 0$ and for any sequence $\{E_n\}$ of pairwise disjoints elements of Σ*

$$\mu(\bigcup_{n=0}^{\infty} E_n) = \sum_{n=0}^{\infty} \mu(E_n).$$

We say that μ is finite if $\mu(X) < +\infty$ and that μ is σ-finite if X is the union of increasing sequence of sets with finite measures.

Definition 1.2 *Let (X, Σ) be a measure space and let $N \in \mathbb{N}$, $N \geq 1$.*

(i) We say that $\mu : \Sigma \to \mathbb{R}^N$ is a measure, if $\mu(\emptyset) = 0$ and for any sequence $\{E_n\}$ of pairwise

disjoints elements of the σ-algebra Σ

$$\mu(\bigcup_{n=0}^{\infty} E_n) = \sum_{n=0}^{\infty} \mu(E_n).$$

If $N = 1$ we say that μ is a real measure, if $N > 1$ we say that μ is a vector measure.

(*ii*) *If μ is a measure, we define its total variation $|\mu|$ as follows*

$$\forall E \in \Sigma \quad |\mu|(E) := \sup\{\sum_{n=0}^{\infty} |\mu(E_n)| : \ E_n \in \Sigma \text{ pairwise disjoints, } E = \bigcup_{n=0}^{\infty} E_n\}.$$

(*iii*) *We say that μ is finite is $|\mu|(X) < +\infty$.*

It can be proven that the total variation $|\mu|$ is a positive measure, more precisely, it is the smallest positive measure such that $|\mu|(E) \geq |\mu(E)|$ for every $E \in \Sigma$. Clearly, if μ is a positive measure, then $|\mu| = \mu$.

We recall the well known definition of absolute continuous and singular measures.

Definition 1.3 (*i*) *Let μ be a positive measure and ν be a vector measure on the measure space (X, Σ). We say that ν is absolutely continuous with respect to μ, and write $\nu << \mu$, if for every $E \in \Sigma$ the following implication holds*

$$\mu(E) = 0 \Rightarrow |\nu|(E) = 0.$$

(*ii*) *If μ, ν are positive measures, we say that they are mutually singular measures and write $\nu \perp \mu$, if there exist E, F such that $\mu(E) = 0$, $\nu(F) = 0$ and $\mu(G) = \mu(G \cap F)$, $\nu(G) = \nu(G \cap E)$ for every $G \in \Sigma$. If μ and ν are real or vector valued, we say that they are mutually singular if $|\mu|$ and $|\nu|$ are so.*

We recall the following classical on a decomposition of a measures.

Theorem 1.1 *Let μ be a positive σ-finite measure and ν an $\mathbb{R}^N$-valued measure ($N \geq 1$). Then there exists a unique pair of vector measures ν^a and ν^s such that $\nu^a << \mu$ and $\nu = \nu^a + \nu^s$. Moreover there exists a unique function $f \in L^1(X;\mathbb{R}^N)$ such that $\nu^a = f\mu$, i.e. $\nu^a(E) = \int_E f d\mu$ for every $E \in \Sigma$. The function f will be called the Radon-Nikodym derivative of ν with respect μ and will be denoted by $\dfrac{\nu}{\mu}$.*

Remark 1.1 *We note that, since each real or vector measure is absolutely continuous with respect to its total variation, then from Theorem 1.1 it follows that there exists a unique $f \in L^1(X;\mathbb{R}^N)$ such that $|f| = 1$ and $f = \dfrac{\mu}{|\mu|}$.*

The next result can be found in [38].

Lemma 1.1 *Let (X,Σ) be a measurable space. Let μ_0, μ_1, and μ_2 be a σ-finite positive measures on (X,Σ) such that*

$$\mu_2 << \mu_1 \quad \textit{and} \quad \mu_1 << \mu_0.$$

Then

(i) $\mu_2 << \mu_0$*;*

(ii) $\dfrac{d\mu_2}{d\mu_0} = \dfrac{d\mu_2}{d\mu_1}\dfrac{d\mu_1}{d\mu_0}$ μ_0*-a.e, where* $\dfrac{\mu_i}{\mu_j}$ *denotes the Radon-Nikodym derivative of* μ_i *with respect* μ_j

For later use we recall the classical Lusin's Theorem.

Lemma 1.2 *Let μ be a measure on a locally compact Hausdorff space X and $A \subset X$ a μ-measurable set. Suppose f is a complex measurable function on X, $\mu(A) < +\infty$, $f(x) = 0$ if $x \notin A$, and $\varepsilon > 0$. Then there exists a function $g \in C_c(X)$ such that*

$$\mu(\{x : f(x) \neq g(x)\}) < \varepsilon.$$

Furthermore, g can be chosen such that

$$\sup_{x\in X} |g(x)| \leq \sup_{x\in X} |f(x)|.$$

An important class of measures is the class of Radon measures.

Definition 1.4 *Let X be a locally compact and separable metric space and $\mathbb{B}(X)$ be the Borel σ-algebra of X. Consider the measure space $(X, \mathbb{B}(X))$.*

(i) A positive measure on $(X, \mathbb{B}(X))$ is a Borel measure.

(ii) A positive measure on each compact subset K of X will be called a positive Radon measure. If μ is an $\mathbb{R}^N$-valued measure defined on all the Borel subset of X s.t. $|\mu|$ is a Radon measure and $|\mu|(X) < +\infty$, we say that μ is an $\mathbb{R}^N$-valued finite Radon measure. We denote by $\mathcal{M}(X;\mathbb{R}^N)$ the space of the all $\mathbb{R}^N$-valued finite Radon measures.

Finally we recall the notion of weak* convergence and strict convergence for Radon measures. We denote by $C_0(X)$ the completion, with respect of the usual norm $\|\varphi\|_\infty := \sup_{x\in X} |\varphi(x)|$, of the space $C_c(X)$ of real continuous functions with compact support defined in X.

Definition 1.5 *Let μ and $\{\mu_n\}$ be $\mathbb{R}^N$-valued finite Radon measures.*

(i) We say that $\{\mu_n\}$ weakly converges to μ (and write $\mu_n \overset{*}{\rightharpoonup} \mu$) if*

$$\lim_{n\to\infty} \int_X \varphi d\mu_n = \int_X \varphi d\mu$$

for every $\varphi \in C_0(X)$;

(ii) We say that $\{\mu_n\}$ strictly converges to μ if $\mu_n \overset{*}{\rightharpoonup} \mu$ and

$$|\mu_n|(X) \to |\mu|(X).$$

We conclude this section by recalling the classical lower semicontinuity and continuity theorems due to Reshetnyak (see [9] for a modern proof).

Theorem 1.2 *Let $f : \Omega \times \mathbb{R}^N \to [0, +\infty)$ be a lower semicontinuous function, convex and positively 1-homogenous in the second variable. Then, for every sequence of measures $\{\mu_n\} \subset \mathcal{M}(\Omega; \mathbb{R}^N)$ weakly* converging to $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ we have*

$$\liminf_{n\to\infty} \int_\Omega f(x, \frac{\mu_n}{|\mu_n|}) d|\mu_n| \geq \int_\Omega f(x, \frac{\mu}{|\mu|}) d|\mu|$$

Theorem 1.3 *Let $f : \Omega \times \mathbb{R}^N \to [0, +\infty)$ be a bounded continuous function, positively 1-homogenous in the second variable. Then, for every sequence of measures $\{\mu_n\} \subset \mathcal{M}(\Omega; \mathbb{R}^N)$ strictly converging to $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ we have*

$$\lim_{n\to\infty} \int_\Omega f(x, \frac{\mu_n}{|\mu_n|}) d|\mu_n| = \int_\Omega f(x, \frac{\mu}{|\mu|}) d|\mu|.$$

1.2 BV-functions

In this section we recall some basic definitions and well known results on $BV(\Omega)$.

Let Ω be an open bounded subset of $\mathbb{R}^N$ and $u \in L^1(\Omega)$. We say that $u \in BV(\Omega)$ if its distributional gradient Du is an $\mathbb{R}^N$-valued Radon measure with bounded total variation $|Du|$ in Ω. Let $u \in L^1(\Omega;\mathbb{R}^M)$, $M \geq 1$; we say that u belongs to the space $BV(\Omega;\mathbb{R}^M)$ if its components $u_i \in BV(\Omega)$ for every $i = 1, ..., M$.

The space $BV(\Omega;\mathbb{R}^M)$ endowed with the following norm

$$\|u\|_{BV(\Omega;\mathbb{R}^M)} := \|u\|_1 + |Du|(\Omega)$$

is a Banach space.

We recall the notion of weak* and strict convergence on $BV(\Omega)$

Definition 1.6 *Let $u \in BV(\Omega;\mathbb{R}^M)$ and $\{u_n\} \subset BV(\Omega;\mathbb{R}^M)$. We say that the sequence $\{u_n\}$ weakly* converges to u if $\{u_n\}$ converges to u in $L^1(\Omega;\mathbb{R}^M)$ and the sequence of measures $\{Du_n\}$ weakly* converges to the measure Du.*

Definition 1.7 *Let $u \in BV(\Omega;\mathbb{R}^M)$ and $\{u_n\} \subset BV(\Omega;\mathbb{R}^M)$. We say that the sequence $\{u_n\}$ strictly converges to u if $\{u_n\}$ converges to u in $L^1(\Omega;\mathbb{R}^M)$ and the sequence of total variations $\{|Du_n|(\Omega)\}$ converges to $|Du|(\Omega)$.*

A compactness property holds on the space $BV(\Omega;\mathbb{R}^M)$ as stated by the following theorem.

Theorem 1.4 *Every sequence* $\{u_n\} \subset BV_{loc}(\Omega;\mathbb{R}^M)$ *such that* $\|u_n\|_{BV(A;\mathbb{R}^M)} \leq M$ *for every open* $A \subset\subset \Omega$ *admits a subsequence* $\{u_{nk}\}$ *converging in* $L^1_{loc}(\Omega;\mathbb{R}^M)$ *to* $u \in BV_{loc}(\Omega;\mathbb{R}^M)$*. If* Ω *has a Lipschitz boundary and* $\|u_n\|_{BV(\Omega;\mathbb{R}^M)} \leq M$*, then* $u \in BV(\Omega;\mathbb{R}^M)$ *and the subsequence* $\{u_{nk}\}$ *weakly* converges to* u*.*

In order to present the canonical decomposition of a BV-function, we give the definition of approximate continuity, approximate jump points and approximate differentiability.

Let $u \in L^1_{loc}(\Omega;\mathbb{R}^M)$. We say that u has an approximate limit in x if there exists $\tilde{u}(x) \in \mathbb{R}^M$ such that

$$\lim_{r\to 0} \fint_{B_r(x)} |u(y) - \tilde{u}(x)|dy = 0.$$

The set C_u of all points where u has an approximate limit is a Borel set.
The function $\tilde{u} : C_u \to \mathbb{R}^M$, called precise representative of u, is a Borel function. We say that u is approximately continuous at x if $x \in C_u$ and $\tilde{u}(x) = u(x)$.

We say that a point $x \notin C_u$ is an approximate jump point if there exist $u^+(x)$, $u^-(x) \in \mathbb{R}^M$ and $\nu_u(x) \in \mathcal{S}^{N-1}$ such that

$$\lim_{r\to 0} \fint_{B_r^+(x;\nu_u(x))} |u(y) - u^+(x)|dy = 0, \quad \lim_{r\to 0} \fint_{B_r^-(x;\nu_u(x))} |u(y) - u^-(x)|dy = 0,$$

where $B_r^+(x;\nu_u(x)) = \{y \in B_r(x) : \langle y - x, \nu_u(x)\rangle > 0\}$ and $B_r^-(x;\nu_u(x)) = \{y \in B_r(x) : \langle y - x, \nu_u(x)\rangle < 0\}$. Also the set J_u of all approximate jump points of u is a Borel set and the function $(u^+(x), u^-(x), \nu_u(x)) : J_u \to \mathbb{R}^M \times \mathbb{R}^M \times \mathcal{S}^{N-1}$ is a Borel function.

Let $x \in C_u$. We say that u is approximately differentiable at x if there exists a matrix

$P \in \mathbb{R}^{M\times N}$ such that

$$\lim_{r\to 0} \fint_{B_r} |u(y) - \tilde{u}(x) - P(y-x)|dy = 0.$$

The matrix P is called the approximate differential at x and it is denoted $\nabla u(x)$. The set $\mathcal{D}_u$ of all points where u is approximately differentiable is a Borel set and the map $\nabla u(x) : D_u \to \mathbb{R}^{M\times N}$ is a Borel map.

We recall that, if $u \in BV(\Omega;\mathbb{R}^M)$, we have $\mathcal{H}^{N-1}(S_u) = 0$, where $S_u = \Omega \setminus (C_u \cup J_u)$ and we can split the measure Du in the following way

$$Du = D^a u + D^s u = D^a u + D^c u + D^j u$$

where

(1.1) $$D^a u = \nabla u \mathcal{L}^N \lfloor D_u,\ D^c u = Du\lfloor (C_u \setminus D_u),\ D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1}\lfloor J_u,$$

where $\nabla u \in L^1(\Omega;\mathbb{R}^{M\times N})$, $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$ and $\otimes$ denotes the tensor product. (see [9] Proposition 3.92).

We recall that, if $S \subset \mathbb{R}^N$ is countably $\mathcal{H}^{N-1}$-rectificable, then there exist for $\mathcal{H}^{N-1}$ a.e. $x \in S$ the approximate tangent plane to S at x, denoted by π_x^S and a unit vector orthogonal to π_x^S, which is called an approximate normal to S $\nu_S(x)$ (see [9], Theorem 2.83). Furthermore (see [9], Proposition 2.85), if $S_i \subset \mathbb{R}^N$ are $\mathcal{H}^{N-1}$-rectificable for every $i = 1, ..., M$ and $S = \bigcap_{i=1}^{M} S_i$, then

(1.2) $$\pi_x^S = \pi_x^{S_i} \quad \mathcal{H}^{N-1} \text{ a.e. } i = 1, ...M.$$

Finally we remark that if $S = J_u$ for some function $u \in BV(\Omega;\mathbb{R}^M)$, then $\nu^{J_u}(x) = \pm\nu_u(x)$. This, together (1.2), yields the following result:

Lemma 1.3 *Let $u \in BV(\Omega;\mathbb{R}^M)$. If $x \in \bigcap_{i=1}^{M} J_{u_i}$, then*

$$\nu_u(x) = \pm\nu_{u_i}(x) \quad \mathcal{H}^{N-1} \text{ a.e. } i = 1, ...M.$$

In the last part of this section, we restrict our attention to the scalar case; i.e. $M = 1$.

For every $u \in BV(\Omega)$ we can define the subgraph of u given by

$$S(u) = \{(x, s) \in \Omega \times \mathbb{R} : s < u^+(x)\}. \tag{1.3}$$

We recall that $S(u)$ is a set with locally finite perimeter in $\Omega\times\mathbb{R}$, i.e. $\chi_{S(u)} \in BV_{loc}(\Omega\times\mathbb{R})$ (see [41] Theorem 3.2.23). We indicate by $\alpha(u) = (\alpha_1(u), .., \alpha_{n+1}(u))$ the distributional derivative of $\chi_{S(u)}$.

For later use we recall the following result.

Proposition 1.1 (see [9], Proposition 3.64**)** *Let $u : \mathbb{R}^N \to \mathbb{R}$ and $\{\varphi_\varepsilon\}$ be a mollifying sequence. Then, if u is approximately continuous at $x \in \mathbb{R}^N$,*

$$(\varphi_\varepsilon * u)(x) \to u(x) \quad \textit{for } \varepsilon \to 0.$$

Moreover, also the following useful result holds true

Proposition 1.2 (see Appendix of [15]) *Let $u \in BV(\Omega)$. Let $M \subset \mathbb{R}$ such that $\mathcal{L}^1(M) = 0$ and let $E = C_u \cap (\tilde{u})^{-1}(M)$. Then $|Du|(E) = 0$.*

For a general survey on BV-functions see, among others, [9, 26, 27, 34, 46].

1.3 Preliminary Lemmas

This brief subsection is devoted to state some technical lemmas which will be used often in the following.

Lemma 1.4 (see [20], Proposition 2.5) *Let E be an open subset of $\mathbb{R}^N$ and G a Borel subset of $\mathbb{R}^d$. Let $g : E \times G \to \mathbb{R}$ be a Borel function in $L^\infty_{loc}(E \times G)$ such that for $\mathcal{L}^N$-almost every $x \in E$ the function $g(x, \cdot)$ is continuous in G. Then there exists a set $M \subset \mathbb{R}^N$ with $\mathcal{L}^N(M) = 0$ such that for every $t \in G$ the function $g(\cdot, t)$ is approximately continuous in $E \setminus M$.*

We recall the following localization lemma due to De Giorgi, Buttazzo and Dal Maso.

Lemma 1.5 (see [23], Lemma 6) *Let $\mu : \mathcal{B}(X) \to [0, \infty]$ be a Radon measure defined on a locally compact Haussdorf space X. Consider a sequence $\{\phi_k\}$ of Borel measurable functions such that for every $k \in \mathbb{N}$ $\phi_k : X \to [0, \infty]$. Then*

$$\int_X \sup_l \phi_l \, d\mu = \sup_{l \in \mathbb{N}} \Big\{ \sum_{k=1}^{l} \int_{A_k} \phi_k \, d\mu : A_k \subset X \ \textit{open and pairwise disjoint} \Big\}.$$

In the same spirit we recall also this result due to De Cicco.

Lemma 1.6 (see[17], Lemma 7) *Let μ be a positive Radon measure on $\Omega \times \mathbb{R}$ and let $\{f_\kappa\}$ be a sequence of nonnegative functions of $L^1(\Omega \times \mathbb{R}; d\mu)$. Set $f := \sup_{\kappa \in \mathbb{N}} f_\kappa \geq 0$. Then for every open subset A of $\Omega \times \mathbb{R}$ we have*

$$\int_A f d\mu = \sup_{\mathcal{D}} \sum_{i \in I} \int_A f_{\kappa_i}(x, s) \varphi_i(x) \psi_i(s) d\mu,$$

where $\mathcal{D}$ is the set of all families $(\kappa_i, \varphi_i, \psi_i)_{i\in I}$ with I finite, $\kappa_i \in \mathbb{N}$, $\varphi_i \in C_0^\infty(\Omega)$, $\psi_i \in C_0^\infty(\mathbb{R})$, $\varphi_i \geq 0$, $\psi_i \geq 0$, $\sum_{i\in I} \varphi_i \otimes \psi_i \leq 1$ and $supp(\varphi_i) \times supp(\psi_i) \subset A$.

1.4 Functionals and their properties

In this section we introduce the functionals, whose properties will be studied in the following.

Let us assume first that f is defined on scalar valued functions. Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N$ be a Borel function such that the map $\xi \to f(x, s, \xi)$ is convex on $\mathbb{R}^N$ for every $(x, s) \in \Omega \times \mathbb{R}$. For every $u \in BV(\Omega)$ and for every $A \in \mathcal{A}(\Omega)$, we consider the following functionals :

$$F(u, A) = \begin{cases} \int_A f(x, u, \nabla u)dx & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{if } u \in BV(\Omega) \setminus W^{1,1}(\Omega); \end{cases} \tag{1.4}$$

$$\begin{aligned} \mathcal{F}(u, A) &= \int_A f(x, u, \nabla u)dx + \int_A f^\infty(x, \tilde{u}, \frac{D^c u}{|D^c(u)|})d|D^c u| \\ &+ \int_{J_u \cap A} \big[\int_{u^-}^{u^+} f^\infty(x, s, \nu_u)ds\big]d\mathcal{H}^{N-1} \end{aligned} \tag{1.5}$$

where $f^\infty(x, s, \xi)$ is the recession function, defined by

$$f^\infty(x, s, \xi) = \lim_{t\to+\infty} \frac{f(x, s, t\xi)}{t} = \sup_{t>0} \frac{f(x, s, t\xi) - f(x, s, 0)}{t}.$$

We note that the previous limit exists for every $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, since the convexity of f implies that the map $t \mapsto \frac{f(x,s,t\xi)-f(x,s,0)}{t}$ is increasing. We recall that f^∞ is Borel function convex and positively homogenous of degree one with respect to ξ.

