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Grassmannians, Lie algebras and quaternionic geometry

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Contents

Introduction

Let G be a compact Lie group and $\mathfrak{g}$ its Lie algebra. The main object of study of this thesis is the Grassmannian

$$\tilde{\mathbb{G}}_3(\mathfrak{g}) = \{\text{oriented 3-dimenisonal subspaces of } \mathfrak{g}\}\,. \tag{1}$$

We shall discuss both the theory for general G and special cases such as $\mathfrak{g} = \mathfrak{so}(4)$ (which is not simple) and $\mathfrak{g} = \mathfrak{su}(3)$.

We shall study the relationship between $\tilde{\mathbb{G}}_3(\mathfrak{g})$ and quaternionic geometry which ultimately derives from the QK moment mapping discussed in Chapter 5, but we first study $\tilde{\mathbb{G}}_3(\mathfrak{g})$ without reference to Quaternion-Kähler manifolds. Some sort of quaternionic structure is already evident in the description of the tangent space of $\tilde{\mathbb{G}}_3(\mathfrak{g})$, even though the dimension of this is a multiple of 3, rather than 4. If $V \in \tilde{\mathbb{G}}_3(\mathfrak{g})$ then we can write $\mathfrak{g} = V \oplus V^{\perp}$ and (using the metric on V)

$$T_V\mathbb{G}_3(\mathfrak{g}) \cong V \otimes V^{\perp}\,. \tag{2}$$

Now V is the standard (and adjoint) representation of $SO(3)$ which also appears in quaternionic geometry as the space $\operatorname{Im}\mathbb{H}$ generated by the imaginary quaternions $\imath,\ \jmath,\ k$, or the corresponding almost complex structures $I,\ J,\ K$. It is this identification of the "tautological" subspace V and $\operatorname{Im}\mathbb{H}$ that underlies many of the constructions in this thesis.

It is well known that much of the quaternionic geometry can ultimately be reduced to the representation theory of $Sp(1) = SU(2)$. In particular, the complexified tangent space of a QK manifold M^{4n} has the form

$$T_xM_{\mathbb{C}} = \Sigma^1 \otimes \mathbb{C}^{2n} \tag{3}$$

where we denote by Σ^k the irreducible complex representaion of $SU(2)$ of dimension $k+1$. In appropriate circumstances the second factor $\mathbb{C}^{2n}$ will itself be a representation of $SU(2)$ and we shall be especially interested in an $SU(2)$ equivariant inclusion

$$\Sigma^1 \otimes \Sigma^{k-1} \subset \Sigma^2 \otimes \Sigma^k\,, \tag{4}$$

which models inclusions

$$M \xhookrightarrow{\Psi} \tilde{\mathbb{G}}_3(\mathfrak{g}) \tag{5}$$

in Swann's theory (developed mainly in [79] and [80]). In this last setting moment mappings μ arising from the action of G are used in order to obtain inclusions of type (5), identifying the image with the unstable manifolds M_u of the gradient flow of an invariant functional f. On the other hand, the quaternionic structure of these last is reconstructed starting from the HyperKäler structure of nilpotent orbits $\mathcal{O}$ in the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and using then an appropriate action of $\mathbb{H}^*$: in this way the various $\mathcal{O}$ appear to be the bundles $\mathcal{U}$ fibring over the corresponding M_u. In this thesis the point of view will be different, in the sense that the quaternionic structure will be described using the map Ψ induced by the moment mappings and exploiting the quaternionic structure inherent in the tangent space $T_V\tilde{\mathbb{G}}_3(\mathfrak{g})$ in correspondence of the critical points for grad f, as in (4).

We give now an outline of the thesis chapter by chapter:

-Chapter 1 contains basic material about homogeneous and symmetric spaces (Section 1), Adjoint orbits (Section 2), together with the discussion of the consimilarity action (Section 3), which will be relevant for Section 4 in Chapter 5; then cohomological properties of compact semisimple Lie goups are discussed (Section 4), included the invariant 3-form which gives rise to the functional f studied in Chapter 3.

-Chapter 2 describes bundles on Grassmann manifolds (Section 1), and introduces twistor-type differential operators existing on the tautological bundle (Sections 2, 3); these operators resemble the well-known twistor operators in QK geometry: in Chapter 6 a correspondence between them will be described; the deRham cohomology of $\tilde{\mathbb{G}}_3(\mathbb{R}^6)$ is calculated in Section 4, where an explicit expression of an invariant 4-form of $\tilde{\mathbb{G}}_3(\mathbb{R}^n)$ for any n is supplied.

-Chapter 3 contains in Section 1 a description of the invariant functional f on $\tilde{\mathbb{G}}_3(\mathfrak{g})$ coming from the standard 3 form of $\mathfrak{g}$, and introduces the Wolf spaces as its absolute maxima; an operator $\boldsymbol{\gamma}$ which represents an obstruction to the orthogonality of vector fields to G-orbits is introduced and used to discuss the invariance of f in an alternative way. In Section 2, a new functional g on $\mathbb{G}_3(\mathfrak{g})$ is introduced, and an expression for its gradient is given, in terms of a "generalized Casimir operator"; the invariance is discussed again using the operator $\boldsymbol{\gamma}$. In Section 3 the Hessian of g is described at critical points corresponding to subalgebras, in therms of $SU(2)$ representations; then we compare it with the Hessian of f. In section 4 the theory discussed in the previous sections is applied to some low-dimensional examples, determining the Hessians of f and g at grad f critical points; it is shown that not all critical submanifolds for grad, g are critical for grad f, exhibiting examples and stating in Section 5 a criterion for individuating a family of such submanifolds.

-Chapter 4 introduces quaternionic geometry, describing firstly the basic facts of the classical theory, including the fundamental examples of the Wolf spaces (Section 1) together with their twistor space (Section 2); then the first bridge with the "Grassmannian interpretation" is built at an algebraic level, using $SU(2)$ representation theory (Section 3); the quaternionic 4-form is introduced, and the previously discussed theory leads to an explicit description of it in the setting of the 8-dimesnional case; this is related to the discussion of the example corresponding to $\mathfrak{su}(3)$ in Section 5 of Chapter 6.

-Chapter 5 introduces the QK moment mapping (Section 1); the relationship with instantons, Nahm's equations and nilpoten orbits theory and Swann's theory are discussed: this latter contains the background regarding the use of moment mappings to obtain the realizations of QK manifolds in $\mathbb{G}_3(\mathfrak{g})$; explicit examples of realizations in the case of some classical Wolf spaces are provided in Section 5, exploiting the knowledge of a trajectory for the flow of $\operatorname{grad} f$ in the case of $\mathfrak{so}(4)$ (Section 3); here the prorportionality of $\operatorname{grad} f$ to $\operatorname{grad} g$ along this trajectory is proved: this will be used later in Chaper 6.

-Chapter 6 consisnts of the main conclusions of the thesis: the Coincidence Theorem is stated and proved, providing a way of "translating" the action of the quaternionic structure on the tangent space T_xM of a QK manifold in the $\mathbb{G}_3(\mathfrak{g})$ setting; the operator $\boldsymbol{\gamma}$ introduced in Chapter 3 is involved in this description of the quaternionic action. The correspondence between the sections in the kernels of the two twistor operators (the "QK" and the "Grassmannian") is described in Section 2; in Section 3 we consider again the gradient of the functional g, studied in Chapter 2: it is proportional to $\operatorname{grad} f$ in several cases, and this fact seems significant in order to relate the quaternionic metric of the unstable manifold with that induced by the ambient Grassmannian; finally the example of $SU(3)$, which for some aspects stands out of the general situation, is discussed in more detail in Section 4.

Notational conventions.

We will adopt the notation $[V]$ and $[\![V]\!] = [V+\overline{V}]$ from [73] to denote the real vector space fixed by an invariant real structure in the complex representation V or $V+\overline{V}$; however we will sometimes omit the brackets for simplicity.

We will denote by exp the exponentiation of matrices and in the Lie algebra context, by *Exp* the exponentiation in the sense of Riemannian geometry.

Antisymmetric and symmetric product of tensors will be usually denoted by $\wedge$ and $\vee$ respectively, but alternative notations will be adopted occasionally (for example in Section 4.3).

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Chapter 1

Homogeneous spaces and Lie algebras

In this Chapter, we shall cover a selection of topics relevant to the thesis. This includes the less well-known action of $SU(3)$ on itself induced from "consimilarity".

1.1 Homogeneous and Symmetric spaces

We introduce here basic facts about G-actions and homogeneous spaces. References are [81], [14], [25].

Let M be a differentiable manifold and G a compact Lie group. A C^∞ map $m : G \times M \to M$ such that

$$m(gh, x) = m(g, m(h, x)), \quad m(e, x) = x \tag{1.1}$$

for all g, h in G and x in M is called an *action on the left of* G *on* M and is called a G*-space*; analogous definitions give rise to actions on the *right.* For simplicity we will denote by $g\,x$ the point $m(g, x)$. Fixed a point x in M, we call the *orbit* $G\,x$ of x under G the subset of M defined by

$$G\,x := \{y \in M \,|y = g\,x\}\,. \tag{1.2}$$

Suppose that the orbit of a point x is the whole manifold M: then we will say that G acts *transitively* on M, which is called a *homogeneous* G*-space.* We define the *isotropy subgroup* (or *stabilizer of* p) $G_p \subset G$ of an orbit $G\,p$ the subgroup

$$G_p := \{g \in G\,|g\,, q = q\} \tag{1.3}$$

for any point $q \in G\,p$; in the transitive case all isotropy subgroups of are conjugate subgroups of G. The differential of the action of an element $g \in G_p$

at the point p determines a representation of G_p in $GL(T_pM)$ which is called the *isotropy representation.* In the case of transitive actions we can identify M with the *coset space* of the group G:

$$M = \frac{G}{G_p} \tag{1.4}$$

for any point $p \in G$. In fact the space G/G_p can be topologized and equipped with the correct differentiable strucure, so that a point $p \in M$ can be identified with a class $g\,G_p$ for some $g \in G$. A map $\psi : M \to N$ between two manifolds with a G-action is cvalled *equivariant* if it satisfies

$$\psi(g\,x) = g\psi(x) \tag{1.5}$$

for any g in G. If the action is not transitive, there exists an orbit with isotropy subgroup G_p such that $gG_pg^{-1} \subset G_q$ for some $g \in G$ and for any other isotropy subgroup G_q; the corresponding orbit is of maximal dimension and is called *principal*; the union of pricipal orbits is an open dense subset of M and the codimension of a principal orbit is called the *cohomogeneity* of the G action. Non-principal orbits $G\,q$ are called *singular* if $\dim G\,q < \dim G\,p$; if $\dim G\,q = \dim G\,p$ but $G_p \subset G_q$ strictly, the orbit $G\,q$ is called *exceprional* and is a discrete cover of the principal orbit $G\,p$.

Examples. Consider the standard representation of $SO(n)$ on $\mathbb{R}^n$; as it preserves the standard euclidean norm, we have

$$G\,S^{n-1} \subset S^{n-1}\,; \tag{1.6}$$

it can be shown that this action is transitive, and the subgroup $G_p \subset SO(n)$ which stabilizes a unit vector (for example $(0, \cdots, 0, 1)$) is isomorphic to $SO(n-1)$; in conclusion we can identify

$$S^{n-1} \cong \frac{SO(n)}{SO(n-1)}\,. \tag{1.7}$$

This type of presentation is not unique: we can in fact analogously consider $SU(n)$ acting on $\mathbb{C}^n$ with its standard hermitian structure; the sphere S^{2n-1} is again preserved, and the action can be again shown to be transitive on it; the stabilizer of a point turns out to be $SU(n-1) \subset SU(n)$, hence

$$S^{2n-1} \cong \frac{SU(n)}{SU(n-1)}\,. \tag{1.8}$$

Other important examples are projective spaces: $\mathbb{CP}^n$ parametrizes the set of complex lines $\mathbb{C} \subset \mathbb{C}^n$ (which are real 2-planes preserved by the standard complex structure J of $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$); then the group $U(n+1)$ acts transitively on such set, and one of the complex lines is fixed by the subgroup $U(1) \times U(n)$, so that

$$\mathbb{CP}^n \cong \frac{U(n+1)}{U(1) \times U(n)} \,. \tag{1.9}$$

analogous considerations lead to the description of real and quaternionic projective spaces:

$$\mathbb{RP}^n \cong \frac{SO(n+1)}{O(n)} \quad , \quad \mathbb{HP}^n \cong \frac{Sp(n+1)}{Sp(1)Sp(n)} \,. \tag{1.10}$$

An important class of homogeneous G-spaces is that of *symmetric spaces*. References for this topic are [35] and [55].

Let (M, g) Riemannian manifold; consider a normal neighborood N_p of a point $p \in M$, where the Exp_p map is a local diffeomorphism with a neighbourhood of 0 in T_pM; we can define a map

$$d\sigma_p : \; T_pM \longrightarrow T_pM \tag{1.11}$$

acting as $-I$; this induces a local involutive diffeomorphism σ_p of the normal neighbourhood N_p in itself called *Cartan involution*, sending a geodesic $\gamma(t)$ through p to $\gamma(-t)$; we have

Definition 1.1. A Riemannian manifold M for which the map σ_p is an isometry for every $p \in M$ is called a *locally symmetric space.*

Theorem 1.1. *Let M be a Riemannian manifold; then the following conditions are equivalent:*

i)M is locally symmetric;
ii)$\nabla R = 0$,

where R is the Riemannian curvature tensor.

The symmetry is defined locally and in general it is not possible to extend it to a global isometry of the manifold M; in this case M is defined a *globally symmetric space*, or simply *symmetric space.* The following proposition relates the two notions:

Proposition 1.2. *Let M be a complete locally symmetric space; if $\pi_1(M) = 0$ then M is globally symmetric.*

Hence we can obtain globally symmetric spaces considering the universal coverings of complete locally symmetric spaces. Symmetric spaces are homogeneous, then if G is the full group of isometries for M we have a presentation

$$M = G/H \tag{1.12}$$

where H is the stabilizer of a point $p; \sigma_p$ induces an involutive automorphism of G, called *Cartan involution*:

$$s_p(g) = \sigma_p \circ g \circ \sigma_p^{-1}\,, \tag{1.13}$$

such that if G_σ denotes the subgroup fixed by s_p and G_σ^0 the connected component of the identity, then

$$G_\sigma^0 \subset H \subset G_\sigma\,; \tag{1.14}$$

at a Lie algebra level s_p induces an automorphism of Lie algenras ds_p with eigenvalues ± 1, being involutive; therefore the decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \tag{1.15}$$

corresponds to the identification of the $+$ eigenspace with $\mathfrak{h}$ and the $-$ eigenspace with $\mathfrak{m} \cong T_pM$. A triple $(\mathfrak{g}, \mathfrak{h}, ds)$ where $\mathfrak{h}$ is a compact Lie subalgebra of the Lie algebra $\mathfrak{g}$ and ds is an involutive Lie algebra automorphism such that $\mathfrak{h}$ coincides with its $+$ eigenspace is called an *orthogonal Lie algebra*; if moreover $\mathfrak{h} \cap \mathfrak{z} = 0$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$, then the algebra is said *effective.* For an orthogonal Lie algebra the following relations hold:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}\,; \tag{1.16}$$

exists a bijective correspondence between orthogonal Lie algebras and symmetric spaces G/H with G simply connected and H connected, where $N \cap H$ is discrete if N is the maximal normal subgroup of G.

Examples All examples discussed about homogeneous spaces (so S^n, $\mathbb{CP}^n$, $\mathbb{RP}^n$, $\mathbb{HP}^n$) are actually symmetric spaces.

Another example which will be particularly relevant for us is that of the real oriented Grassmannians $\tilde{\mathbb{G}}_k(\mathbb{R}^n)$: these represent the set of k-dimensional subspaces in $\mathbb{R}^n$; as symmetric spaces they have a presentation

$$\tilde{\mathbb{G}}_k(\mathbb{R}^n) = \frac{SO(n)}{SO(k) \times SO(n-k)}\,; \tag{1.17}$$

the group $SO(n)$ acts transitively on the set of orthonormal frames of $\mathbb{R}^n$, but a k-plane V is identified by a k-tuple of orthonormal vectors, which can be completed by an $(n-k)$-tuple in $V^\perp$; therefore $SO(k)$ and $SO(n-k)$ act on $V, V^\perp$ respectively, and their tensor product coincides with the stabilizer of the point V.

Observation. The same symmetric space can have different presentations as a homogeneous space: not all of them give a decomposition (1.15) compatible with the involution, satisfying therefore (1.16); consider for insatnce the two presentations of S^{2n-1} given in (1.7) and (1.8): the former is symmetric, the latter is not.

A symmetric space $M = G/K$ can be embedded as a totally geodesic submanifold of the Lie group G with the Riemannian metric induced by the Killing form by the map

$$g\,H \longrightarrow g^s g^{-1} \tag{1.18}$$

called *Cartan embedding*, where g^s is the image of g under the Cartan involution s at a point p. Recall now from [17] that a Cartan subalgebra $\mathfrak{h}$ whose intersection with $\mathfrak{k}$ is a Cartan subalgebra for $\mathfrak{k}$ is called *fundamental* for the symmetric decomposition, and its root system can be decomposed as:

$$\Delta = I_{\mathfrak{k}} + I_{\mathfrak{m}} + II \tag{1.19}$$

where α belongs to $I_{\mathfrak{k}}$ (or in $I_{\mathfrak{m}}$) if $\alpha|\mathfrak{h}_{\mathfrak{m}} = 0$ and $\mathfrak{g}_\alpha$ lies in $\mathfrak{k}_{\mathbb{C}}$ (or in $\mathfrak{m}_{\mathbb{C}}$), while $\alpha \in II$ if $\alpha|\mathfrak{h}_{\mathfrak{m}} \neq 0$.

We can use the roots of type $I_{\mathfrak{m}}$ to obtain minimal immersions of 2-spheres in any symmetric space G/K: consider $\alpha \in I_{\mathfrak{m}}$, then in $\mathfrak{g}_{\mathbb{C}}$ the triple $\{\mathfrak{g}_\alpha,\ \mathfrak{g}_{-\alpha},\ [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]\}$ spans an $\mathfrak{sl}(2,\mathbb{C})$ subalgebra, containng an $\mathfrak{su}(2)$ as the stable set for the appropriate real structure; the semisimple element $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ intersects $\mathfrak{k}$ in a 1-dimensional subalgebra $\mathfrak{u}(1)$; therefore the corresponding homogeneous space is

$$S^2 = \frac{SU(2)}{U(1)}\,. \tag{1.20}$$

The immersion $i_\alpha : SU(2) \subset G$ induces an equivariant immersion of symmetric spaces

$$\begin{tikzcd} SU(2) \arrow[r, hook, "i_\alpha"] \arrow[d] & G \arrow[d] \\ S^2 \arrow[r, hook, "\phi_\alpha"'] & G/K \end{tikzcd} \tag{1.21}$$

and the following proposition holds (see always [17]):

Proposition 1.3. *Let G/K be a compact simply connected symmetric space; then if $\pi_2(G/K)$ is non-trivial, it is generated by the class $[\phi_\alpha]$ for some $\alpha \in I_{\mathfrak{m}}$.*

1.2 Adjoint orbits

We recall here some definitions and lemmas which will be useful in the sequel: we recall that given the action of a compact group H of isometries on a Riemannian manifold M, we define a *section* Γ a smooth submanifold which intersects transversally all the H-orbits; an H-action admitting a section is called *polar*, and if the section is flat it is called *hyperpolar*. Sections are always totally geodesic submanifolds; in this sense the geodesic γ used for $\mathbb{HP}^n$ was a section. Fixed a point $p \in M$, we have the H-orbit through p, and the stabilizer H_p fixing p acts on T_pM, preserving the subspace tangent to the orbit $T_p(H \cdot p)$, the *isotropy representation*, and its orthogonal complement $\nu_p(G \cdot p)$, called *slice representation*; the following useful lemma holds:

Lemma 1.4. *The cohomogenity of the H-action on M equals that of the H_p-action on the slice representation.*

Remark. We notice that in this language what we used to call the isotropy representation of a homogeneous space $M = G/K$ paradoxically coincides with the slice representation.

References about hyperpolar actions on symmetric spaces are [36], [37], [56], [68] and [12]. From [68] we quote the following lemma, that gives a sufficient criterion to individuate sections:

Lemma 1.5. *If Γ is a compact, connected, flat, totally geodesic submanifold of a Riemannian H-manifold M and Γ is orthogonal to some H-orbit at one point, then Γ meets all H- orbits orthogonally. If in addition the dimension of Γ is equal to the cohomogeneity of the action on M, then Γ is a section and the H action on M is hyperpolar.*

Examining the proof that $T_p\Gamma \subset \nu_p$ (with equality if p belongs to a principal orbit) and Γ can be obtained from $Exp(T_p\Gamma)$ for any $p \in \Gamma$. In consequence of this, if we pick a point p, we choose a vector $y \in \nu_p$ such that $\gamma(t) = Exp(ty)$ is a closed geodesic, then it is automathically a section, for cohomogeneity 1 actions. As we shall see, using sections is convenient to determine the behaviour of equivariant maps. In particular recall (see [14, ChapterIV,Theorem 3.1]) that the set of principal orbits in the orbit space M/G of a given compact G-manifold M is open, dense and connected, and in the case of cohomogeneity 1 actions it corresponds to the whole M/G if it is an S^1, or equivalently when the orbits are all principal; otherwise (and this is the case we are most interested in) to the interior $(0,1)$ when $M/G \cong [0,1]$.

A well known action of a Lie group on itself is the Adjoint action; this is defined in the followig way: if $g,\, h \in G$ then

$$Ad_g \cdot h := ghg^{-1}\,. \tag{1.22}$$

Clearly the unit e is a fixed point for this action, and the differential induced on $T_eG = \mathfrak{g}$ gives rise to the *Adjoint representation*, or in other words an inclusion $G/Z \subset GL(\mathfrak{g})$, where Z is the center of G. The principal orbit for this action have the form

$$\frac{G}{T^n}\,, \tag{1.23}$$

where T^n is a maximal torus for G, and they are called *flag manifolds*; the motivation for this name relies on the fact that for $SU(n)$ they represent the manifold of complete flags in $\mathbb{C}^n$, that is of all sequences of complex vector subspaces

$$0 \subset V^1 \subset \cdots \subset V^{n-1} \subset \mathbb{C}^n \tag{1.24}$$

where $\dim V^k = k$, with jumps of 1 dimension. Principal orbits form an open dense subset in $\mathfrak{g}$; the following Theorem, due to Bott, describes the rôle played by maximal Abelian subalgenras:

Theorem 1.6. *Each orbit for the Adjoint action of a centerless compact connected Lie group G intersects a Cartan subalgebra $\mathfrak{t}$ in a finite non-empty set.*

Example 1.1. Consider again the case of $SU(n)$: from an elementary point of view, Linear Algebra tells us that every skew-Hermitian matrix can be put in diagonal form

$$\begin{pmatrix} \imath\lambda_{k_1} & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \imath\lambda_{k_n} \end{pmatrix} \tag{1.25}$$

with $\lambda_{k_i} \in \mathbb{R}$ and $\sum_i k_i = 0$ by conjugation with matrices in $SU(n)$: in the language developed above this corresponds to say that a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{su}(n)$ is a global section for the Adjoint representation of $SU(n)$ on $\mathfrak{su}(n)$, which therefore is a polar action and has cohomogeneity equal to its rank; as $\mathfrak{t}$ inherits the flat metric from the ambient Euclidean space $\mathfrak{g}$, the Adjoint action is hyperpolar. Exponentiating, we obtain the same type of principal orbits for the action of $SU(n)$ on itself, and the torus $T^n = \exp \mathfrak{t}$ is again a section thanks to the surjectivity of exp: we shall see in section

1.3 another type of action of $SU(n)$ on itself with a rather different orbit structure. Singular orbits are given by symmetric spaces of type

$$\frac{SU(n)}{S(U(q_1) \times \cdots \times U(q_r))} \tag{1.26}$$

with $\sum q_i = n$; these represent the manifolds of *partial flags* in $\mathbb{C}^n$, analogous to (1.24) but with dimensional jumps given by q_i: in fact the principal orbits (1.23) correspond to $q_i = 1$ for all i. At the other extreme are complex Grassmannians, for which $r = 2$. Analogous situation holds for other classical compact semisimple Lie groups.

Another description of flag manifolds is obtained passing to the complexified group $G_{\mathbb{C}}$; consider the following type of subalgebras $\mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}}$: $\mathfrak{p}$ is called *Borel subalgebra* if it is a maximal solvable subalgebra; it is called *parabolic* if it contains a Borel subalgebra. If we fix a Cartan decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha} \mathfrak{g}_{\alpha} \tag{1.27}$$

then an example of Borel subalgebra is given by

$$\mathfrak{p} = \mathfrak{t}_{\mathbb{C}} + \sum_{\alpha > 0} \mathfrak{g}_{\alpha} \tag{1.28}$$

and it can be shown that any other Borel subalgebra is conjugate to this one.

Parabolic subalgebras are obtained by adding any negative simple root spaces:

$$\mathfrak{p} = \mathfrak{t}_{\mathbb{C}} + \sum_{\alpha > 0} \mathfrak{g}_{\alpha} + \sum_{\substack{\beta = \sum n_\alpha \alpha \\ \alpha \in I \subset \Delta^+}} \mathfrak{g}_{-\beta} \tag{1.29}$$

where $I \subset \Delta^+$ denotes any subset of the set of simple positive roots and $n_\alpha \in \mathbb{Z}^+$. Also in this case it can be shown that every parabolic subalgebra is conjugate to one of this type. Homogeneous spaces obtained by quotient as

$$\frac{G_{\mathbb{C}}}{P} \tag{1.30}$$

where $P = \exp \mathfrak{p}$ can be shown to be compact, as they can be realized as orbits of the compact group:

$$\frac{G_{\mathbb{C}}}{P} \cong \frac{G}{C(T^k)}\,, \tag{1.31}$$

where $C(T^k)$ is the centralizer of a k-torus with $k \leq \operatorname{rank} G$.

1.3 The consimilarity action

We are going now to consider the following group action $\mathbf{c}$ of $GL(n,\mathbb{C})$ on itself, called *consimilarity action*:

$$\mathbf{c}(A)\cdot B := AB\bar{A}^{-1}\,; \tag{1.32}$$

this action can be restricted to $SU(n) \subset GL(n,\mathbb{C})$, so that $SU(n)$ acts on itself, as in this case

$$AB\bar{A}^{-1} = ABA^t \tag{1.33}$$

is in $SU(n)$ if A, B are. This action is a special case of a family of actions of a Lie group G on itself, described in [36]; these are constructed using an automorphism σ of G and are called *σ-actions.*

First of all we prove

Lemma 1.7. *The consimilarity action of $SU(n)$ on itself is an isometric action with respect to the Killing metric on $SU(n)$.*

Proof. Consider a curve $B(t) \subset SU(3)$ such that $B(0) = B_0$ and $\dot{B}(0) = w$; then

$$\frac{d}{dt}(AB(t)A^t)|_{t=0} = AWA^t\,; \tag{1.34}$$

therefore if $w_1, w_2 \in T_{B_0}SU(n)$ then

$$\langle w_1, w_2\rangle_{B_0} = \langle B_0^{-1}w_1, B_0^{-1}w_2\rangle_e = \mathrm{Tr}(B_0^{-1}w_1B_0^{-1}w_2)\,; \tag{1.35}$$

with $B_0^{-1}w_i \in \mathfrak{su}(n)$; analogously

$$\begin{aligned}\langle Aw_1A^t, Aw_2A^t\rangle_{AB_0A^t} &= \langle (AB_0A^t)^{-1}Aw_1A^t, (AB_0A^t)^{-1}Aw_2A^t\rangle_e \\ &= \langle (A^t)^{-1}B_0^{-1}A^{-1}Aw_1A^t, (A^t)^{-1}B_0^{-1}A^{-1}Aw_2A^t\rangle_e \\ &= \mathrm{Tr}((A^t)^{-1}B_0^{-1}w_1A^t, (A^t)^{-1}B_0^{-1}w_2A^t) \\ &= \mathrm{Tr}(B_0^{-1}w_1B_0^{-1}w_2)\,;\end{aligned} \tag{1.36}$$

the assertion follows. ■

We can therefore use the machinery of smooth Riemannian actions to describe the orbit structure of $SU(n)$ as an $SU(n)$-space under consimilarity action; from now on we concentrate on the case $n = 3$. Let us consider the orbit of the identity $e = I$ under this action:

Proposition 1.8. *The $SU(3)$ orbit $\mathcal{S}$ of I under the consimilarity action is the 5-dimensional symmetric space*

$$\frac{SU(3)}{SO(3)}\,. \tag{1.37}$$

Proof. Consider the map

$$\xi : \frac{SU(3)}{SO(3)} \longrightarrow \mathcal{S} \tag{1.38}$$

acting as

$$A\,SO(3) \longmapsto AA^t \ ; \tag{1.39}$$

it is well defined, as

$$AB(AB)^t = ABB^tA^t = AA^t \tag{1.40}$$

if $B \in SO(3)$; it is clearly surjective; it is also injective as if $AA^t = CC^t$ then

$$C^{-1}A = C^t(A^t)^{-1} = \big((C^{-1}A)^t\big)^{-1} \tag{1.41}$$

so that $C^{-1}A \in SO(3)$, or in other words

$$A\,SO(3) = C\,SO(3)\,. \blacksquare \tag{1.42}$$

From now on we shall write

$$SSU(3) := \frac{SU(3)}{SO(3)} \tag{1.43}$$

as an abbreviation.

Observation. Consider the totally geodesic immersion

$$\zeta : \ A\,SO(3) \longmapsto A^{\sigma}A^{-1} \tag{1.44}$$

as seen in (1.18), where the involution σ is given by

$$\sigma(A) = \overline{A}\,; \tag{1.45}$$

then we have:

$$\zeta = \sigma \circ \xi\,. \tag{1.46}$$

We are intersted in calculating the cohomogeneity of this action and the generic orbit type, and in determining if possible the singular orbits. The answer to these questions is given in

Proposition 1.9. *The consimilarity action of $SU(3)$ on itself is a cohomogeneity one action, with principal orbit*

$$SU(3)/U(1)\,; \tag{1.47}$$

the two singular orbits are given by

$$SSU(3) \quad \textit{and} \quad S^5\,. \tag{1.48}$$

Proof. Cconsider the point I and its stabilizer $SO(3)$; the tangent space at I is $\mathfrak{su}(3)$, which can be decomposed in two orthogonal summands, the isotropy and the slice representations:

$$\mathfrak{su}(3) = \tau \oplus \nu\,; \tag{1.49}$$

for what we said before the isotropy representation of the orbit must be 5 dimensional, and it must be irreducible; moreover the action $\mathbf{c}$ clearly preserves $\mathfrak{so}(3) \subset \mathfrak{su}(3)$, as it coincides with the restriction of the Adjoint representation on $SO(3)$; it is also possible to show directly that the differential of the action at the identity $d\,\mathbf{c}$, acts on $w \in \mathfrak{su}(3)$ as

$$d\,\mathbf{c}(w) = w + w^t \tag{1.50}$$

and the kernel is given precisely by the antisymmetric matrices in $\mathfrak{su}(3)$, which give an $\mathfrak{so}(3)$; the image is the complementary subspace $\mathfrak{so}(3)^\perp$. Therefore decomposition (1.49) becomes

$$\mathfrak{su}(3) = \tau \oplus \nu = \mathfrak{so}(3)^\perp \oplus \mathfrak{so}(3) = [\Sigma^4] \oplus [\Sigma^2] \tag{1.51}$$

as $SO(3)$ representations. The cohomogeneity of $\mathbf{c}$ is the same as the cohomogeneity of $\nu = \Sigma^2$, which is 1; moreover the exponentiation of the Σ^2 bundle over $SSU(3)$ gives a tubular neigborhood of this latter, which turns out to be a singular orbit; we expect therefore to find a 7-dimensional principal orbit and another singular one, for what discussed previously; as exponentiation of ν in a Riemannian G-space is equivariant, the stabilizer of the principal orbit will be the same as that of the slice represenation, which for example along the direction corresponding to the matrix

$$w = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3) = \Sigma^2 \tag{1.52}$$

coincides with w itself; therefore the principal orbit is

$$\mathcal{P} = \frac{SU(3)}{U(1)} \tag{1.53}$$

with $U(1) = \exp(tu)$. The generic stabilizer $U(1)$ has to be contained in both the singular stabilizers, and in fact $U(1) \subset SO(3)$; we are now looking for the second singular orbit, and to do that we exponentiate w in order to get a closed geodesic, which will intersect orthogonally all the orbits by (1.5):

$$B(t) = \exp(tw) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{1.54}$$

Set

$$B_s = B(\pi/4) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.55}$$

and let us consider the differential $d\,\mathbf{c}$ when acting at B_s: if $w \in \mathfrak{su}(3)$ then

$$d\,\mathbf{c}(w) = wB_s + B_s w^t\,; \tag{1.56}$$

then the kernel of $d\,\mathbf{c}$ consists of the span of the elements

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \imath & 0 \\ \imath & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \imath & 0 & 0 \\ 0 & -\imath & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.57}$$

which is the subalgebra $\mathfrak{su}(2) \subset \mathfrak{su}(3)$ corresponding to the maximal root. Therefore the corrsponding orbit has the form

$$\frac{SU(3)}{SU(2)} = S^5 \tag{1.58}$$

in a non-symmetric presentation. ■

This can be expressed alternatively by saying that each element $A \in SU(3)$ is consimilar to $B(t)$ for some t. We have therefore the following double fibration associated to this orbit structure, as $SU(2) \supset U(1) \subset SO(3)$:

$$\begin{tikzcd} & SU(3)/U(1) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ SSU(3) & & S^5 \end{tikzcd} , \tag{1.59}$$

where the projections π_1 and π_2 are obtained in the following way: given a point p in a principal orbit $SU(3)/U(1)$ exists a unique normal vector u, as it is a hypersurface in $SU(3)$; therefore exists a unique geodesic γ which passes through p and is tangent to u; moreover γ is a section and can be obtained from B_s through the consimilarity action. As γ must inersect the singular orbits, if we follow it in the two directions, we will define $\pi_1(p)$ as the first intesection $\gamma \cap SSU(3)$ and $\pi_2(p)$ as the first intersection $\gamma \cap S^5$.

Consider again the singular orbit $SSU(3)$: it is a non-inner symmetric space, and the symmetric decomposition of the Lie algebra $\mathfrak{su}(3)$ is given by

$$\mathfrak{su}(3) = \mathfrak{k} + \mathfrak{m} = \mathfrak{so}(3) \oplus [\Sigma^4]\,; \tag{1.60}$$

let σ denote the induced group involution; then a σ-stable Cartan subalgebra $\mathfrak{t}$ is given by the span of the elements

$$w = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \tilde{w} = \begin{pmatrix} \imath & 0 & 0 \\ 0 & \imath & 0 \\ 0 & 0 & -2\imath \end{pmatrix}\,; \tag{1.61}$$

we notice that if $\mathfrak{t}'$ denotes the Cartan subalgebra generated by diagonal matrices, then

$$\mathfrak{t}' = B_s \mathfrak{t} B_s^{-1}\,; \tag{1.62}$$

with respect to (1.60), we have the decomposition

$$\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}} + \mathfrak{t}_{\mathfrak{m}} \tag{1.63}$$

which is expressed in terms of matrices by (1.61). Therefore if we consider the usual roots $\Delta \ni \alpha'$ in $\mathfrak{t}'^*_{\mathbb{C}}$, then

$$\alpha = \alpha' \circ Ad_{B_s^{-1}}\,; \tag{1.64}$$

moreover we notice that $\mathfrak{t} \cap \mathfrak{so}(3)$ is spanned by w, which is obviously a maximal abelian subalgebra for $\mathfrak{so}(3)$. Consider now in particular the maximal root α'_0 with respect to $\mathfrak{t}'$, which is the dual to

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.65}$$

It is known that $\pi_2(SSU(3)) = \mathbb{Z}_2$ (see [17, Page 38]); we want to find a root $\alpha \in I_{\mathfrak{m}}$ in order to identifiy the generator S^2 of π_2; consider the restriction of the map ξ to the subgroup $SU(2)$ corresponding to the long root α'_0: recall Proposition 1.3 and diagram (1.21); then we have

Proposition 1.10. *The map ξ defined in (1.38) coincides with $\phi_{\alpha'_0}$ and its image is the generator of $\pi_2(SSU(3))$.*

Proof. In view of (1.65) and (1.61), we have that $\alpha|\mathfrak{t}_{\mathfrak{m}} = 0$ only for $\alpha' = \alpha'_0$, so $I_{\mathfrak{k}} + I_{\mathfrak{m}} = \alpha'_0$; moreover the corresponding root space is given by

$$\mathfrak{g}_{\alpha'_0} = t \begin{pmatrix} \imath & 0 & 0 \\ 0 & -\imath & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{m}\,; \tag{1.66}$$

therefore $I_{\mathfrak{m}} = \alpha_0$ and the $SU(2)/U(1)$ corresponding to the highest root generates $\pi_2(SSU(2))$. In particular the semisimple element of $\mathfrak{su}(2)$ corresponds to w, hence the thesis.■

Going back now to the consimilarity action, let us consider the normal bundles ν_1 and ν_2 respectively at I and at B_s: we are interested in determining the intersections

$$\exp \nu_1 \cap S^5 \quad \text{and} \quad \exp \nu_2 \cap SSU(3)\,, \tag{1.67}$$

or in other words the "limit points" of the geodesics emanating from a point in one of the singular orbits when they meet the other the first time.

Proposition 1.11. *The intersection* $\exp \nu_2 \cap SSU(3)$ *is the* S^2 *generating* $\pi_2(SSU(3))$.

Proof. The left multiplications on a Lie group are isometries with respect to the metric induced by the Killing form by definition, therefore they respect exponentiation:

$$\begin{aligned} \exp \nu_2 &= B_s B_s^{-1} \exp \nu_2 \\ &= B_s \exp(B_s^{-1} \nu_2) \\ &= B_s \mathrm{Exp}\,(\mathfrak{su}(2)) \\ &= B_s SU(2) = SU(2)\,; \end{aligned} \tag{1.68}$$

hence we have to determine $SU(2) \cap SSU(3)$; moreover as exp is equivariant, then the intersection is also given by $\exp(S^2_{\pi/4})$, whose image is a homogeneous manifold of the form $SU(2)/K$ with $K \supset U(1)$ where $\mathfrak{u}(1) = tw$, because of equivariance; but clearly we have

$$S^2 = \xi(SU(2)) \subset SU(2) \cap SSU(3) \tag{1.69}$$

as in fact $\xi(SU(2)) \subset SU(2)$; hence it must be $K = U(1)$ and the conclusion follows.■

This can be interpreted in terms of the double fibration (1.59) by saying that

$$\pi_1(\pi_2^{-1} B_s) = S^2\,. \tag{1.70}$$

Analogous considerations lead to the following

Proposition 1.12. *We have*

$$\exp \nu_1 \cap S^5 = S^2 = SO(3)/U(1) = \{AB_sA^t \,|A \in SO(3)\}\,, \tag{1.71}$$

with $\mathfrak{u}(1) = tw$ *.*

Again in terms of (1.59)

$$\pi_2(\pi_1^{-1}I) = S^2\,. \tag{1.72}$$

We want now to discuss a link with the Adjoint action: we can define a map $\Phi : SU(3) \longrightarrow SU(3)$ acting as

$$\Pi(A) = A\bar{A}\,; \tag{1.73}$$

first of all we observe that the map is equivariant with respect to the consimilarity action on the left and the $Ad_{SU(3)}$ action on the right:

$$\Phi(BAB^t) = BAB^t(\bar{(}BAB^t) = BAB^{-1}\,, \tag{1.74}$$

therefore orbits are sent to orbits; in particular it is immediate to show that

$$\Phi(SSU(3)) = I \quad \text{and} \quad \Phi(S^5) = \mathbb{CP}^2 \tag{1.75}$$

and in fact

$$\Phi(B(t)) = B(t)^2 = \begin{pmatrix} \cos^2 t - \sin^2 t & 2\cos t \sin t & 0 \\ -2\cos t \sin t & \cos^2 t - \sin^2 t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.76}$$

$$= \begin{pmatrix} \cos 2t & \sin 2t & 0 \\ -\sin 2t & \cos 2t & 0 \\ 0 & 0 & 1 \end{pmatrix} =: \tilde{B}(t)\,. \tag{1.77}$$

We introduce now the $Ad_{SU(3)}$-invariant subspace $H \subset SU(3)$ defined as

$$H = \{A \in SU(3)\,|\,\mathrm{Im}\,\mathrm{Tr}(A) = 0\}\,; \tag{1.78}$$

clearly $\Phi(SU(3)) \subset H$ as

$$\overline{\mathrm{Tr}(A\overline{A})} = \mathrm{Tr}(\overline{A}A) = \mathrm{Tr}(A\overline{A})\,, \tag{1.79}$$

but moreover we have:

Lemma 1.13. *The subset* $H \subset SU(3)$ *is a connected and algebraic set, smooth everywhere excepted at* I*; moreover*

$$\Phi(SU(3)) = H\,. \tag{1.80}$$

Proof. We can characterize the elements of H as those which can be diagonalized in the form

$$D(\theta) = \begin{pmatrix} e^{\imath\theta} & 0 & 0 \\ 0 & e^{-\imath\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.81}$$

proving that it is arc-connected; this fact can be interpreted by saying that $D(\theta)$ is the intersection $T^2 \cap H$ where T^2 is the diagonal maxima torus; therefore the $Ad_{SU(3)}$ action restricted to H is a cohomogeneity 1 action. the condition $\mathrm{Tr}(A) = \mathrm{Tr}(\overline{A})$ can be put in an algebraic form considering the characteristic polynomial of a generic diagonalized element in $SU(3)$:

$$T(\lambda) = -\lambda^3 + (e^{\imath\theta} + e^{-\imath\phi} + e^{\imath(\phi-\theta)})\lambda^2 + (e^{-\imath\theta} + e^{\imath\phi} + e^{\imath(-\phi+\theta)})\lambda + 1 \tag{1.82}$$

so that H is the zero locus of the function

$$P(a_{11}, \cdots\ a_{33}) = a_{11} + a_{22} + a_{33} - a_{11}a_{22} - a_{22}a_{33} - a_{22}a_{33}\,; \tag{1.83}$$

the gradient of this last is given in terms of θ, ϕ by

$$\mathrm{grad}P(\phi,\theta) = (\cos\theta - \cos(\theta-\phi), -\cos\phi + cos(\theta-\phi)) \tag{1.84}$$

which is zero if and only if $\theta \in \{0, 2/3\pi, -2/3\pi\}$ and $\phi \in \{0, -2/3\pi, 2/3\pi\}$ respectively, or in other words at the center $Z(SU(3)) = (1, \zeta, \zeta^2)$ with $\zeta^3 = 1$, and $Z(SU(3)) \cap H = I$. We finally prove the surjectivity of Φ: for this it would be sufficient proving that

$$\tilde{B}(t) = C\,D(\theta(t))\,C^{-1} \tag{1.85}$$

for some $C \in SU(3)$; but the eigenvalues of $\tilde{B}(t)$ are easily seen to be $1, 2t, -2t$, therefore we are done putting $\theta(t) = 2t$.■

1.4 Lie algebra cohomology

First recall that the Lie algebra $\mathfrak{g}$ of G corresponds to vector fields which are invariant under left translations; in the same way $\mathfrak{g}^*$ corresponds to the space of left invariant differential forms $A_L(G)$. Exterior differentiation d is compatible with any diffeomorphism $\phi : G \to G$ in the sense that if $\alpha \in A_L(G)$

$$\phi^*(d\alpha) = d\phi^*\alpha\,, \tag{1.86}$$

and in particular this is true for left translations; so

$$dA_L(G) \subset A_L(G)\,, \tag{1.87}$$

or in other words the space $A_L(G)$ is stable under d. We will denote by $d_{\mathfrak{g}}$ the restriction of d to $A_L(G) \cong \mathfrak{g}^*$, so that $(\bigwedge \mathfrak{g}^*, d_{\mathfrak{g}})$ is a differential graded algebra; if $X_i,\ i = 1...k{+}1$ are left invariant vector fields and if $\alpha \in \bigwedge^k \mathfrak{g}^*$, then $d_{\mathfrak{g}}\alpha$ behaves in the following way:

$$d_{\mathfrak{g}}\alpha(X_1, ..., X_{k+1}) = \sum_{i<j} (-1)^{i+j} \alpha([X_p\,,\, X_q], X_1, .., \hat{X}_i, .., \hat{X}_j, .., X_{k+1})\,; \tag{1.88}$$

it is clear that $(d_{\mathfrak{g}})^2 = 0$, so we can form the cohomology groups

$$H^p(\mathfrak{g}) = H^p_L(G)\,. \tag{1.89}$$

Denoting by θ the representation induced on $\mathfrak{g}^*$ by the adjoint representation, we can extend it to a representation $\theta^\wedge$ of $\mathfrak{g}$ on $\bigwedge \mathfrak{g}^*$; the subscript $\theta = 0$ will denote the subspace killed by $\theta^\wedge$, a.k.a. the space of *biinvariant*, or simply *invariant*, forms; in other words

$$(\bigwedge \mathfrak{g}^*)_{\theta=0} := \{\alpha \in \bigwedge \mathfrak{g}^* \,|\, \theta^\wedge(\alpha) = 0\}\,. \tag{1.90}$$

In case that G is connected, thanks to the surjectivity of the exp map, the space $(\bigwedge \mathfrak{g}^*)_{\theta=0}$ coincides with the space $(\bigwedge \mathfrak{g}^*)_I$ of forms fixed by the extension of the Ad representation of G on $\mathfrak{g}$. The space $(\bigwedge \mathfrak{g}^*)_{\theta=0}$ is again stable under d, but there is more; in fact

$$d\left((\bigwedge \mathfrak{g}^*)_{\theta=0}\right) = 0\,. \tag{1.91}$$

This can be shown in the following way: if ν denotes the inversion map of g, so that $\nu(a) = a^{-1}$, then

$$\nu^*\alpha = (-1)^p\alpha\,, \tag{1.92}$$

for any invariant form α of degree p; then

$$(-1)^{p+1} d\alpha = \nu^* d\alpha = d\nu^*\alpha = (-1)^p d\alpha\,, \tag{1.93}$$

so $d\alpha$ must be 0. Thanks to this fact the inclusion of $A_I(G)$ in $A_L(G)$ induces a homomorpism of algebras

$$(\bigwedge \mathfrak{g}^*)_{\theta=0} \cong\ A_I(G) \longrightarrow H_L(G)\,. \tag{1.94}$$

The consequence is that under the hypoteses of the compactness and connectedness of G we can reduce the problem of computing the cohomology of G to that of computing the cohomology of $\mathfrak{g}$ and, better, to the knowledge of $(\bigwedge \mathfrak{g}^*)_{\theta=0}$, as the following proposition states:

Proposition 1.14. *If the Lie group G is compact and connected, then all the maps in the following commutative diagram are isomorphisms of algebras:*

$$\begin{tikzcd} A_I(G) \arrow[r] \arrow[d, "\cong"'] & H_L(G) \arrow[r] \arrow[d, "\cong"] & H(G) \ . \\ (\bigwedge \mathfrak{g}^*)_{\theta=0} \arrow[r] & H(\mathfrak{g}) & \end{tikzcd} \tag{1.95}$$

Identifying $(\bigwedge \mathfrak{g}^*)_{\theta=0}$ is not in general an obvious task, excepted in some cases: for example when G is abelian.

