## Tesi di Dottorato

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# A Model Hamiltonian for Condensed Matter Physics 

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# Università degli Studi di Roma "La Sapienza" Facoltà di Scienze Matematiche Fisiche e Naturali 

Dottorato di Ricerca in Matematica XIII Ciclo

## A Model Hamiltonian for Condensed Matter Physics

Domenico Finco

## Introduction

In this thesis we give a rigorous construction of a (Rayleigh) model for the dynamics of a quantum mechanical test particle interacting with condensed matter via point interactions.
From the very beginning point interactions have been widely used in solid state physics: the model of Kronig and Penney, see [1], described the motion of an electron in a one dimensional crystal; the crystal was represented by a periodic array of point interactions.
In this work we have considered a different situation: condensed matter have been represented by N quantum harmonic oscillators with the same mass and proper frequency. It's clear that this model is much more difficult since we are dealing with a many-body problem.
At a formal level, the model was introduced in physics by Fermi in his seminal paper [F] for the analysis of the scattering of slow neutrons from the nuclei of a target of condensed matter. Since the interaction between the neutron and the nuclei of the target is strong and very short range with respect to the wavelength of the neutron, Fermi introduced a formal hamiltonian where the interaction is modelized by the singular delta-potential, also called point interaction. Then he analyzed the three cases of fixed, free and harmonically bounded nuclei, computing in each case the scattering cross section in the Born approximation.
Later, these kind of models have been extensively used in nuclear physics to describe the lowenergy dynamics of neutrons and then to investigate the structure of crystals and in general of condensed matter ([LA], [D], [LO], [DO]).
Indeed point interactions provide non trivial models indexed by a minimal set of parameters: the strengths and the positions of the interaction centers. It is then natural, in these cases, to find very simple representations for studied objects, such as the resolvent and the unitary group.
This remains true even for more complicate models, that is if one considers time dependent parameters or non linear forces, see [Fi] for a survey on these generalizations.
It should be noted that in all the applications one can find in the physical literature the computation of the relevant quantities, are only given in the Born approximation in dimension two and three. This is due to the fact that the second order term diverges if one formally applies perturbation theory to these cases.
From the mathematical point of view this means that point interactions cannot be considered as small perturbation of the free laplacian, even in the sense of quadratic forms.
Then it naturally arises the problem to give (if possible) a rigorous meaning to the formal
hamiltonian as a self adjoint and bounded below operator in the appropriate Hilbert space. In particular this would open the possibility to have a well defined perturbation theory and then to compute the relevant quantities at any order of approximation.
This program started in the sixties when Berezin and Faddeev gave the first rigorous definition ([BF]) for the hamiltonian of a particle undergoing the influence of a single delta interaction and it has been successfully completed (see e.g. $[\mathrm{AGH}-\mathrm{KH}]$ ) for the case of one particle subject to point interactions placed at fixed positions.
The case of a system of $N$ particles interacting via a two-body local point interaction is more difficult. In the case of three particles it was shown in ([MF1],[MF2]) that the most natural self adjoint realization of the hamiltonian is in fact unbounded from below and then physically unreasonable.
Surprisely enough, the same collapse phenomenon occurs in the case of one particle interacting via a point interaction with a system of $N$ non interacting particles, if $N$ is sufficiently large ([DFT], [Mi]).
This means that a satisfactory model for the dynamics of a neutron interacting with free nuclei is still lacking.
Here we approach the last case treated in [F], i.e. the case of a test particle (neutron) interacting via point interaction with a system of $N$ particles (nuclei) harmonically bounded around their equilibrium positions. The corresponding formal hamiltonian can be written as

$$
\begin{align*}
H_{\gamma} & =-\frac{\hbar^{2}}{2 M} \Delta_{x}+\sum_{i=1}^{N}\left(-\frac{\hbar^{2}}{2 m} \Delta_{y_{i}}+\frac{1}{2} m \omega^{2}\left(y_{i}-y_{i}^{0}\right)^{2}\right)+\alpha \sum_{i=1}^{N} \delta\left(x-y_{i}\right) \\
& \equiv H_{0}+\alpha \sum_{i=1}^{N} \delta\left(x-y_{i}\right) \tag{1}
\end{align*}
$$

In (1) we have denoted by $x$ the coordinate of the test particle and by $M$ its mass, by $y_{i}$ the coordinates of the oscillators, by $m$ their mass, by $\omega$ their frequency and by $y_{i}^{0}$ their equilibrium positions.
The hamiltonian $H_{0}$ refers to the free system, i.e. the system of the test particle and the oscillators without point interactions.
The parameter $\alpha \in \mathbb{R}$ plays the role of the strength of the point interaction but in (1) it should be understood only at a formal level since, as we have pointed out, point interactions cannot be treated as an additive interaction.
In chapter one we have studied hamiltonian (1) in dimension one using the theory of quadratic forms and we have constructed integral representations for the resolvent and for the unitary group. Since we are dealing with point interactions with codimension one, the definition of the hamiltonian is straightforward.
In chapter two we have studied hamiltonian (1) in dimension three; the model is much more singular due to the higher codimension of the support of the interaction and a renormalization
of the quadratic form is required in order to define a meaningful quadratic form. The operator and a representation for the resolvent have been constructed when $N=1$.
In chapter three we have studied hamiltonian (1) in dimension two; a renormalization is still required but the model is less singular than the three dimensional case, indeed the hamiltonian (1) has been constructed for every $N$ and a representation for the resolvent has been given.

In chapter four we have considered an application of this model. We have considered in one dimension, a light test particle interacting with one heavy harmonic oscillator placed at the origin and we have derived an asymptotic formula for the evolution in the limit of small mass ratio.
In such a scaling of physical parameters, an adiabatic decoupling of the degrees of freedom of the test particle and of the harmonic oscillator is expected. We have considered a superposition of two spatially separated wave packets as the initial state for the heavy particle and we have studied how the interaction with the light particle produces (partial) decoherence on the heavy one.
Decoherence induced by interaction with the environment is a key argument for understanding the classical behaviour of macro objects, see $[\mathrm{KJ}]$ and ref.cit. In our simple model the light particle represents the environment, while the heavy one is the "macroscopic" system. It worths noticing that in this model decoherence emerges as purely quantum effect, due to a scattering process and that no assumptions on the environment are required. This model should be the first step tosards a more realistic model where the environment is represented by $N$ light particles.
A similar model has been considered in [DuFiT], where the "macroscopic" system has been represented by a similar state of a free particle. Notice that in [DuFiT] the decohence process can happen only once because of the free motion of the unpertubed system; in our model the periodic motion of the system can lead, in principle, to the iteration of the decoherence phenomenon.

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## Chapter 1

## The One Dimensional Case

In this chapter we give a rigorous definition of the hamiltonian (1) as a self-adjoint and semibounded operator in dimension one and we give representations for the resolvent and for the unitary group.
By means of the theory of quadratic forms, the construction of the operator is quite straightforward: by Sobolev trace theorems we are concerned with an infinitesimal perturbation in Kato sense of the free quadratic form. Using standard perturbative arguments we prove closure and semiboundedness of the quadratic form in theorem (1.1.1).
Next, in theorems (1.2.1) and (1.2.2), we construct the operator and the resolvent, proving the decomposition of the elements of the domain in a regular and a singular part which must fulfill a suitable boundary condition, see (1.15).
In theorem (1.3.1) we give an integral representation of the unitary group by means of a system of N coupled integral equations.
It is easy to see that if one puts $\alpha=0$ in (1.2.1), (1.2.2), (1.3.1) one recovers the unperturbed operator, resolvent and propagator: as already pointed out, even if the interaction is singular, i.e. it is supported by a set of zero measure, we could construct perturbative expansions in $\alpha$ which is the true coupling constant of the system. As we shall see in the next chapters, this is no longer true in two and three dimensional case where the role of $\alpha$ is different.

### 1.1 The Quadratic Form

In this section we introduce the quadratic form $F_{\alpha}$ associated to (1) in dimension one and we prove that it's closed and bounded from below.
Let's consider the quadratic form $F_{0}$ associated to $H_{0}$ :

$$
\begin{equation*}
F_{0}[u]=\int d x d y_{1} \ldots d y_{N}\left\{\frac{\hbar^{2}}{2 M}\left|\partial_{x} u\right|^{2}+\sum_{i=1}^{N}\left(\frac{\hbar^{2}}{2 m}\left|\partial_{y_{i}} u\right|^{2}+\frac{1}{2} m \omega^{2}\left(y_{i}-y_{i}^{0}\right)^{2}|u|^{2}\right)\right\} \tag{1.1}
\end{equation*}
$$

$$
\mathscr{D}\left(F_{0}\right)=\left\{u\left(x, y_{1}, \ldots, y_{N}\right) \in L^{2}\left(\mathbb{R}^{N+1}\right) \text { s.t. } F_{0}[u]<+\infty\right\}
$$

It is well known that $F_{0}$ is a closed positive quadratic form and that:

$$
\mathscr{D}\left(F_{0}\right)=\left\{u \in L^{2}\left(\mathbb{R}^{N+1}\right) \text { s.t. } u \in H^{1}\left(\mathbb{R}^{N+1}\right), y_{i} u \in L^{2}\left(\mathbb{R}^{N+1}\right) i=1 \ldots N\right\}
$$

We shall often use $y$ without under script to indicate $\left(y_{1}, \ldots, y_{N}\right)$, and we shall omit summations over the index $i$ to make formulas less heavy. We shall deserve the square modulus only to complex quantities; $y^{2}$ stands for $y_{1}^{2}+\ldots+y_{N}^{2}$. For instance, with these conventions, (1.1) is written as:

$$
\begin{equation*}
F_{0}[u]=\int d x d y\left\{\frac{\hbar^{2}}{2 M}\left|\partial_{x} u\right|^{2}+\frac{\hbar^{2}}{2 m}\left|\partial_{y} u\right|^{2}+\frac{1}{2} m \omega^{2}\left(y-y^{0}\right)^{2}|u|^{2}\right\} \tag{1.2}
\end{equation*}
$$

We can now introduce the quadratic form $F_{\alpha}$ :

$$
\begin{gather*}
F_{\alpha}[u]=\int d x d y\left\{\frac{\hbar^{2}}{2 M}\left|\partial_{x} u\right|^{2}+\frac{\hbar^{2}}{2 m}\left|\partial_{y} u\right|^{2}+\frac{1}{2} m \omega^{2}\left(y-y^{0}\right)^{2}|u|^{2}\right\}+\alpha \sum_{i=1}^{N} \int d y\left|u\left(y_{i}, y_{1}, \ldots, y_{N}\right)\right|^{2}  \tag{1.3}\\
\equiv F_{0}[u]+\alpha F_{\text {int }}[u] \\
\mathscr{D}\left(F_{\alpha}\right)=\left\{u\left(x, y_{1}, \ldots, y_{N}\right) \in L^{2}\left(\mathbb{R}^{N+1}\right) \text { s.t. } F_{\alpha}[u]<+\infty\right\}
\end{gather*}
$$

The main properties of $F_{\alpha}$ are proved in the following theorem:
Theorem 1.1.1 The quadratic form $\left(F_{\alpha}, \mathscr{D}\left(F_{\alpha}\right)\right)$ is closed and bounded from below on $\mathscr{D}\left(F_{\alpha}\right)=$ $\mathscr{D}\left(F_{0}\right)$.

## Proof

By KLMN theorem, see [RSII], it is sufficient to show that $F_{\text {int }}$ is infinitesimal with respect to $F_{0}$ in Kato sense. By Sobolev trace theorems we have:

$$
F_{\text {int }}[u] \leq c\|u\|_{H^{\frac{1}{2}}}^{2} \leq c \varepsilon\|u\|_{H^{1}}^{2}+c \frac{1}{\varepsilon}\|u\|_{L^{2}}^{2}
$$

Since $\|u\|_{H^{1}}^{2} \leq c F_{0}[u]$ the theorem holds.

### 1.2 The Operator and the Resolvent

In this section we construct the operator $H_{\alpha}$ corresponding to the quadratic formula of the resolvent $\left(H_{\alpha}+\lambda\right)^{-1}$.
Let's introduce some more notation. Let $G^{\lambda}\left(x, y ; x^{\prime}, y^{\prime}\right)$ be the integral kernel of the Green's function of the free system; an integral representation for it have been derived in appendix (A).

For given functions $q=\left(q_{1} \ldots q_{N}\right), q_{i}: \pi_{i} \rightarrow \mathbb{C} i=1 \ldots N$, which we call charges, we shall consider the potential produced by $q_{i}$ in $\mathbb{R}^{N+1}$

$$
\left(G^{\lambda} q_{i}\right)(x, y)=\int_{\pi_{i}} d y^{\prime} G^{\lambda}\left(x, y ; y_{i}^{\prime}, y^{\prime}\right) q_{i}\left(y^{\prime}\right)
$$

and also its restriction on $\pi_{j}, i \neq j$

$$
\left(G_{j}^{\lambda} q_{i}\right)(y)=\left(G^{\lambda} q_{i}\right)\left(y_{j}, y\right)=\int_{\pi_{i}} d y^{\prime} G^{\lambda}\left(y_{j}, y ; y_{i}^{\prime}, y^{\prime}\right) q_{i}\left(y^{\prime}\right)
$$

Now we can state the main theorem of this section.
Theorem 1.2.1 The resolvent $\left(H_{\alpha}+\lambda\right)^{-1}$ has the following representation: there exists $\lambda_{0}>0$ s.t.for $\lambda>\lambda_{0}$ and for any $f \in L^{2}\left(\mathbb{R}^{N+1}\right)$ there are $N$ charges $\left(q_{1}, \ldots, q_{N}\right)$ solutions of the following system

$$
\begin{equation*}
q_{i}(y)+\alpha\left(\sum_{j=1}^{N} G_{i}^{\lambda} q_{j}(y)+G^{\lambda} f\left(y_{i}, y\right)\right)=0 \quad i=1 \ldots N \tag{1.4}
\end{equation*}
$$

and the following representation holds

$$
\begin{equation*}
\left(H_{\alpha}+\lambda\right)^{-1} f=G^{\lambda} f+G^{\lambda} q \tag{1.5}
\end{equation*}
$$

## Proof

If $u=\left(H_{\alpha}+\lambda\right)^{-1} f$ and $v \in C_{0}^{\infty}\left(\mathbb{R}^{N+1}\right)$ by the first representation theorem of quadratic forms, one has

$$
F_{\alpha}[u, v]+\lambda(u, v)=(f, v)
$$

It's natural to start from the ansatz

$$
R f=G^{\lambda} f+G^{\lambda} q
$$

which is the typical structure of the resolvent when point interactions are involved, and to look for conditions on $q$ in order to satisfy

$$
\begin{equation*}
F_{\alpha}[R f, v]+\lambda(R f, v)=(f, v) \tag{1.6}
\end{equation*}
$$

Equation (1.6) can be explicitly written as

$$
\begin{align*}
& \int d x d y\left[\frac{\hbar^{2}}{2 M} \overline{\partial_{x} G^{\lambda} f} \partial_{x} v+\frac{\hbar^{2}}{2 m} \overline{\partial_{y} G^{\lambda} f} \partial_{y} v+\frac{1}{2} m \omega^{2}\left(y-y^{0}\right)^{2} \overline{G^{\lambda} f} v+\lambda \overline{G^{\lambda} f} v\right]+ \\
& \quad+\int d x d y\left[\frac{\hbar^{2}}{2 M} \overline{\partial_{x} G^{\lambda} q} \partial_{x} v+\frac{\hbar^{2}}{2 m} \overline{\partial_{y} G^{\lambda} q} \partial_{y} v+\frac{1}{2} m \omega^{2}\left(y-y^{0}\right)^{2} \overline{G^{\lambda} q} v+\lambda \overline{G^{\lambda} q} v\right]+ \\
& +\alpha \sum_{i=1}^{N} \int d y\left(\overline{G^{\lambda} f\left(y_{i}, y\right)+G^{\lambda} q\left(y_{i}, y\right)}\right) v\left(y_{i}, y\right)=\int d x d y \bar{f} v \tag{1.7}
\end{align*}
$$

Since $G^{\lambda} f \in \mathscr{D}\left(H_{0}\right)$, we can integrate by parts the first piece of (1.7) and we obtain

$$
\begin{aligned}
\int d x d y\left[\frac{\hbar^{2}}{2 M} \overline{\partial_{x} G^{\lambda} q} \partial_{x} v+\frac{\hbar^{2}}{2 m} \overline{\partial_{y} G^{\lambda} q} \partial_{y} v\right. & \left.+\frac{1}{2} m \omega^{2}\left(y-y^{0}\right)^{2} \overline{G^{\lambda} q} v+\lambda \overline{G^{\lambda} q} v\right]+ \\
& +\alpha \sum_{i=1}^{N} \int d y\left(\overline{G^{\lambda} f\left(y_{i}, y\right)+G^{\lambda} q\left(y_{i}, y\right)}\right) v\left(y_{i}, y\right)=0
\end{aligned}
$$

Using the regularity of $v$ we can integrate by parts again obtaining:

$$
\begin{equation*}
\sum_{i=1}^{N} \int d y\left(\overline{q_{i}(y)+\alpha G^{\lambda} f\left(y_{i}, y\right)+\alpha G^{\lambda} q\left(y_{i}, y\right)}\right) v\left(y_{i}, y\right)=0 \tag{1.8}
\end{equation*}
$$

In order to satisfy (1.8) for any $v$, each term of the sum must be zero, then we get (1.4).
Now we prove that the domain of $R$ is $L^{2}\left(\mathbb{R}^{N+1}\right)$, that is for every $f$, exists $\lambda_{0}$ such that for $\lambda>\lambda_{0}(1.4)$ admits a solution $q$ such that $G^{\lambda} q_{i} \in L^{2}\left(\mathbb{R}^{N+1}\right) \forall i$.
If we prove

$$
\begin{equation*}
\sup _{x, y} \int d y G^{\lambda}\left(x, y ; y_{i}^{\prime}, y^{\prime}\right) \leq c_{1} \quad \sup _{y^{\prime}} \int d x d y G^{\lambda}\left(x, y ; y_{i}^{\prime}, y^{\prime}\right) \leq c_{2} \tag{1.9}
\end{equation*}
$$

Schur's test, see [HS], implies

$$
\begin{equation*}
\left\|G^{\lambda} q\right\|_{L^{2}\left(\mathbb{R}^{N+1}\right)} \leq \sqrt{c_{1} c_{2}}\|q\|_{L^{2}\left(\mathbb{R}^{N}\right)} \tag{1.10}
\end{equation*}
$$

It is straightforward to see that (1.9) holds if we take:

$$
\begin{equation*}
c_{1}=\frac{1}{\hbar \omega} \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}-\frac{1}{2}} \quad c_{2}=\frac{1}{\sqrt{2 \pi}} \frac{1}{\hbar \omega}\left(\frac{M \omega}{\hbar}\right)^{\frac{1}{2}} \int_{0}^{1} d \nu \frac{\nu^{\frac{\lambda}{\hbar \omega}-\frac{1}{2}}}{\left(\ln \frac{1}{\nu}\right)^{\frac{1}{2}}} \tag{1.11}
\end{equation*}
$$

By (1.10), it is then sufficient to show that (1.4) admits a square integrable solution.
Using again Schur's test it is easy to see that for $\forall i, j=1, \ldots N$ one has:

$$
\begin{equation*}
\int d y\left|G^{\lambda} q_{i}\left(y_{j}, y\right)\right|^{2} \leq\left(\frac{1}{\sqrt{2 \pi}} \frac{1}{\hbar \omega}\left(\frac{M \omega}{\hbar}\right)^{\frac{1}{2}} \int_{0}^{1} d \nu \frac{\nu^{\frac{\lambda}{\hbar \omega}-\frac{1}{2}}}{\left(\ln \frac{1}{\nu}\right)^{\frac{1}{2}}}\right) \int d y\left|q_{i}(y)\right|^{2} \tag{1.12}
\end{equation*}
$$

System (1.4) can be written in a more compact form as a fixed point equation in the Hilbert space $\bigoplus_{i=1}^{N} L^{2}\left(\pi_{i}\right)$ : if we put $\left(T_{f}^{\lambda} q\right)_{i}(y)=G^{\lambda} f\left(y_{i}, y\right)+\sum_{j} G^{\lambda} q_{j}\left(y_{i}, y\right)$ system (1.4) is equivalent to the equation

$$
\begin{equation*}
q+\alpha T_{f}^{\lambda} q=0 \tag{1.13}
\end{equation*}
$$

By estimate (1.12), $T_{f}^{\lambda}$ becomes a contraction as $\lambda \rightarrow+\infty$, then exists $\lambda_{0}>0$ s.t. for $\lambda>\lambda_{0}$ (1.13) has a solution.

Now we prove that the operator $R$ is symmetric: the condition $(R f, g)=(f, R g)$ is equivalent to $\left(G^{\lambda} q^{f}, g\right)=\left(f, G^{\lambda} q^{g}\right)$ which can be written as

$$
\begin{equation*}
\sum_{i=1}^{N} \int d y \overline{q_{i}^{f}(y)} G^{\lambda} g\left(y_{i}, y\right)=\sum_{i=1}^{N} \int d y \overline{G^{\lambda} f\left(y_{i}, y\right)} q_{i}^{g}(y) \tag{1.14}
\end{equation*}
$$

If we solve (1.4) with respect to $G^{\lambda} f\left(y_{i}, y\right)$ and to $G^{\lambda} g\left(y_{i}, y\right)$ and we substitute into (1.14) we immediately obtain an identity.
Since we have proved that $R$ is everywhere defined and symmetric, it is a bounded self adjoint operator operator by Hellinger-Toeplitz theorem, see [RSI].
The last step to conclude the proof of the theorem is showing that $R$ has empty kernel, which is equivalent to saying that $H_{\alpha}$ is densely defined. If we suppose $R f=0$ then we have $R f\left(y_{i}, y\right)=$ $0 \forall i=1 \ldots N$. Notice that the range of $R$ is contained in $H^{1}(\mathbb{R})$ and then we are allowed to take the traces of $R f$ over $\pi_{i}$. System (1.4) immediately implies $q=0$ and then $G^{\lambda} f=0$ which means $f=0$.

The knowledge of the resolvent is equivalent to the knowledge of the operator itself; next theorem is just a different rephrasing of the previous one.

Theorem 1.2.2 The domain and the action of $H_{\alpha}$ are the following:

$$
\begin{gather*}
\mathscr{D}\left(H_{\alpha}\right)=\left\{u \in L^{2}\left(\mathbb{R}^{N+1}\right) \text { s.t. } u=\varphi^{\lambda}+G^{\lambda} q, \varphi^{\lambda} \in \mathscr{D}\left(H_{0}\right), q_{i}+\alpha T r_{\pi_{i}} u=0 \forall i=1 \ldots N\right\}  \tag{1.15}\\
\left(H_{\alpha}+\lambda\right) u=\left(H_{0}+\lambda\right) \varphi^{\lambda} \tag{1.16}
\end{gather*}
$$

## Proof

The key observation is that $u \in \mathscr{D}\left(H_{\alpha}\right)$ if and only $u=\left(H_{\alpha}+\lambda\right)^{-1} f$. If we put $\varphi^{\lambda}=G^{\lambda} f$, (1.15) and (1.16) follow immediately from the previous theorem.

### 1.3 The Unitary Group

In this section we give an integral representation for the unitary group $e^{-i \frac{t}{\hbar} H_{\alpha}}$.
The free propagator is simply the product between the free propagator and the Mehler kernel:

$$
e^{-i \frac{t}{\hbar} H_{0}}\left(x, y ; x^{\prime}, y^{\prime}\right)=\left(\frac{M}{2 \pi i \hbar t}\right)^{\frac{1}{2}} e^{i \frac{M}{2 \hbar t}\left(x-x^{\prime}\right)^{2}}\left(\frac{m \omega}{\pi \hbar\left(1-e^{-2 i \omega t}\right)}\right)^{\frac{N}{2}} e^{-\frac{m \omega}{2 i \hbar \tan (\omega t)}\left(y^{2}+y^{\prime 2}\right)} e^{\frac{m \omega}{i \hbar \sin (\omega t)} y y^{\prime}}
$$

In order to formulate the main theorem of this section we introduce the operators $W_{j k}^{t}$ which are the restrictions of $e^{-i \frac{t}{\hbar} H_{0}}$ over the planes $\pi_{j}, \pi_{k}$.

$$
W_{j, k}^{t}\left(y ; y^{\prime}\right)=\left(\frac{M}{2 \pi i \hbar t}\right)^{\frac{1}{2}} e^{i \frac{M}{2 \hbar t}\left(y_{j}-y_{k}\right)^{2}}\left(\frac{m \omega}{\pi \hbar\left(1-e^{-2 i \omega t}\right)}\right)^{\frac{N}{2}} e^{-\frac{m \omega}{2 i \hbar \tan (\omega t)}\left(y^{2}+y^{\prime 2}\right)} e^{\frac{m \omega}{i \hbar \sin (\omega t)} y y^{\prime}}
$$

Theorem 1.3.1 Let $\psi_{0} \in \mathscr{D}\left(H_{\alpha}\right)$ a regular initial datum, then $\psi(t)$ has the following representation: there are $N$ charges $q_{j}(t)$ depending on time such that

$$
\begin{equation*}
\psi(t ; x, y)=\left(e^{-i \frac{t}{\hbar} H_{0}} \psi_{0}\right)(x, y)+\frac{i}{\hbar} \int_{0}^{t} d s \sum_{j=1}^{N} \int d y^{\prime} e^{-i \frac{t-s}{\hbar} H_{0}}\left(x, y ; y_{j}^{\prime}, y^{\prime}\right) q_{j}\left(s ; y^{\prime}\right) \tag{1.17}
\end{equation*}
$$

The charges $q_{i}(t)$ are solution of the following system:

$$
\begin{equation*}
q_{j}(t)+\alpha \operatorname{Tr}_{\pi_{j}}\left(e^{-i \frac{t}{\hbar} H_{0}} \psi_{0}\right)+\frac{i \alpha}{\hbar} \sum_{k=1}^{N} \int_{0}^{t} d s W_{j k}^{t-s} q_{k}(s)=0 \quad \forall j=1 \ldots N \tag{1.18}
\end{equation*}
$$

## Proof

We start from the ansatz (1.17); if we diagonalize the free evolution with a Fourier transform and a projection on the basis of the harmonic oscillator we obtain:

$$
\begin{gather*}
\psi(t ; k, \underline{n})=e^{-i\left(\frac{k^{2}}{2 M \hbar}+|\underline{n}| \omega\right) t} \psi_{0}(k, \underline{n})+\frac{i}{\hbar} \int_{0}^{t} d s e^{-i\left(\frac{k^{2}}{2 M \hbar}+|\underline{n}| \omega\right)(t-s)} \sum_{j=1}^{N} \check{q}_{j}(s ; k, \underline{n})  \tag{1.19}\\
\check{q}_{j}(t ; k, \underline{n})=\frac{1}{(2 \pi \hbar)^{\frac{1}{2}}} \int d y^{\prime} e^{-i \frac{k y_{j}^{\prime}}{\hbar}} e_{\underline{n}}\left(y^{\prime}\right) q_{j}\left(t ; y^{\prime}\right) \tag{1.20}
\end{gather*}
$$

In the previous formula we have used standard notation on multi-indexes: $\underline{n}=\left(n_{1}, \ldots, n_{N}\right)$ and $|\underline{n}|=n_{1}+\ldots+n_{N}$; we have denoted the n-th eigenvector of the one dimensional harmonic oscillator with $e_{n}$ while $e_{\underline{n}}(y)$ stands for $e_{n_{1}}\left(y_{1}\right) e_{n_{2}}\left(y_{2}\right) \ldots e_{n_{N}}\left(y_{N}\right)$.
With an integration by parts we obtain:

$$
\begin{align*}
& \psi(t ; k, \underline{n})=e^{-i\left(\frac{k^{2}}{2 M \hbar}+|\underline{n}| \omega\right) t} \varphi_{0}^{\lambda}(k, \underline{n})-\frac{1}{\frac{k^{2}}{2 M}+|\underline{n}| \hbar \omega+\lambda} \int_{0}^{t} d s e^{-i\left(\frac{k^{2}}{2 M \hbar}+|\underline{n}| \omega\right)(t-s)} \sum_{i=j}^{N} \stackrel{\check{q}}{j}(s ; k, \underline{n})+ \\
& \quad+\frac{i}{\hbar} \frac{\lambda}{\frac{k^{2}}{2 M}+|\underline{n}| \hbar \omega+\lambda} \int_{0}^{t} d s e^{-i\left(\frac{k^{2}}{2 M \hbar}+|\underline{n}| \omega\right)(t-s)} \sum_{j=1}^{N} \check{q}_{j}(s ; k, \underline{n})+\frac{1}{\frac{k^{2}}{2 M}+|\underline{n}| \hbar \omega+\lambda} \sum_{j=1}^{N} \check{q}_{j}(s ; k, \underline{n}) \tag{1.21}
\end{align*}
$$

If $q$ are regular functions the first three terms on the r.h.s. belong to $\mathscr{D}\left(H_{0}\right)$ while the last one is $G^{\lambda} q(t)$ which doesn't belong to $\mathscr{D}\left(H_{0}\right)$; we impose the boundary condition contained in (1.15) such that $\psi(t) \in \mathscr{D}\left(H_{\alpha}\right)$, then we obtain (1.20). It is now straightforward to verify that (1.17) satisfies the time dependent Schroedinger equation: using respectively (1.17) and (1.21) we have

$$
\begin{align*}
& i \hbar \frac{d \psi}{d t}(t)=\left(\frac{k^{2}}{2 M}+|\underline{n}| \hbar \omega\right) e^{-i\left(\frac{k^{2}}{2 M \hbar}+|\underline{n}| \omega\right) t} \psi_{0}(k, \underline{n})-\sum_{j=1}^{N} \check{q}_{j}(s ; k, \underline{n})+ \\
& \quad+\frac{i}{\hbar}\left(\frac{k^{2}}{2 M}+|\underline{n}| \hbar \omega\right) \int_{0}^{t} d s e^{-i\left(\frac{k^{2}}{2 M \hbar}+|\underline{n}| \omega\right)(t-s)} \sum_{j=1}^{N} \check{q}_{j}(s ; k, \underline{n})+  \tag{1.22}\\
& H_{\alpha} \psi(t ; k, \underline{n})=\left(\frac{k^{2}}{2 M}+|\underline{n}| \hbar \omega\right) e^{-i\left(\frac{k^{2}}{2 M \hbar}+|\underline{n}| \omega\right) t} \varphi_{0}^{\lambda}(k, \underline{n})-\frac{\frac{k^{2}}{2 M}+|\underline{n}| \hbar \omega}{\frac{k^{2}}{2 M}+|\underline{n}| \hbar \omega+\lambda} . \\
& \begin{array}{l}
\int_{0}^{t} d s e^{-i\left(\frac{k^{2}}{2 M \hbar}+|\underline{n}| \omega\right)(t-s)} \sum_{i=1}^{N} \stackrel{\circ}{q}_{i}(s ; k, \underline{n})+\frac{i}{\hbar} \lambda \frac{\left(\frac{k^{2}}{2 M}+|\underline{n}| \hbar \omega\right)}{\frac{k^{2}}{2 M}+|\underline{n}| \hbar \omega+\lambda} \int_{0}^{t} d s e^{-i\left(\frac{k^{2}}{2 M \hbar}+|\underline{n}| \omega\right)(t-s)} \sum_{j=1}^{N} \check{q}_{j}(s ; k, \underline{n})+ \\
\quad-\frac{\lambda}{\frac{k^{2}}{2 M}+|\underline{n}| \hbar \omega+\lambda} \sum_{j=1}^{N} \check{q}_{j}(s ; k, \underline{n}) \quad(1.23)
\end{array}
\end{align*}
$$

The r.h.s. of (1.22) and (1.23) are equal and then $\psi(t)$ is the solution of the time dependent Schroedinger equation. The last step to conclude the proof of the theorem is proving that (1.18) has a global solution. System (1.18) can be written in a more compact form as a linear equation in $C\left(0, T ; \oplus_{i=1}^{N} L^{2}\left(\pi_{i}\right)\right)$.

$$
\begin{equation*}
q(t)+\frac{i \alpha}{\hbar} \int_{0}^{t} d s W^{t-s} q(s)+\alpha \Psi(t)=0 \tag{1.24}
\end{equation*}
$$

We have put

$$
\begin{equation*}
(\Psi(t))_{j}=\operatorname{Tr}_{\pi_{j}} e^{-i \frac{t}{\hbar} H_{0}} \psi_{0} \quad\left(W^{t-s} q(s)\right)_{j}=\sum_{k=1}^{N} W_{j k}^{t-s} q_{k}(s) \tag{1.25}
\end{equation*}
$$

In order to prove the local existence of $q(t)$ it sufficient to prove the following estimate:

$$
\begin{equation*}
\left\|W_{j k}^{t}\right\| \leq C \frac{1}{\sqrt{t}} \tag{1.26}
\end{equation*}
$$

If (1.26) holds then exists $T_{0}$ s.t. for $T<T_{0} \int_{0}^{t} d s W^{t-s} q(s)$ becomes a contraction and (1.24) has a solution. The solution can be extended for any $T$ by iteration. Estimate (1.26) is trivial if $j=k$, otherwise it follows from standard results on Fourier Integral Operators, see [Fu].