We define also

$$\hat{f}(x, s, \xi, t) = \begin{cases} f(x, s, \frac{\xi}{t})t & t > 0, \\ f^\infty(x, s, \xi) & t = 0. \end{cases} \tag{1.6}$$

It is easy to verify that $\hat{f}$ is a Borel function and that for each $(x,s) \in \Omega \times \mathbb{R}$ the map $(\xi,t) \mapsto \hat{f}(x,s,\xi,t)$ is convex and positively homogeneous of degree one.

We assume, for any Borel function, the following convention:

$$\fint_a^b h(t)dt = \begin{cases} \frac{1}{b-a}\displaystyle\int_a^b h(t)dt & a \neq b, \\ h(a) & a = b. \end{cases} \tag{1.7}$$

We notice also that, taking into account (1.7), the functional (1.5) can be rewritten as

$$\mathcal{F}(u,A) = \int_A f(x,u,\nabla u)dx + \int_A \Big[\fint_{u^-}^{u^+} f^\infty(x,s,\frac{D^s u}{|D^s(u)|})ds\Big] d|D^s u|. \tag{1.8}$$

Finally, let us recall the following lemma, due to Dal Maso, which states that functional $\mathcal{F}$ can be written as an integral over the subgraph of its entry u.

Lemma 1.7 ([13], Lemma 2.2**)** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a Borel function such that, for each $(x,s) \in \Omega \times \mathbb{R}$, the map $\xi \mapsto f(x,s,\xi)$ is convex on $\mathbb{R}^N$. Then*

$$\mathcal{F}(u,\Omega) = \int_{\Omega\times\mathbb{R}} \hat{f}(x,s,\frac{\alpha(u)}{|\alpha(u)|})d|\alpha(u)|(x,s).$$

If f is defined on vector valued function, it does not depend on s, i.e. $f : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ and is quasiconvex with respect to the second variable ξ we consider the following functionals

$$F(u,A) := \begin{cases} \displaystyle\int_A f(x,\nabla u)dx & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^M), \\ +\infty & \text{if } u \in BV(\Omega;\mathbb{R}^M) \setminus W^{1,1}(\Omega;\mathbb{R}^M), \end{cases} \tag{1.9}$$

$$\begin{aligned} \mathcal{F}(u,A) &= \int_A f(x,\nabla u)dx + \int_A f^\infty(x,\frac{D^c u}{|D^c(u)|})|D^c u| \\ &+ \int_{J_u\cap A} f^\infty(x,(u^+ - u^-)\otimes\nu_u)d\mathcal{H}^{N-1}, \end{aligned} \tag{1.10}$$

for every $u \in BV(\Omega;\mathbb{R}^M)$ and for every $A \in \mathcal{A}(\Omega)$.

If f is quasiconvex f^∞, is defined in slightly different way:

$$f^\infty(x,\xi) = \limsup_{t\to+\infty} \frac{f(x,\xi,t)}{t},$$

and it turns out to be quasiconvex and positively homogenous of degree one.

Finally the recession function satisfies this useful property.

Proposition 1.3 *Let $f : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty]$ be a convex function. Then*

$$\frac{f(x,t\xi)}{t} \leq f^\infty(x,\xi) + \frac{f(x,0)}{t}.$$

1.5 Approximation of convex functions

One of the main tool, used in the present thesis, in order to prove the lower semicontinuity of the functional in (1.5) is an approximation result for convex functions due to De Giorgi. This result states that any convex function $f : \mathbb{R}^d \to \mathbb{R}$ can be approximated by mean of a sequence of affine functions $a_\alpha + \langle b_\alpha, \xi\rangle$, where

$$a_\alpha := \int_{\mathbb{R}^d} f(\xi)\big((d+1)\alpha(\xi) + \langle \nabla\alpha(\xi), \xi\rangle\big)d\xi \tag{1.11}$$

$$b_\alpha := -\int_{\mathbb{R}^d} f(\xi)\nabla\alpha(\xi)d\xi, \tag{1.12}$$

with $\alpha \in C_0^1(\mathbb{R}^d)$ a nonnegative function such that $\int_{\mathbb{R}^d} \alpha(\xi)d\xi = 1$. The main feature of this approximation is that the coefficients a_α and b_α explicitly depend on f. In particular, when f depends also on x, s the explicit formulas (1.11) and (1.12) permit to deduce regularity properties of De Giorgi's coefficients, from proper hypotheses satisfied by f. We recall therefore the De Giorgi's approximation theorem.

Theorem 1.5 (see [22], Theorem 1**)** *Let $f:\mathbb{R}^d\to\mathbb{R}$ be a convex function and a_α, b_α be defined as in (1.11) and (1.12). Then the following properties hold:*

(i) $f(\xi)\geq a_\alpha+\langle b_\alpha,\xi\rangle$ *for any* $\xi\in\mathbb{R}^d$*;*

(ii) $f(\xi)=\sup_{\beta\in\mathcal{B}}[a_\beta+\langle b_\beta,\xi\rangle]$ *for* $\xi\in\mathbb{R}^d$*,*

where $\mathcal{B}:=\{\beta:\beta(z):=\kappa^d\alpha(\kappa(q-z)),\kappa\in\mathbb{N},q\in\mathbb{Q}^d\ z\in\mathbb{R}^d\}$;

(iii) $f(\xi)=\lim_{j\to\infty}f_j(\xi)$ *for any* $\xi\in\mathbb{R}^d$*, where* $f_j(\xi):=\sup_{i\leq j}\{a_{\beta_i}+\langle b_{\beta_i},\xi\rangle\}$ *for any* $\xi\in\mathbb{R}^d$*,*

where β_i *is an ordering of the class* $\mathcal{B}$*.*

1.6 Relaxation and Γ-convergence

Let F be the functional defined in (1.4). For every $u\in BV(\Omega;\mathbb{R}^M)$, we can define the relaxed functional $\overline{F}$ of F, with respect to the L^1-topology, given by

$$\overline{F}(u,\Omega)=\inf\{\liminf_{n\to\infty}F(u_n,\Omega):u_n\in W^{1,1}(\Omega;\mathbb{R}^M),u_n\to u\text{ in }L^1(\Omega;\mathbb{R}^M)\}. \tag{1.13}$$

We recall that $\overline{F}$ is the greatest lower semicontinuous functional not greater than F. Moreover the following characterization holds:

for every $u\in BV(\Omega;\mathbb{R}^M)$ and every $\{u_n\}\in W^{1,1}(\Omega;\mathbb{R}^M)$, such that $u_n\to u$ strongly in $L^1(\Omega;\mathbb{R}^M)$,

$$\overline{F}(u,\Omega)\leq\liminf_{h\to\infty}F(u_n,\Omega), \tag{1.14}$$

for every $u \in BV(\Omega;\mathbb{R}^M)$ there exists $\{\overline{u}_n\} \in W^{1,1}(\Omega;\mathbb{R}^M)$, such that $\overline{u}_n \to u$ strongly in $L^1(\Omega;\mathbb{R}^M)$,

$$\overline{F}(u,\Omega) = \lim_{h\to\infty} F(\overline{u}_n,\Omega). \tag{1.15}$$

We recall also the definition of $\Gamma-$convergence. We say that a sequence $\{F_n\}$ of the type (1.4) Γ-converges to a functional F^Γ (and we write $F^\Gamma := \Gamma - \lim F_n$) with respect to the $L^1(\Omega)$-topology, if the following two properties hold:

for every $u \in BV(\Omega;\mathbb{R}^M)$ and every $\{u_n\} \in W^{1,1}(\Omega;\mathbb{R}^M)$, such that $u_n \to u$ strongly in $L^1(\Omega;\mathbb{R}^M)$,

$$F^\Gamma(u,\Omega) \leq \liminf_{n\to\infty} F_n(u_n,\Omega), \tag{1.16}$$

for every $u \in BV(\Omega;\mathbb{R}^M)$ there exists $\{\overline{u}_n\} \in W^{1,1}(\Omega;\mathbb{R}^M)$, such that $\overline{u}_n \to u$ strongly in $L^1(\Omega;\mathbb{R}^M)$,

$$F^\Gamma(u,\Omega) = \lim_{n\to\infty} F_n(\overline{u}_n,\Omega). \tag{1.17}$$

We recall that F^Γ, if it exists, is L^1-lower semicontinuous on $BV(\Omega;\mathbb{R}^M)$. Moreover, Γ convergent is compact, i.e. from every sequence of functionals, it is possible to find a Γ-convergent subsequence. Finally if $F_n \equiv F$ for every $n \in \mathbb{N}$, then $F^\Gamma = \overline{F}$.

For further properties of the relaxation and Γ-convergence we refer to [11, 14, 24, 25].

Chapter 2

Classical and recent theory

This chapter is devoted to a brief history of classical and more recent theory of the L^1-lower semicontinuity and relaxation on $BV(\Omega;\mathbb{R}^M)$. In the following, Ω will be an open bounded subset of $\mathbb{R}^N$, $M, N \geq 1$.

2.1 The scalar case

The main problems we will be interested in was inspired by the study of the classical functional of Calculus of Variations

$$F(u,\Omega) = \int_\Omega f(x,u,\nabla u)dx. \qquad u \in W^{1,1}(\Omega). \tag{2.1}$$

One of the basic question is to find a suitable extension of this functional which permits to evaluate it also on singular functions, for instance, those having null gradient almost everywhere but are being not constant. For example we can consider the functional

$$H(u,\Omega) = \int_\Omega \sqrt{1+|\nabla u|^2}dx,$$

that represents the area of the graph of a function $u : \Omega \to \mathbb{R}$. If we evaluate H on a piecewise constant function u, we obtain that the area of its graph of u is equals $\mathcal{L}^N(\Omega)$, which contradicts the fact that u is not constant.

A further difficulty in this kind of problem is the failure of the Direct Methods of the Calculus of Variations. Indeed, even if the functional (2.1) were lower semicontinuous with respect to the L^1-topology, i.e. the following inequality holds

$$F(u,\Omega) \leq \liminf_{n\to\infty} F(u_n,\Omega) \quad \forall u_n, u \in W^{1,1}(\Omega), \quad u_n \to u \text{ in } L^1(\Omega), \tag{2.2}$$

the lack of reflexivity of the space $W^{1,1}(\Omega)$ (which does not ensure anymore the compactness of the minimizing sequence) does not permit to apply the Direct Methods of the Calculus of Variations in order to find the minima of the functional in (2.1). Therefore it is necessary to extend the functional F to a proper larger space, in which we may have its meaningful definition and the compactness of minimizing sequences. In this context the natural candidate to play this role is the space $BV(\Omega)$ of the functions of bounded variation. Indeed, as already seen in Theorem 1.4, from a bounded sequence in $BV(\Omega)$ it is possible to extract a subsequence which converges, with respect to the L^1-topology, to a function still belonging to $BV(\Omega)$. This leads to extend the functional in (2.1) as in (1.4) so that

$$\inf_{W^{1,1}(\Omega)} F(u,\Omega) = \inf_{BV(\Omega)} F(u,\Omega).$$

Unfortunately, there is no hope that this new functional is L^1-lower semicontinuous, since $W^{1,1}(\Omega)$ is dense in $BV(\Omega)$ with respect to the L^1-topology.

In order to avoid this obstacle we can introduce the relaxed functional of (1.4), as defined in (1.13) with $M = 1$.

Firstly we would understand if this new functional is a "good extension" of F, namely if the following equality:

$$\overline{F}(u, \Omega) = F(u, \Omega) \quad \forall u \in W^{1,1}(\Omega) \tag{2.3}$$

holds.

The validity of (2.3) is equivalent to prove the L^1-lower semicontinuity of the functional in (2.1). Indeed, if it is lower semicontinuous, (2.3) follows immediately since, by recalling (1.15), we get $F \leq \overline{F}$ and the opposite inequality $F \geq \overline{F}$ holds for every $u \in W^{1,1}(\Omega)$.

The problem of providing sufficient conditions to ensure L^1-lower semicontinuity of the functional in (2.1) is known in the literature as the Serrin's problem. We present here its classical formulation.

Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be such that

$$\begin{cases} (i) & f(x, s, \cdot) \text{ is convex for all } (x, s) \in \Omega \times \mathbb{R}; \\ (ii) & f \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^N). \end{cases} \tag{2.4}$$

Even if it could seem reasonable that, under hypothesis (2.4), the functional F were lower semicontinuous with respect to the L^1-topology, it is well known that, under hypothesis (2.4), this is not true, as shown by a celebrated counterexample due to Aronszaj'n ([44]).

Sufficient conditions so that (2.2) holds are provided by the following theorem due to Serrin.

Theorem 2.1 (see [45], Theorem 12**)** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be a function satisfying*

(2.4). Assume also at least one of the following conditions holds:

$$\begin{cases} (i) & f(x,s,\cdot) \text{ is strictly convex for all } (x,s) \in \Omega \times \mathbb{R}; \\ (ii) & \lim_{|\xi| \to \infty} f(x,s,\xi) = +\infty; \\ (iii) & \text{the functions } \nabla_x f, \nabla_\xi f, \nabla_x \nabla_\xi f \text{ exists and are continuous in } \Omega \times \mathbb{R} \times \mathbb{R}^N \end{cases}$$

Then F is lower semicontinuous on $W^{1,1}(\Omega)$ with respect to the L^1-topology.

However the problem of understanding if the relaxed functional admits on $BV(\Omega)$ an integral representation is left open by the previous Theorem. The first result in this direction is due to Goffmann and Serrin .

Theorem 2.2 (see [35],Theorem 5) *Let $f : \mathbb{R}^N \to [0, \infty)$ be a convex function then*

$$\overline{F}(u, \Omega) = \int_\Omega f(\nabla u) dx + \int_\Omega f^\infty (\frac{D^s u}{|D^s u|}) d|D^s u| \quad \forall u \in BV(\Omega), \tag{2.5}$$

where, as usual, $D^s u$ stands for the singular part of the measure Du, while f^∞ is the recession function of f given by $f^\infty(\xi) := \lim_{t \to \infty} \dfrac{f(t\xi)}{t}$.

Previous result states that in order to extend the functional F to the space $BV(\Omega)$, it is necessary to take into account the vertical variations of u, represented by the singular part of the measure Du. It is, in some sense, not surprising that this process happens by means of the recession function f^∞, which describes the behavior of f at ∞.

Concerning the general case, in which the integrand f depends also on s and ξ, we may conjecture that, under suitable hypotheses on f, we still have a representation formula of the type (2.5). Precisely the expected formula is

$$\overline{F}(u, \Omega) = \mathcal{F}(u, \Omega) \qquad \forall u \in BV(\Omega), \tag{2.6}$$

where the functional $\mathcal{F}$ is defined in (1.5).

Clearly, the validity of (2.6) implies that $\mathcal{F}$ must be L^1-lower semicontinuous on $BV(\Omega)$. On the contrary, if $\mathcal{F}$ is L^1-lower semicontinuous on $BV(\Omega)$, recalling that $\overline{F}$ is the greatest L^1-lower semicontinuous functional not greater than F, we get the inequality $\mathcal{F} \le \overline{F}$.

A first result in this direction is due to Dal Maso (see [13]), who states the validity of this formula (2.6) by means of two theorems. In the first one the author provides the inequality $\mathcal{F} \le \overline{F}$, by proving that the functional (1.5) is L^1-lower semicontinuous on $BV(\Omega)$. In the second one, the opposite inequality $\mathcal{F} \ge \overline{F}$ is obtained.

Theorem 2.3 (see [13], Theorem 3.1**)** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be a locally bounded Borel function such that*

$$
(2.7) \qquad \begin{cases} (i) & f(x,s,\cdot) \text{ is convex on } \mathbb{R}^N \text{ for every } (x,s) \in \Omega \times \mathbb{R}; \\ \\ (ii) & \text{there exists a Borel subset } B \subset \Omega \times \mathbb{R}, \text{ with } \mathcal{H}^N((\Omega \times \mathbb{R}) \setminus B) = 0, \\ & \text{such that } f \text{ is lower semicontinuous on } B \times \mathbb{R}^N \\ \\ (iii) & \text{there exists a constant } \lambda > 0 \text{ such that} \\ & f(x,s,\xi) \ge \lambda|\xi| \text{ for all } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N. \end{cases}
$$

Then the functional (1.5) is lower semicontinuous on $BV(\Omega)$ with respect to the L^1-topology, i.e.

$$
(2.8) \qquad \mathcal{F}(u,\Omega) \le \liminf_{n\to\infty} \mathcal{F}(u_n,\Omega) \quad \forall u, u_n \in BV(\Omega) \quad u_n \to u \text{ in } L^1(\Omega).
$$

Proof.

We give a sketch of the proof. For more details we remaind the interested reader to the original proof of Dal Maso.

It is possible to show, by (ii) and the local boundeness of f, that $\hat{f}$ (defined in (1.6)) is lower semicontinuous on $B \times \Omega \times \mathbb{R}^N$. We want to prove (2.8), when

$$\liminf_{n \to +\infty} \mathcal{F}(u_n, \Omega) < +\infty, \tag{2.9}$$

otherwise the conclusion is trivial. Suppose that $\{u_n\}_{n \in \mathbb{N}} \subset BV(\Omega)$ and $u \to u$ in $L^1(\Omega)$.

Thanks to (2.9) and (*iii*) one can show that the sequence $\{\alpha(u_n)\}$ of the derivative measures of the characteristic functions of the graphs of $S(u_n)$ weakly* converges in the sense measures to the derivative measure of the characteristic function $\alpha(u)$ of $S(u)$. By (1.8) and Lemma 1.7 we have

$$\mathcal{F}(u_n, \Omega) = \int_{\Omega \times \mathbb{R}} \hat{f}(x, s, \frac{\alpha(u_n)}{|\alpha(u_n)|}) d|\alpha(u_n)|(x, s),$$

and

$$\mathcal{F}(u, \Omega) = \int_{\Omega \times \mathbb{R}} \hat{f}(x, s, \frac{\alpha(u)}{|\alpha(u)|}) d|\alpha(u)|(x, s).$$

Then, since $\hat{f}$ is lower semicontinuous, convex and positively 1-homogenous in the last variable, the thesis follows by Theorem 1.2. ■

Theorem 2.4 (see [13], Theorem 3.2**)** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be a Borel function such that*

$$
(2.10) \quad \begin{cases}
(i) & f(x, s, \cdot) \text{ is convex on } \mathbb{R}^N \text{ for every } (x, s) \in \Omega \times \mathbb{R}; \\
\\
(ii) & \text{for } \mathcal{H}^N - a.e.\ (x_0, s_0) \text{ and for every } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\
& |f(x, s, \xi) - f(x_0, s_0, \xi)| \leq \varepsilon(1 + |\xi|) \\
& \text{for all } (x, s) \in \Omega \times \mathbb{R} \text{ with } |x - x_0| + |s - s_0| < \delta \text{ and for all } \xi \in \mathbb{R}^N; \\
\\
(iii) & \text{there exists a constant } \Lambda > 0 \text{ such that} \\
& f(x, s, \xi) \leq \lambda(1 + |\xi|) \text{ for all } (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N.
\end{cases}
$$

Then

$$\mathcal{F}(u, \Omega) \geq \overline{F}(u, \Omega) \qquad \forall u \in BV(\Omega).$$

Proof.