Example 1.2. If G is abelian then it must be an n-dimensional torus; in this case the adjoint representation θ is trivial, so that

$$(\bigwedge \mathfrak{g}^*)_{\theta=0} = \bigwedge \mathfrak{g}^* \, ; \tag{1.96}$$

so the Betti numbers are just the dimensions of the various $\bigwedge^p \mathfrak{g}^*$ for $p = 0 ... n$:

$$b_p = \dim\, H^p(G) = \dim \bigwedge^p \mathfrak{g}^* = \binom{n}{p} = \frac{n!}{p!(n-p)!} \, ; \tag{1.97}$$

the Poincaré polynomial is equal to $(1+t)^n$. We observe that thanks to the inclusion $(\bigwedge \mathfrak{g}^*)_{\theta=0} \subset \bigwedge \mathfrak{g}^*$ for any other compact connected Lie group G we will have $P_G(t) \leq (1+t)^n$.

In general however it is possible to say something about the low Betti numbers; we start defining a canonical linear map

$$\rho : \ (\bigvee^2 \mathfrak{g}^*)_{\theta=0} \longrightarrow (\bigwedge^3 \mathfrak{g}^*)_{\theta=0} \tag{1.98}$$

in the following way: let $\Xi \in (\bigvee^2 \mathfrak{g}^*)_{\theta=0}$, then $\rho(\Xi) = \Phi$ is defined as

$$\Phi(x, y, z) = \Xi([x\, ,\, y], z) \, ; \tag{1.99}$$

the invariance of Ξ implies that Φ is skew-symmetric, and the Jacobi identity that it is invariant. We denote with f the image of the Killing form $\langle\, ,\, \rangle$ through ρ.

Proposition 1.15. *Let $\mathfrak{g}$ be a Lie algebra such that $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$; then ρ is an isomorphism.*

Thanks to this we can deduce the following facts:

Proposition 1.16. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ be the decomposition of a semisimple Lie algebra in terms of simple ideals; then:*

$i)$ $b_1 = b_2 = 0$;
$ii)$ $b_3 \geq m$;
$iii)$ if the ground field $\Gamma = \mathbb{C}$ or if $\Gamma = \mathbb{R}$
and the Killing form is negative definite, then $b_3 = m$.

Proof. It is sufficent to consider the case $m = 1$. For i we just need to observe that the semisimplicity of the Lie algebra implies the surjectivity of the bilinear map

$$[\,,\,]: \ (x, y) \longrightarrow [x\,,\,y]\,; \tag{1.100}$$

then the exterior derivative of a left-invariant 1-form α is given by

$$(d_{\mathfrak{g}}\,\alpha)(x, y) = -\alpha([x\,,\,y])\,, \tag{1.101}$$

so $d_{\mathfrak{g}}\,\omega = 0$ implies $\omega = 0$, or in other words there are no nonzero closed left-invariant 1-forms. Regarding the second equality, we have

$$(d_{\mathfrak{g}}\,\beta)(x, y, z) = -\beta([x\,,\,y], z) + \beta([x\,,\,z], y) - \beta([y\,,\,z], x)\,; \tag{1.102}$$

recall that $(\bigwedge \mathfrak{g}^*)_{\theta=0}$ is isomorphic to the cohomology algebra, so if $b_1 > 0$ there should be an invariant 2-form β; but invariance implies that

$$0 = (\theta^*(x)\beta)\,(y, z) = -\beta([x\,,\,y], z) - \beta(y, [x\,,\,z]) \tag{1.103}$$

so that

$$(d_{\mathfrak{g}}\,\beta)(x, y, z) = -\beta([y\,,\,z], x)\,. \tag{1.104}$$

Now the surjectivity of $[\,,\,]$ implies the result as before.

Regarding iii, we are in the hypotheses of proposition (1.15), so the Killing form gives origin to the invariant 3 form f. Finally, for iii, if the ground field is $\mathbb{C}$, the existence of 2 invariant elements ϕ, ψ in $\mathfrak{g} \otimes \mathfrak{g}^*$, which must be both nondegenerate for Schur's lemma, would imply that a linear combination $\phi + \lambda\psi$ is degenerate, which is possible only if $ker(\phi + \lambda\psi) = \mathfrak{g}$, always by Schur's lemma; so ϕ, ψ are proportional. Then the Killing form is the only invariant element in $\bigvee^2 \mathfrak{g} \subset \mathfrak{g}^* \otimes \mathfrak{g}$, up to scalars. The same argument holds for $\Gamma = \mathbb{R}$, considering that ψ, ϕ are self adjoint with respect to the Killing form, and so they have real eigenvalues. ■

The cohomology of a compact semisimple Lie group is the same as that of a product of spheres

$$S^{g_1} \times \cdots \times S^{g_r} \tag{1.105}$$

as stated in the following theorem:

Theorem 1.17. *The Poincaré polynomial of* $(\bigwedge \mathfrak{g}^*)_{\theta=0}$ *has the form*

$$f = (1 + t^{g_1}) \cdots (1 + t^{g_r}) \,, \tag{1.106}$$

where the exponents g_i *are odd and satisfy*

$$\sum_{i=0}^{r} g_i = \dim \mathfrak{g} \,. \tag{1.107}$$

We list here the Poincaré polynomials for the classical Lie groups (see [32, Vol. III]).

Examples. The Poincaré polynomial of $SU(n)$ is given by

$$\prod_{p=2}^{n} (1 + t^{2p-1}) \,; \tag{1.108}$$

for $SO(2n+1)$ and $SO(2n)$ we have

$$\prod_{p=1}^{n} (1 + t^{4p-1}) \quad \text{and} \quad (1 + t^{2n-1}) \prod_{p=2}^{n-1} (1 + t^{2p-1}) \,; \tag{1.109}$$

finally for $Sp(n)$

$$\prod_{p=1}^{n} (1 + t^{4p-1}) \,. \tag{1.110}$$

Chapter 2

Geometry of Grassmannians

In this chapter we discuss some basic facts about real oriented Grassmannians and then we introduce the twistor-type differential operator on the tautological bundle and on its normal bundle. Finally we discuss the cohomology of Grassmannians of 3-planes, concentrating on the 4^{th} de Rham class, in general and in the case of $\tilde{\mathbb{G}}_3(\mathbb{R}^6)$ and $\tilde{\mathbb{G}}_3(\mathbb{R}^8)$.

2.1 Bundles on Grassmannians

Consider an n-dimensional real vector space $\mathbb{R}^n$ equipped with an inner product $\langle\, ,\, \rangle$ and the Grassmannian $\tilde{\mathbb{G}}_k(\mathbb{R}^n)$. The dimension of the real Grassmannian is

$$\dim\, \tilde{\mathbb{G}}_k(\mathbb{R}^n) = \frac{n(n-1)}{2} - \frac{k(k-1)}{2} - \frac{(n-k)(n-k-1)}{2} \tag{2.1}$$

$$= \frac{2k(n-k)}{2} = k(n-k)\,. \tag{2.2}$$

As one can deduce from its homogeneous space presentation (see (1.17); a local chart around a point V is obtained in the following way: choose an orthonormal (ON from now on) basis $v_1, \cdots, v_k$ of V and an ON basis $w_1, \cdots, w_{n-k}$ of $V^\perp$; then the open set of k-planes V' in $\mathbb{R}^n$ which project isomorphically onto V, or in other words such that $V' \cap V^\perp = 0$, can be identified homeomorphically with the space of linear homomorphisms $\mathrm{Hom}(V, V^\perp)$, assigning to V' the unique homomorphism T such that

$$V' = \mathrm{span}\{v_i + T(v_i),\ i = 1, ..., k\}\,; \tag{2.3}$$

actually this local chart allows to identify in the same way $T_V\tilde{\mathbb{G}}_k(\mathbb{R}^d)$:

Proposition 2.1. *The tangent space of* $\tilde{\mathbb{G}}_k(\mathbb{R}^n)$ *at* V *can also be identified with* $\mathrm{Hom}(V, V^{\perp})$.

Proof. Consider two ON bases of V and $V^{\perp}$ as before and an element T_{ij} such that $T_{ij}(v_i) = w_j$; consider the curve

$$\gamma(t) = \mathrm{span}\{v_1, \cdots, v_i + t\, T_{ij}(v_i), \cdots v_k\} \tag{2.4}$$
$$= \mathrm{span}\{v_1, \cdots, v_i + t\, w_i, \cdots v_k\}\,; \tag{2.5}$$

the derivative at V is given by $\gamma'(t)|_{t=0} = (0, \cdots, w_j, \cdots, 0)$; clearly each T_{ij} gives rise to a curve through V with a linearly independent tangent vector at V, so for dimensional reasons the result follows.■

Therefore the tangent space at V will be identified with $V^* \otimes V^{\perp}$; the presence of a metric on V, induced from the ambient space $\mathbb{R}^n$, will allow us to write $V \otimes V^{\perp}$, using contraction via the metric for the isomorphism $V \cong V^{\perp}$.

We will be interested to study some differential operators and sections of vector bundles on $\tilde{\mathbb{G}}_k(\mathbb{R}^n)$, so we start by describing the natural objects induced by the euclidean structure of $\mathbb{R}^n$. We have the splitting of the trivial bundle $\tilde{\mathbb{G}}_k(\mathbb{R}^n) \times \mathbb{R}^n$ in two subbundles: the tautological one and its orthogonal complement:

$$\begin{tikzcd} \mathbf{V} \oplus \mathbf{V}^{\perp} \arrow[r, "\cong"] \arrow[dr, "p_1"'] & \tilde{\mathbb{G}}_k(\mathbb{R}^n) \times \mathbb{R}^n \arrow[d, "p_2"] \\ & \tilde{\mathbb{G}}_k(\mathbb{R}^n) \end{tikzcd} \quad . \tag{2.6}$$

Here the $\perp$ is given by the metric on $\mathbb{R}^n$; the presence of this metric allows to define coonnections on these two subbundles just by composition of the d with the two projections π and $\pi^{\perp}$. For example

$$\nabla^{\mathbf{V}} s = \pi\, d\, s\,, \tag{2.7}$$

where $s \in \Gamma(\mathbf{V})$ and d is the derivation in $\mathbb{R}^n$; to prove that this is a connection we have to show that $\nabla^{\mathbf{V}} as = (da)s + a\nabla^{\mathbf{V}} s$ with a a function:

$$\nabla^{\mathbf{V}} as = \pi d(as) = \pi\left((da)s + a(ds)\right) \tag{2.8}$$
$$= (da)s + a\pi(ds) = (da)s + a\nabla^{\mathbf{V}} s \tag{2.9}$$

as required. Moreover this connection is compatible with the metric induced on the fibres of $\mathbf{V}$ by their ambient space $\mathbb{R}^n$: in fact if $s, t \in \Gamma(\mathbf{V})$ and $X \in T_V\tilde{\mathbb{G}}_k(\mathbb{R}^n)$ we have

$$\begin{aligned} X\langle s\,,\,t\rangle =&\langle Xs\,,\,t\rangle + \langle s\,,\,Xt\rangle = \langle \pi Xs\,,\,t\rangle + \langle s\,,\,\pi Xt\rangle \\ =&\langle \nabla^{\mathbf{V}}s\,,\,t\rangle + \langle s\,,\,\nabla^{\mathbf{V}}t\rangle\,. \end{aligned}$$

On the other hand we obtain the corresponfing II fundamental forms projecting in the opposite way:

$$II:\ \Gamma(\mathbf{V}) \longrightarrow \Gamma(T^* \otimes \mathbf{V}^\perp)$$

which sends s to $\pi^\perp ds$; analogously $II^\perp$ sends $s \in \Gamma(\mathbf{V}^\perp)$ in πds. These last two are both tensors, in fact if $s \in \Gamma(\mathbf{V}^\perp)$ for example and a is a function, we get

$$\pi d(as) = \pi(d(a)s + ad(s)) = \pi a d(s) = a\pi ds \tag{2.10}$$

so that we can think to $II^\perp$ as a section of the bundle

$$\mathbf{V}^\perp \otimes \Big(T^*\tilde{\mathbb{G}}_k(\mathbb{R}^n) \otimes \mathbf{V}\Big) \cong \mathrm{Hom}\Big(\mathbf{V}^\perp\,,\,T^*\tilde{\mathbb{G}}_k(\mathbb{R}^n) \otimes \mathbf{V}\Big) \tag{2.11}$$

(identifying $\mathbf{V} \cong \mathbf{V}^*$ via the metric); if it turns out to be injective on every fibre, it determines the immersion of $\mathbf{V}^\perp$ as a subbundle of $T^*\tilde{\mathbb{G}}_k(\mathbb{R}^n)\otimes\mathbf{V}$; we will prove this in the next section, where we shall also introduce a family of elements in $\Gamma(\mathbf{V})$ and $\Gamma(\mathbf{V}^\perp)$ which will be object of interest.

2.2 Sections obtained by projection

We want to use the standard connections and tensors introduced in the previous section to construct new differential operators on the tautological bundle $\mathbf{V}$ and on its orthogonal complement $\mathbf{V}^\perp$. First of all, given an element $A \in \mathbb{R}^n$ we can associate to it two sections of the bundles $\mathbf{V}$ and $\mathbf{V}^\perp$ just using the projections: $s_A = \pi A$ and $s_A^\perp = \pi^\perp A$ with $A = s_A + s_A^\perp$; now A is constant,

$$0 = dA = ds_A + ds_A^\perp \tag{2.12}$$

so that

$$ds_A = -ds_A^\perp\,; \tag{2.13}$$

so in the language developed before

$$\nabla^{\mathbf{V}}s_A = \pi ds_A = -\pi ds_A^\perp = -II^\perp s_A^\perp\,.$$

These equations imply that

$$d\,s_A = -II^{\perp} s_A^{\perp} + II\, s_A\,. \tag{2.14}$$

We now prove the injectivity of $II^{\perp}$:

Proposition 2.2. *The section $II^{\perp}$ is injective on the fibres.*

Proof. To prove the assertion we need to show that, fixed a point $V \in \tilde{\mathbb{G}}_k(\mathbb{R}^n)$, for every $w_0 \in V^{\perp}$ exists at least an element in $T_V\tilde{\mathbb{G}}_k(\mathbb{R}^n)$ such that $II^{\perp}(w_0)$ applied to it has nonzero V projection. Without loss of generality we can impose $\|w_0\| = 1$; then we choose any $v \in V$ with $\|v\| = 1$ and our candidate for the proof is $v \otimes w_0$. In fact consider the curve in the Grassmannian

$$V(\theta) = \langle \sin\theta\, w_0 + \cos\theta\, v, v_2, v_3 \rangle$$

where the two elements v_2 and v_3 are such that v, v_2, v_3 is an orthonormal basis of $V(0) = V$; the tangent vector at $\theta = 0$ of this curve is $v \otimes w_0$. Now we need to find a section $s \in \Gamma(\mathbf{V}^{\perp})$ such that $s(V) = w_0$, and then differentiate it along the curve $V(\theta)$; such a section is provided by $s_{w_0}^{\perp}$, which restdicted to $V(\theta)$ becomes

$$s_{w_0}^{\perp}(\theta) = \sin\theta\,(\sin\theta\, w_0 + \cos\theta\, v)$$

and

$$\frac{d}{d\theta}\, s_{w_0}^{\perp}(\theta)\,|_{\theta=0} = \left(\cos^2\theta - \sin^2\theta\right)\, v + 2\sin\theta\cos\theta\, w_0\,|_{\theta=0} = v$$

so that the V projection coincides with the chosen v and is not zero.■

Observation. We have proved something more: fixed w_0, the image

$$II^{\perp}(w_0)$$

is nonzero on the elements $v \otimes w_0$ for *any* $v \in V$ nonzero.

For convenience we will put together the homomorphisms II and $II^{\perp}$ in the following way:

$$i:\ \Gamma(\mathbb{R}^n) \longrightarrow \Gamma(T^* \otimes \mathbb{R}^n\ ) \tag{2.15}$$

acting like:

$$i(S) = II(\pi S) - II^{\perp}(\pi^{\perp} S)\,. \tag{2.16}$$

in a way which is consistent with equation (2.14); in this way we have

$$ds_A = i(A) \tag{2.17}$$

and

$$ds_A^{\perp} = -i(A)\,. \tag{2.18}$$

The image of $II^{\perp}$ corresponds to elements of the type

$$\sum_{i=1}^{3} \lambda\, \sigma \otimes v_i \otimes v_i \tag{2.19}$$

with $\sigma \in \mathbf{V}^{\perp}$ and $\lambda \in \mathbb{C}$; in fact if we consider the decomposition as $SO(k)$ modules we get

$$\mathbf{V}^{\perp} \otimes \mathbf{V} \otimes \mathbf{V} \cong \underbrace{\mathbf{V}^{\perp} \otimes \mathbb{R}}_{\alpha} + \underbrace{\mathbf{V}^{\perp} \otimes (....)}_{\beta} \tag{2.20}$$

so that precisely one copy of $\mathbf{V}^{\perp}$ appears: once that we find a nontrivial $SO(k) \times SO(n-k)$-equivariant way of putting $\mathbf{V}^{\perp}$ inside this bundle, it is injective (by the Schur Lemma) and essentially unique (up automorphisms of modules); now the expression (2.19) provides the needed copy.

Exactly the same argument using the decomposition as $SO(n-k)$ modules of the bundles says that we can find exactly one copy of $\mathbf{V}$ inside $\mathbf{V} \otimes \mathbf{V}^{\perp} \otimes \mathbf{V}^{\perp} \cong T^*\tilde{\mathbb{G}}_k(\mathbb{R}^n) \otimes \mathbf{V}^{\perp}$. The reason is essentially that only one trivial factor exists in the decomposition ot $V \otimes V$, where V is the standard $SO(n)$ representation for any n. We want now to be more precise about these statements, and calculate explicitly the value of λ, as we see in the next proposition (the tensor products are omitted).

Proposition 2.3. *Let $A \in \mathbb{R}^n$ so that $A = u + y$ with $u \in V$ and $y \in V^{\perp}$ at the point V; let v_j and w_i denote the basis elements of V and $V^{\perp}$ at V; then*

$$d\, s_A|_V = \sum_i y v_i v_i + \sum_j u w_j w_j \tag{2.21}$$

and obviously

$$d\, s_A^{\perp}|_V = -\sum_i y v_i v_i - \sum_j u w_j w_j\,; \tag{2.22}$$

so $\lambda = 1$.

Proof. As usual we differentiate along a curve passing through V and with tangent vector $X = v_1 w_1$; we choose the curve $\mathrm{span}\{v_1 + t w_1, v_2, ... v_k\}$;let $u = \sum a_i v_i$ and at the point $V(t)$

$$A = b_1(v_1 + t w_1) + b_2 v_2 + b_3 v_3 + y'$$

with $y' \in V(t)^{\perp}$, but at $V(0)$

$$A = \sum a_i v_i + y \tag{2.23}$$

so that doing the inner products of A with the v_is we get the equations

$$\begin{aligned} a_1 &= b_1 + \langle v_1 \, , \, y' \rangle \\ a_2 &= b_2 \\ \ldots \\ a_k &= b_k \, ; \end{aligned}$$

but on the other hand

$$\langle y' \, , \, v_1 + t w_1 \rangle = 0 \, ; \tag{2.24}$$

the inner product $\langle A \, , \, w_1 \rangle$ and multiplication by t gives the equation

$$t \langle y \, , \, w_1 \rangle = t^2 b_1 + t \langle y' \, , \, w_1 \rangle \tag{2.25}$$

where the term in t^2 can be omitted as we are interested in the 1 st order terms; we notice that the left hand term is independent of t; what we get is (forgetting order higher than 1)

$$s_A(t) = (a_1 - t\langle y \, , \, w_1 \rangle)(v_1 + t w_1) + b_2 v_2 + b_3 v_3 \tag{2.26}$$

so that

$$X(s_A) = \frac{d}{dt}(s_A(t))|_{t=0} = \langle y \, , \, w_1 \rangle v_1 + a_1 w_1 \, ; \tag{2.27}$$

so varying the tangent vectors we obtain

$$\begin{aligned} d\, s_A &= \sum_{i,j} v_i w_j \otimes (\langle y \, , \, w_j \rangle v_i + a_j w_j) \\ &= \sum_i y v_i v_i + \sum_j u w_j w_j \end{aligned}$$

as claimed. An analogous calculation for $s_A^{\perp}$ gives

$$d\, s_A^{\perp} = -\sum_i y v_i v_i - \sum_j u w_j w_j \tag{2.28}$$

as expected from equation (2.18). ■

Observation. These expressions imply that

$$II(v) = \sum_j v w_j w_j \tag{2.29}$$

and

$$II^{\perp}(w) = -\sum_i w v_i v_i \,, \tag{2.30}$$

with $v \in \Gamma(\mathbf{V})$ and $w \in \Gamma(\mathbf{V}^{\perp})$, in accordance with (2.16); the opposite sign is consistent with the equation

$$0 = d\langle v \,,\, w\rangle = \langle II(v) \,,\, w\rangle + \langle v \,,\, II^{\perp}(w)\rangle$$

which expresses the fact that II and $II^{\perp}$ are adjoint linear operators.

2.3 The Twistor equation

Proposition 2.3 shows that $\nabla^{\mathbf{V}} s_A$ is of the form seen in (2.19), or alternatively that if we call p the projection on the β summand in the decomposition (2.20) and define $D \equiv p \circ \nabla^{\mathbf{V}}$, the section s_A satisfies the *twistor-type* equation

$$\boxed{D\, s_A = 0\,.} \tag{2.31}$$

Symmetrically we can define another operator $D^{\perp}$ such that

$$\boxed{D^{\perp}\, s_A^{\perp} = 0\,.} \tag{2.32}$$

Now if we choose an orthonormal basis $e_1, ..., e_d$ of $\mathbb{R}^n$, every section S of the flat bundle $\tilde{\mathbb{G}}_k(\mathbb{R}^n) \times \mathbb{R}^n$ is nothing else than a d-tuple of functions

$$f_j : \tilde{\mathbb{G}}_k(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \tag{2.33}$$

so that

$$S = \sum f_j e_j \,; \tag{2.34}$$

applying the exterior derivative on $\mathbb{R}^n$ (which is a connection on the flat bundle) we obtain

$$dS = \sum df_j \otimes e_j \tag{2.35}$$

and if $1 \wedge i$ denotes an element in $\mathrm{Hom}\big(T^* \otimes \mathbb{R}^n, (\bigotimes^2 T^*) \otimes \mathbb{R}^n\big)$ (where $\mathbb{R}^n = \tilde{\mathbb{G}}_k(\mathbb{R}^n) \times \mathbb{R}^n$ and $T^* = T^*\tilde{\mathbb{G}}_k(\mathbb{R}^n)$ to simplify the notation) acting in the obvious way, we obtain

$$1 \wedge i\,(dS) = \sum df_j \wedge i\,(e_j)\,; \tag{2.36}$$

on the other hand

$$d\sum f_j\, i(e_j) = \sum df_j \wedge i(e_j) + f_j\, di(e_j)\,, \tag{2.37}$$

so if we can show that

$$di(e_j) = 0 \quad \forall j \tag{2.38}$$

we obtain the commutativity of the following diagram:

$$\begin{tikzcd} & \mathbb{R}^n \arrow[r, "d"] \arrow[d, "i"'] & T^* \otimes \mathbb{R}^n \arrow[d, "1\wedge i"] \\ \mathbb{R}^n \arrow[r, "d"] & T^* \otimes \mathbb{R}^n \arrow[r, "d"] & \Lambda^2 T^* \otimes \mathbb{R}^n \end{tikzcd} \quad ; \tag{2.39}$$

but equation (2.17) implies:

$$di(e_j) = dds_{e_j} = 0\,, \tag{2.40}$$

because the e_j are constant. Previously we showed that i is an injective map (because II and $II^\perp$ are, see Proposition 2.2); if we can show that also $1\wedge i$ is injective (and it happens to be in most part of cases, as we will see) looking at diagram (2.39) we can deduce the following facts: if $s \in \Gamma(\mathbf{V})$ satisfies $Ds = 0$ this is equivalent to say that $ds = i(s+\sigma)$ for some $\sigma \in \Gamma(\mathbf{V}^\perp)$; but obviously $dds = 0$, so $d(s+\sigma)$ must be 0 too, so a constant element $A \in \mathbb{R}^n$. This implies the main result of this Section:

Theorem 2.4. *A section $s \in \Gamma(\mathbf{V})$ satisfies the twistor equation $Ds = 0$ if and only if exists another section $\sigma \in \Gamma(\mathbf{V}^\perp)$ such that $s + \sigma = A$ is a constant section of $\mathbb{R}^n$, provided $k > 1$ and $n - k > 1$.*

In other words sections of type s_A are the only solutions of equation (2.31), under these hypotheses.

Observation. This means that the two equations (2.31) and (2.32) impose very strong conditions on the sections of $\mathbf{V}$ and $\mathbf{V}^\perp$, and one of them is sufficient, for instance, to reconstruct an element in $\mathbb{R}^n$ from its projection on $\mathbf{V}$. All theese considerations are obtained only using differentiation, therefore they have an essencially local nature; this implies that the spaces of sections in ker D are finite-dimensional even locally, a fact that distinguishes this type of operators from other well-known such as the Laplacian Δ.

The missing piece to prove Theorem 2.4 is injectivity of $1 \wedge i$. To prove that we start defining another map which will be useful in the sequel:

$$c : \Gamma(T^* \otimes \mathbb{R}^n) \longrightarrow \Gamma(\mathbb{R}^n) \tag{2.41}$$

acting as a contraction in the following way:

$$c\Big(\sum_{ijk} a^{ijk}\ v_i w_j v_k + \sum_{lmo} b^{lmo}\ w_l v_m w_o\Big) = \sum_{ij} a^{iji}\ w_j + \sum_{lm} b^{lml}\ v_m\,. \tag{2.42}$$

The same map acts also on $\tau \in (\bigotimes^q T^*)\otimes\mathbb{R}^n$ in the following way: if $\tau = \tau'\otimes S$ with $\tau' \in \bigotimes^{q-1} T^*$ and $S \in T^* \otimes \mathbb{R}^n$ then

$$c(\tau) = \tau' \otimes c(S) \tag{2.43}$$

and then extending linearly.
We are now in position to prove the previously stated assertion, which concludes the proof of Theorem 2.4:

Lemma 2.5. *The map $1 \wedge i$ is injective, provided $k > 1$ and $n - k > 1$.*

Proof. Given two bases v^i of V and w^j of $V^\perp$ an element in $T^* \otimes \mathbb{R}^n$ is described by

$$\tau = \sum_{ijh} a^{ijh}\ v_i w_j v_h + \sum_{lmo} b^{lmo} v_l w_m w_o\,; \tag{2.44}$$

now we will prove that $c \circ 1 \wedge i$ is injective, so that $1 \wedge i$ must be.
So we get

$$\begin{aligned} 1 \wedge i\ (\tau) &= \sum_{ijh\mu} a^{ijh}\,(v_i w_j \wedge v_h w_\mu) w_\mu + \sum_{lmo\nu} b^{lmo}(v_l w_m \wedge w_o v_\nu) v_\nu \\ &= \sum_{ijh\mu} a^{ijh}\,(v_i w_j \otimes v_h w_\mu - v_h w_\mu \otimes v_i w_j) w_\mu \\ &\quad + \sum_{lmo\nu} b^{lmo}(v_l w_m \otimes w_o v_\nu - v_o w_\nu \otimes w_l v_m) v_\nu \end{aligned}$$

and applying the contraction

$$\begin{aligned} c(1 \wedge i\ (\tau)) &= \sum_{ijh\mu} a^{ijh}\,(v_i w_j \otimes v_h - v_h w_\mu \otimes v_i \delta^j_\mu) \\ &\quad + \sum_{lmo\nu} b^{lmo}(v_l w_m \otimes w_o - v_o w_\nu \otimes w_l \delta^m_\nu)\,; \end{aligned}$$

now imposing that it's zero, we get the following couples of equations:

$$\begin{cases} (n-k)\, a^{ijh} - a^{hji} = 0 \\ (n-k)\, a^{hji} - a^{ijh} = 0 \end{cases}$$

and

$$\begin{cases} k\, b^{lmo} - b^{oml} = 0 \\ k\, b^{oml} - b^{lmo} = 0 \end{cases}$$

which imply

$$(n-k)^2\, a^{ijh} = a^{ijh}$$

and

$$k^2\, b^{lmo} = b^{lmo}$$

which are absurd if $k > 1$ and $n - k > 1$. ■

2.4 Invariant forms on Grassmannians

Cohomology of a compact symmetric space M can be computed using invariant forms: in fact it can be shown in the same way as for compact Lie groups that

$$H^*(M) \cong H^*_I(M)\,, \tag{2.45}$$

in other words the complex of invariant forms gives rise to a cohomology ring $H^*_I(M)$ which is isomorphic to the DeRham one. Moreover:

Proposition 2.6. *For a symmetric space M we have*

$$H^*(M) \cong (\bigwedge \mathfrak{m})_{\,\theta=0}\,. \tag{2.46}$$

Proof. Thanks to the relation

$$[\mathfrak{m}\,,\,\mathfrak{m}] \subset \mathfrak{h} \tag{2.47}$$

the differential d applied to invariant forms is identically zero; then

$$H^k_I(M) \cong (\bigwedge^k \mathfrak{h}^{\perp})_{\,\theta=0}\,; \tag{2.48}$$

the result follows from equation (2.45). ■

Example 2.1. We can calculate directly the cohomology of some low-dimensional symmetric space using equation (2.48): for example $\mathbb{G}_3(\mathbb{R}^6)$ has the isotropy representation isomorphic to $\Sigma^2_+ \otimes \Sigma^2_-$, where $\Sigma^2_\pm$ are the spin representations of the factors $SU(2)_\pm$ in $SO(4) \cong SU(2)_+ SU(2)_-$; then if $2t, 0, -2t$ and $2s, 0, -2s$ are the weights of the corresponding $\mathfrak{sl}(2,\mathbb{C})$ representations, combining them and then grouping the weight spaces properly, we can decompose the complexified exterior algebra at a point in irreducible summands in the following way (thanks to the metric we forget about distinguishing between tangent and cotangent space):

$$\bigwedge^2 \Sigma_+^2 \otimes \Sigma_-^2 \cong \Sigma_+^4 \otimes \Sigma_-^2 + \Sigma_+^2 \otimes \Sigma_-^4 + \Sigma_+^2 + \Sigma_-^2 \, ; \tag{2.49}$$

$$\bigwedge^3 \Sigma_+^2 \otimes \Sigma_-^2 \cong \Sigma_+^6 + \Sigma_-^6 + \Sigma_+^4 \otimes \Sigma_-^4 + \Sigma_+^4 \otimes \Sigma_-^2 + \tag{2.50}$$

$$\Sigma_+^2 \otimes \Sigma_-^4 + \Sigma_+^2 \otimes \Sigma_-^2 + \Sigma_+^2 + \Sigma_-^2 \, ; \tag{2.51}$$

$$\bigwedge^4 \Sigma_+^2 \otimes \Sigma_-^2 \cong \Sigma_+^6 \otimes \Sigma_-^2 + \Sigma_+^2 \otimes \Sigma_-^6 + \Sigma_+^4 \otimes \Sigma_-^4 + \Sigma_+^4 \otimes \Sigma_-^2 + \tag{2.52}$$

$$\Sigma_+^2 \otimes \Sigma_-^4 + \Sigma_+^4 + \Sigma_-^4 + 2\,\Sigma_+^2 \otimes \Sigma_-^2 + \mathbb{C} \, . \tag{2.53}$$

In conclusion we have $b_1 = b_2 = b_3 = b_6 = b_7 = b_8 = 0$ and $b_0 = b_4 = b_5 = b_9 = 1$. With a bit more of work we can obtain an explicit expression for the invariant 4-form Φ that generates $H^4(\mathbb{G}_3(\mathbb{R}^6))$; in fact consider two bases of the $\Sigma_\pm^2$ given by $\{Y_+, H_+, X_+\}$ and $\{Y_-, H_-, X_-\}$ satisfying

$$[Y_\pm, H_\pm] = 2Y_\pm \, , \; [Y_\pm, X_\pm] = -H_\pm \, , \; [H_\pm, X_\pm] = 2X_\pm \, ; \tag{2.54}$$

then we introduce the following notation:

$$\alpha_1 := X_+ \otimes X_- \, , \; \alpha_2 := X_+ \otimes H_- \, , \; \alpha_3 := X_+ \otimes Y_- \quad \text{etc.}, \tag{2.55}$$

which are a basis for $T_x\mathbb{G}_3(\mathbb{R}^6) \otimes \mathbb{C}$; then restricting the representation to the Cartan subalgebra inside $\mathfrak{sl}(2,\mathbb{C})_+ \times \mathfrak{sl}(2,\mathbb{C})_-$ we have the following correspondence of weights:

$$\begin{array}{|c|c|c|} \hline \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \alpha_4 & \alpha_5 & \alpha_6 \\ \hline \alpha_7 & \alpha_8 & \alpha_9 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline 2t+2s & 2t & 2t-2s \\ \hline 2s & 0 & -2s \\ \hline -2t+2s & -2t & -2t-2s \\ \hline \end{array} \quad ;$$

$\bigwedge^4 \Sigma_+^2 \otimes \Sigma_-^2$ is 126-dimensional, nevertheless the invariant form must be contained in the 10-dimensional subspace of weight 0, so we restrict our attention to this last one; a basis is given by

$$\beta_1 := \alpha_{1289} \, , \; \beta_2 := \alpha_{2378} \, , \; \beta_3 := \alpha_{1469} \, , \; \beta_4 := \alpha_{3647} \, , \; \beta_5 := \alpha_{2846} \, , \tag{2.56}$$

$$\beta_6 := \alpha_{1937} \, , \; \beta_7 := \alpha_{1568} \, , \; \beta_8 := \alpha_{3548} \, , \; \beta_9 := \alpha_{7526} \, , \; \beta_{10} := \alpha_{9524} \, ; \tag{2.57}$$

so the invariant form is a linear combination

$$\Omega = \sum_{i=1}^{10} a_i \beta_i \, . \tag{2.58}$$

To obtain the coefficients we impose the condition $ad_Y\Phi = 0,$ which implies the following set of equations:

$$\begin{cases} -2a_1 + a_6 = 0\,,\ -2a_2 + a_6 = 0\,,\ a_3 - 2a_5 + 2a_{10} = 0, \\ a_3 - 2a_7 = 0\,,\ a_4 + 2a_9 = 0\,,\ a_4 - 2a_8 + 2a_5 = 0, \\ a_5 + a_9 + a_7 = 0\,,\ a_8 + a_{10} = 0 \end{cases}$$

with solutions

$$\begin{aligned} &a_1 = a_2 = a_6/2\,, \quad a_3 = a_4 = -2a_{10}\,, \quad a_5 = 0\,, \\ &a_7 = a_8 = -a_{10}\,, \quad a_9 = a_{10}\,. \end{aligned}$$

Imposing the additional condition $ad_{Y'}\Phi = 0$ we have the following system:

$$\begin{cases} a_1 - 2a_5 - 2a_{10} = 0\,,\ a_1 + 2a_7 = 0\,,\ a_2 + 2a_8 = 0, \\ a_2 + 2a_5 - 2a_9 = 0\,,\ 2a_3 + a_6 = 0\,,\ -2a_4 - a_6 = 0, \\ a_5 - a_7 + a_8 = 0\,,\ a_9 - a_{10} = 0 \end{cases}$$

and the intersection of the solutions gives, imposing $a_1 = 1,$

$$a_1 = 1, a_2 = 1, a_3 = -1, a_4 = -1, a_5 = 0, a_6 = 2, \tag{2.59}$$

$$a_7 = 1/2, a_8 = -1/2, a_9 = 1/2, a_{10} = 1/2\,. \tag{2.60}$$

We want now to go back to the real exterior algebra, so we introduce the real structures of $\Sigma^2_\pm$: the real subrepresentations preserved by the real compact groups $SU(2)_\pm$ have as orthonormal bases $\{e_1,\ e_2,\ e_3\}$ and $\{f_1,\ f_2,\ f_3\}$ satisfying

$$[e_1\,,\ e_j] = \sqrt{2}e_k \qquad [f_1\,,\ f_j] = \sqrt{2}f_k \tag{2.61}$$

for ijk consequent indices (modulo 3); the relations with the bases in the complexified representations are expressed by:

$$X = \frac{1}{\sqrt{2}}(e_1 - \imath e_2)\,,\ Y = -\frac{1}{\sqrt{2}}(e_1 + \imath e_2)\,,\ H = -\imath\sqrt{2}e_3\,, \tag{2.62}$$

and analogous equations for X', H', Y'; definig a basis of the real tangent space as

$$w_1 := e_1 \otimes f_1\,,\ w_2 := e_1 \otimes f_2\,,\ w_3 := e_1 \otimes f_3 \quad \text{etc.}, \tag{2.63}$$

and substituting in the expression (2.58) using relations (2.62), we obtain, up to a constant, the form

$$\Omega = w^{3652} - w^{3614} - w^{5478} + w^{1278} + w^{5214} \tag{2.64}$$

$$+ w^{1937} + w^{5968} + w^{2938} + w^{4967}\,. \tag{2.65}$$

In this way we can in theory always obtain the Poincaré polynomial and an explicit expression for the corresponding invariant forms of any compact symmetric space, but of course calculations become more messy as dimension increases. Nevertheless in the case of Grassmannians of 3-planes in $\mathbb{R}^n$, which are our main object of interest, some more considerations can lead to determine invariant 4-forms in a more straightforward way: in fact the complexified isotropy representation in that case is

$$\Sigma^2 \otimes W = (\mathbb{R}^3 \otimes \mathbb{R}^{n-3}) \otimes \mathbb{C} \tag{2.66}$$

with Σ^2 and $W = \mathbb{C}^{n-3}$ the standard representations of $SO(3)$ and $SO(n-3)$; then

$$\bigwedge^2(\Sigma^2 \otimes W) \supset \big(\bigwedge^2 \Sigma^2\big) \otimes S^2W \supset \Sigma^2 \otimes \langle k \rangle \cong \Sigma^2 \,, \tag{2.67}$$

where k denotes the invariant symmetric tensor corresponding to the Killing form; therefore we have

$$\bigwedge^4(\Sigma^2 \otimes W) \supset \Sigma^0 \tag{2.68}$$

spanned by

$$\Omega = \sum_{i=1}^{3} \omega_i \wedge \omega_i \,, \tag{2.69}$$

where the ω_i are a basis for $\big(\bigwedge^2 \Sigma^2\big) \otimes S^2W$; we will call this form the *Nagatomo 4-form* ([64]). Thus in this type of Grassmannians we always have an invariant 4 form, and hence $b_4 \geq 1$ (see [32] and the observation below). We can describe the Nagatomo 4-form of $\mathbb{G}_3(\mathbb{R}^n)$ for any n in the following way: we fix an ON basis e_1, e_2, e_3 of $\mathbb{R}^3 = [\Sigma^2]$ as before, and an ON basis $f_1, \cdots f_k$ of $\mathbb{R}^k$, with $k = n-3$; therefore the tangent space $T_x\mathbb{G}_3(\mathbb{R}^k)$ is generated by

$$w_1 = e_1 \otimes f_1, \; \cdots \;, \; w_{3k} = e_3 \otimes f_k \,; \tag{2.70}$$

from now on we will adopt the convention of separating indices by a comma, to avoid confusion due to double digits indices. In these terms we have:

Proposition 2.7. *Let $w_1, \cdots, w_{3k}$ the ON basis of the isotropy representation of $\mathbb{G}_3(\mathbb{R}^n)$ given in (2.70); then the Nagatomo 4-form Ω in terms of this basis is*

$$\Omega = \sum_{l=0}^{2} \sum_{i=1+lk}^{k+lk} \sum_{j=i+1}^{k+lk} w^{i,\, i+k,\, j,\, j+k} \tag{2.71}$$

where indices are cyclic modulo $3k$.

Proof. In terms of the given basis we have

$$\omega_1 = e^{12} \otimes (\sum_{i=1}^{k} f^i \otimes f^i) = \sum_{i=1}^{k} w^{i,\,i+k} \tag{2.72}$$

$$\omega_2 = e^{23} \otimes (\sum_{i=1}^{k} f^i \otimes f^i) = \sum_{i=k+1}^{2k} w^{i,\,i+k} \tag{2.73}$$

$$\omega_3 = e^{31} \otimes (\sum_{i=1}^{k} f^i \otimes f^i) = \sum_{i=2k+1}^{3k} w^{i,\,i+k} \tag{2.74}$$

and referrng to (2.69) we obtain

$$\omega_1 \wedge \omega_1 = \sum_{i=1}^{k} \sum_{j=i+1}^{k} w^{i,\,i+k,\,j,\,j+k} \tag{2.75}$$

up to a constant; analogous expressions are obtained for ω_2, ω_3 and the assertion follows summing together. ■

Observation. In [32, vol. III] we can find tables with the Poincaré polynomials of all classical symmetric spaces. In particular for $\tilde{\mathbb{G}}_3(\mathbb{R}^n)$ we always have $b_4 = 1$, excepted for $n = 7$, where $b_4 = 2$. Therefore then Nagatomo form is the only invariant 4-form on this type of Grassmannians, for $n \neq 7$.