## Chapter 2

## The Three Dimensional Case

In this chapter we introduce the quadratic form $F_{\alpha}$ in dimension three. Even if we are still concerned by a perturbation of the free quadratic form $F_{0}$ supported by a null set, i.e. the planes $\pi_{i}$, the situation is very different from the one previous case. The perturbation has now support on a set of codimension three and this gives much stronger singularities.
It is then necessary a renormalization procedure to construct a well-posed quadratic form. We adapt to our case the techniques developed in $[\mathrm{T}]$ and $[\mathrm{DFT}]$ which lead to definition (2.1.1). As a matter of fact, the domain of the quadratic form consists of functions in $\mathscr{D}\left(F_{0}\right)$ plus the potentials $G^{\lambda} q$. It is then clear that we are no longer concerned with a bounded perturbation of $F_{0}$; indeed one can see that if one puts $\alpha=0$ the unperturbed quadratic form is not recovered, that is $\alpha$ is no longer the coupling constant of the system.
In section one we introduce the quadratic form; it is required some technical work, see propositions (2.1.2), (2.1.3), (2.1.4), (2.1.5), to prove that $F_{\alpha}$ is well defined.
Closure and semiboundedness of the quadratic form are proved in the case $N=1$, see (2.1.7) and (2.1.8).
In section two, theorem (2.2.1) characterizes the domain and the action of the operator.

### 2.1 The Quadratic Form

In this section we introduce the quadratic form $F_{\alpha}$, we show that it is well defined and we prove it is closed and positive if $N=1$. The quadratic form $F_{\alpha}$ is defined as follows.

## Definition 2.1.1

$$
\begin{gathered}
\mathscr{D}\left(F_{\alpha}\right)=\left\{u \in L^{2}\left(\mathbb{R}^{3 N+3}\right) \text { s.t. } \exists q \in \mathscr{D}\left(\Phi_{\alpha}^{\lambda}\right), \varphi^{\lambda} \equiv u-G^{\lambda} q \in \mathscr{D}\left(F_{0}\right)\right\} \\
F_{\alpha}[u]=\mathcal{F}^{\lambda}[u]+\Phi_{\alpha}^{\lambda}[u] \\
\mathcal{F}^{\lambda}[u]=\int d x d y\left\{\frac{\hbar^{2}}{2 M}\left|\nabla_{x} \varphi^{\lambda}\right|^{2}+\frac{\hbar^{2}}{2 m}\left|\nabla_{y} \varphi^{\lambda}\right|^{2}+\lambda\left|\varphi^{\lambda}\right|^{2}-\lambda|u|^{2}+\frac{1}{2} m \omega^{2}\left(y-y^{0}\right)^{2}\left|\varphi^{\lambda}\right|^{2}\right\}
\end{gathered}
$$

$$
\begin{gathered}
\mathscr{D}\left(\Phi_{\alpha}^{\lambda}\right)=\left\{\left(q_{1}, \ldots, q_{N}\right), q_{i} \in L^{2}\left(\pi_{i}\right) \text { s.t. } \Phi^{\lambda}[q]<+\infty\right\} \\
\Phi_{\alpha}^{\lambda}[q]=\alpha \sum_{i=1}^{N} \int d y\left|q_{i}(y)\right|^{2}-2 \Re \sum_{i<j} \int d y \overline{q_{i}(y)}\left(G_{i}^{\lambda} q_{j}\right)(y)+ \\
+\sum_{i=1}^{N} \int d y a_{i}^{\lambda}(y)\left|q_{i}(y)\right|^{2}+\frac{1}{2} \sum_{i=1}^{N} \int d y d y^{\prime} G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)\left|q_{i}(y)-q_{i}\left(y^{\prime}\right)\right|^{2}
\end{gathered}
$$

In the previous definition we have introduced the functions $a_{i}^{\lambda}(y)$ given by:

$$
\begin{align*}
& a_{i}^{\lambda}(y)=\frac{1}{(2 \pi)^{\frac{3}{2}}} \omega^{\frac{1}{2}} \frac{1}{\hbar^{\frac{5}{2}}}\left(\frac{m M}{m+M}\right)^{\frac{3}{2}}\left\{\frac{1}{2}+\int_{0}^{1} d \nu \frac{1}{(1-\nu)^{\frac{3}{2}}}\left[1-\nu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1}\left(\frac{2}{1+\nu^{2}}\right)^{\frac{3(N-1)}{2}}\right.\right. \\
&\left(\frac{m+M}{\frac{1+\nu^{2}}{2} \frac{\ln \frac{1}{\nu}}{1-\nu} m+\frac{1+\nu}{2} M}\right)^{\frac{3}{2}} \exp \left(-\frac{11-\nu^{2}}{2} \frac{m \omega}{1+\nu^{2}} \frac{N}{\hbar} \sum_{\substack{j=1 \\
j \neq i}}^{N}\left|y_{j}-y_{j}^{0}\right|^{2}\right) \\
&\left.\left.\exp \left(-\frac{\omega}{\hbar} \frac{\frac{1}{4} m^{2}+\frac{1-\nu}{1+\nu} \frac{1}{2 \ln \frac{1}{\nu}} m M}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} m+\frac{1}{2 \ln \frac{1}{\nu}} M}\left|y_{i}-y_{i}^{0}\right|^{2}\right)\right]\right\} \tag{2.1}
\end{align*}
$$

In order to give a heuristic motivation for this definition one can proceed as follows. The naive quadratic form associated to $H_{\gamma}$ is:

$$
\tilde{F}_{\gamma}[u]=F_{0}[u]+\sum_{j} \gamma \int d x d y|u|^{2} \delta\left(x-y_{j}\right)
$$

which can be written as

$$
\begin{align*}
\tilde{F}_{\gamma}[u]=F_{0}[u- & \left.G^{\lambda} q\right]+\lambda \int d x d y\left|u-G^{\lambda} q\right|^{2}-\lambda \int d x d y|u|^{2}+\left(\left(H_{0}+\lambda\right) G^{\lambda} q, u\right)+ \\
& +\left(u,\left(H_{0}+\lambda\right) G^{\lambda} q\right)-\left(\left(H_{0}+\lambda\right) G^{\lambda} q, G^{\lambda} q\right)+\sum_{j} \gamma \int d x d y|u|^{2} \delta\left(x-y_{j}\right) \tag{2.2}
\end{align*}
$$

for any $\lambda>0$ and any charge $q$. Now we fix $q_{i}=-\left.\gamma u\right|_{\pi_{i}}$ and denote $\varphi^{\lambda}=u-G^{\lambda} q$. Then we use the fact the $\left(H_{0}+\lambda\right) G^{\lambda} q=q$ and we finally obtain

$$
\tilde{F}_{\gamma}[u]+\lambda \int d x d y|u|^{2}=F_{0}\left[\varphi^{\lambda}\right]+\lambda \int d x d y\left|\varphi^{\lambda}\right|^{2}+\tilde{\Phi}_{\gamma}[q]
$$

$$
\begin{equation*}
\tilde{\Phi}_{\gamma}[q]=\sum_{i} \int_{\pi_{i}} d y \bar{q}_{i}(y)\left[-\frac{q_{i}(y)}{\gamma}-G_{i}^{\lambda} q_{i}(y)\right]-2 \Re \sum_{i<j} \int d y \overline{q_{i}}(y) G_{i}^{\lambda} q_{j}(y) \tag{2.3}
\end{equation*}
$$

It's clear that the form $\tilde{\Phi}_{\gamma}[q]$ on the charges is ill defined due to the presence of the first term of the r.h.s. of (2.3) (diagonal terms) since the potential $G^{\lambda} q_{i}$ is divergent on $\pi_{i}$. Its singularity around $y \in \pi_{i}$ is only due to the value of $q_{i}$ in $y$, i.e. it has the same singularity of the potential generated by the uniform charge distribution $q_{i}(y)$ on $U_{\varepsilon}$, a small neighborhood of $y$. Then we can write

$$
G^{\lambda} q_{i}(x, y)=q_{i}(y) \zeta_{1_{\varepsilon}}(x, y)+\tau_{q_{i}}(x, y)
$$

where $\zeta_{1_{\varepsilon}}$ is the divergent part of $G^{\lambda} 1_{\varepsilon}$ and $\tau_{q_{i}}(x, y)$ remains finite in the limit $x \rightarrow y_{i}$; we have denoted the characteristic function of $U_{\varepsilon}$ with $1_{\varepsilon}$. Moreover we define $G_{i, r e n}^{\lambda} q_{i}(y)=\lim _{x \rightarrow y_{i}} \tau_{q_{i}}(x, y)$. The idea is to separate the divergent part and the finite part of the quadratic form and the to compensate the divergence through a renormalization of the coupling constant $\gamma$. We only outline here the procedure.
Define $\pi_{i}^{\delta}=\left\{(x, y)\right.$ s.t. $\left.\frac{1}{\sqrt{2}}\left|x-y_{i}\right| \leq \delta\right\}$ and consider a charge distribution $q_{i}^{\delta}$ on $\pi_{i}^{\delta}$ such that $q_{i}^{\delta} \rightarrow q_{i}$ and $G^{\lambda} q_{i}^{\delta} \rightarrow G^{\lambda} q_{i}$ for $\delta \rightarrow 0$. Moreover $G^{\lambda} q_{i}^{\delta}(x, y)=q_{i}^{\delta}(y) \zeta^{\delta}(x, y)+\tau_{q_{i}}^{\delta}(x, y)$ where $\zeta^{\delta} \rightarrow \zeta_{1_{\varepsilon}}$ and $\tau_{q_{i}}^{\delta} \rightarrow \tau_{q_{i}}$ for $\delta \rightarrow 0$. Introducing a new coupling constant $\gamma=\gamma^{\delta}(x, y)$ the ill defined diagonal term can now be replaced by

$$
\begin{gather*}
\lim _{\delta \rightarrow 0} \int_{\pi_{i}^{\delta}} \overline{q_{i}^{\delta}}(x, y)\left[-\frac{q_{i}^{\delta}(x, y)}{\gamma^{\delta}(x, y)}-G^{\lambda} q_{i}^{\delta}(x, y)\right]= \\
\lim _{\delta \rightarrow 0} \int_{\pi_{i}^{\delta}} \overline{q_{i}^{\delta}}(x, y)\left[-\frac{q_{i}^{\delta}(x, y)}{\gamma^{\delta}(x, y)}-q_{i}^{\delta}(x, y) \zeta^{\delta}(x, y)\right]-\int_{\pi_{i}^{\delta}} \overline{q_{i}^{\delta}}(x, y) \tau_{q_{i}}^{\delta}(x, y) \tag{2.4}
\end{gather*}
$$

In order to get a finite limit in (2.4) it is sufficient to define

$$
-\frac{1}{\gamma^{\delta}(x, y)}=\zeta^{\delta}(x, y)+\alpha, \quad \alpha \in \mathbb{R}
$$

so that the diagonal term reduces to

$$
\begin{equation*}
\alpha \int_{\pi_{i}}\left|q_{i}\right|^{2}-\int_{\pi_{i}} \overline{q_{i}}(y) G_{i, r e n}^{\lambda} q_{i}(y) \tag{2.5}
\end{equation*}
$$

We shall now rewrite the renormalized potential $G_{i, \text { ren }}^{\lambda} q_{i}$ in a more convenient form

$$
\begin{aligned}
G_{i, r e n}^{\lambda} q_{i}(y) & =\lim _{x \rightarrow y_{i}}\left[G^{\lambda} q_{i}(x, y)-q_{i}(y) \zeta_{1 \varepsilon}(x, y)\right]=\lim _{x \rightarrow y_{i}}\left[\int_{\left|y^{\prime}-y\right|>\varepsilon} d y^{\prime} q_{i}\left(y^{\prime}\right) G^{\lambda}\left(x, y ; y_{i}^{\prime}, y^{\prime}\right)+\right. \\
& +\int_{\left|y^{\prime}-y\right|<\varepsilon} d y^{\prime} G^{\lambda}\left(x, y ; y_{i}^{\prime}, y^{\prime}\right)\left(q_{i}\left(y^{\prime}\right)-q_{i}(y)\right)+ \\
& \left.+q_{i}(y)\left(\int_{\left|y^{\prime}-y\right|<\varepsilon} d y^{\prime} G^{\lambda}\left(x, y ; y_{i}^{\prime}, y^{\prime}\right)-\zeta_{1_{\varepsilon}}(x, y)\right)\right]
\end{aligned}
$$

Then the second term of (2.5) becomes:

$$
\begin{aligned}
-\int d y \overline{q_{i}(y)} G_{i, r e n}^{\lambda} q_{i}(y)= & -\int_{\left|y^{\prime}-y\right|>\varepsilon} d y d y^{\prime} \overline{q_{i}(y)} q_{i}\left(y^{\prime}\right) G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)+ \\
& -\int_{\left|y^{\prime}-y\right|<\varepsilon} d y d y^{\prime} \overline{q_{i}(y)}\left(q_{i}\left(y^{\prime}\right)-q_{i}(y)\right) G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)+ \\
& -\int d y\left|q_{i}(y)\right|^{2} \tau_{1_{\varepsilon}}\left(y_{i}, y\right) \\
= & \int d y d y^{\prime} \overline{q_{i}(y)}\left(q_{i}(y)-q_{i}\left(y^{\prime}\right)\right) G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)+ \\
& +\int d y\left|q_{i}(y)\right|^{2}\left(-\tau_{1_{\varepsilon}}\left(y_{i}, y\right)-\int_{\left|y^{\prime}-y\right|>\varepsilon} d y^{\prime} G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)\right) \\
= & \frac{1}{2} \int d y d y^{\prime}\left|q_{i}(y)-q_{i}\left(y^{\prime}\right)\right|^{2} G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)+\int d y a_{i}^{\lambda}(y)\left|q_{i}(y)\right|^{2}
\end{aligned}
$$

Where we have denoted

$$
a_{i}^{\lambda}(y)=-\lim _{x \rightarrow y_{i}}\left(\tau_{1_{\varepsilon}}(x, y)+\int_{\left|y^{\prime}-y\right|>\varepsilon} d y^{\prime} G^{\lambda}\left(x, y ; y_{i}^{\prime}, y^{\prime}\right)\right)
$$

and $\tau_{1_{\varepsilon}}$ is the finite part of the potential generated by $1_{\varepsilon}$.
It remains to analyze $a_{i}^{\lambda}(y)$. To be concrete we fix $i=1$, put $y^{0}=0$ and introduce the coordinates $\xi$ and $\eta$ defined in the following way.

$$
\left\{\begin{aligned}
\xi & =\frac{1}{\sqrt{2}}\left(x-y_{1}\right) \\
\eta & =\frac{1}{\sqrt{2}}\left(x+y_{1}\right)
\end{aligned}\right.
$$

With this choice $\pi_{1}$ is defined by $\xi=0$. We have to study the behavior for small $\xi$ of

$$
\begin{equation*}
\left(G^{\lambda} 1_{\varepsilon}\right)\left(\xi, \eta, y_{2}, \ldots, y_{N}\right)=\int_{U_{\varepsilon}} d \eta^{\prime} d y_{2}^{\prime} \ldots d y_{N}^{\prime} G^{\lambda}\left(\xi, \eta, y_{2}, \ldots, y_{N} ; 0, \eta^{\prime}, y_{2}^{\prime}, \ldots, y_{N}^{\prime}\right) \tag{2.6}
\end{equation*}
$$

The shape of $U_{\varepsilon}$ doesn't matter, indeed $a_{i}^{\lambda}$ doesn't depend on $\varepsilon$; we choose

$$
U_{\varepsilon}=\left\{\left|\eta^{\prime}-\eta\right|<\varepsilon,\left|y_{2}^{\prime}-y_{2}\right|<\varepsilon, \ldots,\left|y_{N}^{\prime}-y_{N}\right|<\varepsilon\right\}
$$

From (A.1) one has that, with simple manipulations, (2.6) can be written as:

$$
\begin{gather*}
\left(G^{\lambda} 1_{\varepsilon}\right)\left(\xi, \eta, y_{2}, \ldots, y_{N}\right)=\frac{1}{(2 \pi)^{\frac{3}{2}}} \frac{1}{\hbar \omega} \frac{1}{\pi^{\frac{3 N}{2}}}\left(\frac{m \omega}{\hbar}\right)^{\frac{3 N}{2}}\left(\frac{M \omega}{\hbar}\right)^{\frac{3}{2}} \int_{0}^{1} d \nu \frac{\nu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1}}{(1+\nu)^{\frac{3 N}{2}}} \frac{1}{(1-\nu)^{\frac{3 N}{2}}}\left(\log \frac{1}{\nu}\right)^{\frac{3}{2}} \\
\prod_{i=2}^{N}\left\{e^{-\frac{m \omega}{\hbar} \frac{1-\nu}{1+\nu} y_{i}^{2}}\left(\frac{m \omega}{\hbar} \frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}}\right)^{-\frac{3}{2}} \int_{\left.\left|y_{i}^{\prime}\right|<\varepsilon\left(\frac{m \omega}{\hbar} \frac{\left(1+\nu^{2}\right)}{2\left(1-\nu^{2}\right)}\right)^{\frac{1}{2}} d y_{i}^{\prime} e^{-y_{i}^{\prime 2}} e^{-y_{i} \cdot y_{i}^{\prime}\left(\frac{m \omega}{\hbar} \frac{2\left(1-\nu^{2}\right)}{\left(1+\nu^{2}\right)}\right)^{\frac{1}{2}} \frac{1-\nu}{1+\nu}}\right\}}^{e^{-\frac{1}{2} \frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar}\left(\eta^{2}-\eta \cdot \xi\right)} e^{-\xi^{2}\left(\frac{1}{4} \frac{1-\nu^{2}}{1+\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{4 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)}\left(\frac{1}{4} \frac{1-\nu^{2}}{1+\nu^{2}} \frac{m \omega}{\hbar} \frac{1}{4 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)^{-\frac{3}{2}}}\right. \\
\int_{\left|\eta^{\prime}\right|<\varepsilon\left(\frac{1}{4} \frac{1-\nu^{2}}{1+\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{4 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)^{\frac{1}{2}} d \eta^{\prime} e^{-\eta^{\prime 2}} \exp \left\{\frac{2 \eta^{\prime} \cdot\left[\left(\frac{1}{\left.\left.4 \ln \frac{1}{\nu} \frac{M \omega}{\hbar}-\frac{1}{2} \frac{\nu}{1-\nu^{2}} \frac{m \omega}{\hbar}\right) \xi+\frac{1}{4} \frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar} \eta\right]}\right.\right.}{\left(\frac{1}{4} \frac{1-\nu^{2}}{1+\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{\left.4 \ln \frac{1}{\nu} \frac{M \omega}{\hbar}\right)^{\frac{1}{2}}}\right\}}\right.} . \begin{array}{l}
\text { (2.7) }
\end{array}
\end{gather*}
$$

With the changes of variable $\nu=1-\nu^{\prime}$ and $\nu^{\prime} \xi^{2}=\nu^{\prime \prime}$ we recover a coulombian singularity for the potential:

$$
\left(G^{\lambda} 1_{\varepsilon}\right)\left(\xi, \eta, y_{2}, \ldots, y_{N}\right) \simeq_{\xi \rightarrow 0} \frac{1}{\pi} \frac{1}{\hbar^{2}} \frac{m M}{m+M} \frac{1}{|\xi|}
$$

This expression can be written in a more useful way as

$$
\left(G^{\lambda} 1_{\varepsilon}\right)\left(\xi, \eta, y_{2}, \ldots, y_{N}\right) \simeq_{\xi \rightarrow 0} \frac{1}{\pi^{\frac{3}{2}}} \frac{\omega^{\frac{1}{2}}}{\hbar^{\frac{5}{2}}}\left(\frac{m M}{m+M}\right)^{\frac{3}{2}} \int_{0}^{\infty} d \nu \frac{1}{\nu^{\frac{3}{2}}} e^{-\frac{\omega}{\hbar} \frac{1}{\nu} \frac{m M}{m+M}|\xi|^{2}}
$$

We separate this integral into two pieces with $\nu$ going from 0 to 1 and from 1 to $+\infty$; this latter term is finite when $\xi \rightarrow 0$ and will give the one half in brace brackets in (2.1), while the former is subtracted from (2.7). Adding to it $\int_{\left|y^{\prime}-y\right|>\varepsilon} d y^{\prime} G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)$ we finally obtain (2.1).

Now we show that definition (2.1.1) is well posed. First we prove that the decomposition $u=\varphi^{\lambda}+G^{\lambda} q$ is well defined and unique; the proof is divided into propositions (2.1.2), (2.1.3) and (2.1.4).

Proposition 2.1.2 If $q_{i} \in L^{2}\left(\mathbb{R}^{3 N}\right)$ then $G^{\lambda} q_{i} \in L^{2}\left(\mathbb{R}^{3 N+3}\right)$.

## Proof

Let's fix $i=1$.

$$
\begin{aligned}
& \int d x d y\left|G^{\lambda} q_{1}(x, y)\right|^{2}=\frac{1}{(2 \pi)^{3}} \frac{1}{(\hbar \omega)^{2}} \frac{1}{\pi^{3 N}}\left(\frac{m \omega}{\hbar}\right)^{3 N}\left(\frac{M \omega}{\hbar}\right)^{3} \\
& \int d x d y \int d z \overline{q_{1}(z)} \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1} \frac{1}{\left(1-\nu^{2}\right)^{\frac{3 N}{2}}} \frac{1}{\left(\ln \frac{1}{\nu}\right)^{\frac{3}{2}}} e^{-\frac{m \omega}{\hbar} \frac{1-\nu}{2(1+\nu)}\left(y^{2}+z^{2}\right)} e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left|x-z_{1}\right|^{2}} \\
& e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}|y-z|^{2}} \int d \omega q_{1}(w) \int_{0}^{1} d \mu \mu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1} \frac{1}{\left(1-\mu^{2}\right)^{\frac{3}{2} N}} \frac{1}{\left(\ln \frac{1}{\mu}\right)^{\frac{3}{2}} e^{-\frac{m \omega}{\hbar} \frac{1-\mu}{2(1+\mu)}\left(y^{2}+w^{2}\right)}} \\
& e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\mu}}\left|x-w_{1}\right|^{2}} e^{-\frac{m \omega}{\hbar} \frac{\mu}{1-\mu^{2}|y-w|^{2}}} \\
& =\frac{1}{(2 \pi)^{3}} \frac{1}{(\hbar \omega)^{2}} \frac{1}{\pi^{3 N}}\left(\frac{m \omega}{\hbar}\right)^{3 N}\left(\frac{M \omega}{\hbar}\right)^{3} \int d z d w \overline{q_{1}}(z) q_{1}(w) \\
& \int_{0}^{1} d \nu \nu \frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1 \frac{1}{\left(1-\nu^{2}\right)^{\frac{3 N}{2}}} \frac{1}{\left(\ln \frac{1}{\nu}\right)^{\frac{3}{2}}} \int_{0}^{1} d \mu \mu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1} \frac{1}{\left(1-\mu^{2}\right)^{\frac{3 N}{2}}} \frac{1}{\left(\ln \frac{1}{\mu}\right)^{\frac{3}{2}}} \\
& \int d x d y e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}|y-z|^{2}} e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left|x-z_{1}\right|^{2}} e^{-\frac{m \omega}{\hbar} \frac{1-\nu}{2(1+\nu)}\left(y^{2}+z^{2}\right)} \\
& \int e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\mu}}\left|x-w_{1}\right|^{2}} e^{-\frac{m \omega}{\hbar} \frac{\mu}{1-\mu^{2}}|y-w|^{2}} e^{-\frac{m \omega}{\hbar} \frac{1-\mu}{2(1+\mu)}\left(y^{2}+w^{2}\right)}
\end{aligned}
$$

It's then sufficient to show that

$$
\begin{aligned}
T(z, w)= & \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1} \frac{1}{\left(1-\nu^{2}\right)^{\frac{3 N}{2}}} \frac{1}{\left(\ln \frac{1}{\nu}\right)^{\frac{3}{2}}} \int_{0}^{1} d \mu \mu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1} \frac{1}{\left(1-\mu^{2}\right)^{\frac{3 N}{2}}} \frac{1}{\left(\ln \frac{1}{\mu}\right)^{\frac{3}{2}}} \\
& \int d x d y e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}|y-z|^{2}} e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left|x-z_{1}\right|^{2}} e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\mu}}\left|x-w_{1}\right|^{2}} e^{-\frac{m \omega}{\hbar} \frac{\mu}{1-\mu^{2}}|y-w|^{2}}
\end{aligned}
$$

is the integral kernel of a bounded operator in $L^{2}\left(\mathbb{R}^{3 N}\right)$.
This kernel can be pointwise estimated in the following way:

$$
\begin{gather*}
T(z, w) \leq\left(\frac{M \omega}{\hbar}\right)^{-\frac{3}{2}}\left(\frac{m \omega}{\hbar}\right)^{-\frac{3 N}{2}} \pi^{\frac{3 N+3}{2}} \int_{0}^{1} d \nu d \mu\left(\frac{\nu}{1-\nu^{2}}+\frac{\mu}{1-\mu^{2}}\right)^{-\frac{3 N+3}{2}} \nu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1} \\
\frac{1}{\left(1-\nu^{2}\right)^{\frac{3 N}{2}}} \frac{1}{\left(\ln \frac{1}{\nu}\right)^{\frac{3}{2}}} \mu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1} \frac{1}{\left(1-\mu^{2}\right)^{\frac{3 N}{2}}} \frac{1}{\left(\ln \frac{1}{\mu}\right)^{\frac{3}{2}}} e^{-\frac{1}{4\left(\frac{1-\nu^{2}}{\nu}+\frac{1-\mu^{2}}{\mu}\right)^{\frac{m \omega}{\hbar}|w-z|^{2}}}} . \tag{2.8}
\end{gather*}
$$

Using the Schur's test, see [HS], and (2.8) we have

$$
\begin{aligned}
& \|T\| \leq \sup _{w} \int d z T(z, w) \leq\left(\frac{M \omega}{\hbar}\right)^{-\frac{3}{2}} \pi^{3 N+\frac{3}{2}} \int_{0}^{1} d \nu d \mu \nu^{\frac{\lambda}{\hbar \omega}-1} \mu^{\frac{\lambda}{\hbar \omega}-1} \frac{1}{\left(\ln \frac{1}{\nu}\right)^{\frac{3}{2}}} \frac{1}{\left(\ln \frac{1}{\mu}\right)^{\frac{3}{2}}} \\
& \left(\frac{\nu}{1-\nu^{2}}+\frac{\mu}{1-\mu^{2}}\right)^{-\frac{3}{2}} \leq\left(\frac{M \omega}{\hbar}\right)^{-\frac{3}{2}} \pi^{3 N+\frac{3}{2}} \int_{0}^{1} d \nu d \mu \frac{\nu^{\frac{\lambda}{\hbar \omega}-1} \mu \frac{\lambda}{\hbar \omega}-1}{\left[\frac{\nu}{1+\nu} \frac{\ln \frac{1}{\nu}}{1-\nu} \ln \frac{1}{\mu}+\frac{\mu}{1+\mu} \frac{\ln \frac{1}{\mu}}{1-\mu} \ln \frac{1}{\nu}\right]^{\frac{3}{2}}}<+\infty
\end{aligned}
$$

and this concludes the proof.

Proposition 2.1.3 If $q_{i} \in \mathscr{D}\left(\Phi_{\alpha}^{\lambda}\right)$ then $G^{\lambda} q_{i} \notin \mathscr{D}\left(F_{0}\right)$.