The proof is based on an application of Theorem 1.3. We give a sketch. Firstly, one assume $u \in BV(\Omega) \cap L^\infty(\Omega)$. By a standard regularization argument one constructs a sequence $\{u_n\} \subset W^{1,1}(\Omega) \cap L^\infty(\Omega)$ converging to u such that $\|u_n\|_\infty \leq \|u\|_\infty$. Moreover, by standard measure theory arguments the sequence $\{\alpha(u_n)\}$ strictly converges on $\Omega \times \mathbb{R}$ to $\alpha(u)$. The continuity requirement (ii) implies the continuity on $\mathbb{R} \times \mathbb{R}^N$ of the map $(x_0, s_0, \xi, t) \mapsto \hat{f}(x_0, s_0, \xi, t)$, so that by using Lemma 1.7 and Theorem 1.3, we get

$$
\begin{aligned}
(2.11)\ \overline{F}(u) \geq \liminf_{n\to\infty} F(u_n, \Omega) \quad &= \quad \lim_{n\to\infty} \mathcal{F}(u_n, \Omega) = \int_{\Omega\times\mathbb{R}} \hat{f}(x, s, \frac{\alpha(u_n)}{|\alpha(u_n)|}) d|\alpha(u_n)|(x, s) \\
&= \quad \int_{\Omega\times\mathbb{R}} \hat{f}(x, s, \frac{\alpha(u)}{|\alpha(u)|}) d|\alpha(u)|(x, s) = \mathcal{F}(u, \Omega),
\end{aligned}
$$

for every $u \in BV(\Omega) \cap L^\infty(\Omega)$. Finally one obtain the inequality $\mathcal{F} \geq \overline{F}$ on the whole space $BV(\Omega)$ by a standard truncation argument. ■

2.1.1 Lack of coercivity

It is worth while to notice that, in the Serrin's Theorem, the coercivity assumption seems not to be a necessary condition in order to obtain lower semicontinuity of the functional (2.1). Therefore one of the main purpose of recent studies on lower semicontinuity and relaxation on $BV(\Omega)$ is to understand if, without coercivity assumption, the representation formula (2.6) still holds. In this direction, very recent developments are due to Fonseca and Leoni. In [29] they prove, by using a blow up method introduced by Fonseca and Muller (see [30, 31]), that, if one removes the coercivity assumption, formula (2.6) continues to hold under suitable uniform continuity (with respect to ξ) of the integrand $f(\cdot,\cdot,\xi)$.

More precisely, in what follows $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to [0,+\infty)$ is a Borel nonnegative function, convex with respect to the last variable satisfying the further following condition: for every $(x_0, s_0) \in \Omega\times\mathbb{R}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x_0, s_0, \xi) - f(x, s, \xi) \leq \varepsilon(1 + f(x, s, \xi)) \tag{2.12}$$

for every $(x, s) \in \Omega\times\mathbb{R}$ such that $|x - x_0| + |s - s_0| \leq \delta$ and for every $\xi \in \mathbb{R}^N$.

The goal is to get the lower bound $\mathcal{F}(u, \Omega) \leq \overline{F}(u, \Omega)$ for every $u \in BV(\Omega)$. To this end, it is enough to show

$$\mathcal{F}(u, \Omega) \leq \liminf_{n\to\infty} F(u_n, \Omega) \quad \forall\{u_n\} \subset W^{1,1}(\Omega),\ \forall u \in BV(\Omega) \quad u_n \to u \text{ in } L^1(\Omega). \tag{2.13}$$

The validity of (2.13) is stated by the next theorem.

Theorem 2.5 (see [29], Theorem 1.1) *Let $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to [0,\infty)$ be as stated before. Then (2.13) holds.*

To achieve the relaxation formula the next step is to state the upper bound: $\overline{F}(u,\Omega) \le \mathcal{F}(u,\Omega)$ for all $u \in BV(\Omega)$. This is proved in the next theorem still due to Fonseca and Leoni.

Theorem 2.6 (see [29], Theorem 1.3**)** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a Borel function convex with respect to ξ. Let $\Lambda > 0$ be such that*

$$0 \le f(x,s,\xi) \le \Lambda(1+|\xi|) \quad \forall (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N. \tag{2.14}$$

Then the following properties hold

(i) if f is Carathèodory or $f(\cdot,\cdot,\xi)$ is upper semicontinuous, then $\overline{F}(u,\Omega \setminus (C_u \bigcup J_u)) \le \int_\Omega f(x,u,\nabla u)dx$;

(ii) if $f^\infty(\cdot,\cdot,\xi)$ is upper semicontinuous then $\overline{F}(u,C_u) \le \int_\Omega f^\infty(x,u,\frac{D^c u}{|D^c u|})d|D^c u|$

(iii) if $f^\infty(\cdot,s,\xi)$ is upper semicontinuous then
$$\overline{F}(u,J_u) \le \int_{J_u \cap \Omega} \Big(\int_{u^-}^{u^+} f^\infty(x,s,(u^+ - u^-)\nu_u)ds \Big) d\mathcal{H}^{N-1}.$$

2.1.2 Lower semicontinuity and summability conditions

A different approach of the study of L^1-lower semicontinuity without coerciveness is introduced in the paper of Gori Maggi and Marcellini (see [36]). Roughly speaking, the authors show that, in order to prove L^1-lower semicontinuity of the functional F in (2.1), it is possible to replace the classical continuity and coerciveness assumption with the weak differentiability of f with respect to x (for further improvements see, among others, also [18, 20, 33]). The result of Gori, Maggi and Marcellini suggests that a similar L^1-lower semicontinuity result can be obtained also for the functional $\mathcal{F}$ in (1.5), which is the BV-counterpart of the functional F in (2.1). As we have seen before, the L^1-lower semicontinuity of $\mathcal{F}$ implies the lower bound $\mathcal{F} \leq \overline{F}$, which is a crucial step in order to prove the integral representation of $\overline{F}$. A first result in this spirit is due to Fusco, Giannetti and Verde (see [32]). They prove the L^1-lower semicontinuity of $\mathcal{F}$, by assuming the continuity of the integrand f in all its variables and a weak differentiability condition of Gori-Maggi-Marcellini's type, i.e.

$$
(2.15) \qquad \begin{cases} (i) & f(\cdot, s, \xi) \in W^{1,1}(\Omega) \quad \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N \text{ and} \\ (ii) & \text{for every bounded set } B \subset \mathbb{R} \times \mathbb{R}^N, \text{ there exists a constant } L(B) \\ & \text{such that } \displaystyle\int_\Omega |\nabla_x f(x,s,\xi)|dx \leq L(B) \quad \forall (s,\xi) \in B. \end{cases}
$$

More precisely, Fusco, Giannetti and Verde state the following result.

Theorem 2.7 (see [32], Theorem 1.1**)** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ be a continuous function, convex with respect to the last variable and and satisfying (2.15). Then the functional $\mathcal{F}$ is lower semicontinuous on $BV(\Omega)$ with respect to the L^1-topology.*

A crucial tool in the proof of the previous theorem is a chain rule formula.

Proposition 2.1 (see [19], Lemma 2.4**)** *Let $b : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function with compact support satisfying the following properties:*

(i) $b(\cdot, s) \in W^{1,1}(\Omega)$ for every $s \in \mathbb{R}$,

(ii) there exists a constant L, such that for every $s \in \mathbb{R}$,

$$
\int_\Omega |\nabla_x b(x,s)|dx \leq L.
$$

Then for every $\varphi \in C_0^1(\Omega)$ and for every $u \in BV(\Omega) \cap L^\infty(\Omega)$ we have:

$$
\begin{aligned}
&- \int_\Omega \Big(\int_0^{u(x)} b(x,s)ds \Big) \nabla\varphi dx = \int_\Omega b(x,u)\varphi \nabla u dx \\
(2.16) \qquad & \\
&+ \int_\Omega \Big(⨍_{u^-}^{u^+} b(x,s)ds \Big) \varphi dD^s u + \int_\Omega \Big(\int_0^{u(x)} \nabla_x b(x,s)ds \Big) \varphi dx.
\end{aligned}
$$

The use of this chain rule in the proof of Theorem 2.7 suggests that in order to improve Theorem 2.7 itself, an improvement of the chain rule is needed. This has been done by De Cicco, Fusco and Verde. In [18, 19] they prove the chain rule formula (2.16) by removing the continuity assumption with respect to x. This leads to obtain a better lower semicontinuity result, in which the continuity of the integrand f with respect to x is not required.

Theorem 2.8 (see [19], Theorem 1.1**)** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be a locally bounded Borel function. Assume that there exists a set $Z \subset \Omega$ with $\mathcal{L}^N(Z) = 0$ such that*

(i) $f(x, s, \cdot)$ is convex in $\mathbb{R}^N$ for every $(x, s) \in (\Omega \setminus Z) \times \mathbb{R}$;

(ii) $f(x, \cdot, \xi)$ is continuous in $\mathbb{R}$ for every $(x, \xi) \in (\Omega \setminus Z) \times \mathbb{R}^N$;

(iii) $f(\cdot, s, \xi) \in W^{1,1}(\Omega)$ for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and the estimate (2.15) holds.

Then the functional $\mathcal{F}$ is lower semicontinuous on $BV(\Omega)$ with respect to the L^1-topology.

2.2 The vectorial case

We now turn our attention to the vectorial case and consider the integrand $f : \Omega \times \mathbb{R}^M \times \mathbb{R}^{M\times N} \to [0,\infty)$ and the corresponding integral functional

$$F(u,\Omega) = \int_\Omega f(x,u,\nabla u)dx, \qquad u \in W^{1,1}(\Omega;\mathbb{R}^M). \tag{2.17}$$

One of the main differences of the vectorial case with respect to the scalar one, is that the convexity assumption is not more the natural assumption in order to study the lower semicontinuity of the functional F defined in (2.17). It is well known (see [12] on this argument) that the quasiconvexity introduced by Morrey (see [42, 43]) is the appropriate condition to deal with functionals defined on vector valued functions. This last fact implies that many of the techniques available in convex analysis for the scalar case, could not be easily extended to the vectorial case.

An important contribution in the quasiconvex setting is due to Acerbi and Fusco. In [1] the authors proved the lower semicontinuity, with respect to the $W^{1,1}$-weak topology, of the functional defined in (2.17), by assuming that the integrand f is Carathéodory, quasiconvex with respect to ξ and growths at most linearly. Another crucial step in the history of the vectorial case is due to Ambrosio and Dal Maso ([8]). In that paper it was proven that, if the integrand f depends only on the variable ξ, is quasiconvex and growths at most linearly, then the relaxed functional of F with respect to the L^1-topology, is given by

$$\overline{F}(u,\Omega) = \int_\Omega f(\nabla u)dx + \int_\Omega f^\infty(\frac{D^s u}{|D^s u|})d|D^s u| \quad \forall u \in BV(\Omega;\mathbb{R}^M),$$

according to the scalar case.

A generalization of this result was provided by Amar and De Cicco, in [3], where they have considered functionals depending on higher order derivatives. Another generalization of the result of Ambrosio and Dal Maso was proved by Fonseca and Muller in [31], where the authors considered integrands depending on the full set of variables. More precisely their result, in the special case of integrand independent of s, is the following

Theorem 2.9 (see [31], Theorem 2.16) *Let $f : \Omega \times \mathbb{R}^{M\times N} \to [0, \infty)$ be a continuous function, quasiconvex in the last variable, satisfying the following hypotheses:*

(*i*) *there exist a constants $\lambda, \Lambda > 0$ such that*

$$\lambda|\xi| \leq f(x, \xi) \leq \Lambda|\xi|$$

for all $(x, \xi) \in \Omega \times \mathbb{R}^{M\times N}$;

(*ii*) *for every compact $K \subset\subset \Omega$ there exists a continuous function ω with $\omega(0) = 0$ such that*

$$|f(x, \xi) - f(x', \xi)| \leq \omega(|x - x'|)(1 + |\xi|)$$

for all (x, ξ), $(x', \xi) \in K \times \mathbb{R}^{M\times N}$. In addition, for every x_0 and for all $\varepsilon \geq 0$ there exists $\delta > 0$ such that if $|x - x_0| \leq \delta$, then

$$f(x, \xi) - f(x_0, \xi) \geq -\varepsilon$$

for every $\xi \in \mathbb{R}^{M\times N}$;

(*iii*) *there exist Λ', $0 \leq m \leq 1$ such that*

$$|f^\infty(x, \xi) - f(x, \xi)| \leq \Lambda'(1 + |\xi|^{1-m})$$

for every $(x,\xi) \in \Omega \times \mathbb{R}^{M\times N}$.

Then

$$\begin{aligned} \overline{F}(u,\Omega) &= \int_\Omega f(x,\nabla u)dx + \int_\Omega f^\infty(x,\frac{D^c u}{|D^c(u)|})d|D^c u| \\ &+ \int_{J_u\cap\Omega} f^\infty(x,(u^+ - u^-)\otimes \nu_u)d\mathcal{H}^{N-1}. \end{aligned} \tag{2.18}$$

More recently Fonseca and Leoni proved that formula (2.18) holds without coerciveness. This is obtained by assuming a uniform (with respect to ξ), continuity condition of the integrand $f(\cdot,\xi)$, as stated in the following theorem (see [29], Theorem 1.7 and 1.9).

Theorem 2.10 *Let* $f : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ *be Borel function, quasiconvex in the second variable, such that:*

(i) there exists a constant $\Lambda > 0$ *such that*

$$0 \le f(x,\xi) \le \Lambda(1+\xi);$$

(ii) for all $x_0 \in \Omega$ *and* $\varepsilon > 0$ *there exists a* $\delta > 0$*, such that*

$$f(x_0,\xi) - f(x,\xi) \le \varepsilon(1 + f(x,\xi))$$

for all $x \in \Omega$ *with* $|x - x_0| \le \delta$ *and for all* $\xi \in \mathbb{R}^{M\times N}$*;*

(iii) $f(\cdot,\xi)$*,* $f^\infty(\cdot,\xi)$ *are upper semicontinuous functions in* Ω *for all* $\xi \in \mathbb{R}^{M\times N}$.

Then, for all $u \in BV(\Omega;\mathbb{R}^M)$*,*

$$\overline{F}(u,\Omega) = \int_\Omega f(x,\nabla u)dx + \int_\Omega f^\infty(x,\frac{D^c u}{|D^c(u)|})d|D^c u| + \int_{J_u\cap\Omega} f^\infty(x,(u^+ - u^-)\otimes\nu_u)d\mathcal{H}^{N-1}.$$

An important tool which widely used in all the these relaxation results is the blow-up method introduced by Fonseca and Muller (see [30, 31]) and a result by Alberti (see [2]), showing that the density of the cantor part of BV-vector valued function is a rank one matrix.

Chapter 3

The scalar case

In this chapter[1] we prove a new L^1-lower semicontinuity result for the functional $\mathcal{F}$ in (1.5), by weakening the regularity assumptions with respect to s, but assuming the continuity with respect to x.

As already pointed out in the introduction the main tools of the proof of the lower semicontinuity theorem are a new chain rule formula (see Theorem 3.1) together with De Giorgi approximation result (see Theorem 1.5).

As an application of this result, we obtain the relaxation formula (2.6) and a Γ-convergence theorem for a sequence of functionals whose integrals pointwise converge. In the following Ω will be an open bounded subset of $\mathbb{R}^N$, $N \geq 1$.

3.1 Chain rule

In this section we improve both the chain rule of [20] and of [32]. Indeed, with respect to [20] we replace the Sobolev space $W^{1,1}(\Omega)$ with the space $BV(\Omega)$, while with respect to [32] we

[1] All the results of this chapter are contained in [39]

do not require any continuity assumption with respect to s.

Theorem 3.1 *Let $b : \Omega\times\mathbb{R}\to\mathbb{R}$ be a bounded Borel function with compact support in $\Omega\times\mathbb{R}$, satisfying the following properties*

(i) $b(\cdot,s)\in W^{1,1}(\Omega)\cap C(\Omega)$ for almost every $s\in\mathbb{R}$,

(ii) $\nabla_x b\in L^1(\Omega\times\mathbb{R})$.

Then for every $\varphi\in C^1_0(\Omega)$ and for every $u\in BV(\Omega)\cap L^\infty(\Omega)$ we have:

$$
\begin{aligned}
& - \int_\Omega\Big(\int_0^{u(x)} b(x,s)ds\Big)\nabla\varphi dx = \int_\Omega b(x,u)\varphi\nabla u dx \\
(3.1) \qquad & \\
& + \int_\Omega\Big(\fint_{u^-}^{u^+} b(x,s)ds\Big)\varphi dD^s u + \int_\Omega\Big(\int_0^{u(x)} \nabla_x b(x,s)ds\Big)\varphi dx.
\end{aligned}
$$

Proof.

Let $\{\psi_\delta\}_{\delta>0}$ be a mollifying sequence in $\mathbb{R}$. Let us define $b_\delta(x,s) = \int_\mathbb{R}\psi_\delta(s-t)b(x,t)dt$. We claim that, for every $\delta>0$, $b_\delta(x,s)$ is a continuous function in $\Omega\times\mathbb{R}$. In order to prove this, we notice the following properties: for every $\delta\in\mathbb{R}$, the function $\psi_\delta(\cdot-t)b(\cdot,t)$ is continuous in Ω for almost every $t\in\mathbb{R}$ thanks to the hypothesis (i) and to the regularity properties of mollifiers. Furthermore, since b has compact support, there exist two compact sets $K\subset\Omega$ and $\Lambda\subset\mathbb{R}$ such that the support of b is contained in $K\times\Lambda$ and the support of the function $b(x,\cdot)$ is contained in Λ for every $x\in K$. Hence we have that, for almost every $s\in\mathbb{R}$, $|\psi_\delta(s-t)b(x,t)|\le\|b\|_\infty\|\psi_\delta\|_\infty\chi_\Lambda(t)\in L^1(\mathbb{R})$. It follows, by the dominated convergence theorem, that $b_\delta(x,s)$ is continuous in $\Omega\times\mathbb{R}$. Let us show that, for every $\delta>0$,

$b_\delta(\cdot, s) \in W^{1,1}(\Omega)$ for every $s \in \mathbb{R}$. Indeed using Tonelli's theorem we get:

$$\begin{aligned}\int_\Omega |b_\delta(x,s)|dx &\leq \int_\Omega dx \int_\mathbb{R} |\psi_\delta(s-t)b(x,t)|dt = \int_{\Omega\times\mathbb{R}} |\psi_\delta(s-t)||b(x,t)|dxdt \\ &= \int_{K\times\Lambda} |\psi_\delta(s-t)||b(x,t)|dxdt \leq \mathcal{L}^N(K)\|b\|_\infty \int_\mathbb{R} |\psi_\delta(s-t)|dt \leq C,\end{aligned}$$

so that $b_\delta(\cdot, s) \in L^1(\Omega)$ for every $s \in \mathbb{R}$. Furthermore the following equality holds in the weak sense for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$,

$$\nabla_x(\int_\mathbb{R} \psi_\delta(s-t)b(x,t)dt) = \int_\mathbb{R} \psi_\delta(s-t)\nabla_x b(x,t)dt. \tag{3.2}$$

In fact, let S be the set of $t \in \mathbb{R}$ such that $b(\cdot, t) \notin W^{1,1}(\Omega)$. By hypothesis (i), $\mathcal{L}^1(S) = 0$. Multiplying by $\varphi \in C_0^1(\Omega; \mathbb{R}^N)$ the righthand side of (3.2), integrating over Ω, and applying Fubini's theorem (taking into account hypothesis (ii)), we get

$$\begin{aligned}\int_\Omega \varphi dx \int_\mathbb{R} \psi_\delta(s-t)\nabla_x b(x,t)dt &= -\int_{\mathbb{R}\setminus S} \psi_\delta(s-t)dt \int_\Omega b(x,t)\mathrm{div}_x\varphi \\ &= -\int_\Omega \mathrm{div}_x\varphi dx \int_\mathbb{R} \psi_\delta(s-t)b(x,t)dt\end{aligned}$$

and (3.2) is proved.