In the case of $\mathbb{G}_3(\mathbb{R}^6)$ we can obtain again in this way the form Ω as in (2.64), up to a constant.

In the case of $\mathbb{G}_3(\mathbb{R}^8)$, which will also be of interest for us, we have $k = 5$ and a basis of $\Sigma^2 \subset \bigwedge^2(\mathbb{R}^3 \otimes \mathbb{R}^5)$ is given by

$$\omega_1 = e^{12} \otimes (\sum_{i=1}^{15} f^i \otimes f^i) = w^{1,6} + w^{2,7} + w^{3,8} + w^{4,9} + w^{5,10} \tag{2.76}$$

$$\omega_2 = e^{23} \otimes (\sum_{i=1}^{15} f^i \otimes f^i) = w^{6,11} + w^{7,12} + w^{8,13} + w^{9,14} + w^{10,15} \tag{2.77}$$

$$\omega_3 = e^{31} \otimes (\sum_{i=1}^{15} f^i \otimes f^i) = w^{11,1} + w^{12,2} + w^{13,3} + w^{14,4} + w^{15,5} \tag{2.78}$$

In this case the invariant 4-form Ω is given by

$$\begin{aligned}\Omega = \sum_{i=1}^{3} \omega_i \wedge \omega_i = {} & w^{1,6,2,7} + w^{1,6,3,8} + w^{1,6,4,9} + w^{1,6,5,10} + w^{2,7,3,8} \\ & + w^{2,7,4,9} + w^{2,7,5,10} + w^{3,8,4,9} + w^{3,8,5,10} + w^{4,9,5,10} \\ & + w^{6,11,7,12} + w^{6,11,8,13} + w^{6,11,9,14} + w^{6,11,10,15} + w^{7,12,8,13} \\ & + w^{7,12,9,14} + w^{7,12,10,15} + w^{8,13,9,14} + w^{8,13,10,15} + w^{9,14,10,15} \\ & + w^{11,1,12,2} + w^{11,1,13,3} + w^{11,1,14,4} + w^{11,1,15,5} + w^{12,2,13,3} \\ & + w^{12,2,14,4} + w^{12,2,15,5} + w^{13,3,14,4} + w^{13,3,15,5} + w^{14,4,15,5} \,.\end{aligned} \tag{2.79}$$

Chapter 3

Functionals on $\tilde{\mathbb{G}}_3(\mathfrak{g})$

In this Chapter we consider Grassmannians of 3-planes of a compact Lie algebra $\mathfrak{g}$: the richness of these latter objects allows to introduce more structure which would not be possible with a simple vector space $\mathbb{R}^n$; in particular the standard 3-form f induces a well-known Morse-Bott function on $\tilde{\mathbb{G}}_3(\mathfrak{g})$, whose gradient flow identifies submanifolds which carry a Quaternionic-Kähler structure, even if this aspect will not be considered until Chapter 5. We shall define here a new functional g which will have significance regarding the quaternionic structure. We compare the Hessians of f and g at the critical points for f, finding explicit expressions for the eigenvalues of $\operatorname{Hess} g$. An early question that presented itself was whether the critical points of g coincide with those of f: a negative answer is provided Section 3.4, presenting explicit examples, and more systematically in Section 3.5.

3.1 The functional f

Consider the Grassmannian of oriented three-planes of a compact semisimple Lie algebra

$$\tilde{\mathbb{G}}_3(\mathfrak{g})$$

equipped with the Riemannian metric coming from the Ad-invariant inner product on the Lie algebra $\langle\ ,\ \rangle$ (minus the Killing form); from now on we will write $\mathbb{G}_3(\mathfrak{g})$ instead of $\tilde{\mathbb{G}}_3(\mathfrak{g})$ to simplify the notation, or simply $\mathbb{G}_3$. The closed 3-form

$$\rho(\langle\ ,\ \rangle) = \langle\, [x, y]\, , z\,\rangle, \tag{3.1}$$

obtained from the metric through the Cartan map induces a function f on the Grassmannian (see 1.98): in fact if v_1, v_2, v_3 is an orthonormal basis of the 3-plane V with respect to the inner product then we put

$$f(V) := \langle [v_1\,,\,v_2]\,,\,v_3 \rangle \tag{3.2}$$

which is well defined: in fact the space V is isomorphic to $[\Sigma^2]$ as an $SO(3)$ representation, so that $\bigwedge^3 V \cong \bigwedge^3 [\Sigma^2] \cong [\Sigma^0]$, and any element (proportional to the volume form) is $SO(3)$ invariant.

Let us calculate the gradient vector field obtained as the dual of df via the metric: we define

$$w_1 = [v_2\,,\,v_3]^{\perp} \tag{3.3}$$
$$w_2 = [v_3\,,\,v_1]^{\perp} \tag{3.4}$$
$$w_3 = [v_1\,,\,v_2]^{\perp} \tag{3.5}$$

so that

$$[v_1, v_2] = \langle\, [v_1, v_2]\,,\, v_3 \,\rangle + w_3 \tag{3.6}$$
$$[v_2, v_3] = \langle\, [v_2, v_3]\,,\, v_1 \,\rangle + w_1 \tag{3.7}$$
$$[v_3, v_1] = \langle\, [v_3, v_1]\,,\, v_2 \,\rangle + w_2 \tag{3.8}$$

Observation. Recall that invariance of $\langle\,,\,\rangle$ implies

$$\langle\, [a, b]\,,\, c \,\rangle = \langle\, a, [b\,, c] \,\rangle\,. \tag{3.9}$$

An explicit expression for df is so obtained in the following lemma:

Lemma 3.1. *We have*

$$(\text{grad } f)_V = \sum_{i=1}^{3} v_i \otimes w_i\,. \tag{3.10}$$

Proof. Let us consider a curve $V(t) \subset \mathbb{G}_3(\mathfrak{g})$ such that $V(0) = V$ defined by

$$(v_1 + tp_1, v_2 + tp_2, v_3 + tp_3)$$

where $p_i \in V^{\perp}$ then we have

$$f(V(t)) = \langle\, [v_1 + tp_1, v_2 + tp_2]\,,\, v_3 + tp_3 \,\rangle = \tag{3.11}$$
$$= f(V) + t(\langle\, [p_1, v_2]\,,\, v_3] \,\rangle + \langle\, [v_1, p_2]\,,\, v_3] \,\rangle + \tag{3.12}$$
$$+ \langle\, [v_1, v_2]\,,\, p_3] \,\rangle) + O(t^2) \tag{3.13}$$

and so, using 3.9

$$\frac{d}{dt} f(V(t))|_{t=0} = \sum_{i=0}^{3} \langle\, w_i, p_i \,\rangle \tag{3.14}$$

and then as

$$\operatorname{grad} f = \sum_{i=1}^{3} v_i \otimes q_i$$

for some $q_i \in V^{\perp}$ then from

$$\sum_{i=1}^{3} p_i \otimes q_i = \sum_{i=1}^{3} w_i \otimes p_i$$

we can desume

$$q_i = w_i$$

as required. ■

A consequence of this is that critical points for the function f are the subspaces for which

$$w_i = 0 \quad \forall\, i\,,$$

that is the 3-dimensional subalgebras of $\mathfrak{g}$, isomorphic to $\mathfrak{su}(2)$. By definition our f is an Ad_G invariant functional, so we know that its gradient is orthogonal to the orbits of the Ad_G action; but we can prove this fact by checking directly the orthogonality condition:

Corollary 3.1. *The vector field* $\operatorname{grad} f$ *is orthogonal to the orbits of Ad_G; so f is invariant under the Ad_G action on $\mathbb{G}_3(\mathfrak{g})$.*

Proof. We observe that the field $\tilde{A}$ in the point p has the form

$$\sum_{i=1}^{3} v_i \otimes [A\,,\, v_i]^{\perp}$$

(this comes from the adjoint representation of $\mathfrak{g}$) ; so

$$\mathfrak{S}\langle\, [A\,,\, v_1]^{\perp}\,,\, [v_2\,,\, v_2]^{\perp}\,\rangle = \tag{3.15}$$
$$= \mathfrak{S}\langle\, [A\,,\, v_1]\,,\, [v_2\,,\, v_3]^{\perp}\,\rangle = \tag{3.16}$$
$$= \mathfrak{S}\langle\, [A\,,\, v_1]\,,\, [v_2\,,\, v_3] - f v_1\,\rangle = \tag{3.17}$$
$$= \mathfrak{S}\langle\, [A\,,\, v_1]\,,\, [v_2\,,\, v_3]\,\rangle - \mathfrak{S}\langle\, [A\,,\, v_1]\,,\, f v_1\,\rangle = \tag{3.18}$$
$$= \mathfrak{S}\langle\, A\,, [\, v_1]\,,\, [v_2\,,\, v_3]]\,\rangle + \mathfrak{S}\langle\, [A\,, [\, v_1\,,\, v_1]\,\rangle = \tag{3.19}$$
$$= 0 \tag{3.20}$$

using the Jacobi identity in the last part. (here $\mathfrak{S}$ means taking the sum after a cyclic permutation of the three indices).

This is a particular case of a more general situation: given a vector field P on the Grassmannian we have the following

Definition 3.1. *We define the* projection map

$$\boldsymbol{\gamma} \ : \ T_p\mathbb{G}_3 \to \mathfrak{g} \tag{3.21}$$

acting as

$$\boldsymbol{\gamma}(P) := \sum_{i=1}^{3} [v_i \, , \, p_i] \tag{3.22}$$

where the p_i are the orthogonal components of the vector field at p.

The map $\boldsymbol{\gamma}(P)$ can be interpreted as a projection on the orbit through a point p, in fact we have the following

Lemma 3.2. *Let $A \in \mathfrak{g}$; let us consider the vector field $\tilde{A}$ associated to the 1-parameter group of diffeomorphisms of $\mathbb{G}_3$ generated by $\exp tA$ through the adjoint action on $\mathfrak{g}$; then a vector field P on $\mathbb{G}_3$ is orthogonal to the orbit at the point p if and only if $\boldsymbol{\gamma}(P) = 0$.*

Proof. The condition of orthogonality of P is expressed by

$$0 = \langle \, \tilde{A} \, , \, P \, \rangle = \sum_{i=1}^{3} \langle \, [A \, , \, v_i]^{\perp} \, , \, P_i \, \rangle = \tag{3.23}$$

$$= \sum_{i=1}^{3} \langle \, [A \, , \, v_i] \, , \, P_i \, \rangle = \sum_{i=1}^{3} \langle \, A \, , [\, v_i \, , \, P_i] \, \rangle = \tag{3.24}$$

$$= \langle \, A \, , \boldsymbol{\gamma}(P) \, \rangle \quad . \quad \blacksquare \tag{3.25}$$

Maximal critical submanifolds

The Ad_G orbit of a 3-dimensional subalgebra is a homogeneous submanifold of the form

$$M = \frac{G}{N(\mathfrak{su}(2))} \, , \tag{3.26}$$

where $N(\mathfrak{su}(2))$ is the normalizer of the subalgebra,i.e. $N(\mathfrak{su}(2)) = \{g \in \mathfrak{g} \,|\, [g, h] \, \in \mathfrak{su}(2) \, \forall h \in \mathfrak{su}(2)\}$. The critical submanifold obtained choosing the copy of $\mathfrak{su}(2) \subset \mathfrak{g}$ corresponding to the maximal root is of particular interest for us; in fact they are isomorphic to compact Wolf sapces (see [83]), the only known examples of compact Quaternion-Kähler manifolds. They are symmetric spaces of positive scalar curvature, and we will describe them more in detail in the next chapter. We want however point out here a property of this type of subalgebras which characterizes their critical submanifolds as the maxima for the flow of grad f.

Observation. The functional f changes sign if we reverse the orientation: therefore every critical manifold C such that $f(C) > 0$ has a specular copy C' for whhich $f(C') < 0$; hence for instance Wolf spaces appear both as absolute maxima and absolute minima.

The property of being a local maximum for f can be deduced by just looking at the roots diagram of $\mathfrak{g}$; to do this, let us recall some facts from the classical theory of roots: a Lie algebra can be completely reconstructed from the data encoded in the corresponding Cartan matrix or, equivalently, in the corresponding Dynkin diagram; these two entities describe the angles between a couple of simple roots (those such that are a basis of the algebra and respect to which every other root can be expressed as a linear combination with integer coefficients of the same sign). These angles are expressed by coefficients (called Cartan numbers)

$$n_{\beta\alpha} := \langle \beta \, , \, \alpha \rangle = 2\frac{\langle \beta \, , \, \alpha \rangle}{\langle \alpha \, , \, \alpha \rangle}$$

for two roots α and β, with $\|\beta\| \geq \|\alpha\|$; these coefficients moreover allow to reconstruct the α string through β, i.e. the coefficients p, q for which

$$\beta - p\,\alpha, \, \cdots \, , \, \beta + q\,\alpha$$

with $n_{\beta\alpha} = p - q$ is still a root; in Table 3.1 are expressed all the possible values of $\langle \alpha \, , \, \beta \rangle$ for any couple of roots, obtained from the relation

$$n_{\beta\alpha} = 2\cos\theta \frac{\|\beta\|}{\|\alpha\|} \Longrightarrow n_{\alpha\beta} n_{\beta\alpha} = 4\cos^2\theta \, . \tag{3.27}$$

We have the following proposition, a version of Theorem 4.2 in [83] adapted to our needs:

Proposition 3.3. *Let* $\mathfrak{su}(2) \subset \mathfrak{g}$ *be a* 3*-dimensional subalgebra corresponding to a root, then the decompositions in* $\mathfrak{su}(2)$ *modules of* $\mathfrak{g}$ *under the restriction of the adjoint representation has the form*

$$\mathfrak{g} \cong \mathfrak{su}(2) + h[\Sigma^0] + k[\Sigma^1] \, , \tag{3.28}$$

with $h, k \in \mathbb{N}$ *if and only if the root is the maximal one.*

Proof. Let β be the maximal root and let $\alpha \neq \beta$ be any posotive root; then $\alpha + \beta$ is not a root, instead $\alpha - \beta$ could be a root; finally $\alpha - 2\beta$ cant't be a root, otherwise $2\beta - \alpha = \beta + (\beta - \alpha) > \beta$ would be a root, which is

$\langle\alpha\,,\,\beta\rangle$	$\langle\beta\,,\,\alpha\rangle$	θ	$\|\beta\|^2/\|\alpha\|^2$
0	0	$\pi/2$	?
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Table 3.1: A list of all the possible values of $\langle\alpha\,,\,\beta\rangle$ and the corresponding angles θ.

impossible. So the Cartan integer $n_{\alpha\beta}$ can be 0 or 1, or in other words the β string through α consists of $\{\alpha\}$ or $\{\alpha, \alpha-\beta\}$; now if we choose a vector X in the negative root space $V_{-\beta}$, its adjoint action ad_X applied to elements $Y \in V_\alpha$ kills them after 1 or 2 applications, depending if $n_{\alpha,\beta}$ is 0 or 1; this implies that the nilpotent part of $\mathfrak{g}_{\mathbb{C}}$ is completely decomposed in summands of type Σ^0 or Σ^1; regarding the Cartan subalgebra $\mathfrak{h}$, it can be decomposed as

$$\mathfrak{h} = H_\beta \oplus H_\beta^\perp\,, \tag{3.29}$$

where $H_\beta^\perp$ consists of trivial $\mathfrak{su}(2)$ representations, so again of type Σ^0. Vice versa, let β be a root satisfying (3.28), supposing it is not maximal; we can order the roots so that β belongs to the closure of the positive Weyl chamber, so that $\langle\alpha, \beta\rangle \geq 0$ for all $\alpha > 0$. Then exists a root $\alpha > 0$ such that $\alpha + \beta$ is still a root; now, if $\langle\alpha, \beta\rangle = 0$ then the β string through α contains at least two elements, if $\langle\alpha, \beta\rangle > 0$ at least 3; in both cases α belogs to a summand of type Σ^k with $k > 1$. ■

Observation. i)This fact, together with the expression for the Hessian of f given later (see (3.69)) shows that the Wolf spaces appear as local maxima (minima) for f;
ii) another consequence of roots theory is that in the decomposition of $\mathfrak{g}$ given by a copy of $\mathfrak{su}(2)$ associated to a root, we can obtain summands Σ^k with k at most 3, because the lenght of any string of roots is at most 4 (see(3.27)).

Actually more can be deduced using the classification of subalgebras due to Dynkin ([26]): in fact the functional f can be related to the *index* of a representation $p : \mathfrak{su}(2) \hookrightarrow \mathfrak{g}$: this is defined as the number n satisfiyng

$$\langle p(u), p(v)\rangle_{\mathfrak{g}} = n\langle u, v\rangle_{\mathfrak{su}(2)} \tag{3.30}$$

for $u, v \in \mathfrak{su}(2)$, where $\langle\cdot,\cdot\rangle_{(\cdot)}$ denotes the Killing form of the two algebras normalized in such a way that the highest root has lenght $\sqrt{2}$; therefore the index is the homotethy factor of the representation p with respect to these metrics. The relation with f is expressed in the following lemma:

Lemma 3.4. *Let $\mathfrak{su}(2) \subset \mathfrak{g}$ be a subalgebra of index n; then*

$$n = \frac{2}{f^2(V)} \tag{3.31}$$

with $V = \mathfrak{su}(2)$.

Proof. Consider an orthonormal basis e_1, e_2, e_3 of $\mathfrak{su}(2)$ satisfying

$$[e_1, e_2] = \sqrt{2}e_3\,; \tag{3.32}$$

the f is calculated using a basis which is orthonormal with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$, so consider the $\mathfrak{g}$-normalized basis

$$\tilde{e_i} = \frac{e_i}{\|e_i\|_{\mathfrak{g}}} = \frac{e_i}{\sqrt{n}}\,; \tag{3.33}$$

then

$$f(\mathfrak{su}(2)) = \langle[\,p(\tilde{e_1}),\ p(\tilde{e_2})], p(\tilde{e_3})\rangle_{\mathfrak{g}} = \frac{\sqrt{2}}{n\sqrt{n}}\langle p(e_3), p(e_3)\rangle_{\mathfrak{g}} \tag{3.34}$$

$$= \sqrt{2}\,\frac{n}{n\sqrt{n}}\,\langle e_3, e_3\rangle_{\mathfrak{su}(2)} = \frac{\sqrt{2}}{\sqrt{n}}\,, \tag{3.35}$$

hence the conclusion. ■

Dynkin showed that the index n is a natural number: a consequence is that a list of the possible values of f in correspondece of the critical submanifolds is given by:

$$\pm\sqrt{2},\ \pm 1,\ \pm\frac{\sqrt{2}}{\sqrt{3}},\ \pm\frac{1}{\sqrt{2}},\ \text{....etc.} \tag{3.36}$$

Dynkin also showed the following theorem:

Theorem 3.5. *Let $\tilde{\mathfrak{g}}$ a subalgebra of $\mathfrak{g}$ of index 1; then the maximal root and its root vectors coincide with the maximal root and root vectors of $\mathfrak{g}$.*

A consequence is the following:

Corollary 3.2. *The functional f attains its absolute maximum (minimum, changing orientation) on the Wolf spaces.*

3.2 The functional g

We define now another functional on the grassmannian which will be useful in the sequel. We define it as

$$g(V) = \sum_{i<j} \|[v_i\,,\, v_j]\|^2\,, \tag{3.37}$$

and observe that it can be written as

$$g(V) = 3\,f^2 + \|\mathrm{grad}\, f\|^2\,. \tag{3.38}$$

We can describe the field grad g and then use the previous results to show that it is orthogonal to the Ad_G orbits; before this we record

Definition 3.2. *Let V be a k-dimenional linear subspace of a Lie algebra $\mathfrak{g}$ with an ON basis $v_1, \cdots v_k$; the* generalized Casimir operator *is a map*

$$C:\ V \longrightarrow \mathfrak{g} \tag{3.39}$$

defined as

$$C(v) = \sum_{i=1}^{k} (ad_{v_i})^2(v)\,. \tag{3.40}$$

This concept generalizes that of the Casimir operatr, which is defined as

$$C(v) = \sum_{i=1}^{k} (\rho_{v_i})^2(v)\,. \tag{3.41}$$

where $\rho\,:\,V\,\rightarrow\,\mathfrak{g}'$ is a representation of a Lie algebra $V\,=\,\mathfrak{g}'$ in $\mathfrak{g}$, and not a simple vector space. This operator is related to the laplace-Beltrami operator; for an application in a representation-theorethic context see [27].

Going back to g, we have

Proposition 3.6. *The gradient of* g *is*

$$\mathrm{grad}\, g = -2 \sum_{i=1}^{3} v_i \otimes C(v_i)^{\perp} \quad , \tag{3.42}$$

where C is the generalized Casimir operator associated to V.

Proof. We have

$$g(V(t)) = \sum_{i<j}^{3} \|[v_i + tP_i \, , \, v_j + tP_j]\|^2 = \tag{3.43}$$
$$= g(V_0) + t(2[v_1 \, , \, P_2] \, , \, [v_1 \, , \, v_2] + [P_1 \, , \, v_2] \, , \, [v_1 \, , \, v_2] + \tag{3.44}$$
$$+ 2[v_2 \, , \, P_3] \, , \, [v_2 \, , \, v_3] + [P_2 \, , \, v_3] \, , \, [v_2 \, , \, v_3] + \tag{3.45}$$
$$+ 2[v_3 \, , \, P_1] \, , \, [v_3 \, , \, v_1] + [P_3 \, , \, v_1] \, , \, [v_3 \, , \, v_1]) + \tag{3.46}$$
$$+ O(t^2) \tag{3.47}$$

and so

$$\frac{d}{dt} g(V(t))|_{t=0} = -2\langle P_1 \, , \, [v_2 \, , \, [v_2 \, , \, v_1]] \rangle - 2\langle P_2 \, , \, [v_1 \, , \, [v_1 \, , \, v_2]] \rangle + \tag{3.48}$$
$$- 2\langle P_3 \, , \, [v_2 \, , \, [v_2 \, , \, v_3]] \rangle - 2\langle P_2 \, , \, [v_3 \, , \, [v_3 \, , \, v_2]] \rangle + \tag{3.49}$$
$$- 2\langle P_1 \, , \, [v_3 \, , \, [v_3 \, , \, v_1]] \rangle - 2\langle P_3 \, , \, [v_1 \, , \, [v_1 \, , \, v_3]] \rangle \tag{3.50}$$
$$= -2 \sum_{i=1}^{3} \langle P_i \, , \, C(v_i) \rangle \quad , \tag{3.51}$$

by definition of the Casimir operator; the conclusion follows as for grad f. ■

Observation. A consequence is that the three dimensional subalgebras of $\mathfrak{g}$ are critical points for g, as well as for f, but not necessarily all of them.

The next step is proving that g is Ad_G invariant; this actually follows directly from the invariance of $\langle , \rangle$ and of $[,]$, but but we prove it here to better exemplifying the use of the operator $\boldsymbol{\gamma}$.

We need the following lemma:

Lemma 3.7. *The following identity holds:*

$$\sum_{i,j=1}^{3} [v_i \, , \, [v_j \, , \, [v_j \, , \, v_i]]] = 0 \quad .$$

Proof. We apply the Jacobi Identity to one of the terms of the sum (after noting that for $i = j$ the term is 0):

$$0 = [v_i \, , \, [v_j \, , \, [v_j \, , \, v_i]]] + [[v_j \, , \, v_i] \, , \, [v_i \, , \, v_j]] + [v_j \, , \, [[v_j \, , \, v_i] \, , \, v_i]] = \tag{3.52}$$
$$= [v_i \, , \, [v_j \, , \, [v_j \, , \, v_i]]] + [v_j \, , \, [v_i \, , \, [v_i \, , \, v_j]]] \quad ; \tag{3.53}$$

the conclusion follows immediately. ■

We are now able to prove the following fact, previously stated.

Theorem 3.8. *We have*

$$\boldsymbol{\gamma}(\mathrm{grad}\, g) = 0 \quad , \tag{3.54}$$

so grad g *is orthogonal to the* Ad_G *orbits and* g *is* Ad_G *invariant.*

Proof. We have

$$\boldsymbol{\gamma}(\mathrm{grad}\, g) = \sum_{i,j=1}^{3} [v_i\,,\, C(v_i)] = \sum_{i,j=1}^{3} [v_i\,,\, [v_j\,,\, [v_j\,,\, v_i]]^{\perp}] \quad ; \tag{3.55}$$

but by definition

$$[v_j\,,\, [v_j\,,\, v_i]]^{\perp} = [v_j\,,\, [v_j\,,\, v_i]] - \sum_{k=1}^{3} \langle\, [v_j\,,\, [v_j\,,\, v_i]], v_k \,\rangle v_k \quad , \tag{3.56}$$

and then

$$\boldsymbol{\gamma}(\mathrm{grad}\, g) = \sum_{i,j=1}^{3} [v_i\,,\, [v_j\,,\, [v_j\,,\, v_i]]] - \sum_{i,j,k=1}^{3} [v_i\,,\, \langle\, [v_j\,,\, [v_j\,,\, v_i]], v_k \,\rangle v_k] \quad ; \tag{3.57}$$

but the first summand is zero by the previous lemma; so we have to check the second summand: more explicitly

$$\begin{aligned}
\sum_{i,j,k=1}^{3} [v_i\,,\, \langle\, [v_j\,,\, [v_j\,,\, v_i]], v_k \,\rangle v_k] &= [v_1\,,\, \langle\, [v_2\,,\, [v_2\,,\, v_1]], v_1 \,\rangle v_1] + \\
&+ [v_1\,,\, \langle\, [v_2\,,\, [v_2\,,\, v_1]], v_2 \,\rangle v_2] + [v_1\,,\, \langle\, [v_2\,,\, [v_2\,,\, v_1]], v_3 \,\rangle v_3] + \\
&+ [v_2\,,\, \langle\, [v_1\,,\, [v_1\,,\, v_2]], v_1 \,\rangle v_1] + [v_1\,,\, \langle\, [v_1\,,\, [v_1\,,\, v_2]], v_2 \,\rangle v_2] + \\
&+ [v_1\,,\, \langle\, [v_1\,,\, [v_1\,,\, v_2]], v_3 \,\rangle v_3] + [v_1\,,\, \langle\, [v_3\,,\, [v_3\,,\, v_1]], v_1 \,\rangle v_1] + \\
&+ [v_1\,,\, \langle\, [v_3\,,\, [v_3\,,\, v_1]], v_2 \,\rangle v_2] + [v_1\,,\, \langle\, [v_3\,,\, [v_3\,,\, v_1]], v_3 \,\rangle v_3] + \\
&+ [v_3\,,\, \langle\, [v_1\,,\, [v_1\,,\, v_3]], v_1 \,\rangle v_1] + [v_3\,,\, \langle\, [v_1\,,\, [v_1\,,\, v_3]], v_2 \,\rangle v_2] + \\
&+ [v_3\,,\, \langle\, [v_1\,,\, [v_1\,,\, v_3]], v_3 \,\rangle v_3] + [v_3\,,\, \langle\, [v_2\,,\, [v_2\,,\, v_3]], v_1 \,\rangle v_1] + \\
&+ [v_3\,,\, \langle\, [v_2\,,\, [v_2\,,\, v_3]], v_2 \,\rangle v_2] + [v_3\,,\, \langle\, [v_2\,,\, [v_2\,,\, v_3]], v_3 \,\rangle v_3] + \\
&+ [v_2\,,\, \langle\, [v_3\,,\, [v_3\,,\, v_2]], v_1 \,\rangle v_1] + [v_2\,,\, \langle\, [v_3\,,\, [v_3\,,\, v_2]], v_2 \,\rangle v_2] + \\
&+ [v_2\,,\, \langle\, [v_3\,,\, [v_3\,,\, v_2]], v_3 \,\rangle v_3] = \\
&= \langle\, [v_1\,,\, v_2]\,,\, [v_2\,,\, v_3] \,\rangle [v_3\,,\, v_1] + \langle\, [v_2\,,\, v_1]\,,\, [v_1\,,\, v_3] \,\rangle [v_3\,,\, v_2] + \\
&+ \langle\, [v_1\,,\, v_3]\,,\, [v_3\,,\, v_2] \,\rangle [v_2\,,\, v_1] + \langle\, [v_3\,,\, v_1]\,,\, [v_1\,,\, v_2] \,\rangle [v_2\,,\, v_3] + \\
&+ \langle\, [v_3\,,\, v_2]\,,\, [v_2\,,\, v_1] \,\rangle [v_1\,,\, v_3] + \langle\, [v_2\,,\, v_3]\,,\, [v_3\,,\, v_1] \,\rangle [v_1\,,\, v_2] = \\
&= 0 \quad . \blacksquare
\end{aligned}$$

3.3 Hessians

We introduce next some basic language from Morse-Bott theory, which will be useful in the sequel; then we will study the critical manifolds for the functional g, in comparison with those of f.

Morse-Bott theory

Morse-Bott theory is a generalization of Morse theory: allowing only the existence of isolated critical points is somewhat restrictive, as for example costant functions are not of Morse type; nor it is the height function of a 2-torus lying horizontally (the critical manifolds are two circles). Morse-Bott theory furnishes a decomposition of the given manifold in terms of cell bundles, relating its topology to that of critical submanifolds.

A critical submanifold $X \subset M$ for a differentiable function $f : M \longrightarrow \mathbb{R}$ is said *nondegenerate* if the 0-eigenspace for the Hessian of f at any point x of X coincides with T_xX. The *index* of a critical manifold X is the dimension of the negative eigenspace of the Hessian of f at $x \in X$.

Definition 3.3. *Let f be a smooth real valued function on a differentiable manifold M; if all its critical manifolds are nondegenerate, then we call it a* Morse-Bott *function.*

The Morse-Bott inequalities allow us to estimate the Betti numbers of the manifold M with the number of critical manifolds of a fixed index:

Theorem 3.9. *The following equality holds:*

$$MB(t) - P(t) = (1 + t)Q(t)\,, \tag{3.58}$$

where $Q(t)$ is a polynomial with nonnegative coefficients.

Definition 3.4. *A Morse-Bott function is called* perfect *if*

$$P(t) = MB(t). \tag{3.59}$$

Theorem 3.9 implies that if $MB(t)$ contains no odd powers, then the Morse-Bott function is authomatically perfect.

The Hessian of g

The functional f is known to be a Morse-Bott function (see (3.69)); instead not much is known about the critical manifolds of the functional g; first of all we determine an expression for the Hessian of g at the critical submanifolds corresponding to subalgebras; in the later sections we will try to find new critical manifolds for g.

Let us calculate the Hessian of g at a critical point for f, in the same spirit of Proposition 2.2 in [50]:

Proposition 3.10. *The eigenvalues of the Hessian of the function* g *at a critical point for the flow of* grad f *are decribed by the second degree polynomial:*

$$\boxed{Hg_{|_{\Sigma^k}} = 2f^2(l^2 - l - 2)} \quad . \tag{3.60}$$

where l is an integer depending on the degree of the $\mathfrak{su}(2)$ representation in $V^\perp$ and $V \otimes V^\perp$.

<u>Proof.</u> We have

$$\begin{aligned} Hg\,(X,Y) = 2\mathfrak{S}\,(&\langle\,[XYv_1\,,\,v_2]\,,\,[v_1\,,\,v_2]\,\rangle + \langle\,[Xv_1\,,\,Yv_2]\,,\,[v_1\,,\,v_2]\,\rangle \\ &+ \langle\,[v_1\,,\,XYv_2]\,,\,[v_1\,,\,v_2]\,\rangle + \langle\,[Xv_1\,,\,v_2]\,,\,[Yv_1\,,\,v_2]\,\rangle \\ &+ \langle\,[v_1\,,\,Xv_2]\,,\,[v_1\,,\,Yv_2]\,\rangle + \langle\,[Yv_1\,,\,Xv_2]\,,\,[v_1\,,\,v_2]\,\rangle \\ &+ \langle\,[Xv_1\,,\,v_2]\,,\,[v_1\,,\,Yv_2]\,\rangle + \langle\,[Yv_1\,,\,v_2]\,,\,[v_1\,,\,Xv_2]\,\rangle)\,, \end{aligned}$$

for 24 terms in total. Defining the operator

$$A_X = -\frac{1}{f}\sum_j ad\,v_j \circ X \circ ad\,v_j \tag{3.61}$$

and grouping the terms in a suitable way we obtain:

$$\begin{aligned} &2\mathfrak{S}\,(\langle\,[XYv_1\,,\,v_2]\,,\,[v_1\,,\,v_2]\,\rangle + \langle\,[v_1\,,\,XYv_2]\,,\,[v_1\,,\,v_2]\,\rangle) = \\ &2\mathfrak{S}\,f(\langle\,[XYv_1\,,\,v_2]\,,\,v_3\,\rangle + \langle\,[v_1\,,\,XYv_2]\,,\,v_3\,\rangle) = \\ -&2\mathfrak{S}\,f(\langle\,Yv_1\,,\,X[v_2\,,\,v_3]\,\rangle + \langle\,Yv_2\,,\,X[v_3\,,\,v_1]\,\rangle) = \\ &\quad -4f^2\sum_i \langle Yv_i\,,\,Xv_i\rangle \end{aligned}$$

using 6 terms; then

$$\begin{aligned} &2\mathfrak{S}\,(\langle\,[Xv_1\,,\,Yv_2]\,,\,[v_1\,,\,v_2]\,\rangle + \langle\,[Yv_1\,,\,Xv_2]\,,\,[v_1\,,\,v_2]\,\rangle) = \\ &2\mathfrak{S}\,f(\langle\,[Xv_1\,,\,Yv_2]\,,\,v_3\,\rangle + \langle\,[Yv_1\,,\,Xv_2]\,,\,v_3\,\rangle) = \\ &2\mathfrak{S}\,f(\langle\,Xv_1\,,\,[Yv_2\,,\,v_3]\,\rangle + \langle\,Yv_1\,,\,[Xv_2\,,\,v_3]\,\rangle) = \\ &2\mathfrak{S}\,f(\langle\,Xv_3\,,\,[Yv_1\,,\,v_2]\,\rangle + \langle\,Yv_1\,,\,[Xv_2\,,\,v_3]\,\rangle) = \\ &2\mathfrak{S}\,f(\langle\,Yv_1\,,\,[v_2\,,\,Xv_3] - [v_3\,,\,Xv_2]\,\rangle = \\ &= -\,2f\sum_i \langle Yv_i\,,\,A_X v_i\rangle\,, \end{aligned}$$

using other 6 terms; finally

$$\begin{aligned}&2\mathfrak{S}\,(\langle\,[Xv_1\,,\,v_2]\,,\,[Yv_1\,,\,v_2]\,\rangle+\langle\,[v_1\,,\,Xv_2]\,,\,[Yv_1\,,\,v_2]\,\rangle)+\\&\quad\langle\,[Xv_1\,,\,v_2]\,,\,[v_1\,,\,Yv_2]\,\rangle+\langle\,[v_1\,,\,Xv_2]\,,\,[v_1\,,\,Yv_2]\,\rangle)=\\&2\mathfrak{S}\,(\langle\,Yv_1\,,\,[v_2\,,\,A_Xv_3]\,\rangle-\langle\,Yv_2\,,\,[v_1\,,\,A_Xv_3]\,\rangle)=\\&2\mathfrak{S}\,f(\langle\,A_Xv_3\,,\,[v_2\,,\,Xv_1]-[v_1\,,\,Xv_2]\,\rangle=\\&=2\sum_i\langle A_Yv_i\,,\,A_Xv_i\rangle\,,\end{aligned}$$

using the last 12 terms; so in conclusion we can write

$$Hg\,(X,Y)=\sum_i -4f^2\langle Yv_i\,,\,Xv_i\rangle-2f\langle Yv_i\,,\,A_Xv_i\rangle+\langle A_Yv_i\,,\,A_Xv_i\rangle\,; \tag{3.62}$$

moreover recall that if v_1, v_2, v_3 is an orthonormal basis for te subalgebra isomorphic to $\mathfrak{su}(2)$ the representation p is not an isometry with respect to the standard Killing forms, in fact

$$B_{\mathfrak{g}}(p^*v_i\,,\,p^*v_j)=-2\,\delta_{ij}\,f(V)^2\,;$$

so the Casimir operator C_ρ associated to the $\mathfrak{su}(2)$ representation ρ obtained from the adjoint action of V inside $\mathfrak{g}$ can be expressed as

$$C_\rho=-\frac{1}{2f(V)^2}\sum_{i=1}^{3}\rho(v_i)\rho(v_i)\,; \tag{3.63}$$

this operator acts on the irreducible $\mathfrak{su}(2)$ representation Σ^k by multiplication for the constant

$$C(k)=\frac{(k+1)^2-1}{8}\,. \tag{3.64}$$

The definition of C_p implies the equality

$$C_{V^*\otimes V^\perp}X=C_{V^\perp}X+XC_V+\frac{1}{f}A_X \tag{3.65}$$

so that we have

$$A_X=f(C_{V^*\otimes V^\perp}-C_{V^\perp}-1)X\,. \tag{3.66}$$

Then, if $X=Y$ is an element of an orthonormal basis for $T_V\mathbb{G}_3=V^*\otimes V^\perp$ belonging to the subrepresentation Σ^k, defining a new parameter as

$$l=C_{V^*\otimes V^\perp}-C_{V^\perp}-1 \tag{3.67}$$

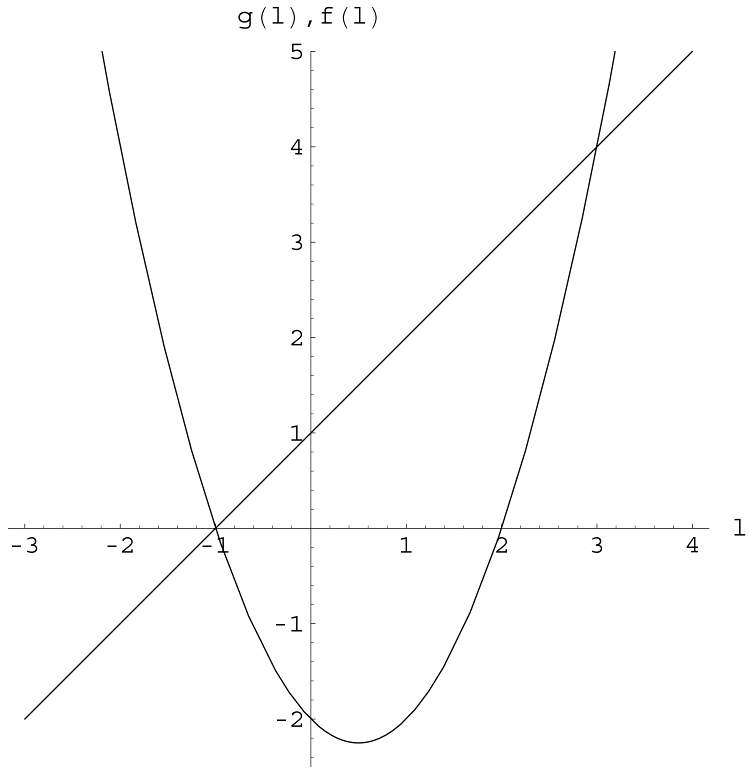

Figure 3.1: The graphs of the Hessians of f and g in function of the parameter l.

depending on the degree of the Σ^k summand, hence the conclusion. ■

The polynomial in (3.60) has -1 and 2 as zeroes; we observe that -1 is a root also for the corresponding polynomial for f: in fact a similar calculation (see [50]) gives

$$Hf\,(X,Y) = -\sum_i f\langle Yv_i\,,\,Xv_i\rangle + \langle Yv_i\,,\,A_X v_i\rangle \tag{3.68}$$

and consequently

$$\boxed{Hf_{|_{\Sigma^k}} = -f(l+1)} \quad . \tag{3.69}$$

Observation: Equation (3.69) implies that f is a Morse-Bott function, see [50].

The following lemma describes some relations between the Hessians of the two functionals:

Lemma 3.11. *Let $V = \mathfrak{su}(2) \subset \mathfrak{g}$ be a critical point for f; then: i)the 0-eigenspace for the Hessian of g contains that of the Hessian of f; the inclusion is strict if and only if $\mathfrak{su}(2)^{\perp} \supset \Sigma^4$;*

ii)if the corresponding critical manifold is a local maximum (minimum) for f then it is also a local maximum (minimum) for g.

Proof. i)Equation (3.60) implies that a sufficient condition for the Hessian of g being 0 is that

$$C_{V^* \otimes V^{\perp}} - C_{V^{\perp}} = 0\,, \tag{3.70}$$

and this happens exactly in the middle term of the tensor product of $\mathfrak{su}(2)$ representations that describe the tangent space at V:

$$\Sigma^2 \otimes \Sigma^k = \Sigma^{k+2} + \Sigma^k + \Sigma^{k-2}\,; \tag{3.71}$$

this is consistent with the fact that critical manifolds for f are also critical for g; but the condition is not necessary, for the presence of the second root 2: for example, as we shall see later, in the $\mathfrak{so}(3)$ critical manifold for $\mathfrak{su}(3)$ other directions have 0 as eigenvalue for the Hessian of g; in fact C_{Σ^6} is the multiplication for 6, so

$$C_{\Sigma^6} - C_{\Sigma^4} - 1 = 6 - 3 - 1 = 2\,. \tag{3.72}$$

This last condition is satisfied if and only if $k = 4$: in fact we are asking that

$$l = C_{V^* \otimes V^{\perp}} - C_{V^{\perp}} - 1 = 2 \tag{3.73}$$

which is equivalent to write

$$l = \frac{(k+2+1)^2 - 1}{8} - \frac{(k+1)^2 - 1)^2}{8} - 1 = 2 \tag{3.74}$$

if and only if

$$\frac{k^2 + 2k + 8}{8} - \frac{k^2 + 2k}{8} - 1 = 2 \tag{3.75}$$

if and only if

$$k = 4\,; \tag{3.76}$$

so we can have another 0 eigenspace for the Hessian of g if and only if

$$V^{\perp} \supset \Sigma^4\,. \tag{3.77}$$

ii)We note first that at a local minimum for f with $f < 0$ the condition

$$l + 1 \geq 0 \tag{3.78}$$

has to be satisfied, but this means that

$$C_{V^* \otimes V^\perp} - C_{V^\perp} \geq 0\,; \tag{3.79}$$

this happens if and only if (see also (3.3))

$$V^\perp = a\Sigma^0 + b\Sigma^1\,,\ a,\, b \in \mathbb{N} \tag{3.80}$$

so that in consequence we have

$$Hg = -4f^2 \quad \text{or} \quad Hg = -\frac{9}{4}f^2\,, \tag{3.81}$$

on the normal bundle, so that V is a local maximum for g. On the other hand we can't have local minima for f when $f > 0$ because for analogous reasons the decompositon of the tangent space should not contain irreducible $\mathfrak{su}(2)$ modules of dimension grater than the ones in $V^\perp$, but this is impossible. We can obtain all local maxima for f just reversing the orientation of a minimum, so that for what we have just said they are maxima also for g, the conclusion follows. ∎

3.4 Low dimensional examples

We are going now to study more in detail the Hessians of the two functionals f, g on the Grassmannians of 3-planes of some low dimensional classical Lie algebras.