## Proof

Fix $i=1$ and put $y^{0}=0$. Since $G^{\lambda} q_{1}$ is divergent near $\pi_{1}$, we shall consider a regularized version of $F_{0}$ restricting the integration to the whole space minus a small cylinder around $\pi_{1}$ and, with integrations by parts, we shall show that it diverges when the small cylinder shrinks to $\pi_{1}$.
Let's show that the following identity holds:

$$
\begin{align*}
& \int_{\mathbb{R}^{3 N+3} \backslash \pi_{1}^{\delta}} d x d y\left\{\frac{\hbar^{2}}{2 M}\left|\nabla_{x} G^{\lambda} q_{1}\right|^{2}+\frac{\hbar^{2}}{2 m}\left|\nabla_{y} G^{\lambda} q_{1}\right|^{2}+\left(\frac{1}{2} m \omega^{2} y^{2}+\lambda\right)\left|G^{\lambda} q_{1}\right|^{2}\right\}= \\
& \left.=-\frac{1}{2} \int d y d y^{\prime} \right\rvert\, q_{1}(y)-\left.q_{1}\left(y^{\prime}\right)\right|^{2} \int_{\partial \pi_{1}^{\delta}} d x^{\prime \prime} d y^{\prime \prime} G^{\lambda}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}, y\right) \frac{\partial G^{\lambda}}{\partial n}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}^{\prime}, y^{\prime}\right)+ \\
&+\int_{\partial \pi_{1}^{\delta}} d x d y\left(G^{\lambda} 1-\zeta_{1}\right) \frac{\partial G^{\lambda}}{\partial n}\left(\left|q_{1}\right|^{2}\right)+\int_{\partial \pi_{1}^{\delta}} d x d y \zeta_{1} \frac{\partial G^{\lambda}}{\partial n}\left(\left|q_{1}\right|^{2}\right) \tag{2.9}
\end{align*}
$$

Formula (2.9) is equivalent to:

$$
\begin{align*}
& \int_{\mathbb{R}^{3 N+3} \backslash \pi_{1}^{\delta}} d x d y\left\{\frac{\hbar^{2}}{2 M}\left|\nabla_{x} G^{\lambda} q_{1}\right|^{2}+\frac{\hbar^{2}}{2 m}\left|\nabla_{y} G^{\lambda} q_{1}\right|^{2}+\left(\frac{1}{2} m \omega^{2} y^{2}+\lambda\right)\left|G^{\lambda} q_{1}\right|^{2}\right\}= \\
&=-\frac{1}{4} \int d y d y^{\prime}\left|q_{1}(y)-q_{1}\left(y^{\prime}\right)\right|^{2}\left(\int_{\partial \pi_{1}^{\delta}} d x^{\prime \prime} d y^{\prime \prime} G^{\lambda}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}, y\right) \frac{\partial G^{\lambda}}{\partial n}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}^{\prime}, y^{\prime}\right)+\right. \\
&\left.+\int_{\partial \pi_{1}^{\delta}} d x^{\prime \prime} d y^{\prime \prime} G^{\lambda}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}^{\prime}, y^{\prime}\right) \frac{\partial G^{\lambda}}{\partial n}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}, y\right)\right)+\int_{\partial \pi_{1}^{\delta}} d x d y G^{\lambda} 1 \frac{\partial G^{\lambda}}{\partial n}\left(\left|q_{1}\right|^{2}\right) \tag{2.10}
\end{align*}
$$

If we further simplify (2.10), we obtain

$$
\begin{align*}
\int_{\mathbb{R}^{3 N+3} \backslash \pi_{1}^{\delta}} d x d y\left\{\frac{\hbar^{2}}{2 M}\left|\nabla_{x} G^{\lambda} q_{1}\right|^{2}+\frac{\hbar^{2}}{2 m}\left|\nabla_{y} G^{\lambda} q_{1}\right|^{2}+\left(\frac{1}{2} m \omega^{2} y^{2}\right.\right. & \left.+\lambda)\left|G^{\lambda} q_{1}\right|^{2}\right\}= \\
& =\int_{\partial \pi_{1}^{\delta}} d x d y \overline{G^{\lambda} q_{1}} \frac{\partial G^{\lambda}}{\partial n}\left(q_{1}\right) \tag{2.11}
\end{align*}
$$

which is true by Stoke's theorem and because we have removed $\pi_{1}^{\delta}$ from the domain of integration.
In order to prove that $F_{0}\left[G^{\lambda} q_{1}\right]=+\infty$ we shall consider the r.h.s. of (2.9) and we shall prove the following steps: the sum of first and the second term converge, up to a multiplicative constant, to $\Phi_{\alpha}^{\lambda}[q]$ as $\delta \rightarrow 0$ while the third term diverges; since by proposition (2.1.2), the l.h.s. of (2.9) differs from $F_{0}\left[G^{\lambda} q_{1}\right]$ only by a bounded term it's clear that $F_{0}\left[G^{\lambda} q_{1}\right]=+\infty$.
Now we show that

$$
\begin{gathered}
\lim _{\delta \rightarrow 0} \int_{\partial \pi_{1}^{\delta}} d x d y\left(G^{\lambda} 1-\zeta_{1}\right)(x, y) \frac{\partial G^{\lambda}}{\partial n}\left(x, y ; y_{1}^{\prime}, y^{\prime}\right)=-\frac{4}{\hbar^{2}} \frac{m M}{m+M} a_{1}^{\lambda}\left(y^{\prime}\right) \\
\lim _{\delta \rightarrow 0} \int_{\partial \pi_{1}^{\delta}} d x^{\prime \prime} d y^{\prime \prime} G^{\lambda}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}, y\right) \frac{\partial G^{\lambda}}{\partial n}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}^{\prime}, y^{\prime}\right)=\frac{4}{\hbar^{2}} \frac{m M}{m+M} G^{\lambda}\left(y_{1}, y ; y_{1}^{\prime}, y^{\prime}\right)
\end{gathered}
$$

Notice that in this case $\frac{\partial}{\partial n}=-\frac{\partial}{\partial|\xi|}$, so the integral kernel of $\frac{\partial G^{\lambda}}{\partial n}$ can be easily obtained from (A.1).

$$
\begin{align*}
& \frac{\partial G^{\lambda}}{\partial n}\left(\xi, \eta, y_{2}, \ldots, y_{n} ; y_{1}^{\prime}, y^{\prime}\right)=\frac{1}{(2 \pi)^{\frac{3}{2}}} \frac{1}{\hbar \omega} \frac{1}{\pi^{\frac{3 N}{2}}}\left(\frac{m \omega}{\hbar}\right)^{\frac{3 N}{2}}\left(\frac{M \omega}{\hbar}\right)^{\frac{3}{2}} \int_{0}^{1} d \nu \frac{\nu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1}}{\left(1-\nu^{2}\right)^{\frac{3 N}{2}}\left(\ln \frac{1}{\nu}\right)^{\frac{3}{2}}} \\
& e^{-\frac{1}{2} \frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar}\left[\frac{1}{2}(\eta-\xi)^{2}+y_{2}^{2}+\ldots y_{N}^{2}+y^{\prime 2}\right]} e^{-\frac{\nu}{1-\nu^{2}} \frac{m \omega}{\hbar}\left(\left|\frac{1}{\sqrt{2}}(\eta-\xi)-y_{1}^{\prime}\right|^{2}+\left|y_{2}-y_{2}^{\prime}\right|^{2}+\ldots+\left|y_{N}-y_{N}^{\prime}\right|^{2}\right)} e^{-\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\left|\frac{1}{\sqrt{2}}(\eta+\xi)-y_{1}^{\prime}\right|^{2}} \\
& \left\{\left[|\xi|+\sqrt{2} \frac{\xi}{|\xi|} \cdot\left(\frac{1}{\sqrt{2}} \eta+y_{1}^{\prime}\right)\right] \frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\left[|\xi|+\sqrt{2} \frac{\xi}{|\xi|} \cdot\left(\frac{1}{\sqrt{2}} \eta-y_{1}^{\prime}\right)\right] \frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right\} \tag{2.12}
\end{align*}
$$

The potential $G^{\lambda} 1$ and its divergent part $\zeta_{1}$ have the following expressions:

$$
\left.\begin{array}{c}
\left(G^{\lambda} 1\right)\left(\xi, \eta, y_{2}, \ldots, y_{n}\right)=\frac{1}{\pi^{\frac{3}{2}}} \frac{\omega^{\frac{1}{2}}}{\hbar^{\frac{5}{2}}}\left(\frac{m M}{m+M}\right)^{\frac{3}{2}} \int_{0}^{1} d \mu \frac{1}{(1-\mu)^{\frac{3}{2}}} \mu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1}\left(\frac{2}{1+\mu^{2}}\right)^{\frac{3(N-1)}{2}} \\
\left(\frac{1+\mu^{2}}{2} \frac{\ln \frac{1}{\mu}}{1-\mu} m+\frac{1+\mu}{2} M\right)^{-\frac{3}{2}} \exp \left(-\frac{1}{2} \frac{1-\mu^{2}}{1+\mu^{2}} \frac{m \omega}{\hbar} \sum_{j=2}^{N} y_{j}^{2}\right) \exp \left(-\frac{\omega}{\hbar} \frac{\frac{1}{4} m^{2}+\frac{1-\mu}{1+\mu} \frac{1}{2 \ln \frac{1}{\mu}} m M}{\frac{1}{2} \frac{1+\mu^{2}}{1-\mu^{2}} m+\frac{1}{2 \ln \frac{1}{\mu}} M} \frac{1}{2} \eta^{2}\right) \\
\exp \left(-4 \frac{\frac{1}{4 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar} \frac{1}{4} \frac{1+\mu^{2}}{1-\mu^{2}} \frac{m \omega}{\hbar}}{\frac{1}{4 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar}+\frac{1}{4} \frac{1+\mu^{2}}{1-\mu^{2}} \frac{m \omega}{\hbar}} \xi^{2}\right) \exp \left(-\frac{\frac{1}{8}\left(\frac{m \omega}{\hbar}\right)^{2}}{\frac{1}{4 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar}+\frac{1}{4} \frac{1+\mu^{2}}{1-\mu^{2}} \frac{m \omega}{\hbar}} \eta \cdot \xi\right) \\
\zeta_{1}
\end{array}\right) \frac{1}{\pi^{\frac{3}{2}}} \frac{\omega^{\frac{1}{2}}}{\hbar^{\frac{5}{2}}}\left(\frac{m M}{m+M}\right)^{\frac{3}{2}} \int_{0}^{+\infty} d \mu \frac{1}{\mu^{\frac{3}{2}}} e^{-\frac{1}{\mu} \frac{\omega}{\hbar} \frac{m M}{m+M} \xi^{2}} .
$$

It is useful to write $G^{\lambda} 1-\zeta_{1}$ in the following way:

$$
\begin{gather*}
\left.\left.\left.\left(G^{\lambda} 1-\zeta_{1}\right)\left(\xi, \eta, y_{2} \ldots y_{N}\right)=\frac{1}{(2 \pi)^{\frac{3}{2}} \frac{\omega^{\frac{1}{2}}}{\hbar^{\frac{5}{2}}}\left(\frac{m M}{m+M}\right)^{\frac{3}{2}}\left\{\int_{1}^{+\infty} d \mu \frac{1}{\mu^{\frac{3}{2}}} e^{-\frac{1}{\mu} \frac{\omega}{\hbar} \frac{m M}{m+M} \xi^{2}}+\int_{0}^{1} d \mu \frac{1}{(1-\mu)^{\frac{3}{2}}}\right.} \begin{array}{c}
{\left[\mu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1}\left(\frac{2}{1+\mu^{2}}\right)^{\frac{3(N-1)}{2}} \frac{1}{\left(\frac{1+\mu^{2}}{2} \frac{\ln \frac{1}{\mu}}{1-\mu} m+\frac{1+\mu}{2} M\right)^{\frac{3}{2}}} \exp \left(-\frac{1}{2} \frac{1-\mu^{2}}{1+\mu^{2}} \frac{m \omega}{\hbar} \sum_{j=2}^{N} y_{j}^{2}\right)\right.} \\
\exp \left(-\frac{\omega}{\hbar} \frac{\frac{1}{4} m^{2}+\frac{1-\mu}{1+\mu} \frac{1}{2 \ln \frac{1}{\mu}} m M}{\frac{1}{2} \frac{1+\mu^{2}}{1-\mu^{2}} m+\frac{1}{2 \ln \frac{1}{\mu}} M} \frac{1}{2} \eta^{2}\right) \exp \left(-4 \frac{\frac{1}{4 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar} \frac{1}{4} \frac{1+\mu^{2}}{1-\mu^{2}} \frac{m \omega}{\hbar}}{4 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar}+\frac{1}{4} \frac{1+\mu^{2}}{1-\mu^{2}} \frac{m \omega}{\hbar} \xi^{2}\right) \\
\exp \left(-\frac{\frac{1}{2} \frac{1-\mu}{1+\mu} \frac{m \omega}{\hbar}}{\frac{1}{2 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar}} \frac{1}{4 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar}+\frac{1}{4} \frac{1+\mu^{2}}{1-\mu^{2}} \frac{m \omega}{\hbar}\right. \\
\hbar
\end{array} \cdot \xi\right)-\exp \left(\frac{1}{1-\mu} \frac{m M}{m+M} \frac{\omega}{\hbar} \xi^{2}\right)\right]\right\}
\end{gather*}
$$

Equation (2.13) shows that $a_{1}^{\lambda}(y)=-2^{\frac{3}{2}}\left(G^{\lambda} 1-\zeta_{1}\right)\left(0, \eta, y_{2} \ldots y_{N}\right)$; the numerical factor $2^{\frac{3}{2}}$ has been absorbed in (2.1) by a change of variable.
The integral kernel $\frac{\partial G^{\lambda}}{\partial n}$ has the following property:

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \int_{\partial \pi_{1}^{\delta}} d x d y f(x, y) \frac{\partial G^{\lambda}}{\partial n}\left(x, y ; y_{1}^{\prime}, y^{\prime}\right)=\frac{4}{\hbar^{2}} \frac{m M}{m+M} f\left(y_{1}^{\prime}, y^{\prime}\right) \tag{2.14}
\end{equation*}
$$

If we prove (2.14) when $f$ is $\left(G^{\lambda}-\zeta_{1}\right)(x, y)$ or $G^{\lambda}\left(x, y ; y_{1}^{\prime \prime}, y^{\prime \prime}\right)$ we conclude the first step of the proof; if we interchange the integral over the parameter $\mu$ with the limit over $\delta$ and the integral
over $(x, y)$ we can restrict ourselves to prove (2.14) when $f$ is a gaussian. Now it's only a matter of long calculations to prove that:

$$
\begin{aligned}
\lim _{\delta \rightarrow 0} \int_{\partial \pi_{1}^{\delta}} d \xi d \eta d y_{2} \ldots d y_{N} e^{-\frac{\omega}{\hbar}\left(a \xi^{2}+b \eta^{2}+c \eta \cdot \xi\right)} e^{-\frac{\omega}{\hbar} d\left(y_{2}^{2}+\ldots+y_{N}^{2}\right)} \frac{\partial G^{\lambda}}{\partial n} & \left(\xi, \eta, y_{2}, \ldots, y_{n} ; y_{1}^{\prime}, y^{\prime}\right) \\
& =\frac{4}{\hbar^{2}} \frac{m M}{m+M} e^{-\frac{\omega}{\hbar} b \eta^{\prime 2}} e^{-\frac{\omega}{\hbar} d\left(y_{2}^{\prime 2}+\ldots+y_{N}^{\prime 2}\right)}
\end{aligned}
$$

Using (2.12) and integrating over $\eta, y_{2}, \ldots, y_{N}$, we have to study

$$
\begin{align*}
& \lim _{\delta \rightarrow 0} \frac{1}{(2 \pi)^{\frac{3}{2}}} \frac{1}{\hbar \omega} \frac{1}{\pi^{\frac{3 N}{2}}}\left(\frac{m \omega}{\hbar}\right)^{\frac{3 N}{2}}\left(\frac{M \omega}{\hbar}\right)^{\frac{3}{2}} \int_{0}^{1} d \nu \frac{\nu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1}}{\left(1-\nu^{2}\right)^{\frac{3 N}{2}}\left(\ln \frac{1}{\nu}\right)^{\frac{3}{2}}}\left[\frac{1}{\pi} \frac{\omega}{\hbar}\left(\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} m+d\right)\right]^{-\frac{3(N-1)}{2}} \\
& \exp \left(-\frac{\omega}{\hbar}\left(y_{2}^{\prime}+\ldots y_{N}^{\prime 2}\right) \frac{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} m d+\frac{1}{4} m^{2}}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} m+d}\right)\left(\frac{1}{4} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{4 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar} \frac{\omega}{\hbar} b\right)^{\frac{3}{2}} \\
& \exp \left\{-\frac{1}{2} \delta^{2}\left[\frac{\frac{1+\nu^{2}}{1-\nu^{2}} \frac{1}{\ln \frac{1}{\nu}}+2 \frac{\omega}{\hbar} b\left(\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)+2 \frac{\omega}{\hbar} c\left(\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}-\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)+\left(\frac{\omega}{\hbar} c\right)^{2}}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b}\right]\right\} \\
& \exp \left\{-y_{1}^{\prime 2}\left[\frac{\frac{1}{4}\left(\frac{m \omega}{\hbar}\right)^{2}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar} \frac{1}{2} \frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar}+2 \frac{\omega}{\hbar} b\left(\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{M \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b}\right]\right\} \\
& \int_{|\xi|=\delta} d \xi \exp \left\{\sqrt{2} \xi \cdot y_{1}^{\prime}\left[\frac{\frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b\left(\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}-\frac{\nu}{1-\nu^{2}} \frac{m \omega}{\hbar}\right)+\frac{\omega}{\hbar} b\left(\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+\frac{\nu}{1-\nu^{2}} \frac{m \omega}{\hbar}\right)}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b}\right]\right\} \\
& \left\{|\xi| \frac{4 \frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b\left(\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)+\frac{\omega}{\hbar} c\left(\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}-\frac{\nu}{1-\nu^{2}} \frac{m \omega}{\hbar}\right)}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b}+\right. \\
& \left.-\sqrt{2} \frac{\xi}{|\xi|} \cdot y_{1}^{\prime} \frac{\frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+\frac{\omega}{\hbar} b\left(\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}-\frac{\nu}{1-\nu^{2}} \frac{m \omega}{\hbar}\right)}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b}\right\} \tag{2.15}
\end{align*}
$$

Making the change of variables $\nu=\nu^{\prime}$ and $\nu^{\prime}=\delta^{2} \nu^{\prime \prime}$ when $\delta \rightarrow 0$ only the most singular terms of (2.15) survives, in particular the surface integral becomes trivial; then we obtain

$$
\frac{8}{\pi^{\frac{1}{2}}} \frac{\omega^{\frac{3}{2}}}{\hbar^{\frac{7}{2}}}\left(\frac{m M}{m+M}\right)^{\frac{5}{2}} \int_{0}^{+\infty} d \nu \frac{1}{\nu^{\frac{5}{2}}} e^{-\frac{1}{\nu}} e^{-2 \frac{\omega}{\hbar} b y_{1}^{\prime 2}} e^{-\frac{\omega}{\hbar} d\left(y_{2}^{\prime 2}+\ldots+y_{N}^{\prime 2}\right)}=\frac{4}{\hbar^{2}} \frac{m M}{m+M} e^{-2 \frac{\omega}{\hbar} b y_{1}^{\prime 2}} e^{-\frac{\omega}{\hbar} d\left(y_{2}^{\prime 2}+\ldots+y_{N}^{\prime 2}\right)}
$$

Since on $\pi_{1} \eta=\sqrt{2} y_{1}$ we have proved (2.14).

The third term on the r.h.s. of (2.9) is equal to:

$$
\begin{equation*}
\frac{1}{\pi} \frac{1}{\hbar^{2}} \frac{m M}{m+M} \frac{1}{\delta} \int_{\partial \pi_{1}^{\delta}} \frac{\partial G^{\lambda}}{\partial n}\left(\left|q_{1}\right|^{2}\right) \tag{2.16}
\end{equation*}
$$

because $\zeta_{1}$ is a function of $\xi$ only. Equation (2.14), considering constant as a particular case of gaussian functions, also implies that

$$
\lim _{\delta \rightarrow 0} \int_{\partial \pi_{1}^{\delta}} \frac{\partial G^{\lambda}}{\partial n}\left(\left|q_{1}\right|^{2}\right)=\frac{1}{\hbar^{2}} \frac{m M}{m+M} \int d y^{\prime}\left|q_{1}\left(y^{\prime}\right)\right|^{2}<+\infty
$$

which shows that (2.16) diverges when $\delta \rightarrow 0$. The proof of the proposition is concluded.

Notice that we have showed that $\Phi_{\alpha}^{\lambda}[q]$ can be looked as a renormalized energy of the potential $G^{\lambda} q$.
Proposition 2.1.4 The decomposition $u=\varphi^{\lambda}+G^{\lambda} q$ is well defined and unique.

## Proof

By proposition (2.1.2) $G^{\lambda} q \in L^{2}\left(\mathbb{R}^{3 N+3}\right)$ so that the decomposition is meaningful in the Hilbert space; uniqueness easily follows from proposition (2.1.3). If by absurd $u=\varphi^{\lambda}+G^{\lambda} q=\tilde{\varphi}^{\lambda}+G^{\lambda} \tilde{q}$ then $0=\varphi^{\lambda}-\tilde{\varphi}^{\lambda}=G^{\lambda}(\tilde{q}-q)$; but then $\tilde{q}-q=0$ because the operator $G^{\lambda}$ has empty kernel.

In definition (2.1.1) it is understood that one must fix $\lambda>0$; the role of this parameter is only to regularize the behavior of the potential at infinity in such a way that it belongs to $L^{2}\left(\mathbb{R}^{3 N+3}\right)$.
Proposition 2.1.5 The quadratic form $\left(F_{\alpha}, \mathscr{D}\left(F_{\alpha}\right)\right)$ doesn't depend on $\lambda$.

## Proof

First we note that $\mathscr{D}\left(F_{\alpha}\right)$ doesn't depend on $\lambda$. If $u=\varphi^{\lambda}+G^{\lambda} q$ then also $u=\varphi^{\lambda^{\prime}}+G^{\lambda^{\prime}} q$ holds because $G^{\lambda} q-G^{\lambda^{\prime}} q \in \mathscr{D}\left(F_{0}\right)$ and then we can put $\varphi^{\lambda^{\prime}} \equiv \varphi^{\lambda}+G^{\lambda} q-G^{\lambda^{\prime}} q$. Now we show that

$$
\begin{gather*}
\mathcal{F}^{\lambda}[u]-\mathcal{F}^{\lambda^{\prime}}[u]=\Phi^{\lambda^{\prime}}[u]-\Phi^{\lambda}[u]  \tag{2.17}\\
\mathcal{F}^{\lambda}[u]-\mathcal{F}^{\lambda^{\prime}}[u]=\int d x d y\left(\frac{\hbar^{2}}{2 M} \overline{\nabla_{x}\left(u-G^{\lambda} q\right)} \cdot \nabla_{x}\left(u-G^{\lambda} q\right)+\frac{\hbar^{2}}{2 m} \overline{\nabla_{y}\left(u-G^{\lambda} q\right)} \cdot \nabla_{y}\left(u-G^{\lambda} q\right)\right)+ \\
-\int d x d y\left(\overline{\nabla_{x}\left(u-G^{\lambda^{\prime}} q\right)} \cdot \nabla_{x}\left(u-G^{\lambda^{\prime}} q\right)+\frac{\hbar^{2}}{2 m} \overline{\nabla_{y}\left(u-G^{\lambda^{\prime}} q\right)} \cdot \nabla_{y}\left(u-G^{\lambda^{\prime}} q\right)\right)+ \\
+\lambda \int d x d y \overline{\left(u-G^{\lambda} q\right)}\left(u-G^{\lambda} q\right)-\lambda^{\prime} \int d x d y \overline{\left(u-G^{\lambda^{\prime}} q\right)}\left(u-G^{\lambda^{\prime}} q\right) \\
+\int d x d y \frac{1}{2} m \omega^{2} y^{2} \overline{\left(u-G^{\lambda} q\right)}\left(u-G^{\lambda} q\right)-\int d x d y \frac{1}{2} m \omega^{2} y^{2} \overline{\left(u-G^{\lambda^{\prime}} q\right)}\left(u-G^{\lambda^{\prime}} q\right)+\left(\lambda^{\prime}-\lambda\right) \int d x d y|u|^{2} \tag{2.18}
\end{gather*}
$$

The first part of the r.h.s. of (2.18) can be written as:

$$
\begin{align*}
& \int d x d y\left(\frac{\hbar^{2}}{2 M} \overline{\nabla_{x}\left(u-G^{\lambda} q\right)} \cdot \nabla_{x}\left(u-G^{\lambda} q\right)+\frac{\hbar^{2}}{2 m} \overline{\nabla_{y}\left(u-G^{\lambda} q\right)} \cdot \nabla_{y}\left(u-G^{\lambda} q\right)\right)+ \\
& \quad-\int d x d y\left(\frac{\hbar^{2}}{2 M} \overline{\nabla_{x}\left(u-G^{\lambda^{\prime}} q\right)} \cdot \nabla_{x}\left(u-G^{\lambda^{\prime}} q\right)+\frac{\hbar^{2}}{2 m} \overline{\nabla_{y}\left(u-G^{\lambda^{\prime}} q\right)} \cdot \nabla_{y}\left(u-G^{\lambda^{\prime}} q\right)\right) \\
& =\int d x d y\left(\frac{\hbar^{2}}{2 M} \overline{\nabla_{x}\left(u-G^{\lambda} q\right)} \cdot \nabla_{x}\left(G^{\lambda^{\prime}} q-G^{\lambda} q\right)+\frac{\hbar^{2}}{2 m} \overline{\nabla_{y}\left(u-G^{\lambda} q\right)} \cdot \nabla_{y}\left(G^{\lambda^{\prime}} q-G^{\lambda} q\right)\right)+ \\
& +\int d x d y\left(\frac{\hbar^{2}}{2 M} \overline{\nabla_{x}\left(G^{\lambda^{\prime}} q-G^{\lambda} q\right)} \cdot \nabla_{x}\left(u-G^{\lambda^{\prime}} q\right)+\frac{\hbar^{2}}{2 m} \overline{\nabla_{y}\left(G^{\lambda^{\prime}} q-G^{\lambda} q\right)} \cdot \nabla_{y}\left(u-G^{\lambda^{\prime}} q\right)\right) \\
& =-\int d x d y\left(\frac{\hbar^{2}}{2 M} \overline{\left(u-G^{\lambda} q\right)} \Delta_{x}\left(G^{\lambda^{\prime}} q-G^{\lambda} q\right)+\frac{\hbar^{2}}{2 m} \overline{\left(u-G^{\lambda} q\right)} \Delta_{y}\left(G^{\lambda^{\prime}} q-G^{\lambda} q\right)\right)+ \\
& \quad-\int d x d y\left(\frac{\hbar^{2}}{2 M} \overline{\Delta_{x}\left(G^{\lambda^{\prime}} q-G^{\lambda} q\right)}\left(u-G^{\lambda^{\prime}} q\right)+\frac{\hbar^{2}}{2 m} \overline{\Delta_{y}\left(G^{\lambda^{\prime}} q-G^{\lambda} q\right)}\left(u-G^{\lambda^{\prime}} q\right)\right) \tag{2.19}
\end{align*}
$$

The integration by parts in (2.19) is allowed since $G^{\lambda} q-G^{\lambda^{\prime}} q \in \mathscr{D}\left(H_{0}\right)$.
Using the identity

$$
-\frac{\hbar^{2}}{2 M} \Delta_{x}\left(G^{\lambda^{\prime}} q-G^{\lambda} q\right)-\frac{\hbar^{2}}{2 m} \Delta_{y}\left(G^{\lambda^{\prime}} q-G^{\lambda} q\right)=-\frac{1}{2} m \omega^{2}\left(y-y^{0}\right)^{2}\left(G^{\lambda^{\prime}} q-G^{\lambda} q\right)+\lambda G^{\lambda} q-\lambda^{\prime} G^{\lambda^{\prime}} q
$$

and (2.19), (2.18) becomes

$$
\mathcal{F}^{\lambda}[u]-\mathcal{F}^{\lambda^{\prime}}[u]=\left(\lambda^{\prime}-\lambda\right) \int d x d y \overline{G^{\lambda} q} G^{\lambda^{\prime}} q
$$

Now we analyze the r.h.s. of (2.17). Let's first concentrate on diagonal terms.

$$
\begin{gather*}
\frac{1}{2} \int d y d y^{\prime}\left(G^{\lambda^{\prime}}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)-G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)\right)\left|q_{i}(y)-q_{i}\left(y^{\prime}\right)\right|^{2}+\int d y\left(a_{i}^{\lambda^{\prime}}(y)-a_{i}^{\lambda}(y)\right)\left|q_{i}(y)\right|^{2}= \\
-\int d y d y^{\prime}\left(G^{\lambda^{\prime}}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)-G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)\right) \overline{q_{i}(y)} q_{i}\left(y^{\prime}\right)+ \\
+\int d y\left[\left(a_{i}^{\lambda^{\prime}}(y)-a_{i}^{\lambda}\right)(y)+\int d y^{\prime}\left(G^{\lambda^{\prime}}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)-G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)\right)\right]\left|q_{i}(y)\right|^{2}= \\
-\int d y d y^{\prime}\left(G^{\lambda^{\prime}}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)-G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)\right) \overline{q_{i}(y)} q_{i}\left(y^{\prime}\right) \tag{2.20}
\end{gather*}
$$

In (2.20) we have used $\left(a_{i}^{\lambda^{\prime}}(y)-a_{i}^{\lambda}(y)\right)+\int d y^{\prime}\left(G^{\lambda^{\prime}}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)-G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)\right)=0$; this identity can be seen from (2.1) and (2.13). So we can write

$$
\begin{align*}
& \Phi^{\lambda}[u]-\Phi^{\lambda^{\prime}}[u]=-\sum_{i=1}^{N} \int d y d y^{\prime}\left(G^{\lambda^{\prime}}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)-G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)\right) \overline{q_{i}(y)} q_{i}\left(y^{\prime}\right)+ \\
&-\sum_{i, j=1}^{N} \int d y \overline{q_{i}(y)}\left(\left(G_{j}^{\lambda^{\prime}}-G_{j}^{\lambda}\right) q_{i}\right)(y) \tag{2.21}
\end{align*}
$$

The first resolvent identity applied to (2.1) and (2.21) proves (2.17).

Proposition (2.1.3) shows that $G^{\lambda} q_{i}$ are the most singular part of $u$; the converse is also true: from the singularities of $u$ around $\pi_{i}$ it is possible to recover $q_{i}$. More precisely we have:

Proposition 2.1.6 The following limit holds in $L^{2}\left(\pi_{i}\right)$.

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \frac{\hbar^{2}}{2} \frac{m+M}{m M} \frac{1}{\delta} \int_{\left|x-y_{i}\right|=\delta} d x u=q_{i} \tag{2.22}
\end{equation*}
$$

## Proof

First, using Sobolev embedding theorems we prove that

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \frac{1}{\delta} \int_{\left|x-y_{i}\right|=\delta} d x u=\lim _{\delta \rightarrow 0} \frac{1}{\delta} \int_{\left|x-y_{i}\right|=\delta} d x G^{\lambda} q_{i} \tag{2.23}
\end{equation*}
$$

and then it's enough to show that

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \frac{1}{\delta} \int_{\left|x-y_{i}\right|=\delta} d x G^{\lambda} q_{i}=\frac{2}{\hbar^{2}} \frac{m M}{m+M} q_{i} \tag{2.24}
\end{equation*}
$$

Since $\varphi^{\lambda}(\cdot, y) \in H^{1}$ for a.e. $y$, its trace over $B_{\delta}=\left\{x\right.$ s.t. $\left.\left|x-y_{i}\right|=\delta\right\}$ belongs to $H^{\frac{1}{2}}$; due to the embedding $H^{\frac{1}{2}}\left(B_{\delta}\right) \hookrightarrow L^{4}\left(B_{\delta}\right)$ we can write the following chain of inequalities

$$
\begin{equation*}
\int_{\left|x-y_{i}\right|=\delta} d x\left|\varphi^{\lambda}(\cdot, y)\right| \leq c\left\|\varphi^{\lambda}(\cdot, y)\right\|_{L^{4}\left(B_{\delta}\right)}\|1\|_{L^{\frac{4}{3}\left(B_{\delta}\right)}} \leq c\left\|\varphi^{\lambda}(\cdot, y)\right\|_{H^{\frac{1}{2}\left(B_{\delta}\right)}} \delta^{\frac{3}{2}} \leq c \delta^{\frac{3}{2}}\left\|\varphi^{\lambda}(\cdot, y)\right\|_{H^{1}\left(\mathbb{R}^{3}\right)} \tag{2.25}
\end{equation*}
$$

If we take the $L^{2}$ norm of the first and the last term in (2.25) we see that the regular part of $u$ gives no contribute to the limit in (2.22). Since $G^{\lambda} q_{j}$ is a regular function on $\pi_{i}$ apart from the intersection with $\pi_{j}$, which has zero measure, then (2.23) holds.