It remains to show that $\nabla_x b_\delta(\cdot, s) \in L^1(\Omega)$ uniformly with respect to $s \in \mathbb{R}$. From (3.2) and hypothesis (ii) we have:

$$\begin{aligned}\int_\Omega |\nabla_x b_\delta(x,s)|dx &\leq \int_\Omega dx \int_{\mathbb{R}\setminus\ S} |\psi_\delta(s-t)||\nabla_x b(x,t)|dt \\ &= \int_{\mathbb{R}\setminus S} \psi_\delta(s-t)dt \int_\Omega |\nabla_x b(x,t)|dx \leq \|\psi_\delta\|_\infty \int_{\Omega\times\mathbb{R}} |\nabla_x b(x,t)|dxdt \leq C.\end{aligned}$$

This implies that $b_\delta(x,s)$ satisfies all the hypotheses of Lemma 2.4 of [32] and so (3.1) holds for $b_\delta(x,s)$, i.e.

$$
\begin{aligned}
& - \int_\Omega \Big(\int_0^{u(x)} b_\delta(x,s)ds\Big)\nabla\varphi dx = \int_\Omega b_\delta(x,u)\varphi\nabla u dx \\
(3.3) \qquad & \\
& + \int_\Omega \Big(\fint_{u^-}^{u^+} b_\delta(x,s)ds\Big)\varphi dD^s u + \int_\Omega \Big(\int_0^{u(x)} \nabla_x b_\delta(x,s)ds\Big)\varphi dx,
\end{aligned}
$$

for every $\varphi \in C_0^1(\Omega)$. Now we pass to the limit as $\delta \to 0$.

Let us consider the first term in (3.3). We remark that $b_\delta(x,s)$ is continuous in $\Omega \times \mathbb{R}$ and there exists $M \subset \mathbb{R}$ with $\mathcal{L}^1(M) = 0$, such that by Lemma 1.4, $b(x,\cdot)$ is approximately continuous in $\mathbb{R} \setminus M$ for every $x \in \Omega$. Then, by Proposition 1.1, $b_\delta(x,s) \to b(x,s)$ for every $x \in \Omega$ and every $s \in \mathbb{R} \setminus M$. It is not difficult to prove that

$$
\begin{aligned}
& \Big| \int_\Omega \Big(\int_0^{u(x)} b_\delta(x,s)ds\Big)\nabla\varphi dx - \int_\Omega \Big(\int_0^{u(x)} b(x,s)ds\Big)\nabla\varphi dx\Big| \\
\le & \int_\Omega \Big(\int_{\mathbb{R}\setminus M} \chi_{[0,u(x)]}|b_\delta(x,s) - b(x,s)|ds\Big)|\nabla\varphi| dx \to 0,
\end{aligned}
$$

since

$$
(3.4) \quad \chi_{[0,u]}|b_\delta(x,s) - b(x,s)||\nabla\varphi| \le (\|b_\delta\|_\infty + \|b\|_\infty)|\nabla\varphi|\chi_H \le 2\|b\|_\infty|\nabla\varphi|\chi_H \in L^1(\Omega\times\mathbb{R}),
$$

for a proper compact set $H \subset \Omega \times \mathbb{R}$ and independent of δ.

Let us consider the second term of (3.3). As we have already remarked $b_\delta(x,s) \to b(x,s)$ for every $x \in \Omega$ and every $s \in \mathbb{R} \setminus M$. Moreover, reasoning as in (3.4), it follows

$$
(3.5) \qquad |b_\delta(x,s) - b(x,s)||\varphi||\nabla u| \le 2\|b\|_\infty|\varphi||\nabla u| \in L^1(\Omega),
$$

for every $\delta > 0$.

Hence by Proposition 1.2, we get

$$\Big|\int_\Omega b_\delta(x,u)\varphi\nabla u dx - \int_\Omega b(x,u)\varphi\nabla u dx\Big| = \Big|\int_{\Omega\setminus(\tilde{u})^{-1}(M)} b_\delta(x,u)\varphi\nabla u dx - \int_{\Omega\setminus(\tilde{u})^{-1}(M)} b(x,u)\varphi\nabla u dx\Big|,$$

and, letting $\delta \to 0$,

$$\Big|\int_{\Omega\setminus(\tilde{u})^{-1}(M)} b_\delta(x,u)\varphi\nabla u dx - \int_{\Omega\setminus(\tilde{u})^{-1}(M)} b(x,u)\varphi\nabla u dx\Big| \to 0,$$

as a consequence of (3.5) and the dominated convergence theorem.

Let us consider the third term of (3.3). Thanks to (1.1) and (1.7), we can rewrite this term as

$$\int_{J_u\cap\Omega}\Big(\fint_{u^-}^{u^+} b_\delta(x,s)ds\Big)\varphi dD^j(u) + \int_{C_u\Omega} b_\delta(x,\tilde{u}(x))\varphi dD^c u. \tag{3.6}$$

Clearly for every $x \in \Omega \cap J_u$ we have

$$\fint_{u^-}^{u^+} |b_\delta(x,s) - b(x,s)|ds \to 0, \text{ as } \delta \to 0.$$

Furthermore, the function $g_\delta(x) = |\varphi(x)|\fint_{u^-}^{u^+} |b_\delta(x,s) - b(x,s)|ds$ satisfies the following estimate

$$0 \le g_\delta(x) \le 2\|b\|_\infty\|\varphi\|_\infty \in L^1(J_u \cap \Omega, |D^j u|)$$

so that, letting $\delta \to 0$, we get

$$\Big|\int_{J_u\cap\Omega}\Big(\fint_{u^-}^{u^+} b_\delta(x,s)ds\Big)\varphi dD^j(u) - \int_{J_u\cap\Omega}\Big(\fint_{u^-}^{u^+} b(x,s)ds\Big)\varphi dD^j(u)\Big| \to 0.$$

As far as the second term of (3.6), for every $t \in \mathbb{R}$ and for every $x \in C_u \cap \Omega \setminus (\tilde{u})^{-1}(M)$, we have

$$|b_\delta(x,s) - b(x,s)||\varphi| \leq 2\|b\|_\infty |\varphi| \in L^1(C_u \cap \Omega, |D^c u|), \tag{3.7}$$

so that, by the dominated convergence theorem and Lemma 1.2, we obtain

$$\int_{\Omega \cap C_u} b_\delta(x, \tilde{u}(x))\varphi dD^c u \to \int_{\Omega \cap C_u} b(x, \tilde{u}(x))\varphi dD^c u,$$

so that

$$\int_\Omega \Big(\fint_{u^-}^{u^+} b_\delta(x,s)ds \Big)\varphi dD^s u \to \int_\Omega \Big(\fint_{u^-}^{u^+} b(x,s)ds \Big)\varphi dD^s u.$$

Let us consider the last term of (3.3). Thanks to the hypothesis (ii), we have that for $\mathcal{L}^N$-almost every $x \in \Omega$ the function $\nabla_x b(x,\cdot) \in L^1(\mathbb{R})$. Therefore, from (3.2), it follows that for $\mathcal{L}^N$-almost every $x \in \Omega$,

$$\nabla_x b_\delta(x,\cdot) = \psi_\delta * \nabla_x b(x,\cdot) \to \nabla_x b(x,\cdot) \text{ in } L^1(\mathbb{R}),$$

as $\delta \to 0$. This implies that, for $\mathcal{L}^N$-almost every $x \in \Omega$, we obtain

$$\lim_{\delta \to 0} \int_0^{u(x)} |\nabla_x b_\delta(x,s) - \nabla_x b(x,s)|ds = 0.$$

In order to conclude, we note that, thanks to the hypothesis (ii),

$$\begin{aligned} |\varphi(x)| \int_0^{u(x)} |\nabla_x b_\delta(x,s)| &\leq \|\varphi\|_\infty \int_{\mathbb{R}} ds \int_{\mathbb{R}} \psi_\delta(s-t)|\nabla_x b(x,t)|dt \\ &= \int_{\mathbb{R}} |\nabla_x b(x,t)|dt \int_{\mathbb{R}} \psi_\delta(s-t)ds = \int_{\mathbb{R}} |\nabla_x b(x,t)|dt \in L^1(\Omega), \end{aligned}$$

for a.e. $x \in \Omega$ and hence

$$\int_\Omega \Big(\int_0^{u(x)} \nabla_x b_\delta(x,s)ds \Big)\varphi dx \to \int_\Omega \Big(\int_0^{u(x)} \nabla_x b(x,s)ds \Big)\varphi dx,$$

as $\delta \to 0$. The proof is now complete. $\blacksquare$

We prove a refinement of previous result, which will be useful in the next section.

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let $b : \Omega \times \mathbb{R} \to \mathbb{R}^N$ be a Borel function with a compact support in $\Omega \times \mathbb{R}$ satisfying the following properties*

(i) there exists $g \in L^1(\mathbb{R})$ such that $|b(x,s)| \le g(s)$ for every $x \in \Omega$ and for every $s \in \mathbb{R}$;

(ii) $b(\cdot, s) \in W^{1,1}(\Omega; \mathbb{R}^N) \cap C(\Omega; \mathbb{R}^N)$ for almost every $s \in \mathbb{R}$,

(iii) $\nabla_x b \in L^1(\Omega \times \mathbb{R})$.

Then for every $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega \langle b(x,u), \nabla u \rangle^+ dx < +\infty;$$

$$\int_\Omega \Big(\fint_{u^-}^{u^+} \langle b(x,s), \frac{D^s u}{|D^s u|} \rangle^+ ds \Big) d|D^s u| < +\infty.$$

and for every $\varphi \in C_0^1(\Omega)$ we have

$$\int_\Omega \langle b(x,u), \nabla u \rangle \varphi dx + \int_\Omega \Big(\fint_{u^-}^{u^+} \langle b(x,s), \frac{D^s u}{|D^s u|} \rangle ds \Big) \varphi d|D^s u|$$

(3.8)

$$= \; -\int_\Omega \langle \int_0^{u(x)} b(x,s) ds, \nabla \varphi \rangle dx - \int_\Omega \Big(\int_0^{u(x)} \mathrm{div}_x b(x,s) ds \Big) \varphi dx.$$

Proof.

Let us define

$$b_h(x,s) = b(x,s)\chi_{A_h}(s) \quad \text{where } A_h = \{ s \in \mathbb{R} : g(s) \le h \}.$$

Clearly $b_h \in L^\infty(\Omega \times \mathbb{R})$ for every $h \in \mathbb{N}$ and $b_h(x,s) \to b(x,s)$ for a.e. $x \in \Omega$ and for a.e $s \in \mathbb{R}$ Therefore (3.8) holds for b_h, i.e.

$$\int_\Omega \langle b_h(x,u), \nabla u\rangle \varphi dx + \int_\Omega \Big(\fint_{u^-}^{u^+} \langle b_h(x,s), \frac{D^s u}{|D^s u|}\rangle ds\Big)\varphi d|D^s u|$$

(3.9)

$$= \; -\int_\Omega \langle \int_0^{u(x)} b_h(x,s)ds, \nabla\varphi\rangle dx - \int_\Omega \Big(\int_0^{u(x)} \mathrm{div}_x b_h(x,s)ds\Big)\varphi dx,$$

for every $\varphi \in C_0^1(\Omega)$ Moreover, $\mathrm{div}_x b_h(x,s) = \chi_{A_h}(s)\mathrm{div}_x b(x,s) \to \mathrm{div}_x b(x,s)$ for a.e. $(x,s) \in \Omega \times \mathbb{R}$. Since $|\mathrm{div}_x b_h(x,s)| \leq |\nabla_x b(x,s)|$ for a.e. $(x,s) \in \Omega \times \mathbb{R}$, and, by (iii), $|\nabla_x b(x,\cdot)| \in L^1(\mathbb{R})$ for a.e. $x \in \Omega$, we get a.e.

$$\varphi(x)\int_0^{u(x)} \mathrm{div}_x b_h(x,s)ds \to \varphi(x)\int_0^{u(x)} \mathrm{div}_x b(x,s)ds.$$

Using again (iii), it follows

$$\Big|\varphi\int_0^{u(x)} \mathrm{div}_x b_h(x,s)ds\Big| \leq |\varphi|\int_\mathbb{R} |\nabla_x b(x,s)|ds \in L^1(\Omega),$$

and hence

$$\int_\Omega \Big(\int_0^{u(x)} \mathrm{div}_x b_h(x,s)ds\Big)\varphi dx \to \int_\Omega \Big(\int_0^{u(x)} \mathrm{div}_x b(x,s)ds\Big)\varphi dx.$$

Let us consider the lefthand side of (3.9) Since $\langle b_h(x,s),\xi\rangle^+$ and $\langle b_h(x,s),\xi\rangle^-$ are increasing sequences which converge to $\langle b(x,s),\xi\rangle^+$ and $\langle b(x,s),\xi\rangle^-$ respectively, from Beppo Levi's theorem and hypothesis (ii), we obtain that

$$\lim_{h\to+\infty}\int_\Omega \langle b_h(x,u), \nabla u\rangle\varphi dx = \int_\Omega \langle b(x,u),\nabla u\rangle \varphi dx.$$

Analogously, using again hypothesis (ii), we get

$$\lim_{h\to+\infty}\int_\Omega \Big(\fint_{u^-}^{u^+}\langle b_h(x,s),\frac{D^s u}{|D^s u|}\rangle ds\Big)\varphi d|D^s u| = \int_\Omega \Big(\fint_{u^-}^{u^+}\langle b(x,s),\frac{D^s u}{|D^s u|}\rangle ds\Big)\varphi d|D^s u|.$$

Therefore passing to the limit, as $h \to +\infty$, in (3.9) we get (3.8). The thesis is achieved. ■

3.2 Lower semicontinuity

By using Theorem 3.1, in the same spirit of [20] and [32], but on the space $BV(\Omega)$ and without continuity with respect to the variable s, we obtain the lower semicontinuity result with respect to the L^1-topology, of the functional $\mathcal{F}$ in (1.5).

Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a Borel function such that:

$$(3.10)\qquad \begin{cases} (i) & f(x,s,\cdot) \text{ is convex on } \mathbb{R}^N \text{ for every } (x,s) \in \Omega\times\mathbb{R};\\[2ex] (ii) & f(\cdot,s,\xi) \in C(\Omega)\cap W^{1,1}_{loc}(\Omega) \text{ for almost every } s\in\mathbb{R} \text{ and for every } \xi\in\mathbb{R}^N;\\[2ex] (iii) & \text{for every bounded set } B\subset \mathbb{R}\times\mathbb{R}^N,\ \text{there exists a costant } L(B)\\ & \text{such that } \displaystyle\int_\Omega |\nabla_x f(x,s,\xi)|dx \le L(B) \text{ for every } (s,\xi)\in B. \end{cases}$$

Theorem 3.3 *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a locally bounded Borel function, satisfying (3.10) and that*

$$(3.11)\qquad f(x,s,0) = 0 \quad \forall (x,s) \in \Omega\times\mathbb{R}.$$

Then the functional (1.5) is lower semicontinuous on $BV(\Omega)$ with respect to the L^1-topology.

Proof.

By Theorem 1.5 there exists a sequence $\{\alpha_\kappa\} \subset C_0^\infty(\Omega)$ with $\alpha_\kappa \ge 0$ and $\int_{\mathbb{R}^N}\alpha_\kappa dx = 1$ such that for any $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ we have

$$f(x, s, \xi) = \sup_{\kappa \in \mathbb{N}} (a_\kappa(x, s) + \langle b_\kappa(x, s), \xi \rangle)^+$$

and

$$f^\infty(x, s, \xi) = \sup_{\kappa \in \mathbb{N}} \langle b_\kappa(x, s), \xi \rangle^+,$$

where, recalling (1.11) and (1.12),

$$\begin{aligned} a_\kappa(x, s) &= \int_{\mathbb{R}^N} f(x, s, \xi)\big((N+1)\alpha_\kappa(\xi) + \langle \nabla \alpha_\kappa(\xi), \xi \rangle\big) d\xi \\ b_\kappa(x, s) &= -\int_{\mathbb{R}^N} f(x, s, \xi) \nabla \alpha_\kappa(\xi) d\xi. \end{aligned} \tag{3.12}$$

Hence, if we set $f_\kappa(x, s, \xi) = (a_\kappa(x, s) + \langle b_\kappa(x, s), \xi \rangle)^+$, we obtain $\hat{f}(x, s, \xi, t) = \sup_\kappa \hat{f}_\kappa(x, s, \xi, t)$. Therefore, applying Lemma 1.6 with f, f_k and μ replaced by $\hat{f}$, $\hat{f}_k$ and $|\alpha(u)|$ respectively, we obtain

$$\begin{aligned} \mathcal{F}(u, \Omega) &= \int_{\Omega \times \mathbb{R}} \hat{f}(x, s, \frac{\alpha(u)}{|\alpha(u)|}) d|\alpha(u)|(x, s) \\ &= \sup_{\mathcal{D}} \sum_{i \in I} \int_{\Omega \times \mathbb{R}} \hat{f}_{\kappa_i}(x, s, \frac{\alpha(u)}{|\alpha(u)|}) \varphi_i(x) \psi_i(u) d|\alpha(u)|(x, s) \\ &= \sup_{\mathcal{D}} \sum_{i \in I} \Big\{ \int_\Omega \psi_i(u) \big(a_{\kappa_i}(x, u) \\ &+ \langle b_{\kappa_i}(x, u), \nabla u \rangle\big)^+ \varphi_i(x) dx \\ &+ \int_\Omega \Big(\fint_{u^-}^{u^+} \psi_i(s) \langle b_{\kappa_i}(x, s), \frac{D^s u}{|D^s u|} \rangle^+ ds \Big) \varphi_i(x) d|D^s u| \Big\}, \end{aligned} \tag{3.13}$$

where the first and the last equality are due to Lemma 1.7 and we used the notation in (1.8).