The Lie algebra $\mathfrak{u}(2)$

We know that critical manifolds for f are also critical for g; it's quite natural to ask ourselves if these are all of them; the answer is no. In fact for example let us consider the case of $\mathfrak{u}(2)$: we have the orthogonal decomposition

$$\mathfrak{u}(2) = \mathbb{R} \oplus \mathfrak{su}(2)\,,$$

where the $\mathbb{R}$ factor (generated by an element h) kills everything under adjoint action; the known critical manifolds inside the Grassmannian $\mathbb{G}_3(\mathfrak{u}(2))$ (isomorphic to S^3) are the ones corresponding to the copy of $\mathfrak{su}(2)$ with the two orientations (with normalizer the whole algebra, so isolated points); these are the absolute maximum and minimum for f, but only local maxima for g; this consideration implies the existence of other critical points. In fact let us consider the 3 dimensional subspace V generated by $h, a_1, a_2,$ where the last two elements are two of the standard generators of $\mathfrak{su}(2)$ properly normalized, it's immediate to check that the image under ad^2 is contained in V itself, so that the orthogonal projection makes $\operatorname{grad} g = 0$ at this point; more precisely:

$$(ad\, h)^2(V) = 0 \quad ; \quad (ad\, a_1)^2(V) = \lambda a_2 \quad ; \quad (ad\, a_1)^2(V) = \lambda a_2 \, ;$$

we can observe that the invariance under the ad^2 action of the basis on itself, differently from the one of the ad action for the f critical points, is sufficient but not necessary, as each component of $\operatorname{grad} g$ has the form

$$[v_i \, , \, [v_i \, , \, v_j]]^{\perp} + [v_k \, , \, [v_k \, , \, v_j]]^{\perp} \, ,$$

so the two summands can annihilate separately or cancel each other. Going back to our critical point, the isotropy of the adjoint action is given exactly by h itself and the missing element a_3, and the orbit under the Adjoint action is the sphere S^2 that parametrizes the subspaces

$$V = \langle \, h \, , \, \sum_{i=1}^{3} \alpha_i a_i \, , \, \sum_{i=1}^{3} \beta_i a_i \, \rangle \, ,$$

with $\sum_{i=1}^{3} \alpha_i^2 = \sum_{i=1}^{3} \beta_i^2 = 1$ and $\sum_{i=1}^{3} \alpha_i \beta_i = 0$, that is all the 3 dimensional subspaces containing h. There are no more critical points for g: in fact the remaining points of the Grassmannian are precisely the 3-dimensional subspaces that do not contain h, excepted $\mathfrak{su}(2)$ with the two orientations; for dimensional reasons, the intersection of such a subspace with $\mathfrak{su}(2)$ is precisely 2 dimensional, so that it is spanned, choosing a suitable basis, by

$$\langle \alpha h + \beta a_1 \, , \, a_2, \, , \, a_3 \rangle \, ,$$

with the a_is orthonormal in $\mathfrak{su}(2)$, and $\alpha^2 + \beta^2 = 1$, $\beta \neq \{0, 1\}$; these are parametrized by $\tilde{\mathbb{G}}_2(\mathfrak{su}(2)) \times (S^1 - (\{(0,1)\} \cup \{(0,-1)\})$; now it's clear that the first component of $\operatorname{grad} g$ cannot be 0, in fact

$$(ad^2 a_2) + (ad^2 a_3)(\alpha h + \beta a_1) = \lambda \beta a_1$$

for some $\lambda \neq 0$, that does not belong to V evidently. The Adjoijnt orbits of theese points are 2 dimensional spheres of ray α; the dependence from the parameter β is expressed by

$$f(\beta) = \pm \frac{1}{\sqrt{2}} \beta \quad , \quad g(\beta) = \frac{1}{2} + \beta^2$$

so that the second derivative of g along the curve

$$\gamma(\beta) = V(\beta) \, , \, \beta \in [0 \, , \, 1]$$

is constantly equal to 2.

	Σ^2	Σ^4	Σ^6
C	1	3	6
Hf	$-\sqrt{2}$	0	$3/\sqrt{2}$
Hg	10	0	0

Table 3.2: The Hessians at $\mathfrak{so}(3) \subset \mathfrak{su}(3)$

	Σ^1	Σ^2	Σ^3
C	3/8	1	15/8
Hf	0	$\sqrt{2}$	$3/\sqrt{2}$
Hg	0	-8	-9

.

Table 3.3: The Hessians at $\mathfrak{su}(2) \subset \mathfrak{su}(3)$

The Lie algebra $\mathfrak{su}(3)$

The following tables describe the behaviour of the two Hessians at the critical points for f: at $\mathfrak{so}(3)$ (with $f = -\frac{1}{\sqrt{2}}$ and $g = \frac{3}{2}$) we have the decomposition

$$T_{\mathfrak{so}(3)}\mathbb{G}_3 = \Sigma^2 \otimes \Sigma^4 = \Sigma^2 + \Sigma^4 + \Sigma^6\,, \tag{3.82}$$

and the eigenvalues of the Hessians are reported in Table 3.2; instead at $\mathfrak{su}(2) \subset \mathfrak{su}(2)$, with $f = -\sqrt{2}$ and $g = 6$, where the decomposition of the tangent space is given by

$$T_{\mathfrak{su}(2)}\mathbb{G}_3 = \Sigma^2 \otimes (\Sigma^0 + 2\Sigma^1) = \Sigma^2 + 2\Sigma^1 + 2\Sigma^3 \tag{3.83}$$

we have the Hessians of the two functions taking the values as in Table 3.3.

The last 0 in the last row of Table 3.2 could be interpreted as a degeneracy of the Hessian of g, so that it would not be a Morse-Bott function, or it could mean that the critical manifold for g strictly contains the one of f. We now recall the decomposition of $\mathfrak{su}(3)$ under the adjoint action restricted to the Cartan subalgebra $\mathfrak{h}$ of the diagonal elements:

$$\mathfrak{su}(3) = \mathfrak{h} \oplus \hat{\mathfrak{g}}_\alpha \oplus \hat{\mathfrak{g}}_\beta \oplus \hat{\mathfrak{g}}_{\alpha+\beta}\,, \tag{3.84}$$

where $\hat{\mathfrak{g}}_\alpha$ denotes the 2 dimensional real space generated by elements of the form $(\mathfrak{g}_\alpha - \mathfrak{g}_{-\alpha})$ and $\imath(\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$, α being a root; the Adjoint action of the maximal torus corresponding to $\mathfrak{h}$ is essentially a $\mathfrak{u}(1)$ action on each of these summands: in fact the kernel of the corresponding root in $\mathfrak{h}$, which is a hyperplane, is tangent in general to a $\mathrm{rank}(\mathfrak{g})-1$ dimensional torus which is the stabilizer of a point of the maximal torus representation.

It is straightforward to prove that the 3-dimensional torus $\mathbb{T}^3$

$$V = \langle \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e^{\imath s} & 0 \\ -e^{-\imath s} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{\imath t} \\ 0 & -e^{-\imath t} & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & e^{\imath u} \\ 0 & 0 & 0 \\ -e^{-\imath u} & 0 & 0 \end{pmatrix} \rangle$$

for $s, t, u \in \mathbb{R}$ identifies 3-dimensional subspaces that are closed under the squared adjoint action mutually applied between the generators, as usual, so that $\mathrm{grad}\, g = 0$ at V; we observe that these subspaces are not subalgebras excepted when the parameters satisfy the condition

$$s + t - u = k\pi \tag{3.85}$$

for which we get the intersection of $\mathbb{T}^3$ with the orbit of $\mathfrak{so}(3)$ under the Adjoint action; the restriction of f to this submanifold is expressed by

$$f(s,t,u) = \frac{1}{\sqrt{2}} \cos{(s + t - u)} \tag{3.86}$$

so that the intersection with the fibres is given by the 2 dimensional tori $s + t - u = const$; instead we have

$$g = \frac{3}{2} \tag{3.87}$$

constantly.

Observation. This explains partially the apparent degeneracy of the Hessian of g (here we have one new degeneracy direction, while the 0-eigenspace for Hg is S^6 which is 7-dimensional as seen in Table 3.2).

The kernel of the adjoint action (that is the "normalizer" of V) at the points of the torus is precisely $\mathfrak{so}(3)$ for the subalgebras, giving the already known critical manifold

$$SSU(3) = SU(3)/SO(3)\,; \tag{3.88}$$

instead it is trivial at all other points: in fact the Cartan subalgebra $\mathfrak{h}$ is never in the kernel, as it acts as a rotation in the spaces $\hat{\mathfrak{g}}_\alpha$; on the other hand the three generators of V preserve V itself if and only if V is a subalgebra; in the end, the elements orthogonal to these last ones, with corresponding parameters

$$s' = s + \pi/2\,,\; t' = t + \pi/2\,,\; u' = u + \pi/2$$

	Σ^2
C	1
Hf	$\sqrt{2}$
Hg	-8

Table 3.4: The Hessians at $\mathfrak{su}(2) \subset \mathfrak{so}(4)$

satisfy the condition $(ad\, v_1(s'))(v_1(s)) \in \mathfrak{h}$ for example, so they don't preserve the space too. So, in conclusion, if $s+t-u \neq k\pi$ the Adjoint orbit is the whole $SU(3)$.

The Lie algebra $\mathfrak{so}(4)$

Inside $\mathfrak{so}(4)$ we can find essentially three copies of $\mathfrak{su}(2)$ (for a total of 6 considering the orientation): two corresponding to the standard decomposition

$$\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \tag{3.89}$$

so that we get the decomposition

$$\Sigma^2 \otimes (\Sigma^0 + \Sigma^0 + \Sigma^0) = \Sigma^2 + \Sigma^2 + \Sigma^2 \tag{3.90}$$

and the Hessians at theese points (which are the maxima and the minima for f), take the values reported in Table 3.4. The two functions assume values $f(\mathfrak{su}(2)) = -\sqrt{2}$ and $g(\mathfrak{su}(2)) = 6$.

The other critical point for f is the diagonal subalgebra $\mathfrak{su}(2)_\Delta$; the values of our functions are respectively $f(\mathfrak{su}(2)_\Delta) = -1$ and $g(\mathfrak{su}(2)_\Delta) = 3$; the decomposition of the tangent space is given by:

$$\Sigma^2 \otimes \Sigma^2 = \Sigma^4 + \Sigma^2 + \Sigma^0 \tag{3.91}$$

and the behaviour of the Hessians is described in Table 3.5. This tells us for example that this cannot be the absolute minimum for g, and suggests that we can find other critical manifolds for it. In fact for example it is easy to check that the following subspaces are critical for g; actually this is just a consequence of the functoriality respect to the immersion of subalgebras, in particular for the immersion of $\mathfrak{u}(2)$: if v_i denotes an orthonormal basis of $\mathfrak{su}(2)$ the space

$$V = \langle (v_1\,,\,0)\,,\,(v_2\,,\,0)\,,\,(0\,,\,v_3)\rangle \tag{3.92}$$

	Σ^0	Σ^2	Σ^4
C	0	1	3
Hf	-1	0	2
Hg	8	0	-4

Table 3.5: The Hessians at $\mathfrak{su}(2)_\Delta \subset \mathfrak{so}(4)$

comes from the immersion

$$\mathfrak{su}(2) \oplus \mathbb{R} \hookrightarrow \mathfrak{su}(2) \oplus \mathfrak{su}(2) \tag{3.93}$$

with values $f(V) = 0$ and $g(V) = 2$; moreover the kernel of the adjoint action on V is given by the abelian subalgebra

$$\mathfrak{u}(1) \times \mathfrak{u}(1) = \langle (v_3\,,\,0)\,,\,(0\,,\,v_3) \rangle \tag{3.94}$$

so that the Adjoint orbit is

$$SO(4)/T^2\,; \tag{3.95}$$

on the other hand the diagonal immersion

$$\mathfrak{su}(2) \oplus \mathbb{R} \overset{\Delta}{\hookrightarrow} \mathfrak{su}(2) \oplus \mathfrak{su}(2) \tag{3.96}$$

has associated the space

$$W = \langle (v_1\,,\,v_1)\,,\,(v_2\,,\,v_2)\,,\,(v_3\,,\,-v_3) \rangle$$

with values $f(W) = 0$ and $g(W) = 3$; in this case the kernel of the adjoint action is

$$\mathfrak{su}(2) = \langle (v_1\,,\,-v_1)\,,\,(v_2\,,\,-v_2)\,,\,(v_3\,,\,v_3) \rangle \tag{3.97}$$

and the Adjoint orbit is then

$$SO(4)/O(3) = \mathbb{RP}^3\,. \tag{3.98}$$

The Lie algebra $\mathfrak{so}(5)$

The decomposition at the maximal critical manifold, corresponding to the long root, is given by

$$\mathfrak{so}(5) = \mathfrak{su}(2) + 2\Sigma^1 + 3\Sigma^0 \tag{3.99}$$

	Σ^0	Σ^2	Σ^4	$\tilde{\Sigma}^2$
C	0	1	3	1
Hf	-1	0	2	$\sqrt{2}$
Hg	8	0	-4	-8

.

Table 3.6: The Hessians at $\mathfrak{su}(2)_\Delta \subset \mathfrak{so}(5)$

and the tangent space is

$$T^{\mathbb{C}}_{\mathfrak{su}(2)_\Delta}\mathbb{G}_3(\mathfrak{so}(5)) \cong \Sigma^2 \otimes (2\Sigma^1 + 3\Sigma^0) = 2(\Sigma^3 + \Sigma^1) + 3\Sigma^2 \qquad (3.100)$$

where $2\Sigma^1$ is tangent to $\mathbb{HP}^1$, and the remaining summands are the stable part: the Hessians in fact behave in the same way described in Table 3.3, because the representations involved are the same (but with different multuplicities), see (3.83).

The decomposition associated to the next to minimal critical manifold, or in other words to the copy $\mathfrak{su}(2)_\Delta$ corresponding to the short root, is given by

$$\mathfrak{so}(5) = \mathfrak{su}(2)_\Delta + \Sigma^0 + 2\Sigma^2 \qquad (3.101)$$

so that

$$T^{\mathbb{C}}_{\mathfrak{su}(2)_\Delta}\mathbb{G}_3(\mathfrak{so}(5)) \cong \Sigma^2 \otimes (\Sigma^0 + 2\Sigma^2) = 2(\Sigma^4 + \Sigma^0) + 3\Sigma^2 \qquad (3.102)$$

and the eigenvalues of the Hessians are described in Table 3.6.

Observation. The $\tilde{\Sigma}^2$ column comes from the Σ^0 summand of the decomposition of $V^\perp$, so that the values of the Hessians are different with respect to the Σ^2 coming from the Σ^4 summand even if they are isomorphic representations, exemplifying the dependence of the parameter l from both $V \otimes V^\perp$ and $V^\perp$ (see (3.67)).

3.5 New critical manifolds for grad g

Critical points for g provide some kind of "generalized subalgebras", in the sense that

$$\operatorname{grad} f = 0 \quad \Longrightarrow \quad \operatorname{grad} g = 0\,; \qquad (3.103)$$

we have already shown in the previous sections that the converse is not true, analyzing in detail examples of compact Lie algebras.

Another approach for finding g critical points that are not subalgebras is looking at the root structure, using the same type of arguments used to prove Proposition 3.3.

Let us consider the case that $\langle \beta , \alpha \rangle = -2$ for two simple roots; this happens precisely in $\mathfrak{so}(2n+1)$, $\mathfrak{sp}(n)$ and $\mathfrak{f}(4)$. This means essentially that $\alpha + \beta$ and $2\alpha + \beta$ are the only roots belonging to the α string through β ($\beta - \alpha$ is never a root if they are both simple). Now we can consider the 3 dimensional space V generated by elemnts of norm 1 taken in the spaces $(\mathfrak{g}_\alpha - \mathfrak{g}_{-\alpha})$, $(\mathfrak{g}_\beta - \mathfrak{g}_{-\beta})$ and $(\mathfrak{g}_{2\alpha+\beta} - \mathfrak{g}_{-(2\alpha+\beta)})$; if we perform the double adjoint action of one generator on the other, keeping in mind that

$$[\mathfrak{g}_\alpha , \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \tag{3.104}$$

we fall in the same subspaces: in fact we have to take in account that the following are not roots:

$$2(\alpha + \beta) , \; \alpha + 2\beta , \; 3\alpha + 2\beta ,$$

the last one because every root of level m can be obtained from one of level $m - 1$ by just adding one simple root; but nor $2(\alpha + \beta)$ nor $3\alpha + \beta$ are roots in our situation. So taking the orthogonal projection as usual the gradient of g annihilates; this is not a subalgebra, as $\alpha + \beta$ stands in $V^\perp$. In particular consider $\mathfrak{sp}(2)$: a subspace of this kind gives origin, as for $\mathfrak{su}(3)$, to a 3 dimesnional torus described in terms of matrices (not normalized) by

$$\begin{pmatrix} 0 & 0 & e^{\imath s} & 0 \\ 0 & 0 & 0 & 0 \\ -e^{-\imath s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\imath t} \\ 0 & 0 & 0 & 0 \\ 0 & -e^{-\imath t} & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & e^{\imath u} & 0 & 0 \\ -e^{-\imath u} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\imath u} \\ 0 & 0 & -e^{\imath u} & 0 \end{pmatrix}$$

corresponding respectively to $\beta, 2\alpha + \beta$ and α. This torus is contained in the fibres $f^{-1}(0)$ and $g^{-1}(1)$, and doesn't contain any subalgebra.

Lie Triple Systems

Another possible approach for trying to understand g critical manifolds is through the so called Lie Triple Systems (from now on LTS), i.e. those subspaces V of a Lie Algebra which satisfy the condition

$$[X , [Y , Z]] \in V \quad \forall X, Y, Z \in V ; \tag{3.105}$$

we quote here a result which links LTS to totally geodesic submanifolds of a symmetric space (see [35]):

Proposition 3.12. *Let $M = G/K$ be a Riemannian gobally symmetric space; let $\mathfrak{n} \subset \mathfrak{m}$ be a LTS; put $N = Exp\,\mathfrak{n}$. Then N has a natural differentiable structure in which it is a totally geodesic submanifold of M with $\mathfrak{n}$ as tangent space. On the other hand if N is a totally geodesic submanifold of M and $\mathfrak{n}$ is the tangent space at some point, then it is a LTS.*

We remark that N is a globally symmetric space too, of the form G'/K', where G' is a closed subgroup of G and $K' = K \cap G'$. We are now in position to state the following criterion for identifying critical submanifolds for grad g which are not critical for grad f, or in other words which do not correspond to subalgebras: recall that grad $g = 0$ is equivalent to the set of three equations

$$[v_i, [v_i, v_k]]^\perp + [v_j, [v_j, v_k]^\perp = 0 \tag{3.106}$$

for $i,\ j,\ k = 1,\ 2,\ 3$ modulo ciclyc permutations; then we have:

Proposition 3.13. *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and let $M = G/K$ be a symmetric space; then any 3-dimensional totally geodesic submanifold N corresponds to a critical submanifold in $\mathbb{G}_3(\mathfrak{g})$ for* grad g *which is not critical for* grad f.

Proof. Proposition (3.12) translates the condition of being a totally geodesic submanifold to the algebraic condition of a LTS; this clearly implies that V satisfies equations (3.106) ; moreover a LTS cannot be a subalgebra, as $V \subset \mathfrak{m}$ implies

$$[u, v] \in \mathfrak{h} \subset V^\perp \tag{3.107}$$

for property (1.16) of symmetric spaces; the Adjoint orbit of the LTS does not depend on the base point chosen in N. ■

For a classification of totally geodesic submanifolds of symmetric spaces see [18] and [19].

We go now do describe some examples: we start with the decomposition

$$\mathfrak{so}(5) = \mathfrak{so}(4) \oplus \mathfrak{m} \tag{3.108}$$

which corresponds to the symmetric space

$$SO(5)/SO(4) = S^4\,; \tag{3.109}$$

the decomposition can be written in terms of the roots structure in the following way (as usual we shall denote $\hat{\mathfrak{g}}_\alpha$ the 2-dimensional subspace generated by $(\mathfrak{g}_\alpha - \mathfrak{g}_{-\alpha})$ and $\imath(\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$ and with $\mathfrak{t}$ the Cartan subalgebra):

$$\mathfrak{so}(5) = (\mathfrak{t} + \hat{\mathfrak{g}}_\beta + \hat{\mathfrak{g}}_{2\alpha+\beta})\ +\ (\hat{\mathfrak{g}}_\alpha + \hat{\mathfrak{g}}_{\alpha+\beta}) \tag{3.110}$$

where α and β are simple roots labelled consistently with the previous section; so

$$T_p S^4 = \hat{\mathfrak{g}}_\alpha + \hat{\mathfrak{g}}_{\alpha+\beta} \tag{3.111}$$

and every 3 dimensional totally geodesic subanifolds is a maximal S^3; but being this a rank 1 symmetric space, that means that the isotropy (Adjoint) representation of $SO(4)$ on $T_p S^4$ is transitive on the unit vectors, and being all 3 dimensional subspaces of the tangent space identified by the orthogonal unit vector, we can essentially obatin all elements in the same Adjoint orbit in this way. We choose just to fix the ideas the space

$$V = \hat{\mathfrak{g}}_\alpha + (\mathfrak{g}_{\alpha+\beta} - \mathfrak{g}_{-(\alpha+\beta)}) \,, \tag{3.112}$$

and one can check directly that it satisfies the required condition. The values of our functions are at this point

$$f(V) = 0 \quad , \quad g(V) = 3 \,. \tag{3.113}$$

In this way we can give a different interpretation of the critical point found in the previous section; in fact we can consider the decomposition

$$\begin{aligned} \mathfrak{sp}(2) &= (\mathfrak{t} + \hat{\mathfrak{g}}_{\alpha+\beta}) \; + \; (\hat{\mathfrak{g}}_\alpha + \hat{\mathfrak{g}}_\beta + \hat{\mathfrak{g}}_{2\alpha+\beta}) \\ &= (\mathbb{R} \oplus \mathfrak{su}(2)) \oplus 2\Sigma^2 \end{aligned}$$

corresponding to the symmetric space

$$\frac{Sp(2)}{SU(2)U(1)} \, ; \tag{3.114}$$

the 3 dimensional subspace

$$V = (\mathfrak{g}_\alpha - \mathfrak{g}_{-\alpha}) + (\mathfrak{g}_\beta - \mathfrak{g}_{-\beta}) + (\mathfrak{g}_{2\alpha+\beta} - \mathfrak{g}_{-(2\alpha+\beta)}) \tag{3.115}$$

satisfies the definition of LTS, so can be thought as the tangent space of a totally geodesic submanifold of this last one.

Now let us go back to $\mathfrak{su}(3)$: we can decompose it as a symmetric pair as

$$\begin{aligned} \mathfrak{su}(3) &= (\mathfrak{t} \oplus \hat{\mathfrak{g}}_{\alpha+\beta}) \; + \; (\hat{\mathfrak{g}}_\alpha + \hat{\mathfrak{g}}_\beta) \\ &= (\mathbb{R} \oplus \mathfrak{su}(2)) \oplus 2\Sigma^1 \end{aligned} \tag{3.116}$$

and for example if we choose the 3 dimensional subspace in $2\Sigma^1$

$$V = \hat{\mathfrak{g}}_\alpha + (\mathfrak{g}_\beta - \mathfrak{g}_{-\beta}) \tag{3.117}$$

we see that it is a LTS, with $f(V) = 0$ and $g(V) = 3$, so this is not contained in the torus found before.

The kernel of the adjoint action at this point is given by the kernel of β inside $\mathfrak{h}$, in fact it kills $(\mathfrak{g}_\beta - \mathfrak{g}_{-\beta})$ and rotates $\hat{\mathfrak{g}}_\alpha$, so the Adjoint orbit is

$$\frac{SU(3)}{U(1)}\,. \tag{3.118}$$

Incidentally we can observe that among the examples found until this moment only some belong to this family; for instance in the 3 dimensional tori found in $\mathfrak{su}(3)$ and $\mathfrak{sp}(2)$ only the 2 dimensional tori corresponding to the conditions $s+t-u = k\pi$ (the subalgebras $\mathfrak{so}(3)$ in $\mathfrak{su}(3)$) and $s-t-2u = k\pi$ in the latter situation are associated to totally geodesic submanifolds in some symmetric space (the Lie group $SU(3)$ itself in the first case, for example). This says that the condition of being a LTS is strictly stronger than asking

$$[v_i, \, , [v_i \, , \, v_j]] \subset V \tag{3.119}$$

with v_i, v_j a pair from an orthonormal basis; nevertheless this point of view suggests the possibility of finding an interpretation of the gradient of g in terms of the curvature tensor of suitable symmetric spaces.

Chapter 4

Quaterion-Kähler spaces

In this chapter we introduce Quaternion-Kähler geometry, showing the main examples in the case of positive scalar curvature: compact Wolf spaces. To give a more rich picture of the background we describe then the twistor and Swann bundles of Wolf spaces, which are important tools in the study of quaternionic geometry, but which houwever will not play a crucial rôle in our exposition. In Section 4.3 we introduce a way of describing the $Sp(n)Sp(1)$ structure which is compatible with the tangent space of $\mathbb{G}_3$ at critical points for f in a purely algebraic way: this is the first link between quaternionic and Grassmannian geometry. In this same spirit, in the last section we give a non-standard description of the $Sp(2)Sp(1)$-invariant 4-form in $\mathbb{R}^8$.

4.1 The Wolf Spaces

We state here the basic definitions and facts about Quaternion-Kaäler manifolds. References for this material are [71], [13], [44].

A Quaternion-Kähler manifold (from now on QK manifold) M of dimesnion $4n$ is a Riemannian manifold characterized by the reduction of the restricted holonomy group Hol^0 to the group $Sp(1)Sp(n) \subset SO(4n)$; the complexified holonomy representation corrsponds to $\mathbb{C}^2 \otimes \mathbb{C}^{2n} \cong H \otimes E$, where H, E are the standard representations of $Sp(1)$ and $Sp(n)$ respectively.

The invariant symplectic 2-forms ω_H, ω_E can be used to describe the (symmetric) metric tensor in terms

$$g = \omega_H \otimes \omega_E \,. \tag{4.1}$$

These two representations have both *a quaternionic structure*, in other words an invariant antilinear endomorphism ϵ such that $\epsilon^2 = -1$; this structure allows to identify $\mathbb{C}^2$ with $\mathbb{H}$ and $\mathbb{C}^{2n}$ with $\mathbb{H}^n$. The tensor product $\epsilon_{HE} = \epsilon_H \otimes$

ϵ_E is instead an involutive invariant antilinear map, defining a *real structure* on $H \otimes E$, so that the real tangent space at a point T_xM is the fixed point set for ϵ_{HE}:

$$T_xM \cong [H \otimes E] = \{a \in H \otimes E | \, \epsilon_{HE}(a) = a\}. \tag{4.2}$$

We have the embedding $\mathfrak{sp}(1) \cong S^2H \hookrightarrow \mathrm{End}(TM_{\mathbb{C}})$, acting on $TM_{\mathbb{C}}$ in the following way:

$$S^2H \otimes (H \otimes E) \hookrightarrow (\underline{H} \otimes H) \otimes (\underline{H} \otimes E) \longrightarrow H \otimes E \tag{4.3}$$

contracting the underlined factors using the invariant form ω_H. In this sense S^2H, also called the *quaternionic bundle*, can be also identified with $\mathfrak{sp}(1)$ as the Adjoint representation of $Sp(1) \subset \mathrm{Hol}^0(M)$, and thanks to the metric also as a subbundle of $\bigwedge^2 TM$. A local trivialization is given by a triple of skew-symmetric endomorphisms $I_1,\ I_2,\ I_3$ satisfying the quaternionic relations:

$$I_iI_j = (-1)^\epsilon I_k\,, \tag{4.4}$$

with the sign depending on the permutation ijk. QK manifolds are Einstein (see [71]), therefore have constant scalar curvature s; we will only consider the case $s > 0$. Alternative definitions of QK manifolds will be provided in Section 4.4.

A fundamental example of compact Quaternion-Kähler manifolds are those described by Wolf in [83]; they are symmetric spaces, and as Alekseevsky showed in [2], they are the only complete homogeneous examples with positive scalar curvature. Moreover LeBrun and Salamon conjectured that there are no others; this conjecture was proven up to dimension 12 (see [69], [39], [40]). We have in part described the structure of these spaces in section 3.1, we will go more in deep now: recall that they are defined as the quotient

$$M = \frac{G}{N(\mathfrak{sp}(1))}\,, \tag{4.5}$$

where $\mathfrak{sp}(1)$ is a subalgebra corresponding to the maximal root and $K := N(\mathfrak{su}(2))$ is its normalizer; there is an exponent of this type for each simple Lie group, classical

$$\mathbb{HP}^n = \frac{Sp(n+1)}{Sp(n)Sp(1)}\,; \quad \mathbb{G}_2(\mathbb{C}^n) = \frac{SU(n)}{S(U(n-2)\times U(2)}\,; \tag{4.6}$$

$$\tilde{\mathbb{G}}_4(\mathbb{R}^n) = \frac{SO(n)}{SO(n-4)SO(4)} \tag{4.7}$$

and exceptional:

$$\frac{G_2}{SO(4)}\,; \quad \frac{F_4}{Sp(3)Sp(1)}\,; \quad \frac{E_6}{SU(6)Sp(1)}\,; \tag{4.8}$$
$$\frac{E_7}{Spin(12)Sp(1)}\,; \quad \frac{E_8}{E_7Sp(1)}\,. \tag{4.9}$$

These spaces are all symmetric and have positive scalar curvature; their non-compact duals give examples of QK symmetric spaces of negative scalar curvature; the theory of negative QK manifolds has rather different developments with respect to the positive case: for instance Alekseevsky found examples which are homogeneous but non-symmetric ([3]). However we will focus our attention exclusively on the positive case.

Going back to our examples, the tangent space at the point at eK can be identified with $\mathfrak{m}$ in the Lie algebra decomposition

$$\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{sp}(1) \oplus \mathfrak{m} \tag{4.10}$$

where $\mathfrak{c}$ denotes the centralizer of $\mathfrak{sp}(1)$ so that K is the adjoint group to the subalgebra $\mathfrak{c} \oplus \mathfrak{sp}(1)$; so K is the stabilizer of a point and the holonomy group of M via the Adjont action on $\mathfrak{m}$; the structure of a QK manifold is given by a field of algebras of skew-symmetric endomorphisms of the tangent spce isomorphic to $\operatorname{Im}\mathbb{H}$: in this case these algebras correspond to the $\mathfrak{sp}(1)$ of the previous decomposition; in fact $\mathfrak{sp}(1)$ acts on $\mathfrak{m}$ by adjoint representation (restricting the brackets of $\mathfrak{g}$) and in this way we have a K-invariant immersion

$$\mathfrak{sp}(1) \hookrightarrow \mathfrak{m} \otimes \mathfrak{m}$$

(where the duality $\mathfrak{m} \cong \mathfrak{m}^*$ is provided by the Killing metric, which induces the G invariant Riemannian metric on M); K-invariance allows to extended the action of $\mathfrak{sp}(1)$ globally on the whole M as a subbundle of $\operatorname{End}(TM)$.

4.2 The Twistor space

We want to discuss now some important homogeneous spaces which are bundles over the Wolf spaces; they can be generalized to bundles over any Quaternion-Kähler manifold, as was shown by Salamon ([71]). Consider the complexification of any compact semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$; let $\mathfrak{h}$ be a Cartan subalgebra, and

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \sum_{\alpha \in \Xi} \mathfrak{g}_\alpha \tag{4.11}$$

the correspondin Cartan decomposition, with $\mathfrak{g}_\alpha$ the root spaces of the roots system Ξ; then $\mathfrak{g}$ is a maximal compact subalgebra of $\mathfrak{g}_\mathbb{C}$, and $[\mathfrak{h}]$ denotes the intersection $\mathfrak{h} \cap \mathfrak{g}$; let ρ denote the highest root and consider the parabolic subalgebra of $\mathfrak{g}_\mathbb{C}$

$$\mathfrak{p} = \mathfrak{h} \oplus \sum_{\langle \alpha, \rho \rangle \geq 0} \mathfrak{g}_\alpha\,; \tag{4.12}$$

then if $P \subset G_\mathbb{C}$ denotes the corresponding adjoint subgroup, we can define the complex homogeneous space of complex dimension n

$$\mathcal{Z} = \frac{G_\mathbb{C}}{P}\,; \tag{4.13}$$

we observe that $\mathfrak{p}$ is precisely the subalgebra of the stabilizers of the complex line $\mathbb{C}\mathfrak{g}_\rho$, so $\mathcal{Z}$ can be interpreted as the $G_\mathbb{C}$ orit under the adjoint action of the point $\mathbb{C}\mathfrak{g}_\rho$ in the projectivised Lie algebra $\mathbb{P}(\mathfrak{g}_\mathbb{C})$; instead the orbit of any nilpotent element $X_\rho \in \mathfrak{g}_\rho$ inside $\mathfrak{g}_\mathbb{C}$ is given by

$$\mathcal{U} = \frac{G_\mathbb{C}}{P_1}\,, \tag{4.14}$$

where P_1 is the adjoint group of the Lie algebra $\mathfrak{p}_1$ defined as

$$\mathfrak{p}_1 := \{a \in \mathfrak{p} | \langle a, H_\rho \rangle = 0\}, \tag{4.15}$$

because the semisimple element H_ρ doesn't stabilize X_ρ; in particular we have the quotient $\mathfrak{p}/\mathfrak{p}_1 \cong \mathbb{C}$, so the projection

$$\mathcal{U} = \frac{G_\mathbb{C}}{P_1} \longrightarrow \frac{G_\mathbb{C}}{P} = \mathcal{Z} \tag{4.16}$$

gives a bundle with fibre $\exp \mathbb{C} = \mathbb{C}^*$; this bundle can be considered as the pricnipal bundle of a complex line bundle over the base space $\mathcal{Z}$. The tangent space at the identity of $\mathcal{U}$ can be identified with

$$T_{eP_1}\mathcal{U} = \{H_\rho\} \oplus \sum_{\langle \alpha, \rho \rangle < 0} \mathfrak{g}_\alpha\,; \tag{4.17}$$

we can define on it an $Ad(P_1)$-invariant cotangent vector

$$Y \longrightarrow \langle X_\rho, Y \rangle \tag{4.18}$$

which thus extends globally to a $G_\mathbb{C}$-invariant holomorphic 1-form θ; it can be shown that θ satisfies

1. $(d\theta)^{n+1} \neq 0\,;$

2. the restriction of θ to the fibre is identically 0;

3. $\tau_z^*\theta = z\theta$ if τ_z is the action of $\mathbb{C}^*$ on $\mathcal{U}$.

We are going now to define a type of structures which are strictly related to the homogeneous spaces we are discussing:

Definition 4.1. *Let C be a complex manifold of dimension $2n+1$; a* complex contact structure *is a family $\{(U_i, \theta_i)\}$ where the U_i is an open covering of C and the θ_i are 1-forms defined on each U_i such that $(d\theta_i)^n \wedge \theta_i \neq 0$ in U_i and $\theta_j = f_{ij}\theta_i$ with f_{ij} some smooth function defined on $U_j \cap U_i$; moreover it's required that this family is the maximal enjoying these properties.*

The functions f_{ij}^{-1} can be viewed as transition functions for a comples line bundle over C, which thus turns out to be naturally associated to the contact structure; its principal bundle $\pi : B \longrightarrow C$ with fibre $\mathbb{C}^*$ can be equipped with a global 1-form obtained gluing together the pullbacks $\pi^*\theta_i$. On the other hand, given a principal $\mathbb{C}^*$ bundle B over a complex manifold C, the presence of a holomorphic 1-form satisfiying conditions $1, 2, 3$ above descends on C to a complex contact structure. Applying these facts to our examples, we can say that $\mathcal{Z}$ is a complex manifold endowed with a complex contact structure given by the holomorphic 1-form θ defined on the principal bundle $\mathcal{U}$.

Now the compact subgroup $G \subset G_{\mathbb{C}}$ acts on $\mathcal{Z}$, giving rise to a compact orbit through eP which can be described as

$$M' = \frac{G}{G \cap P}\,; \tag{4.19}$$

the subalgebra of the stabilizer is given by

$$\mathfrak{g} \cap \mathfrak{p} = \mathfrak{g} \cap \big(\mathfrak{h} \oplus \sum_{\langle\alpha,\rho\rangle=0} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})\big) \tag{4.20}$$

as $\mathfrak{g}$ doesn't intersect a single root space $\mathfrak{g}_\alpha$; the following lemma shows that actually the G orbit has the same dimension of the ambient space, so that in conclusion $M' = \mathcal{Z}$ and this last is compact:

Lemma 4.1. *The following equality holds:*

$$\dim_{\mathbb{R}} G_{\mathbb{C}} - \dim_{\mathbb{R}} G = \dim_{\mathbb{R}} P - \dim_{\mathbb{R}} G \cap P \tag{4.21}$$

<u>Proof.</u> The summand on the left side equals $\frac{1}{2}\dim_{\mathbb{R}} G_{\mathbb{C}}$; on the right side we can subdivide in two parts: the Cartan subalgebra and the root spaces involved. Clearly $\dim_{\mathbb{R}} \mathfrak{h} = 2$

$dim_{\mathbb{R}}\, \mathfrak{h} \cap \mathfrak{g}$; regarding the root spaces involved in $\mathfrak{p}$ we can divide them in two types: those for which $\langle \alpha, \rho \rangle > 0$ and those for which $\langle \alpha, \rho \rangle = 0$; we observe that all positive roots satisfy one of these conditions ($\mathfrak{p}$ is parabolic!). In the first case if $\mathfrak{g}_\alpha \subset \mathfrak{p}$ then $\mathfrak{g}_{-\alpha} \not\subset \mathfrak{p}$, and

$$\dim_{\mathbb{R}} \sum_{\langle \alpha, \rho \rangle > 0} \mathfrak{g}_\alpha = \frac{1}{2} \dim_{\mathbb{R}} \sum_{\langle \alpha, \rho \rangle \neq 0} \mathfrak{g}_\alpha \, ; \tag{4.22}$$

in the second case $\mathfrak{g}_{\pm\alpha} \subset \mathfrak{p}$ and

$$\dim_{\mathbb{R}} \sum_{\langle \alpha, \rho \rangle = 0} (\mathfrak{g}_\alpha + \mathfrak{g}_\alpha) = 2 \dim_{\mathbb{R}} \left(\mathfrak{g} \cap \sum_{\langle \alpha, \rho \rangle = 0} (\mathfrak{g}_\alpha + \mathfrak{g}_\alpha) \right) . \tag{4.23}$$

The conclusion follows. ■

Hence, denoting $\mathfrak{l} = \mathfrak{p} \cap \mathfrak{g}$ and L the corresponding subgroup, we can write in compact presentation

$$\mathcal{Z} = \frac{G_{\mathbb{C}}}{P} \cong \frac{G}{L} \, . \tag{4.24}$$

We note now that

$$\mathfrak{l} = \mathfrak{c} \oplus \{iH_\rho\} \subset N(\mathfrak{sp}(1)) \, , \tag{4.25}$$

so we have again a fibration

$$\mathcal{Z} \longrightarrow M \, , \tag{4.26}$$

with fibre the quotient

$$\frac{Sp(1)}{U(1)} \cong S^2 \, , \tag{4.27}$$

whose tangent space at the identity in $\mathcal{Z}$ is given by

$$\mathfrak{g} \cap (\mathfrak{g}_\rho + \mathfrak{g}_{-\rho}) \, . \tag{4.28}$$

The following theorem, due to Wolf ([83],Th.6.1), states more precisely the relationship between contact and quaternionic structures in the homogeneous context:

Theorem 4.2. *There is a one-to-one correspondence between compact simply connected homogeneous contact complex manifolds $\mathcal{Z}$ and compact QK symmetric spaces M. This correspondence is given by a bundle $\pi : \mathcal{Z} \longrightarrow M$ with fibres 2-spheres.*

The following diagram syntethizes the relationships between the various spaces involved in the previous description:

$$\begin{tikzcd}
 & G_{\mathbb{C}}/P_1 \arrow[r, leftrightarrow, "\cong"'] \arrow[d] & \mathcal{U} \arrow[r, hook] \arrow[d, "\mathbb{C}^*"] & \mathfrak{g}_{\mathbb{C}} \\
G/L \arrow[r, leftrightarrow, "\cong"'] \arrow[dr] & G_{\mathbb{C}}/P \arrow[r, leftrightarrow, "\cong"'] \arrow[d] & \mathcal{Z} \arrow[r, hook] \arrow[d, "S^2"] & \mathbb{P}(\mathfrak{g}_{\mathbb{C}}) \\
 & G/N(\mathfrak{su}(2)) \arrow[r, leftrightarrow, "\cong"'] & M \arrow[r, hook] & \mathbb{G}_3
\end{tikzcd} \quad . \tag{4.29}$$

Example. The main example to consider is quaternionic projective space:

$$\mathbb{H}^{n+1}\backslash\{0\} \longrightarrow \mathbb{CP}^{2n+1} \longrightarrow \mathbb{HP}^n \tag{4.30}$$

Observation. The twistor bundle $\mathcal{Z}$ is non-trivial in general, and for compact QK manifolds with $s > 0$ it does not admit any global integrable section (see [70], [7]), or in other words any global complex structure compatible with the quaternionic structure; in the case of $\mathbb{HP}^n$ Massey proved in [63] that no global almost complex structure exists at all. In the case of the complex Grassmannians $\mathbb{G}_2(\mathbb{C}^n)$, it is well known that they carry a global complex structure J; in the case J belongs to $(S^2H)^{\perp}$ inside $\mathrm{End}(TM)$, so that it is not a section of $\mathcal{Z}$.

4.3 The $Sp(1)Sp(n)$ structure

Let $h, \hat{h}$ denote a unitary basis of H, in such a way that $\omega_H(h, \hat{h}) = 1$; with respect to this basis we have

$$\omega_H = h \wedge \hat{h} = \frac{1}{2}(h\hat{h} - \hat{h}h)\,. \tag{4.31}$$

We can in terms of $h, \hat{h}$ dtermine a basis of S^2H:

$$\begin{array}{rcl} I_1 & = & \imath(h \vee \hat{h}) \\ I_2 & = & h^2 + \hat{h}^2 \\ I_3 & = & \imath(h^2 - \hat{h}^2) \end{array} \tag{4.32}$$

are orthogonal of norm $\sqrt{2}$ with respect to the metric $\omega_H \otimes \omega_H$ induced on S^2H; they satisfy the same relations of quaternions:

$$I_k^2 = -1 \quad , \quad I_i I_j = sgn_{(ijk)} I_k \tag{4.33}$$

with $sgn_{(ijk)}$ the sign of the permutation; the composition is obtained by contracting again with ω_H just as in (4.3).

Consider now the case where the $Sp(1)$ representation inside $Sp(1)Sp(n)$ is such that the projection on the $Sp(n)$ factor is nonzero: this means that the E representation is nontrivial under this $Sp(1)$ action.