Now we introduce the following two integral operators:

$$
\begin{array}{r}
T_{\delta}\left(y ; y^{\prime}\right)=\frac{1}{(2 \pi)^{\frac{3}{2}}} \frac{1}{\hbar \omega} \frac{1}{\pi^{\frac{3 N}{2}}}\left(\frac{m \omega}{\hbar}\right)^{\frac{3 N}{2}}\left(\frac{M \omega}{\hbar}\right)^{\frac{3}{2}} \frac{1}{\delta} \int_{\left|x-y_{i}\right|=\delta} d x \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1} \frac{1}{\left(1-\nu^{2}\right)^{\frac{3 N}{2}}} \frac{1}{\left(\ln \frac{1}{\nu}\right)^{\frac{3}{2}}} . \\
e^{-\frac{1}{2} \frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar}\left(y^{2}+y^{\prime 2}\right)} e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left|x-y_{1}^{\prime}\right|^{2}} e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}\left|y-y^{\prime}\right|^{2}}(2.26) \\
T_{\delta}^{\prime}\left(y ; y^{\prime}\right)=\frac{1}{(2 \pi)^{\frac{3}{2}}} \frac{1}{\hbar \omega} \frac{1}{\pi^{\frac{3 N}{2}}}\left(\frac{m \omega}{\hbar}\right)^{\frac{3 N}{2}}\left(\frac{M \omega}{\hbar}\right)^{\frac{3}{2}} \frac{1}{\delta} \int_{\left|x-y_{i}\right|=\delta} d x \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1} \frac{1}{\left(1-\nu^{2}\right)^{\frac{3 N}{2}}} \frac{1}{\left(\ln \frac{1}{\nu}\right)^{\frac{3}{2}}} . \\
e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left|x-y_{1}^{\prime}\right|^{2}} e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}\left|y-y^{\prime}\right|^{2}} \tag{2.27}
\end{array}(2.27) .
$$

Notice that $T_{\delta}\left(y ; y^{\prime}\right) \leq T_{\delta}^{\prime}\left(y ; y^{\prime}\right)$ and that, with a straightforward integration, $\int d y^{\prime} T_{\delta}^{\prime}\left(y ; y^{\prime}\right) \leq$ $\frac{2}{\hbar 2} \frac{m M}{m+M}$; then $\left\{T_{\delta}\right\}$ and $\left\{T_{\delta}^{\prime}\right\}$ are two equibounded family of bounded operators.
In order to prove $(2.24)$, since $\frac{1}{\delta} \int_{\left|x-y_{i}\right|=\delta} d x G^{\lambda} q_{i}=\int d y^{\prime} T_{\delta}\left(y ; y^{\prime}\right) q_{1}\left(y^{\prime}\right)$, it is sufficient to prove that $T_{\delta}^{\prime}$ is an approximation of the unity, see $[\mathrm{R}]$, and that

$$
\begin{equation*}
\int d y^{\prime} T_{\delta}\left(y ; y^{\prime}\right) \rightarrow \frac{2}{\hbar^{2}} \frac{m M}{m+M} \quad \text { uniformly on compact sets } \tag{2.28}
\end{equation*}
$$

Indeed, if these two statements holds, consider $q \in C_{0}^{\infty} ;(2.24)$ follow from

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \int d y\left|\int d y^{\prime} T_{\delta}\left(y ; y^{\prime}\right)\left(q(y)-q\left(y^{\prime}\right)\right)\right|^{2}=0 \tag{2.29}
\end{equation*}
$$

which is true since

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \int d y\left|\int d y^{\prime} T_{\delta}\left(y ; y^{\prime}\right)\left(q(y)-q\left(y^{\prime}\right)\right)\right|^{2} \leq \lim _{\delta \rightarrow 0} \int d y\left[\int d y^{\prime} T_{\delta}^{\prime}\left(y ; y^{\prime}\right)\left|q(y)-q\left(y^{\prime}\right)\right|\right]^{2}=0 \tag{2.30}
\end{equation*}
$$

The second limit in (2.30) is 0 because we can interchange the limit and the integral over $y$ by dominate convergence theorem, and because $T_{\delta}^{\prime}$ is an approximation of the unity.
Now we prove that $T_{\delta}^{\prime}\left(y ; y^{\prime}\right)$ is an approximation of the identity; first with a change of variable we see that $T_{\delta}^{\prime}$ is a convolution operator.

$$
\begin{array}{r}
T_{\delta}^{\prime}\left(y ; y^{\prime}\right)=\frac{1}{(2 \pi)^{\frac{3}{2}}} \frac{1}{\hbar \omega} \frac{1}{\pi^{\frac{3 N}{2}}}\left(\frac{m \omega}{\hbar}\right)^{\frac{3 N}{2}}\left(\frac{M \omega}{\hbar}\right)^{\frac{3}{2}} \frac{1}{\delta} \int_{|x|=\delta} d x \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1} \frac{1}{\left(1-\nu^{2}\right)^{\frac{3 N}{2}}} \frac{1}{\left(\ln \frac{1}{\nu}\right)^{\frac{3}{2}}} . \\
e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left(x+y_{1}-y_{1}^{\prime}\right)^{2}} e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}\left(y-y^{\prime}\right)^{2}}(2.3 \tag{2.31}
\end{array}
$$

We have already noticed that the $L^{1}$ norm of the kernel has an upper bound uniform with respect to $\delta$. With further changes of variable we see that $T_{\delta}^{\prime}\left(y ; y^{\prime}\right)$ can be written in the following way which resembles the usual expression for an approximation of the identity.

$$
\begin{align*}
& T_{\delta}^{\prime}\left(y ; y^{\prime}\right)=\frac{1}{(2 \pi)^{\frac{3}{2}}} \frac{1}{\hbar \omega} \frac{1}{\pi^{\frac{3 N}{2}}}\left(\frac{m \omega}{\hbar}\right)^{\frac{3 N}{2}}\left(\frac{M \omega}{\hbar}\right)^{\frac{3}{2}} \frac{1}{\delta^{3 N}} \int_{|x|=1} d x \int_{0}^{\frac{1}{\delta^{2}}} d \nu \frac{\left(1-\delta^{2} \nu\right)^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1}}{(\nu)^{\frac{3 N}{2}}\left(2-\delta^{2} \nu\right)^{\frac{3 N}{2}}} \\
&\left(\frac{\delta^{2}}{\ln \frac{1}{1-\delta^{2} \nu}}\right)^{\frac{3}{2}} e^{-\frac{M \omega}{\hbar} \frac{\delta^{2}}{\ln \frac{1}{1-\delta^{2} \nu}}\left(x+y_{1}-y_{1}^{\prime}\right)^{2}} e^{-\frac{m \omega}{\hbar} \frac{1-\delta^{2} \nu \frac{1}{2-\delta^{2} \nu}\left(y-y^{\prime}\right)^{2}}{}} \tag{2.32}
\end{align*}
$$

The integral kernel $T_{\delta}^{\prime}\left(y-y^{\prime}\right)$ is pointwise positive and its integral is a constant; using (2.31) it is straighfroward to verify that the integral outside a compact containing the origin goes to zero as delta goes to zero In order to conclude the proof, we have only to prove (2.28); using the same change of variable of (2.32) we obtain:

$$
\begin{align*}
\int d y p T_{\delta}\left(y ; y^{\prime}\right)= & \frac{1}{(2 \pi)^{\frac{3}{2}}} \frac{1}{\hbar \omega} \frac{1}{\pi^{\frac{3 N}{2}}}\left(\frac{m \omega}{\hbar}\right)^{\frac{3 N}{2}}\left(\frac{M \omega}{\hbar}\right)^{\frac{3}{2}} \int_{|x|=1} d x \int d y^{\prime} \int_{0}^{\frac{1}{\delta^{2}}} d \nu \frac{\left(1-\delta^{2} \nu\right)^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1}}{(\nu)^{\frac{3 N}{2}}\left(2-\delta^{2} \nu\right)^{\frac{3 N}{2}}} \\
& \left(\frac{\delta^{2}}{\ln \frac{1}{1-\delta^{2} \nu}}\right)^{\frac{3}{2}} e^{-\frac{m \omega}{\hbar} \frac{1}{2} \frac{\delta^{2} \nu}{2-\delta^{2} \nu}\left(y^{2}+\left(\delta y+y^{\prime}\right)^{2}\right)} e^{-\frac{M \omega}{\hbar} \frac{\delta^{2}}{\ln \frac{1}{1-\delta^{2} \nu}}\left(x-y_{1}^{\prime}\right)^{2}} e^{-\frac{m \omega}{\hbar} \frac{1-\delta^{2} \nu \frac{1}{2} \delta^{\prime 2}}{2} y^{2}} \tag{2.33}
\end{align*}
$$

If we exchange the limit and the integrals by dominate convergence theorem and we compute the obtained integrals, we prove (2.28)

Now we turn our attention to the most important issue of this section: the study of the closure and of the boundedness from below of the quadratic form $F_{\alpha}$. The real object to study is $\Phi_{\alpha}^{\lambda}$, the part of $F_{\alpha}$ containing most of the information about the interaction. Once we have proved that $\Phi_{\alpha}^{\lambda}$ is closed and positive it is quite straightforward proving that $F_{\alpha}$ is closed and bounded from below as we show in the next theorem.

Theorem 2.1.7 Let $\Phi_{\alpha}^{\lambda}$ be closed and positive, then $F_{\alpha}$ is closed and bounded from below.

## Proof

It is sufficient to prove that $F_{\alpha}^{\lambda}[u]=F_{\alpha}[u]+\lambda\|u\|^{2}$ is positive and closed. Positivity of $F_{\alpha}^{\lambda}[u]$ is trivial, let's discuss closure.
If $\left\{u_{n}=\varphi_{n}^{\lambda}+G^{\lambda} q_{n}\right\} \subset \mathscr{D}\left(F_{\alpha}\right)$ is a sequence converging to $u$ in $L^{2}\left(\mathbb{R}^{3 N+3}\right)$ s.t. $\lim _{n, m} F_{\alpha}^{\lambda}\left[u_{n}-\right.$ $\left.u_{m}\right]=0$, one has

$$
\lim _{n, m \rightarrow \infty}\left(F_{0}+\lambda\right)\left[\varphi_{n}^{\lambda}-\varphi_{m}^{\lambda}\right]=0 \quad \lim _{n, m \rightarrow \infty} \Phi_{\alpha}^{\lambda}\left[q_{n}-q_{m}\right]=0
$$

Since $\left(F_{0}+\lambda\right)[\varphi] \geq \lambda\|\varphi\|^{2}$ and $\left\|G^{\lambda} q\right\|^{2} \geq c\|q\|^{2}$ these conditions, toegheter with $u_{n} \xrightarrow{L^{2}} u$, imply $\varphi_{n}^{\lambda} \xrightarrow{L^{2}} \varphi^{\lambda}$ and $q_{n} \xrightarrow{L^{2}} q$.
By the closure of $F_{0}$ and of $\Phi_{\alpha}^{\lambda}$, it follows that $\varphi^{\lambda} \in \mathscr{D}\left(F_{0}\right), q \in \mathscr{D}\left(\Phi_{\alpha}^{\lambda}\right)$ and

$$
\lim _{n \rightarrow \infty}\left(F_{0}+\lambda\right)\left[\varphi_{n}^{\lambda}-\varphi^{\lambda}\right]=0 \quad \lim _{n \rightarrow \infty} \Phi_{\alpha}^{\lambda}\left[q_{n}-q\right]=0
$$

It is clear then that $u \in \mathscr{D}\left(F_{\alpha}\right)$ and $F_{\alpha}^{\lambda}\left[u_{n}-u\right] \rightarrow 0$.
Now we prove the positivity and the closure of $\Phi_{\alpha}^{\lambda}$ when $N=1$.
Lemma 2.1.8 Let $N=1$, if $\lambda \geq \lambda_{0}$ then $\Phi_{\alpha}^{\lambda}$ is closed and positive.

## Proof

If $N=1, \Phi_{\alpha}^{\lambda}$ reduces to:

$$
\Phi^{\lambda}[q]=\alpha \int d y|q(y)|^{2}+\int d y a^{\lambda}(y)|q(y)|^{2}+\frac{1}{2} \int d y d y^{\prime} G^{\lambda}\left(y, y ; y^{\prime}, y^{\prime}\right)\left|q(y)-q\left(y^{\prime}\right)\right|^{2}
$$

If $\alpha$ is non negative then $\Phi_{\alpha}^{\lambda}$ is positive since $a^{\lambda}(y)>0$; if $\alpha$ is negative it sufficient to observe that $\lim _{\lambda \rightarrow+\infty} \inf _{y} a^{\lambda}(y)=+\infty$ : if we take $\lambda$ such that $\inf _{y} a^{\lambda}+\alpha>c$ then $\Phi_{\alpha}^{\lambda}[q]>c\|q\|^{2}$.
In order to prove the closure of $\Phi_{\alpha}^{\lambda}$ we can mimic the Fischer-Riesz proof of the completeness of $L^{2}$. Given a Cauchy sequence $\left\{q_{n}\right\} \subset \mathscr{D}\left(\Phi_{\alpha}^{\lambda}\right)$ we can pick a subsequence, still denoted by $\left\{q_{n}\right\}$, such that $\Phi_{\alpha}^{\lambda}\left[q_{n}-q_{n+1}\right]<2^{-n}$ and prove the convergence of such a subsequence.
Let's consider $Q_{N}(y)=\sum_{i=1}^{N}\left|q_{i}(y)-q_{i+1}(y)\right|$ and $Q(y)=\sum_{i=1}^{\infty}\left|q_{i}(y)-q_{i+1}(y)\right|$; by the monotone convergence theorem we have $Q \in L^{2}\left(\mathbb{R}^{3}\right)$ and $Q<+\infty$ a.e..
In order to prove that $Q \in \mathscr{D}\left(\Phi_{\alpha}^{\lambda}\right)$ we notice that $\Phi_{\alpha}^{\lambda}\left[Q_{N}\right]$ is uniformly bounded with respect to $N$, indeed one has:

$$
\sqrt{\Phi_{\alpha}^{\lambda}\left[Q_{N}\right]} \leq \sum_{i=1}^{N} \sqrt{\Phi_{\alpha}^{\lambda}\left[\left|q_{i}-q_{i+1}\right|\right]} \leq c
$$

By the Fatou lemma we get $Q \in \mathscr{D}\left(\Phi_{\alpha}^{\lambda}\right)$.
Then $q_{1}+\sum_{i}\left(q_{i+1}-q_{i}\right)$ is absolutely convergent to a sum $q$ with $\left|q-q_{1}\right| \leq Q$ so that $q \in \mathscr{D}\left(\Phi_{\alpha}^{\lambda}\right)$. It is straightforward to show that $\Phi_{\alpha}^{\lambda}\left[q-q_{n}\right] \rightarrow 0$.

We state the main result of this section which follows from (2.1.7) and (2.1.8) for completeness.
Theorem 2.1.9 For $\lambda>\lambda_{0}$ and $N=1$ the quadratic form $F_{\alpha}$ is closed and bounded from below.

If $N \geq 2$ the positivity and the closure of $\Phi_{\alpha}^{\lambda}$ are not obvious: we can divide $\Phi_{\alpha}^{\lambda}$ into diagonal terms $\Phi_{\alpha, \text { diag }}^{\lambda}$ and non diagonal terms $\Phi_{\alpha, \text { diag }}^{\lambda}$

$$
\Phi_{\alpha, d i a g}^{\lambda}[q]=\alpha \sum_{i=1}^{N} \int d y\left|q_{i}(y)\right|^{2}+\sum_{i=1}^{N} \int d y a_{i}^{\lambda}(y)\left|q_{i}(y)\right|^{2}+\frac{1}{2} \sum_{i=1}^{N} \int d y d y^{\prime} G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)\left|q_{i}(y)-q_{i}\left(y^{\prime}\right)\right|^{2}
$$

$$
\Phi_{\alpha, \text { ndiag }}^{\lambda}[q]=-2 \Re \sum_{i<j} \int d y \overline{q_{i}(y)}\left(G_{j}^{\lambda} q_{i}\right)(y)
$$

It's clear, by the same argument used in the previous lemma, that for $\lambda>\lambda_{0}$ diagonal terms are closed and positive; non diagonal terms don't have a definite sign.
In order to prove that $\Phi_{\alpha}^{\lambda}$ is positive one should prove that the diagonal terms dominate, in some sense to be precised, the non diagonal terms.
There are physical models with zero range interactions, in which this vague condition is not satisfies. It is well known, see [Mi] [DFT] [MF1] [MF2], that the three body problem with delta potentials has an unbounded from below hamiltonian; that is if one consider the local and self-adjoint extension of $\tilde{H}_{0}$ defined by

$$
\begin{gathered}
\mathscr{D}\left(\tilde{H}_{0}\right)=\left\{u \in C_{0}^{\infty}\left(\mathbb{R}^{9} \backslash \Sigma\right), \Sigma=\cup_{i<j} \sigma_{i j}, \sigma_{i j}=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{9} \text { s.t. } x_{i}=x_{j}\right\}\right\} \\
\tilde{H}_{0} u=\left(-\frac{\hbar^{2}}{2 m_{1}} \Delta_{x_{1}}-\frac{\hbar^{2}}{2 m_{2}} \Delta_{x_{2}}-\frac{\hbar^{2}}{2 m_{3}} \Delta_{x_{3}}\right) u
\end{gathered}
$$

they are all unbounded from below.
In [DFT] [M], it has been proved that also a system composed of a test particle interacting with $N$ free particles with delta potentials has an unbounded from below hamiltonian if $N$ is large enough.
It's clear that in these models diagonal terms don't dominate the non diagonal terms otherwise we would obtain a semibounded operator.
Nevertheless we think that $\Phi_{\alpha}^{\lambda}$ is positive for any $N$, at least for suitable values of the physical parameters. We give only some remarks in this direction: in [DFT], a sequence of states $\left\{u_{n}=G^{\lambda} q_{n}\right\}$ such that $\widehat{F_{\alpha}}\left[G^{\lambda} q_{n}\right] \rightarrow-\infty$ for $n \rightarrow+\infty$, has been constructed for the three body problem, $\widehat{F_{\alpha}}$ is the quadratic form corresponding to $F_{\alpha}$. Such a sequence is not admissible for $\Phi_{\alpha}^{\lambda}$, i.e. $q_{n} \notin \mathscr{D}\left(\Phi_{\alpha}^{\lambda}\right)$ : this is due to the slow decay at infinity of $q_{n}$. The functions $a_{i}^{\lambda}$ diverge at infinity with a power law:

$$
a_{i}^{\lambda}(y) \simeq_{y \rightarrow \infty} \frac{1}{(2 \pi)^{\frac{3}{2}} \hbar^{3}} m^{\frac{1}{2}}\left(\frac{m M}{m+M}\right)^{\frac{3}{2}}\left(\int_{0}^{+\infty} d \nu \frac{1}{\nu^{\frac{3}{2}}}\left(1-e^{-\frac{\nu}{2}}\right)\right) \omega|y|
$$

Then a necessary condition for $q \in \mathscr{D}\left(\Phi_{\alpha}^{\lambda}\right)$ is $|y|^{\frac{1}{2}} q(y) \in L^{2}$. This condition is not satisfied for the sequence $\left\{q_{n}\right\}$.
The divergence at infinity of $a_{i}^{\lambda}$ is due to presence of the harmonic potential: in the three body problem no such a condition on the behavior of the charges at infinity is required.
We expect that, in a regime where the harmonic potential gives a large contribute to the energy of the state, i.e. large frequency limit or large separation of the equilibrium position limit, the quadratic form $\Phi_{\alpha}^{\lambda}$ should be positive: we expect the non diagonal terms to become small in Kato sense with respect to diagonal ones; the taking big enough $\lambda, \Phi_{\alpha}^{\lambda}$ should become positive.

### 2.2 The Operator and the Resolvent

In this section we construct, in the case $N=1$, the operator $H_{\alpha}$ and the resolvent $\left(H_{\alpha}+\lambda\right)^{-1}$. Let's call $\Gamma_{\alpha}^{\lambda}$ the positive self adjoint operator associated to $\Phi_{\alpha}^{\lambda}$.

Theorem 2.2.1 The domain and the action of $H_{\alpha}$ are the following:

$$
\begin{gathered}
\mathscr{D}\left(H_{\alpha}\right)=\left\{u \in L^{2}\left(\mathbb{R}^{6}\right) \text { s.t. } u=\varphi^{\lambda}+G^{\lambda} q, \varphi^{\lambda} \in \mathscr{D}\left(H_{0}\right), q \in \mathscr{D}\left(\Gamma_{\alpha}^{\lambda}\right), \Gamma_{\alpha}^{\lambda} q=\operatorname{Tr}_{\pi} \varphi^{\lambda}\right\} \\
\left(H_{\alpha}+\lambda\right) u=\left(H_{0}+\lambda\right) \varphi^{\lambda}
\end{gathered}
$$

The resolvent has the following representation:

$$
\left(H_{\alpha}+\lambda\right)^{-1} f=G^{\lambda} f+G^{\lambda} q \quad q \text { solution of } \Gamma_{\alpha}^{\lambda} q=\operatorname{Tr}_{\pi} G^{\lambda} f
$$

## Proof

A function $u$ belongs to $\mathscr{D}\left(H_{\alpha}\right)$ if and only if it exists $w \in L^{2}$, which is by definition $H_{\alpha} u$, such that

$$
\begin{equation*}
F_{\alpha}[u, v]=(w, v) \quad \forall v \in \mathscr{D}\left(F_{\alpha}\right) \tag{2.34}
\end{equation*}
$$

In particular for $v \in C_{0}^{\infty}$, which means $q_{v} \equiv 0$, (2.34) reduces to:

$$
\begin{equation*}
\int d x d y\left[\frac{\hbar^{2}}{2 M} \overline{\partial_{x} \varphi_{u}^{\lambda}} \partial_{x} v+\frac{\hbar^{2}}{2 m} \overline{\partial_{y} \varphi_{u}^{\lambda}} \partial_{y} v+\frac{1}{2} m \omega^{2}\left(y-y^{0}\right)^{2} \overline{\varphi_{u}^{\lambda}} v+\lambda \overline{\varphi_{u}^{\lambda}} v\right]=\int d x d y \overline{\left(H_{\alpha}+\lambda\right) u} v \tag{2.35}
\end{equation*}
$$

Equation (2.35) implies

$$
\begin{equation*}
\varphi_{u}^{\lambda} \in \mathscr{D}\left(H_{0}\right) \quad\left(H_{\alpha}+\lambda\right) u=\left(H_{0}+\lambda\right) \varphi_{u}^{\lambda} \tag{2.36}
\end{equation*}
$$

If we now take a generic element of the form domain as test function we obtain:

$$
\begin{aligned}
& \int d x d y\left[\frac{\hbar^{2}}{2 M} \overline{\partial_{x} \varphi_{u}^{\lambda}} \partial_{x} \varphi_{v}^{\lambda}+\frac{\hbar^{2}}{2 m} \overline{\partial_{y} \varphi_{u}^{\lambda}} \partial_{y} \varphi_{v}^{\lambda}+\left(\frac{1}{2} m \omega^{2}\left(y-y^{0}\right)^{2}\right.\right.\left.+\lambda) \overline{\varphi_{u}^{\lambda}} \varphi_{v}^{\lambda}\right] \\
&+ \\
&+\Phi_{\alpha}^{\lambda}\left[q_{u}, q_{v}\right]=\int d x d y \overline{\left(H_{\alpha}+\lambda\right) u} v
\end{aligned}
$$

Using (2.36), we obtain

$$
\Phi_{\alpha}^{\lambda}\left[q_{u}, q_{v}\right]=\int d x d y \overline{\left(H_{0}+\lambda\right) \varphi_{u}^{\lambda}} G^{\lambda} q_{v}
$$

which is equivalent to

$$
\Phi_{\alpha}^{\lambda}\left[q_{u}, q_{v}\right]=\int d x d y \overline{T r_{\pi} \varphi_{u}^{\lambda}} q_{v}
$$

The previous equation says

$$
\begin{equation*}
q_{u} \in \mathscr{D}\left(\Gamma_{\alpha}^{\lambda}\right) \quad \Gamma_{\alpha}^{\lambda} q_{u}=\operatorname{Tr}_{\pi} \varphi_{u}^{\lambda} \tag{2.37}
\end{equation*}
$$

Equation (2.37) is a boundary condition, which must be satisfied by the elements of the domain. The boundary condition can be written also as $q_{u}=\left(\Gamma_{\alpha}^{\lambda}\right)^{-1} T r_{\pi} \varphi_{u}^{\lambda}$; the advantage of this reformulation is that $\left(\Gamma_{\alpha}^{\lambda}\right)^{-1}$ is a bounded operator and then it's clear that for any $\varphi_{u}^{\lambda}, q_{u}$ is well defined and continuously depends on $\varphi_{u}^{\lambda}$. This statement follows from lemma (2.1.8) when we observed that for $\lambda>\lambda_{0}$ one has $\Phi_{\alpha}^{\lambda}[q]>c\|q\|^{2}$

## Chapter 3

## The Two Dimensional Case

In this chapter we introduce the quadratic form in dimension two. The situation is similar to the three dimensional case: a renormalization of the quadratic form is still required in order to achieve a meaningful quadratic form since we are not concerned with a bounded perturbation of $F_{0}$.
In section we introduce $F_{\alpha}$; propositions (3.1.2) (3.1.3) (3.1.4) and (3.1.5) proves that definition (3.1.1) is well put. Closure and semiboundednes of the quadratic form are proved for any $N$ since in the two dimensional case we can give an $L^{2}$ estimate of non diagonal term of $\Phi_{\alpha}^{\lambda}$, see lemma (3.1.7). In section two, theorem (3.2.1) characterizes the domain and the action of the operator.

### 3.1 The Quadratic Form

In this section we introduce the quadratic form $F_{\alpha}$ in two dimension and we prove that it is closed and bounded from below for any $N$. The definition of $F_{\alpha}$ is similar to the three dimanional case.

## Definition 3.1.1

$$
\begin{gathered}
\mathscr{D}\left(F_{\alpha}\right)=\left\{u \in L^{2}\left(\mathbb{R}^{2 N+2}\right) \text { s.t. } \exists q \in \mathscr{D}\left(\Phi^{\lambda}\right), \varphi^{\lambda} \equiv u-G^{\lambda} q \in \mathscr{D}\left(F_{0}\right)\right\} \\
F_{\alpha}[u]=\mathcal{F}^{\lambda}[u]+\Phi_{\alpha}^{\lambda}[u] \\
\mathcal{F}^{\lambda}[u]=\int d x d y\left\{\frac{\hbar^{2}}{2 M}\left|\nabla_{x} \varphi^{\lambda}\right|^{2}+\frac{\hbar^{2}}{2 m}\left|\nabla_{y} \varphi^{\lambda}\right|^{2}+\lambda\left|\varphi^{\lambda}\right|^{2}-\lambda|u|^{2}+\frac{1}{2} m \omega^{2}\left(y-y^{0}\right)^{2}\left|\varphi^{\lambda}\right|^{2}\right\} \\
\mathscr{D}\left(\Phi_{\alpha}^{\lambda}\right)=\left\{\left(q_{1}, \ldots, q_{N}\right), q_{i} \in L^{2}\left(\pi_{i}\right) \text { s.t. } \Phi_{\alpha}^{\lambda}[q]<+\infty\right\}
\end{gathered}
$$

$$
\begin{aligned}
& \Phi_{\alpha}^{\lambda}[q]=\alpha \sum_{i=1}^{N} \int d y\left|q_{i}(y)\right|^{2}-2 \Re \sum_{i<j} \int d y \overline{q_{i}(y)}\left(G_{i}^{\lambda} q_{j}\right)(y)+ \\
&+\sum_{i=1}^{N} \int d y a_{i}^{\lambda}(y)\left|q_{i}(y)\right|^{2}+\frac{1}{2} \sum_{i=1}^{N} \int d y d y^{\prime} G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)\left|q_{i}(y)-q_{i}\left(y^{\prime}\right)\right|^{2}
\end{aligned}
$$

In the two dimensional case the functions $a_{i}^{\lambda}(y)$ have the following expression:

$$
\begin{gather*}
a_{i}^{\lambda}(y)=\frac{1}{(2 \pi)} \frac{1}{\hbar^{2}} \frac{m M}{m+M}\left\{C+\int_{0}^{1} d \nu \frac{1}{(1-\nu)^{2}}\left[1-\nu^{\frac{\lambda}{\hbar \omega}+N-1}\left(\frac{2}{1+\nu^{2}}\right)^{(N-1)}\right.\right. \\
\left(\frac{m+M}{\frac{1+\nu^{2}}{2} \frac{\ln \frac{1}{\nu}}{1-\nu} m+\frac{1+\nu}{2} M}\right) \exp \left(-\frac{1}{2} \frac{1-\nu^{2}}{1+\nu^{2}} \frac{m \omega}{\hbar} \sum_{\substack{j=1 \\
j \neq i}}^{N}\left|y_{j}-y_{j}^{0}\right|^{2}\right) \\
\left.\left.\quad \exp \left(-\frac{\omega}{\hbar} \frac{\frac{1}{4} m^{2}+\frac{1-\nu}{1+\nu} \frac{1}{2 \ln \frac{1}{\nu}} m M}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} m+\frac{1}{2 \ln \frac{1}{\nu}} M}\left|y_{i}-y_{i}^{0}\right|^{2}\right)\right]\right\}  \tag{3.1}\\
C=-\left(\int_{0}^{1} d \nu \frac{1}{\nu} e^{-\frac{1}{\nu}}+\int_{1}^{\infty} d \nu \frac{1}{\nu^{2}} \ln \frac{1}{\nu} e^{-\frac{1}{\nu}}\right)
\end{gather*}
$$

The heuristic argument at the basis of definition (3.1.1) is the same of the three dimensional case: the potential generated by a charge distribution is still divergent near the planes $\pi_{i}$ and then a renormalization of the quadratic form is required. Nevertheless the singularity of the potential is of logarithmic type, much weaker than the coulombian type of the previous case; as we shall see later this fact makes easier the study of the closure of the quadratic form. The analysis of $a_{i}^{\lambda}(y)$ follows the same line of the three dimensional case. To be concrete we fix $i=1$, put $y^{0}=0$ and introduce the coordinates $\xi$ and $\eta$.

$$
\left\{\begin{aligned}
\xi & =\frac{1}{\sqrt{2}}\left(x-y_{1}\right) \\
\eta & =\frac{1}{\sqrt{2}}\left(x+y_{1}\right)
\end{aligned}\right.
$$

With this choice $\pi_{1}$ is defined by $\xi=0$. We have to study the behavior for small $\xi$ of

$$
\begin{equation*}
\left(G^{\lambda} 1_{\varepsilon}\right)\left(\xi, \eta, y_{2}, \ldots, y_{N}\right)=\int_{U_{\varepsilon}} d \eta^{\prime} d y_{2}^{\prime} \ldots d y_{N}^{\prime} G^{\lambda}\left(\xi, \eta, y_{2}, \ldots, y_{N} ; 0, \eta^{\prime}, y_{2}^{\prime}, \ldots, y_{N}^{\prime}\right) \tag{3.2}
\end{equation*}
$$

The shape of $U_{\varepsilon}$ doesn't matter, indeed $a_{i}^{\lambda}$ doesn't depend on $\varepsilon$; we choose

$$
U_{\varepsilon}=\left\{\left|\eta^{\prime}-\eta\right|<\varepsilon,\left|y_{2}^{\prime}-y_{2}\right|<\varepsilon, \ldots,\left|y_{N}^{\prime}-y_{N}\right|<\varepsilon\right\}
$$

From (A.1) one has that, with simple manipulations, (3.2) can be written as:

$$
\begin{gather*}
\left(G^{\lambda} 1_{\varepsilon}\right)\left(\xi, \eta, y_{2}, \ldots, y_{N}\right)=\frac{1}{2 \pi} \frac{1}{\hbar \omega} \frac{1}{\pi^{N}}\left(\frac{m \omega}{\hbar}\right)^{N}\left(\frac{M \omega}{\hbar}\right) \int_{0}^{1} d \nu \frac{\nu^{\frac{\lambda}{\hbar \omega}+N-1}}{(1+\nu)^{N}} \frac{1}{(1-\nu)^{N}\left(\log \frac{1}{\nu}\right)} \\
\prod_{i=2}^{N}\left\{e^{-\frac{m \omega}{\hbar} \frac{1-\nu}{1+\nu} y_{i}^{2}}\left(\frac{m \omega}{\hbar} \frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}}\right)^{-1} \int_{\left.\left|y_{i}^{\prime}\right|<\varepsilon\left(\frac{m \omega}{\hbar} \frac{\left(1+\nu^{2}\right)}{2\left(1 \nu^{2}\right)}\right)^{\frac{1}{2}} d y_{i}^{\prime} e^{-y_{i}^{\prime 2}} e^{-y_{i} \cdot y_{i}^{\prime}\left(\frac{m \omega}{\hbar} \frac{2\left(1-\nu^{2}\right)}{\left(1+\nu^{2}\right)}\right)^{\frac{1}{2}} \frac{1-\nu}{1+\nu}}\right\}}^{e^{-\frac{1}{2} \frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar}\left(\eta^{2}-\eta \cdot \xi\right)} e^{-\xi^{2}\left(\frac{1}{4} \frac{1-\nu^{2}}{1+\nu^{2}} \frac{m \omega}{\hbar} \frac{1}{4 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)}\left(\frac{1}{4} \frac{1-\nu^{2}}{1+\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{4 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)^{-1}}\right. \\
\int_{\left|\eta^{\prime}\right|<\varepsilon\left(\frac{1}{4} \frac{1-\nu^{2}}{1+\nu^{2}} \frac{m \omega}{\hbar} \frac{1}{4 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)^{\frac{1}{2}} d \eta^{\prime} e^{-\eta^{\prime 2}} \exp \left\{\frac{2 \eta^{\prime} \cdot\left[\left(\frac{1}{\left.\left.4 \ln \frac{1}{\nu} \frac{M \omega}{\hbar}-\frac{1}{4} \frac{1-\nu^{2}}{1+\nu^{2}} \frac{m \omega}{\hbar}\right) \xi+\frac{1}{4} \frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar} \eta\right]}\right.\right.}{\left(\frac{1}{4} \frac{1-\nu^{2}}{1+\nu^{2}} \frac{m \omega}{\hbar} \frac{1}{4 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)^{\frac{1}{2}}}\right\}} .
\end{gather*}
$$

With the changes of variable $\nu=1-\nu^{\prime}$ and $\nu^{\prime} \xi^{2}=\nu^{\prime \prime}$ we recover a logarithmic singularity for the potential:

$$
\begin{equation*}
\left(G^{\lambda} 1_{\varepsilon}\right)\left(\xi, \eta, y_{2}, \ldots, y_{N}\right) \simeq_{\xi \rightarrow 0}-\frac{1}{\pi} \frac{1}{\hbar^{2}} \frac{m M}{m+M} \ln \left(\xi^{2} \frac{\omega}{\hbar} \frac{m M}{m+M}\right) \tag{3.4}
\end{equation*}
$$

If we subtract (3.3) from (3.4) and we add $\int_{\left|y^{\prime}-y\right|>\varepsilon} d y^{\prime} G^{\lambda}\left(y_{i}, y ; y_{i}^{\prime}, y^{\prime}\right)$, we obtain (3.1).
The proof that (2.1.1) is well defined is divided into several propositions; the proofs differ from the three dimensional case only by minor modifications.
First we prove that the decomposition $u=\varphi^{\lambda}+G^{\lambda} q$ is well defined and unique.
Proposition 3.1.2 If $q_{i} \in L^{2}\left(\mathbb{R}^{2 N}\right)$ then $G^{\lambda} q_{i} \in L^{2}\left(\mathbb{R}^{2 N+2}\right)$.