Let us define

$$(3.14)\qquad \begin{aligned} G_i(u) \;&:=\; \int_\Omega \psi_i(u)\big(a_{\kappa_i}(x,u)+\langle b_{\kappa_i}(x,u),\nabla u\rangle\big)^+\varphi_i(x)dx \\ &+\; \int_\Omega \Big(\fint_{u^-}^{u^+} \psi_i(s)\langle b_{\kappa_i}(x,s),\frac{D^s u}{|D^s u|}\rangle^+ ds\Big)\varphi_i(x)d|D^s u|. \end{aligned}$$

We remark that, by (ii) of (3.10) and (3.12), $a_{\kappa_i}(\cdot,s)$ is continuous for almost every $s\in\mathbb{R}$. By Scorza-Dragoni theorem it is possible to find an increasing sequence K_h of compact subsets of $\mathbb{R}$ such that, if we set $E:=\bigcup_{h\in\mathbb{N}}K_h$, $\mathcal{L}^1(\mathbb{R}\setminus E)=0$, and for every $\kappa_i\in\mathbb{N}$ $a_{\kappa_i}\in C^0(\Omega\times K_h)$. We remark that, by hypothesis (3.11), we have $a_{\kappa_i}\le 0$, hence, by Proposition 1.2 it follows that

$$\begin{aligned} G_i(u) \;&=\; \int_\Omega \chi_E(u)\psi_i(u)\big(a_{\kappa_i}(x,u)+\langle b_{\kappa_i}(x,u),\nabla u\rangle\big)^+\varphi_i(x)dx \\ &+\; \int_\Omega \Big(\fint_{u^-}^{u^+} \chi_E(s)\psi_i(s)\langle b_{\kappa_i}(x,s),\frac{D^s u}{|D^s u|}\rangle^+ ds\Big)\varphi_i(x)d|D^s u| \\ &=\; \sup_{h\in\mathbb{N}}\Big\{\int_\Omega \chi_{K_h}(u)\psi_i(u)\big(a_{\kappa_i}(x,u)+\langle b_{\kappa_i}(x,u),\nabla u\rangle\big)^+\varphi_i(x)dx \\ &+\; \int_\Omega \Big(\fint_{u^-}^{u^+} \chi_{K_h}(s)\psi_i(s)\langle b_{\kappa_i}(x,s),\frac{D^s u}{|D^s u|}\rangle^+ ds\Big)\varphi_i(x)d|D^s u|\Big\}. \end{aligned}$$

As $\mathcal{L}^N$ and $|D^s u|$ are mutually singular measures,

$$(3.15)\qquad \begin{aligned} G_i(u) \;&=\; \sup_{h\in\mathbb{N}}\sup_{0\le\eta\le 1}\Big\{\int_\Omega \chi_{K_h}(u)\psi_i(u)(a_{\kappa_i}(x,u)\eta(x)\varphi_i(x)dx \\ &+\; \int_\Omega \chi_{K_h}(u)\langle\psi_i(u)b_{\kappa_i}(x,u),\nabla u\rangle\eta(x)\varphi_i(x)dx \\ &+\; \int_\Omega \Big(\fint_{u^-}^{u^+} \chi_{K_h}(s)\eta(x)\langle\psi_i(s)b_{\kappa_i}(x,s),\frac{D^s u}{|D^s u|}\rangle ds\Big)\varphi_i(x)d|D^s u|\Big\}; \end{aligned}$$

where $\eta\in C_0^\infty(\Omega)$. Since $a_{\kappa_i}\in C^0(\Omega\times K_h)$ and $a_{\kappa_i}\le 0$, the function $\chi_{K_h}(s)\psi_i(s)a_{\kappa_i}(x,s)$ is lower semicontinuous with respect to $s\in\mathbb{R}$. Therefore, as a consequence of Fatou's lemma, the first term in (3.15) is lower semicontinuous with respect to the L^1-topology. Now we prove the lower semicontinuity with respect to the L^1-topology of the last two terms of (3.15). Since $u_n \to u$ strongly in $L^1(\Omega)$, without loss of generality, we may assume that $u_n \to u$ almost everywhere in Ω. Let us define

$$\begin{aligned} H(u_n) \quad &:= \quad \int_\Omega \chi_{K_h}(u_n)\langle \psi_i(u_n) b_{\kappa_i}(x,u_n), \nabla u_n\rangle \eta(x)\varphi_i(x)dx \\ &+ \quad \int_\Omega \Big(\fint_{u_n^-}^{u_n^+} \chi_{K_h}(s)\eta(x)\langle \psi_i(s) b_{\kappa_i}(x,s), \frac{D^s u_n}{|D^s u_n|}\rangle ds \Big) \varphi_i(x) d|D^s u_n|. \end{aligned}$$

We claim that the scalar function $\eta(x)\psi_i(s)b^j_{\kappa_i}(x,s)$ satisfies for $1 \le j \le n$ all the hypotheses of Theorem 3.1. Indeed $\eta(x)\psi_i(s)b^j_{\kappa_i}(x,s)$ has compact support in $\Omega \times \mathbb{R}$ and it is bounded in $\Omega \times \mathbb{R}$, since $f \in L^\infty_{loc}(\Omega \times \mathbb{R} \times \mathbb{R}^N)$. Moreover, by (ii) of (3.10) and the dominated convergence theorem, it follows that $\eta(\cdot)\psi_i(s)b^j_{\kappa_i}(\cdot,s)$ is continuous for almost every $s \in \mathbb{R}$. Finally by (ii) and (iii) of (3.10), we have that $\eta(\cdot)\psi_i(s)b^j_{\kappa_i}(\cdot,s)$ belongs to $W^{1,1}(\Omega)$ with $\nabla_x\big(\eta(x)\psi_i(s)b^j_{\kappa_i}(x,s)\big) \in L^1(\Omega \times \mathbb{R})$. Furthermore, thanks to the presence of the characteristic function χ_{K_h} in the definition of the functional $H(u_n)$, we may assume, without loss of generality, for every $n \in \mathbb{N}$ $u_n \in L^\infty(\Omega)$.

Therefore, by applying Theorem 3.1, we get

$$\begin{aligned} \liminf_{n\to+\infty} H(u_n) \quad &= \quad \lim_{n\to+\infty} \Big\{ -\int_\Omega \big(\int_0^{u_n(x)} \mathrm{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds \big)\varphi_i dx \\ &- \quad \int_\Omega \langle \int_0^{u_n(x)} b_{\kappa_i}(x,s)\chi_{K_h}(s)\eta(x)\psi_i(s)ds, \nabla\varphi_i\rangle dx \Big\}. \end{aligned}$$

From (iii) of (3.10) and from the absolute continuity of the integral it follows that as $n \to +\infty$

$$\lim_{n\to+\infty} \int_0^{u_n(x)} \mathrm{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds = \int_0^{u(x)} \mathrm{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds.$$

Moreover,

$$\begin{aligned} & \Big|\varphi_i(x)\int_0^{u_n(x)} \operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds\Big| \\ \text{(3.16)} \qquad & \\ \leq \quad & \|\varphi_i\|_\infty \int_{\mathbb{R}} |\operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))|ds \in L^1(\Omega), \end{aligned}$$

so that

$$\lim_{n\to\infty}\int_\Omega \varphi_i(x)\int_0^{u_n(x)} \operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds = \varphi_i(x)\int_0^{u_(x)} \operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds.$$

Analogously we get

$$\lim_{n\to+\infty}\int_\Omega \langle \int_0^{u_n(x)} b_{\kappa_i}(x,s)\chi_{K_h}(s)\eta(x)\psi_i(s)ds, \nabla\varphi_i\rangle dx = \int_\Omega \langle \int_0^{u(x)} b_{\kappa_i}(x,s)\chi_{K_h}(s)\eta(x)\psi_i(s)ds, \nabla\varphi_i\rangle dx.$$

Therefore letting $n \to +\infty$ in (3.16) we obtain

$$\begin{aligned} \liminf_{n\to+\infty} H(u_n) \quad = \quad & -\int_\Omega \big(\int_0^{u(x)} \operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds\big)\varphi_i dx \\ \text{(3.17)} \qquad & \\ - \quad & \int_\Omega \langle \int_0^{u(x)} b_{\kappa_i}(x,s)\chi_{K_h}(s)\eta(x)\psi_i(s)ds, \nabla\varphi_i\rangle dx. \end{aligned}$$

Hence, applying Theorem 3.1 to (3.17), we obtain the lower semicontinuity of the second and the third term of (3.15). This implies that G_i, being the supremum of lower semicontinuous functions is lower semicontinuous itself, so that, by (3.13) and (3.14), $\mathcal{F}$ is lower semicontinuous too.

The thesis is then achieved. ■

Remark 3.1 *It is not very difficult to verify that Theorem 3.3 continues to hold under a weaker assumption than* (iii) *of (3.10), which is the following*

$$\nabla_x f \in L^1_{loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n). \tag{3.18}$$

Indeed in the proof of the previous theorem we only need to know that $\eta(\cdot)\psi_i(s)b^j_{\kappa_i}(\cdot, s)$ *belongs to* $W^{1,1}(\Omega)$ *with* $\nabla_x\big(\eta(x)\psi_i(s)b^j_{\kappa_i}(x,s)\big) \in L^1(\Omega\times\mathbb{R})$, *and it is guaranteed by hypothesis (3.18).*

In the same spirit of the papers of De Giorgi Buttazzo and Dal Maso (see [23]) and Ambrosio (see [7]) we give a further lower semicontinuity result, where assumption (3.11) is replaced by a weaker one.

Theorem 3.4 *Let* $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ *be a locally bounded Borel function satisfying (3.10) such that:*

(a) $f(x, \cdot, 0)$ *is lower semicontinuous on* $\mathbb{R}$ *for* $\mathcal{L}^N$ *a.e* $x \in \Omega$

(b) *there exists a Borel function*

$$\lambda : \Omega \times \mathbb{R} \to \mathbb{R}^N,$$

with $\lambda(x, s) \in \partial_\xi f(x, s, 0)$ *for every* $(x, s) \in \Omega \times \mathbb{R}$, *such that*

(i) $g(s) = \sup_{x\in\Omega} |\lambda(x, s)| \in L^1_{loc}(\mathbb{R})$

(ii) $\lambda(\cdot, s) \in C(\Omega; \mathbb{R}^N)$ *for* $\mathcal{L}^1$ *a.e* $s \in \mathbb{R}$

(iii) $\lambda(\cdot, s) \in W^{1,1}_{loc}(\Omega; \mathbb{R}^N)$ *for* $\mathcal{L}^1$ *a.e* $s \in \mathbb{R}$ *with* $\nabla_x\lambda \in L^1_{loc}(\Omega \times \mathbb{R})$.

Then the functional (1.5) is lower semicontinuous in $BV(\Omega)$ *with respect to the strong* L^1-*topology.*

Proof.

Without loss of generality, we may suppose that there exists a constant $C > 0$ such that $f(x,s,\xi) = 0$ for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, with $|s| \geq C$. Indeed, in the general case, we can write

$$f(x,s,\xi) = \sup_{k \in \mathbb{N}} f(x,s,\xi)\chi_{(-k,k)}(s).$$

Moreover since $\lambda(x,s) \in \partial_p f(x,s,0)$ and that $f \geq 0$, it follows that $f(x,s,\xi) \geq \langle \lambda(x,s), \xi \rangle^+$ for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. Hence we may assume that $\lambda(x,s) = 0$ for every $x \in \Omega$ and $s \in \mathbb{R}$, with $|s| \geq C$. Besides, since f is locally bounded, λ is locally bounded, too. Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ be defined by

$$g(x,s,\xi) = f(x,s,\xi) - f(x,s,0) - \langle \lambda(x,s), \xi \rangle.$$

Then for every $\varphi \in C_0^\infty(\Omega)$ and for every open set $A \subset\subset \Omega$ we have

$$\mathcal{F}_A(f,u,\varphi) = \mathcal{F}_A(g,u,\varphi) + G_A(f,u,\varphi) + H_A(\lambda,u,\varphi), \tag{3.19}$$

where

$$\begin{aligned}
\mathcal{F}_A(f,u,\varphi) &= \int_A f(x,u,\nabla u)\varphi dx + \int_A \Big[\fint_{u^-}^{u^+} f^\infty(x,s,\frac{D^s u}{|D^s(u)|}) ds \Big] \varphi d|D^s u|, \\
G_A(f,u,\varphi) &= \int_A f(x,u,0)\varphi dx \\
H_A(\lambda,u,\varphi) &= \int_A \langle \lambda(x,u), \nabla u \rangle \varphi dx + \int_A \Big[\fint_{u^-}^{u^+} \langle \lambda(x,s), \frac{D^s u}{|D^s(u)|} \rangle ds \Big] \varphi d|D^s u|.
\end{aligned}$$

Let $u_n \to u \in BV(\Omega)$ strongly in $L^1(\Omega)$. Without loss of generality, we may suppose that $u_n \to u$ almost everywhere in Ω and that $F(u_n) \leq M$, for every $n \in \mathbb{N}$. Since the function $g\varphi$

satisfies all the hypotheses of Theorem 3.3 we obtain that

$$\mathcal{F}_A(g,u,\varphi) \le \liminf_{n\to+\infty} \mathcal{F}_A(g,u_n,\varphi). \tag{3.20}$$

Moreover, by hypothesis (a) and Fatou's lemma it follows that

$$G_A(f,u,\varphi) \le \liminf_{n\to+\infty} G_A(f,u_n,\varphi). \tag{3.21}$$

Since $f(x,s,\xi) \ge \langle\lambda(x,s),\xi\rangle^+$ we have, for every $n\in\mathbb{N}$,

$$\int_A \langle\lambda(x,u_n),\nabla u_n\rangle^+ dx \le F(u_n) \le M \tag{3.22}$$

and

$$\int_A \Big[\fint_{u_n^-}^{u_n^+} \langle\lambda(x,s),\frac{D^s u_n}{|D^s u_n|}\rangle^+ ds\Big] d|D^s u_n| \le F(u_n) \le M. \tag{3.23}$$

We remark that, since λ is locally bounded, we have

$$\int_A \langle\lambda(x,u),\nabla u\rangle^+ dx \le M \tag{3.24}$$

and

$$\int_A \Big[\fint_{u^-}^{u^+} \langle\lambda(x,s),\frac{D^s u}{|D^s u|}\rangle^+ ds\Big] d|D^s u| \le M. \tag{3.25}$$

Furthermore, if we define

$$\tilde\lambda(x,s) = \begin{cases} \lambda(x,s) & (x,s)\in supp\varphi\times[-C,C], \\ 0 & (x,s)\notin supp\varphi\times[-C,C]. \end{cases} \tag{3.26}$$

The function $\tilde\lambda$ satisfies all the hypotheses of Lemma 3.2. Then using, by (3.22) and (3.23), Lemma 3.2, we get

$$\liminf_{n\to+\infty} H_A(\lambda,u_n,\varphi) = \lim_{n\to+\infty}\Big\{-\int_\Omega\langle\int_0^{u_n(x)}\lambda(x,s)ds,\nabla\varphi\rangle dx - \int_\Omega\Big(\int_0^{u_n(x)}\mathrm{div}_x\lambda(x,s)ds\Big)\varphi dx\Big\};$$

so that, by (3.24) and (3.25), using again Lemma 3.2

$$\lim_{n\to+\infty} H_A(\lambda,u_n,\varphi) \;=\; -\int_\Omega \langle \int_0^{u(x)} \lambda(x,s)ds, \nabla\varphi\rangle dx - \int_\Omega \big(\int_0^{u(x)} \mathrm{div}_x\lambda(x,s)ds\big)\varphi dx$$

(3.27)

$$= \; H_A(\lambda,u,\varphi).$$

Therefore from (3.19), (3.20), (3.21) and (3.27) we have

$$\mathcal{F}_A(f,u,\varphi) \le \liminf_{n\to+\infty} \mathcal{F}_A(f,u_n,\varphi) \le \liminf_{n\to+\infty} \mathcal{F}_\Omega(f,u_n,\varphi).$$

Then, since A is arbitrary, the functional $u \to \mathcal{F}_\Omega(f,u,\varphi)$ is lower semicontinuous. The conclusion follows by

$$\mathcal{F}(u,\Omega) = \sup\{\mathcal{F}_\Omega(f,u,\varphi) : \varphi \in C_0^\infty(\Omega),\ 0 \le \varphi \le 1\}. \quad \blacksquare$$

3.3 Applications

In this section, as a consequence of our lower semicontinuity results, we firstly give an integral representation theorem for the relaxation functional of the functional in (1.13), then we prove a Γ-limit result for a sequence of functionals $\{F_n\}$ of the same type.

3.3.1 Relaxation

In this subsection, given F as in (1.4), we will show that the following representation holds

$$\mathcal{F}(u,\Omega) = \overline{F}(u,\Omega) \quad \forall u \in BV(\Omega) \tag{3.28}$$

where $\overline{F}$ and $\mathcal{F}$ are defined in (1.13) and(1.5), respectively. In order to get (3.28), we use a result due to Fonseca and Leoni. To this aim we assume that $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to [0,+\infty)$ is a

Borel function such that

$$0 \le f(x,s,\xi) \le C(1+|\xi|) \quad \text{for all } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N. \tag{3.29}$$

Theorem 3.5 (see [29], Theorem 1.6**)** *Let* $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ *be a Borel function convex with respect to* ξ *for every* $(x,s) \in \Omega \times \mathbb{R}$*, and continuous with respect to* x *for every* $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$*. Assume that* f *satisfies (3.29), and* $f^\infty(\cdot,s,\xi)$ *is upper semicontinuous in* Ω *for every* $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$*. Then*

$$\overline{F}(u,\Omega) \le \mathcal{F}(u,\Omega).$$

Remark 3.2 *Following the proof of Theorem* 1.6 *in [29], it is not difficult to see that the Theorem 3.5 holds even if the hypothesis that* f *is continuous with respect to* x *for every* $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$*, is replaced by*

$$|f(x,s_1,0) - f(x,s_2,0)| \le C\rho(s_1 - s_2),$$

for every $x \in \Omega$ *and* s_1*,* $s_2 \in \mathbb{R}$*, where* ρ *is a modulus of continuity, i.e. a nonnegative, increasing and continuous function* ρ *such that* $\rho(0) = 0$*, or by the assumption that for every* $s \in \mathbb{R}$ *exists* $N \subset \Omega$ *such that* $\mathcal{H}^{N-1}(N) = 0$ *and* $f(\cdot,s,0)$ *is approximately continuous in* $\Omega \setminus N$ *(these conditions are in particular implied by* $f(x,s,0) = 0$*).*

Theorem 3.6 *Let* $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ *be a Borel function, which satisfies hypotheses (3.10),(3.29) and* (*a*)*,* (*b*) *of Theorem 3.4. Assume that* $f^\infty(\cdot,s,\xi)$ *is upper semicontinuous in* Ω *for every* $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$*. Then* $\mathcal{F}(u,\Omega) = \overline{F}(u,\Omega)$*.*

Proof.

Since $\overline{F}$ is the greatest lower semicontinuous functional not greater than F, $\mathcal{F} \leq F$ and, by Theorem 3.4, $\mathcal{F}$ is L^1-lower semicontinuous, it follows that

$$\mathcal{F}(u, \Omega) \leq \overline{F}(u, \Omega).$$

The opposite inequality is stated in Theorem 3.5. ■

3.3.2 Γ-convergence

In this subsection, in the same spirit of [4, 13], we state a Γ-convergence result for a sequence of integral functionals of type (1.4), whose integrands f_n pointwise converge to an integrand f, which is not necessarily continuous with respect to s nor coercive.

Theorem 3.7 *Let $f_n : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ be a sequence of Borel functions such that*

$$0 \leq f_n(x, s, \xi) \leq \Lambda(1 + |\xi|) \quad \textit{for every } (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \tag{3.30}$$

where $0 < \Lambda < +\infty$ is a fixed constant. For every $u \in BV(\Omega)$ we define

$$F_n(u, \Omega) = \begin{cases} \displaystyle\int_\Omega f_n(x, u, \nabla u)dx & \textit{if } u \in W^{1,1}(\Omega) \\ +\infty & \textit{if } u \in BV(\Omega) \setminus W^{1,1}(\Omega). \end{cases} \tag{3.31}$$

Assume that $\{f_n\}$ converges pointwise to a locally bounded Borel function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ satisfying all the hypotheses of Theorem 3.6.

Finally, let $\{\varepsilon_n\}$ be an infinitesimal sequence, such that

$$(1 + \varepsilon_n)f_n(x, s, \xi) \geq f(x, s, \xi) - \varepsilon_n \quad \textit{for every } (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \forall n \in \mathbb{N}. \tag{3.32}$$

Then for every $u \in BV(\Omega)$, we have

$$F^{\Gamma}(u,\Omega) := \Gamma - \lim F_n(u,\Omega) = \mathcal{F}(u,\Omega).$$

Proof.