In this case it becomes significant analyzing the quaternionic action from the point of view of these new $Sp(1)$ representations; first of all we adopt the following notation: we have the symmetrization map S acting on tensors as

$$S(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!} \sum_{\pi^n} x_{\pi^n(1)} \otimes \cdots \otimes x_{\pi^n(n)} \tag{4.34}$$

where π varies in the group of permutations on n elements; the map extends linearly. We give then the following definition: we denote as

$$\{\cdot,\cdot\} : \ \Sigma^k \otimes \Sigma^h \longrightarrow \Sigma^{h+k} \tag{4.35}$$

the symmetrization of the two factors, more explicitly if

$$\alpha = \sum_{\pi^k} \alpha_{\pi^k(1)} \otimes \cdots \otimes \alpha_{\pi^k(k)} \in \Sigma^k \quad , \quad \beta = \sum_{\pi^h} \beta_{\pi^h(1)} \otimes \cdots \otimes \beta_{\pi^h(h)} \in \Sigma^h \tag{4.36}$$

then

$$\{\alpha \otimes \beta\} = \sum_{\pi^k,\pi^h} S\big(\alpha_{\pi^k(1)} \otimes \cdots \otimes \alpha_{\pi^k(k)} \otimes \beta_{\pi^h(1)} \otimes \cdots \otimes \beta_{\pi^h(h)}\big)\,. \tag{4.37}$$

In particular we denote by σ the map $\{\cdot,\cdot\}$ when the first index is 2:

$$\sigma := \{\cdot,\cdot\} : \ \Sigma^2 \otimes \Sigma^i \longrightarrow \Sigma^{i+2} \ . \tag{4.38}$$

Consider now for simplicity the case that E corresponds to an irreducible $Sp(1)$ representation; then

$$T_x M_{\mathbb{C}} \cong \Sigma^1 \otimes \Sigma^{i-1} \tag{4.39}$$

and using Clebsch-Gordan relation, we obtain

$$T_x M_{\mathbb{C}} \cong \ \Sigma^i + \Sigma^{i-2} \hookrightarrow \Sigma^{i+2} + \Sigma^i + \Sigma^{i-2} \ \cong \Sigma^2 \otimes \Sigma^i\,; \tag{4.40}$$

more precisely $T_x M_{\mathbb{C}}$ coincides with the kernel of the symmetrization

$$\sigma : \Sigma^2 \otimes \Sigma^i \to \Sigma^{i+2}\,. \tag{4.41}$$

Example 4.1. There are essentially three ways (up to conjugation) of sending $Sp(1)$ inside $Sp(2)$: two correspond to the roots, but in these cases the decomposition of the standard $Sp(2)$ representation $\mathbb{C}^4$ is not irreducible; in fact

$$E = \mathbb{C}^4 = \Sigma^0 + \Sigma^0 + \Sigma^1 \tag{4.42}$$

for the long root, as comparing with the known decomposition of the adjoint representation one has

$$\mathfrak{sp}(2) = S^2(\mathbb{C}^4) = S^2(2\Sigma^0 + \Sigma^1) = \Sigma^2 + 2\Sigma^1 + 3\Sigma^0\,; \tag{4.43}$$

for the short root we have instead

$$E = \mathbb{C}^4 = \Sigma^1 + \Sigma^1 \tag{4.44}$$

as in fact

$$\mathfrak{sp}(2) = S^2(\mathbb{C}^4) = S^2(2\Sigma^1) = 3\Sigma^2 + \Sigma^0\,. \tag{4.45}$$

There is a third embedding, corresponding to the $\mathfrak{sl}(2,\mathbb{C})$ triple

$$X = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix}, \tag{4.46}$$

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{4.47}$$

obtained using the recipe in [21], for which

$$E = \mathbb{C}^4 = \Sigma^3\,. \tag{4.48}$$

Observation. This last can be interpreted in the following way: recall that the decomposition of the Lie algebra $\mathfrak{g}_2$ with respect to $\mathfrak{so}(4) \subset \mathfrak{g}_2$ is given by

$$\Sigma^2_+ + \Sigma^2_- + \Sigma^1_- \otimes \Sigma^3_+\,, \tag{4.49}$$

where $\Sigma^k_\pm$ denote the representations of the $\mathfrak{sp}(1)$ corresponding to the long $(+)$ or to the short$(-)$ root; so considering the diagonal embedding

$$\mathfrak{sp}(1)_\Delta \hookrightarrow \mathfrak{so}(4) = \mathfrak{sp}(1)_+ + \mathfrak{sp}(1)_- \hookrightarrow \mathfrak{sp}(1)_+ + \mathfrak{sp}(2)\;, \tag{4.50}$$

consistently with the $Sp(1)Sp(2)$ structure of the Wolf space

$$\frac{G_2}{SO(4)}\,, \tag{4.51}$$

we have a description of its tangent space in the EH formalism as $H \otimes E \cong \Sigma^1 \otimes \Sigma^3$, corresponding to the representation in (4.48).

The action of $S^2H \cong \Sigma^2$ on $T_xM_{\mathbb{C}}$ can be therefore expressed suitably exploiting this new formulation, involving the Σ^2 factor instead of the $\Sigma^1 = H$; to understand more deeply this Σ^2-approach we need to define more explicitly the invariant immersion (4.40). Let us define the map as

$$Q:\ \Sigma^1 \otimes \Sigma^{i-1} \xrightarrow{\omega_H \otimes} \underline{\Sigma^1} \otimes \underline{\underline{\Sigma^1}} \otimes \underline{\Sigma^1} \otimes \underline{\underline{\Sigma^{i-1}}} \xrightarrow{\{\cdot,\cdot\}} \Sigma^2 \otimes \Sigma^i \tag{4.52}$$

acting in the following way: if

$$Y = h \otimes \beta + \hat{h} \otimes \hat{\beta} \in \Sigma^1 \otimes \Sigma^{i-1}, \quad \beta, \hat{\beta} \in \Sigma^{i-1} \tag{4.53}$$

then

$$Q(Y) = \frac{1}{2}\{hh\}\{\hat{h}\beta\} + \frac{1}{4}(h\hat{h} + \hat{h}h)\big(\{\hat{h}\hat{\beta}\} - \{h\beta\}\big) - \frac{1}{2}\{\hat{h}\hat{h}\}\{h\hat{\beta}\} \tag{4.54}$$

obtained, after tensorizing with the invariant element ω_H, by symmetrization of the tensorial factors respecting the simple or double underlining marks in (4.52).

Our next aim is to express the quaternionic action in terms of this description: a first guess in this sense is that for $Q(Y) = \sum v_i \otimes p_i$ then

$$Q(I_1Y) = v_2 \otimes p_3 + v_3 \otimes p_2\,, \tag{4.55}$$

mimicking the adjoint representation of $\mathfrak{su}(2)$ on itself; but this is not corrct, as at the second step

$$Q(I_1^2Y) = -v_2 \otimes p_2 - v_3 \otimes p_3\,, \tag{4.56}$$

which is not $-Id$. Something more is needed to "reconstruct" the missing term $-v_1 \otimes p_1$.

The next proposition gives the correct answer in order to express the quaternionic action from the Σ^2 viewpoint:

Proposition 4.3. *Let $Y \in T_xM = \Sigma^1 \otimes \Sigma^i$; if $Q(Y) = \sum v_i \otimes p_i$ then*

$$\boxed{Q(I_1Y) = v_1 \otimes \tfrac{1}{4}\sigma(Y) + v_2 \otimes p_3 - v_3 \otimes p_2\,.} \tag{4.57}$$

Proof. We have the definition of $Q(Y)$ as in (4.54): then if we identify v_i with the basis I_i defined in (4.32), grouping the terms properly we obtain

$$\begin{aligned} p_1 &= -\frac{\imath}{4}\big(\{\hat{h}\hat{\beta}\} - \{h\beta\}\big) && (4.58)\\ p_2 &= \frac{1}{4}\big(\{\hat{h}\beta\} - \{h\hat{\beta}\}\big) && (4.59)\\ p_3 &= -\frac{\imath}{4}\big(\{\hat{h}\beta\} + \{h\hat{\beta}\}\big)\,; && (4.60) \end{aligned}$$

the quaternionic action of I_1 on Y is given, in the Σ^1 context, by

$$I_1 Y = -\imath h \otimes \beta + \imath\hat{h} \otimes \hat{\beta}\,; \tag{4.61}$$

so we obtain

$$Q(I_1Y) = -\frac{\imath}{2}\{hh\}\{\hat{h}\beta\} + \frac{\imath}{4}(h\hat{h} + \hat{h}h)\big(h\beta + \hat{h}\hat{\beta}\big) - \frac{\imath}{2}\{\hat{h}\hat{h}\}\{h\beta\} \tag{4.62}$$

and in the form $Q(I_1Y) = \sum_{i=1}^{3} v_i \otimes q_i^1$ we have

$$\begin{aligned} q_1^1 &= \frac{\imath}{4}\big(\{h\beta\} + \{\hat{h}\hat{\beta}\}\big) && (4.63)\\ q_2^1 &= -\frac{i}{4}\big(\{\hat{h}\beta\} + \{h\hat{\beta}\}\big) && (4.64)\\ q_3^1 &= -\frac{1}{4}\big(\{\hat{h}\beta\} - \{h\hat{\beta}\}\big)\,; && (4.65) \end{aligned}$$

the conclusion follows by the definition of σ and comparing the two sets of equalities. ■

In the same way we obtain for the other quaternionic elements

$$\begin{aligned} I_2Y &= -\hat{h} \otimes \beta + h \otimes \hat{\beta} && (4.66)\\ I_3Y &= \imath\hat{h} \otimes \beta + \imath h \otimes \hat{\beta} && (4.67) \end{aligned}$$

so that

$$\begin{aligned} Q(I_2Y) &= \tfrac{1}{2}\{hh\}\{\hat{h}\hat{\beta}\} - \tfrac{1}{2}\{h\hat{h}\}\big(\hat{h}\beta + h\hat{\beta}\big) + \tfrac{1}{2}\{\hat{h}\hat{h}\}\{h\beta\} && (4.68)\\ Q(I_3Y) &= \tfrac{\imath}{2}\{hh\}\{\hat{h}\hat{\beta}\} + \tfrac{\imath}{2}\{h\hat{h}\}\big(\hat{h}\beta - h\hat{\beta}\big) - \tfrac{\imath}{2}\{\hat{h}\hat{h}\}\{h\beta\} && (4.69) \end{aligned}$$

which imply the equalities

$$q_j^i = \eta_{ijk}\, p_k - \delta_i^j\, \frac{1}{4}\, \sigma(Y)\,, \tag{4.70}$$

where $\eta_{ijk} = sgn_{ijk}$ if $i \neq j$, otherwise $\eta_{iik} = 0$; moreover

$$p_i = -\frac{1}{4}\sigma(I_iY)\,. \tag{4.71}$$

We can therefore state the quaternionic relations in terms of this description: for example

$$Q(I_1^2 Y) = Q(I_1 I_1 Y) = -v_1 \otimes \frac{1}{4}\sigma(I_1 Y) - v_2 \otimes p_2 - v_3 \otimes p_3 \tag{4.72}$$
$$= -v_1 \otimes p_1 - v_2 \otimes p_2 - v_3 \otimes p_3 \tag{4.73}$$
$$= -Q(Y) \tag{4.74}$$

and also

$$Q(I_1 I_2 Y) = -v_1 \otimes \frac{1}{4}\sigma(I_2 Y) - v_2 \otimes q_3^2 - v_3 \otimes q_2^2 \tag{4.75}$$
$$= -v_1 \otimes p_2 + v_2 \otimes p_1 - v_3 \otimes \frac{1}{4}\sigma(Y) \tag{4.76}$$
$$= Q(I_3 Y) \tag{4.77}$$

as expected.

4.4 The quaternionic 4-form in 8 dimensions

The local triple of almost complex structures I_i generate together with the metric a local triple of symplectic forms

$$\omega_i \in \bigwedge^2 T_x M\,, \tag{4.78}$$

spanning a subbundle of the bundle of 2-forms isomorphic to S^2H. The condition $Hol^0(M) \subset Sp(1)Sp(n)$ can be expressed by the condition that the subbundle S^2H is preserved by covariant differentiation, more precisely

$$\begin{aligned} \nabla_Y \omega_1 &= \alpha_3(Y)\omega_2 + \alpha_2(Y)\omega_3 \\ \nabla_Y \omega_2 &= -\alpha_3(Y)\omega_1 + \alpha_1(Y)\omega_3 \\ \nabla_Y \omega_3 &= \alpha_2(Y)\omega_1 - \alpha_1(Y)\omega_2\,; \end{aligned} \tag{4.79}$$

where α_i are local 1-forms and Y is any tangent vector. We can define a global 4-form using these local 2-forms:

$$\frac{1}{2}\,\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3 \tag{4.80}$$

and the following conditions are equivalent:

$$\nabla_Y S^2H \subset S^2H \quad \Longleftrightarrow \quad \nabla_Y \Omega = 0 \tag{4.81}$$

for any $Y \in T_xM$. Swann proved in [78] that if dim $M \geq 12$ then the condition $\nabla_Y \Omega = 0$ is equivalent to

$$d\Omega = 0\,; \tag{4.82}$$

Salamon showed an example of an 8-demensional compact nilmanifold endowed with 4-form Ω with stabilizer $Sp(2)Sp(1)$, which is closed but non-parallel ([74]).

We want now to give a description of the quaternionic 4-form Ω using the point of view of the previous section: suppose for example that we have as in (4.48)

$$T_xM_{\mathbb{C}} \cong \Sigma^1 \otimes \Sigma^3 \cong \Sigma^2 \oplus \Sigma^4\,; \tag{4.83}$$

then the bundle S^2H can be detected finding the Σ^2 summands in $\bigwedge^2 T_xM$:

$$\bigwedge^2 TM_x \cong \bigwedge^2 \Sigma^2 \oplus \Sigma^2 \otimes \Sigma^4 \oplus \bigwedge^2 \Sigma^4 \tag{4.84}$$

$$\cong \underset{v}{\Sigma^2} \oplus (\Sigma^6 \oplus \underset{m}{\Sigma^4} \oplus \Sigma^2) \oplus (\Sigma^6 \underset{h}{\oplus} \Sigma^2)\,; \tag{4.85}$$

the elements of the three groups of summands will be called *vertical, mixed* and *horizontal.* We fix a basis of the vertical space to be e^{23}, e^{31}, e^{12}, with respect to a basis e^1, e^2, e^3 of the Σ^2 summand in (4.83); a basis of the Σ^4 component of the tangent space is given by

$$e^8 = \frac{1}{\sqrt{3}}(2 \cdot 11 - 22 - 33) \tag{4.86}$$

$$e^7 = (22 - 33) \tag{4.87}$$

$$e^4 = (12 + 21) \tag{4.88}$$

$$e^6 = (23 + 32) \tag{4.89}$$

$$e^5 = (31 + 13)\,, \tag{4.90}$$

where the notation "11" stands for $e^1 \otimes e^1$, underlining in this way the identity

$$\Sigma^4 \cong S^2_0\,\Sigma^2\,. \tag{4.91}$$

We want now to identify a basis for the horizontal part; the copy of Σ^2 contained in $\bigwedge^2 \Sigma^3$ is also a submodule of $\mathrm{End}(\Sigma^4)$ and its elements can be described in terms of invariant operations, as the following example shows:

$$\beta^1(e^4) = \beta^1(12{+}21) = (1{*}1)\vee 2{+}(1{*}2)\vee 1 = 0{+}3\vee 1 = (31{+}13) = e^5\,, \tag{4.92}$$

where $*$ is the isomorphism $\bigwedge^2 \Sigma^2 \cong \Sigma^2$ and $\vee$ is symmetrization; this description is unique up to a constant, thanks to Schur's lemma an the presence of a unique copy of Σ^2 in $\mathrm{End}(\Sigma^4)$. Going on in the same way we obtain:

$$\beta^1(e^6) = \beta^1(23+32) = (1*2)\vee 3 + (1*3)\vee 2 = 2(33-22) = -2e^7\,; \tag{4.93}$$

$$\beta^1(e^5) = \beta^1(31+13) = (1*3)\vee 1 + 0 = -(21+12) = -e^4\,; \tag{4.94}$$

$$\beta^1(e^7) = \beta^1(22-33) = (1*2)\vee 2 - (1*3)\vee 3 = 2(32+23) = 2e^6\,; \tag{4.95}$$

$$\beta^1(e^8) = \beta^1(\frac{1}{\sqrt{3}}(2\cdot 11 - 22 - 33)) = \frac{1}{\sqrt{3}}(0 - (1*2)\vee 2 - (1*3)\vee 3) \tag{4.96}$$

$$= -(32+23) + (23+32) = 0\,. \tag{4.97}$$

Analogous calculations determine the following basis of the horizontal 2-forms

$$\beta^1 = -\,2e^{76} - e^{45} \tag{4.98}$$

$$\beta^2 = \sqrt{3}e^{85} + e^{75} + e^{46} \tag{4.99}$$

$$\beta^3 = e^{74} - \sqrt{3}e^{84} + e^{65}\,. \tag{4.100}$$

Now the elements of the basis of the quaternionic bundle $S^2H \subset \bigwedge^2(S^2 + S^4)$ are decomposed as

$$\omega_i = \omega_i^h + \omega_i^m + \omega_i^v \tag{4.101}$$

with

$$\omega_i^h = \beta^i \tag{4.102}$$

$$\omega_i^m = \sum_{j=4}^{8} \langle e^i\,,\, e^j \rangle \tag{4.103}$$

$$\omega_i^v = e^{jk}\,, \tag{4.104}$$

where $\langle\,,\,\rangle$ denotes the contraction of tensors. We are going now to determine the mixed term in the same spirit of what we did to find the β^i, in other words findig an invariant submodule in $\Sigma^2 \otimes \Sigma^4$ isomorphic to Σ^2; subsequently we will find constants a, b, c such that $\langle \omega_i, \omega_i >= -I$, if $\omega_i = a\,\omega_i^h + b\,\omega_i^m + c\,\omega_i^v$.

Proposition 4.4. *In this setting the quaternionic 4-form Ω is*

$$\begin{aligned}\frac{1}{2}\Omega &= \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3 \\ &= -\sqrt{3}e^{7653} - 2e^{7623} - e^{7681} - \sqrt{3}e^{7642} \\ &- e^{4523} - 2e^{4581} + \sqrt{3}e^{8142} + \sqrt{3}e^{8153} \\ &+ e^{3175} - e^{8275} - e^{8246} + e^{7241} \\ &+ e^{6341} + e^{6385} - e^{7483} + e^{6512}\,.\end{aligned}$$

Proof.We subdivide the proof in 3 parts:

1)The 2-form ω_1.
Starting with ω_1^m we have

$$\omega_1^m = \langle 1\,,\ \frac{1}{\sqrt{3}}(2\cdot 11 - 22 - 33) + (22 - 33) \tag{4.105}$$

$$+ (12 + 21) + (23 + 32) + (31 + 13)\rangle \tag{4.106}$$

$$= \frac{2}{\sqrt{3}}\, e^{81} + 0 + e^{42} + 0 + e^{53} \tag{4.107}$$

so that in total

$$\omega_1 = a(\underset{h}{-2e^{76}} - e^{45}) + b(\underset{m}{\frac{2}{\sqrt{3}}\, e^{81} + e^{42} + e^{53}}) + c(\underset{v}{e^{23}})\,, \tag{4.108}$$

for some constants $a, b, c;$now we calculate $\langle \omega^1\,,\, \omega^1\rangle$:

$$\begin{aligned}\langle \omega_1\,,\, \omega_1\rangle = \langle &a(-2e^{76} - e^{45}) + b(\frac{2}{\sqrt{3}}\, e^{81} + e^{42} + e^{53}) + c(e^{23}) \\ &,\ a(-2e^{76} - e^{45}) + b(\frac{2}{\sqrt{3}}\, e^{81} + e^{42} + e^{53}) + c(e^{23})\rangle\end{aligned}$$

that more explicitely becomes

$$\begin{aligned}&(-2a(76 - 67) - a(45 - 54) + \\ &\frac{2}{\sqrt{3}}b(81 - 18) + b(42 - 24) + b(53 - 35) + c(23 - 32))^2\end{aligned}$$

and contracting we obtain

$$\begin{aligned}&-4a^2(77 + 66) - a^2(55 + 44) - \frac{4}{3}b^2(11 + 88) - b^2(44 + 22) - b^2(55 + 33) \\ &- c^2(22 + 33) + ab(52 + 25) - bc(25 + 52) - ab(43 + 34) + bc(43 + 34)\end{aligned}$$

where the last four summands cancel each other if $a = c$. Moreover if we impose the conditions

$$4a^2 = \frac{4}{3}b^2 = a^2 + b^2$$

the remaining terms have equal norms; in particular if we choose $a = \frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$) we obtain

$$(\omega_1)^2 = -I\,.$$

2)The 2-form ω_2.
We can do the same calculation for ω^2: first of all we have

$$\omega_2^m = -\frac{1}{\sqrt{3}}e^{82} + e^{72} + e^{63} + e^{41}$$

so that

$$\omega_2 = a(\sqrt{3}e^{85} + e^{75} + e^{46}) + b(-\frac{1}{\sqrt{3}}e^{82} + e^{72} + e^{63} + e^{41}) + c(e^{31})$$

or more explicitly

$$\begin{aligned}\omega_2 =& c(31-13) - \frac{b}{\sqrt{3}}(82-28) + b(72-27) + b(63-36)\\ &+ b(41-14) + a\sqrt{3}(85-58) + a(75-57) + a(46-64)\,,\end{aligned}$$

so that squaring and contracting we obtain

$$\begin{aligned}\langle\omega_2\,,\,\omega_2\rangle =& -c^2(33+11) - \frac{b^2}{3}(88+22) - b^2(77+22) - b^2(66+33)\\ &- b^2(44+11) - 3a^2(88+55) - a^2(77+55) - a^2(44+66)\\ &- bc(34+43) + bc(16+61) + \frac{b^2}{\sqrt{3}}(87+78) + ab(25+52)\\ &- ab(25+52) + ba(34+43) - ab(16+61) - a^2\sqrt{3}(87+78)\,,\end{aligned}$$

and if we impose the conditions seen for ω_1, i.e. $a = c = \frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$ we have

$$\langle\omega_2\,,\,\omega_2\rangle = -I.$$

3)The 2-form ω_3
In the same manner we obtain for ω_3

$$\omega_3^m = -\frac{1}{\sqrt{3}}e^{83} - e^{73} + e^{62} + e^{51}$$

so that

$$\begin{aligned}\omega_3 &= a(e^{74} - \sqrt{3}e^{84} + e^{65}) + b(-\frac{1}{\sqrt{3}}e^{83} - e^{73} + e^{62} + e^{51}) + c(e^{12}) \\ &= a(74-47) - a\sqrt{3}(84-48) + a(65-56) - \frac{b}{\sqrt{3}}(83-38) \\ &- b(73-37) + b(62-26) + b(51-15) + c(12-21)\,;\end{aligned}$$

again contracting:

$$\begin{aligned}\langle\omega_3\,,\,\omega_3\rangle = &-a^2(77+44) - 3a^2(88+44) - a^2(66+55) - \frac{b^2}{3}(88+33) \\ &- b^2(77+33) - b^2(66+22) - b^2(55+11) - c^2(11+22) \\ &+ a^2\sqrt{3}(78+87) + ab(43+34) - ab(43+34) - ab(52+25) \\ &+ ab(61+16) - \frac{b^2}{\sqrt{3}}(87+78) - bc(61+16) + bc(52+25)\end{aligned}$$

and as usual for $a = c = \frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$ we obtain

$$\langle\omega_3\,,\omega_3\rangle = -I\,.$$

Collecting together what we have found we obtain the 4-form Ω in terms of the chosen basis. ■

Chapter 5

Moment mappings and realizations

In this Chapter we recall basic facts about QK moment maps, showing in the simple case of a compact simple Lie group G acting transitively on a Wolf space G/K how μ can be used to relate QK geometry to Grassmannians. Then we discuss some well-known theory coming from Nilpotent orbits, with results due to Kronheimer about the existence of HyperKäler metrics on them; then an account of results is given from Swann's theory, which is for many aspects the most natural background for our approach. After that we present the description of an explicit trajectory for the flow of $\operatorname{grad} f$ in the case of $\mathfrak{so}(4)$, together with the proportionality of $\operatorname{grad} f$ and $\operatorname{grad} g$ along it. This is used in the last two sections in order to describe explicitly the realizations of some classical Wolf spaces in $\mathbb{G}_3(\mathfrak{g})$, for $\mathfrak{g}$ the Lie algebra of a non-transitive group of isometries (in particular of cohomogeneity 1).

5.1 The moment mapping

Let G be a compact group acting on a differentiable manifold M preserving some structure present on it, as for example a symplectic structure; then it is possible to define a *moment map*

$$\mu : M \longrightarrow \mathfrak{g}^* \tag{5.1}$$

which under suitable hypotheses turns out to be G-equivariant, with respect to the Adjoint action of G on $\mathfrak{g}$. The main use of moment maps is that of operating *reductions*, which means considering a fibre of μ, which is G-invariant, and then considering the quotient space under the G action: this new manifold M' inherits the structure of the first manifold M, offering a systemathic way of obtaining new examples of symplectic manifolds (see [33]).

This procedure has been generalized to other contexts: the Hyperkähler case ([42]) and the QK case ([29] and [30]) for instance; the latter is the case of our interest. The main difference in the QK case is that μ turns out to be a section

$$\mu \in \Gamma(\mathfrak{g} \otimes S^2 H) \tag{5.2}$$

instead of a function; therefore only the zero-locus is well defined.

Observation. We point out that the moment map in the QK setting is automathically equivariant.

In [29] Galicki and Lawson proved that μ is the only section of $\mathfrak{g} \otimes S^2 H$ satisfying

$$\nabla\langle \mu,\, A\rangle = \sum (I_i \tilde{A})^\flat \otimes I_i, \tag{5.3}$$

where $A \in \mathfrak{g}$, $\tilde{A}$ is the corresponding Killing vector field, I_i is the local basis of the bundle $S^2 H$ and $\flat$ is the duality isomorphism of TM and T^*M induced by the metric. Moerover μ satisfies

$$\mu_A := \langle \mu,\, A\rangle = c\,\pi_{S^2 H}(\nabla(\tilde{A})^\flat) \tag{5.4}$$

where $\pi_{S^2 H}$ is the projection on the quaternionic bundle, seen as a subbundle of $\bigwedge^2 T^*M$, and c is a constant depending on the scalar curvature. As QK manifolds are Einstein, then the scalar curveture s is constant, therefore we will consider $c = 1$ for simplicity; we observe that the 2-tensor $\nabla\tilde{A}$ is skew-symmetric (see [55]).

In this context, Battaglia studied in [10], [11] the case where $G = S^1$ using Morse-theorethic methods, obtaining ([11])

Theorem 5.1. *The complex Grassmannian $\mathbb{G}_2(\mathbb{C}^n)$ is the only positive QK manifoldthat can be obtained as a QK quotient by a circle action.*

We will make a rather different use of the QK moment map μ: in fact we will consider Lie groups G of arbitrary dimension, but with the aim of obtaining immersions of a QK manifold in the Grassmannian $\mathbb{G}_3(\mathfrak{g})$; this is possible because in local coordinates we have

$$\mu = \sum_{i=1}^{3} \omega_i \otimes A_i \tag{5.5}$$

where ω_i are a local orthonormal basis for $S^2 H$ and A_i belong to $\mathfrak{g}$; if the A_i are linearly independent, they span a 3-dimensional subdpace $V \subset \mathfrak{g}$

which is independent from the trivialization, as the structure group for S^2H is $SO(3)$. This determines a map

$$\Psi : M \longrightarrow \mathbb{G}_3(\mathfrak{g}) \tag{5.6}$$

which is defined only at the points $x \in M$ for which $A_i(x)$ are independent; therefore our hope is that the zero locus of μ is as small as possible. As we shall see in the examples discussed in Sections 5.4 and 5.5, the zero locus will be empty, so no reduction would be possible at all in the cases we discuss. In this sense we have (see [80, Proposition 3.5]).

Proposition 5.2. *The set M_0 is an open dense subset of the union $\bigcup S$ of G orbits on M such that* $\dim S \geq 3$.

In this section we will exemplify our viewpoint obtaining the realization of the classical Wolf spaces G/K in $\mathbb{G}_3(\mathfrak{g})$, using the moment map induced by the action of the transitive group G. Let us denote by $\mathfrak{g}$ the lie algebra of G and let us consider the symmetric decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \tag{5.7}$$

with $\mathfrak{k}$ the Lie algebra of $K = N(Sp(1))$. The quaternionic subalgebra $\mathfrak{sp}(1)$ corresponds to a 3 dimensional space of Killing vector fields on M which vanish at the point eK; let us choose an orthonormal basis A_1, A_2, A_3 which correspond to Killing vector fields $\tilde{I}_i$; a well known result from the theory of symmetric spaces (see [13]) states that the covariant derivative of a Kvf $\tilde{X}$ can be expressed at a point in the following way:

$$\nabla_Y \tilde{X}|_{eK} = [X, Y]_{\mathfrak{m}}\,, \tag{5.8}$$

where $Y \in \mathfrak{m}$: from what we said before, the projection π_{S^2H} applied to the 2-form $\langle [X\,,\,\cdot\,]\,,\,\cdot\rangle \in \bigwedge^2 \mathfrak{m}$ acts as the identity. In the case of the action of the isometry group G on the Wolf space M, we can ask ourselves what are the three v_is at a given point; in other words we are trying to solve the equation

$$\pi_{S^2H}(\nabla\tilde{X}) = \sum_{i=1}^{3} \langle v_i\,,\, X\rangle \omega_i \tag{5.9}$$

in the v_is for any $X \in \mathfrak{g}$; now we observe that $\omega_i = \langle [A_i\,,\,\cdot\,]\,,\,\cdot\rangle$, and thanks to equation (5.8), putting $X = A_i$ the (5.9) becomes

$$\langle [A_i\,,\,\cdot\,]\,,\,\cdot\rangle = \sum_{i=1}^{3} \langle v_i\,,\, A_i\rangle \langle [A_i\,,\,\cdot\,]\,,\,\cdot\rangle\ ; \tag{5.10}$$

on the other hand choosing $X \in \mathfrak{m}$ the covariant derivative $\nabla\tilde{X}$ vanishes at eK, so equation (5.9) becomes

$$0 = \sum_{i=1}^{3} \langle v_i\,,\, X\rangle\langle[A_i\,,\,\cdot]\,,\,\cdot\rangle \tag{5.11}$$

and analogously if we choose $X \in \mathfrak{c}$ the left hand side vanishes when we project on S^2H. In conclusion it must be

$$v_i = A_i \quad i = 1...3\,. \tag{5.12}$$

In conclusion we can deduce the following result:

Proposition 5.3. *Let M be a Wolf space of the form G/K, with G a simple Lie group; let μ be the corresponding moment map and Ψ the induced map with values in $\mathbb{G}_3(\mathfrak{g})$; then Ψ embeds M in $\mathbb{G}_3(\mathfrak{g})$ and $\Psi(M)$ is precisely the critical manifold for the flow of* grad f *corresponding to the absolute maximum.*

5.2 Nilpotent orbits and Swann's theory

Let $\mathfrak{g}^*_{\mathbb{C}}$ denote a complex simple Lie coalgebra; the orbit $\mathcal{O}_X$ of any element X under the coadjoint action of $G_{\mathbb{C}}$ is naturally endowed with the Kirillov-Kostant-Soriau complex nondegenerate holomorphic 2-form ω, acting at the point X in the following way:

$$\omega_X(A_Y, A_Z) = \langle X, [Y, Z]\rangle \tag{5.13}$$

on tangent vectors A_Y, A_Z obtained by infinitesimal action of $Y, Z \in \mathfrak{g}_C$; because of the duality induced by the Killing form we can refer to the Lie algebra $\mathfrak{g}$ in place of the coalgebra. The presence of such *complex symplectic* structure is typical of HyperKähler manifolds, where $\omega = \omega_J + \imath\omega_K$ is holomorphic with respect to the first complex structure I. It is therfore natural to ask if exists any metric on $\mathcal{O}$ making it a HK manifold in such a way that the induced complex symplectic structure is just the one in (5.13). Kronheimer showed in [58] the existence of such metrics on orbits of nilpotent elements identifying them whit the moduli spaces of the solutions of a system of ODE called *Nahm equations* (see also [24]); the proof is based on Hitchin's work about HyperKäler metrics on moduli spaces ([41]).

Let us now describe briefly how Nahm equations arise: in the space of connections on a vector bundle V over a manifold M, those which are critical points of the Yang-Mills functional

$$YM(\nabla) = \frac{1}{2}\int_M \mathrm{Tr}(F \wedge *F) \tag{5.14}$$

are called *instantons*, and have importance in particle physics ($*$ is the Hodge operator of the metric on M and F is the curvature 2-form of the connection ∇). Equivalently, the curvature form F of such critical points satisfies the (anti)-self-dual equations

$$F = \pm * F\,. \tag{5.15}$$

Consider now the 4-dimensional Euclidean space $\mathbb{R}^4 = S^3 \times \mathbb{R}$; the Lie group $SU(2)$ is topologically S^3 and its Lie algebra $\mathfrak{su}(2)$ can be identified with a family of left invariant vector fields globally trivializing the tangent bundle TS^3, with a basis given by e_1, e_2, e_3; consider now the trivial principal G bundle $P = (S^3 \times \mathbb{R}) \times G$ for some compact Lie group G; then any linear map $L(t) : \mathfrak{su}(2) \to \mathfrak{g}$, depending on time $t \in \mathbb{R}$, determines a global $\mathfrak{g}$-valued 1-form, that is a connection form on a trivial G-bundle over $S^3 \times \mathbb{R}, SU(2)$-invariant under its left multiplication. It is always possible to find a gauge transform for which the connection takes the form

$$dt + \alpha^1 A_1 + \alpha^2 A_2 + \alpha^3 A_3\,, \tag{5.16}$$

with $\alpha^i \in \mathfrak{su}(2)^*$ satisfying $d\alpha_i = 2\epsilon_{ij}\alpha_1 \wedge \alpha_j$ and $A_i \in \mathfrak{g}$.

Consider now the Chern-Simons functional $\phi : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined as

$$\phi(A_1, A_2, A_3) = \sum_{i=1}^{3}\langle A_i, A_i\rangle + \langle A_1, [A_2, A_3]\rangle \tag{5.17}$$

whose gradient flow equations $\dot{A} = -\mathrm{grad}\,\phi$ are just the Nahm equations quoted above:

$$\begin{array}{rcl} \dot{A}_1 & = & -2A_1 - [A_2, A_3] \\ \dot{A}_2 & = & -2A_2 - [A_3, A_1] \\ \dot{A}_3 & = & -2A_3 - [A_1, A_2] \quad ; \end{array} \tag{5.18}$$

these equations are equivalent to the anti-self-duality condition (5.15) for the connection identified by a map L defined as

$$L(e_1) = A_i\,,\; i = 1, \cdots, 3\,; \tag{5.19}$$

in other words each trajectory for the gradient field $\mathrm{grad}\,\phi$ in $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ represents an instanton for the trivial bundle discussed above.

Critical points for grad ϕ turn out to be Lie algebra homomorphisms $L : \mathfrak{su}(2) \to \mathfrak{g}$: the functional ϕ is strictly related to the functional f on $\mathbb{G}_3(\mathfrak{g})$; compare in fact (5.18) with the gradient of f (3.1).

The link with the nilpotent orbits is obtained complexifying equations (5.18) and passing from $\mathfrak{su}(2)$ subalgebras in $\mathfrak{g}$ to $\mathfrak{sl}(2,\mathbb{C})$ subalgebras in $\mathfrak{g}_\mathbb{C}$.

Nilpotent orbits

Nilpotent orbits in a complex semisimple Lie algebra have been completely classified (see [21]). The nilpotent variety $\mathcal{N} \subset \mathfrak{g}_\mathbb{C}$ consisting of all the nilpotent elements in the algebra is an algebraic variety; the orbits $\mathcal{O}$ have a partial ordering described by the relation

$$\mathcal{O}_1 < \mathcal{O}_2 \quad \Leftrightarrow \quad \overline{\mathcal{O}}_2 \supset \mathcal{O}_1\,; \tag{5.20}$$

the element 0 gives the only 0-dimensional orbit, contained in the closure of all other ones. Just above of this in the ordering we have the *minimal* nilpotent orbit $\mathcal{O}_{min}$, the one generated by a nilpotent element X_ρ in a highest root space $\mathfrak{g}_\rho$; going one step more up we find the *next-to-minimal* orbits. At the other extreme of the diagram we find the *principal* orbit $\mathcal{O}_{princ}$, the one of maximal dimension, which is an open subset of $\mathcal{N}$. Nilpotent orbits are in correspondence with three dimensional Lie subalgebras, isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ and presented in standard form as a triple $\{Y, H, X\}$, with X, Y nilpotent and H semisimple; Jacobson-Morozov theorem states partly this correspondence:

Theorem 5.4. *Let $\mathfrak{g}_\mathbb{C}$ be a complex semisimple Lie algebra. if X is a nonzero nilpotent element of $\mathfrak{g}$, then there exists a standard triple for $\mathfrak{g}$ whose nilpositive element is X.*

Actually the correspondence is bijective, as two standard triple having in common a nilpotent element X are in the same $Ad_{G_\mathbb{C}}$ orbit, and analogously if they share the same semisimple element H (see [21] and [57]). Moreover, semisimple elements have the special property of satisfying

$$\langle H, \alpha \rangle \in \{0, 1, 2\} \tag{5.21}$$

for any α simple root; thus we can identify the triple, up to conjugacy, by just labelling every node of the Dynkyn diagram with $0, 1$ or 2, obtaining a so called *weighted Dynkin diagram* (wDd).

Observation. In this way we obtain at most $3^{\operatorname{rank}\mathfrak{g}}$ possible choices, proving that nilpotent orbits are finite in number; we note that for semisimple orbits the situation is much different: fo example semisimple orbits in $\mathfrak{sl}(2,\mathbb{C})$ are

$$\boxed{0 \;\text{——}\; \mathcal{O}_{min} \;\text{——}\; \mathcal{O}_{princ}}$$

Figure 5.1: Hasse diagram for $\mathfrak{sl}(3)$

$$\begin{tikzcd}
 & Ad_{G_{\mathbb{C}}}\{\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}\} \arrow[dr] & \\
\mathcal{O}_X \arrow[ur, leftrightarrow, "1:1"] \arrow[rr] \arrow[dr, leftrightarrow, "1:1"'] & & \mathbf{wDd} \\
 & \mathcal{O}_H \arrow[ur] &
\end{tikzcd} \tag{5.22}$$

Figure 5.2: Nilpotent orbits,3-dimensional Lie subalgebras and weighted Dynkin diagrams.

parametrized by $\mathbb{C}/\mathbb{Z}_2$. However this upper bound is rather rough, not all $3^{\operatorname{rank}\mathfrak{g}}$ are attached to any standard triples: for example in Table 5.1 is shown the Hasse diagram for $\mathfrak{sl}(3)$, with 3 orbits, whereas $3^{\operatorname{rank}\mathfrak{sl}(3)} = 3^2 = 9$.

We have thus a correspondence between nilpotent orbits, conjugacy classes of 3-dimensional subalgebras and weighted Dinkyn diagrams (see Figure 5.2). In the case of classical Lie algebras, nilpotent orbits can be identified by a *partition*: for example for $\mathfrak{sl}(n)$ nilpotent orbits are in bijective correspondence with the partitions of n; in this way the minimal nilpotent orbit has associate partition: $2, \underbrace{1, \cdots, 1}_{n-2}$, also denoted by $[2, 1^{n-2}]$. For the other classical algebras, $\mathfrak{sp}(n)$ and $\mathfrak{so}(n)$, which are subalgebras of $\mathfrak{sl}(N)$ for N big enough, nilpotent orbits are in bijective correspondence with appropriate subsets of all the partitions of N.

Unstable manifolds in $\mathbb{G}_3$

Wolf spaces can be identified with maximal critical submanifolds of the Grassmannian $\mathbb{G}_3(\mathfrak{g})$ for the flow of grad f, as we have seen in Chapter 3; on the other hand their Swann bundle $\mathcal{U}$ appears as the minimal nilpotent orbit $\mathcal{O}_{min}$ in the complexified algebra $\mathfrak{g}_{\mathbb{C}}$; $\mathcal{U}$ has a HyperKäler structure ([79]) and Kronheimer's results extend this property to any nilpotent orbit. It is therefore natural to ask if any of them fibres over a QK manifold, with fibre $\mathbb{H}^*/\mathbb{Z}_2$. Swann prooved that this is in fact the case: nilpotent orbits always admit an action of $\mathbb{H}^*$, so that $\mathcal{O}/\mathbb{H}^*$ is a QK manifold.

The quotient manifold can be identified again with a submanifold of the Grassmannian $\mathbb{G}_3(\mathfrak{g})$ using the functional f, but in general, for orbits which are not minimal, it is not merely a critical submanifold for $\operatorname{grad} f$ (which often has the wrong dimension), but an unstable submanifold connecting two critical sets, one of which contains the copy of $\mathfrak{su}(2)$ corresponding to any nilpotent element $X \in \mathcal{O}$ (see the Jacobson-Morosov theorem above). Let us recall the main results in this sense: consider the Chern-Simons functional ϕ, together with a map L as seen in (5.19) which defines a critical point (and hence an $\mathfrak{su}(2)$ representation in $\mathfrak{g}$); let $\mathcal{O}_L$ be the nilpotent orbit associated to $L(\mathfrak{su}(2))$ (the orbit of a nilpotent element in $L(\mathfrak{su}(2))_{\mathbb{C}}$; then we have ([80, Theorem 4.1 b), c)]):

Theorem 5.5. *Let $L : \mathfrak{su}(2) \to \mathfrak{g}$ be a critical point for the functional ϕ and let M^L denote the set of ϕ trajectories $A(t)$ such that $\lim_{t\to\infty} A(t) = 0$ and $\lim_{t\to-\infty} A(t)$ is in the same Adjoint orbit of $L(\mathfrak{su}(2))$; then M^L admits a free action of $\mathbb{H}^*/\mathbb{Z}_2$ and the quotient M_L is a QK manifold; moreover for any standard $\mathbb{C}^* \subset \mathbb{H}^*$, the quotient $M^L/\mathbb{C}^*$ is isomorphic both to $\mathbb{P}(\mathcal{O}_L)$ and to the twistor space $\mathcal{Z}_L$ of M_L.*

The link between the Chern-Simons functional ϕ and the functional f discussed in Chapter 3 is contained in the next result ([80, Theorem 5.2]:

Theorem 5.6. *Let $\mathcal{O}_L$ be a nilpotent orbit, and let M_L be its $\mathbb{H}^*$ quotient; then M_L is isomorphic to the f unstable submanifold M'_L of $\mathbb{G}_3(\mathfrak{g})$ associated to the critical set containing $L(\mathfrak{su}(2))$.*

Let us denote by $\mathcal{Z}'_L$ the twistor space of M'_L; then the isomorphisms just discussed are expressed by

$$\begin{tikzcd} \mathcal{Z}_L \arrow[rr, leftrightarrow, "\cong"] \arrow[dr, leftrightarrow, "\cong"'] & & \mathcal{Z}'_L \arrow[dl, leftrightarrow, "\cong"] \\ & \mathbb{P}(\mathcal{O}_L) & \end{tikzcd} \quad . \tag{5.23}$$

What QK manifolds can be obtained starting from a nilpotent orbit? The next result provides, using a moment map construction in the complex contact setting, an answer to this question:

Theorem 5.7. *Let M be a QK manifold whose isometry group is compact; suppose that its twistor space $\mathcal{Z}$ is homogeneous with respect to a subgroup C of the complexified group $G_{\mathbb{C}}$. Then M is locally isometric to M_L,* *where $\mathcal{O}_L$ is the nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ obtained in Theorem 5.5; in particular M has positive scalar curvature. Conversely, every nilpotent orbit in $\mathfrak{g}_C$ gives rise to a QK manifold of positive scalar curvature, with twistor space homogeneous under $G_{\mathbb{C}}$.*

Observation. The QK structure is reconstructed using the Inverse Twistor Construction ([59]).

Consider now the complexification $V_{\mathbb{C}}$ of a 3-dimensional space $V \in \mathbb{G}_3$, then the isotropic cone $\mathcal{C}$ is defined by equation

$$x^2 + y^2 + z^2 = 0 \tag{5.24}$$

with respect to an orthonormal basis e_1, e_2, e_3 of V; let us denote by $\mathcal{F} \subset \mathbb{G}_3$ the subset of 3-planes such that $\mathcal{C}$ consists of nilpotent elements; then we have (see [80, Section 3])

Proposition 5.8. *If $\mathcal{Z}$ has an open $G_{\mathbb{C}}$ orbit then $\Psi(M) \subset \mathcal{F}$.*

Here the map $\Psi : M \to \mathbb{G}_3(\mathfrak{g})$ is that induced by the QK moment map as discussed in Section 5.1; this property will be significant in Chapter 6, where the existence of an open $G_{\mathbb{C}}$ orbit on $\mathcal{Z}$ will be considered a basic hypothesis.

5.3 A trajectory for $\mathfrak{so}(4)$

In this section we study in more detail the example of $\mathfrak{g} = \mathfrak{so}(4)$. We wish to thank A.F. Swann for taking to our attention the importance of this example ([77]).