## Proof

Let's fix $i=1$.

$$
\begin{aligned}
& \int d x d y\left|G^{\lambda} q_{1}(x, y)\right|^{2}=\frac{1}{(2 \pi)^{2}} \frac{1}{(\hbar \omega)^{2}} \frac{1}{\pi^{2 N}}\left(\frac{m \omega}{\hbar}\right)^{2 N}\left(\frac{M \omega}{\hbar}\right)^{2} \\
& \int d x d y \int d z \overline{q_{1}(z)} \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+N-1} \frac{1}{\left(1-\nu^{2}\right)^{N}} \frac{1}{\left(\ln \frac{1}{\nu}\right)} e^{-\frac{m \omega}{\hbar} \frac{1-\nu}{2(1+\nu)}\left(y^{2}+z^{2}\right)} e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left|x-z_{1}\right|^{2}} \\
& e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}|y-z|^{2}} \int d \omega q_{1}(w) \int_{0}^{1} d \mu \mu^{\frac{\lambda}{\hbar \omega}+N-1} \frac{1}{\left(1-\mu^{2}\right)^{N}} \frac{1}{\left(\ln \frac{1}{\mu}\right)} e^{-\frac{m \omega}{\hbar} \frac{1-\mu}{2(1+\mu)}\left(y^{2}+w^{2}\right)} \\
& e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\mu}}\left|x-w_{1}\right|^{2}} e^{-\frac{m \omega}{\hbar} \frac{\mu}{1-\mu^{2}|y-w|^{2}}}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{1}{(2 \pi)^{2}} \frac{1}{(\hbar \omega)^{2}} \frac{1}{\pi^{2 N}}\left(\frac{m \omega}{\hbar}\right)^{2 N}\left(\frac{M \omega}{\hbar}\right)^{2} \int d z d \omega \overline{q_{1}}(z) q_{1}(w) \\
& \quad \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+N-1} \frac{1}{\left(1-\nu^{2}\right)^{N}\left(\ln \frac{1}{\nu}\right)} \int_{0}^{1} d \mu \mu^{\frac{\lambda}{\hbar \omega}+N-1} \frac{1}{\left(1-\mu^{2}\right)^{N}} \frac{1}{\left(\ln \frac{1}{\mu}\right)} \\
& \quad \int d x d y e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}|y-z|^{2}} e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left|x-z_{1}\right|^{2}} e^{-\frac{m \omega}{\hbar} \frac{1-\nu}{2(1+\nu)}\left(y^{2}+z^{2}\right)} \\
& \quad e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\mu}}\left|x-w_{1}\right|^{2}} e^{-\frac{m \omega}{\hbar} \frac{\mu}{1-\mu^{2}}|y-w|^{2}} e^{-\frac{m \omega}{\hbar} \frac{1-\mu}{2(1+\mu)}\left(y^{2}+w^{2}\right)}
\end{aligned}
$$

It's then sufficient to show that

$$
\begin{aligned}
T(z, w)=\int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+N-1} & \frac{1}{\left(1-\nu^{2}\right)^{N}} \frac{1}{\left(\ln \frac{1}{\nu}\right)} \int_{0}^{1} d \mu \mu^{\frac{\lambda}{\hbar \omega}+N-1} \frac{1}{\left(1-\mu^{2}\right)^{N}} \frac{1}{\left(\ln \frac{1}{\mu}\right)} \\
& \int d x d y e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}|y-z|^{2}} e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left|x-z_{1}\right|^{2}} e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\mu}}\left|x-w_{1}\right|^{2}} e^{-\frac{m \omega}{\hbar} \frac{\mu}{1-\mu^{2}}|y-w|^{2}}
\end{aligned}
$$

is the integral kernel of a bounded operator in $L^{2}\left(\mathbb{R}^{2 N}\right)$. This kernel can be pointwise estimated in the following way:

$$
\begin{align*}
T(z, w) \leq\left(\frac{M \omega}{\hbar}\right)^{-1} & \left(\frac{m \omega}{\hbar}\right)^{-N} \pi^{N+1} \int_{0}^{1} d \nu d \mu\left(\frac{\nu}{1-\nu^{2}}+\frac{\mu}{1-\mu^{2}}\right)^{-(N+1)} \nu^{\frac{\lambda}{\hbar \omega}+N-1} \\
& \frac{1}{\left(1-\nu^{2}\right)^{N}} \frac{1}{\left(\ln \frac{1}{\nu}\right)} \mu^{\frac{\lambda}{\hbar \omega}+N-1} \frac{1}{\left(1-\mu^{2}\right)^{N}} \frac{1}{\left(\ln \frac{1}{\mu}\right)} e^{-\frac{1}{4\left(\frac{1-\nu^{2}}{\nu}+\frac{1-\mu^{2}}{\mu}\right)^{\frac{m \omega}{\hbar}|w-z|^{2}}}} \tag{3.5}
\end{align*}
$$

Using the Schur's test, and (3.5) we have

$$
\begin{aligned}
& \|T\| \leq \sup _{w} \int d \omega T(z, w) \leq\left(\frac{M \omega}{\hbar}\right)^{-1} \pi^{N+1} \int_{0}^{1} d \nu d \mu \nu^{\frac{\lambda}{\hbar \omega}-1} \mu^{\frac{\lambda}{\hbar \omega}-1} \frac{1}{\left(\ln \frac{1}{\nu}\right)} \frac{1}{\left(\ln \frac{1}{\mu}\right)} \\
& \left(\frac{\nu}{1-\nu^{2}}+\frac{\mu}{1-\mu^{2}}\right)^{-1} \leq\left(\frac{M \omega}{\hbar}\right)^{-1} \pi^{N+1} \int_{0}^{1} d \nu d \mu \nu^{\frac{\lambda}{\hbar \omega}-1} \mu^{\frac{\lambda}{\hbar \omega}-1} \frac{1}{\left[\frac{\nu}{1+\nu} \frac{\ln \frac{1}{\nu}}{1-\nu} \ln \frac{1}{\mu}+\frac{\mu}{1+\mu} \frac{\ln \frac{1}{\mu}}{1-\mu} \ln \frac{1}{\nu}\right]}<+\infty
\end{aligned}
$$

and this concludes the proof.

Proposition 3.1.3 If $q_{i} \in \mathscr{D}\left(\Phi_{\alpha}^{\lambda}\right)$ then $G^{\lambda} q_{i} \notin \mathscr{D}\left(F_{0}\right)$.

## Proof

Fix $i=1$ and put $y^{0}=0$. Since $G^{\lambda} q_{1}$ is divergent near $\pi_{1}$, we shall consider a regularized version of $F_{0}$ restricting the integration to the whole space minus a small cylinder around $\pi_{1}$ and, with integrations by parts, we shall show that it diverges when the small cylinder shrinks to $\pi_{1}$.
Let's show that the following identity holds:

$$
\begin{align*}
& \int_{\mathbb{R}^{2 N+2} \backslash \pi_{1}^{\delta}} d x d y\left\{\frac{\hbar^{2}}{2 M}\left|\nabla_{x} G^{\lambda} q_{1}\right|^{2}+\frac{\hbar^{2}}{2 m}\left|\nabla_{y} G^{\lambda} q_{1}\right|^{2}+\left(\frac{1}{2} m \omega^{2} y^{2}+\lambda\right)\left|G^{\lambda} q_{1}\right|^{2}\right\}= \\
& \left.=-\frac{1}{2} \int d y d y^{\prime} \right\rvert\, q_{1}(y)-\left.q_{1}\left(y^{\prime}\right)\right|^{2} \int_{\partial \pi_{1}^{\delta}} d x^{\prime \prime} d y^{\prime \prime} G^{\lambda}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}, y\right) \frac{\partial G^{\lambda}}{\partial n}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}^{\prime}, y^{\prime}\right)+ \\
&+\int_{\partial \pi_{1}^{\delta}} d x d y\left(G^{\lambda} 1-\zeta_{1}\right) \frac{\partial G^{\lambda}}{\partial n}\left(\left|q_{1}\right|^{2}\right)+\int_{\partial \pi_{1}^{\delta}} d x d y \zeta_{1} \frac{\partial G^{\lambda}}{\partial n}\left(\left|q_{1}\right|^{2}\right) \tag{3.6}
\end{align*}
$$

Formula (3.6) is equivalent to:

$$
\begin{align*}
& \int_{\mathbb{R}^{2 N+2} \backslash \pi_{1}^{\delta}} d x d y\left\{\frac{\hbar^{2}}{2 M}\left|\nabla_{x} G^{\lambda} q_{1}\right|^{2}+\frac{\hbar^{2}}{2 m}\left|\nabla_{y} G^{\lambda} q_{1}\right|^{2}+\left(\frac{1}{2} m \omega^{2} y^{2}+\lambda\right)\left|G^{\lambda} q_{1}\right|^{2}\right\}= \\
&=-\frac{1}{4} \int d y d y^{\prime}\left|q_{1}(y)-q_{1}\left(y^{\prime}\right)\right|^{2}\left(\int_{\partial \pi_{1}^{\delta}} d x^{\prime \prime} d y^{\prime \prime} G^{\lambda}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}, y\right) \frac{\partial G^{\lambda}}{\partial n}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}^{\prime}, y^{\prime}\right)+\right. \\
&\left.+\int_{\partial \pi_{1}^{\delta}} d x^{\prime \prime} d y^{\prime \prime} G^{\lambda}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}^{\prime}, y^{\prime}\right) \frac{\partial G^{\lambda}}{\partial n}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}, y\right)\right)+\int_{\partial \pi_{1}^{\delta}} d x d y G^{\lambda} 1 \frac{\partial G^{\lambda}}{\partial n}\left(\left|q_{1}\right|^{2}\right) \tag{3.7}
\end{align*}
$$

If we further simplify (3.7), we obtain

$$
\begin{align*}
\int_{\mathbb{R}^{2 N+2} \backslash \pi_{1}^{\delta}} d x d y\left\{\frac{\hbar^{2}}{2 M}\left|\nabla_{x} G^{\lambda} q_{1}\right|^{2}+\frac{\hbar^{2}}{2 m}\left|\nabla_{y} G^{\lambda} q_{1}\right|^{2}+\left(\frac{1}{2} m \omega^{2} y^{2}\right.\right. & \left.+\lambda)\left|G^{\lambda} q_{1}\right|^{2}\right\}= \\
& =\int_{\partial \pi_{1}^{\delta}} d x d y \overline{G^{\lambda} q_{1}} \frac{\partial G^{\lambda}}{\partial n}\left(q_{1}\right) \tag{3.8}
\end{align*}
$$

which is true by Stoke's theorem and because we have removed $\pi_{1}^{\delta}$ from the domain of integration.
In order to prove that $F_{0}\left[G^{\lambda} q_{1}\right]=+\infty$ we shall consider the r.h.s. of (3.6) and we shall prove the following steps: the sum of first and the second term converge, up to multiplicative constant, to $-\frac{4}{\hbar 2} \frac{m M}{m+M} \Phi_{\alpha}^{\lambda}[q]$ as $\delta \rightarrow 0$ while the third term diverges; since by proposition (3.1.2), the l.h.s. of (3.6) differs from $F_{0}\left[G^{\lambda} q_{1}\right]$ only by a bounded term it's clear that $F_{0}\left[G^{\lambda} q_{1}\right]=+\infty$.
Now we show that

$$
\lim _{\delta \rightarrow 0} \int_{\partial \pi_{1}^{\delta}} d x d y\left(G^{\lambda} 1-\zeta_{1}\right)(x, y) \frac{\partial G^{\lambda}}{\partial n}\left(x, y ; y_{1}^{\prime}, y^{\prime}\right)=-\frac{4}{\hbar^{2}} \frac{m M}{m+M} a_{1}^{\lambda}\left(y^{\prime}\right)
$$

$$
\lim _{\delta \rightarrow 0} \int_{\partial \pi_{1}^{\delta}} d x^{\prime \prime} d y^{\prime \prime} G^{\lambda}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}, y\right) \frac{\partial G^{\lambda}}{\partial n}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}^{\prime}, y^{\prime}\right)=\frac{4}{\hbar^{2}} \frac{m M}{m+M} G^{\lambda}\left(x^{\prime \prime}, y^{\prime \prime} ; y_{1}^{\prime}, y^{\prime}\right)
$$

In this case $\frac{\partial}{\partial n}=-\frac{\partial}{\partial|\xi|}$, then the integral kernel of $\frac{\partial G^{\lambda}}{\partial n}$ can be easily obtained from (A.1).

$$
\begin{align*}
& \frac{\partial G^{\lambda}}{\partial n}\left(\xi, \eta, y_{2}, \ldots, y_{n} ; y_{1}^{\prime}, y^{\prime}\right)=\frac{1}{2 \pi} \frac{1}{\hbar \omega} \frac{1}{\pi^{N}}\left(\frac{m \omega}{\hbar}\right)^{N}\left(\frac{M \omega}{\hbar}\right) \int_{0}^{1} d \nu \frac{\nu^{\frac{\lambda}{\hbar \omega}+N-1}}{\left(1-\nu^{2}\right)^{N}\left(\ln \frac{1}{\nu}\right)} \\
& e^{-\frac{1}{2} \frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar}\left[\frac{1}{2}(\eta-\xi)^{2}+y_{2}^{2}+\ldots y_{N}^{2}+y^{\prime 2}\right]} e^{-\frac{\nu}{1-\nu^{2}} \frac{m \omega}{\hbar}\left(\left|\frac{1}{\sqrt{2}}(\eta-\xi)-y_{1}^{\prime}\right|^{2}+\left|y_{2}-y_{2}^{\prime}\right|^{2}+\ldots+\left|y_{N}-y_{N}^{\prime}\right|^{2}\right) e^{-\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\left|\frac{1}{\sqrt{2}}(\eta+\xi)-y_{1}^{\prime}\right|^{2}}} \\
& \left\{\left[|\xi|+\sqrt{2} \frac{\xi}{|\xi|} \cdot\left(\frac{1}{\sqrt{2}} \eta+y_{1}^{\prime}\right)\right] \frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\left[|\xi|+\sqrt{2} \frac{\xi}{|\xi|} \cdot\left(\frac{1}{\sqrt{2}} \eta-y_{1}^{\prime}\right)\right] \frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right\} \tag{3.9}
\end{align*}
$$

The potential $G^{\lambda} 1$ and its divergent part $\zeta_{1}$ have the following expressions:

$$
\begin{gathered}
\left(G^{\lambda} 1\right)\left(\xi, \eta, y_{2}, \ldots, y_{n}\right)=\frac{1}{2 \pi} \frac{1}{\hbar \omega} \frac{m \omega}{\hbar} \frac{M \omega}{\hbar} \int_{0}^{1} d \mu \frac{1}{\left(1-\mu^{2}\right)\left(\ln \frac{1}{\mu}\right)} \mu^{\frac{\lambda}{\hbar \omega}+\frac{3 N}{2}-1}\left(\frac{2}{1+\mu^{2}}\right)^{N-1} \\
\left(\frac{1+\mu^{2}}{2} \frac{\ln \frac{1}{\mu}}{1-\mu} m+\frac{1+\mu}{2} M\right)^{-1} \exp \left(-\frac{1}{2} \frac{1-\mu^{2}}{1+\mu^{2}} \frac{m \omega}{\hbar} \sum_{j=2}^{N} y_{j}^{2}\right) \exp \left(-\frac{\omega}{\hbar} \frac{\frac{1}{4} m^{2}+\frac{1-\mu}{1+\mu} \frac{1}{2 \ln \frac{1}{\mu}} m M}{\frac{1}{2} \frac{1+\mu^{2}}{1-\mu^{2}} m+\frac{1}{2 \ln \frac{1}{\mu}}} \frac{1}{2} \eta^{2}\right) \\
\exp \left(-4 \frac{\frac{1}{4 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar} \frac{1}{4} \frac{1+\mu^{2}}{1-\mu^{2}} \frac{m \omega}{\hbar}}{\frac{1}{4 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar}+\frac{1}{4} \frac{1+\mu^{2}}{1-\mu^{2}} \frac{m \omega}{\hbar}} \xi^{2}\right) \exp \left(-\frac{\frac{1}{8}\left(\frac{m \omega}{\hbar}\right)^{2}}{\left.\frac{1}{4 \ln \frac{1}{\mu} \frac{M \omega}{\hbar}+\frac{1}{4} \frac{1+\mu^{2}}{1-\mu^{2}} \frac{m \omega}{\hbar}} \eta \cdot \xi\right)}\right. \\
\zeta_{1}=\frac{1}{\pi} \frac{1}{\hbar^{2}} \frac{m M}{m+M} \ln \left(\frac{1}{\xi^{2}} \frac{\hbar}{\omega} \frac{m+M}{m M}\right)
\end{gathered}
$$

It is useful to write $G^{\lambda} 1-\zeta_{1}$ in the following way:

$$
\begin{align*}
& \left(G^{\lambda} 1-\zeta_{1}\right)\left(\xi, \eta, y_{2} \ldots y_{N}\right)=\frac{1}{\pi} \frac{1}{\hbar^{2}} \frac{m M}{m+M}\left\{C+\int_{0}^{1} d \mu \frac{1}{(1-\mu)^{\frac{3}{2}}}\left[\mu^{\frac{\lambda}{\hbar \omega}+N-1}\left(\frac{2}{1+\mu^{2}}\right)^{N-1}\right.\right. \\
& \frac{1}{\left(\frac{1+\mu^{2}}{2} \frac{\ln \frac{1}{\mu}}{1-\mu} m+\frac{1+\mu}{2} M\right)} \exp \left(-\frac{1}{2} \frac{1-\mu^{2}}{1+\mu^{2}} \frac{m \omega}{\hbar} \sum_{j=2}^{N} y_{j}^{2}\right) \exp \left(-\frac{\omega}{\hbar} \frac{\frac{1}{4} m^{2}+\frac{1-\mu}{1+\mu} \frac{1}{2 \ln \frac{1}{\mu}} m M}{\frac{1}{2} \frac{1+\mu^{2}}{1-\mu^{2}} m+\frac{1}{2 \ln \frac{1}{\mu}}} \frac{1}{2} \eta^{2}\right) \\
& \left.\left.\exp \left(-4 \frac{\frac{1}{4 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar} \frac{1}{4} \frac{1+\mu^{2}}{1-\mu^{2}} \frac{m \omega}{\hbar}}{\frac{1}{4 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar}+\frac{1}{4} \frac{1+\mu^{2}}{1-\mu^{2}} \frac{m \omega}{\hbar}} \xi^{2}\right) \exp \left(-\frac{\frac{11-\mu}{2} \frac{m \omega}{1+\mu} \frac{1}{\hbar} \frac{1}{2 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar}}{\frac{1}{4 \ln \frac{1}{\mu}} \frac{M \omega}{\hbar}+\frac{1}{4} \frac{1+\mu^{2}}{1-\mu^{2}} \frac{m \omega}{\hbar}} \eta \cdot \xi\right)-\exp \left(\frac{1}{1-\mu} \frac{m M}{m+M} \frac{\omega}{\hbar} \xi^{2}\right)\right]\right\} \tag{3.10}
\end{align*}
$$

Equation (3.10) shows that $a_{1}^{\lambda}(y)=-2\left(G^{\lambda} 1-\zeta_{1}\right)\left(0, \eta, y_{2} \ldots y_{N}\right)$, the numerical factor 2 is absorbed in (3.1) by a change of variable.
The integral kernel $\frac{\partial G^{\lambda}}{\partial n}$ has the following property:

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \int_{\partial \pi_{1}^{\delta}} d x d y f(x, y) \frac{\partial G^{\lambda}}{\partial n}\left(x, y ; y_{1}^{\prime}, y^{\prime}\right)=\frac{4}{\hbar^{2}} \frac{m M}{m+M} f\left(y_{1}^{\prime}, y^{\prime}\right) \tag{3.11}
\end{equation*}
$$

If we prove (3.11) when $f$ is $\left(G^{\lambda}-\zeta_{1}\right)(x, y)$ or $G^{\lambda}\left(x, y ; y_{1}^{\prime \prime}, y^{\prime \prime}\right)$ we conclude the first step of the proof; if we interchange the integral over the parameter $\mu$ with the limit over $\delta$ and the integral over $(x, y)$ we can restrict ourselves to prove (3.11) when $f$ is a gaussian. Now it's only a matter of long calculations to prove that:

$$
\begin{aligned}
\lim _{\delta \rightarrow 0} \int_{\partial \pi_{1}^{\delta}} d \xi d \eta d y_{2} \ldots d y_{N} e^{-\frac{\omega}{\hbar}\left(a \xi^{2}+b \eta^{2}+c \eta \cdot \xi\right)} e^{-\frac{\omega}{\hbar} d\left(y_{2}^{2}+\ldots+y_{N}^{2}\right)} \frac{\partial G^{\lambda}}{\partial n} & \left(\xi, \eta, y_{2}, \ldots, y_{n} ; y_{1}^{\prime}, y^{\prime}\right) \\
& =\frac{4}{\hbar^{2}} \frac{m M}{m+M} e^{-\frac{\omega}{\hbar} b \eta^{\prime 2}} e^{-\frac{\omega}{\hbar} d\left(y_{2}^{\prime 2}+\ldots+y_{N}^{\prime 2}\right)}
\end{aligned}
$$

Using (3.9) and integrating over $\eta, y_{2}, \ldots, y_{N}$, we have to study

$$
\begin{align*}
& \lim _{\delta \rightarrow 0} \frac{1}{\pi} \frac{1}{\hbar \omega} \frac{1}{\pi^{N}}\left(\frac{m \omega}{\hbar}\right)^{N}\left(\frac{M \omega}{\hbar}\right) \int_{0}^{1} d \nu \frac{\nu^{\frac{\lambda}{\hbar \omega}+N-1}}{\left(1-\nu^{2}\right)^{N}\left(\ln \frac{1}{\nu}\right)}\left[\frac{1}{\pi} \frac{\omega}{\hbar}\left(\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} m+d\right)\right]^{-(N-1)} \\
& \exp \left(-\frac{\omega}{\hbar}\left(y_{2}^{\prime}+\ldots y_{N}^{\prime 2}\right) \frac{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} m d+\frac{1}{4} m^{2}}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} m+d}\right)\left(\frac{1}{4} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{4 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar} \frac{\omega}{\hbar} b\right)^{\frac{3}{2}} \\
& \exp \left\{-\frac{1}{2} \delta^{2}\left[\frac{\frac{1+\nu^{2}}{1-\nu^{2}} \frac{1}{\ln \frac{1}{\nu}}+2 \frac{\omega}{\hbar} b\left(\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)+2 \frac{\omega}{\hbar} c\left(\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}-\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)+\left(\frac{\omega}{\hbar} c\right)^{2}}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b}\right]\right\} \\
& \exp \left\{-y_{1}^{\prime 2}\left[\frac{\frac{1}{4}\left(\frac{m \omega}{\hbar}\right)^{2}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar} \frac{1}{2} \frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar}+2 \frac{\omega}{\hbar} b\left(\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{M \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b}\right]\right\} \\
& \int_{|\xi|=\delta} d \xi \exp \left\{\sqrt{2} \xi \cdot y_{1}^{\prime}\left[\frac{\frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b\left(\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}-\frac{\nu}{1-\nu^{2}} \frac{m \omega}{\hbar}\right)+\frac{\omega}{\hbar} b\left(\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+\frac{\nu}{1-\nu^{2}} \frac{m \omega}{\hbar}\right)}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b}\right]\right\} \\
& \left\{|\xi| \frac{4 \frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b\left(\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\right)+\frac{\omega}{\hbar} c\left(\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}-\frac{\nu}{1-\nu^{2}} \frac{m \omega}{\hbar}\right)}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b}+\right. \\
& \left.-\sqrt{2} \frac{\xi}{|\xi|} \cdot y_{1}^{\prime} \frac{\frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+\frac{\omega}{\hbar} b\left(\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}-\frac{\nu}{1-\nu^{2}} \frac{m \omega}{\hbar}\right)}{\frac{1}{2} \frac{1+\nu^{2}}{1-\nu^{2}} \frac{m \omega}{\hbar}+\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}+2 \frac{\omega}{\hbar} b}\right\} \tag{3.12}
\end{align*}
$$

Making the change of variables $\nu=\nu^{\prime}$ and $\nu^{\prime}=\delta^{2} \nu^{\prime \prime}$ when $\delta \rightarrow 0$ only the most singular terms of (3.12) survives, in particular the surface integral becomes trivial; then we obtain

$$
\begin{aligned}
& \frac{4}{\hbar \omega}\left(\frac{\omega}{\hbar}\right)^{2}\left(\frac{m M}{m+M}\right)^{2} \int_{0}^{+\infty} d \nu \frac{1}{\nu^{2}} e^{-\frac{1}{\nu} \frac{m M}{\hbar} \frac{m M}{m+M}} e^{-2 \frac{\omega}{\hbar} b y_{1}^{\prime 2}} e^{-\frac{\omega}{\hbar} d\left(y_{2}^{\prime 2}+\ldots+y_{N}^{\prime 2}\right)}= \\
& \frac{4}{\hbar^{2}} \frac{m M}{m+M} e^{-2 \frac{\omega}{\hbar} b y_{1}^{\prime 2}} e^{-\frac{\omega}{\hbar} d\left(y_{2}^{\prime 2}+\ldots+y_{N}^{\prime 2}\right)}
\end{aligned}
$$

Since on $\pi_{1} \eta=\sqrt{2} y_{1}$ we have proved (3.11).
The third term on the r.h.s. of (2.9) is equal to:

$$
\begin{equation*}
\frac{1}{\pi} \frac{1}{\hbar^{2}} \frac{m M}{m+M} \ln \left(\frac{1}{\delta^{2}} \frac{\hbar}{\omega} \frac{m+M}{m M}\right) \int_{\partial \pi_{1}^{\delta}} \frac{\partial G^{\lambda}}{\partial n}\left(\left|q_{1}\right|^{2}\right) \tag{3.13}
\end{equation*}
$$

because $\zeta_{1}$ is a function of $\xi$ only. Equation (3.11), considering constant as a particular case of gaussian functions, also implies that

$$
\lim _{\delta \rightarrow 0} \int_{\partial \pi_{1}^{\delta}} \frac{\partial G^{\lambda}}{\partial n}\left(\left|q_{1}\right|^{2}\right)=\frac{1}{\hbar^{2}} \frac{m M}{m+M} \int d y^{\prime}\left|q_{1}\right|^{2}<+\infty
$$

which shows that (3.13) diverges when $\delta \rightarrow 0$. The proof of the proposition is concluded.

Proposition 3.1.4 The decomposition $u=\varphi^{\lambda}+G^{\lambda} q$ is well defined and unique.

## Proof

By proposition (3.1.2) $G^{\lambda} q \in L^{2}\left(\mathbb{R}^{2 N+2}\right)$ so that the decomposition is meaningful; uniqueness follows from proposition (3.1.3). If by absurd $u=\varphi^{\lambda}+G^{\lambda} q=\tilde{\varphi}^{\lambda}+G^{\lambda} \tilde{q}$ then $0=\varphi^{\lambda}-\tilde{\varphi}^{\lambda}=$ $G^{\lambda}(\tilde{q}-q)$; but then $\tilde{q}-q=0$ because the operator $G^{\lambda}$ has empty kernel.

Proposition 3.1.5 The quadratic form $\left(F_{\alpha}, \mathscr{D}\left(F_{\alpha}\right)\right)$ doesn't depend on $\lambda$.

## Proof

The proof is identical to the three dimensional case since in (2.1.5) we have used only identities independent from the dimension.

Proposition 3.1.6 The following limit holds in $L^{2}\left(\pi_{i}\right)$.