By the compactness of Γ-convergence, we may assume that, up to a subsequence, there exists $\Gamma - \lim F_n$. Firstly, we will prove that $\Gamma - \lim F_n \geq \mathcal{F}$. Given $u \in BV(\Omega)$, by (3.32), for every $n \in \mathbb{N}$ we obtain that $F_n(u,\Omega) \geq F(u,\Omega) - \varepsilon_n[\mathcal{L}^N(\Omega) + F_n(u,\Omega)]$, where F is defined in (1.4). By (1.17), we have that for every $u \in BV(\Omega)$, there exists $\overline{u}_n \to u$ strongly in $L^1(\Omega)$, such that

$$(\Gamma - \lim_{n\to\infty} F_n)(u,\Omega) = \lim_{n\to\infty} F_n(\overline{u}_n,\Omega).$$

We may assume that the previous limit is finite (otherwise the conclusion is trivial). Therefore, taking into account Theorem 3.6, it follows

$$\begin{aligned}(\Gamma - \lim_{n\to\infty}, F_n)(u,\Omega) = \lim_{n\to\infty} F_n(\overline{u}_n,\Omega) &\geq \liminf_{n\to\infty} F(\overline{u}_n,\Omega) - \lim_{n\to\infty} \varepsilon_n[\mathcal{L}^N(\Omega) + F_n(\overline{u}_n,\Omega)] \\ &\geq \overline{F}(u,\Omega) = \mathcal{F}(u,\Omega).\end{aligned}$$

In order to prove the opposite inequality, we note that, by dominated convergence theorem, we have

$$\lim_{n\to\infty} F_n(u,\Omega) = F(u,\Omega) \quad \text{for every } u \in W^{1,1}(\Omega).$$

Hence, by (1.16)

$$(\Gamma - \lim_{n\to\infty} F_n)(u,\Omega) \leq F(u,\Omega) \quad \text{for every } u \in BV(\Omega).$$

So that, by the lower semicontinuity of the $\Gamma - \lim$ and Theorem 3.6, it follows

$$(\Gamma - \lim_{n\to\infty} F_n)(u, \Omega) \leq \overline{F}(u, \Omega) = \mathcal{F}(u, \Omega) \quad \text{for every } u \in BV(\Omega).$$

Since this independent from the subsequence, we obtain that the whole sequence F_n Γ- converges to F. Then the thesis is achieved. ■

Chapter 4

Vectorial case

In this chapter we study L^1-lower semicontinuity and relaxation properties of the following functionals:

$$F(u,\Omega) = \int_\Omega f(x,\nabla u)dx; \qquad u \in W^{1,1}(\Omega;\mathbb{R}^M). \tag{4.1}$$

and its natural extension to the larger space $BV(\Omega;\mathbb{R}^M)$ given by

$$\begin{aligned} \mathcal{F}(u,\Omega) &= \int_\Omega f(x,\nabla u)dx + \int_\Omega f^\infty(x,\frac{D^c u}{|D^c(u)|})d|D^c u| \\ &+ \int_{J_u\cap\Omega} f^\infty(x,(u^+ - u^-)\otimes\nu_u)d\mathcal{H}^{N-1}. \end{aligned} \tag{4.2}$$

Here, as in the previous chapter, Ω will be an open bounded subset of $\mathbb{R}^N$, $N \geq 1$

4.1 Mathematical tools

4.1.1 Euclidean structure of the matrix space $\mathbb{R}^{M\times N}$

In the following we denote by $\mathbb{R}^{M\times N}$ matrices which is isomorphically equivalent to the vectorial space $\mathbb{R}^{MN}$ (the space $\mathbb{R}^{MN}$ of MN-dimensional vectors).

Indeed, let $A \in \mathbb{R}^{M\times N}$, $M, N \geq 1$. We define the linear operator $T : \mathbb{R}^{M\times N} \to \mathbb{R}^{MN}$ by

$$T(A) = a = (a_1, ..., a_{MN}) = (A^{1,j}, ..., A^{M,j})_{j=1,...,N}. \tag{4.3}$$

It is not difficult to see that T is an isomorphism and that his inverse is given by

$$T^{-1}(a) = A = (A^{i,j})^{i=1,...,M}_{j=1,...,N} \quad \text{with } A^{i,j} = a^{(i-1)N+j}. \tag{4.4}$$

By means the operator T we can define the scalar product and the norm on the space $\mathbb{R}^{M\times N}$. Precisely given A, $B \in \mathbb{R}^{M\times N}$ we have

$$\begin{aligned} \langle A, B\rangle_{\mathbb{R}^{M\times N}} := \langle T(A), T(B)\rangle_{\mathbb{R}^{MN}} &= \langle a, b\rangle_{\mathbb{R}^{MN}} = \sum_{k=1}^{MN} a_k b_k \\ &= \sum_{i=1}^{M}\sum_{j=1}^{N} A^{i,j}B^{i,j} = \sum_{i=1}^{M}\langle A^i, B^i\rangle_{\mathbb{R}^N}; \end{aligned} \tag{4.5}$$

and

$$\|A\|_{\mathbb{R}^{M\times N}} := \|T(A)\|_{\mathbb{R}^{MN}} = \Big(\sum_{k=1}^{MN} |a_k|^2\Big)^{\frac{1}{2}} = \Big(\sum_{i=1}^{M}\sum_{j=1}^{N} |A^{i,j}|^2\Big)^{\frac{1}{2}};$$

where

$$A^i = (A^{i,1}, ..., A^{i,N}) \quad \text{and } B^i = (B^{i,1}, ..., B^{i,N}).$$

We recall the definition of the tensor product. If $a \in \mathbb{R}^M$, $b \in \mathbb{R}^N$, then the tensor product $a \otimes b$ is a $M \times N$ matrix defined by $(a \otimes b)^{i,j} = a_i b_j$. We remark that if $A \in \mathbb{R}^{M\times N}$, $a \in \mathbb{R}^M$ and $b \in \mathbb{R}^N$ then, by (4.5), the following property holds

$$\langle A, a \otimes b\rangle_{\mathbb{R}^{M\times N}} = \langle T(A), T(a \otimes b)\rangle_{\mathbb{R}^{MN}} = \sum_{i=1}^{M}\langle A^i, a_i b\rangle_{\mathbb{R}^N}, \tag{4.6}$$

where $\langle \cdot, \cdot\rangle_X$ denotes the scalar product into the Euclidean space X.

4.1.2 Chain rule

In this subsection, we state a generalization of the Leibniz rule for the derivation of a product of two functions. This result can be obtained as a simplified version of the chain rule due to De Cicco, Fusco and Verde (see [19]), in which the functions involved depend only the spatial variable x.

Lemma 4.1 (see [19], Theorem 1.1**)** *Let* $b : \mathbb{R}^N \to \mathbb{R}^N$ *be a bounded function with compact support in* $\mathbb{R}^N$ *such that*

(i) $b \in W^{1,1}(\mathbb{R}^N; \mathbb{R}^N)$*,*

(ii) b *is approximately continuous* $\mathcal{H}^{N-1}$*-a.e. in* $\mathbb{R}^N$*.*

Then for every $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and for every* $\varphi \in C_0^1(\mathbb{R}^N)$*, the following formula holds*

$$\int_{\mathbb{R}^N} \langle b(x), \nabla\varphi(x)\rangle_{\mathbb{R}^N}\ u(x)dx = -\int_{\mathbb{R}^N} \mathrm{div}(b(x))u(x)\varphi(x)dx - \int_{\mathbb{R}^N} \langle b(x), \nabla u(x)\rangle_{\mathbb{R}^N}\ \varphi(x)dx$$
$$-\int_{\mathbb{R}^N} \langle b(x), \frac{D^c u}{|D^c u|}(x)\rangle_{\mathbb{R}^N}\ \varphi(x)d|D^c(u)| - \int_{J_u} \langle b(x), (u^+ - u^-) \otimes \nu_u(x)\rangle_{\mathbb{R}^N}\ \varphi(x)d\mathcal{H}^{N-1}$$

Remark 4.1 *Note that hypothesis* (ii) *is needed in order to identify* b *with its precise representative* $\mathcal{H}^{N-1}$*-a.e. in* $\mathbb{R}^N$*, otherwise the previous formula is false.*

4.2 Lower Semicontinuity

The aim of this section is to obtain L^1-lower semicontinuity for the functional in (4.2), by means of the chain rule formula, stated in Lemma 4.1, and De Giorgi's approximation Theorem (see Theorem 1.5). This strategy seems to be new in the vectorial setting. However, for its application, convexity of the integrand f with respect to ξ is crucial, whereas it is well known that this request is not optimal in the vectorial case (see, for instance, [12]). Nevertheless this approach allows us to weaken regularity hypothesis on the integrand f. Thus no continuity assumption with respect to the variable x is required. Furthermore our approach can be regarded as a first step in order to attack the polyconvex case.

We prove the following theorem, which states the lower semicontinuity of the functional $\mathcal{F}$ along equibounded sequences of vectorial BV-functions. The proof is based on an argument introduced in [32].

Let $f : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ be a locally bounded Borel function which satisfies the following conditions:

$$
(4.7) \quad \begin{cases} (i) & f(\cdot,\xi) \in W^{1,1}_{loc}(\Omega;\mathbb{R}^M) \text{ for every } \xi \in \mathbb{R}^{M\times N}; \\[2ex] (ii) & \text{for every bounded set } B \subset \mathbb{R}^{M\times N}, \text{ there exists a constant } L(B) \text{ such that} \\ & \displaystyle\int_\Omega |\nabla_x f(x,\xi)|dx \leq L(B) \text{ for every } \xi \in B; \\[2ex] (iii) & \exists G \subset \Omega \text{ with } \mathcal{H}^{N-1}(G) = 0 \text{ such that } \forall \xi \in \mathbb{R}^{M\times N} \\[2ex] & f(\cdot,\xi) \text{ is approximately continuous in } \Omega \setminus G. \end{cases}
$$

Theorem 4.1 *Sia $f : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ be a locally bounded function, convex in the second variable and satisfying (4.7). Then the functional (4.2) is lower semicontinuous on $BV(\Omega;\mathbb{R}^M)$ with respect to the L^1-topology along sequences $\{u_n\} \subset BV(\Omega;\mathbb{R}^M)$ such that $\|u_n\|_\infty \leq C$.*

Proof.

Let be $\Omega' \subset\subset \Omega$. There exists two compact sets K_1, K_2 such that

$$K_1 \subset \Omega' \cap C_u, \quad K_2 \subset \Omega' \setminus C_u. \tag{4.8}$$

By Hausdorff property we may find two disjoint open sets Ω_1, $\Omega_2 \subset \Omega'$ such that

$$K_1 \subset \Omega_1, \quad K_2 \subset \Omega_2. \tag{4.9}$$

So that, if $\{u_n\} \subset BV(\Omega;\mathbb{R}^M)$ is a sequence converging in $L^1(\Omega;\mathbb{R}^M)$ to $u \in BV(\Omega;\mathbb{R}^M)$, then

$$\liminf_{n\to\infty} \mathcal{F}(u_n,\Omega) \geq \liminf_{n\to\infty} \mathcal{F}(u_n,\Omega_1) + \liminf_{n\to\infty} \mathcal{F}(u_n,\Omega_2). \tag{4.10}$$

Therefore we will treat separately the two terms of the righthand side of the last inequality. Let us consider $\liminf\limits_{n\to\infty} \mathcal{F}(u_n,\Omega_1)$.

We define $g : \Omega \times \mathbb{R}^{MN} \to [0,\infty)$ and $g^\infty : \Omega \times \mathbb{R}^{MN} \to [0,\infty)$ given by

$$g(x,\underline{\xi}) = f(x, T^{-1}(\underline{\xi})),$$

and

$$g^\infty(x,\underline{\xi}) = f^\infty(x, T^{-1}(\underline{\xi})),$$

where T^{-1} is the linear operator defined in (4.4), and f^∞ is the recession function of $f(x,\cdot)$. Since f is convex with respect to ξ, g is convex with respect to $\underline{\xi}$. Therefore by Theorem 1.5 there exists a sequence $\{\alpha_\kappa\} \subset C_0^\infty(\mathbb{R}^{MN})$ with $\alpha_\kappa \geq 0$ and $\int_{\mathbb{R}^{MN}} \alpha_\kappa dx = 1$ such that for any $(x,\underline{\xi}) \in \Omega \times \mathbb{R}^{MN}$ we have

$$g(x,\underline{\xi}) = \sup_{\kappa\in\mathbb{N}} (a_\kappa(x) + \langle b_\kappa(x), \underline{\xi}\rangle_{\mathbb{R}^{MN}})^+, \tag{4.11}$$

and

$$g^\infty(x,\underline{\xi}) = \sup_{\kappa\in\mathbb{N}} (\langle b_\kappa(x), \underline{\xi}\rangle_{\mathbb{R}^{MN}})^+, \tag{4.12}$$

where recalling (1.11) and (1.12) with $d = MN$, a_κ, b_κ are defined by

$$a_\kappa(x) := \int_{\mathbb{R}^{MN}} g(x,\underline{\xi})\big((MN+1)\alpha_\kappa(\underline{\xi}) + \langle\nabla\alpha_\kappa(\underline{\xi}), \underline{\xi}\rangle_{\mathbb{R}^{MN}}\big)d\underline{\xi}, \tag{4.13}$$

$$b_\kappa(x) := -\int_{\mathbb{R}^{MN}} g(x,\underline{\xi})\nabla\alpha_\kappa(\underline{\xi})d\underline{\xi}. \tag{4.14}$$

By the definition of g and the assumption made on f we easily get that g is locally bounded and satisfies the following conditions:

$$\begin{cases} (i) & g(\cdot,\underline{\xi}) \in W^{1,1}_{loc}(\Omega) \text{ for every } \xi \in \mathbb{R}^{MN}; \\ (ii) & \text{for every bounded set } B \subset \mathbb{R}^{MN}, \text{ there exists a costant } L(B) \text{ such that} \\ & \displaystyle\int_\Omega |\nabla_x g(x,\underline{\xi})|dx \le L(B) \text{ for every } \underline{\xi} \in B. \\ (iii) & \exists G \subset \Omega \text{ with } \mathcal{H}^{N-1}(G) = 0 \text{ such that } \forall \underline{\xi} \in \mathbb{R}^{MN} \\ & g(\cdot,\underline{\xi}) \text{ is approximately continuous in } \Omega\setminus G. \end{cases} \tag{4.15}$$

By the definition of g and T and using (4.11) and (4.12), we obtain the following approximation for the functions f and f^∞:

$$f(x,\xi) = \sup_{\kappa\in\mathbb{N}}(a_\kappa(x) + \langle b_\kappa(x), T(\xi)\rangle_{\mathbb{R}^{MN}})^+ = \sup_{\kappa\in\mathbb{N}}(a_\kappa(x) + \langle T^{-1}(b_\kappa(x)), \xi\rangle_{\mathbb{R}^{M\times N}})^+, \tag{4.16}$$

$$f^\infty(x,\xi) = \sup_{\kappa\in\mathbb{N}}(\langle b_\kappa(x), T(\xi)\rangle_{\mathbb{R}^{MN}})^+ = \sup_{\kappa\in\mathbb{N}}(\langle T^{-1}(b_\kappa(x)), \xi\rangle_{\mathbb{R}^{M\times N}})^+. \tag{4.17}$$

Let K be a finite set of index. Let A_k be a finite family of disjoint open sets with the closure contained in Ω_1 and, for any $k \in K$, $\eta_{k,l}$ be a sequence in $C^1_0(A_k)$, with $0 \le \eta_{k,l} \le 1$ for all

k,l. Finally let $\{\varphi_r\}$ be a sequence in $C^1_0(\Omega_1)$ with $0 \le \varphi_r \le 1$ for all r. By (4.16), and (4.17) we have

$$\begin{aligned}
\liminf_{n\to+\infty} \mathcal{F}(u_n,\Omega_1) &\ge \sum_{k\in K} \liminf_{n\to+\infty} \int_{\Omega_1} a_k \eta_{k,l}(x)\varphi_r(x)dx \\
&+ \sum_{k\in K} \liminf_{n\to+\infty} \Big\{ \int_{\Omega_1} \langle b_k(x)\eta_{k,l}(x), T(\nabla u_n(x))\rangle_{\mathbb{R}^{MN}}\ \varphi_r(x)dx \\
&+ \int_{\Omega_1} \langle b_k(x)\eta_{k,l}(x), T(\frac{D^c u_n}{|D^c u_n|}(x))\rangle_{\mathbb{R}^{MN}}\ \varphi_r(x)d|D^c u_n| \\
&+ \int_{J_{u_n}} \langle b_k(x)\eta_{k,l}(x), T((u_n^+(x)-u_n^-(x))\otimes \nu_{u_n}(x))\rangle_{\mathbb{R}^{MN}}\ \varphi_r(x)d\mathcal{H}^{N-1}\Big\}.
\end{aligned}$$

Now let us set $B_k := T^{-1}(b_k)$, so that, thanks to (4.5) and (4.6), we obtain

$$\begin{aligned}
\liminf_{n\to+\infty} \mathcal{F}(u_n,\Omega_1) &\ge \sum_{k\in K} \liminf_{n\to+\infty} \int_{\Omega_1} a_k \eta_{k,l}(x)\ \varphi_r(x)dx \\
&+ \sum_{i=1}^{M}\sum_{k\in K} \liminf_{n\to+\infty} \Big\{ \int_{\Omega_1} \langle B^i_k(x)\eta_{k,l}(x), \nabla u^i_n(x)\rangle_{\mathbb{R}^{N}}\ \varphi_r(x)dx \\
&+ \int_{\Omega_1} \langle B^i_k(x)\eta_{k,l}(x), \frac{D^c u^i_n}{|D^c u_n|}(x)\rangle_{\mathbb{R}^{N}}\ \varphi_r(x)d|D^c u_n| \\
&+ \int_{J_{u_n}} \langle B^i_k(x)\eta_{k,l}(x), ({u^i_n}^+ - {u^i_n}^-)\nu_{u_n}(x)\rangle_{\mathbb{R}^{N}}\ \varphi_r(x)d\mathcal{H}^{N-1}\Big\}.
\end{aligned} \tag{4.18}$$

In order to apply Lemma 4.1 we notice that

$$D^c u^i_n = \frac{D^c u^i_n}{|D^c u^i_n|}|D^c u^i_n| \quad \text{and} \quad |D^c u^i_n| = \frac{|D^c u^i_n|}{|D^c u_n|}|D^c u_n|.$$

Therefore from Lemma 1.1 it follows that

$$\frac{D^c u^i_n}{|D^c u_n|} = \frac{D^c u^i_n}{|D^c u^i_n|}\frac{|D^c u^i_n|}{|D^c u_n|} \quad |D^c u_n|\text{-a.e. in } \Omega. \tag{4.19}$$

Furthermore, fixed $i = 1, ..., M$, since $J_{u^i_n}$ is contained, up to the $\mathcal{H}^{N-1}$-negligible set $\bigcup_{i=1}^{M} S_{u^i_n} \setminus J_{u^i_n}$, in the disjoint union of $J_{u^i_n} \setminus \bigcup_{j\ne i} J_{u^j_n}$ and $\bigcup_{j\ne i}(J_{u^i_n} \cap J_{u^j_n})$, we can choose $\nu_{u_n} = \pm\nu_{u^i_n}$ on every set $J_{u_n^i} \setminus \bigcup_{j\neq i} J_{u_n^j}$, and, thanks to Lemma 1.3, $\nu_{u_n} = \pm\nu_{u_n^i}$ for $\mathcal{H}^{N-1}$-a.e $x \in \bigcup_{j\neq i}(J_{u_n^i} \cap J_{u_n^j})$. Let us assume $\nu_{u_n} = +\nu_{u_n^i}$ and $(u_n^+)^i = (u_n^i)^+$ for $i = 1, ..., M$, otherwise it is enough to change the sign. From this fact and equality (4.19) we can rewrite (4.18) as

$$
\begin{aligned}
\liminf_{n\to+\infty} \mathcal{F}(u_n, \Omega_1) \geq & \sum_{k\in K} \int_{\Omega_1} a_k \eta_{k,l}(x)\ \varphi_r(x) dx \\
& + \sum_{i=1}^{M} \sum_{k\in K} \liminf_{n\to+\infty} \Big\{ \int_{\Omega_1} \langle B_k^i(x)\eta_{k,l}(x), \nabla u_n^i(x)\rangle_{\mathbb{R}^N}\ \varphi_r(x) dx \\
& + \int_{\Omega_1} \langle B_k^i(x)\eta_{k,l}\varphi_r(x), \frac{D^c u_n^i}{|D^c u_n^i|}(x)\rangle_{\mathbb{R}^N}\ d|D^c u_n^i| \\
& + \int_{J_{u_n^i}} \langle B_k^i(x)\eta_{k,l}(x), \big((u_n^i)^+ - (u_n^i)^-\big)\nu_{u_n^i}(x)\rangle_{\mathbb{R}^N}\ \varphi_r(x) d\mathcal{H}^{N-1} \Big\}.
\end{aligned}
\tag{4.20}
$$