We introduce for $\mathfrak{so}(4) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$ the following way of representing 3 dimensional subspaces: let e_1, e_2, e_3 and f_1, f_2, f_3 be orthonormal bases of the subalgebras $\mathfrak{su}(2)_+$ and $\mathfrak{su}(2)_-$ respectively; then a subspace V not intersecting the $\mathfrak{su}(2)_-$ subalgebra can be described through an element $\mathbf{X} \in \mathrm{Hom}(\mathfrak{su}(2)_+, \mathfrak{su}(2)_-)$ so that a basis is given by $e_i + \mathbf{X}e_i$; with this notation the space V can be identified in $\Lambda^3\mathfrak{so}(4)$ by

$$\begin{aligned}\gamma =&(e_1 + \mathbf{X}e_1) \wedge (e_2 + \mathbf{X}e_2) \wedge (e_3 + \mathbf{X}e_3) \\ =& \, e_1 \wedge e_2 \wedge e_3 + \mathfrak{S}(e_1 \wedge \mathbf{X}e_2 \wedge \mathbf{X}e_3) \\ &+\mathfrak{S}(\mathbf{X}e_1 \wedge e_2 \wedge e_3) + \mathbf{X}e_1 \wedge \mathbf{X}e_2 \wedge \mathbf{X}e_3 \,,\end{aligned} \tag{5.25}$$

whose norm is given by

$$\begin{aligned}\|\gamma\|^2 =& \, 1 + \mathfrak{S}|\mathbf{X}e_1 \wedge \mathbf{X}e_2|^2 + \mathfrak{S}|\mathbf{X}e_1|^2 + |\mathbf{X}e_1 \wedge \mathbf{X}e_2 \wedge \mathbf{X}e_3|^2 \\ =& \, \det(I + \mathbf{X}\mathbf{X}^T) = c^{\mathbf{X}\mathbf{X}^T}(-1)\end{aligned}$$

(with the last expression we mean the "total Chern class" of the matrix $\mathbf{X}\mathbf{X}^T$ in a formal sense); in this way, as f is linear on $\Lambda^3\mathfrak{so}(4)$ and as it is nonzero only in the e_{123} and f_{123} directions, it can be expressed in function of $\mathbf{X}$ as

$$f(\mathbf{X}) = f(e_{123})\langle\frac{\gamma}{\|\gamma\|}\,,\, e_{123}\rangle + f(f_{123})\langle\frac{\gamma}{\|\gamma\|}\,,\, f_{123}\rangle \tag{5.26}$$

$$= \frac{\sqrt{2}(1+\det(\mathbf{X}))}{(\det(I+\mathbf{X}\mathbf{X}^T))^{\frac{1}{2}}}\,. \tag{5.27}$$

Now if we suppose that $\mathbf{X}$ is an invertible matrix, defining $\check{\mathbf{X}} = \det(\mathbf{X})\mathbf{X}^{-1}$, we can differentiate and calculate $\operatorname{grad} f$ using this alternative description:

$$\frac{d}{dt}\,f(\mathbf{X}+tY) = \frac{d}{dt}\frac{\sqrt{2}(1+\det(\mathbf{X}+tY))}{(\det(I+(\mathbf{X}+tY)(\mathbf{X}+tY)^T))^{\frac{1}{2}}} \tag{5.28}$$

and considering the case where $A = I + \mathbf{X}\mathbf{X}^T$ is invertible one has

$$\begin{aligned}\frac{d}{dt}(\det(I+(\mathbf{X}+tY)(\mathbf{X}+tY)^T)) =& \frac{d}{dt}\,\det(I+\mathbf{X}\mathbf{X}^T \\ &+t(\mathbf{X}Y^T+Y\mathbf{X}^T)+t^2(YY^T)) \\ &= \operatorname{Tr}(\check{A}\,(\mathbf{X}Y^T+Y\mathbf{X}^T))\,,\end{aligned}$$

and on the other hand

$$\frac{d}{dt}\,\det(\mathbf{X}+tY) = \operatorname{Tr}(\tilde{\mathbf{X}}Y)\,; \tag{5.29}$$

so putting everything together we obtain

$$\begin{aligned}\operatorname{grad} f|_{\mathbf{X}}(Y) =& \frac{\sqrt{2}}{c^{\mathbf{X}\mathbf{X}^T}(-1)}\left(\operatorname{Tr}(\check{\mathbf{X}}Y)\,(c^{\mathbf{X}\mathbf{X}^T}(-1))^{\frac{1}{2}} + \right. \\ &\left. -\,(1+\det\mathbf{X})\,\frac{\operatorname{Tr}(\check{A}\,(\mathbf{X}Y^T+Y\mathbf{X}^T))}{2(c^{\mathbf{X}\mathbf{X}^T}(-1))^{\frac{1}{2}}}\right)\,;\end{aligned}$$

moreover as $\operatorname{Tr}(BC) = \operatorname{Tr}(CB)$ for any matrices B and C and as A is symmetric, we can write

$$\operatorname{grad} f|_{\mathbf{X}} = \frac{\sqrt{2}}{c^{\mathbf{X}\mathbf{X}^T}(-1)}\left((c^{\mathbf{X}\mathbf{X}^T}(-1))^{\frac{1}{2}}\check{\mathbf{X}} - \frac{1+\det\mathbf{X}}{(c^{\mathbf{X}\mathbf{X}^T}(-1))^{\frac{1}{2}}}\check{A}\mathbf{X}^T\right) \tag{5.30}$$

acting on tangent vectors by the inner product defined by $\operatorname{Tr}(\operatorname{grad} f\cdot Y)$; in particular for $\mathbf{X} = I$ we obtain the diagonal subalgebra $\mathfrak{su}(2)_\Delta$, where it's immediate to check that grad $f = 0$ as we well know. Now we can differentiate again around this point to obtain an analogous expression for the hessian. In fact using the same arguments we obtain the following expression of Hess f as a quadratic form:

$$\operatorname{Hess} f|_{\mathbf{X}=I}(Y) = \frac{c_2^Y(-1) - c_2^{(\frac{Y^T+Y}{2})}(-1) - \operatorname{Tr}(\frac{YY^T}{2}) + \frac{1}{2}(\operatorname{Tr}(Y))^2}{2}\,; \tag{5.31}$$

now it's immediate to check that the decomposition of the 9 dimensional tangent space in $\mathfrak{su}(2)$ modules corresponds to the following type of matrices:

$$\Sigma^0 = \operatorname{span}\{I\} \tag{5.32}$$
$$\Sigma^2 = \Lambda^3\mathbb{R}^3 \tag{5.33}$$
$$\Sigma^4 = S_0^2(\mathbb{R}^3)\,. \tag{5.34}$$

Observation. If we consider $\mathbf{X}$ belonging to $SO(3)$ we see that f is constant; as the critical set M at $\mathbf{X} = I$ is isomorphic to $\mathbb{RP}^3$ we can in this way identify it with $SO(3)$, and the tangent space is just $\mathfrak{so}(3) = \Lambda^3\mathbb{R}^3$.

We start now to move in the Σ^0 direction, along the curve

$$\mathbf{X}(t) = t\,I \quad , \; t \geq 0\,; \tag{5.35}$$

we notice that $V(\mathbf{X}(0)) = \mathfrak{su}(2)_+, V(x(1)) = \mathfrak{su}(2)_\Delta$ and $\lim_{t\to\infty} V(\mathbf{X}(t)) = \mathfrak{su}(2)_-$; so we obtain

$$f(\mathbf{X}(t)) = \frac{\sqrt{2}\;(1+t^3)}{(1+t^2)^{\frac{3}{2}}} \tag{5.36}$$

and

$$\operatorname{grad} f(\mathbf{X}(t)) = \frac{\sqrt{2}\,(-1+t)\,t}{(1+t^2)^{\frac{5}{2}}}\; I\,; \tag{5.37}$$

so this is an integral curve of the flow of grad f but with the wrong parametrization. Orthonormal bases of $V(\mathbf{X}(t))$ and of $V^\perp(\mathbf{X}(t))$ are given by

$$\left\{\frac{e_i + tf_i}{\sqrt{1+t^2}}\right\}_{i=1\ldots3} \quad \text{and} \quad \left\{\frac{f_i - te_i}{\sqrt{1+t^2}}\right\}_{i=1\ldots3}\,; \tag{5.38}$$

moreover $\dot{\mathbf{X}}(t) = I$, which represents the homomorphism snding $e_i + tf_i$ to f_i, whose projection on $V^\perp$ is given by

$$\left\langle f_i\,,\, \frac{f_i - te_i}{\sqrt{1+t^2}}\right\rangle \frac{f_i - te_i}{\sqrt{1+t^2}} = \left(\frac{1}{\sqrt{1+t^2}}\right)\frac{f_i - te_i}{\sqrt{1+t^2}} \tag{5.39}$$

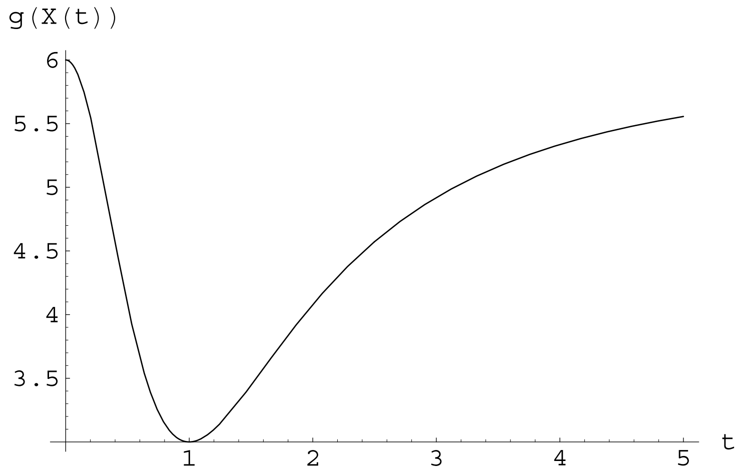

Figure 5.3: The graph of $g(\mathbf{X}(t))$

so that as an element of $V \otimes V^{\perp}$, with respect to the given orthonormal basis,

$$\dot{\mathbf{X}}(t) = \frac{1}{1+t^2}\, I\,; \tag{5.40}$$

so with respect to this basis we obtain

$$\operatorname{grad} f(\mathbf{X}(t)) = \frac{\sqrt{2}\,(-1+t)\,t}{(1+t^2)^{\frac{3}{2}}}\, I\,. \tag{5.41}$$

Now recall that

$$g = 3f^2 + \|\operatorname{grad} f\|^2 \tag{5.42}$$

we get

$$g(\mathbf{X}(t)) = \frac{6\ (1+t^4)}{(1+t^2)^2} \tag{5.43}$$

and one can check that the only critical points are just at $t = 0$ and $t = 1$ (see Figure 5.3); this proves that $\operatorname{grad} g$ cannot be 0 on the other orbits met by $\mathbf{X}(t)$; but as the $SO(4)$ action on the unstable manifold is of cohomogeneity one, this means that the behaviour of both f and g and essentially of every Ad invariant function h on the Grassmannian (in the sense of their values and gradients) on this submanifold can be completely reconstructed from just one trajectory, just because

$$d\,Ad_g(\operatorname{grad} h|_x) = \operatorname{grad} h|_{gx} \quad \forall g \in SO(4)\,.$$

In other words g cannot have any other critical point on the whole unstable manifold M_Δ of f, moreover $\operatorname{grad} g$ cannot be orthogonal to $\operatorname{grad} f$ on it, as the $Ad_{SO(4)}$ action preserves the scalar product. All this mechanism holds in general for any G simple group, so if we know $\langle \operatorname{grad} f\,,\, \operatorname{grad} g\rangle$ on one trajectory emanting from this kind of critical manifold, that means that it lies on the f unstable Quaternionic-Kähler submanifold $M(\mathcal{O})$ corresponding to a next-to-minimal nilpotent orbit $\mathcal{O}$ (see [22]), then we know it on the whole submanifold. In particular we have the following

Proposition 5.9. *The unstable manifolds emanating from the critical set $C_\Delta \subset \mathbb{G}_3(\mathfrak{so}(4))$ for the flows of* $\operatorname{grad} f$ *and* $\operatorname{grad} g$ *are the same:*

$$M^f_\Delta = M^g_\Delta\,. \tag{5.44}$$

Proof. Recall the usual way of representing the gradients of f and g and compare them on the trajectory $\mathbf{X}(t)$: we have

$$\left[\frac{e_i + t f_i}{\sqrt{1+t^2}}\,,\, \frac{e_j + t f_j}{\sqrt{1+t^2}}\right] = \frac{[e_i, e_j] + t^2 [f_i, f_j]}{1+t^2}$$

and recalling that

$$[e_i, e_j] = \pm\sqrt{2} e_k \quad \text{and} \quad [\mathrm{f_i}, \mathrm{f_j}] = \pm\sqrt{2}\mathrm{f_k}$$

(the sign depending on the parity of the permutation ijk), we can go on calculating the j-th components of the gradients; so

$$\begin{aligned}
-\frac{\operatorname{grad} g_j}{2} &= \left[\frac{e_i + t f_i}{\sqrt{1+t^2}}, \frac{[e_i, e_j] + t^2[f_i, f_j]}{1+t^2}\right]^\perp + \left[\frac{e_k + t f_k}{\sqrt{1+t^2}}, \frac{[e_k, e_j] + t^2[f_k, f_j]}{1+t^2}\right]^\perp \\
&= \left(\frac{[e_i, [e_i, e_j]] + t^3[f_i, [f_i, f_j]]}{(1+t^2)^{\frac{3}{2}}}\right)^\perp + \left(\frac{[e_k, [e_k, e_j]] + t^3[f_k, [f_k, f_j]]}{(1+t^2)^{\frac{3}{2}}}\right)^\perp \\
&= -4\left(\frac{e_j + t^3 f_j}{(1+t^2)^{\frac{3}{2}}}\right)^\perp = -4\left\langle \frac{e_j + t^3 f_j}{(1+t^2)^{\frac{3}{2}}}\,,\, \frac{f_j - t e_j}{\sqrt{1+t^2}}\right\rangle \frac{f_j - t e_j}{\sqrt{1+t^2}} \\
&= 4\left(\frac{t - t^3}{(1+t^2)^2}\right) \frac{f_j - t e_j}{\sqrt{1+t^2}}
\end{aligned}$$

and as we have seen

$$\operatorname{grad} f_j = \sqrt{2}\left(\frac{t^2 - t}{(1+t^2)^{\frac{3}{2}}}\right) \frac{f_j - t e_j}{\sqrt{1+t^2}}\,. \tag{5.45}$$

Then we have

$$\operatorname{grad} g = 4\sqrt{2}\left(\frac{1+t}{\sqrt{1+t^2}}\right) \operatorname{grad} f \tag{5.46}$$

M	G	M	G
$\mathbb{HP}^n$	$Sp(n)$	$\tilde{\mathbb{G}}_4(\mathbb{R}^n)$	$SO(n-1)$
$\mathbb{HP}^n$	$SU(n+1)$	$\tilde{\mathbb{G}}_4(\mathbb{R}^7)$	G_2
$\mathbb{HP}^n$	$\prod_i Sp(k_i)$	$G_2/SO(4)$	$SU(3)$
$\mathbb{G}_2(\mathbb{C}^n)$	$SU(n-1)$	$F_4/Sp(3)Sp(1)$	$Spin(9)$
$\mathbb{G}_2(\mathbb{C}^{2n})$	$Sp(n)$	$E_6/SU(6)Sp(1)$	F_4

Table 5.1: Cohomogeneity 1 actions of semisimple Lie Goups on compact QK manifolds

on $\mathbf{X}(t)$ and in consequence on all M_Δ; hence the conclusion. ■

Observation. This example is fundamental for us in the sense that $\mathfrak{so}(4)$ can be found as a subalgebra in many other Lie algebras $\mathfrak{g}$, so that equivariance arguments can help to generalize what is known in this case to the ambient algebras. This will be exploited in the next Sections in order to identify the image of the map Ψ in $\mathbb{G}_3(\mathfrak{g})$. Kobak and Swann used the abundance of copies of $\mathfrak{so}(4)$ to find Hyperkhäler potential on cohomogeneity 2 nilpotent orbits (see [52, Theorem 4.2]).

5.4 Realizations in cohomogeneity 1: $\mathbb{HP}^n$

We reprise the theme of Section 5.1 asking what happens if we apply the same ideas when a subgroup H of G acts on M: what is $\Psi_H(M)$ for the induced Ψ_H? First we observe that in this case $\mathfrak{h} \subset \mathfrak{g}$ and there is an inclusion

$$\mathbb{G}_3(\mathfrak{h}) \hookrightarrow \mathbb{G}_3(\mathfrak{g}) \; ; \tag{5.47}$$

moreover because of the equivariance of Ψ_H the H orbit through eK is sent to the intersection $\Psi(M) \cap \mathbb{G}_3(\mathfrak{h})$. Before starting, we display in Table 5.1 a list of all cohomogeneity 1 actions on compact QK manifolds, see [22, Theorem 7.4], for a detailed proof, and also [56] for a classification of cohomogeneity 1 and hyperpolar actions.

In this section we will make use of some properties of rank one symmetric spaces; we collect the main results, taken from [35, Chap.VII,Prop.10.2 and Theor.10.3], in the following proposition.

Proposition 5.10. *Let M be a compact Riemannian globally symmetric space. Then M has a simply closed geodesic.If M is rank one, then all geodesics are simply closed and have the same lenght.Moreover in this case, let*

$2L$ denote the common lenght of the geodesics: if p is any point in M then $Exp : T_pM \to M$ is a diffeomorphism of the open ball $\|X\| < L$ in T_pM onto the complement $M - A_p$, where $A_p = Exp(\{\|X\| = L\})$ is called the antipodal set *for p. This last, with the Riemannian structure induced by that of M, is a Riemannian globally symmetric space of rank one, and a totally geodesic submanifold of M.*

We pass to a fundamental example.

$SO(4)$ acting on $\mathbb{HP}(1)$

The group $SO(4) \subset Sp(2)$ is the stabilizer of a point in the standard presentation of the quaternionic projective space:

$$\mathbb{HP}^1 = \frac{Sp(2)}{Sp(1)Sp(1)};$$

we can fix a point (the class of e) and decompose the Lie algebra of $Sp(2)$ as usual:

$$\mathfrak{sp}(2) = \mathfrak{so}(4) \oplus \mathfrak{m}$$

where $\mathfrak{m} = [E \otimes H]$ quaternionically; this means that if we decompose $\mathfrak{so}(4)$ as $\mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$ then E and H represent the standard complex representations of the two summands (that extend globally to the Spin bundles). Now if we act on $\mathbb{HP}(1)$ by the action of elements $g \in Sp(2)$ we get analogous decompositions, so that at the point corresponding to the class of g we get

$$\mathfrak{sp}(2) = Ad_g\mathfrak{so}(4) \oplus Ad_g\mathfrak{m}\,. \tag{5.48}$$

We can now decompose elements of $\mathfrak{so}(4)$ projecting them on the summands of the new decomposition: so if $X \in \mathfrak{so}(4)$ then

$$X = \pi_{1,g}X + \pi_{2,g}X \tag{5.49}$$

where $\pi_{i,g}$ is the projection on the i-th summand in (5.48); we observe that the quaternionic structure at the point corresponding to g is provided by $Ad_g\mathfrak{su}(2)_-$ by adjoint action on $Ad_g\mathfrak{m}$. We have a representation of $\mathfrak{so}(4)$ on the module $T_{gSO(4)}\mathbb{HP}^1 \cong Ad_g\mathfrak{m}$ via the adjoint action, but the π_2 component has no effect in consequence of the symmetric space structure; so the projection on $Ad_g\mathfrak{so}(4)$ is the significant part, and in particular the subspace

$$\pi_{1,g}^{-1}\left(Ad_g\mathfrak{su}(2)_-\right) \cap \mathfrak{so}(4) \tag{5.50}$$

acts as $\mathrm{Im}\mathbb{H}$ on $Ad_g\mathfrak{m}$; so it becomes interesting understanding (5.50) in order to explain the interaction between the action of the fixed $SO(4)$ and quaternionic bundle. We have the following proposition:

Proposition 5.11. *Let Ψ be the map induced by te moment map μ of the action of $SO(4)$ on $\mathbb{HP}^1$ with values in $\mathbb{G}_3(\mathfrak{so}(4))$; then $\Psi(\mathbb{HP}^1) = M_\Delta \cup \mathfrak{su}(2)_+ \cup \mathfrak{su}(2)_-$, where M_Δ is the unstable manifold associated to the critical manifold containing $\mathfrak{su}(2)_\Delta$; in particular:*

$$\Psi(N) = \mathfrak{su}(2)_- \tag{5.51}$$
$$\Psi(S) = \mathfrak{su}(2)_+ \tag{5.52}$$
$$\Psi(\gamma(\pi/4)) = \mathfrak{su}(2)_\Delta \,. \tag{5.53}$$

Proof: We choose now the following copy of $U(1) \subset Sp(2)$:

$$g(t) = \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix} = \exp \begin{pmatrix} 0 & t & 0 & 0 \\ -t & 0 & 0 & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & -t & 0 \end{pmatrix} , \tag{5.54}$$

where the matrix on the right is denoted by $t\,u$. If we identify the north pole N of $S^4 \cong \mathbb{HP}^1$ with the class of e, this subgroup (we observe that $U(1) \not\subset SO(4)$) moves along a geodesic $\gamma(t)$ connecting N $(t=0)$ with the south pole S $(t = \pi/2)$ passing through the equator $(t = \pi/4)$, and then backwards to N $(t = \pi)$. The curve obtained is a section with respect to the action of $SO(4)$, in fact it is obtained by exponentiation of a straight line passing through the origin in the tangent space at the point N, which is the standard representation of $SO(4)$ on $\mathbb{R}^4$; the $SO(4)$ equivariance of the map Exp implies that $\gamma(t)$ is transverse to each orbit (the principal ones are copies of S^3 and the singular ones are N and S) and intersects all of them. Moreover the stabilizer of the $SO(4)$ action is constant along the curve on points that are different from N and S, and coincides with $SO(3)_\Delta$, both along $\gamma(t)$ in $\mathbb{HP}^1$ and along $\mathfrak{u}(1)$ for the isotropy representation. So restricting to the interval $t \in (0, \pi/2)$, the nonvanishing Kvf are given by $\pi_{2,g}\mathfrak{so}(4)$ and some calculations shows in fact that

$$\pi_{2,g}(\mathfrak{su}(2)_\Delta) \equiv 0\,;$$

on the contrary the only Kvf whose covariant derivative is nonzero at the points $\gamma(t)$ are precisely $\pi_{1,g}^{-1}(Ad_g\mathfrak{so}(4))$; in particular if $X \in \mathfrak{so}(4)$ then

$$\nabla_{(\cdot)}\tilde{X} = [\pi_{1,g(t)}X \,,\, \cdot\,] \tag{5.55}$$

at the given point; but the only vectors whose induced antisymmetric endomorphism is contained in the quaternionic algebra are those given by (5.50), as we already observed.

Now we recall the notation used in section 5.3 to study the flow of grad f in $\mathfrak{so}(4)$: e_i and f_i denote orthonormal bases of $\mathfrak{su}(2)_+$ and $\mathfrak{su}(2)_-$ respectively; as $\mathfrak{so}(4)$ is a subalgebra of $\mathfrak{sp}(2)$ corresponding to the longest root, the elements of the two copies of $\mathfrak{su}(2)$ correspond to the following matrices:

$$e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \imath & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\imath & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad f_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \imath & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\imath \end{pmatrix} , \tag{5.56}$$

$$e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad f_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{5.57}$$

and

$$e_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & \imath & 0 \\ 0 & 0 & 0 & 0 \\ \imath & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad f_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \imath \\ 0 & 0 & 0 & 0 \\ 0 & \imath & 0 & 0 \end{pmatrix} ; \tag{5.58}$$

so if $e_i(t)$ and $f_i(t)$ denote an orthonormal basis of $Ad_{g(t)}\mathfrak{so}(4)$, we get via the Killing metric:

$$\langle e_i\,,\, f_j(t)\rangle = \delta^i_j \sin^2 t \tag{5.59}$$
$$\langle e_i\,,\, e_j(t)\rangle = \delta^i_j \cos^2 t \tag{5.60}$$
$$\langle f_i\,,\, e_j(t)\rangle = \delta^i_j \sin^2 t \tag{5.61}$$
$$\langle f_i\,,\, f_j(t)\rangle = \delta^i_j \cos^2 t\,; \tag{5.62}$$

the conclusion is that at the point $\gamma(t)$

$$\pi_{g(t)\mathfrak{su}(2)_-}(e_i) = \sin^2 t\; f_i(t) \quad , \quad \pi_{g(t)\mathfrak{su}(2)_-}(f_i) = \cos^2 t\; f_i(t)\,, \tag{5.63}$$

or in terms of Kvf

$$\pi_{S^2H}(\nabla\tilde{e}_i) = \sin^2 t\; f_i(t) \quad , \quad \pi_{S^2H}(\nabla\tilde{f}_i) = \cos^2 t\; f_i(t) \tag{5.64}$$

and the moment map for the action of $SO(4)$ on $\mathbb{HP}^1$ along $\gamma(t)$ is given by

$$\boxed{\mu(\gamma(t)) = \textstyle\sum_i \omega_i \otimes\, (\cos^2 t\; f_i + \sin^2 t\; e_i)\,,} \tag{5.65}$$

up to a constant. This is the only information that we need to reconstruct the moment map on the whole $\mathbb{HP}^1$, as γ intersects all the orbits and the moment map is equivariant.

We can now interpret these facts in terms of the induced map Ψ: if we look for a moment to equation (5.65) and we go back to (5.35), we can see that the span of $v_i(t)$ consists exactly of the 3-planes $V(t)$ along the trajectory that generates the unstable manifold for grad f in $\mathbb{G}_3(\mathfrak{so}(4))$: the conclusion follows. ■

Observation. The map Ψ is not injective. The points corresponding to t and $\pi - t$ are sent to the same 3-plane; so the principal orbits of type S^3 in $\mathbb{HP}^1$ are sent to the orbits of type $\mathbb{RP}^3$ in M_Δ. The map Ψ becomes injective on the orbifold $\mathbb{HP}^1/\mathbb{Z}_2$.

Generalizing to $\mathbb{HP}^n$

We go now to consider $\mathbb{HP}^n$ under the action of $Sp(n)Sp(1)$, the subgroup of $Sp(n+1)$ which stabilizes a point, obtaining the following proposition which extends 5.11:

Proposition 5.12. *Let $G = Sp(n)Sp(1) \subset Sp(n+1)$ be the stabilizer of a point N of $\mathbb{HP}^n$; then the map Ψ induced by the action of G with values in the Grassmannian $\mathbb{G}_3(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)_-)$ sends $\mathbb{HP}^n$ in $M'_\Delta \cup \mathbb{HP}(n-1) \cup \mathfrak{su}(2)_-$; the map Ψ is compatible with the standard inclusion of $\mathbb{HP}^1$ in $\mathbb{HP}^n$ as the following commutative diagram shows:*

$$\begin{array}{ccc} \mathbb{HP}^n & \xrightarrow[2:1]{\Psi_1} & \mathbb{G}_3(n,1) \\ \uparrow & & \uparrow \\ \mathbb{HP}^1 & \xrightarrow[\Psi_2]{2:1} & \mathbb{G}_3(\mathfrak{so}(4))\,; \end{array} \tag{5.66}$$

moreover we have the same equations as in Proposition 5.11, where S belongs to A_N.

Proof. At a Lie algebra level we have the following decompositions:

$$\mathfrak{sp}(n+1) = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)_- \oplus \mathfrak{n} \tag{5.67}$$

where $\mathfrak{n}$ is as usual the quaternionic isotropy representation $[E \otimes H]$, of cohomogeneity one; then,also:

$$\mathfrak{sp}(n+1) = \mathfrak{sp}(n-1) \oplus \mathfrak{sp}(1)_+ \oplus \mathfrak{n}' \oplus \mathfrak{sp}(1)_- \oplus \mathfrak{n}\,; \tag{5.68}$$

we added the subscripts $+, -$ to distinguish the copies of $\mathfrak{sp}(1)$, and also to be consistent with the previous paragraph: in fact we can find a copy of $\mathfrak{sp}(2) \subset \mathfrak{sp}(n+1)$ such that

$$\mathfrak{sp}(2) \cap (\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)_-) = \mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_- \tag{5.69}$$

$$[\mathfrak{sp}(2)\,,\,\mathfrak{sp}(n-1)] = 0 \tag{5.70}$$

$$\langle \mathfrak{sp}(2)\,,\,\mathfrak{n}' \rangle = 0\,, \tag{5.71}$$

so that we can exploit the results for $\mathbb{HP}^1$: in fact the Lie algebra of the principal stabilizer of the isotropy representation is precisely $\mathfrak{sp}(n-1) \oplus \mathfrak{su}(2)_\Delta$, where $\mathfrak{su}(2)_\Delta \subset \mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_-$, and if we take the element in (5.54) we obtain a geodesic as for $\mathbb{HP}^1$, with antipodal point for $t = \pi/2$; evolving $\mathfrak{sp}(n+1)$ under adjoint action, we obtain the usual effect on the copy of $\mathfrak{sp}(2)$ obviously, and moreover $\mathfrak{sp}(n-1)$ is left fixed, because of condition (5.70); regarding $\mathfrak{n}'$, in view of (5.71) and as $\mathfrak{sp}(2)$ is preserved by $Ad_{g(t)}$, we can conclude that it remains orthogonal to the latter under this action, so that in particular $\langle Ad_{g(t)}\mathfrak{n}'\,,\,\mathfrak{sp}(1)_-\rangle \equiv 0$. So reasoning exactly as for $\mathbb{HP}^1$ we see that the only Kvf along $\gamma(t)$ with nonzero covariant derivative are those given by $\mathfrak{sp}(n-1) \oplus \mathfrak{su}(2)_\Delta$, but the projection on $\mathfrak{sp}(1)_-$ kills the elements in $\mathfrak{sp}(n-1)$.

The conclusion is that the image in the Grassmannian $\mathbb{G}_3(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))$ (from now on in this section denoted as $\mathbb{G}_3(n,1)$ for brevity) under the Ψ induced by the moment map of this action is given by the 3 planes obtained in $\mathfrak{so}(4) = \mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_-$ under the inclusion

$$\mathfrak{so}(4) \subset \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)_- \subset \mathfrak{sp}(n+1)\,. \tag{5.72}$$

Now consider the critical manifolds in $\mathbb{G}_3(n,1)$: as the algebra is not simple, we get 2 absolute maxima, a point corresponding to $\mathfrak{sp}(1)_-$ and a copy of $\mathbb{HP}(n-1)$ corresponding to $\mathfrak{sp}(n)$; another critical point is given by $\mathfrak{su}(2)_\Delta$, included in $\mathfrak{so}(4)$ as in (5.72); we describe the decomposition of the tangent space of the Grassmannian at $\mathfrak{su}(2)_\Delta$, in terms of the representations of the whole stabilizer $Sp(n-1)SO(3)_\Delta$:

$$\begin{aligned}\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)_- &= \mathfrak{sp}(n-1) \oplus \mathfrak{sp}(1)_+ \oplus \mathfrak{n}' \oplus \mathfrak{sp}(1)_- \\ &= [S^2E \otimes \mathbb{C}] + [\Sigma^2_\Delta] + [E \otimes H_+] + [\Sigma^2]\end{aligned}$$

with E, H_+ are the standard $\mathfrak{sp}(n-1)$ and $\mathfrak{sp}(1)_+$ representations as usual, and the Σ^2 is the antidiagonal representation in $\mathfrak{so}(4)$; so, in terms of complex representations,

$$T_{\mathfrak{su}(2)_\Delta}\mathbb{G}_3(n,1) = \Sigma^2_\Delta \otimes (E \otimes \Sigma^1 + S^2E \otimes \Sigma^0 + \Sigma^2_\Delta + \Sigma^2) \tag{5.73}$$
$$= E \otimes (\Sigma^3 + \Sigma^1) + S^2E \otimes \Sigma^2 + (\Sigma^4 + \Sigma^2 + \Sigma^0) \tag{5.74}$$

so that the tangent space to the $4n-1$ dimensional critical manifold C_Δ is given by

$$T_{\mathfrak{su}(2)_\Delta}C_\Delta = E \otimes \Sigma^1 + \Sigma^2 \tag{5.75}$$

and the unstable bundle by

$$W = \Sigma^0\,; \tag{5.76}$$

this implies that C_Δ is a principal orbit too, in particular the $\mathbb{Z}_2$ quotient of S^{4n-1}, and moreover that the cohomogeneity of the unstable manifold M'_Δ is one, as usual when the corresponding nilpotent orbit is next-to-minimal.

Remark. In fact if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two simple Lie algebras, $\mathcal{O}_1$ and $\mathcal{O}_2$ the minimal nilpotent orbits of $\mathfrak{g}_i \otimes \mathbb{C}$ (of cohomogeneity 1 with respect to the G action), then

$$\mathrm{cohom}_{G_1G_2}\mathcal{O}_1 \times \mathcal{O}_2 = \mathrm{cohom}_{G_1}\mathcal{O}_1 + \mathrm{cohom}_{G_2}\mathcal{O}_2 = 2 \tag{5.77}$$

and $\mathcal{O}_1 \times \mathcal{O}_2$ is next-to-minimal in the partial order for both minimal orbits (in fact contains them).

We have in conclusion the following commutative diagram of inclusions, thanks to the compatibility of grad f with the inclusion of subalgebras:

$$\begin{tikzcd} M'_\Delta \arrow[r, hook, "i"] & \mathbb{G}_3(n,1) \\ M_\Delta \arrow[u, hook] \arrow[r, hook, "i"'] & \mathbb{G}_3(\mathfrak{so}(4))\,. \arrow[u, hook] \end{tikzcd}$$

which implies the result. ■

Observations. 1)The antipodal set $A_N = Exp(S^{4n-1}(\pi/2))$ of the point N is sent $Sp(n)Sp(1)$-equivariantly to $\mathbb{HP}(n-1)$, and if we leave $SO(4)$ acting on the tangent space we get an $S^3 \subset S^{4n-1}$ which is sent in the same point by the exponential map; so we have the following composition of equivariant maps:

$$\Psi \circ Exp : S^{4n-1} \longrightarrow \mathbb{HP}(n-1)\,, \tag{5.78}$$

whose fibre contains S^3; then we have the following sequence of fibrations:

$$S^{4n-1} = \frac{Sp(n)Sp(1)}{Sp(n-1)SU(2)_\Delta} \xrightarrow{Exp} A_N \xrightarrow{\Psi} \frac{Sp(n)Sp(1)}{Sp(n-1)SO(4)}\ ; \tag{5.79}$$

for dimensional reasons we conclude that the antipodal set is precisely $\mathbb{HP}(n-1)$, a well known fact (see [19]).

2)It is tempting to restrict to the subgroup $Sp(n) \subset Sp(n+1)$; its action is still of cohomogeneity 1 on $\mathbb{HP}^n$, so it a good candidate for an immersion compatible with the next-to-minimal unstable manifolds. However looking more carefully we can see that in this case the moment map becomes

$$\mu(\gamma(t)) = \frac{1}{\lambda}\sum_i \omega_i \otimes \sin^2 t\ e_i\,, \tag{5.80}$$

so it degenerates at the point N, moreover $\Psi(\gamma(t))$ for $t \neq 0$ is constantly $\mathfrak{su}(2)_+$, and the whole manifold is squashed onto the maximal critical $\mathbb{HP}(n-1)$ in $\mathbb{G}_3(\mathfrak{sp}(n))$; then there is no intersection of $\Psi(\mathbb{HP}^n)$ with M_Δ, and in consequence no hope of using the dynamics of grad f to describe the quaternionic structure. We will see in the next section that in this sense the action of $Sp(n)$ will be more efficient on the complex Grassmannians.

3)The $\mathbb{Z}_2$ action on $\mathbb{HP}^n$ is the one induced by the symmetry at the point N. In this case M_Δ is isomorphic to (see [80]):

$$\frac{\mathbb{HP}^n\backslash(N \cup \mathbb{HP}(n-1))}{\mathbb{Z}_2}\,. \tag{5.81}$$

Remark. In Chapter 6 the conformality of $\hat{\Phi}$ will be considered as a general hypothesis; in the examples here in discussion the conformal factor is given by the function

$$\boxed{\boldsymbol{\lambda}^2(t) = \sin^4 t + \cos^4 t\,,} \tag{5.82}$$

shown in figure 5.4.

5.5 Realizations in cohomogeneity $1\colon \mathbb{G}_2(\mathbb{C}^{2n})$ and $\tilde{\mathbb{G}}_4(\mathbb{R}^n)$

We go now to discuss analogous immersions for complex Grassmannians, which are rank two symmetric spaces; this implies that we have no hope of

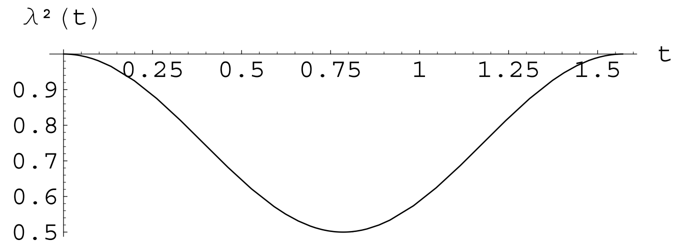

Figure 5.4: The graph of $\boldsymbol{\lambda}^2(t)$

getting a local immersion as unstable next-to-minimal manifolds using the stabilizer of a point as we did for $\mathbb{HP}^n$; instead we have to choose amongst the actions available from Table 5.1. We can apply now what we know about sections in the case of $Sp(2)$ acting on the 8-dimensional Grassmannian $\mathbb{G}_2(\mathbb{C}^4)$; as usual we obtain:

Proposition 5.13. *Let Ψ be the map induced by te moment map μ of the action of $Sp(2)$ on $\mathbb{G}_2(\mathbb{C}^4)$ with values in $\mathbb{G}_3(\mathfrak{sp}(2))$; then $\Psi(\mathbb{G}_2(\mathbb{C}^4)) = M_\Delta \cup \mathbb{HP}^1$, where M_Δ is the unstable manifold associated to the critical manifold containing $\mathfrak{su}(2)_\Delta \subset \mathfrak{so}(4) \subset \mathfrak{sp}(2)$; $\mathbb{HP}^1$ is mapped to the maximal critical manifold and the other singular orbit Q_F is mapped to C_Δ; in particular equations (5.51), (5.52) and (5.53) hold. Moreover we have the following commutative diagram:*

$$\begin{tikzcd}
\mathbb{G}_2(\mathbb{C}^4) \arrow[r, "\Psi_1", "2:1"'] & \mathbb{G}_3(\mathfrak{sp}(2)) \\
\mathbb{HP}^1_\nu \arrow[u, hook] \arrow[r, "2:1", "\Psi_2"'] & \mathbb{G}_3(\mathfrak{so}(4)) \,. \arrow[u, hook]
\end{tikzcd} \tag{5.83}$$

<u>Proof.</u> First of all we notice that $\mathfrak{sp}(2)$ can be naturally embedded in $\mathfrak{su}(4)$, in such a way that if $\mathfrak{so}(4) \subset \mathfrak{sp}(2)$ then $\mathfrak{sp}(2) \cap (\mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_- \oplus \mathfrak{u}(1)) = \mathfrak{so}(4)$; we choose the basis defined in (5.56), (5.57) and (5.58) for $\mathfrak{so}(4)$, so if

$$\mathbb{G}_2(\mathbb{C}^4) = \frac{SU(4)}{S(U(2) \times U(2))} \tag{5.84}$$

then the orbit passing through $N = e\,K$ (with $K = S(U(2) \times U(2))$) is

$$\frac{Sp(2)}{Sp(1)Sp(1)} = \mathbb{HP}^1 \,. \tag{5.85}$$

Remark. The subalgebras $\mathfrak{su}(2)_+$ and $\mathfrak{su}(2)_-$ spanned by (5.56), (5.57) and

(5.58) are $SU(4)$-conjugate to copies of $\mathfrak{su}(2)$ embedded respectively as

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \tag{5.86}$$

for $A, B \in \mathfrak{su}(2)$, via the element

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \tag{5.87}$$

thus both $\mathfrak{su}(2)_\pm$ correspond to the minimal nilpotent orbit also if we see them inside $\mathfrak{su}(4)$.

The slice representation ν_N is $\mathbb{R}^4$ with the standard $SO(4)$ action, and is spanned by

$$k_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad k_2 = \begin{pmatrix} 0 & 0 & 0 & \imath \\ 0 & 0 & -\imath & 0 \\ 0 & -\imath & 0 & 0 \\ \imath & 0 & 0 & 0 \end{pmatrix} \tag{5.88}$$

and

$$k_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad , \quad k_4 = \begin{pmatrix} 0 & \imath & 0 & 0 \\ \imath & 0 & 0 & 0 \\ 0 & 0 & 0 & \imath \\ 0 & 0 & \imath & 0 \end{pmatrix} ; \tag{5.89}$$

we choose the geodesic $\beta(t)$ identified by k_4, that is

$$g(t) = \exp(t\, k_4) = \begin{pmatrix} \cos t & \imath \sin t & 0 & 0 \\ \imath \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \imath \sin t \\ 0 & 0 & \imath \sin t & \cos t \end{pmatrix} ; \tag{5.90}$$

clearly it is closed and orthogonal to $\mathbb{HP}^1$ at $\beta(0)$,so it is a section. Following the evolution of the quaternionic structure and projecting on it the antysimmetric endomorphisms generated by $\mathfrak{sp}(2)$ as we did before, we get an expression for the moment map $\mu(\beta(t))$ which is identical to (5.65). Recall that the maximal critical manifold for the flow of grad f in $\mathbb{G}_3(\mathfrak{sp}(2))$ is $\mathbb{HP}^1$; moreover the next-to-minimal critical manifold C_Δ corresponds to the diagonal subalgebra $\mathfrak{su}(2)_\Delta \subset \mathfrak{so}(4)$, and at the level of nilpotent elements to the Jordan-type partition $\{2^2\}$ (see [21] and [52]); the critical submanifold C_Δ is the

6-dimensional homogeneous manifold

$$\frac{Sp(2)}{U(2)_\Delta}\,; \tag{5.91}$$

this $\mathfrak{su}(2)_\Delta$ is the one associated to the short root, and its centralizer is the $\mathfrak{u}(1)$ corresponding to the matrix called u in (5.54). Now $\mathfrak{sp}(2)$ decomposes in terms of $SU(2)\times U(1)$ representations as:

$$\begin{aligned}
\mathfrak{sp}(2,\mathbb{C}) &= S^2(\Sigma^1\otimes(A_m+A_{-m})) && (5.92)\\
&= \Sigma^2\otimes(A_{2m}+A_{-2m}+A_m\otimes A_{-m}) && (5.93)\\
&+\Sigma^0\otimes(A_m\otimes A_{-m}) && (5.94)\\
&= \Sigma^2+\Sigma^0+\Sigma^2\otimes(A_{2m}+A_{-2m}), && (5.95)
\end{aligned}$$

where A_m is some $U(1)$ representation; the decomposition of the Grassmannian's tangent space at C_Δ in terms of the whole isotropy group is:

$$\begin{aligned}
T_{\mathfrak{su}(2)_\Delta}\mathbb{G}_3(\mathfrak{sp}(2)) &= \Sigma^2\otimes(\Sigma^0+\Sigma^2\otimes(A_{2m}+A_{-2m})) && (5.96)\\
&= \Sigma^2\oplus(\Sigma^4+\Sigma^2+\Sigma^0)\otimes(A_{2m}+A_{-2m}) && (5.97)
\end{aligned}$$

as we have seen in (3.102) in terms of $SU(2)$ representations; the unstable bundle is

$$W=[\Sigma^0\otimes(A_{2m}+A_{-2m})]\cong\mathbb{R}^2 \tag{5.98}$$

which is of cohomogeneity 1 under the $U(1)$ action.
Looking more carefully, we can notice that

$$\mathfrak{so}(4)\oplus\nu_N\cong\mathfrak{sp}(2)_\nu\,, \tag{5.99}$$

so we have two copies of $\mathfrak{sp}(2)$ in $\mathfrak{su}(4)$, satisfying

$$\mathfrak{sp}(2)\cap\mathfrak{sp}(2)_\nu=\mathfrak{so}(4)\,; \tag{5.100}$$

geometrically this corresponds to the existence of another projective line, transversal to the first one and denoted by $\mathbb{HP}^1_\nu$; it is clear from the definitions that $\mathbb{HP}^1_\nu\supset\beta(t)$; the action of $Sp(2)$ on it reduces to the well known action of $SO(4)$.
As we saw previously, the spheres of ray $0<r<\pi/2$ in ν_N are mapped diffeomorphically onto $\mathbb{HP}^1_\nu\backslash\{N,S\}$, while the sphere of ray $\pi/2$ is sent to the single point S: so the differential of the exponential degenerates on it, and S belongs to a singular orbit. In $\mathbb{G}_2(\mathbb{C}^4)$ the first singular orbit corresponding to the points $\{0\}$ is $\mathbb{HP}^1$ and $\mathfrak{su}(2)_\Delta$ is the centralizer of k_4 in the slice representation, and as we pointed out before the differential $dExp$ is regular

for $t < \pi/2$; so the Kvf corresponfing to the antidiagonal representation in $\mathfrak{so}(4)$ are preserved under it; in conclusion the 7-dimensional principal orbits have type

$$\frac{Sp(2)}{SU(2)_\Delta}\,; \tag{5.101}$$

we notice that the evolution of the generator of the $\mathfrak{u}(1)$

$$k_9 = \begin{pmatrix} \imath & 0 & 0 & 0 \\ 0 & -\imath & 0 & 0 \\ 0 & 0 & \imath & 0 \\ 0 & 0 & 0 & -\imath \end{pmatrix}\,, \tag{5.102}$$

belonging to the algebra $\mathfrak{so}(4) \oplus \mathfrak{u}(1)$ of the stabilizer on N, along β is given by:

$$Ad_{g(t)}k_9 = \begin{pmatrix} \imath\cos 2t & \sin 2t & 0 & 0 \\ -\sin 2t & -\imath\cos 2t & 0 & 0 \\ 0 & 0 & \imath\cos 2t & \sin 2t \\ 0 & 0 & -\sin 2t & -\imath\cos 2t \end{pmatrix}\,; \tag{5.103}$$

this belongs to $\mathfrak{sp}(2)$ if and only if $t = \pi/4 + k\,\pi/2$; indeed for these values we get $Ad_{g(\pi/4)}k_9 = u$ and the orbit takes the form

$$Q_F = \frac{Sp(2)}{U(2)_\Delta}\,; \tag{5.104}$$

where F stands for *focal*: in fact $u \in \mathfrak{sp}(2)$ represents a Jacobi field which vanishes at these values of the parameter; if $\mathcal{N}$ denotes the normal bundle of $\mathbb{HP}^1$ and $\mathcal{B} \subset \mathcal{N}$ is the sphere bundle of ray $\pi/4$ inside it, then Q_F is nothing else than $Exp(\mathcal{B})$, and u is the kernel of $d\,Exp$ there.
Going on with the evolution we see that

$$Ad_{g(\pi/2)}(\mathfrak{so}(4) \oplus \mathfrak{u}(1)) \cap \mathfrak{so}(4) = \mathfrak{so}(4)\,; \tag{5.105}$$

so the second singular orbit (5.104) corresponds to $\{1\}$ in the quotient space $[0,1]$, and at $t = \pi/2$ the geodesic β meets again $\mathbb{HP}^1$, but at a different point ($\mathfrak{su}(2)_+$ and $\mathfrak{su}(2)_-$ are swapped); then it goes back to Q_F, and hence again to $\mathbb{HP}^1$, closing at $t = \pi$.
In fact the points $\Psi(\beta(\pi/4 + t))$ and $\Psi(\beta(\pi/4 - t))$ are on the same $Ad_{Sp(2)}$-orbit in $\mathbb{G}_3(\mathfrak{sp}(2))$, more precisely

$$\Psi(\beta(\pi/4 + t)) = Ad_{\beta(\pi/2)}\Psi(\beta(\pi/4 - t))\,; \tag{5.106}$$

the restriction of f to the trajectory $\Psi(\beta(t))$ is given by

$$f(\Psi(\beta(t))) = \frac{\sqrt{2}\left(\cos^6 t + \sin^6 t\right)}{\left(\cos^4 t + \sin^4 t\right)^{\frac{3}{2}}}\,, \tag{5.107}$$

with nonzero derivative in the interval $(0, \pi/4)$, hence injective: so must be Ψ when restricted to the union of orbits intersecting β in this interval.