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \hbar^{2} \frac{m+M}{m M} \frac{1}{\delta \ln \frac{1}{\delta^{2}}} \int_{\left|x-y_{i}\right|=\delta} d x u=q_{i} \tag{3.14}
\end{equation*}
$$

## Proof

First, using Sobolev embedding theorems we prove that

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \frac{1}{\delta \ln \frac{1}{\delta^{2}}} \int_{\left|x-y_{i}\right|=\delta} d x u=\lim _{\delta \rightarrow 0} \frac{1}{\delta \ln \frac{1}{\delta^{2}}} \int_{\left|x-y_{i}\right|=\delta} d x q_{i} \tag{3.15}
\end{equation*}
$$

and then it's enough to show that

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \hbar^{2} \frac{m+M}{m M} \frac{1}{\delta \ln \frac{1}{\delta^{2}}} \int_{\left|x-y_{i}\right|=\delta} d x q_{i}=q_{i} \tag{3.16}
\end{equation*}
$$

Since $\varphi^{\lambda}(\cdot, y) \in H^{1}$ for a.e. $y$, its trace over $B_{\delta}=\left\{x\right.$ s.t. $\left.\left|x-y_{i}\right|=\delta\right\}$ belongs to $H^{\frac{1}{2}}$; due to the embedding $H^{\frac{1}{2}}\left(B_{\delta}\right) \hookrightarrow L^{\infty}\left(B_{\delta}\right)$ we can write the following chain of inequalities

$$
\begin{equation*}
\int_{\left|x-y_{i}\right|=\delta} d x\left|\varphi^{\lambda}(x, y)\right| \leq c\left\|\varphi^{\lambda}(\cdot, y)\right\|_{L^{\infty}\left(B_{\delta}\right)}\|1\|_{L^{1}\left(B_{\delta}\right)} \leq c\left\|\varphi^{\lambda}(\cdot, y)\right\|_{H^{\frac{1}{2}\left(B_{\delta}\right)}} \delta \leq c \delta\left\|\varphi^{\lambda}(\cdot, y)\right\|_{H^{1}\left(\mathbb{R}^{2}\right)} \tag{3.17}
\end{equation*}
$$

If we take the $L_{y}^{2}$ norm of the first and the last term in (3.17) Then the regular part of $u$ gives no contribute to the limit in (3.14). Since $G^{\lambda} q_{j}$ is a regular function on $\pi_{i}$ apart from the intersection with $\pi_{j}$, which has zero measure, then (3.15) holds.
Now we introduce the following two integral operators:

$$
\begin{align*}
& T_{\delta}\left(y ; y^{\prime}\right)=\frac{1}{2 \pi} \frac{1}{\hbar \omega} \frac{1}{\pi^{N}}\left(\frac{m \omega}{\hbar}\right)^{N}\left(\frac{M \omega}{\hbar}\right) \frac{1}{\delta \ln \frac{1}{\delta^{2}}} \int_{\left|x-y_{i}\right|=\delta} d x \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+N-1} \frac{1}{\left(1-\nu^{2}\right)^{N}} \frac{1}{\left(\ln \frac{1}{\nu}\right)} . \\
& e^{-\frac{1}{2} \frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar}\left(y^{2}+y^{\prime 2}\right)} e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left(x-y_{1}^{\prime}\right)^{2}} e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}\left(y-y^{\prime}\right)^{2}}  \tag{3.18}\\
& T_{\delta}^{\prime}\left(y ; y^{\prime}\right)=\frac{1}{2 \pi} \frac{1}{\hbar \omega} \frac{1}{\pi^{N}}\left(\frac{m \omega}{\hbar}\right)^{N}\left(\frac{M \omega}{\hbar}\right) \frac{1}{\delta \ln \frac{1}{\delta^{2}}} \int_{\left|x-y_{i}\right|=\delta} d x \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+N-1} \frac{1}{\left(1-\nu^{2}\right)^{N}} \frac{1}{\left(\ln \frac{1}{\nu}\right)} . \\
& e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left(x-y_{1}^{\prime}\right)^{2}} e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}\left(y-y^{\prime}\right)^{2}} \tag{3.19}
\end{align*}
$$

Notice that $T_{\delta}\left(y ; y^{\prime}\right) \leq T_{\delta}^{\prime}\left(y ; y^{\prime}\right)$ and that, with a straightforward integration, $\int d y^{\prime} T_{\delta}^{\prime}\left(y ; y^{\prime}\right) \leq$ $\frac{1}{\hbar 2} \frac{m M}{m+M}$; then $\left\{T_{\delta}\right\}$ and $\left\{T_{\delta}^{\prime}\right\}$ are two equibounded family of bounded operators.

In order to prove (2.24), since $\frac{1}{\delta \ln \frac{1}{\delta^{2}}} \int_{\left|x-y_{i}\right|=\delta} d x G^{\lambda} q_{i}=\int d y^{\prime} T_{\delta}\left(y ; y^{\prime}\right) q_{1}\left(y^{\prime}\right)$, it is sufficient to prove that $T_{\delta}^{\prime}$ is an approximation of the unity, see $[\mathrm{R}]$, and that

$$
\begin{equation*}
\int d y^{\prime} T_{\delta}\left(y ; y^{\prime}\right) \rightarrow \frac{1}{\hbar^{2}} \frac{m M}{m+M} \quad \text { uniformly on compact sets } \tag{3.20}
\end{equation*}
$$

Indeed, if these two statements holds, consider $q \in C_{0}^{\infty} ;(2.24)$ follows from

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \int d y\left|\int d y^{\prime} T_{\delta}\left(y ; y^{\prime}\right)\left(q(y)-q\left(y^{\prime}\right)\right)\right|^{2}=0 \tag{3.21}
\end{equation*}
$$

which is true since

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \int d y\left|\int d y^{\prime} T_{\delta}\left(y ; y^{\prime}\right)\left(q(y)-q\left(y^{\prime}\right)\right)\right|^{2} \leq \lim _{\delta \rightarrow 0} \int d y\left[\int d y^{\prime} T_{\delta}^{\prime}\left(y ; y^{\prime}\right)\left|q(y)-q\left(y^{\prime}\right)\right|\right]^{2}=0 \tag{3.22}
\end{equation*}
$$

The second limit in (3.22) is 0 because we can interchange the limit and the integral over $y$ by dominate convergence theorem, and because $T_{\delta}^{\prime}$ is an approximation of the unity.
Now we prove that $T_{\delta}^{\prime}\left(y ; y^{\prime}\right)$ is an approximation of the identity; first with a change of variable we see that $T_{\delta}^{\prime}$ is a convolution operator.

$$
\begin{array}{r}
T_{\delta}^{\prime}\left(y ; y^{\prime}\right)=\frac{1}{2 \pi} \frac{1}{\hbar \omega} \frac{1}{\pi^{N}}\left(\frac{m \omega}{\hbar}\right)^{N}\left(\frac{M \omega}{\hbar}\right) \frac{1}{\delta \ln \frac{1}{\delta^{2}}} \int_{|x|=\delta} d x \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+N-1} \frac{1}{\left(1-\nu^{2}\right)^{N}} \frac{1}{\left(\ln \frac{1}{\nu}\right)} \\
e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left(x+y_{i}-y_{1}^{\prime}\right)^{2}} e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}\left(y-y^{\prime}\right)^{2}} \tag{3.23}
\end{array}
$$

We have already noticed that the $L^{1}$ norm of the kernel has an upper bound uniform with respect to $\delta$. With futher change of variable we see that $T_{\delta}^{\prime}\left(y ; y^{\prime}\right)$ can be written in the following way which resembles the usual expression for an approximation of the identity.

$$
\begin{align*}
T_{\delta}^{\prime}\left(y ; y^{\prime}\right)= & \frac{1}{2 \pi} \frac{1}{\hbar \omega} \frac{1}{\pi}\left(\frac{m \omega}{\hbar}\right)\left(\frac{M \omega}{\hbar}\right) \frac{1}{\delta^{2 N} \ln \frac{1}{\delta^{2}}} \int_{|x|=1} d x \\
& \int_{0}^{\frac{1}{\delta^{2}}} d \nu \frac{\left(1-\delta^{2} \nu\right) \frac{\lambda}{\hbar \omega}+N-1}{\nu^{N}\left(2-\delta^{2} \nu\right)^{N}}\left(\frac{\delta^{2}}{\ln \frac{1}{1-\delta^{2} \nu}}\right) e^{-\frac{M \omega}{\hbar} \frac{\delta^{2}}{\ln \frac{1}{1-\delta^{2} \nu}}\left(x+y_{1}-y_{1}^{\prime}\right)^{2}} e^{-\frac{m \omega}{\hbar} \frac{1-\delta^{2} \nu \frac{1}{2-\delta^{2} \nu}\left(y-y^{\prime}\right)^{2}}{}} \tag{3.24}
\end{align*}
$$

We have already noticed that $T_{\delta}^{\prime}\left(y-y^{\prime}\right)$ is positive and its integral is a constant; using (3.23) it is easy to see that the integral of $T_{\delta}^{\prime}$ outside every compact containing the origin goes to zero. In order to conclude the proof, we have only to prove (3.20); with the same change of variable
of (3.24) we see that

$$
\begin{align*}
& \int \operatorname{dyp} T_{\delta}\left(y ; y^{\prime}\right)=\frac{1}{2 \pi} \frac{1}{\hbar \omega} \frac{1}{\pi^{N}}\left(\frac{m \omega}{\hbar}\right)^{N}\left(\frac{M \omega}{\hbar}\right) \int_{|x|=1} d x \int d y^{\prime} \\
& \int_{0}^{\frac{1}{\delta^{2}}} d \nu \frac{\left(1-\delta^{2} \nu\right)^{\frac{1}{\hbar \omega}+N-1}}{(\nu)^{N}\left(2-\delta^{2} \nu\right)^{N}}\left(\frac{\delta^{2}}{\ln \frac{1}{1-\delta^{2} \nu}}\right) e^{-\frac{m \omega}{\hbar} \frac{1}{2} \frac{\delta^{2} \nu}{2-\delta^{2} \nu}\left(y^{2}+\left(\delta y+y^{\prime}\right)\right)} e^{-\frac{M \omega}{\hbar} \frac{\delta^{2}}{\ln \frac{1}{1-\delta^{2} \nu}}\left(x-y_{1}^{\prime}\right)^{2}} e^{-\frac{m \omega}{\hbar} \frac{1-\delta^{2} \nu}{2-\delta^{2} \nu} \frac{1}{\nu} y^{\prime 2}} \tag{3.25}
\end{align*}
$$

If we exchange the limit and the integrals by dominate convergence theorem and we compute the obtained integrals, we prove (2.28)

Now we study the closure and semiboundedness of $F_{\alpha}$. Theorem (2.1.7) holds also in dimension two: its proof is based only on the form of $F_{\alpha}$ since no information about the dimension has been used. Then we are still concerned with the study of $\Phi_{\alpha}^{\lambda}$.
Lemma 3.1.7 If $\lambda>\lambda_{0}$ then $\Phi_{\alpha}^{\lambda}$ is closed and positive on

$$
\mathscr{D}\left(\Phi_{\alpha, \text { diag }}^{\lambda}\right)=\left\{q \in L^{2}\left(\mathbb{R}^{2 N}\right) \text { s.t. } \Phi_{\alpha, \text { diag }}^{\lambda}[q]<+\infty\right\}
$$

## Proof

Let's define $C_{\alpha}^{\lambda}=\alpha+\inf _{y} a_{i}^{\lambda}(y)$; for $\lambda>\lambda_{0} \Phi_{\alpha, \text { diag }}^{\lambda}$ is closed, see the argument used in (2.1.8) and positive since

$$
\Phi_{\alpha, \text { diag }}^{\lambda} \geq C_{\alpha}^{\lambda}\|q\|^{2}
$$

and $\lim _{\lambda \rightarrow+\infty} C_{\alpha}^{\lambda}=+\infty$
It is sufficient to prove the $L^{2}$ boundedness of non diagonal terms. Indeed if we prove

$$
\begin{equation*}
\left|\int d y \overline{q_{i}} G^{\lambda} q_{j}\right| \leq C\left\|q_{i}\right\|\left\|q_{j}\right\| \tag{3.26}
\end{equation*}
$$

then we have

$$
\left|\Phi_{\alpha, \text { ndiag }}^{\lambda}[q]\right| \leq \frac{C(N-1)}{C_{\alpha}^{\lambda}} \Phi_{\alpha, \text { diag }}^{\lambda}[q]
$$

If we define $\lambda_{0}$ such that $\frac{C N}{C_{\alpha}^{\lambda_{0}}}<1$, by KLMN theorem we have that $\Phi_{\alpha}^{\lambda}$ is closed and positive on $\mathscr{D}\left(\Phi_{\alpha, d i a g}^{\lambda}\right)$.
Now we prove (3.26); take $i=1 j=2$ to be concrete. Since

$$
\begin{equation*}
G^{\lambda}\left(y_{1}, y ; y_{2}^{\prime}, y^{\prime}\right) \leq \frac{1}{2 \pi} \frac{1}{\hbar \omega} \frac{1}{\pi^{N}}\left(\frac{m \omega}{\hbar}\right)^{N} \frac{M \omega}{\hbar} \int_{0}^{1} d \nu \frac{\nu^{\frac{\lambda}{\hbar \omega}+N-1}}{\left(1-\nu^{2}\right)^{N} \ln \frac{1}{\nu}} e^{-\frac{\nu}{1-\nu^{2}} \frac{m \omega}{\hbar}\left(y-y^{\prime}\right)^{2}} e^{-\frac{1}{2 \ln \frac{1}{\nu}} \frac{M \omega}{\hbar}\left(y_{1}-y_{2}^{\prime}\right)^{2}} \tag{3.27}
\end{equation*}
$$

it is sufficient to prove that the r.h.s. of (3.27) is a bounded operator. Making a Fourier transform, we are concerned with the estimate of the following term.

$$
\begin{align*}
& \left\lvert\, \frac{2^{N-5}}{\hbar \omega} \int d k_{1} d k_{2} d k_{2}^{\prime} d k_{3} \ldots d k_{N} \overline{q_{1}\left(k_{1}, \ldots, k_{N}\right)} q_{2}\left(k_{1}+k_{2}-k_{2}^{\prime}, k_{2}^{\prime}, k_{3}, \ldots, k_{N}\right)\right. \\
& \left.\int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}-1} e^{-\frac{1}{2} \ln \frac{1}{\nu} \frac{\hbar}{M \omega}\left(k_{2}-k_{2}^{\prime}\right)^{2}} e^{-\frac{1-\nu^{2}}{4 \nu} \frac{\hbar}{m \omega}\left(k_{2}^{2}+\left(k_{1}+k_{2}-k_{2}^{\prime}\right)^{2}\right)} \right\rvert\, \\
& \leq \frac{2^{N-5}}{\hbar \omega} \int d k d k_{2}^{\prime}\left|q_{1}\left(k_{1}, k_{2}+k_{2}^{\prime}, k_{3}, \ldots, k_{N}\right)\right|\left|q_{2}\left(k_{1}+k_{2}, k_{2}^{\prime}, k_{3}, \ldots, k_{N}\right)\right| \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}-1} . \\
& e^{-\frac{1-\nu}{2} \frac{\hbar}{m^{*} \omega}\left[k_{2}^{2}+\left(k_{2}+k_{2}^{\prime}\right)^{2}+\left(k_{1}+k_{2}\right)^{2}\right]} \tag{3.28}
\end{align*}
$$

In the previous formula we have introduced $m^{*}=\max \{m, M\}$. With the following change of variables

$$
\left\{\begin{array}{l}
k_{1}=-2 p_{1}+p_{2} \\
k_{2}=p_{1}-p_{2}-p_{3} \\
k_{2}^{\prime}=-2 p_{1}+p_{3}
\end{array}\right.
$$

equation (3.28) becomes

$$
\begin{aligned}
& \frac{92^{N-5}}{\hbar \omega} \int d p_{1} d p_{2} d p_{3} d k_{3} \ldots d k_{N}\left|q_{1}\left(-2 p_{1}+p_{2},-p_{1}-p_{2}, k_{3}, \ldots, k_{N}\right) \| q_{2}\left(-p_{1}-p_{3},-2 p_{1}+p_{3}, k_{3}, \ldots, k_{N}\right)\right| \cdot \\
& \qquad \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}-1} e^{-\frac{1-\nu}{2} \frac{\hbar}{m^{*} \omega}\left[3 p_{1}^{2}+2 p_{2}^{2}+2 p_{3}^{2}+2 p_{2} \cdot p_{3}\right]} \\
& \leq \frac{92^{N-5}}{\hbar \omega} \int d p_{1} d p_{2} d p_{3} d k_{3} \ldots d k_{N}\left|\tilde{q}_{1}\left(p_{1}, p_{2}, k_{3}, \ldots, k_{N}\right) \| \tilde{q}_{2}\left(p_{1}, p_{3}, k_{3}, \ldots, k_{N}\right)\right| \cdot \\
& \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}-1} e^{-(1-\nu) \frac{\hbar}{m^{*} \omega}\left[p_{2}^{2}+p_{3}^{2}+p_{2} \cdot p_{3}\right]}
\end{aligned}
$$

We have put $q_{1}\left(-2 p_{1}+p_{2},-p_{1}-p_{2}, k_{3}, \ldots, k_{N}\right)=\tilde{q}_{1}\left(p_{1}, p_{2}, k_{3}, \ldots, k_{N}\right)$ and $q_{2}\left(-p_{1}-p_{3},-2 p_{1}+\right.$ $p_{3}, k_{3}, \ldots, k_{N}=\tilde{q}_{2}\left(p_{1}, p_{3}, k_{3}, \ldots, k_{N}\right)$. Now by the Cauchy-Schwartz inequality, in order to conclude the proof, it is sufficient to prove that

$$
S\left(p_{2} ; p_{3}\right)=\int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}-1} e^{-(1-\nu) \frac{\hbar}{m^{*} \omega}\left[p_{2}^{2}+p_{3}^{2}+p_{2} \cdot p_{3}\right]}
$$

is a bounded operator from $L^{2}\left(\mathbb{R}^{2}\right)$ to $L^{2}\left(\mathbb{R}^{2}\right)$. Using Schur's test we have

$$
\begin{aligned}
\|S\| \leq \sup _{p_{2}}\left|p_{2}\right|^{\frac{1}{2}} \int d p_{3} & \int_{0}^{1} d \nu \nu \nu^{\frac{\lambda}{\hbar \omega}-1} e^{-(1-\nu))^{\frac{\hbar}{*} \omega}}\left[p_{2}^{2}+p_{3}^{2}+p_{2} \cdot p_{3}\right]
\end{aligned} \frac{1}{\left|p_{3}\right|^{\frac{1}{2}}}
$$

and this concludes the proof.
We state for completeness the theorem following from the previous lemma and (2.1.7).
Theorem 3.1.8 The quadratic form $F_{\alpha}$ is closed and bounded from below for $\lambda>\lambda_{0}$.
The main difference between the two dimensional case and the three dimensional case, as regards closure and semiboundedness of $F_{\alpha}$, is the $L^{2}$ boundedness of $\Phi_{\alpha, \text { ndiag }}^{\lambda}$ which is due to the much weaker typical singularity of the potential.

### 3.2 The Operator and the Resolvent

In this section we construct, the operator $H_{\alpha}$ and the resolvent $\left(H_{\alpha}+\lambda\right)^{-1}$. Let's call $\Gamma_{\alpha}^{\lambda}$ the positive self adjoint operator associated to $\Phi_{\alpha}^{\lambda}$. Unless we specify more informations on $\Gamma_{\alpha}^{\lambda}$ and on its properties the construction is the same of the three dimensional case.

Theorem 3.2.1 The domain and the action of $H_{\alpha}$ are the following:

$$
\begin{gathered}
\mathscr{D}\left(H_{\alpha}\right)=\left\{u \in L^{2}\left(\mathbb{R}^{2 N+2}\right) \text { s.t. } u=\varphi^{\lambda}+G^{\lambda} q, \varphi^{\lambda} \in \mathscr{D}\left(H_{0}\right), q \in \mathscr{D}\left(\Gamma_{\alpha}^{\lambda}\right), \Gamma_{\alpha}^{\lambda} q=\operatorname{Tr}_{\pi} \varphi^{\lambda}\right\} \\
\left(H_{\alpha}+\lambda\right) u=\left(H_{0}+\lambda\right) \varphi^{\lambda}
\end{gathered}
$$

The resolvent has the following representation:

$$
\left(H_{\alpha}+\lambda\right)^{-1} f=G^{\lambda} f+G^{\lambda} q \quad q \text { solution of } \Gamma_{\alpha}^{\lambda} q=\operatorname{Tr}_{\pi} G^{\lambda} f
$$

## Proof

A function $u$ belongs to $\mathscr{D}\left(H_{\alpha}\right)$ if and only if it exists $w \in L^{2}$, which is by definition $H_{\alpha} u$, such that

$$
\begin{equation*}
F_{\alpha}[u, v]=(w, v) \quad \forall v \in \mathscr{D}\left(F_{\alpha}\right) \tag{3.29}
\end{equation*}
$$

In particular for $v \in C_{0}^{\infty}$, which means $q_{v} \equiv 0$, (3.29) reduces to:

$$
\begin{equation*}
\int d x d y\left[\frac{\hbar^{2}}{2 M} \overline{\partial_{x} \varphi_{u}^{\lambda}} \partial_{x} v+\frac{\hbar^{2}}{2 m} \overline{\partial_{y} \varphi_{u}^{\lambda}} \partial_{y} v+\frac{1}{2} m \omega^{2}\left(y-y^{0}\right)^{2} \overline{\varphi_{u}^{\lambda}} v+\lambda \overline{\varphi_{u}^{\lambda}} v\right]=\int d x d y \overline{\left(H_{\alpha}+\lambda\right) u} v \tag{3.30}
\end{equation*}
$$

Equation (3.30) implies

$$
\begin{equation*}
\varphi_{u}^{\lambda} \in \mathscr{D}\left(H_{0}\right) \quad\left(H_{\alpha}+\lambda\right) u=\left(H_{0}+\lambda\right) \varphi_{u}^{\lambda} \tag{3.31}
\end{equation*}
$$

If we now take a generic element of the form domain as test function we obtain:

$$
\begin{aligned}
\int d x d y\left[\frac{\hbar^{2}}{2 M} \overline{\partial_{x} \varphi_{u}^{\lambda}} \partial_{x} \varphi_{v}^{\lambda}+\frac{\hbar^{2}}{2 m} \overline{\partial_{y} \varphi_{u}^{\lambda}} \partial_{y} \varphi_{v}^{\lambda}+\left(\frac{1}{2} m \omega^{2}\left(y-y^{0}\right)^{2}\right.\right. & \left.+\lambda) \overline{\varphi_{u}^{\lambda}} \varphi_{v}^{\lambda}\right] \\
& + \\
& +\Phi_{\alpha}^{\lambda}\left[q_{u}, q_{v}\right]=\int d x d y \overline{\left(H_{\alpha}+\lambda\right) u v}
\end{aligned}
$$

Using (3.31), we obtain

$$
\Phi_{\alpha}^{\lambda}\left[q_{u}, q_{v}\right]=\int d x d y \overline{\left(H_{0}+\lambda\right) \varphi_{u}^{\lambda}} G^{\lambda} q_{v}
$$

which is equivalent to

$$
\Phi_{\alpha}^{\lambda}\left[q_{u}, q_{v}\right]=\int d x d y \overline{T r_{\pi} \varphi_{u}^{\lambda}} q_{v}
$$

The previous equation says

$$
\begin{equation*}
q_{u} \in \mathscr{D}\left(\Gamma_{\alpha}^{\lambda}\right) \quad \Gamma_{\alpha}^{\lambda} q_{u}=\operatorname{Tr}_{\pi} \varphi_{u}^{\lambda} \tag{3.32}
\end{equation*}
$$

Equation (3.32) is a boundary condition, written as integral equation, which must be satisfied by the elements of the domain. Even in the two dimensional case the boundary condition can be written in a more convenient form as $q_{u}=\left(\Gamma_{\alpha}^{\lambda}\right)^{-1} \operatorname{Tr}_{\pi} \varphi_{u}^{\lambda}$ since $\left(\Gamma_{\alpha}^{\lambda}\right)^{-1}$ is a bounded operator.

## Chapter 4

## Approximate Dynamics for Small Mass Ratio

In this chapter we study an application of our model: we consider a one dimensional light particle incident over an heavy harmonic oscillator and we show that the interaction produces a partial decoherence between different components of the wave function of the heavy particle. In section one we derive an asymptotic formula for the evolution in the limit of small mass ratio, see theorem (4.1.1); as it is expected, we find an adiabatic decoupling between the light particle and the heavy particle. This formula has been used to obtain a representation form for the reduced density matrix of the harmonic oscillator.
In section two, with additional assumptions on the physical parameters of the system, we find a more precise form for the reduced density matrix, see theorem (4.1.1); formula (4.69) shows the decoherence effect on the heavy particle.

### 4.1 Approximated Evolution

In this section we study the one dimensional time dependent Schroedinger equation:

$$
\left\{\begin{array}{l}
\imath \hbar \frac{\partial \psi}{\partial t}(t)=H \psi(t)  \tag{4.1}\\
H=-\frac{\hbar^{2}}{2 M} \Delta_{x}-\frac{\hbar^{2}}{2 m} \Delta_{y}+\frac{1}{2} m \omega^{2} y^{2}+\alpha \delta(x-y) \\
\psi(0 ; x, y)=\psi(x, y)=g(x) f(y)
\end{array}\right.
$$

We take as initial state the product of two functions belonging to Schwartz class and we derive an approximate solution of (4.1) in the limit of small mass ratio $\varepsilon=\frac{M}{m}$. In such an asymptotic regime it is expected an adiabatic decoupling between the fast degrees of freedom of the system (i.e. the incident particle) and the slow ones, a situation very similar to the Born-Oppenheimer approximation; the main difference lies on the absence of bound states for the light particle.

We introduce some notation in order to formulate the main result of this section. Let $\mu$ and $\nu$ be the reduced and the total mass of the system. It will be useful to use baricentral coordinates.

$$
\left\{\begin{array}{l}
x_{1}=x-y \\
x_{2}=\frac{M x+m y}{m+M}
\end{array}\right.
$$

Let $U_{\gamma}^{m, x_{0}}\left(t ; x, x^{\prime}\right)$ be the integral kernel of the propagator of $-\frac{\hbar^{2}}{2 m} \Delta+\gamma \delta\left(\cdot-x_{0}\right)$; in $[\mathrm{S}]$ it is proved that

$$
\begin{equation*}
\left(U_{\gamma}^{m, x_{0}}(t) \phi\right)(x)=\int d x^{\prime} U^{m}\left(t ; x-x^{\prime}\right) \phi\left(x^{\prime}+x_{0}\right)-\frac{m \gamma}{\hbar^{2}} \int_{0}^{+\infty} e^{-\frac{m \gamma}{\hbar^{2}} u} \int d x^{\prime} U^{m}\left(t ; u+|x|+\left|x^{\prime}\right|\right) \phi\left(x^{\prime}+x_{0}\right) \tag{4.2}
\end{equation*}
$$

In (4.2) we have denoted the kernel of the free propagator with $U^{m}(t ; x)$.

$$
U^{m}(t ; x)=\left(\frac{m}{2 \pi i \hbar t}\right)^{\frac{1}{2}} e^{-\frac{m}{2 i \hbar t} x^{2}}
$$

We shall omit $x_{0}$ as superscript for $U_{\gamma}^{m, x_{0}}\left(t ; x, x^{\prime}\right)$ whenever is 0 . We define also the integral operator

$$
\begin{gather*}
\left(W_{+}^{\gamma, x_{0}} h\right)(k)=\frac{1}{(2 \pi)^{\frac{1}{2}}} \int d x h(x)\left(e^{-i k x}+\mathscr{R}_{\gamma}(k) e^{-i x_{0} k} e^{i|k|\left|x-x_{0}\right|}\right)  \tag{4.3}\\
\mathscr{R}_{\gamma}(k)=-\frac{\gamma}{\gamma-i|k|}
\end{gather*}
$$

The integral kernel of (4.3) is the generalized eigenfunction of $-\frac{1}{2} \Delta+\gamma \delta\left(\cdot-x_{0}\right)$ and $\mathscr{R}_{\gamma}(k)$ is the reflection coefficient; see $[\mathrm{AGH}-\mathrm{KH}]$. Let $\Omega_{+}^{\gamma, x_{0}}$ be the corresponding wave operator.

$$
\left(\Omega_{+}^{\gamma, x_{0}} h\right)(x)=\left[\left(W_{+}^{\gamma, x_{0}}\right)^{-1} \mathcal{F} h\right](x)
$$

Let $V^{m, \omega}\left(t ; x, x^{\prime}\right)$ be the Mehler kernel for the unitary propagator of the harmonic oscillator:

$$
V^{m, \omega}\left(t ; x, x^{\prime}\right)=\left[\frac{m \omega}{\pi \hbar\left(1-e^{-2 i \omega t}\right)}\right]^{\frac{1}{2}} e^{-\frac{m \omega}{\hbar} \frac{x^{2}+x^{\prime 2}}{2 i \tan (\omega t)}} e^{\frac{m \omega}{\hbar} \frac{x x^{\prime}}{i \sin (\omega t)}}
$$

We put $\tilde{\omega}=\frac{\omega}{\sqrt{1+\varepsilon}}$. With the above notation we are ready to formulate the main theorem of this section.
Theorem 4.1.1 Fix $t$ such that $\sqrt{\varepsilon} t \ll 1$, then for every $0<\delta<\frac{1}{2}$ the following estimate holds:

$$
\begin{gathered}
\left\|\psi(t)-\psi_{a}(t)\right\| \leq C \varepsilon^{\frac{1}{2}-\delta} \\
\psi_{s}(t ; x, y)\left(\frac{M}{i \hbar t}\right)^{\frac{1}{2}} e^{i \frac{M}{2 \hbar t} x^{2}} \int d y^{\prime} V^{m, \omega}\left(t ; y, y^{\prime}\right) W_{+}^{\alpha_{0}, y} g\left(\frac{m x}{\hbar t}\right)
\end{gathered}
$$

The proof of the theorem will be divided into three lemmas.

Lemma 4.1.2 Fix $t$ such that $\sqrt{\varepsilon} t \ll 1$, then for every $0<\delta<\frac{1}{2}$ the following estimate holds:

$$
\begin{gathered}
\left\|\psi(t)-\psi_{1}(t)\right\| \leq C \varepsilon^{\frac{1}{2}-\delta} \\
\psi_{1}(t ; x, y)=\int d y^{\prime} V^{\nu, \tilde{\omega}}\left(t ; \frac{M x+m y}{m+M}, y^{\prime}\right) f\left(y^{\prime}\right) \int d x^{\prime} U_{\alpha}^{\mu}\left(t ; x-y, x^{\prime}\right) g\left(x^{\prime}+y^{\prime}\right)
\end{gathered}
$$

Where $C$ is explicitly given as a function of $t, M, \alpha, g, f$.