We claim that the function $x \mapsto B_k^i(x)\eta_{k,l}(x)$ satisfies, for all $k \in K$ and $i = 1, ..., M$, the hypotheses of Lemma 4.1. Indeed, since g is locally bounded, it has compact support and it is bounded in Ω_1 . Moreover from (i) and (ii) of hypothesis (4.15), it follows that $B_k^i\eta_{k,l}$ belongs to $W^{1,1}(\Omega)$. Finally from (iii) of hypothesis (4.15) and Fubini's Theorem it follows that $B_k^i\eta_{k,l}$ satisfies hypothesis (ii) of Lemma 4.1. Therefore, by applying Lemma 4.1, we get

$$
\begin{aligned}
\lim_{n\to\infty} \Big\{ \int_{\Omega_1} & \langle B_k^i(x)\eta_{k,l}(x), \nabla u_n^i(x)\rangle_{\mathbb{R}^N}\ \varphi_r(x) dx + \int_{\Omega_1} \langle B_k^i(x)\eta_{k,l}(x), \frac{D^c u_n^i}{|D^c u_n^i|}(x)\rangle_{\mathbb{R}^N}\ \varphi_r(x) d|D^c u_n^i| \\
& + \int_{J_{u_n^i}} \langle B_k^i(x)\eta_{k,l}(x), \big((u_n^i)^+ - (u_n^i)^-\big)\nu_{u_n^i}(x)\rangle_{\mathbb{R}^N}\ \varphi_r(x) d\mathcal{H}^{N-1} \Big\} \\
& = \lim_{n\to\infty} \Big\{ - \int_{\Omega_1} \langle B_k^i(x)\eta_{k,l}(x), \nabla\varphi_r(x)\rangle_{\mathbb{R}^N}\ u_n^i(x) dx \\
& - \int_{\Omega_1} \operatorname{div}(B_k^i(x)\eta_{k,l}(x)) u_n^i(x)\varphi_r(x) dx \Big\}.
\end{aligned}
\tag{4.21}
$$

Thus, letting $n \to +\infty$, by using the dominated convergence theorem, the equiboundeness of sequence $\{u_n\}$ and Lemma 4.1 again, we have, summing with respect to i, k,

$$
\begin{aligned}
\liminf_{n\to+\infty} \mathcal{F}(u_n,\Omega_1) \ \geq\ & \sum_{k\in K}\int_{\Omega_1} [a_k\eta_{k,l}(x)dx + \sum_{i=1}^{M}\langle B_k^i(x)\eta_{k,l}(x), \nabla u^i(x)\rangle_{\mathbb{R}^N}]\ \varphi_r(x)dx \\
+\ & \sum_{k\in K}\sum_{i=1}^{M}\int_{\Omega_1}\langle B_k^i(x)\eta_{k,l}(x), \frac{D^c u^i}{|D^c u^i|}(x)\rangle_{\mathbb{R}^N}\ \varphi_r(x)d|D^c u^i| \\
+\ & \sum_{k\in K}\sum_{i=1}^{M}\int_{J_{u^i}}\langle B_k^i(x)\eta_{k,l}(x), \big((u^i)^+ - (u^i)^-\big)\nu_{u^i}(x)\rangle_{\mathbb{R}^N}\ \varphi_r(x)d\mathcal{H}^{N-1}\Big\}.
\end{aligned} \tag{4.22}
$$

By Lemma 1.2 we may find a sequence $\varphi_r \in C_0^1(\Omega_1)$ with $0 \leq \varphi_r \leq 1$ such that $\varphi_r(x) \to \chi_{C_u\cap\Omega_1}(x)$ for $|Du|$-a.e. $x \in \Omega_1$. Hence, letting $r \to \infty$ in (4.22), we get

$$
\begin{aligned}
\liminf_{n\to+\infty} \mathcal{F}(u_n,\Omega_1) \ \geq\ & \sum_{k\in K}\int_{\Omega_1} [a_k\eta_{k,l}(x)dx + \sum_{i=1}^{M}\langle B_k^i(x)\eta_{k,l}(x), \nabla u^i(x)\rangle_{\mathbb{R}^N}]dx \\
+\ & \sum_{k\in K}\sum_{i=1}^{M}\int_{\Omega_1}\langle B_k^i(x)\eta_{k,l}(x), \frac{D^c u^i}{|D^c u^i|}(x)\rangle_{\mathbb{R}^N}\ d|D^c u^i|.
\end{aligned}
$$

By using again the definition of the operator T, (4.5) and (4.19) we have

$$
\begin{aligned}
\liminf_{n\to+\infty} \mathcal{F}(u_n,\Omega_1) \ \geq\ & \sum_{k\in K}\int_{\Omega_1} [a_k\eta_{k,l}(x) + \langle b_k(x)\eta_{k,l}(x), T(\nabla u(x))\rangle_{\mathbb{R}^{MN}}]dx \\
+\ & \sum_{k\in K}\int_{\Omega_1}\langle b_k(x)\eta_{k,l}(x), T\big(\frac{D^c u}{|D^c u|}(x)\big)\rangle_{\mathbb{R}^{MN}}\ d|D^c u|.
\end{aligned}
$$

Now for, any $k \in K$, we take $\eta_{k,l}(x)$ converging to $\chi_{D_k}(x) + \chi_{C_k}(x)$ for $|Du|$-a.e. $x \in A_k$, where

$$
D_k = \{x \in A_k \cap \mathcal{D}_u : a_k(x) + \langle b_k(x), T(\nabla u(x))\rangle_{\mathbb{R}^{MN}} > 0\},
$$

$$
C_k = \{x \in A_k \cap (C_u \setminus \mathcal{D}_u) : \langle b_k(x), T(\frac{D^c u}{|D^c u|}(x))\rangle_{\mathbb{R}^{MN}} > 0\}.
$$

So that, letting $l \to \infty$, we get

$$\begin{aligned}\liminf_{n\to+\infty} \mathcal{F}(u_n,\Omega_1) &\geq \sum_{k\in K}\int_{A_k}[a_k+\langle b_k(x),T(\nabla u(x))\rangle_{\mathbb{R}^{MN}}]^+dx\\ &+ \sum_{k\in K}\int_{A_k}[\langle b_k(x),T\big(\frac{D^cu}{|D^cu|}(x)\big)\rangle_{\mathbb{R}^{MN}}]^+d|D^cu|.\end{aligned}$$

Hence, by applying Lemma 1.5 with $\mu = |Du|$ and

$$\phi_k(x) = [a_k+\langle b_k(x),T(\nabla u(x))\rangle_{\mathbb{R}^{MN}}]^+\chi_{\mathcal{D}_u}(x) + [\langle b_k(x),T\big(\frac{D^cu}{|D^cu|}(x)\big)\rangle_{\mathbb{R}^{MN}}]^+\chi_{C_u\setminus\mathcal{D}_u}(x),$$

we get from (4.16) and (4.17)

$$\liminf_{n\to\infty}\mathcal{F}(u_n,\Omega_1) \geq \int_{\Omega_1} f(x,\nabla u)dx + \int_{\Omega_1} f^\infty(x,\frac{D^cu}{|D^c(u)|})|D^cu|. \tag{4.23}$$

Now let us consider $\liminf\limits_{n\to+\infty}\mathcal{F}(u_n,\Omega_2)$.

As above we fix a finite family A_k of disjoint open sets with the closure contained in Ω_2. For any $k \in K$, let $\eta_{k,l}$ be a sequence in $C_0^1(A_k)$, with $0 \leq \eta_{k,l} \leq 1$ for all k, l. Finally let $\{\varphi_r\}$ be a sequence in $C_0^1(\Omega_2)$ with $0 \leq \varphi_r \leq 1$ for all $r \in \mathbb{R}$. In this case we let $\varphi_r(x) \mapsto \chi_{J_u\cap\Omega_2}(x)$ $|Du|$-a.e. in Ω_2. Thus, by using again (4.5), (4.6) and the choice of the normal ν_u, we arrive to

$$\liminf_{n\to\infty}\mathcal{F}(u_n,\Omega_2) \geq \sum_{k\in K}\int_{A_k}\langle b_k(x)\eta_{k,l}(x),T((u^+(x)-u^-(x))\otimes\nu_u(x)\rangle_{\mathbb{R}^{MN}}d\mathcal{H}^{N-1}.$$

Let S_m be an increasing sequence of Borel sets such that $\bigcup\limits_{m=1}^{+\infty} S_m = \mathbb{R}^N$ and $\mathcal{H}^{N-1}\lfloor J_u(S_m) < +\infty$ for any m. We can use again Lemma 1.2 to find, for any $k \in K$ and $m \in \mathbb{N}$, a sequence $\eta_{k,l}(x) \mapsto \chi_{U_k\cap S_m}(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_2$, where

$$U_k = \{x \in A_k \cap J_u : \langle b_k(x),T((u^+(x)-u^-(x))\otimes\nu_u(x))\rangle_{\mathbb{R}^{MN}} > 0\}.$$

Hence, letting $k \to +\infty$, we get

$$\liminf_{n\to\infty} \mathcal{F}(u_n, \Omega_2) \geq \sum_{k\in K} \int_{A_k \cap S_m} [\langle b_k(x), T((u^+(x) - u^-(x)) \otimes \nu_u(x))\rangle_{\mathbb{R}^{MN}}]^+ d\mathcal{H}^{N-1}.$$

Thus, letting $m \to +\infty$,

$$\liminf_{n\to\infty} \mathcal{F}(u_n, \Omega_2) \geq \sum_{k\in K} \int_{A_k} [\langle b_k(x), T((u^+(x) - u^-(x)) \otimes \nu_u(x))\rangle_{\mathbb{R}^{MN}}]^+ d\mathcal{H}^{N-1},$$

and then, by applying Lemma 1.5 with $\mu = \mathcal{H}^{N-1} \lfloor J_u$, we obtain

$$\liminf_{n\to\infty} \mathcal{F}(u_n, \Omega_2) \geq \int_{J_u \cap \Omega_2} f^\infty(x, (u^+(x) - u^-(x)) \otimes \nu_u(x)) d\mathcal{H}^{N-1}. \tag{4.24}$$

Hence, from (4.9), (4.10), (4.23), (4.24) we conclude that

$$\begin{aligned} \liminf_{n\to\infty} \mathcal{F}(u_n, \Omega) &\geq \mathcal{F}(u, K_1) + \mathcal{F}(u, K_2) \\ &= \int_{K_1} f(x, \nabla u) dx + \int_{K_1} f^\infty(x, \frac{D^c u}{|D^c(u)|}) d|D^c u| \\ &+ \int_{J_u \cap K_2} f^\infty(x, (u^+ - u^-) \otimes \nu_u) d\mathcal{H}^{N-1}. \end{aligned}$$

Finally, taking into account (4.8), by letting $K_1 \uparrow \Omega' \cap C_u$, $K_2 \uparrow \Omega' \setminus C_u$, and then $\Omega' \uparrow \Omega$, we get the result. ■

In the next we prove the lower semicontinuity theorem of $\mathcal{F}$ along sequences $\{u_n\} \subset BV(\Omega; \mathbb{R}^M)$ equibounded in $L^{\frac{N}{N-1}}(\Omega; \mathbb{R}^M)$.

Theorem 4.2 *Let $f : \Omega \times \mathbb{R}^{M\times N} \to [0, \infty)$ be a locally bounded function, convex in the second variable and satisfying (i) and (iii) of (4.7). Suppose that, in addition:*

$$\forall B \subset \mathbb{R}^{M\times N} \text{ bounded } \exists\, L(B) \text{ such that } \int_\Omega |\nabla_x f(x, \xi)|^N dx \leq L(B) \quad \forall \xi \in B. \tag{4.25}$$

Then the functional (4.2) is lower semicontinuous on $BV(\Omega;\mathbb{R}^M)$ with respect to the L^1-topology along sequences $\{u_n\} \subset BV(\Omega;\mathbb{R}^M)$ such that $\|u_n\|_{\frac{N}{N-1}} \leq C$.

Proof.

Firstly we notice that from (4.25) it follows that the function $g = f \circ T^{-1}$ satisfies the following property:

$$\forall B \subset \mathbb{R}^{MN} \ \exists \ L(B) \text{ such that } \int_\Omega |\nabla_x g(x,\underline{\xi})|^N dx \leq L(B) \quad \forall \underline{\xi} \in B. \tag{4.26}$$

Then we proceed as in the proof of Theorem 4.1 till the equality (4.21); i.e.

$$\begin{aligned}
\lim_{n\to\infty} \Big\{ \int_{\Omega_1} & \langle B_k^i(x)\eta_{k,l}(x), \nabla u_n^i(x)\rangle_{\mathbb{R}^N} \ \varphi_r(x)dx \\
+ & \int_{\Omega_1} \langle B_k^i(x)\eta_{k,l}(x), \frac{D^c u_n^i}{|D^c u_n^i|}(x)\rangle_{\mathbb{R}^N}\varphi_r(x)d|D^c u_n^i| \\
+ & \int_{J_{u_n^i}} \langle B_k^i(x)\eta_{k,l}(x), \big((u_n^i)^+ - (u_n^i)^-\big)\nu_{u_n^i}\rangle_{\mathbb{R}^N}\varphi_r(x)d\mathcal{H}^{N-1}\Big\} \\
= & \lim_{n\to\infty} \Big\{ - \int_{\Omega_1} \langle B_k^i(x)\eta_{k,l}(x), \nabla\varphi_r(x)\rangle_{\mathbb{R}^N} u_n^i(x)dx \\
- & \int_{\Omega_1} \operatorname{div}(B_k^i(x)\eta_{k,l}(x))u_n^i(x)\varphi_r(x)dx\Big\}.
\end{aligned} \tag{4.27}$$

We claim that for every $i = 1, ..., M$ the function $\operatorname{div}B_k^i \in L^N(\Omega)$. Let $\varphi \in C_0^1(\Omega)$:

$$\begin{aligned}
\int_\Omega \operatorname{div}B_k^i\varphi dx &= -\int_\Omega \langle\nabla\varphi, B_k^i\rangle_{\mathbb{R}^N}dx = -\int_\Omega \langle\nabla\varphi, \big(T^{-1}(b_k)\big)^i\rangle_{\mathbb{R}^N}dx \\
&= -\int_\Omega \langle\nabla\varphi, \Big(T^{-1}\big(-\int_{\mathbb{R}^{MN}} g(x,\underline{\xi})\nabla_{\underline{\xi}}\alpha_k(\underline{\xi})d\underline{\xi}\big)\Big)^i\rangle_{\mathbb{R}^N}dx \\
&= -\int_\Omega \langle\nabla\varphi, \int_{\mathbb{R}^{MN}} g(x,\underline{\xi})\big(T^{-1}(\nabla_{\underline{\xi}}\alpha_k(\underline{\xi}))\big)^i d\underline{\xi}\rangle_{\mathbb{R}^N}dx \\
&= -\int_\Omega \langle\nabla\varphi, \int_{\mathbb{R}^{MN}} g(x,\underline{\xi})\nabla_{\xi^i}\alpha_k(\underline{\xi})d\underline{\xi}\rangle_{\mathbb{R}^N}dx \\
&= \int_\Omega \varphi \int_{\mathbb{R}^{MN}} \langle\nabla_x g(x,\underline{\xi}), \nabla_{\xi^i}\alpha_k(\underline{\xi})\rangle_{\mathbb{R}^N}d\underline{\xi}dx.
\end{aligned}$$

Therefore we have the following representation formula for the distributional divergence of B_k^i:

$$\text{div} B_k^i = \int_{\mathbb{R}^{MN}} \langle \nabla_x g(x, \underline{\xi}), \nabla_{\xi^i} \alpha_k(\underline{\xi}) \rangle_{\mathbb{R}^N} d\underline{\xi} \quad \text{in } \mathcal{D}'(\Omega). \tag{4.28}$$

By applying Hölder's inequality to the equation(4.28), we obtain

$$|\text{div} B_k^i|^N \leq (\mathcal{L}^{MN}(supp\alpha_k))^{N-1} \int_{supp\alpha_k} |\nabla_x g(x, \underline{\xi})|^N d\underline{\xi}$$

and and, by Tonelli's Theorem, taking into account also (4.26), with $B = supp\alpha_k$, we get

$$\begin{aligned} \|\text{div} B_k^i\|_{L^N} &\leq (\mathcal{L}^{MN}(supp\alpha_k))^{\frac{N}{N-1}} \Big(\int_{supp\alpha_k} d\underline{\xi} \int_\Omega |\nabla_x g(x, \underline{\xi})|^N dx \Big)^{\frac{1}{N}} \\ &\leq (\mathcal{L}^{MN}(supp\alpha_k))^{\frac{N-1}{N}} L^{\frac{1}{N}} (\mathcal{L}^{MN}(supp\alpha_k))^{\frac{1}{N}} (\mathcal{L}^N(\Omega))^{\frac{1}{N}} < +\infty. \end{aligned}$$

Therefore, for every $i = 1, ..., M$ the function $\text{div} B_k^i \in L^N(\Omega)$. Furthermore, by the equi-boundeness of $\{u_n\}$ in $L^{\frac{N}{N-1}}$, it follows that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $L^{\frac{N}{N-1}}$. Hence passing to the limit in (4.27) and using again Lemma 4.1, we get

$$\begin{aligned} \liminf_{n \to +\infty} \mathcal{F}(u_n, \Omega_1) &\geq \sum_{k \in K} \int_{\Omega_1} [a_k \eta_{k,l}(x) dx + \sum_{i=1}^M \langle B_k^i(x) \eta_{k,l}(x), \nabla u^i(x) \rangle_{\mathbb{R}^N}] \; \varphi_r(x) dx \\ &+ \sum_{k \in K} \sum_{i=1}^M \int_{\Omega_1} \langle B_k^i(x) \eta_{k,l}(x), \frac{D^c u^i}{|D^c u^i|}(x) \rangle_{\mathbb{R}^N} \; \varphi_r(x) d|D^c u^i| \\ &+ \sum_{k \in K} \sum_{i=1}^M \int_{J_{u^i}} \langle B_k^i(x) \eta_{k,l}(x), \big((u^i)^+ - (u^i)^-\big) \nu_{u^i}(x) \rangle_{\mathbb{R}^N} \; \varphi_r(x) d\mathcal{H}^{N-1} \Big\}. \end{aligned}$$

The thesis is achieved, by proceeding exactly as in the proof of Theorem 4.1. ■

It is possible to remove the constraint $\|u_n\|_{\frac{N}{N-1}} \leq C$, by dealing with an open bounded set Ω with Lipschitz boundary and assuming the coercivity of the integrand f, as stated in the next result.

Corollary 4.1 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ be a locally bounded function, convex in the second variable, satisfying (i) and (iii) of (4.7)and (4.25). Suppose that there exists a constant $\lambda > 0$ such that:*

$$f(x,\xi) \geq \lambda|\xi| \quad \textit{for all } x \in \Omega. \tag{4.29}$$

Then the functional (4.2) is lower semicontinuous on $BV(\Omega;\mathbb{R}^M)$ with respect to the L^1-topology.

Proof.

Let $\{u_n\} \subset BV(\Omega;\mathbb{R}^M)$ converging in L^1 to $u \in BV(\Omega;\mathbb{R}^M)$, with respect to the L^1-topology. Let us suppose that there exists a constant L such that $\liminf\limits_{n\to+\infty} \mathcal{F}(u_n,\Omega) \leq L$ otherwise the conclusion is trivial.

By the continuous imbedding of $BV(\Omega;\mathbb{R}^M)$ in $L^{\frac{N}{N-1}}(\Omega;\mathbb{R}^M)$ and by (4.29) we get:

$$L \geq \liminf_{n\to\infty} \mathcal{F}(u_n,\Omega) \geq \lambda|Du_n| \geq C\|u_n\|_{\frac{N}{N-1}} - \lambda\|u_n\|_1,$$

so that, recalling that $u_n \to u$ strongly in $L^1(\Omega;\mathbb{R}^M)$, we have

$$\|u_n\|_{\frac{N}{N-1}} \leq \frac{L + \lambda\|u_n\|_1}{C} = C_1,$$

where the constant C_1 does not depend on n. Hence the result follows from Theorem 4.2. ■

4.3 Relaxation

The aim of this section is to give an integral representation formula for the lower semicontinuous envelope of the functional

$$(4.30) \qquad F(u,\Omega) := \begin{cases} \displaystyle\int_\Omega f(x,\nabla u)dx & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^M), \\ +\infty & \text{if } u \in BV(\Omega;\mathbb{R}^M) \setminus W^{1,1}(\Omega;\mathbb{R}^M), \end{cases}$$

defined by

$$(4.31) \qquad \overline{F}(u,\Omega) = \inf\{\liminf_{n\to\infty} F(u_n,\Omega) : u_n \in W^{1,1}(\Omega;\mathbb{R}^M), u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^M)\}.$$

More precisely, our purpose is to state the following equality:

$$(4.32) \qquad \overline{F}(u,\Omega) = \mathcal{F}(u,\Omega) \quad \text{for every } u \in BV(\Omega;\mathbb{R}^M);$$

where $\mathcal{F}$ is defined in (4.2).