Remark. $\mathbb{HP}^1$ and $\mathbb{HP}^1_\nu$ are embedded isometrically, quaternionically and (hence) totally geodesically in $\mathbb{G}_2(\mathbb{C}^4)$.

Observations. 1)This time we have a $\mathbb{Z}_2$ action corresponding to the isometry sending v in $-v$ on the fibre of the normal bundle $\mathcal{N}$ of $\mathbb{HP}^1$; then we have the following identification:

$$M_\Delta \cong \frac{\mathbb{G}_2(\mathbb{C}^4)\backslash\mathbb{HP}^1}{\mathbb{Z}_2}\,. \tag{5.108}$$

2)In [6] geodesics that are sections (called *normal*) are used to study cohomogeneity 1 actions; there it is introduced the concept of the *twist* of such geodesics (which actually is independent of the chosen one), defined as:

$$tw(\beta) := |\{\beta \cap P_0\}|\,, \tag{5.109}$$

where P_0 is one of the singular orbits; in the case of $\mathbb{G}_2(\mathbb{C}^4)$ with $Sp(2)$ action we have $tw(\beta) = 2$; moreover

$$tw(\beta) = \frac{1}{2}|\mathbf{W}| \tag{5.110}$$

(Theorem 6.1 in the cited paper), where $\mathbf{W}$ is the *generalized Weyl Group*, equal to $N(\beta)/K$, where $N(\beta)$ is the subgroup preserving β globally and K is the principal stabilizer, which preserves β pointwisely; in this case $\mathbf{W}$ is the dihedral group generated by reflections at the points $\beta(0)$ and $\beta(\pi/4)$, so it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Instead in the case of $\mathbb{HP}^1$ under the $SO(4)$ action we had $tw(\gamma) = 1$ and $\mathbf{W} = \mathbb{Z}_\mathbf{2}$.

Generalizing to $\mathbb{G}_2(\mathbb{C}^{2n})$

The inclusion $\mathfrak{sp}(2) \subset \mathfrak{sp}(n) \subset \mathfrak{su}(2n)$ suggests how to generalize the previous results to complex Grassmannians of dimension multiple of 8: in the same spirit of what we did with projective spaces, we have a decomposition

$$\mathfrak{su}(2n) = \mathfrak{sp}(n) \oplus \mathfrak{n} \tag{5.111}$$

and both $\mathfrak{sp}(2)$ and $\mathfrak{sp}(2)_\nu$ can be immersed in $\mathfrak{sp}(n)$ preserving the same relations seen for $\mathbb{G}_2(\mathbb{C}^4)$; in particular if we evolve along $\exp(tk_4)$, the decomposition

$$\mathfrak{su}(2n) = \mathfrak{sp}(2)_\nu \oplus \mathfrak{n}' \tag{5.112}$$

is preserved, and as $\mathfrak{su}(2)_- \subset \mathfrak{so}(4) \subset \mathfrak{sp}(2)_\nu$ then the only elements which have nonzero projection on $\mathfrak{su}(2)_-$ are the same as for $\mathbb{G}_2(\mathbb{C}^4)$: in other words we obtain again the same expression for $\mu(\beta(t))$ as in (5.65). The $\mathfrak{sp}(n)$ orbit through $\exp(\pi/4 \cdot k_4)$ is again $Exp(\mathcal{B})$ in the notation used before; this is the other singular orbit, which in this case is of type

$$Q_F = \frac{Sp(n)}{Sp(n-2) \times U(2)_\Delta}\,, \tag{5.113}$$

so that again $Q_F \cong C_\Delta$ up to a covering (see [22, Theorem 6.8]). Thus we can extend Proposition 5.13:

Proposition 5.14. *We have the exactly the same results as for Propostion (5.13), with the substitution of* $\mathbb{G}_2(\mathbb{C}^4)$ *with* $\mathbb{G}_2(\mathbb{C}^{2n})$, $\mathfrak{sp}(2)$ *with* $\mathfrak{sp}(n)$, $\mathbb{HP}^1$ *with* $\mathbb{HP}(n-1)$.

The Real Grassmannians $\tilde{\mathbb{G}}_4(\mathbb{R}^n)$

We see from Table 5.1 that $SO(n-1)$ is the only group acting with cohomogeneity 1 on $\tilde{\mathbb{G}}_4(\mathbb{R}^n)$. We note the following low-dimensional isomorphisms, which relate this case with the previous ones:

$$\mathbb{HP}^1 = \frac{Sp(2)}{Sp(1)Sp(1)} = \frac{SO(5)}{SO(4)} = S^4 \tag{5.114}$$

$$\mathbb{G}_2(\mathbb{C}^4) = \frac{SU(4)}{S(U(2)\times U(2))} = \frac{SO(6)}{SO(4)\times SO(2)} = \tilde{\mathbb{G}}_4(\mathbb{R}^6)\,; \tag{5.115}$$

the second one suggest to translate what discussed in the previous subsection for $\mathbb{G}_2(\mathbb{C}^4)$ in terms of antisymmetric matrices; we obtain more in general:

Proposition 5.15. *Let* Ψ *be the map induced by te moment map* μ *of the action of* $SO(n-1)$ *on* $\tilde{\mathbb{G}}_4(\mathbb{R}^n)$ *with values in* $\mathbb{G}_3(\mathfrak{so}(n-1))$; *then* $\Psi(\tilde{\mathbb{G}}_4(\mathbb{R}^n)) = M_\Delta \cup \tilde{\mathbb{G}}_4(\mathbb{R}^{n-1})$, *where* M_Δ *is the unstable manifold associated to the critical manifold containing* $\mathfrak{su}(2)_\Delta \subset \mathfrak{so}(4) \subset \mathfrak{so}(n-1)$; $\tilde{\mathbb{G}}_4(\mathbb{R}^{n-1})$ *is mapped to the maximal critical manifold; in particular equations (5.51), (5.52) and (5.53) hold. Moreover we have the following commutative diagram:*

$$\begin{tikzcd} \tilde{\mathbb{G}}_4(\mathbb{R}^n) \arrow[r, "\Psi_1", "2:1"'] & \mathbb{G}_3(\mathfrak{so}(n-1)) \\ \mathbb{HP}^1_\nu \arrow[u, hook] \arrow[r, "2:1", "\Psi_2"'] & \mathbb{G}_3(\mathfrak{so}(4)) \arrow[u, hook] \end{tikzcd} \quad . \tag{5.116}$$

Proof. The following matrix explains pictorically the inclusions of $\mathfrak{su}(3) \subset \mathfrak{so}(4)$ and ν_N in $\mathfrak{so}(6) \cong \mathfrak{su}(4)$:

$$\begin{pmatrix} \bullet & \bullet & \bullet & \circ & 0 & \star \\ \bullet & \bullet & \bullet & \circ & 0 & \star \\ \bullet & \bullet & \bullet & \circ & 0 & \star \\ \circ & \circ & \circ & \circ & 0 & \star \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \end{pmatrix} \tag{5.117}$$

where $\bullet$ represent elements in $\mathfrak{so}(3)$, $\circ$ elements in $\mathfrak{so}(4)\backslash\mathfrak{so}(3)$, or in other words the antidiagonal elements, and finally the $\star$ the slice representation ν_N. As observed before, $\mathfrak{so}(4) \oplus \nu_N = \mathfrak{sp}(2)_\nu$. Now the pattern is clear also in higher dimension: in fact $\mathfrak{so}(4)$ and ν_N (which is always 4 dimensional) are embedded in the same way as in (5.117), just adding more columns and rows of zeroes to the ones existing in this case. To exemplify the situation, mantaining the notation used in the previous sections, we point out that:

$$k_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & -2 & \dots & 0 \end{pmatrix}, e_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We choose to evolve again along the geodsic β with $\beta'(0) = k_4$, the most natural thing to do to extend the 8 dimensional case; the form of the moment map obatined is the usual one, exactly for the same arguments used before. In $\mathfrak{so}(n)$ exist several next-to-minimal nilpotent orbits, and correspondingly several next-to-minimal unstable manifolds, but the image of our Ψ results to be the one with related partition $\{3, 1^n\}$. Now the principal stabilizer is given by $SO(3)_\Delta \times SO(n-5)$, and the $4n-17$ dimensional principal orbit (for $n \geq 6$) is of type

$$\frac{SO(n-1)}{SO(n-5) \times SO(3)_\Delta}; \tag{5.118}$$

this time all elements in $\mathfrak{so}(n-1)$ with nonzero entries at $(4, 4+h)$ for $h = 1...n-5$, represent a family of Jacobi fields which intersect the stabilizing subalgebra $\mathfrak{so}(4) \times \mathfrak{so}(n-4)$ along β with the same periodicity as u did in the complex Grassmannians (see (5.103) and (5.104)), at $t = \pi/4 + k\,\pi/2$; in consequence these values correspond again to the second singular orbit, which has form

$$Exp(\mathcal{B}) = \mathbb{G}_3(\mathbb{R}^{n-1})_F = \frac{SO(n-1)}{SO(n-4) \times SO(3)_\Delta}, \tag{5.119}$$

which again coincides with C_Δ up to a covering (see [22]); the conclusion is the usual one. ■

Observations. 1) Again we have a $\mathbb{Z}_2$ action sending v in $-v$ in $\mathcal{N}$, inducing the equivariant diffeomorphism

$$M_\Delta \cong \frac{\tilde{\mathbb{G}}_4(\mathbb{R}^n)\backslash\tilde{\mathbb{G}}_4(\mathbb{R}^{n-1})}{\mathbb{Z}_2}\,. \tag{5.120}$$

2) Again we have $tw(\beta) = 2$ and $\mathbf{W} = \mathbb{Z}_\mathbf{2} \times \mathbb{Z}_\mathbf{2}$.

Brylinski and Kostant classification

Something is missing in the previous sections. In fact Complex Grassmannians of dimension $8n+4$ are not treated together with the other classical Wolf Spaces; we see from Table 5.1 that $SU(n)$ is the only group acting with cohomogeneity 1 on these spaces, and if we try to perform the same type of calculations as for other spaces on some low-dimensional example, we do not get the same type of answer: the image of Ψ seems not to be contained in the next-to-minimal unstable manifold as usual. This is not by chance: an explanation for this phenomenon comes from Brylinski-Kostant classification of shared nilpotent orbits: in fact in Table 5.2, taken from [49], but see also [15], we see that in the third column the algebra $\mathfrak{su}(n)$ is missing (for $n \neq 3$); if we had a finite covering map from an open subset of $\mathbb{G}_2(\mathbb{C}^{2n+1})$ to a next-to -minimal unstable manifold, then this would lift to a similar map at the level of the Swann bundles, which are respectively minimal and next-to-minimal nilpotent orbits; then $\mathfrak{su}(n)$ should appear in the table.

Wolf Space($\mathfrak{g}'$)	$\mathfrak{g}'$	$\mathfrak{g}$	k	dim $\mathcal{O}$
$\tilde{\mathbb{G}}_4(\mathbb{R}^7)$	$\mathfrak{so}(7)$	$\mathfrak{g}_2$	1	8
$\tilde{\mathbb{G}}_4(\mathbb{R}^{2n+2})$	$\mathfrak{so}(2n+2)$	$\mathfrak{so}(2n+1)$	2	$4n-2$
$\mathbb{G}_2(\mathbb{C}^{2n})$	$\mathfrak{sl}(2n)$	$\mathfrak{sp}(2n)$	2	$4n-2$
$E_6/SU(6)Sp(1)$	$\mathfrak{e}_6$	$\mathfrak{f}_4$	2	22
$G_2/SO(4)$	$\mathfrak{g}_2$	$\mathfrak{su}(3)$	3	6
$\tilde{\mathbb{G}}_4(\mathbb{R}^{2n+1})$	$\mathfrak{so}(2n+1)$	$\mathfrak{so}(2n)$	2	$4n-4$
$F_4/Sp(3)Sp(1)$	$\mathfrak{f}_4$	$\mathfrak{so}(9)$	2	16
$F_4/Sp(3)Sp(1)$	$\mathfrak{f}_4$	$\mathfrak{so}(8)$	4	16
$\tilde{\mathbb{G}}_4(\mathbb{R}^8)$	$\mathfrak{so}(8)$	$\mathfrak{g}_2$	6	10
$\mathbb{HP}^{n-1}$	$\mathfrak{sp}(n)$	$\oplus_1^k \mathfrak{sp}(n_i)$	2^{k-1}	$2n$

Table 5.2: Shared nilpotent orbits. The number k is the degree of the covering $\mathcal{O}' \to \mathcal{O}$.

Chapter 6

Latent quaternionic geometry

In this final Chapter we describe in detail the quaternionic structure as it appears through the "Grassmannian filter": the word "latent" expresses the idea that this strucure is in some sense already present in $\mathbb{G}_3(\mathfrak{g})$, but manifest itself only on the appropriate submanifolds.

6.1 The Coincidence Theorem

Suppose that the moment map μ induced by the action of a compact semi-simple Lie group G on M has the form

$$\mu(x) = \sum_i I_i \otimes B_i \,, \tag{6.1}$$

where $B_i = \boldsymbol{\lambda}(x) v_i$, for v_i an orthonormal basis of $V = \operatorname{span}\{B_1, B_2, B_3\}$; in other words, μ induces a conformal map between the S^2H bundle and the restriction of the tautological bundle $\mathbf{V}$ to $\Psi(M)$; this hypothesis is not excessively restrictive: in fact it was shown by Swann that if the twistor space $\mathcal{Z}$ has an open $G^{\mathbb{C}}$ orbit, then the image of μ is contained in the set $\mathcal{F} \subset \mathbb{G}_3$ of thoes 3-planes V such that the null cone of the complexification $V_{\mathbb{C}}$ consists of nilpotent elements (see Proposition 5.8); in this case

$$\mu_{\mathbb{C}}(I_2 + \imath I_3) = B_2 + \imath B_3 \in \mathcal{N} \,; \tag{6.2}$$

but nilpotent elements in $\mathfrak{g}_{\mathbb{C}}$ are isotropic with respect to the Killing form: in fact thanks to Engel's theorem their adjoint representation can be given in terms of strictly upper triangular matrices, with respect to a suitable basis, and the product of such matrices is still strictly upper triangular and hence traceless; in other words

$$0 = \langle B_2 + \imath B_3 \,,\, B_2 + \imath B_3 \rangle = \|B_1\|^2 - \|B_2\|^2 + 2\imath \langle B_2 \,,\, B_3 \rangle \,, \tag{6.3}$$

which implies $B_2 \perp B_3$ and $\|B_2\| = \|B_3\|$, conditions that are equivalent to conformality. We assume along the whole section that this condition holds for the moment map μ.

Recall that μ satisfies the *twistor equation*

$$\overset{M}{\nabla}\mu_A = s\sum_{i=1}^{3} I_i\tilde{A}^\flat \otimes I_i\,, \tag{6.4}$$

where $\tilde{A}$ is the Killing vector field generated by A in $\mathfrak{g}$, $\flat$ means passing to the corresponding 1-form via the metric and s is the scalar curvature, which is constant as the metric is Einstein: for simplicity we can put $s = 1$. On the other hand on $\mathbf{V}$ we have defined the sections s_A and the natural connection $\overset{\mathbb{G}}{\nabla}$ so that, as seen in Chapter 2, more precisely (2.19) and Proposition 2.3

$$\overset{\mathbb{G}}{\nabla}s_A = \sum_{i=1}^{3} s_A^\perp \otimes v_i \otimes v_i\,. \tag{6.5}$$

In general, given a differentiable embedding $\Psi : M \to N$ of manifolds, and an isomorphism $\hat{\Phi}$ between vector bundles $E \to F$ on the manifold M and N respectively, the second one equipped with a connection $\overset{F}{\nabla}$, we can define the *pullback connection* $\hat{\Psi}^*\overset{F}{\nabla}$ acting in the following way on elements σ of $\Gamma(E)$:

$$(\Psi^*\overset{F}{\nabla})_Y(\sigma) := \hat{\Psi}^{-1}(\overset{F}{\nabla}_{(\Psi_*Y)}(\hat{\Psi}\sigma)) \tag{6.6}$$

where $Y \in T_xM$. In fact

Lemma 6.1. *The operator $\hat{\Psi}^*\overset{F}{\nabla}$ defined on $\Gamma(F)$ is a connection.*

Proof. We check that it satisfies the definition of a connection: it is clearly linear in the Y, and as $\hat{\Phi}$ is a homomorphism, for $\lambda(x)$ a differentiable function on M, we have

$$\begin{aligned}
(\Psi^*\overset{F}{\nabla})_Y(\lambda\sigma) &= \hat{\Psi}^{-1}\big(\overset{F}{\nabla}_{(\Psi_*Y)}\big(\hat{\Psi}\lambda\sigma\big)\big) && (6.7)\\
&= \hat{\Psi}^{-1}\big((\Psi_*Y)(\lambda)\hat{\Psi}\sigma + \lambda(\overset{F}{\nabla}_{(\Psi_*Y)}\sigma)\big) && (6.8)\\
&= Y(\lambda)\sigma + \lambda\hat{\Psi}^{-1}\big(\overset{F}{\nabla}_{(\Psi_*Y)}\sigma\big) && (6.9)\\
&= Y(\lambda)\sigma + \lambda\big(\Psi^*\overset{F}{\nabla}\big)\sigma\,. \blacksquare && (6.10)
\end{aligned}$$

We want to apply this construction in our case, with the map $\Psi : M \to \mathbb{G}_3$ induced by μ, $N = \mathbb{G}_3$, $E = S^2H$, $F = \mathbf{V}$, to prove the *Coincidence Theorem*, which relates, at a fixed point $x \in M$, the action of the quaternionic structure on 1-forms induced by G (the duals of the Killing vector fields) with special cotangent vectors on the Grassmannian $\mathbb{G}_3$.

Before starting, we say in advance that the map $\Psi : M \to \mathbb{G}_3$ can be lifted to an isomorphis between the quaternionic bundle S^2H and the tautological bundle $\mathbf{V}$; this question will be discussed in more detail in Section 6.2, but we need it in the next proof in order to use the pullback connection.

Theorem 6.2. *Let $M, \mathfrak{g}, \mathbb{G}_3, \mu$ be defined as usual, with*

$$\mu = \sum_{i=1}^{3} I_i \otimes B_i \tag{6.11}$$

where $B_i = \boldsymbol{\lambda} v_i, \boldsymbol{\lambda}$ a differentaiable G-invariant function on M and v_i an orthonormal basis of a point $V \in \mathbb{G}_3$; let us choose $A \in V^{\perp} \subset \mathfrak{g}$; then at the point x such that $\Psi(x) = V$, for Ψ induced by μ as usual, we have

$$\boxed{\tfrac{1}{\boldsymbol{\lambda}} I_i \tilde{A}^{\flat} = \Psi^*(A \otimes v_i)\,,} \tag{6.12}$$

where $A \otimes v_i \in T^*_x\mathbb{G}_3$. Moreover we have $\|\mu\|^2 = 3\boldsymbol{\lambda}^2$.

<u>Proof.</u> Let $\hat{\Phi}$ denote the conformal lift of the map μ so that

$$\hat{\Phi}(I_i) = B_i\,; \tag{6.13}$$

then

$$\hat{\Phi}(\mu_A) = \hat{\Phi}\left(\sum_{i=1}^{3} I_i \langle B_i\,,\, A\rangle\right) = \hat{\Phi}\left(\boldsymbol{\lambda}\sum_{i=1}^{3} I_i \langle v_i\,,\, A\rangle\right) \tag{6.14}$$

$$= \boldsymbol{\lambda}\sum_{i=1}^{3} \hat{\Phi}(I_i)\langle v_i\,,\, A\rangle = \boldsymbol{\lambda}^2 \sum_{i=1}^{3} v_i \langle v_i\,,\, A\rangle \tag{6.15}$$

$$= \boldsymbol{\lambda}^2 s_A\,; \tag{6.16}$$

then applying the $\hat{\Phi}^*\overset{\mathbb{G}}{\nabla}$ connection of S^2H to μ_A we obtain

$$(\hat{\Phi}^*\overset{\mathbb{G}}{\nabla})\mu_A = \hat{\Phi}^{-1}\left(\overset{\mathbb{G}}{\nabla}(\hat{\Phi}(\mu_A))\right) = \hat{\Phi}^{-1}\left(\overset{\mathbb{G}}{\nabla}(\boldsymbol{\lambda}^2 s_A)\right) \tag{6.17}$$

$$= \frac{d(\boldsymbol{\lambda}^2)}{\boldsymbol{\lambda}^2}\mu_A + \boldsymbol{\lambda}^2\hat{\Phi}^{-1}\left(\overset{\mathbb{G}}{\nabla}s_A\right) \tag{6.18}$$

$$= \frac{d(\boldsymbol{\lambda}^2)}{\boldsymbol{\lambda}^2}\mu_A + \boldsymbol{\lambda}^2\hat{\Phi}^{-1}\left(\sum_{i=1}^{3} s_A^{\perp}\otimes v_i\otimes v_i\right) \tag{6.19}$$

$$= \frac{d(\boldsymbol{\lambda}^2)}{\boldsymbol{\lambda}^2}\mu_A + \boldsymbol{\lambda}\sum_{i=1}^{3}\Psi^*(s_A^{\perp}\otimes v_i)\otimes I_i\,; \tag{6.20}$$

on the other hand the difference of two connections on the same vector bundle is a tensor, so given any section $\sigma \in S^2H$ which vanishes at a point $x \in M$

$$(\overset{M}{\nabla} - \Psi^*\overset{\mathbb{G}}{\nabla})\sigma(x) = 0\,; \tag{6.21}$$

but this is precisely the case for the section μ_A at the point x such that $\hat{\Phi}(S^2H_x) = V$, because $A \in V^{\perp}$ by hypothesis; in other words

$$\overset{M}{\nabla}\mu_{A\,|x} = (\Psi^*\overset{\mathbb{G}}{\nabla})\mu_{A\,|x}\,; \tag{6.22}$$

thanks to the calculations in (6.17) and using the twistor equation (6.4), we can deduce

$$\sum_{i=1}^{3} I_i\tilde{A}^{\flat}\otimes I_i = \boldsymbol{\lambda}\sum_{i=1}^{3}\Psi^*(s_A^{\perp}\otimes v_i)\otimes I_i\,; \tag{6.23}$$

the result follows considering that $s_A^{\perp} = A$ at V. ■

The Coincidence Theorem leads to various ways of relating elements in $T_xM, T_V\mathbb{G}_3$ and the quaternionic elements I_i. In fact

Corollary 6.1. *Let $Y \in T_xM$ such that*

$$\Psi_*Y = \sum v_i \otimes P_i\,; \tag{6.24}$$

for $P_i \in V^{\perp}$ with $V = \Psi(x)$; let us consider a tangent vector of the form $X = (1/\boldsymbol{\lambda})\sum_i I_i\tilde{P}_i$ in T_xM; then

$$X^{\flat} = \Psi^*\big((\Psi_*Y)^{\flat}\big)\,. \tag{6.25}$$

Proof. Using the definitions and (6.12) we obtain

$$(\Psi_* Y)^\flat\big(\Psi_* Z\big) = \langle \sum v_i \otimes P_i\,,\ \Psi_* Z\rangle_{\mathbb{G}_3} \tag{6.26}$$

$$= \frac{1}{\boldsymbol{\lambda}}\langle \sum I_i \tilde{P}_i\,,\ Z\rangle_M \tag{6.27}$$

$$= X^\flat\,, \tag{6.28}$$

for any $Z \in T_x M$, hence the conclusion. ■

Observation. We are using the $\flat$ homomorphism in two different meanings, depending on what metric we are considering: in one case that on $T_V\mathbb{G}_3$ and in the other in $T_x M$.

Another consequence of the Coincidence Theorem 6.2 is

Corollary 6.2. *Under the hypotheses of Theorem 6.2 we have*

$$\begin{aligned}\Psi_* I_1 X = &\frac{1}{\boldsymbol{\lambda}}\sum_j \langle X\,,\ \tilde{A}_j\rangle_M\, v_1 \otimes A_j \\ &- \frac{1}{\boldsymbol{\lambda}}\sum_j \langle X\,,\ I_3\tilde{A}_j\rangle_M\, v_2 \otimes A_j \\ &+ \frac{1}{\boldsymbol{\lambda}}\sum_j \langle X\,,\ I_2\tilde{A}_j\rangle_M\, v_3 \otimes A_j\,.\end{aligned} \tag{6.29}$$

with $\langle,\rangle_M$ the metric on M; analogous statements are valid for I_2, I_3.

Proof. We have

$$\Psi_* X = \sum_{i,j} \langle \Psi_* X\,,\ v_i \otimes A_j\rangle_{\mathbb{G}_3}\, v_i \otimes A_j \tag{6.30}$$

$$= \frac{1}{\boldsymbol{\lambda}}\sum_{i,j} \langle X\,,\ I_i\tilde{A}_j\rangle_M\, v_i \otimes A_j \tag{6.31}$$

and applying one of the quaternionic endomorphisms we obtain immediately (6.29). ■

Observation. We notice that the first summand in (6.29) does not depend on I_1.

The equivariance of the moment map μ implies that Kvf on M are sent to Kvf on $\mathbb{G}_3$: in other words if $\tilde{A}$ is induced by $A \in \mathfrak{g}$ on M, then

$$\Psi_* \tilde{A} = \sum_{i=1}^{3} v_i \otimes [A\,,\ v_i]^\perp\,. \tag{6.32}$$

Let now $\alpha = \sum_{i=1}^{3} v_i \otimes p_i \in T_x^*\mathbb{G}_3$ and let A_r be an orthonormal basis of $V^\perp$; then

$$\sum_{r=1}^{d-3}\langle \Psi^*\alpha,\, \tilde{A}_r\rangle A_r = \sum_{r=1}^{d-3}\langle \alpha,\, \Psi_*\tilde{A}_r\rangle A_r = \sum_{i,r}\langle p_i,\, [v_i\,,\, A_r]^\perp\rangle A_r \tag{6.33}$$

$$= \sum_{i,r}\langle\, p_i,\, [v_i\,,\, A_r]\,\rangle A_r = \sum_{i,r}\langle [p_i,\, v_i]\,,\, A_r\rangle A_r \tag{6.34}$$

$$= \sum_{i}[p_i,\, v_i]^\perp\,. \tag{6.35}$$

We can therefore define a mapping

$$\rho:\ T_x^*M \longrightarrow V^\perp \tag{6.36}$$

by $\rho(\sigma) = \sum_r \langle \sigma\,,\, \tilde{A}_r\rangle A_r$; so if $\alpha \in T_x\mathbb{G}_3$, then $\Psi^*\alpha \in T_xM$, and the composition $\boldsymbol{\gamma} = \rho\circ\Psi^*$ is a map

$$\tilde{\boldsymbol{\gamma}}:\ T_x^*\mathbb{G}_3 \longrightarrow V^\perp\ . \tag{6.37}$$

defined by $\boldsymbol{\gamma}(\alpha) = \sum_i [v_i\,,\, p_i]^\perp$, obtained from the projection map $\boldsymbol{\gamma}$ defined in Chapter 3, definition 3.1, by

$$\tilde{\boldsymbol{\gamma}} = \pi^\perp \circ \boldsymbol{\gamma}\,; \tag{6.38}$$

recall that $\boldsymbol{\gamma}(\alpha) = 0$ if $\langle \alpha\,,\, \tilde{A}\rangle = 0$ for al $A \in \mathfrak{g}$, or in other words $\boldsymbol{\gamma}$ detects if α is ortogonal to the G-orbit; in consequence if α is orthogonal, then $\tilde{\boldsymbol{\gamma}}(\alpha) = 0$.

Proposition 6.3. *Let $Y \in T_xM$ so that*

$$\Psi_*Y = v_1\otimes p_1 + v_2\otimes p_2 + v_3\otimes p_3\,; \tag{6.39}$$

then we have

$$\boxed{\Psi_*I_1Y = \tfrac{1}{\boldsymbol{\lambda}}\, v_1\otimes\rho(Y^\flat) -\ v_2\otimes p_3 + v_3\otimes p_2\,.} \tag{6.40}$$

Proof. Consider any $A \in V^\perp$, then

$$\langle p_1\,,\, A\rangle_{\mathbb{G}} = \langle \Psi_*Y\,,\, A\otimes v_1\rangle_{\mathbb{G}} = \frac{1}{\boldsymbol{\lambda}}\langle I_1\tilde{A}^\flat\,,\, Y\rangle \tag{6.41}$$

$$= \frac{1}{\boldsymbol{\lambda}}\langle I_1\tilde{A}\,,\, Y\rangle_M = -\frac{1}{\boldsymbol{\lambda}}\langle \tilde{A}\,,\, I_1Y\rangle_M \tag{6.42}$$

$$= -\frac{1}{\boldsymbol{\lambda}}\langle I_1Y^\flat\,,\, \tilde{A}\rangle\,, \tag{6.43}$$

where $\langle\,,\,\rangle_{M,\mathbb{G}}$ denote the respective metrics, while $\langle\,,\,\rangle$ without subscript is merely the contraction of a cotangent and tangent vector; then considering (6.41) and (6.36)

$$p_1 = \sum_r \langle p_1\,,\,A_r\rangle_{\mathbb{G}} = -\frac{1}{\boldsymbol{\lambda}}\sum_r \langle I_1 Y^\flat\,,\,\tilde{A}_r\rangle \tag{6.44}$$

$$= -\frac{1}{\boldsymbol{\lambda}}\rho(I_1 Y^\flat) \tag{6.45}$$

and analogously

$$p_i = -\frac{1}{\boldsymbol{\lambda}}\rho(I_i Y^\flat)\,, \quad i = 2,3\,; \tag{6.46}$$

in consequence

$$\Psi_* I_1 Y = \frac{1}{\boldsymbol{\lambda}}\,v_1 \otimes \rho(Y^\flat) - \frac{1}{\boldsymbol{\lambda}}\,v_2 \otimes \rho(I_3 Y^\flat) + \frac{1}{\boldsymbol{\lambda}}\,v_3 \otimes \rho(I_2 Y^\flat) \tag{6.47}$$

$$= \frac{1}{\boldsymbol{\lambda}}\,v_1 \otimes \rho(Y^\flat) - \,v_2 \otimes p_3 + v_3 \otimes p_2\,. \blacksquare \tag{6.48}$$

Clearly analogous assertions are valid for I_2 and I_3.

Remarks. i) Proposition 6.3 predicts that if Y is perpendicular to the G-orbit on M, then

$$\rho(Y^\flat) = 0\,, \tag{6.49}$$

thanks to the definition of ρ. We can therefore expect this situation when we go to analyze the action of the quaternionic structure on the unstable manifolds of $\mathbb{G}_3$; this was be shown in a purely algebraic way on the tangent space of $\mathbb{G}_3$ at the f-critical submanifolds in Section 4.3;

ii) a striking feature of (6.40) is that in the expression obtained the first summand is independent from I_1; the operators ρ, $\boldsymbol{\gamma}$ appear to be the essential ingredient to reconstruct the quaternionic action; the other summands $-v_2 \otimes p_3 + v_3 \otimes p_2$ are obtained by just mimicking the adjoint representation of $\mathfrak{su}(2)$, as already observed in Section 4.3; this is not sufficient however to obtain

$$I_i^2 = -Id\,. \tag{6.50}$$

iii) the nature of the operator $\boldsymbol{\gamma}$ is strictly related to the Lie algebra $\mathfrak{g}$; this is relevant to the general philosophy of the thesis: in fact $\boldsymbol{\gamma}$ is one of the key links between the Grassmannians, Lie algebras and the quaternionic structures;

iv) compare 6.40 with 4.57.

6.2 The two Twistor equations

Let us consider as usual a group of isometries G preserving the quaternionic structure; in this case a Klling vector field X satisfies the condition

$$L_X\Omega = 0\,; \tag{6.51}$$

as discussed in Section 5.1 we define μ_A a section of S^2H, with μ the moment map for the G action, which satisfies the equation

$$d\mu_A = i(\tilde{A})\,\Omega\,, \tag{6.52}$$

where $\tilde{A}$ is the Killing vector field generate by A. Another way of describing the sections coming from the moment map is expressed by

$$\mu_A = \pi_{S^2H}(\nabla\tilde{A}) \tag{6.53}$$

up to a costant; this was exploited in Sections 5.4 and 5.5 to obtain examples of realizations of QK manifolds in $\mathbb{G}_3(\mathfrak{g})$.

This latter point of view can be related to the following differential operators (the symbol denoting the spaces of sections is omitted): the *Dirac operator*

$$\delta : S^2H \xrightarrow{\nabla} E\otimes H\otimes S^2H \hookrightarrow (E\otimes \underline{H})\otimes(H\otimes \underline{H}^*) \longrightarrow T^* \tag{6.54}$$

where the underlined terms are contracted and $T^* = E\otimes H$ as discussed in Section 4.1; the *QK twistor operator*, defined as follows:

$$\mathcal{D} : S^2H \xrightarrow{\nabla} E\otimes H\otimes S^2H \xrightarrow{sym} E\otimes S^3H\ , \tag{6.55}$$

where we symmetrize after covariant differentiation. In [71][Lemma 6.5] Salamon proved that sections of S^2H belonging to ker $\mathcal{D}$ are in bijection with Killing vector fields preserving the QK structure; this means that if σ is in ker $\mathcal{D}$ then $\delta(\sigma)$ is dual to a Killing vector field $\tilde{A}$, and on the other hand $\sigma = \mu_A$, or in other words

$$\mathcal{D}\,\mu_A = 0 \tag{6.56}$$

and all elements in ker $\mathcal{D}$ are of this form.

Recall now what discussed for Grassmannians in Chapter 2: there we introduced another differential operator D on the tautological bundle $\mathbf{V}$ over $\mathbb{G}_3(\mathfrak{g})$ (see Section 2.3); elements in its kernel were proved to be precisely the sections s_A obtained by projection from the trivial bundle with fibre $\mathfrak{g}$. We want to relate the kernels of $\mathcal{D}$ and D theough the map Ψ insuced by μ: first of all let us restrict to a subset $M_0 \subset M$ such that Ψ is well defined, in the sense that μ identifies a 3-dimensional subspace in $\mathfrak{g}$; in local coordinates

$$\mu = \sum_{i=1}^{3} \omega_i \otimes B_i \tag{6.57}$$

and this is equvalent to say that the B_i are linearly independent; one can ask if this restriction leads to a too small (and so not so interesting) subset, but this is not the case as seen in Proposition 5.2 (or see [80, Proposition 3.5]).

We can construct through the map Ψ a pullback bundle on M_0 as diagram

$$\begin{tikzcd} \Psi^*(\mathbf{V}) \arrow[r, "\hat{\Psi}"] \arrow[d, "p_V^*"'] & \mathbf{V} \arrow[d, "p_V"] \\ M_0 \arrow[r, "\Psi"'] & \mathbb{G}_3(\mathfrak{g}) \end{tikzcd} \tag{6.58}$$

shows, where $\hat{\Psi}$ is the lifting of Ψ; the pullback of a bundle is unique up to isomorphism of bundles (see [82, Cap.I, Prop. 2.15]), so any vector bundle $W \longrightarrow M_0$ such that exists a map of bundles

$$\hat{\Phi} : W \longrightarrow \mathbf{V} \tag{6.59}$$

which is injective on the fibres and makes commutative diagram (6.58) is isomorphic to $\Psi^*(\mathbf{V})$. We want to construct such a map for the bundle S^2H, using th moment map μ.

Lemma 6.4. *We have the following isomorphism of bndles on M_0:*

$$S^2H \cong \Psi^*(\mathbf{V}), \tag{6.60}$$

and the digram

$$\begin{tikzcd} S^2H \arrow[r, "\hat{\Phi}"] \arrow[d, "p_V^*"'] & \mathbf{V} \arrow[d, "p_V"] \\ M_0 \arrow[r, "\Psi"'] & \mathbb{G}_3(\mathfrak{g}) \end{tikzcd} \tag{6.61}$$

commutes, with $\hat{\Phi} = \hat{\Psi}$ modulo isomorphism of bundles.

Proof. The morphism of bundles

$$\hat{\Phi} : S^2H \longrightarrow \mathbf{V} \tag{6.62}$$

is obtained sending

$$\big(x, \omega_i(x)\big) \longmapsto \big(\mathrm{span}\{B_1(x), B_2(x), B_3(x)\}, B_i(x)\big) \tag{6.63}$$

(see (6.57)) and extending linearly on the fibres; this is essentially the contraction of a vector $v \in S^2H_x$ with the S^2H component of $\mu(x)$ using the metric, so it doesn't depend on the trivialization (the structure group preserves the metric) and is injective on the fibres by the definition of M_0. ■

We want to point out that $\hat{\Phi}$ is not an isometry of Riemannian bundles in general; nevertheless we can retain the hypotheses considered in Section 6.1 when discussing Theorem 6.2: we can therefore assume that $\hat{\Psi}$ is a conformal map of Riemannian bundles, considering S^2H and $\mathbf{V}$ equipped with the natural metrics coming respectively from M and from $\mathbb{G}_3$. In other words the B_i in (6.57) are orthogonal and $\|B_1\| = \|B_2\| = \|B_3\|$.

In this case we can exploit the freedom composing $\hat{\Phi}$ with a bundle automorphism of S^2H; we can for instance operate a dilation

$$\xi(x, w) = (x, \frac{w}{\|B_i\|}), \tag{6.64}$$

which is independent of the trivialization; in this way

$$\hat{\Xi}(\omega_i) := \hat{\Phi} \circ \xi(\omega_i) = \frac{B_i}{\|B_i\|}, \tag{6.65}$$

and so an orthonormal basis is sent to another orthonormal basis: this is therefore an isometry of the two bundles compatible with the map Ψ induced by μ.

We can now state the main result of this section.

Proposition 6.5. *There exists a lift $\hat{\Psi}$ of the map Ψ such that*

$$\hat{\Psi}(\mu_A) = s_A\,, \tag{6.66}$$

determining a map

$$\ker\,\mathcal{D} \longrightarrow \ker\,D\,. \tag{6.67}$$

<u>Proof.</u> We are looking for a lift $\hat{\Psi}$ such that the diagram

$$\begin{tikzcd} S^2H \arrow[r, "\hat{\Psi}"] & \mathbf{V} \\ M_0 \arrow[u, "\mu_A"] \arrow[r, "\Psi"'] & \mathbb{G}_3(\mathfrak{g})\,. \arrow[u, "s_A"'] \end{tikzcd} \tag{6.68}$$

commutes; recall the usual local description (6.57) of μ, and let us define $\hat{\Psi}$ so that

$$\hat{\Psi}(\omega_i) = \frac{B_i}{\|B_i\|^2}\,, \tag{6.69}$$

obtaind by composing $\hat{\Phi}$ with the dilation ξ^2 (see 6.64); this is again a lift of Ψ; consider as usual $\mu_A \in \Gamma(S^2H)$ satisfiying the twistor equation; then

$$\hat{\Psi}(\mu_A) = \hat{\Psi}\big(\sum_i \omega_i \langle B_i\,,\, A\rangle\big) \tag{6.70}$$

$$= \sum_i \frac{B_i}{\|B_i\|^2}\langle B_i\,,\, A\rangle \tag{6.71}$$

$$= \pi_V A = s_A\,. \tag{6.72}$$

The section s_A is obtained invariantly by projection, therefore it is independent from the chosen basis, so no ambiguity comes in case that Ψ is not injective: if $\Psi(x) = \Psi(x') = V$ then

$$\hat{\Psi}\big(\mu_A(x)\big) = \hat{\Psi}\big(\mu_A(x')\big) = s_A\,. \blacksquare \tag{6.73}$$

The situation can be summarized in diagram (6.74):

$$\begin{tikzcd} & \boxed{A \in \mathfrak{g}} \arrow[dl, dash] \arrow[dr, dash] & \\ \boxed{s_A \in \ker\, D} \arrow[rr, dash] & & \boxed{\mu_A \in \ker\, \mathcal{D}} \end{tikzcd} \tag{6.74}$$

.