## Proof

The proof of this lemma will be divided into two steps: let $\hat{\psi}_{1}$ and $\psi_{1}$ be the solution of the following time-dependent Schroedinger equations:

$$
\begin{aligned}
& \left\{\begin{array}{l}
i \hbar \frac{\partial \hat{\psi}_{1}}{\partial t}(t)=\hat{H}_{1} \hat{\psi}_{1}(t) \\
\hat{H}_{1}=-\frac{\hbar^{2}}{2 \mu} \Delta_{x_{1}}+\alpha \delta\left(x_{1}\right)+\frac{1}{2} \mu \varepsilon \tilde{\omega}^{2}\left(\frac{\varepsilon}{1+\varepsilon}\right) x_{1}^{2}-\frac{\hbar^{2}}{2 \nu} \Delta_{x_{2}}+\frac{1}{2} \nu \tilde{\omega}^{2} x_{2}^{2} \\
\hat{\psi}_{1}\left(0 ; x_{1}, x_{2}\right)=\hat{\psi}_{1}\left(x_{1}, x_{2}\right)=g\left(x_{1}+x_{2}\right) f\left(x_{2}\right)
\end{array}\right. \\
& \left\{\begin{array}{l}
i \hbar \frac{\partial \psi_{1}}{\partial t}(t)=H_{1} \psi_{1}(t) \\
H_{1}=-\frac{\hbar^{2}}{2 \mu} \Delta_{x_{1}}+\alpha \delta\left(x_{1}\right)-\frac{\hbar^{2}}{2 \nu} \Delta_{x_{2}}+\frac{1}{2} \nu \tilde{\omega}^{2} x_{2}^{2} \\
\psi_{1}\left(0 ; x_{1}, x_{2}\right)=\psi_{1}\left(x_{1}, x_{2}\right)=g\left(x_{1}+x_{2}\right) f\left(x_{2}\right)
\end{array}\right.
\end{aligned}
$$

In order to prove (4.1.2) it is sufficient to estimate $\|\psi(t)-\hat{\psi}(t)\|$ and $\left\|\psi_{1}(t)-\hat{\psi}_{1}(t)\right\|$ separately. It is useful to write (4.1) using $\left(x_{1}, x_{2}\right)$ as coordinates.

$$
\left\{\begin{array}{l}
i \hbar \frac{\partial \psi}{\partial t}(t)=H \psi(t) \\
H=-\frac{\hbar^{2}}{2 \mu} \Delta_{x_{1}}+\alpha \delta\left(x_{1}\right)-\frac{\hbar^{2}}{2 \nu} \Delta_{x_{2}}+\frac{1}{2} \nu \tilde{\omega}^{2}\left[x_{2}^{2}-2 \frac{\varepsilon}{1+\varepsilon} x_{1} x_{2}+\left(\frac{\varepsilon}{1+\varepsilon}\right)^{2} x_{2}^{2}\right] \\
\psi\left(0 ; x_{1}, x_{2}\right)=\psi\left(x_{1}, x_{2}\right)=g\left(\frac{1}{1+\varepsilon} x_{1}+x_{2}\right) f\left(x_{2}-\frac{\varepsilon}{1+\varepsilon} x_{1}\right)
\end{array}\right.
$$

We proceed as follows:

$$
\begin{aligned}
\left\|\psi(t)-\hat{\psi}_{1}(t)\right\| & =\left\|e^{-i \frac{t}{\hbar} H} \psi-e^{-i \frac{t}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}\right\| \\
& \leq\left\|e^{-i \frac{t}{\hbar} H} \psi-e^{-i \frac{t}{\hbar} H} \hat{\psi}_{1}\right\|+\left\|e^{-i \frac{t}{\hbar} H} \hat{\psi}_{1}-e^{-i \frac{t}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}\right\| \\
& =\left\|\psi-\hat{\psi}_{1}\right\|+\left\|e^{-i \frac{t}{\hbar} H} \hat{\psi}_{1}-e^{-i \frac{t}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}\right\|
\end{aligned}
$$

The estimate of $\left\|\psi-\hat{\psi}_{1}\right\|$ can be obtained in the following way:

$$
\begin{align*}
\left\|\psi-\hat{\psi}_{1}\right\|^{2}= & \int d x_{1} d x_{2}\left|g\left(\frac{1}{1+\varepsilon} x_{1}+x_{2}\right) f\left(x_{2}-\frac{\varepsilon}{1+\varepsilon} x_{1}\right)-g\left(x_{1}+x_{2}\right) f\left(x_{2}\right)\right|^{2} \\
\leq & 2 \int d x_{1} d x_{2}\left|g\left(\frac{1}{1+\varepsilon} x_{1}+x_{2}\right)\right|^{2}\left|f\left(x_{2}-\frac{\varepsilon}{1+\varepsilon} x_{1}\right)-f\left(x_{2}\right)\right|^{2}+ \\
& +2 \int d x_{1} d x_{2}\left|f\left(x_{2}\right)\right|^{2}\left|g\left(\frac{1}{1+\varepsilon} x_{1}+x_{2}\right)-g\left(x_{1}+x_{2}\right)\right|^{2} \\
= & \int d x_{1}\left|g\left(x_{1}\right)\right|^{2} \int d x_{2}\left|f\left(\left(1+\frac{\varepsilon}{1+\varepsilon}\right) x_{2}-\varepsilon x_{1}\right)-f\left(x_{2}\right)\right|^{2}+ \\
& +2 \int d x_{2}\left|f\left(x_{2}\right)\right|^{2} \int d x_{1}\left|g\left(\frac{1}{1+\varepsilon} x_{1}+\frac{\varepsilon}{1+\varepsilon} x_{2}\right)-g\left(x_{1}\right)\right|^{2} \tag{4.4}
\end{align*}
$$

The action of translations and dilatations on regular functions can be estimated as follows:

$$
\begin{equation*}
\int d x|h(a x+b y)-h(x)|^{2} \leq c(1-a)^{2}\left(\|h\|^{2}+\int d x x^{2}\left|h^{\prime}(x)\right|^{2}\right)+c b^{2} y^{2}\left\|h^{\prime}\right\|^{2} \tag{4.5}
\end{equation*}
$$

Using (4.5) in (4.4) we prove that $\left\|\psi-\hat{\psi}_{1}\right\| \leq C \varepsilon$.
Let's turn our attention to $\left\|e^{-i \frac{t}{\hbar} H} \hat{\psi}_{1}-e^{-i \frac{t}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}\right\|$. Using Duhamel's formula for the representation of the unitary group

$$
e^{-i \frac{t}{\hbar} H} \hat{\psi}_{1}-e^{-i \frac{t}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}=-\frac{i}{\hbar} \int_{0}^{t} d s e^{-i \frac{(t-s)}{\hbar} H}\left(H-\hat{H}_{1}\right) e^{-i \frac{s}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}
$$

it is sufficient to show that the following inequality holds:

$$
\begin{equation*}
\int d x_{1} d x_{2}\left|x_{1} x_{2}\left(e^{-i \frac{s}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}\right)\left(x_{1}, x_{2}\right)\right|^{2} \leq c \tag{4.6}
\end{equation*}
$$

Indeed, if (4.6) holds, we can immediately complete the proof of step one in the following way.

$$
\begin{align*}
\left\|e^{-i \frac{t}{\hbar} H} \hat{\psi}_{1}-e^{-i \frac{t}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}\right\| & \leq \frac{1}{\hbar} \int_{0}^{t} d s\left\|e^{-i \frac{(t-s)}{\hbar} H}\left(H-\hat{H}_{1}\right) e^{-i \frac{s}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}\right\| \\
& =\frac{1}{\hbar} \int_{0}^{t} d s\left\|\left(H-\hat{H}_{1}\right) e^{-i \frac{s}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}\right\| \\
& \leq c \frac{\varepsilon}{1+\varepsilon} \frac{t}{\hbar} \nu \tilde{\omega}^{2} \tag{4.7}
\end{align*}
$$

In order to prove (4.6) we introduce $V_{\alpha}^{m, \omega}(t)$, the unitary group generated by $-\frac{\hbar^{2}}{2 m} \Delta_{x}+\frac{1}{2} m \omega^{2} x^{2}+$ $\alpha \delta(x)$. In appendix (B) it is proved that the following representation holds for $V_{\alpha}^{m, \omega}(t)$ :

$$
\left\{\begin{array}{l}
\left(V_{\alpha}^{m, \omega} \phi\right)(t ; x)=\left(V^{m, \omega} \phi\right)(t ; x)+\frac{i}{\hbar} \int_{0}^{t} d s V^{m, \omega}(t-s ; x, 0) q(s)  \tag{4.8}\\
q(t)+\alpha\left(V^{m, \omega} \phi\right)(t ; 0)+i \frac{\alpha}{\hbar} \int_{0}^{t} d s V^{m, \omega}(t-s ; 0,0) q(s)=0
\end{array}\right.
$$

Estimate (4.6) can be proved in this way:

$$
\begin{align*}
& \int d x_{1} d x_{2}\left|x_{1} x_{2}\left(e^{-i \frac{t}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}\right)\left(x_{1}, x_{2}\right)\right|^{2} \leq \\
& \quad \leq 2 \int d x_{1} d x_{2}\left|x_{1} x_{2} \int d x_{2}^{\prime} V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right) \int d x_{1}^{\prime} V^{\mu, \sqrt{\varepsilon} \tilde{\omega}}\left(t ; x_{1}, x_{1}^{\prime}\right) g\left(x_{1}^{\prime}+x_{2}^{\prime}\right)\right|^{2}+ \\
& \quad+2 \int d x_{1} d x_{2}\left|\frac{\alpha}{\hbar} x_{1} x_{2} \int d x_{2}^{\prime} V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right) \int_{0}^{t} d s V^{\mu, \sqrt{\varepsilon} \tilde{\omega}}\left(t-s ; x_{1}, 0\right) q_{\varepsilon}\left(s ; x_{2}^{\prime}\right)\right|^{2} \tag{4.9}
\end{align*}
$$

We have added a dependence on $x_{2}^{\prime}$ to $q_{\varepsilon}(t)$ because it solves the following integral equation.

$$
\begin{align*}
& q_{\varepsilon}(t)+\alpha\left[\frac{\mu \sqrt{\varepsilon} \tilde{\omega}}{\pi \hbar\left(1-e^{-2 i \sqrt{\varepsilon} \tilde{\omega} t}\right)}\right]^{\frac{1}{2}} \int d x_{1}^{\prime} e^{-\frac{\mu \sqrt{\varepsilon} \tilde{\omega}}{2 i \hbar \tan (\sqrt{\varepsilon} \tilde{\omega})} x_{1}^{\prime 2}} g\left(x_{1}^{\prime}+x_{2}^{\prime}\right)+ \\
& \quad+\frac{i}{\hbar} \alpha \int_{0}^{t} d s\left[\frac{\mu \sqrt{\varepsilon} \tilde{\omega}}{\pi \hbar\left(1-e^{-2 i \sqrt{\varepsilon} \tilde{\omega}(t-s)}\right)}\right]^{\frac{1}{2}} q_{\varepsilon}(s)=0 \tag{4.10}
\end{align*}
$$

The "initial datum" of (4.10) has a parametric dependence on $x_{2}^{\prime}$, and the same is true for the solution of the equation.
Now we introduce two identities which will be used several times in the following:

$$
\begin{align*}
& y \int d y^{\prime} V^{m, \omega}\left(t ; y, y^{\prime}\right) f\left(y^{\prime}\right)=\int d y^{\prime} V^{m, \omega}\left(t ; y, y^{\prime}\right)\left(\cos (\omega t) y^{\prime} f\left(y^{\prime}\right)-\frac{i \hbar}{m \omega} \sin (\omega t) f^{\prime}\left(y^{\prime}\right)\right)  \tag{4.11}\\
& \frac{d}{d y} \int d y^{\prime} V^{m, \omega}\left(t ; y, y^{\prime}\right) f\left(y^{\prime}\right)=\int d y^{\prime} V^{m, \omega}\left(t ; y, y^{\prime}\right)\left(\cos (\omega t) f^{\prime}\left(y^{\prime}\right)-\frac{m \omega}{i \hbar} \sin (\omega t) y^{\prime} f\left(y^{\prime}\right)\right) \tag{4.12}
\end{align*}
$$

In order to prove (4.11) and (4.12) it is sufficient to carefully integrate by parts.
We shall estimate the two terms on the r.h.s. of (4.9) separately: the first one can be handled using (4.11)

$$
\begin{aligned}
& x_{1} x_{2} \int d x_{2}^{\prime} V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right) \int d x_{1}^{\prime} V^{\mu, \sqrt{\varepsilon} \tilde{\omega}}\left(t ; x_{1}, x_{1}^{\prime}\right) g\left(x_{1}^{\prime}+x_{2}^{\prime}\right)= \\
& \int d x_{1}^{\prime} d x_{2}^{\prime} V^{\mu, \sqrt{\varepsilon} \tilde{\omega}}\left(t ; x_{1}, x_{1}^{\prime}\right) V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) \frac{i \hbar \sin (\mu \sqrt{\varepsilon} \tilde{\omega} t)}{\mu \sqrt{\varepsilon} \tilde{\omega}} \frac{i \hbar \sin (\nu \tilde{\omega} t)}{\nu \tilde{\omega}} \\
& {\left[\frac{\nu \tilde{\omega}}{2 i \hbar \tan (\tilde{\omega} t)} x_{2}^{\prime}\left(\frac{\mu \sqrt{\varepsilon} \tilde{\omega}}{2 i \hbar \tan (\sqrt{\varepsilon} \tilde{\omega} t)} x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right)-g^{\prime}\left(x_{1}^{\prime}+x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right)\right)+\right.} \\
& \left.\left.\left.-\frac{\mu \sqrt{\varepsilon} \tilde{\omega}}{2 i \hbar \tan (\sqrt{\varepsilon} \tilde{\omega} t)} x_{1}^{\prime}\left(g^{\prime}\left(x_{1}^{\prime}+x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right)+g\left(x_{1}^{\prime}+x_{2}^{\prime}\right) f^{\prime}\left(x_{2}^{\prime}\right)\right)+g^{\prime \prime}\left(x_{1}^{\prime}+x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right)\right)+g^{\prime}\left(x_{1}^{\prime}+x_{2}^{\prime}\right) f^{\prime}\left(x_{2}^{\prime}\right)\right)\right]
\end{aligned}
$$

Since $f$ and $g$ belong to Schwartz class each term in the square bracket has bounded $L^{2}$ norm then by unitarity we have proved the first half (4.6).

Now, we estimate the norm of

$$
\begin{equation*}
x_{1} \int_{0}^{t} d s V^{\mu, \sqrt{\varepsilon} \tilde{\omega}}\left(t-s ; x_{1}, 0\right)\left[x_{2} \int d x_{2}^{\prime} V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right) q_{\varepsilon}\left(s ; x_{2}^{\prime}\right)\right] \tag{4.13}
\end{equation*}
$$

In $[\mathrm{AT}]$ it is proved that:

$$
\begin{gather*}
q(t) \in L^{\infty}(0, T) \Rightarrow \int_{0}^{t} d s U^{\mu}(t-s ; x) q(s) \in L^{2}(\mathbb{R})  \tag{4.14}\\
q(t) \in H^{\frac{1}{4}}(0, T) \Rightarrow x \int_{0}^{t} d s U^{\mu}(t-s ; x, 0) q(s) \in L^{2}(\mathbb{R}) \tag{4.15}
\end{gather*}
$$

These two propositions can be extended in order to be used in our case, in particular, due to the the expression of (4.13), we need a vector valued version of (4.14) and (4.15). We shall use these two simple generalizations:

$$
\begin{gather*}
q(t) \in L^{\infty}\left(0, T ; L^{2}(\mathbb{R})\right) \Rightarrow \int_{0}^{t} d s V^{\mu, \omega}(t-s ; x) q(s) \in L^{2}\left(\mathbb{R}^{2}\right)  \tag{4.16}\\
q(t) \in H^{\frac{1}{4}}\left(0, T ; L^{2}(\mathbb{R})\right) \Rightarrow x \int_{0}^{t} d s V^{\mu, \varepsilon^{\frac{1}{2}} \tilde{\omega}}(t-s ; x, 0) q(s) \in L^{2}\left(\mathbb{R} ; L^{2}(\mathbb{R})\right)=L^{2}\left(\mathbb{R}^{2}\right)
\end{gather*}
$$

Then we need a regularity condition on $q_{\varepsilon}$ to ensure:

$$
\begin{equation*}
x_{2} \int d x_{2}^{\prime} V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right) q_{\varepsilon}\left(s ; x_{2}^{\prime}\right) \in H^{\frac{1}{4}}\left(0, T ; L^{2}(\mathbb{R})\right) \tag{4.17}
\end{equation*}
$$

Using again (4.11), it is sufficient to show

$$
\begin{equation*}
q_{\varepsilon}\left(s ; x_{2}^{\prime}\right) \in H^{\frac{1}{4}}\left(0, T ; W^{1, \infty}(\mathbb{R})\right) \tag{4.18}
\end{equation*}
$$

Indeed if (4.18) holds one has:

$$
\begin{aligned}
& x_{2} \int d x_{2}^{\prime} V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right) q\left(s ; x_{2}^{\prime}\right)=\int d x_{2}^{\prime} V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) \frac{i \hbar \sin (\nu \tilde{\omega} t)}{\nu \tilde{\omega}} . \\
& {\left[\frac{\nu \tilde{\omega}}{2 i \hbar \tan (\tilde{\omega} t)} x_{2}^{\prime} f\left(x_{2}^{\prime}\right) q\left(s ; x_{2}^{\prime}\right)-f^{\prime}\left(x_{2}^{\prime}\right) q\left(s ; x_{2}^{\prime}\right)-f\left(x_{2}^{\prime}\right) q^{\prime}\left(s ; x_{2}^{\prime}\right)\right] }
\end{aligned}
$$

All the terms in square bracket belong to $H^{\frac{1}{4}}\left(0, T ; L^{2}(\mathbb{R})\right)$ and then by unitarity (4.17) holds. It remains to prove (4.18) to conclude step one of the lemma. The initial datum of (4.10) is a regular function of both time and space variables; in particular it belongs to $C\left(0, T ; L^{\infty}(\mathbb{R})\right)$. Using a simple fixed point argument it is possible to show that $q_{\varepsilon}\left(t ; x_{2}^{\prime}\right) \in L^{2}\left(0, T ; L^{\infty}(\mathbb{R})\right)$ for small enough $T$; then the solution con be extended for any $T$ by iteration.

Exploiting the Abel-like regularization properties of the kernel in (4.10), see [GV], we find $q_{\varepsilon}\left(t ; x_{2}^{\prime}\right) \in H^{\frac{1}{4}}\left(0, T ; L^{\infty}(\mathbb{R})\right)$. If we derive both sides of (4.10) with respect to $x_{2}^{\prime}$, we see that $q_{\varepsilon}^{\prime}$ satisfies the same equation with $g^{\prime}$ as new initial datum. Then we can repeat the previous argument to prove $q_{\varepsilon}^{\prime}\left(t, x_{2}^{\prime}\right) \in H^{\frac{1}{4}}\left(0, T ; L^{\infty}(\mathbb{R})\right)$.
Now we estimate $\left\|e^{-i \frac{t}{\hbar} H} \hat{\psi}_{1}-e^{-i \frac{t}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}\right\|$.
Using (4.8) and the corresponding representation for $U_{\alpha}^{m}$

$$
\left\{\begin{array}{l}
\left(U_{\alpha}^{m} \phi\right)(t ; x)=\left(U^{m} \phi\right)(t ; x)+\frac{i}{\hbar} \int_{0}^{t} d s U^{m}(t-s ; x, 0) q(s) \\
q(t)+\alpha\left(U^{m} \phi\right)(t ; 0)+i \frac{\alpha}{\hbar} \int_{0}^{t} d s U^{m}(t-s ; 0,0) q(s)=0
\end{array}\right.
$$

we can write:

$$
\begin{align*}
e^{-i \frac{t}{\hbar} H} \hat{\psi}_{1}-e^{-i \frac{t}{\hbar} \hat{H}_{1}} \hat{\psi}_{1}= & \int d x_{1}^{\prime} d x_{2}^{\prime}\left(V^{\mu, \sqrt{\varepsilon} \tilde{\omega}}\left(t ; x_{1}-x_{1}^{\prime}\right)-U^{\mu}\left(t ; x_{1}, x_{1}^{\prime}\right)\right) V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) g\left(x_{1}^{\prime}+x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right)+ \\
+ & \frac{1}{\hbar}\left[\int_{0}^{t} d s\left(V^{\mu, \sqrt{\varepsilon} \tilde{\omega}}\left(t-s ; x_{1}, 0\right)-U^{\mu}\left(t-s ; x_{1}, 0\right)\right) .\right. \\
& \left.\int d x_{2}^{\prime} V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right) q_{\varepsilon}\left(s ; x_{2}^{\prime}\right)\right]+ \\
& +\frac{i}{\hbar} \int_{0}^{t} d s U^{\mu}\left(t-s ; x_{1}, 0\right) \int d x_{2}^{\prime} V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right)\left(q_{\varepsilon}\left(s ; x_{2}^{\prime}\right)-q\left(s ; x_{2}^{\prime}\right)\right) \tag{4.19}
\end{align*}
$$

We will separately estimate the three terms on the r.h.s. of (4.19). Now we consider the first term on r.h.s. of (4.19): in order to prove that it is of order $\varepsilon^{2}$ it is sufficient to show that

$$
\sup _{x_{2}^{\prime}} \int d x_{1}\left|\int d x_{1}^{\prime}\left(V^{\mu, \sqrt{\varepsilon} \tilde{\omega}}\left(t ; x_{1}, x_{1}^{\prime}\right)-U^{\mu}\left(t ; x_{1}, x_{1}^{\prime}\right)\right) g\left(x_{1}^{\prime}+x_{2}^{\prime}\right)\right|^{2} \leq C \varepsilon^{2}
$$

Using Duhamel's formula as in (4.7), the estimate reduces to proving $\int d x_{1}\left|x_{1}^{2} U^{\mu} g\left(t ; x_{1}+x_{2}\right)\right|^{2} \leq$ $C$. Usin (4.11) twice we obtain:

$$
\begin{aligned}
x_{1}^{2} U^{\mu} g\left(t ; x_{1}+x_{2}\right)=\int d x_{1}^{\prime} U^{\mu}\left(t ; x_{1}, x_{1}^{\prime}\right)\left(x_{1}^{\prime 2} g\left(x_{1}^{\prime}+x_{2}^{\prime}\right)+\right. & 2 \frac{i t}{m} x_{1}^{\prime} g^{\prime}\left(x_{1}^{\prime}+x_{2}^{\prime}\right)+ \\
& \left.+\frac{i t}{m} g\left(x_{1}^{\prime}+x_{2}^{\prime}\right)+\frac{i t}{m} g^{\prime \prime}\left(x_{1}^{\prime}+x_{2}^{\prime}\right)\right)
\end{aligned}
$$

Each term in brackets has finite norm bounded by a constant non depending on $x_{2}^{\prime}$.
Now we consider the second term of (4.19). We denote $\int d x_{2}^{\prime} V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) f\left(x_{2}^{\prime}\right) q_{\varepsilon}\left(s ; x_{2}^{\prime}\right)$ with $\hat{q}_{\varepsilon}(s)$ in order to make following formulas more compact. With a Fourier transform, we have to evaluate the norm of

$$
\begin{equation*}
\frac{1}{(2 \pi \hbar)^{\frac{1}{2}}} \int_{0}^{t} d s e^{-i \frac{k^{2}}{2 \mu \hbar} s} \hat{q}_{\varepsilon}(t-s)-\frac{1}{(2 \pi \hbar)^{\frac{1}{2}}} \int_{0}^{t} d s\left(\frac{2 \tan (\sqrt{\varepsilon} \tilde{\omega} s)}{1-e^{-2 i \sqrt{\varepsilon} \tilde{\omega} s}}\right)^{\frac{1}{2}} e^{-i \frac{k^{2}}{2 \mu \hbar} \frac{\tan (\sqrt{ } \tilde{\omega} \tilde{})}{\sqrt{\varepsilon} \tilde{\omega}}} \hat{q}_{\varepsilon}(t-s) \tag{4.20}
\end{equation*}
$$

Since $\left(\frac{2 \tan (\sqrt{\varepsilon} \tilde{\omega} s)}{1-e^{-2 i \sqrt{\tilde{\omega}} \tilde{s}}}\right)^{\frac{1}{2}}=\left(\frac{e^{i \sqrt{\varepsilon} \tilde{\omega} s}}{\cos (\sqrt{\varepsilon} \tilde{\omega} s)}\right)^{\frac{1}{2}}$, this quantity differs from one by a term of order $\sqrt{\varepsilon}$ uniformly in $s$. Then we can simply estimate

$$
\begin{equation*}
\int_{0}^{t} d s e^{-i \frac{k^{2}}{2 \mu \hbar} s} \hat{q}_{\varepsilon}(t-s)-\int_{0}^{t} d s e^{-i \frac{k^{2}}{2 \mu \hbar} \frac{\tan (\sqrt{\varepsilon} \tilde{\omega} s)}{\sqrt{\varepsilon} \omega}} \hat{q}_{\varepsilon}(t-s) \tag{4.21}
\end{equation*}
$$

It is useful to make the following change of variable in the second term of (4.21)

$$
\begin{align*}
& \int_{0}^{t} d s e^{-i \frac{k^{2}}{2 \mu \hbar} \frac{\tan (\sqrt{\varepsilon} \tilde{\omega} s)}{\sqrt{\varepsilon} \tilde{\omega}}} \hat{q}_{\varepsilon}(t-s)=\int_{0}^{\frac{\tan (\sqrt{\varepsilon} \tilde{\omega} s)}{\sqrt{\varepsilon} \tilde{\omega}}} e^{-i \frac{k^{2}}{2 \mu \hbar} s} \hat{q}_{\varepsilon}\left(t-\frac{\arctan (\sqrt{\varepsilon} \tilde{\omega} s)}{\sqrt{\varepsilon} \tilde{\omega}}\right) \frac{1}{1+\varepsilon s^{2}}= \\
& \int_{0}^{\frac{\tan (\sqrt{\varepsilon} \tilde{\omega} s)}{\sqrt{\varepsilon} \tilde{\omega}}} e^{-i \frac{k^{2}}{2 \mu \hbar} s} \hat{q}_{\varepsilon}\left(t-\frac{\arctan (\sqrt{\varepsilon} \tilde{\omega} s)}{\sqrt{\varepsilon} \tilde{\omega}}\right)\left(\frac{1}{1+\varepsilon s^{2}}-1\right)+ \\
&+\int_{0}^{\frac{\tan (\sqrt{\varepsilon} \tilde{\omega} s)}{\sqrt{\varepsilon} \tilde{\omega}}} e^{-i \frac{k^{2}}{2 \mu \hbar} s}\left[\hat{q}_{\varepsilon}\left(t-\frac{\arctan (\sqrt{\varepsilon} \tilde{\omega} s)}{\sqrt{\varepsilon} \tilde{\omega}}\right)-\hat{q}_{\varepsilon}(t-s)\right]+ \\
&+\int_{0}^{\frac{\tan (\sqrt{\varepsilon} \tilde{\omega} s)}{\sqrt{\varepsilon} \tilde{\omega}}} e^{-i \frac{k^{2}}{2 \mu \hbar} s} \hat{q}_{\varepsilon}(t-s) \tag{4.22}
\end{align*}
$$

The $L^{2}$ norm of the first term on the r.h.s. of (4.22) can be estimated with $c \varepsilon t^{2}\left\|\hat{q}_{\varepsilon}\right\|_{\infty}$ by (4.16). In order to estimate the second term notice that, fully exploiting the regularizing properties of (4.10), one has $\hat{q}_{\varepsilon} \in H^{1-\delta}(0, T)$ for every positive $\delta$, and then $\hat{q}_{\varepsilon} \in C^{\frac{1}{2}-\delta}(0, T)$ due to Sobolev embeddings theorems, see [A]. Then the norm of this term can be estimated by $c\left\|\hat{q}_{\varepsilon}\left(t-\frac{\arctan (\sqrt{\varepsilon} \tilde{\epsilon} \cdot)}{\sqrt{\varepsilon} \tilde{\omega}}\right)-\hat{q}_{\varepsilon}(t-\cdot)\right\|_{\infty} \leq c\left(\varepsilon t^{2}\right)^{\frac{1}{2}-\delta}$. Then we have only to estimate

$$
\begin{aligned}
\int d k\left|\int_{t}^{\frac{\tan (\sqrt{\varepsilon} \tilde{\omega} s)}{\sqrt{\varepsilon} \tilde{\omega}}} e^{-i \frac{k^{2}}{2 \mu \hbar} s^{\prime}} \hat{q}_{\varepsilon}(t-s)\right|^{2}= & 2 \Re \int_{t<s^{\prime}<s<\frac{\tan (\sqrt{\varepsilon} \tilde{\omega} s)}{\sqrt{\varepsilon} \bar{\omega}}} d s d s^{\prime} \frac{\overline{\hat{q}}_{\varepsilon}(s) \hat{q}_{\varepsilon}\left(s^{\prime}\right)}{\sqrt{s-s^{\prime}}} \leq \\
& \leq 2\left\|\hat{q}_{\varepsilon}\right\|_{\infty}^{2} \int_{t<s^{\prime}<s<\frac{\tan (\sqrt{\varepsilon} \tilde{\omega} s)}{\sqrt{\varepsilon} \omega}} d s d s^{\prime} \frac{1}{\sqrt{s-s^{\prime}}} \leq c\left\|\hat{q}_{\varepsilon}\right\|_{\infty}^{2}\left(\varepsilon t^{2}\right)^{\frac{3}{2}}
\end{aligned}
$$

Let's now estimate the last term of (4.19). Let's introduce $\tilde{q}\left(t, x_{2}^{\prime}\right)=q_{\varepsilon}\left(t, x_{2}^{\prime}\right)-q\left(t, x_{2}^{\prime}\right)$. If we prove

$$
\sup _{t}\|\tilde{q}(t ; \cdot)\|_{\infty} \leq C \varepsilon
$$

the proof of the lemma is completed by (4.16). Notice that the integral equation satisfied by $q$ has the same properties of (4.10), any argument made about $q_{\varepsilon}$ can be repeated for $q$; in
particular both functions belong to $C\left(0, T ; L^{2}(\mathbb{R})\right)$ and so $\tilde{q}$. The equation satisfied by $\tilde{q}$ is

$$
\begin{aligned}
& \tilde{q}\left(t ; x_{2}^{\prime}\right)+\alpha \int d x_{1}^{\prime}\left\{\left[\frac{\mu \sqrt{\varepsilon} \tilde{\omega}}{\pi \hbar\left(1-e^{-2 i \sqrt{\varepsilon} \tilde{\omega} t}\right)}\right]^{\frac{1}{2}} e^{-\frac{\mu \sqrt{\varepsilon} \tilde{\omega}}{2 i \hbar \tan (\sqrt{\varepsilon} \tilde{\omega} t)} x_{1}^{\prime 2}}-\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{-\frac{\mu}{2 i \hbar t} x_{1}^{\prime 2}}\right\} g\left(x_{1}^{\prime}+x_{2}^{\prime}\right)+ \\
&+i \frac{\alpha}{\hbar} \int_{0}^{t} d s\left[\left(\frac{\mu \sqrt{\varepsilon} \tilde{\omega}}{\pi \hbar\left(1-e^{-2 i \sqrt{\varepsilon} \tilde{\omega}(t-s)}\right)}\right)^{\frac{1}{2}}-\left(\frac{\mu}{2 \pi \hbar i(t-s)}\right)^{\frac{1}{2}}\right] \tilde{q}\left(s ; x_{2}^{\prime}\right)=0 \\
&=\alpha T g\left(t ; x_{2}^{\prime}\right)+i \frac{\alpha}{\hbar} \int_{0}^{t} d s K(t-s) \tilde{q}\left(s ; x_{2}^{\prime}\right)=0
\end{aligned}
$$

Since $\|K(\cdot)\|_{\infty} \leq C \varepsilon$ and $\sup _{t}\|T g(t, \cdot)\|_{\infty} \leq C \varepsilon$ we obtain $\sup _{t}\|\tilde{q}(t, \cdot)\|_{\infty} \leq C \varepsilon$.

Since a small value of $M$ in the interacting unitary group $U_{\alpha}^{\mu}(t)$ is equivalent to a large value of time, in the next lemma we use a typical scattering estimate to approximate $U_{\alpha}^{\mu}(t)$.

Lemma 4.1.3 The following estimate holds:

$$
\begin{gathered}
\left\|\psi_{1}(t)-\psi_{2}(t)\right\| \leq C \varepsilon \\
\psi_{2}\left(t ; x_{1}, x_{2}\right)=\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{\mu}{2 \hbar t} x_{1}^{2}} \int d x_{2}^{\prime} f\left(x_{2}^{\prime}\right) V^{\nu, \tilde{\omega}}\left(t ; x_{2}, x_{2}^{\prime}\right) \int d x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}^{\prime}\right)\left(e^{-i \frac{\mu}{\hbar t} x_{1} x_{1}^{\prime}}-\frac{e^{-i \frac{\mu}{\hbar t}\left|x_{1} \| x_{1}^{\prime}\right|}}{1-i \frac{\hbar}{\alpha t}\left|x_{1}\right|}\right)
\end{gathered}
$$

and $C$ is explicitly given as a function of $t, m, M, \alpha, g$.