We precise that this result will be obtained on the one hand by using the lower semicontinuity Theorems proven in the previous section, on the other hand by adapting some known techniques (see [5, 28, 29]).

The first step is the inequality: $\mathcal{F} \leq \overline{F}$. A first result in this direction is an immediate consequence of Corollary 4.1.

Proposition 4.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ be a locally bounded function convex in the second variable satisfying (i) and (iii) of (4.7),(4.25) and (4.29). Then*

$$\mathcal{F}(u,\Omega) \leq \overline{F}(u,\Omega) \quad \textit{for every } u \in BV(\Omega;\mathbb{R}^M).$$

Proof.

By Corollary 4.1, $\mathcal{F}$ is lower semicontinuous and, by definition, $\mathcal{F}(u,\Omega) \leq F(u,\Omega)$ for every $u \in BV(\Omega;\mathbb{R}^M)$. Then the thesis follows by recalling that $\overline{F}$ is the greatest lower semicontinuous functional not greater than F. ■

Actually, the inequality stated in Proposition 4.1 above can be obtained in $BV(\Omega;\mathbb{R}^M) \cap L^\infty(\Omega;\mathbb{R}^M)$, also without assuming (4.25). To this purpose, let f be a function satisfying the following linear growth conditions:

$$f(x,\xi) \geq \lambda|\xi| \quad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^{M\times N}, \tag{4.33}$$

with $\lambda > 0$;

$$f(x,\xi) \leq \Lambda(1+|\xi|) \quad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^{M\times N}, \tag{4.34}$$

with $\Lambda > 0$.

To our purpose we define, in the spirit of [28], the following functional:

$$\begin{aligned} \overline{F}_\infty(u,\Omega) &:= \\ &= \inf\{\liminf_{n\to\infty} F(u_n) : u_n \in W^{1,1}(\Omega;\mathbb{R}^M), u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^M),\ \|u_n\|_\infty \leq C\}. \end{aligned} \tag{4.35}$$

The following lemma holds. We adapt a truncation argument introduced in [28].

Lemma 4.2 *Let $\Omega \subset \mathbb{R}^N$ an open bounded set with Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ be a Borel function satisfying (4.33). Then*

$$\overline{F}_\infty(u,\Omega) = \overline{F}(u,\Omega) \quad \forall u \in BV(\Omega;\mathbb{R}^M) \cap L^\infty(\Omega;\mathbb{R}^M).$$

Proof.

It is enough to show $\overline{F} \geq \overline{F}_\infty$. for all $u \in BV(\Omega;\mathbb{R}^M) \cap L^\infty(\Omega;\mathbb{R}^M)$. If $\overline{F}(u,\Omega) = +\infty$ the result is obvious, thus there is no loss of generality in assuming $\overline{F}(u,\Omega) \leq L < +\infty$. Then, by the properties of relaxation (see 1.12), there exists $\{u_n\} \subset W^{1,1}(\Omega;\mathbb{R}^M)$ such that

$$\overline{F}(u,\Omega) = \lim_{n\to\infty} F(u_n,\Omega) \leq L < +\infty. \tag{4.36}$$

Let $i \in \mathbb{N}$. Let us define a smooth truncation function $\varphi_i \in C^1_0(\Omega;\mathbb{R}^M)$ given by:

$$\varphi_i(z) := \begin{cases} z & \text{if } |z| < e^i \\ 0 & \text{if } |z| \geq e^{i+1}; \end{cases} \quad |\nabla\varphi_i(z)| \leq 1. \tag{4.37}$$

Let us set $w_n^i := \varphi_i(u_n)$. We have $w_n^i \subset W^{1,1}(\Omega;\mathbb{R}^M)$ and

$$\begin{cases} \|w_n^i\|_\infty \leq e^i \\ \nabla w_n^i(x) = \nabla\varphi_i(u_n(x))\nabla u_n(x). \end{cases} \tag{4.38}$$

Since $u \in L^\infty(\Omega;\mathbb{R}^M)$ it is possible to choose i large enough so that $u = \varphi_i(u)$. Then by (4.37) we have, for such any i, that

$$\|w_n^i - u\|_1 = \|\varphi_i(u_n) - \varphi_i(u)\|_1 \leq \|u_n - u\|_1. \tag{4.39}$$

By (4.34), (4.37) and (4.38) we have

$$\int_\Omega f(x,\nabla w_n^i) \leq \int_{\{|u_n|<e^i\}} f(x,\nabla u_n)dx + \Lambda \int_{\{e^i \leq |u_n| < e^{i+1}\}} (1+|\nabla u_n|)dx + \Lambda \mathcal{L}^N\{|u_n| \geq e^{i+1}\}.$$

By Chebyshev's inequality we get

$$\int_\Omega f(x,\nabla w_n^i) \leq \int_{\{|u_n|<e^i\}} f(x,\nabla u_n)dx + \Lambda \int_{\{e^i \leq |u_n| < e^{i+1}\}} \big(1+|\nabla u_n|\big)dx + \frac{\Lambda}{e^i}\|u_n\|_1.$$

Now let $l \in \mathbb{N}$. By summing for $i = 1, 2..., l$, we obtain

$$\frac{1}{l}\sum_{i=1}^{l}\int_{\Omega} f(x, \nabla w_n^i)dx \leq \int_{\Omega} f(x, \nabla u_n)dx + \Lambda\frac{\|u_n\|_1}{l}\sum_{i=1}^{l}\frac{1}{e^i} + \frac{\Lambda}{l}\sum_{i=1}^{l}\int_{\{e^i \leq |u_n| < e^{i+1}\}}(1 + |\nabla u_n|)dx$$

Notice that, if we define $E_i := \{e^i \leq |u_n| < e^{i+1}\}$, then E_i are disjoint set; therefore by (4.33) and (4.36), we get

$$\sum_{i=1}^{l}\int_{E_i}(1 + |\nabla u_n|)dx = \int_{\bigcup_i^l E_i}(1 + |\nabla u_n|)dx \leq \int_{\Omega}(1 + |\nabla u_n|)dx \leq L < +\infty. \tag{4.40}$$

Then

$$\frac{1}{l}\sum_{i=1}^{l}\int_{\Omega} f(x, \nabla w_n^i)dx \leq \int_{\Omega} f(x, \nabla u_n)dx + \frac{K(n, l)}{l};$$

with $K(n, l) := \Lambda\Big(\|u_n\|_1\sum_{i=1}^{l}\frac{1}{e^i} + L\Big)$ and $\lim_{l\to\infty}\lim_{n\to\infty}\frac{K(n, l)}{l} = 0$.

Furthermore we may find some $i_n \in \{1, ..., l\}$ such that

$$\int_{\Omega} f(x, \nabla w_n^{i_n})dx \leq \int_{\Omega} f(x, \nabla u_n)dx + \frac{K(n, l)}{l}. \tag{4.41}$$

Let us recall that by (4.39) $w_n^{i_n} \to u$ in $L^1(\Omega; \mathbb{R}^M)$ and by (4.38) $\|w_n^{i_n}\|_\infty \leq e^l$. Then by (4.36) and (4.41)

$$\overline{F}(u) + \lim_{n\to\infty}\frac{K(n, l)}{l} \geq \liminf_{n\to\infty} F(u_n, \Omega) + \lim_{n\to\infty}\frac{K(n, l)}{l} \geq \liminf_{n\to\infty} F(w_n^{i_n}) \geq \overline{F}_\infty(u, \Omega).$$

The conclusion follows letting l tend to $+\infty$. $\blacksquare$

As a consequence of the previous lemma we have the following theorem.

Theorem 4.3 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^{M\times N} \to [0, \infty)$ be a Borel function, convex in the second variable and satisfying (4.7) and (4.33). Then*

$$\mathcal{F}(u, \Omega) \leq \overline{F}(u, \Omega) \quad \textit{for every } u \in BV(\Omega; \mathbb{R}^M) \cap L^\infty(\Omega; \mathbb{R}^M).$$

Proof.

Let $u \in BV(\Omega;\mathbb{R}^M) \cap L^\infty(\Omega;\mathbb{R}^M)$. By Theorem 4.1 for every sequence $\{u_n\} \subset W^{1,1}(\Omega;\mathbb{R}^M)$ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^M)$ with $\|u_n\|_\infty \leq C$, we get

$$\mathcal{F}(u,\Omega) \leq \liminf_{n\to\infty} \mathcal{F}(u_n,\Omega) = \liminf_{n\to\infty} F(u_n,\Omega),$$

so that

$$\mathcal{F}(u,\Omega) \leq \overline{F}_\infty(u,\Omega).$$

Hence the thesis follows by Lemma 4.2. ■

The second step is the lim sup inequality: $\overline{F} \leq \mathcal{F}$. In what follows $\mathcal{A}(\Omega)$ denotes the family of all open subset of Ω and, in the spirit of [5], we will assume the following conditions:

$$\exists G \subset \Omega \text{ with } \mathcal{H}^{N-1}(G) = 0 \text{ such that } \forall \xi \in \mathbb{R}^{M\times N}$$

$$f^\infty(\cdot,\xi) \text{ is approximately continuous in } \Omega \setminus G. \tag{4.42}$$

Theorem 4.4 *Let $f : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ be a Borel function convex in the second variable, satisfying (4.33) and (4.34). Then $\overline{F}(u,\cdot)$ is the trace of a finite Radon measure on $\mathcal{A}(\Omega)$ and for every $u \in BV(\Omega;\mathbb{R}^M)$; we have*

(i)

$$\overline{F}(u,\Omega \setminus (C_u \cup J_u)) \leq \int_\Omega f(x,\nabla u)dx$$

(ii) if (4.42) holds, then

$$\overline{F}(u,C_u) \leq \int_\Omega f^\infty(x,\frac{D^c u}{|D^c u|})d|D^c u|;$$

(iii) if $f^\infty(\cdot,\xi)$ is upper semicontinous, then

$$\overline{F}(u,J_u) \leq \int_{J_u\cap\Omega} f^\infty(x,(u^+-u^-)\otimes\nu_u)d\mathcal{H}^{N-1}.$$

Proof.

It is known (see Theorem 4.1.2 of [10]) that $\overline{F}(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon Measure and

$$0 \leq \mathcal{F}(u,A) \leq C(\mathcal{L}^N(A)+|Du|(A)). \quad \forall A\in\mathcal{A}(\Omega).$$

Therefore, following [29] (Theorem 1.3), it is enough to prove:

$$\begin{array}{ll}
(i) & \dfrac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) \leq f\big(x_0,\nabla u(x_0)\big) \quad \text{for } \mathcal{L}^N-\text{almost every } x_0\in\Omega,\\
(ii) & \dfrac{d\overline{F}(u,\cdot)}{d|D^cu|}(x_0) \leq f^\infty\big(x_0,\dfrac{D^cu}{|D^cu|}(x_0)\big) \quad \text{for } |D^cu|-\text{almost every } x_0\in\Omega,\\
(iii) & \dfrac{d\overline{F}(u,\cdot)}{d\mathcal{H}^{N-1}}(x_0) \leq f^\infty\big(x_0,(u^+(x_0)-u^-(x_0))\otimes\nu_u(x_0)\big) \quad \text{for } \mathcal{H}^{N-1}-\text{almost every } x_0\in J_u.
\end{array}$$

The proof of (i) follows, with minor modifications, the proof of (i) in Theorem 1.3 of [29]. Consider the coercive functional associated to $\overline{F}$ defined by $\overline{F}_1(u,A)=\overline{F}(u,A)+|Du|(A)$. By Theorem 3.7 of [10] we have that

$$\frac{d\overline{F_1}(u,\cdot)}{d\mathcal{L}^N}(x_0)=\frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0)+|\nabla u(x_0)|=f_1(x_0,\nabla u(x_0)) \text{ for } \mathcal{L}^N-\text{almost every } x_0\in\Omega,$$

where

$$f_1(x_0,\xi):=\limsup_{\varepsilon\to0}\frac{1}{\varepsilon^N}\inf\{\overline{F}_1(v,Q(x_0,\varepsilon)) : v\in BV(Q(x_0,\varepsilon);\mathbb{R}^M), v=\xi(x-x_0) \text{ on } \partial Q(x_0,\varepsilon)\},$$

where $Q(x_0,\varepsilon):=x_0+\varepsilon Q$ with $Q=(-\frac{1}{2},\frac{1}{2})^N$. Thus, to complete the proof of (i) it is enough to show that

$$f_1(x_0,\xi)\leq f(x_0,\xi)+|\xi|$$

for $\mathcal{L}^N$ a.e $x_0 \in \Omega$ and for every $\xi \in \mathbb{R}^{M\times N}$.

Since f is convex, then the function $f(x,\cdot)$ is continuous for $\mathcal{L}^N$ a.e. $x \in \Omega$. Therefore, by applying Lemma 1.4 to the function $g : \Omega \times \mathbb{R}^{MN} \to [0,\infty)$ defined by $g(x,\underline{\xi}) = f(x, T^{-1}(\underline{\xi}))$, we may find a $\mathcal{L}^N$-null set N_0 independent of $\underline{\xi}$ such that $g(\cdot,\underline{\xi})$ is approximately continuous in $\Omega \setminus N_0$. This implies that $f(\cdot,\xi)$ is approximately continuous in $\Omega \setminus N_0$ uniformly with respect to ξ. Therefore we may assume that $f(\cdot,\xi)$ is approximately continuous in x_0. By choosing as a test function $v = \xi(x - x_0)$ we get

$$f_1(x_0,\xi) - |\xi| \le \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^N}\overline{F}(\xi(x-x_0), Q(x_0,\varepsilon)) \le \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f(x,\xi)dx = f(x_0,\xi).$$

The proof of (ii) follows, with minor modifications, the proof of (ii) in Theorem 1.3 of [29]. By Lemma 3.9 of [10], for $|D^c u|$-a.e $x_0 \in \Omega$ there exists a sequence $\{t_\varepsilon^{(k)}\}$ such that

$$t_\varepsilon^{(k)} \to \infty, \quad \varepsilon t_\varepsilon^{(k)} \to 0^+ \quad \text{as } \varepsilon \to 0^+ \tag{4.43}$$

and

$$\begin{aligned} &\frac{d\overline{F_1}(u,\cdot)}{d|D^c u|}(x_0) = \frac{d\overline{F}(u,\cdot)}{d|D^c(u)|}(x_0) + |a| \\ = \;& \lim_{k\to\infty}\limsup_{\varepsilon\to 0^+} \frac{\inf\{\overline{F}_1(v, Q_\nu^{(k)}(x_0,\varepsilon)), v = t_\varepsilon^{(k)} a\otimes\nu(x-x_0) \text{ on } \partial Q_\nu^{(k)}(x_0,\varepsilon)\}}{k^{N-1}\varepsilon t_\varepsilon^{(k)}}, \end{aligned}$$

where $\dfrac{D^c u}{|D^c u|}(x_0) = a\otimes\nu$, $a = a_u(x_0)$, $\nu = \nu_u(x_0)$, $Q_\nu^{(k)}(x_0,\varepsilon) := x_0 + \varepsilon Q_\nu^{(k)}$ with

$$Q_\nu^{(k)} = R_\nu\Big((-\frac{k}{2},\frac{k}{2})^{N-1} \times (-\frac{1}{2},\frac{1}{2})\Big),$$

where R_ν denotes a rotation such that $R_\nu e_N = \nu$. Take $x_0 \in \Omega \setminus G$ so that all the limit above exist and are finite. Choose as a test function $v(x) = t_\varepsilon^{(k)} a \otimes \nu(x - x_0)$. Then

$$
\begin{aligned}
\frac{d\overline{F}(u,\cdot)}{d|D^c(u)|} + |a| &\leq \lim_{k\to\infty} \limsup_{\varepsilon\to 0^+} \frac{\overline{F}_1(t_\varepsilon^{(k)} a \otimes \nu(x - x_0), Q_\nu^{(k)}(x_0,\varepsilon))}{k^{N-1}\varepsilon^N t_\varepsilon^{(k)}}, \\
&\leq \lim_{k\to\infty} \limsup_{\varepsilon\to 0^+} \frac{1}{k^{N-1}\varepsilon^N t_\varepsilon^{(k)}} \int_{Q_\nu^{(k)}(x_0,\varepsilon)} f(x, t_\varepsilon^{(k)} a \otimes \nu) dx + |a|.
\end{aligned}
\tag{4.44}
$$

Then by Proposition 1.3 and the right inequality of (4.33), we get

$$
\frac{f(x, t_\varepsilon^{(k)} a \otimes \nu)}{t_\varepsilon^{(k)}} \leq f^\infty(x, a\otimes\nu) + \frac{\Lambda}{t_\varepsilon^{(k)}}.
$$

Hence, by (4.42), (4.44) and (4.43)

$$
\begin{aligned}
\frac{d\overline{F}(u,\cdot)}{d|D^c u|}(x_0) \leq \lim_{k\to\infty} \limsup_{\varepsilon\to 0^+} \left[\frac{1}{k^{N-1}\varepsilon^N} \int_{Q_\nu^{(k)}(x_0,\varepsilon)} f^\infty(x, a\otimes\nu)dx + \frac{\Lambda}{t_\varepsilon^{(k)}}\right] &= f^\infty(x_0, a\otimes\nu) \\
&= f^\infty(x_0, \frac{D^c u}{|D^c u|}(x_0));
\end{aligned}
$$

(iii) is proved in part (iii) of the proof of Theorem 1.3 of [29].

By combining the previous theorem with Corollary 4.1 we obtain a relaxation result on $BV(\Omega;\mathbb{R}^M)$ for discontinuous integrand.

Theorem 4.5 *Let $\Omega \subset \mathbb{R}^N$ be a bounded set with Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ be a Borel function convex in the second variable satisfying (i) and (iii) of (4.7),(4.25), (4.33), (4.34), (4.42) and such that $f^\infty(\cdot,\xi)$ is upper semicontinous. Then*

$$
\mathcal{F}(u,\Omega) = \overline{F}(u,\Omega) \quad \textit{for every } u \in BV(\Omega;\mathbb{R}^M).
$$

While, by dealing with the space $BV(\Omega;\mathbb{R}^M) \cap L^\infty(\Omega;\mathbb{R}^M)$, we have, as a consequence of Theorems 4.3 and 4.4, the following result.

Theorem 4.6 *Let $\Omega \subset \mathbb{R}^N$ be a bounded set with Lipschitz Boundary. Let $f : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ be a Borel function convex in the second variable satisfying (4.7), (4.33), (4.34),(4.42) and such that $f^\infty(\cdot,\xi)$ is upper semicontinous. Then*

$$\mathcal{F}(u,\Omega) = \overline{F}(u,\Omega) \quad \text{for every } u \in BV(\Omega;\mathbb{R}^M) \cap L^\infty(\Omega;\mathbb{R}^M).$$

Remark 4.2 *Note that, even if only about the regularity in the spatial variable x, Theorems* 4.3 *and* 4.4 *improve the relaxation result of [29], since no continuity assumptions with respect to x are assumed. Furthermore, we emphasize that the convexity assumption, which is not natural in the case $M, N > 1$, becomes realistic in the case $N = 1$, where we deal with $\Omega = (a,b)$ and $u \in BV\big((a,b);\mathbb{R}^M\big)$, even if $M > 1$, since in this special case convexity and quasiconvexity are equivalent. This is also true if the function f has the following form:*

$$f(x,\xi) = h(x,\|\xi\|) \quad \text{with } h : \Omega \times (0,+\infty) \to (0,+\infty) \text{ convex with respect to } \xi.$$

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