Observations. i) We can interpret μ as a collection of $d = \dim\, \mathfrak{g}$ sections of S^2H: if A_i are an orthonormal basis for $\mathfrak{g}$ the moment map μ is completely determined by the μ_{A_i}. Locally we get

$$B_i = \sum_j a_i^j\, A_j \tag{6.75}$$

so that

$$\mu_{A_i} = \sum_j a_i^j\, \omega_j\,; \tag{6.76}$$

So for instance, if locally a section $\sigma \in \Gamma(S^2H)$ is given by

$$\sigma = \sum_i c^i \omega_i \tag{6.77}$$

then

$$\hat{\Phi}(\sigma) = \sum_i c^i B_i \,; \tag{6.78}$$

with respect to the basis A_i of $\mathfrak{g}$ the local description of the morphism $\hat{\Phi}$ is encoded in the $(3 \times (d-3))$ matrix of the coefficients a^i_j seen in (6.75);

ii) this construction is similar to that used to prove the so called "classification Theorem" for vector bundles (see for example [82, Chap III, Prop. 4.2]); in fact μ can be interpreted as a collection of $N = \dim(\mathfrak{g})$ sections of the bundle S^2H; nevertheless here the sections used to construct the map to the appropriate Grassmannian are chosen in order to satisfy the twistor equation (6.56), coming from the quaternionic structure, and not with the criterion of ensuring their maximal rank point by point; this lack is cured restricting to the subset M_0;

iii) consider an element $A \in \mathfrak{g}$ and the associated section $s_A \in \Gamma(\mathbf{V})$; at the point $V \in \mathbb{G}_3(\mathfrak{g})$ we have

$$A = \pi_V A + \pi_{V^\perp} A \tag{6.79}$$

to which we can associate pointwise the section μ_{s_A}; actually $\mu_{s_A} = \mu_A$ point by point, because the summand $\pi_{V^\perp}$ does not affect μ_A:

$$\sum \omega_i \,\langle A\,,\, B_i\rangle = \sum \omega_i \langle \pi_V A + \pi_{V^\perp} A\,,\, B_i\rangle \tag{6.80}$$

$$= \sum \omega_i \,\langle \pi_V A\,,\, B_i\rangle \,. \tag{6.81}$$

6.3 The interpretation of the functional g

As discussed in Section 5.3, in the case of Lie algebras $\mathfrak{g}$ such that

$$\mathfrak{g} \supset \mathfrak{so}(4) \tag{6.82}$$

the next-to-minimal unstable manifold for $\operatorname{grad} f$ is always of cohomogeneity 1 and the relation

$$\operatorname{grad} f = \mathbf{a} \operatorname{grad} g \tag{6.83}$$

holds on it, with $\mathbf{a}$ a function described in Proposition 5.9. Consider now the realizations constructed in Sections 5.4 and 5.5, which are essentially based on the inclusion

$$\mathbb{HP}^1 \subset M \tag{6.84}$$

for the various Wolf spaces M; this inclusion is stricly related to (6.82).

Recall that if $\mathfrak{so}(4) = \text{span}\{e_i, f_i,\ i = 1\cdots 3\}$, then the moment map is expressed by

$$\mu = \sum \omega_i \otimes \left(\sin^2 t e_i + \cos^2 t f_i\right), \tag{6.85}$$

(see (5.65) ;therefore the conformality factor coming from the general hypotheses of Section 6.1 is given in these cases by

$$\boldsymbol{\lambda}^2(t) = \sin^4 t + \cos^4 t \tag{6.86}$$

along the trajectory

$$V(t) = \text{span}\{\sin^2 te_i + \cos^2 tf_i,\ i = 1\cdots 3\}, \tag{6.87}$$

which is another way of describing the trajectory (5.35) discussed in Section 5.3. We have

Lemma 6.6. *Consider the unstable next-to-minimal submanifolds in* $\mathbb{G}_3(\mathfrak{g})$, *for* $\mathfrak{g} = \mathfrak{so}(n)$, $\mathfrak{sp}(n)$ *or* $\mathfrak{sp}(n)\mathfrak{sp}(1)$; *there the equality*

$$\mathbf{a} = -\frac{2\sqrt{2}}{\boldsymbol{\lambda}}. \tag{6.88}$$

holds.

Proof. Let us calculate explicitly the objects of our interest along $V(t)$, adopting the notation $\sin^2 t = u$ and $\cos^2 t = w$:

$$(\text{grad}\, f)_i = \frac{1}{\boldsymbol{\lambda}^3}\langle\left[u\, e_j + w\, f_j\,,\, u\, e_k + w\, f_k\right], \tag{6.89}$$

$$- w\, e_i + u\, f_i\rangle \tag{6.90}$$

$$= \frac{\sqrt{2}}{\boldsymbol{\lambda}^3}\, wu\,(w-u)\,, \tag{6.91}$$

and

$$(\text{grad}\, g)_i = -\frac{2}{\boldsymbol{\lambda}^4}\Big(\big[u\, e_j + w\, f_j, \big[u\, e_j + w\, f_j\,,\, u\, e_i + w\, f_i\big]\big]^\perp \tag{6.92}$$

$$+ \big[u\, e_k + w\, f_k, \big[u\, e_k + w\, f_k\,,\, u\, e_i + w\, f_i\big]\big]^\perp \tag{6.93}$$

$$= -\frac{4}{\boldsymbol{\lambda}^4}\, wu\,(w^2 - u^2)\,; \tag{6.94}$$

so we have

$$\mathbf{a} = -\frac{4\,wu\,(w^2-u^2)}{\boldsymbol{\lambda}^4}\frac{\boldsymbol{\lambda}^3}{wu\,(w-u)\sqrt{2}} \tag{6.95}$$

$$= -\frac{2\sqrt{2}(w+u)}{\boldsymbol{\lambda}} = -\frac{2\sqrt{2}}{\boldsymbol{\lambda}}\,.\ \blacksquare \tag{6.96}$$

We are interested in discussing the interplay between the functions $\mathbf{a}$, $\boldsymbol{\lambda}$ and the quaternionic structure on a QK manifold M. Let us denote by I_i a basis spanning $\operatorname{Im}\mathbb{H}$ with the usual relations

$$I_i I_j = (-1)^{\epsilon} I_k\,, \tag{6.97}$$

with ϵ depending on the permutation ijk.

A conjecture about the description of the I_i in the Grassmannian language leads to define

$$\hat{I}_i(A) := v_i \times A + c\, v_i \otimes \boldsymbol{\gamma}(A)^{\perp}\,, \tag{6.98}$$

where $\times$ means the $\mathbb{R}^3$ exterior product on the V component of A and c is some function; this definition is modelled on the basis of (6.40), but it does not take care of the effect due to Ψ_* when passing between the tangent spaces.

Nevertheless it is instructive to determine the action of such endomorphisms on $\operatorname{grad} f$: in fact asking that

$$\hat{I}^2 = -Id\,, \tag{6.99}$$

under the hypothesis that $\operatorname{grad} g = \mathbf{a}\operatorname{grad} f$, for instance we have

$$\hat{I}_i(\operatorname{grad} f) = -\widetilde{v}_i\,; \tag{6.100}$$

moreover

$$\hat{I}_i^2(\operatorname{grad} f) = -\operatorname{grad} f \tag{6.101}$$

or more explicitly

$$\hat{I}_i\big(\hat{I}_i(\operatorname{grad} f)\big) = v_i \otimes c\,\boldsymbol{\gamma}\big(I_i(\operatorname{grad} f)\big)^{\perp} - v_j \otimes w_j - v_k \otimes w_k \tag{6.102}$$

$$= -\operatorname{grad} f\,; \tag{6.103}$$

in particular this should imply that

$$-c\,\boldsymbol{\gamma}(\widetilde{v}_i)^{\perp} = -w_i\,; \tag{6.104}$$

expliciting the action of $\boldsymbol{\gamma}$ we obtain

$$\boldsymbol{\gamma}(\widetilde{v}_i)^\perp = \boldsymbol{\gamma}\big(v_j \otimes [v_i, v_j]^\perp + v_k \otimes [v_i, v_k]^\perp\big)^\perp \tag{6.105}$$
$$= \big(([v_j, [v_i, v_j]] - \langle [v_i, v_j], v_k\rangle [v_j, v_k]) \tag{6.106}$$
$$+ ([v_k, [v_i, v_k]] - \langle [v_i, v_k], v_j\rangle [v_k, v_j])\big)^\perp \tag{6.107}$$
$$= -C(v_i)^\perp - 2f(V)\, w_i \tag{6.108}$$
$$= \frac{(\operatorname{grad} g)_i}{2} - 2f(V)\,(\operatorname{grad} f)_i\,, \tag{6.109}$$

where $C(\cdot)$ is the generalized Casimir operator described in 3.39, and $(\operatorname{grad} g)_i$ are the $V^\perp$ factors of the i-th component of the gradients. Hence, from (6.104) and (6.109) we have

$$c\,\big(\operatorname{grad} g - 4f(V)\operatorname{grad} f\big) = 2\operatorname{grad} f\,, \tag{6.110}$$

so that the function c should satisfy

$$c = \frac{2}{(\mathbf{a}\ - 4f)}\,. \tag{6.111}$$

In this way we obtain (6.101): the $\boldsymbol{\gamma}$ component recovers at the second step what is lost by the action of $\times$ at the first step.

Observation. With this choice of c the quaternionic relations hold for $\hat{I}_i$, so they behave like a quaternionic structure, at least on

$$\operatorname{span}\{\widetilde{v}_1,\, \widetilde{v}_2,\, \widetilde{v}_3,\, \operatorname{grad} f\} \subset T_V\mathbb{G}_3 \tag{6.112}$$

which so should turn out to be to be the quaternionic span of $\operatorname{grad} f$ at V.

Proportionality of $\operatorname{grad} f$ and $\operatorname{grad} g$ is also helpful in determining what happens to vectors X in T_xM when they pass through Ψ_*: recall that

$$\operatorname{grad} f = \sum_i v_i \otimes w_i \tag{6.113}$$

with w_i defined in (6.40); then using the Coincidence Theorem 6.2 (or more precisely Corollary 6.1) we can define

$$Z := \big(\Psi^*(\operatorname{grad} f)^\flat\big)_\sharp = \frac{1}{\boldsymbol{\lambda}} \sum I_i\, \tilde{w}_i\,, \tag{6.114}$$

where $\flat$ in $\operatorname{Hom}(T_xM,\, T_x^*M)$ and in $\operatorname{Hom}(T_V\mathbb{G}_3,\, T_V^*\mathbb{G}_3)$ denotes the duality isomorphism defined by the respective metrics, and $\sharp = \flat^{-1}$; we observe that as the two dualities depend on the metrics, the diagram

$$
\begin{tikzcd}
T^*_x M \arrow[d, "\sharp"'] & T^*_V \mathbb{G}_3 \arrow[l, "\Psi^*"'] \arrow[d, "\sharp"] \\
T_x M \arrow[r, "\Psi_*"'] & T_V \mathbb{G}_3
\end{tikzcd}
\tag{6.115}
$$

is not commutative in general. Therefore no relationship exists a priori between a tangent vector X in T_xM and

$$\natural(X) := (\Psi^*(\Psi_* X)^\flat)_\sharp \quad \in T_x M\,; \tag{6.116}$$

nevertheless, in case of proportionality between $\operatorname{grad} f$ and $\operatorname{grad} g$ we have

Proposition 6.7. *Let us consider*

$$Z = \frac{1}{\boldsymbol{\lambda}} \sum_i I_i \widetilde{w}_i\,; \tag{6.117}$$

as seen in (6.114); if $\operatorname{grad} f = \mathbf{a} \operatorname{grad} g$ *at* V *then*

$$\sum_i I_i \natural(\widetilde{w}_i) = \kappa\, Z \tag{6.118}$$

for an appropriate invariant function $\kappa(V)$.

Proof. Writing explicitly

$$\sum_i I_i \natural\big(\tilde{w}_i\big) = \sum_{i,j} I_i \big(\Psi^*\,(v_j \otimes [w_i\,,\,v_j]^\perp)^\flat\big)_\sharp \tag{6.119}$$

and then using Theorem 6.2 (or Corollary 6.1)

$$
\begin{aligned}
&\frac{1}{\boldsymbol{\lambda}} \sum_{i,j} I_i\, I_j\, \widetilde{[w_i\,,\,v_j]}^\perp = \frac{1}{\boldsymbol{\lambda}}\Big(- \sum_{i=j} \widetilde{[w_i\,,\,v_j]}^\perp + I_1 \left(\widetilde{[w_2\,,\,v_3]}^\perp - \widetilde{[w_3\,,\,v_2]}^\perp\right) \\
&+ I_2 \left(\widetilde{[w_3\,,\,v_1]}^\perp - \widetilde{[w_1\,,\,v_3]}^\perp\right) + I_3 \left(\widetilde{[w_1\,,\,v_2]}^\perp - \widetilde{[w_2\,,\,v_1]}^\perp\right)\Big) \\
&= \frac{1}{\boldsymbol{\lambda}}\Big(- \mathfrak{S}\, [\widetilde{[v_2\,,\,v_3]^\perp}\,,\,v_1]^\perp + I_1 \left([\widetilde{[v_3\,,\,v_1]^\perp}\,,\,v_3]^\perp - [\widetilde{[v_1\,,\,v_2]^\perp}\,,\,v_2]^\perp\right) \\
&+ I_2 \left([\widetilde{[v_1\,,\,v_2]^\perp}\,,\,v_1]^\perp - [\widetilde{[v_2\,,\,v_3]^\perp}\,,\,v_3]^\perp\right) \\
&+ I_3 \left([\widetilde{[v_2\,,\,v_3]^\perp}\,,\,v_2]^\perp - [\widetilde{[v_3\,,\,v_1]^\perp}\,,\,v_1]^\perp\right)\Big)\,,
\end{aligned}
$$

and writing more explicitly the projection on $V^\perp$ and using the Jacobi identity we obtain for the first summand

$$\mathfrak{S}\,[\widetilde{[v_2\,,\,v_3]^\perp}\,,\,v_1]^\perp = \mathfrak{S}\,[\widetilde{[v_2\,,\,v_3]}\,,\,v_1]^\perp = 0 \tag{6.120}$$

(recall that $\mathfrak{S}$ means summing over cyclic indices); for the remaining part of the expression we obtain

$$\begin{aligned}
&\frac{1}{\boldsymbol{\lambda}}\Big(I_1\Big([\widetilde{[v_3\,,\,v_1]}\,,\,v_3]^\perp - [\widetilde{[v_1\,,\,v_2]}\,,\,v_2]^\perp - 2f\widetilde{w}_1\Big)\\
&+ I_2\Big([\widetilde{[v_2\,,\,v_1]}\,,\,v_1]^\perp - [\widetilde{[v_2\,,\,v_3]}\,,\,v_3]^\perp - 2f\widetilde{w}_2\Big)\\
&+ I_3\Big([\widetilde{[v_2\,,\,v_3]}\,,\,v_2]^\perp - [\widetilde{[v_3\,,\,v_1]}\,,\,v_1]^\perp - 2f\widetilde{w}_3\Big)\Big)\\
&= \frac{1}{\boldsymbol{\lambda}}\sum I_i(-\widetilde{C(v_i)}^\perp - 2f\widetilde{w}_i) = \frac{\mathbf{a}-4f}{2\boldsymbol{\lambda}}\sum I_i\widetilde{w}_i\,,
\end{aligned}$$

hence the conclusion. ■

6.4 The case of $\mathfrak{su}(3)$

The case of $\mathfrak{su}(3)$ is different in more than a sense from examples discussed in Sections 5.4 and 5.5: first of all, it is related the inclusion $\mathfrak{su}(3) \subset \mathfrak{g}_2$, where $\mathfrak{g}_2$ is the Lie algebra of the exceptional Lie group G_2; the lack of a matricial description, at least to our knowledge, of this latter does not allow to perform the same type of calculations and to obtain an explicit expression for μ. Moreover we have that $\mathfrak{so}(4) \not\subset \mathfrak{su}(3)$, therefore it would not be possible in this case to use the trajectory studied in Section 5.3, even if as we shall see ina moment a trajectory is known also in this case.

The unstable manifold in $\mathbb{G}_3(\mathfrak{su}(3))$ has been studied in detail in [50], nevertheless we want to give here an alternative description of the rich geometry present in this example.

It is known that the unstable manifold M for $\operatorname{grad} f$ departing from the critical set

$$\frac{SSU(3)}{\mathbb{Z}_3} = \frac{SU(3)}{SO(3)\times\mathbb{Z}_3} \tag{6.121}$$

is locally isomorphic to the exceptional Wolf space

$$\frac{G_2}{SO(4)}\,; \tag{6.122}$$

more precisely (see [50]) there exists a $3:1$ covering

$$\Psi : \frac{G_2}{SO(4)} \setminus \mathbb{CP}^2 \longrightarrow M\ . \tag{6.123}$$

An explicit trajectory for the grad f flow has been already described in [50], an expression is given by

$$\Gamma(x,y) = \langle \begin{pmatrix} 0 & x^3 & 0 \\ -x^3 & 0 & y^3 \\ 0 & -y^3 & 0 \end{pmatrix}, - \begin{pmatrix} 0 & \imath x^3 & 0 \\ \imath x^3 & 0 & \imath y^3 \\ 0 & \imath y^3 & 0 \end{pmatrix}, \tag{6.124}$$

$$\sqrt{(x^2+y^2)} \begin{pmatrix} \imath x^2 & 0 & 0 \\ 0 & \imath(y^2-x^2) & 0 \\ 0 & 0 & -\imath y^2 \end{pmatrix} \rangle \tag{6.125}$$

with the condition that $2(x^6+y^6)=1$; the flow goes from the critical manifold $C_r = SSU(3)/\mathbb{Z}_3$ to the maximal critical manifold, the classical Wolf space $\mathbb{CP}^2$; the intersection points of the trajectory $\Gamma(x,y)$ with these two orbits, wich turn out to be the singular orbits of the cohomogeneity one action of $SU(3)$ restricted to M, corrspond to the conditions $x=y=2^{-1/3}$ and $y=0$, $x=2^{-1/6}$; the intersection points are the 3-dimensional subalgebras spanned by

$$w_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \ w_2 = - \begin{pmatrix} 0 & \imath & 0 \\ \imath & 0 & \imath \\ 0 & \imath & 0 \end{pmatrix}, \ w_3 = \begin{pmatrix} \imath & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\imath \end{pmatrix} \tag{6.126}$$

in C_r and by

$$h_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ h_2 = - \begin{pmatrix} 0 & \imath & 0 \\ \imath & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ h_3 = \begin{pmatrix} \imath & 0 & 0 \\ 0 & -\imath & 0 \\ 0 & 0 & 0 \end{pmatrix}; \tag{6.127}$$

in $\mathbb{CP}^2$. In this last case the subalgebra is conjugate to the standard real $\mathfrak{so}(3)$ via the element

$$A_\Gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & \imath \\ 0 & -\imath & 0 \\ -1 & 0 & -\imath \end{pmatrix}; \tag{6.128}$$

the two subalgebras (6.126) and (6.127) will be denoted by W_r and W_h respectively in the sequel.

We point out that Nagatomo desribed in [65] the same $\mathbb{CP}^2$ as the singualr set of an ASD bundle over $G_2/SO(4)$.

The correspondence between points in a principal orbit, which is of type

$$\frac{SU(3)}{U(1)} \tag{6.129}$$

with $U(1)$ corresponding to

$$\begin{pmatrix} \imath & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\imath \end{pmatrix} \in \mathfrak{u}(1)\,, \tag{6.130}$$

and the limit points along $\Gamma(x, y)$ in the singular orbits gives rise to a double fibration of the same type described for the consimilarity action on $SU(3)$ in (1.59):

$$\begin{tikzcd} & SU(3)/U(1) \arrow[dl, "\tilde{\pi}_1"'] \arrow[dr, "\tilde{\pi}_2"] & \\ SSU(3) & & \mathbb{CP}^2 \end{tikzcd} \quad ; \tag{6.131}$$

the only real difference here in the presence of a $\mathbb{CP}^2$ substituting S^5. If we consider the Hopf fibration

$$\begin{tikzcd} S^1 \arrow[r, hook] & S^5 \arrow[d, "\pi_H"] \\ & \mathbb{CP}^2 \end{tikzcd} \tag{6.132}$$

we can ask if the equations

$$\begin{cases} \tilde{\pi}_1 \circ \tilde{\pi}_2^{-1}(x) = \pi_1 \circ (\pi_H \circ \pi_2)^{-1}(x) \\ \tilde{\pi}_2 \circ \tilde{\pi}_1^{-1}(x) = \pi_H \circ \pi_2 \circ \pi_1^{-1}(x) \end{cases} \tag{6.133}$$

hold; recall that π_1, π_2 are the projections analogous to $\tilde{\pi}_1$, $\tilde{\pi}_2$ in (1.59). This depends on the intersection $U(1) = N(SU(2)) \cap SO(3)$ of the singular stabilizers. Ispecting the subalgebras (6.126) and (6.127), we see that this is not the case; in conclusion if the $\mathbb{CP}^2$ in (6.131) and (6.132), then the S^5 in (6.132) is not the S^5 in (1.59). We want however to give an interpretation of the double fibration (6.131) induced by Γ.

We recall here some facts about Lagrangian and Special Lagrangian subspaces; a reference for this material is [34, Section III].

Let $\mathbb{R}^6 = \mathbb{C}^3$ be the standard $SU(3)$ representation endowed with the standard symplectic 2-form

$$\omega = e^{12} + e^{34} + e^{56}\,, \tag{6.134}$$

almost complex structure

$$Je_1 = e_2, \quad Je_3 = e_4, \quad Je_5 = e_6 \tag{6.135}$$

and $(3,0)$-form $\psi = \psi^+ + \imath\psi^-$:

$$\psi^+ = e^{135} - e^{146} - e^{236} - e^{245} \tag{6.136}$$
$$\psi^- = -e^{246} + e^{136} + e^{235} + e^{145}\,; \tag{6.137}$$

recall that a subspace $V \subset \mathbb{R}^{2n}$ such that $V = V^\perp$ with respect to ω is called a *Lagrangian subspace*; this condition implies $\dim V = n$; if in addition $\psi^+|_V = \mathrm{vol}_V$ or equivalently $\psi^-|_V = 0,$ then it is called a *Special Lagrangian subspace* (SL from now on); other equivalent conditions carachterizing SL subspaces are given by

$$\omega \wedge \kappa = 0 \quad , \quad \psi^- \wedge \kappa = 0\,, \tag{6.138}$$

where κ is the n-form dual to V under the metric (in the sense that if $V = \mathrm{span}\{v_1, \cdots, v_k\}$ then $\kappa = v^1 \wedge \cdots \wedge v^k$). The following lemma holds:

Lemma 6.8. *The sets of Lagrangian and SL subspaces in $\mathbb{R}^{2n}$ are parametrized respectively by the symmetric spaces*

$$\frac{U(n)}{O(n)} \quad \textit{and} \quad \frac{SU(n)}{SO(n)}\,. \tag{6.139}$$

Recall some isomorphisms of representations, introducing together some notation:

$$\Lambda^2 := \bigwedge\nolimits^2 \mathbb{R}^3 \cong \mathfrak{so}(3) \tag{6.140}$$
$$\bigwedge\nolimits^2 \mathbb{R}^6 \cong \mathfrak{so}(6) \tag{6.141}$$

and also

$$\Lambda_0^{1,1} := [\![\bigwedge\nolimits_0^{1,1} \mathbb{C}^6]\!] \cong \mathfrak{su}(3) \subset \mathfrak{so}(6) \tag{6.142}$$
$$\Lambda^{2,0} := [\![\bigwedge\nolimits^{2,0} \mathbb{C}^6]\!]\,, \tag{6.143}$$

where $\mathbb{C}^6 = \mathbb{R}^6_{\mathbb{C}}$, so that

$$\bigwedge\nolimits^2 \mathbb{R}^6 = \Lambda^{1,1} \oplus \Lambda^{2,0} = \omega \oplus \Lambda_0^{1,1} \oplus \Lambda^{2,0}\,; \tag{6.144}$$

we denote by $\pi^{1,1}$ the projection on the first two summands and by $\pi_0^{1,1}$ that on the second summand, and noting that

$$J^{\wedge 2} = \begin{cases} +1 & \text{on } \Lambda^{1,1} \\ -1 & \text{on } \Lambda^{2,0} \end{cases} \tag{6.145}$$

we obtain the expression

$$\pi^{1,1} = Id + J^{\wedge 2}\,. \tag{6.146}$$

Consider now the well-known decomposition

$$\bigwedge^2 \mathbb{R}^4 = \bigwedge^2_+ \mathbb{R}^4 \oplus \bigwedge^2_- \mathbb{R}^4 = \Lambda^2_+ \oplus \Lambda^2_- \tag{6.147}$$

corresponding to the ± 1 eigenvalue of the $*$ operator, where the two summands are called *self-dual* and *anti-self-dual* forms; recalling that

$$\Lambda^2_- = \Lambda^{1,1}_0 \tag{6.148}$$

we can identify the projection on the second summand π_- with $\pi^{1,1}_0$; recall that $\mathbb{F}$ coincides with the *twistor bundle* of $\mathbb{CP}^2$:

$$\begin{tikzcd} S^2 \arrow[r, hook] & S^2(\Lambda^2_-) =: \mathbb{F} \arrow[d, "p_0"] \\ & \mathbb{CP}^2 \end{tikzcd} \tag{6.149}$$

where S^2 is the unitary sphere bundle contained in the 3-dimensional anti-self dual bundle over $\mathbb{CP}^2$ and p_0 is the projection on a complex line L_0; the fibre of this twistor projection consists of the complex planes containing L_0, parametrized by the complex lines in the plane $L_0^\perp$: this can be seen more clearly by describing the total space $\mathbb{F}$ of the bundle as a homogeneous space under $SU(3)$: this turns out to be the flag manifold $SU(3)/T^2$, describing the complete flags in $\mathbb{C}^3$; a projection under a $U(1)$ action relates $\mathbb{F}$ to our generic orbit $SU(3)/U(1)$:

$$\frac{SU(3)}{U(1)} \xrightarrow{\pi_F} \mathbb{F}\,. \tag{6.150}$$

The choice of a unit element σ of Λ^2_- identifies an additional decomposition of the $L_0^\perp$, as one can deduce from the following proposition (see [1]):

Proposition 6.9. *The twistor bundle $\mathbb{F}$ of $\mathbb{CP}^2$ can be identified with $\mathbb{P}\big(T^{(1,0)}\mathbb{CP}^2\big)$.*

<u>Proof.</u> Let σ be an anti-self-dual 2-form belonging to a fibre over a point L_0; this can be interpreted as an almost-complex structure J_σ via the metric; consider the self-dual form obtained by restriction of the standard symplectic 2-form

$$\omega' := \omega|_{\mathbb{R}^4}\,; \tag{6.151}$$

then, normalizing the metric so that $|\sigma| = |\omega'| = 1$, the form

$$\frac{\omega' + \sigma}{2} \tag{6.152}$$

is always a J-invariant decomposable 2-form (see [76]), so of type

$$\frac{\omega' + \sigma}{2} = \eta \wedge J\eta \tag{6.153}$$

for some 1-form η; it therfore identifies a complex line $L_1 \subset \mathbb{R}^4$ defined as the 2-dimensional subspace $\mathbb{R}^4$ dual to $\eta \wedge J\eta$; this 2-plane is invariant at the same time for the standard J and also for J_σ. At present we have two complex lines L_0 and L_1 contained in $\mathbb{C}^3$ so that we obtain a decomposition

$$\mathbb{C}^3 = L_0 + L_1 + L_2 \tag{6.154}$$

where also L_2 is complex. This decomposition identifies a complete flag in $\mathbb{C}^3$

$$0 \subset L_0 \subset L_0 + L_1 \subset L_0 + L_1 + L_2 = \mathbb{C}^3\,, \tag{6.155}$$

therefore a point in the flag manifold $\mathbb{F}$; the tangent space $T_{L_0}\mathbb{CP}^2$ is isomorphic to $\mathrm{Hom}(L_0, L_0^\perp) \cong \overline{L_0} \otimes L_0^\perp$, so identifying L_1 is equivalent to identifying

$$\mathbb{C}(\overline{L_0} \otimes L_1) \subset \overline{L_0} \otimes L_0^\perp\,, \tag{6.156}$$

hence a point in $\mathbb{P}\big(T^{(1,0)}_{L_0}\mathbb{CP}^2\big)$. ■

Let us analyze more in detail the situatuion: there are three possible projections from a point of $\mathbb{F}$ to $\mathbb{CP}^2$, depending on what complex line we choose in (6.154):

$$\begin{tikzcd} \mathbb{CP}^2 & & \\ & \mathbb{F} \arrow[ul, "p_1"'] \arrow[r, "p_0"] \arrow[dl, "p_2"] & \mathbb{CP}^2 \\ \mathbb{CP}^2 & & \end{tikzcd} \tag{6.157}$$

at the level of the Lie algebra $\mathfrak{su}(3)$ we have

$$\mathfrak{su}(3) = \mathfrak{t} \oplus A_0 \oplus A_1 \oplus A_2 \tag{6.158}$$

where A_i are the irreducible T^2 representations which can be identified with $[\![\overline{L_j} L_k]\!]$ for $j, k \in \{0, 1, 2\}$; these can be oriented in 2 ways each, obtaining in total $2^3 = 8$ orientations for $\mathbb{F}$, each of which determines a different almost-complex structure compatible with the Killing metric; only 6 of these are integrable: in fact the complexification of the tangent space $T_x\mathbb{F}$ corresponds to the 6 standard roots spaces $\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}, \mathfrak{g}_{\pm(\alpha+\beta)}$ for α, β positive simple roots, and the 2 choices corresponding to the holomorphic tangent bundle

$$T_x^{(1,0)}\mathbb{F} = \pm\{\alpha, \beta, -(\alpha+\beta)\} \tag{6.159}$$

are not closed under brackets, a condition equivalent to integrability. The natural almost complex structure induced by the twistor fibration is such that p_0 is not a holomorphic nor an antiholomorphic projection. A question arises at this point: is the following diagram commutative

$$\begin{tikzcd} SU(3)/U(1) \arrow[r, "\pi_F"] \arrow[dr, "\tilde{\pi}_2"'] & \mathbb{F} \arrow[d, "p_i"] \\ & \mathbb{CP}^2 \end{tikzcd} \tag{6.160}$$

if we choose one of the p_i in (6.157)? The following theorem containes the answer:

Theorem 6.10. *Exists an $SU(3)$-equivariant isomorphism between the spaces*

$$\frac{G_2}{SO(4)} \setminus \mathbb{CP}^2 \xrightarrow{\cong} SU(3) \setminus S^5\,; \tag{6.161}$$

moreover the Wolf space $G_2/SO(4)$ can be obtained from $SU(3)$ by substitution of S^5 with $\mathbb{CP}^2$ via the projection map

$$p_1 \circ \pi_F : \frac{SU(3)}{U(1)} \longrightarrow \mathbb{CP}^2 \tag{6.162}$$

where p_1 is defined in (6.157) and π_F in (6.160).

We postpone the proof for the moment; we want first a more explicit way of associating to a SL subspace a corresponding subalgebra $\mathfrak{so}(3) \subset \mathfrak{su}(3)$; from now on we will freely pass from V to V^* using the metric, so that for instance $V^* \otimes V \cong V \otimes V$.

Proposition 6.11. *Let $V \subset \mathbb{R}^6$ be a SL subspace with respect to the standard symplectic structure; then the composition of $SU(3)$ equivariant mappings*

$$V \xrightarrow{*} \bigwedge^2 V^* \xrightarrow{\pi_0^{1,1}} \mathfrak{su}(3) \tag{6.163}$$

induces an $SU(3)$ equivariant map

$$\Phi : \frac{SU(3)}{SO(3)} \xrightarrow{3:1} \mathbb{G}_3(\mathfrak{su}(3)) \tag{6.164}$$

so that the image of Φ is the critical submanifold C_r.

Proof. The space of SL subspaces of $\mathbb{R}^n$ is parametrized by the symmetric space

$$\frac{SU(3)}{SO(3)}, \tag{6.165}$$

as seen in Lemma 6.8, therefore we will prove the assertion for a specific V and the $SU(3)$-equivariance of the map $*$ and of $\pi_0^{1,1}$ will imply the resut for any SL subspace V; it is sufficient therefore that $\pi_0^{1,1}(V)$ is a subalgebra isomorphic to $\mathfrak{so}(3)$ which stabilizes V itself. We choose

$$V = \mathrm{span}\{e_1, e_3, e_5\} \tag{6.166}$$

and therefore

$$\bigwedge^2 V^* = \mathrm{span}(e^{13},\, e^{35},\, e^{51}) \tag{6.167}$$

so that

$$\pi^{1,1}(e^{13}) = e^{13} + e^{24} \tag{6.168}$$
$$\pi^{1,1}(e^{35}) = e^{35} + e^{46} \tag{6.169}$$
$$\pi^{1,1}(e^{51}) = e^{51} + e^{62} \tag{6.170}$$

which generate a subspace $W \subset \mathfrak{u}(3)$ that is already orthogonal to ω, so that $\pi^{1,1}(V) = \pi_0^{1,1}(V)$ and $W \subset \mathfrak{su}(3)$; moreover it is closed under contraction, so isomorphic to $\mathfrak{so}(3)$ because Schur's Lemma guarantees that the contraction and the brackets are the same thing up to a constant: in fact $\mathfrak{su}(3)$ is irreducible seen as adjoint representation, and the space $\bigwedge^2 \mathfrak{su}(3)$ contains exactly 1 copy of $\mathfrak{su}(3)$ itself. The subspace W can be seen as a space of skew-symmetric endomorphisms of $\mathbb{R}^6$ which clearly preserves V; also we note that

$$\Phi(\zeta V) = \Phi(\zeta^2 V) = W \tag{6.171}$$

with $\zeta \in Z$, the center of $SU(3)$, follows from equivariance of Ψ, but can be checked directly. ∎

Therefore we can identify the restriction to C_r of the tautological bundle $\mathbf{V}$ of $\mathbb{G}_3(\mathfrak{su}(3))$ with the bundle of SL subspaces whose fibre is stabilized by the isotropy subgroup $SO(3)$, thanks to the isomorphism of the standard and the Adjoint representation V and $\mathfrak{so}(3)$; let us choose for example an element $v \in V$, and extend it to an ON basis $v = v_1,\ v_2,\ v_3$ is an ON basis for V; this corresponds to choosing an element both in $\mathfrak{so}(3)$ and in the SL subspace stabilized by $\mathfrak{so}(3)$, and more precisely:

Lemma 6.12. *Let $v_1 \in V$ as before: then $\sigma := \pi^{1,1}(*v_1)$ generates the stabilizer $U(1)$ of v_1.*

Proof. In fact

$$\sigma = v^{23} + (Jv^2) \wedge (Jv^3) \tag{6.172}$$

which stabilizes v_1 acting by contraction. ■

Observation. In this case the decomposable form discussed in Proposition 6.9 is given by

$$\frac{\omega' + \sigma}{2} = \frac{1}{2}\big(v^2 \wedge Jv^2 + v^3 \wedge Jv^3 + v^{23} + (Jv^2) \wedge (Jv^3)\big) \tag{6.173}$$

$$= \frac{1}{2}(v^2 - Jv^3) \wedge (Jv^2 + v^3)\,. \tag{6.174}$$

We identify therefore with this choice the homogeneous space

$$\frac{SU(3)}{U(1)}\,, \tag{6.175}$$

corresponding to the principal orbit along the flow line identified by v_1; we observe that elements in $\mathfrak{so}(3)$ have always $\det(A) = 0$ (as matrices), so they are not conjugate to the singular elements, which are of type

$$\begin{pmatrix} \imath & 0 & 0 \\ 0 & \imath & 0 \\ 0 & 0 & -2\imath \end{pmatrix}\,; \tag{6.176}$$

hence these elements individuate a unique maximal torus $T^2 \supset U(1)$ and a point of the flag manifold $\mathbb{F}$; the 2-form σ also stabilizes Jv_1, for the same reason seen in Lemma 6.12, hence all the complex line they span $L_0 := \mathrm{span}(v_1, Jv_1)$ so that we get a decomposition

$$\mathbb{C}^3 = L_0 \oplus \mathbb{C}^2 = \mathbb{C} \oplus \mathbb{R}^4\,. \tag{6.177}$$

In the specific case of W_r (see (6.126)) it stabilizes the subspace

$$V_r = \mathrm{span}\{e_1 - e_5,\ e_2 + e_6,\ e_3\}\,; \tag{6.178}$$

moreover we can interpret the basis w_1, w_2, w_3 as skew-symmetric endomorphisms of $\mathbb{C}^3 = \mathbb{R}^6$ represented by the 2-forms:

$$\begin{aligned} w_1 &= -e^{13} + e^{53} - e^{24} + e^{64} && (6.179)\\ w_2 &= -e^{14} + e^{23} - e^{36} + e^{45} && (6.180)\\ w_3 &= e^{12} - e^{56} && (6.181) \end{aligned}$$

using the approach discussed in Proposition 6.11 we have in fact:

Lemma 6.13. *Let V_r and W_r as before; then*

$$\pi_0^{1,1}(V_r) = W_r\,. \tag{6.182}$$

Proof. We have

$$\bigwedge^2 V_r^* = \mathrm{span}\{e^{12} + e^{16} - e^{52} - e^{56},\, e^{23} + e^{63},\, e^{31} - e^{35}\}\,; \tag{6.183}$$

we do the projection:

$$\begin{aligned} \pi^{1,1}(e^{12} + e^{16} - e^{52} - e^{56}) &= e^{12} - e^{56} = w_3 && (6.184)\\ \pi^{1,1}(e^{23} + e^{63}) &= -e^{14} + e^{23} - e^{36} + e^{45} = w_2 && (6.185)\\ \pi^{1,1}(e^{31} - e^{35}) &= -e^{13} + e^{53} - e^{24} + e^{64} = w_1 && (6.186) \end{aligned}$$

hence the result.■

We can finally prove Theorem 6.10:

Proof. The assertion regarding the local $SU(3)$-invariant isomorphism between $G_2/SO(4)$ and $SU(3)$ follows by just observing that both

$$\frac{G_2}{SO(4)}\backslash\mathbb{CP}^2 \quad \text{and} \quad SU(3)\backslash S^5 \tag{6.187}$$

are tubular neighbourhoods of the singular orbit $SSU(3)$ obtained by exponentiation of the same homogeneous vector bundle given by the standard 3-dimensional representation of $SO(3)$. For the second assertion we have to prove that diagram (6.160) is commutative if we choose p_1 in diagram (6.157); consider $w_3 \in V_r$: this corresponds to

$$\begin{aligned} v_1 &= e_3 && (6.188)\\ v_2 &= e_1 - e_5 && (6.189)\\ v_3 &= e_2 + e_6 && (6.190) \end{aligned}$$

referring to (6.178), therefore

$$e^3 \wedge Je^3 = e^{34}\,, \tag{6.191}$$

hence $L_0 = \mathrm{span}\{e_3, e_4\}$; moreover

$$\big((e^1 - e^5) - J(e^2 + e^6)\big) \wedge J\big((e^1 - e^5) - J(e^2 + e^6)\big)\} \tag{6.192}$$
$$\big(e^1 - e^5 + e^1 + e^5)\big) \wedge \big(e^2 - e^6) + e^2 + e^6)\big)\} \tag{6.193}$$
$$= 4e^{12} \tag{6.194}$$

so that $L_1 = \mathrm{span}\{e_1, e_2\}$ and finally

$$L_2 = \mathrm{span}\{e_5, e_6\}\,; \tag{6.195}$$

comparing this last with (6.126) and recalling that both the projections $\tilde{\pi}_2$ and $p_1 \circ \pi_F$ are $SU(3)$ equivariant, if they coincide at a point then they have to coincide globally, hence the assertion follows. ■

Observation. Let us denote by τ the element orthogonal to σ in $\mathfrak{t}$:

$$\tau := -2e^{34} + e^{12} + e^{56}\,. \tag{6.196}$$

In this case, (6.158) becomes

$$\mathfrak{su}(3) = \mathfrak{t} \oplus A_1 \oplus A_2 \oplus A_2 \oplus \tag{6.197}$$
$$= \mathrm{span}\{\sigma,\, \tau\} + [\![\overline{L_0}L_2]\!] + [\![\overline{L_0}L_1]\!] + [\![\overline{L_1}L_2]\!] \tag{6.198}$$
$$= \mathrm{span}\{\sigma,\, \tau\} + \mathrm{span}\{e^{35} - e^{64}, e^{36} + e^{45}\} \tag{6.199}$$
$$+ \mathrm{span}\{e^{13} - e^{42}, e^{14} - e^{23}\} \tag{6.200}$$
$$+ \mathrm{span}\{e^{15} + e^{62}, e^{16} + e^{25}\}\,. \tag{6.201}$$

The Coincidence Theorem allows to identify the quaternionic span of $\mathrm{grad}\, f$ on the unstable manifolds; we want in this case to identify the element of the Lie algebra which generate the Kvf spanning the imaginary part $\mathrm{Im}\,\mathbb{H}\cdot \mathrm{grad}\, f$ of this span in this specific example, in terms of the decomposition previously discussed: we observe that the matrix (6.125) belongs to $\mathfrak{t}$, while the two elements w_1, w_2 span a 2-dimensional space B_1 lying in $A_1 \oplus A_2$; the first two components of $\Gamma(x, y)$ instead span a subspace $C_1(x, y)$ which along the geodesic varies from B_1 to A_2.

Lemma 6.14. *The quaternionic span* $\mathrm{Im}\,\mathbb{H} \cdot \mathrm{grad}\, f$ *at the point* $\Gamma(x, y)$ *is given by the Killing fields generated by*

$$\mathrm{span}\{\tau\} \oplus C_1\,. \tag{6.202}$$

Proof. The proof follows from the definitions just given and by observing that σ is the stabilizer of the generic orbits along Γ, the Killing field generated by (6.125) equals that of the orthogonal projection on τ. ■

An immediate consequence is

Corollary 6.3. *In the previous notation, we have*

$$\tilde{A}_0 \subset (\mathbb{H} \cdot \operatorname{grad} f)^{\perp} \tag{6.203}$$

at any point of $M \subset \mathbb{G}_3(\mathfrak{su}(3))$, where $\tilde{A}_0$ denotes the Killing vector fileds generated by A_0.

The quaternionic 4-form

Recall what discussed in Section 4.4: there a quaternionic 4-form is described in purely algebraic terms (Proposition 4.4); recall that the critical submanifold $SSU(3) \subset \mathbb{G}_3(\mathfrak{su}(3))$ corresponds to the subalgebras $\mathfrak{so}(3)$ which give a decomposition

$$\mathfrak{su}(3) = \mathfrak{so}(3) \oplus [\Sigma^4] \tag{6.204}$$

so that the tangent space $T_V\mathbb{G}_3(\mathfrak{su}(3))$ for $V = \mathfrak{so}(3)$ can be described as

$$T_V\mathbb{G}_3(\mathfrak{su}(3))_{\mathbb{C}} = \Sigma^2 \otimes \Sigma^4\,; \tag{6.205}$$

comparing this with 4.83 we can conclude that Ω corresponds to the quaternionic 4-form on $G_2/SO(4)$ to the tangent space restricted to $SSU(3)$.

6.5 Still open questions

There are questions which have been left open, in the hope of finding more complete answers in the future.

i) First of all undersanding more precisely diagram 6.115, which expresses the interplay between μ_*, μ^*, the the metric induced by the ambient Grassmannian $\mathbb{G}_3(\mathfrak{g})$ and the QK metric.

ii) It would be also significant in this sense understanding more in deep the rôle played by the conformal factor $\boldsymbol{\lambda}$ and the proportionality factor $\mathbf{a}$, which are clearly related in many cases (see Lemma 6.6).

iii) Another question is determining completely the quaternionic 4-form Ω described in 4.4 in this Grassmannian description, as the part we have described corresponds to the restriction to the submanifold $SSU(3)$ (in fact $d\Omega \neq 0$). More in general: $\mathbb{G}_3(\mathfrak{g})$ has always $b_4 = 1$ for $\mathfrak{g}$ simple; what is the relationship between the invariant 4-form generating the cohomology and the quaternionic 4-form on the unstable submanifolds?

iv) Finally it would be interesting classifying completely the "generalized subalgebras" V satisfying $\operatorname{grad} g_V = 0$.

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