## Proof

Using the isometric character of $V^{\nu, \tilde{\omega}}(t)$ and the explicit expression of $U_{\alpha}^{\mu}(t)$, see (4.2), we have:

$$
\begin{gather*}
\left\|\psi_{1}(t)-\psi_{2}(t)\right\|^{2} \leq 2 \int d x_{1} d x_{2}\left|f\left(x_{2}\right) \int d x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}\right)\left(U^{\mu}\left(t ; x_{1}-x_{1}^{\prime}\right)-\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{\mu}{2 h t} x_{1}^{2}-i \frac{\mu}{h t} x_{1} x_{1}^{\prime}}\right)\right|^{2} \\
+2 \int d x_{1} d x_{2} \left\lvert\, f\left(x_{2}\right) \int d x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}\right)\left(\frac{\mu \alpha}{\hbar^{2}} \int_{0}^{\infty} d u e^{-\frac{\mu \alpha}{\hbar^{2}} u} U^{\mu}\left(t ; u+\left|x_{1}\right|+\left|x_{1}^{\prime}\right|\right)+\right.\right. \\
\left.-\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} \frac{1}{1-i \frac{\hbar}{\alpha t}\left|x_{1}\right|} e^{i \frac{\mu}{2 h t} x_{1}^{2}-i \frac{\mu}{\hbar t}\left|x_{1}\right|\left|x_{1}^{\prime}\right|}\right)\left.\right|^{2} \tag{4.23}
\end{gather*}
$$

The first term of (4.23) can be easily estimated in the following way:

$$
\begin{aligned}
& 2 \int d x_{1} d x_{2}\left|f\left(x_{2}\right) \int d x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}\right)\left(U^{\mu}\left(t ; x_{1}-x_{1}^{\prime}\right)-\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{\mu}{2 h t} x_{1}^{2}-i \frac{\mu}{\hbar t} x_{1} x_{1}^{\prime}}\right)\right|^{2} \\
& =\frac{\mu}{\pi \hbar t} \int d x_{2}\left|f\left(x_{2}\right)\right|^{2} \int d x_{1}\left|\int d x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}^{\prime}\right)\left(e^{i \frac{\mu}{2 h t} x_{1}^{\prime 2}}-1\right) e^{-i \frac{\mu}{\hbar t} x_{1} x_{1}^{\prime}}\right|^{2} \\
& \leq \int d x_{2}\left|f\left(x_{2}\right)\right|^{2} \int d x_{1}^{\prime}\left|g\left(x_{1}^{\prime}+x_{2}\right)\right|^{2}\left|e^{i \frac{\mu}{2 \hbar t} x_{1}^{\prime 2}}-1\right|^{2} \\
& \leq \frac{\varepsilon^{2} m^{2}}{2(1+\varepsilon)^{2} \hbar^{2} t^{2}} \int d x_{2}\left|f\left(x_{2}\right)\right|^{2} \int d x_{1}^{\prime}\left|g\left(x_{1}^{\prime}+x_{2}\right)\right|^{2} x_{1}^{\prime 4}
\end{aligned}
$$

Concerning the second term, we introduce $\alpha_{0}=\frac{\alpha m}{\hbar^{2}}$ and the change of variables

$$
v=\frac{\mu \alpha}{\hbar^{2}} u \quad y_{1}=\frac{\mu}{\hbar t} x_{1}
$$

and use the identity

$$
\int_{0}^{\infty} d v e^{-v+i \frac{\hbar\left|x_{1}\right|}{\alpha t} v}=\frac{1}{1-i \frac{\hbar\left|x_{1}\right|}{\alpha t}}
$$

It can be estimated in the following way:

$$
\begin{align*}
& 2 \int d x_{1} d x_{2} \left\lvert\, f\left(x_{2}\right) \int d x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}\right)\left(\frac{\mu \alpha}{\hbar^{2}} \int_{0}^{\infty} d u e^{-\frac{\mu \alpha}{\hbar^{2}} u} U^{\mu}\left(t ; u+\left|x_{1}\right|+\left|x_{1}^{\prime}\right|\right)+\right.\right. \\
& \left.\quad-\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} \frac{1}{1-i \frac{\hbar}{\alpha t}\left|x_{1}\right|} e^{i \frac{\mu}{2 h t} x_{1}^{2}-i \frac{\mu}{\hbar t}\left|x_{1}\right|\left|x_{1}^{\prime}\right|}\right)\left.\right|^{2} \\
& =\frac{1}{\pi} \int d x_{2}\left|f\left(x_{2}\right)\right|^{2} \int d y_{1} \left\lvert\, \int d x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}\right) e^{i\left|y_{1}\right|\left|x_{1}^{\prime}\right|}\left(e^{i \frac{\mu}{2 \hbar t} x_{1}^{\prime 2}} \int_{0}^{\infty} d v e^{-v+i \frac{\hbar\left|x_{1}\right|}{\alpha t} v} e^{i \frac{(1+\varepsilon) M}{2 \hbar t \alpha_{0}^{2}} v^{2}+i \frac{M}{\hbar t \alpha_{0}}\left|x_{1}^{\prime}\right| v}+\right.\right. \\
& \left.\quad-\int_{0}^{\infty} d v e^{-v+i \frac{\left(1+\varepsilon| | y_{1} \mid\right.}{\alpha_{0}} v}\right)\left.\right|^{2} \\
& \leq \frac{2}{\pi} \int d x_{2}\left|f\left(x_{2}\right)\right|^{2} \int d y_{1} \left\lvert\, \int d x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}\right) e^{i\left|y_{1}\right|\left|x_{1}^{\prime}\right|} e^{i \frac{\mu}{2 h t} x_{1}^{\prime 2}} \cdots\right. \\
& \left.\quad \int_{0}^{\infty} d v e^{-v+i \frac{\left(1+\varepsilon\left|y_{1}\right|\right.}{\alpha_{0}} v}\left(e^{i \frac{(1+\varepsilon) M}{2 \hbar t \alpha_{0}^{2}} v^{2}+i \frac{M}{\hbar t \alpha_{0}}\left|x_{1}^{\prime}\right| v}-1\right)\right|^{2}+ \\
& +\frac{2}{\pi} \int d x_{2}\left|f\left(x_{2}\right)\right|^{2} \int d y_{1}\left|\int d x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}\right) e^{i\left|y_{1}\right|\left|x_{1}^{\prime}\right|}\left(e^{i \frac{\mu}{2 h t} x_{1}^{\prime 2}}-1\right) \frac{1}{1-i \frac{1+\varepsilon}{\alpha_{0}}\left|y_{1}\right|}\right|^{2} \tag{4.24}
\end{align*}
$$

To estimate the first term of (4.24) we proceed as follows:

$$
\begin{aligned}
& \frac{2}{\pi} \int d x_{2}\left|f\left(x_{2}\right)\right|^{2} \int d y_{1}\left|\int d x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}\right) e^{i\left|y_{1}\right|\left|x_{1}^{\prime}\right|} e^{i \frac{\mu}{2 h t} x_{1}^{\prime 2}} \int_{0}^{\infty} d v e^{-v+i \frac{(1+\varepsilon)\left|y_{1}\right|}{\alpha_{0}} v}\left(e^{i \frac{(1+\varepsilon) M}{2 \hbar t \alpha_{0}^{2}} v^{2}+i \frac{M}{\hbar t \alpha_{0}}\left|x_{1}^{\prime}\right| v}-1\right)\right|^{2} \\
& \leq \frac{1}{2 \pi}\left(\frac{\mu}{\hbar t}\right)^{2} \int d x_{2}\left|f\left(x_{2}\right)\right|^{2} \int d y_{1} \frac{1}{1+\left(\frac{1+\varepsilon}{\alpha_{0}}\right)^{2} y_{1}^{2}}\left(\int d x_{1}^{\prime} x_{1}^{\prime 2}\left|g\left(x_{1}^{\prime}+x_{2}\right)\right|\right)^{2} \\
& \leq \frac{\varepsilon^{2} m^{2} \alpha_{0}}{2(1+\varepsilon)^{3} \hbar^{2} t^{2}} \int d x_{2}\left|f\left(x_{2}\right)\right|^{2}\left(\int d x_{1}^{\prime} x_{1}^{\prime 2}\left|g\left(x_{1}^{\prime}+x_{2}\right)\right|\right)^{2}
\end{aligned}
$$

It is convenient to integrate the second term of (4.24) by parts.

$$
\begin{aligned}
& \frac{2}{\pi} \int d x_{2}\left|f\left(x_{2}\right)\right|^{2} \int d y_{1}\left|\int d x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}\right) e^{i\left|y_{1}\right|\left|x_{1}^{\prime}\right|}\left(e^{i \frac{\mu}{2 h t} x_{1}^{\prime 2}}-1\right) \frac{1}{1-i \frac{1+\varepsilon}{\alpha_{0}}\left|y_{1}\right|}\right|^{2} \\
& \left.=\frac{2}{\pi} \int d x_{2}\left|f\left(x_{2}\right)\right|^{2} \int d y_{1} \frac{1}{1+\left(\frac{1+\varepsilon}{\alpha_{0}}\right)^{2}\left|y_{1}\right|^{2}} \right\rvert\, \int d x_{1}^{\prime} g\left(x_{1}^{\prime}+x_{2}\right) e^{i\left|y_{1}\right|\left|x_{1}^{\prime}\right|} e^{i \frac{\mu}{2 h t} x_{1}^{\prime 2}} \\
& \left.\quad \int_{0}^{\infty} d v e^{-v+i \frac{(1+\varepsilon)\left|y_{1}\right|}{\alpha_{0}} v} e^{i \frac{(1+\varepsilon) M}{2 \hbar t \alpha_{0}^{2}} v^{2}+i \frac{M}{\hbar t \alpha_{0}}\left|x_{1}^{\prime}\right| v} \frac{(1+\varepsilon) \mu}{\hbar t \alpha_{0}}\left(\frac{1+\varepsilon}{\alpha_{0}} v+\left|x_{1}^{\prime}\right|\right)\right|^{2} \\
& \leq \frac{2(1+\varepsilon) \mu^{2}}{\pi \hbar^{2} t^{2} \alpha_{0}} \int d x_{2}\left|f\left(x_{2}\right)\right|^{2} \int d y \frac{1}{1+y^{2}}\left(\int d x_{1}^{\prime}\left|g\left(x_{1}^{\prime}+x_{2}\right)\right| \int_{0}^{\infty} d v e^{-v}\left(\frac{1+\varepsilon}{\alpha_{0}} v+\left|x_{1}^{\prime}\right|\right)\right)^{2} \\
& \leq \frac{2 \varepsilon^{2} m^{2}}{(1+\varepsilon) \hbar^{2} t^{2} \alpha_{0}} \int d x_{2}\left|f\left(x_{2}\right)\right|^{2}\left(\int d x_{1}^{\prime}\left|g\left(x_{1}^{\prime}+x_{2}\right)\right|\left(\frac{1+\varepsilon}{\alpha_{0}}+\left|x_{1}^{\prime}\right|\right)\right)^{2}
\end{aligned}
$$

Notice that one must assume that $\alpha$ is of order $\varepsilon^{-1}$, that is $\alpha_{0}=\alpha m$, in order to have an effect of finite order on the wave function.

Lemma 4.1.4 The following estimate holds:

$$
\begin{equation*}
\left\|\psi_{2}(t)-\psi_{a}(t)\right\| \leq C \sqrt{\varepsilon} \tag{4.25}
\end{equation*}
$$

## Proof

The two wave functions have the following expressions:

$$
\begin{aligned}
& \psi_{2}(t ; x, y)=\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{\mu}{2 \hbar t}(x-y)^{2}} \int d y^{\prime} f\left(y^{\prime}\right) V^{\nu, \tilde{\omega}}\left(t ; \frac{m y+M x}{m+M}, y^{\prime}\right) \\
& \int d x^{\prime} g\left(x^{\prime}+y^{\prime}\right)\left(e^{-i \frac{\mu}{\hbar t} x^{\prime}(x-y)}-\frac{e^{i \frac{\mu}{\hbar t}\left|x^{\prime}\right||x-y|}}{1-i \frac{\hbar}{\alpha t}|x-y|}\right)
\end{aligned}
$$

$$
\psi_{a}(t ; x, y)=\left(\frac{M}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{M}{2 \hbar t} x^{2}} \int d y^{\prime} f\left(y^{\prime}\right) V^{m, \omega}\left(t ; y, y^{\prime}\right) e^{-i \frac{M}{\hbar t} x y^{\prime}} \int d x^{\prime} g\left(x^{\prime}+y^{\prime}\right)\left(e^{-i \frac{M}{\hbar t} x^{\prime} x}-\frac{e^{i \frac{M}{\hbar t}|x|\left|x^{\prime}\right|}}{1-i \frac{M \hbar}{\alpha_{0} t}|x|}\right)
$$

If we make the change of variable

$$
\left\{\begin{array}{l}
\sqrt{\mu} x=z \\
\sqrt{\nu} y=w
\end{array}\right.
$$

we arrive at

$$
\begin{array}{r}
\psi_{2}(t ; z, w)=\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{1}{2 \hbar t}\left(z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right)^{2}} \int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) V^{1, \tilde{\omega}}\left(t ; \frac{1}{1+\varepsilon} w+\sqrt{\varepsilon} z, y^{\prime}\right) \\
\int d x^{\prime} g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right)\left(e^{-i \frac{\sqrt{\mu}}{\hbar t} x^{\prime}\left(z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right)}-\frac{e^{i \frac{\mu}{\hbar t}\left|x^{\prime}\right|\left|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right|}}{1-i \frac{\hbar}{\alpha t}\left|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right|}\right) \\
\begin{array}{r}
\psi_{a}(t ; z, w)=\left(\frac{M}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{1}{2 \hbar t} z^{2}} \int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) V^{1, \tilde{\omega}}\left(t ; w, y^{\prime}\right) e^{-i \frac{\sqrt{\varepsilon}}{(1+\varepsilon)^{2}} \frac{1}{\hbar t} z \frac{1}{\sqrt{\nu}} y^{\prime}} \\
\int d x^{\prime} g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right)\left(e^{-i \frac{\sqrt{\nu}}{\hbar t} x^{\prime} z}-\frac{e^{i \frac{\mu}{\hbar t}\left|x^{\prime}\right||z|}}{1-i \frac{\hbar}{\alpha t}|z|}\right)
\end{array}
\end{array}
$$

We shall estimate estimate separately the difference between the scattered and the free part of the wave functions. Let's consider the free parts. First we prove that the free part of $\psi_{2}$ is near to

$$
\psi^{\star}(t ; z, w)=\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{1}{2 h t} z^{2}} \int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) V^{1, \tilde{\omega}}\left(t ; w, y^{\prime}\right) \int d x^{\prime} g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right) e^{-i \frac{\sqrt{\mu}}{\hbar t} x^{\prime} z}
$$

In order to prove this statement it is sufficient to estimate these two norms:

$$
\begin{align*}
& \int d z d \omega \frac{1}{\sqrt{\mu \nu}} \left\lvert\, \int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) V^{1, \tilde{\omega}}\left(t ; \frac{1}{1+\varepsilon} w+\sqrt{\varepsilon} z, y^{\prime}\right)\right. \\
& {\left[e^{i \frac{1}{2 h t}\left(z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right)^{2}} \int d x^{\prime} g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right) e^{-i \frac{\sqrt{\mu}}{\hbar t} x^{\prime}\left(z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right)}-e^{i \frac{1}{2 h t} z^{2}} \int d x^{\prime} g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right) e^{-i \frac{\sqrt{\mu}}{\hbar t} x^{\prime} z}\right]_{(4.26}^{2} }  \tag{4.26}\\
& \int d z d \omega \frac{1}{\sqrt{\mu \nu}} \left\lvert\,\left[\int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) V^{1, \tilde{\omega}}\left(t ; \frac{1}{1+\varepsilon} w+\sqrt{\varepsilon} z, y^{\prime}\right)+\right.\right. \\
&\left.-\int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) V^{1, \tilde{\omega}}\left(t ; w, y^{\prime}\right)\right]\left.e^{i \frac{1}{2 h t} z^{2}} \int d x^{\prime} g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right) e^{-i \frac{\sqrt{\mu}}{\hbar t} x^{\prime} z}\right|^{2} \tag{4.27}
\end{align*}
$$

Using (4.5) with (4.11) and (4.12), it is straightforward to see that (4.26) and (4.27) are of order $\varepsilon$. In the same way it can be proved that the difference between $\psi^{\star}$ and $\psi^{\sharp}$ is order $\sqrt{\varepsilon}$ with

$$
\psi^{\sharp}(t ; z, w)=\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{1}{2 \hbar t} z^{2}} \int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) V^{1, \tilde{\omega}}\left(t ; w, y^{\prime}\right) e^{-i \frac{\sqrt{\varepsilon}}{(1+\varepsilon)^{2}} \frac{1}{\hbar t} z \frac{1}{\sqrt{\nu}} y^{\prime}} \int d x^{\prime} g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right) e^{-i \frac{\sqrt{\mu}}{\hbar t} x^{\prime} z}
$$

Coming back to $(x, y)$ variables we have

$$
\begin{aligned}
& \psi^{\sharp}(t ; x, y)=\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{\mu}{2 h t} x^{2}} \int d y^{\prime} f\left(y^{\prime}\right) V^{\nu, \omega}\left(t ; y, y^{\prime}\right) e^{-i \frac{M}{\hbar t} x y^{\prime}} \int d x^{\prime} g\left(x^{\prime}+y^{\prime}\right) e^{-i \frac{\mu}{\hbar t} x^{\prime} x}= \\
& =\left(\frac{1}{1+\varepsilon} \frac{M}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{1}{1+\varepsilon} \frac{M}{2 \hbar t} x^{2}}\left(\frac{m \omega}{\pi \hbar\left(1-e^{-2 i \tilde{\omega} t}\right)} \sqrt{1+\varepsilon}\right)^{\frac{1}{2}} \int d y^{\prime} f\left(y^{\prime}\right) e^{-\frac{m \omega}{2 i \hbar \tan (\tilde{\omega} t)}\left(y^{2}+y^{\prime 2}\right) \sqrt{1+\varepsilon}} \\
& e^{\frac{m \omega}{i \hbar \sin (\tilde{\omega} t)} y y^{\prime} \sqrt{1+\varepsilon}} \int d x^{\prime} g\left(x^{\prime}\right) e^{-i \frac{1}{1+\varepsilon} \frac{M}{\hbar t} x^{\prime} x}
\end{aligned}
$$

Using (4.5) with (4.11) and (4.12), one proves that $\psi^{\sharp}$ is near to $\psi^{\natural}$.

$$
\begin{gathered}
\psi^{\natural}(t ; x, y)=\left(\frac{M}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{M}{2 \hbar t} x^{2}}\left(\frac{m \omega}{\pi \hbar\left(1-e^{-2 i \tilde{\omega} t}\right)}\right)^{\frac{1}{2}} \int d y^{\prime} f\left(y^{\prime}\right) e^{-\frac{m \omega}{2 i \hbar \tan (\tilde{\omega} t)}\left(y^{2}+y^{\prime 2}\right)} e^{\left.\frac{m \omega}{i \hbar \sin (\tilde{\omega} t}\right) y y^{\prime}} \\
\int d x^{\prime} g\left(x^{\prime}\right) e^{-i \frac{M}{\hbar t} x^{\prime} x}=\left(\frac{M}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{M}{2 \hbar t} x^{2}} \int d y^{\prime} f\left(y^{\prime}\right) V^{m, \omega}\left(\frac{t}{\sqrt{1+\varepsilon}} ; y, y^{\prime}\right) \int d x^{\prime} g\left(x^{\prime}\right) e^{-i \frac{M}{\hbar t} x^{\prime} x}
\end{gathered}
$$

This term is near order $\varepsilon$ to the free part of $\psi_{a}$ by Stone's theorem.
Now we turn our attention to the scattered part of $\psi_{2}$, first we prove that it is near order $\varepsilon$ to

$$
\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} \int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) V^{1, \tilde{\omega}}\left(t ; w, y^{\prime}\right) \int d x^{\prime} g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right) \frac{e^{i \frac{\sqrt{\mu}}{\hbar t}|z|\left|x^{\prime}\right|}}{1-i \frac{\hbar}{\alpha t \sqrt{\mu}}|z|}
$$

In order to prove this statement it is sufficient to estimate these two differences:

$$
\begin{gather*}
\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} \int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) V^{1, \tilde{\omega}}\left(t ; w, y^{\prime}\right)\left[\frac{1}{1-i \frac{\hbar}{\alpha \sqrt{\mu t}}|z|} \int d x^{\prime} g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right) e^{i \frac{\sqrt{\mu}}{\hbar t}|z|\left|x^{\prime}\right|}+\right. \\
\left.-\frac{1}{1-i \frac{\hbar}{\alpha \sqrt{\mu t}}\left|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right|} \int d x^{\prime} g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right) e^{i \frac{\sqrt{\mu}}{\hbar t}\left|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right|\left|x^{\prime}\right|}\right]  \tag{4.28}\\
\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}}\left[\int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) V^{1, \tilde{\omega}}\left(t ; \frac{1}{1+\varepsilon} w+\sqrt{\varepsilon} z, y^{\prime}\right)-\int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) V^{1, \tilde{\omega}}\left(t ; w, y^{\prime}\right)\right] \\
\frac{1}{1-i \frac{\hbar}{\alpha \sqrt{\mu} t}\left|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right|} \int d x^{\prime} g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right) e^{i \frac{\sqrt{\mu}}{\hbar t}\left|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right|\left|x^{\prime}\right|} \tag{4.29}
\end{gather*}
$$

Now we introduce an auxiliary function $F_{0}\left(w, x^{\prime}\right)$.

$$
F_{0}\left(w, x^{\prime}\right)=\int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right) V^{1, \tilde{\omega}}\left(t ; \frac{1}{1+\varepsilon} w+\sqrt{\varepsilon} z, y^{\prime}\right)
$$

The following estimate holds:

$$
\begin{equation*}
\int \frac{d z d \omega}{\sqrt{\mu \nu}}\left|\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}}\left(\frac{1}{1-i \frac{\hbar}{\alpha \sqrt{\mu t}}\left|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right|}-\frac{1}{1-i \frac{\hbar}{\alpha \sqrt{\mu t}}|z|}\right) \int d x^{\prime} e^{i \frac{\sqrt{\mu}}{\hbar t}|z|\left|x^{\prime}\right|} F_{0}\left(w, x^{\prime}\right)\right|^{2} \leq c \varepsilon \tag{4.30}
\end{equation*}
$$

Using the estimate

$$
\left(\frac{\hbar}{\alpha t \sqrt{\mu}}\right)^{2} \frac{\left(\left|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right|-|z|\right)^{2}}{\left(1+\left(\frac{\hbar}{\alpha t \sqrt{\mu}}\right)^{2} z^{2}\right)\left(1+\left(\frac{\hbar}{\alpha t \sqrt{\mu}}\right)^{2}\left(z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right)^{2}\right)} \leq c\left(\frac{\hbar}{\alpha t \sqrt{\mu}}\right)^{2} \frac{w^{2}}{1+\left(\frac{\hbar}{\alpha t \sqrt{\mu}}\right)^{2} z^{2}}
$$

in order to prove (4.30) it is sufficient to prove

$$
\begin{align*}
& \left|w \int d x^{\prime} e^{i \frac{\sqrt{\mu}}{\hbar t}|z|\left|x^{\prime}\right|} F_{0}\left(w, x^{\prime}\right)\right| \leq c_{1}  \tag{4.31}\\
& \left|w^{2} \int d x^{\prime} e^{\left.i \frac{\sqrt{\mu}}{\hbar t}|z| x^{\prime} \right\rvert\,} F_{0}\left(w, x^{\prime}\right)\right| \leq c_{2} \tag{4.32}
\end{align*}
$$

Indeed, if (4.31) and (4.32) hold, then (4.30) is less or equal than

$$
\varepsilon\left(\frac{\hbar}{\alpha t \sqrt{\mu}}\right)^{2} \frac{1}{2 \pi \hbar t} \int \frac{d z d \omega}{\sqrt{\mu \nu}}\left(\frac{\hbar}{\alpha t \sqrt{\mu}}\right)^{2} \frac{1}{1+\left(\frac{\hbar}{\alpha t \sqrt{\mu}}\right)^{2} z^{2}} \frac{c_{1}+c_{2}}{(1+|w|)^{2}}=C \varepsilon^{2}
$$

Now we prove (4.31): using (4.5) with (4.11) and (4.12) it is straightforward to see that

$$
\begin{aligned}
&\left|w \int d x^{\prime} e^{i \frac{\sqrt{\mu}}{\hbar t}|z|\left|x^{\prime}\right|} F_{0}\left(w, x^{\prime}\right)\right| \leq \nu \int d x|g(x)| \int d y|y f(y)|+\frac{\hbar}{\omega} \int d x\left|g^{\prime}(x)\right| \int d y|f(y)|+ \\
&+\frac{\hbar}{\omega} \int d x|g(x)| \int d y\left|f^{\prime}(y)\right|
\end{aligned}
$$

Using (4.11) twice, it is possible to prove (4.32).
Now we prove that

$$
\begin{equation*}
\int \frac{d z d \omega}{\sqrt{\mu \nu}}\left|\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} \frac{1}{1-i \frac{\hbar}{\alpha \sqrt{\mu t}}\left|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right|} \int d x^{\prime}\left(e^{i \frac{\sqrt{\mu}}{h t}|z|\left|x^{\prime}\right|}-e^{i \frac{\sqrt{\mu}}{\hbar t}\left|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right|\left|x^{\prime}\right|}\right) F_{0}\left(w, x^{\prime}\right)\right|^{2} \leq c \varepsilon^{2} \tag{4.33}
\end{equation*}
$$

Using the inequality

$$
\left|e^{i \frac{\sqrt{\mu}}{\hbar t}|z|\left|x^{\prime}\right|}-e^{i \frac{\sqrt{\mu}}{\hbar t}\left|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon} w\right|\left|x^{\prime}\right|}\right| \leq \frac{\sqrt{\varepsilon}}{1+\varepsilon} \frac{\sqrt{\mu}}{\hbar t}|w|\left|x^{\prime}\right|
$$

in (4.33), we arrive at

$$
\begin{equation*}
\varepsilon \frac{\mu}{2 \pi \hbar t} \frac{\mu}{(\hbar t)^{2}} \int \frac{d z d w}{\sqrt{\mu \nu}} \frac{1}{1+\left(\frac{\hbar}{\alpha t \sqrt{\mu}}\right)^{2} z^{2}}\left(\int d x^{\prime}\left|w x^{\prime} F_{0}\left(w, x^{\prime}\right)\right|\right)^{2} \tag{4.34}
\end{equation*}
$$

In order to prove that (4.34) is order $\varepsilon^{2}$ it is then sufficient to prove

$$
\begin{align*}
& \left|w x^{\prime} \int d x^{\prime} e^{i \frac{\sqrt{\mu}}{\hbar t}\left|z \| x^{\prime}\right|} F_{0}\left(w, x^{\prime}\right)\right| \leq c_{3}  \tag{4.35}\\
& \left|w^{2} x^{\prime} \int d x^{\prime} e^{i \frac{\sqrt{\mu}}{\hbar t}|z|\left|x^{\prime}\right|} F_{0}\left(w, x^{\prime}\right)\right| \leq c_{4} \tag{4.36}
\end{align*}
$$

The proof of (4.35) and (4.36) is similar to the proof of (4.31) and (4.32). From (4.30) and (4.33) it follows that (4.28) is order $\varepsilon$.

In order to estimate (4.29) it is sufficient to estimate

$$
\begin{equation*}
\int \frac{d z d w}{\sqrt{\mu \nu}}\left(\frac{\mu}{2 \pi \hbar t}\right) \frac{1}{1+\left(\frac{\hbar}{\alpha \sqrt{\mu t}}\right)^{2} z^{2}}\left|\int d x^{\prime} e^{i \frac{\sqrt{\mu}}{\hbar t}|z|\left|x^{\prime}\right|}\left(F_{0}\left(w+\sqrt{\varepsilon} z, x^{\prime}\right)-F_{0}\left(w, x^{\prime}\right)\right)\right|^{2} \tag{4.37}
\end{equation*}
$$

It is convenient to divide the domain of integration of the $z$ variable in two regions and use

$$
\begin{align*}
& \int d w\left|\int d x^{\prime} e^{i \frac{\sqrt{\mu}}{\hbar t}|z|\left|x^{\prime}\right|}\left(F_{0}\left(w+\sqrt{\varepsilon} z, x^{\prime}\right)-F_{0}\left(w, x^{\prime}\right)\right)\right|^{2} \leq \\
& \varepsilon z^{2} \int d w\left(\int d x^{\prime}\left|\frac{\partial}{\partial w} F_{0}\left(w+\sqrt{\varepsilon} z, x^{\prime}\right)-F_{0}\left(w, x^{\prime}\right)\right|\right)^{2} \text { if } \sqrt{\varepsilon}|z| \leq 1  \tag{4.38}\\
& \int d w\left|\int d x^{\prime} e^{i \frac{\sqrt{\mu}}{\hbar t}|z|\left|x^{\prime}\right|}\left(F_{0}\left(w+\sqrt{\varepsilon} z, x^{\prime}\right)-F_{0}\left(w, x^{\prime}\right)\right)\right|^{2} \leq \\
& 2 \int d w\left(\int d x^{\prime}\left|F_{0}\left(w+\sqrt{\varepsilon} z, x^{\prime}\right)-F_{0}\left(w, x^{\prime}\right)\right|\right)^{2} \text { if } \sqrt{\varepsilon}|z|<1 \tag{4.39}
\end{align*}
$$

In order to prove that the quantities appearing at the r.h.s of (4.38) and (4.39) are finite, one must proceed as in the estimate of (4.30) It is now straightforward to prove that the integral
over $z$ is order $\varepsilon$ and conclude the estimate of (4.29). Now we introduce a second auxiliary function

$$
F_{\varepsilon}\left(w, z, x^{\prime}\right)=\int d y^{\prime} f\left(\frac{1}{\sqrt{\nu}} y^{\prime}\right) g\left(x^{\prime}+\frac{1}{\sqrt{\nu}} y^{\prime}\right) e^{-i \frac{\sqrt{\varepsilon}}{1+\varepsilon} \frac{1}{\hbar t} y^{\prime} z} V^{1, \tilde{\omega}}\left(t ; \frac{1}{1+\varepsilon} w+\sqrt{\varepsilon} z, y^{\prime}\right)
$$

Using (4.5) with (4.11) and (4.12), it is straightforward to prove that:

$$
\begin{equation*}
\int \frac{d z d w}{\sqrt{\mu \nu}}\left(\frac{\mu}{2 \pi \hbar t}\right) \frac{1}{1+\left(\frac{\hbar}{\alpha \sqrt{\mu t}}\right)^{2} z^{2}}\left|\int d x^{\prime} e^{i \frac{\sqrt{\mu}}{\hbar t}|z|\left|x^{\prime}\right|}\left(F_{\varepsilon}\left(w, z, x^{\prime}\right)-F_{0}\left(w, x^{\prime}\right)\right)\right|^{2} \leq c \varepsilon \tag{4.40}
\end{equation*}
$$

Coming back to $(x, y)$ variables, so far we have proved that the scattered part of $\psi_{2}$ is near to

$$
\psi_{a}^{b}(t ; x, y)=\left(\frac{\mu}{2 \pi \hbar i t}\right)^{\frac{1}{2}} e^{i \frac{\mu}{2 \hbar t} x^{2}} \int d y^{\prime} f\left(y^{\prime}\right) V^{\nu, \tilde{\omega}}\left(t ; y, y^{\prime}\right) e^{-i \frac{\mu}{\hbar t} x y^{\prime}} \int d x^{\prime} g\left(x^{\prime}+y^{\prime}\right) \frac{e^{i \frac{\mu}{\hbar t}|x|\left|x^{\prime}\right|}}{1-i \frac{\mu \hbar}{\alpha_{0} t}|x|}
$$

Repeating the arguments used at the end of the estimate of the free part, we prove that $\psi^{b}$ is near order $\varepsilon$ to the free part of $\psi_{a}$ and this concludes the proof.

## Proof of theorem 4.1.1

It is an obvious consequence of lemmas 4.1.2, 4.1.3, 4.1.4.
The result of theorem 4.1.1 can be more conveniently rephrased in terms of reduced density matrix for the heavy particle, which is defined by the integral operator $\hat{\rho}(t)$ in $L^{2}(\mathbb{R})$ given by the kernel

$$
\begin{equation*}
\hat{\rho}\left(t ; y, y^{\prime}\right)=\int d x \psi(t ; x, y) \bar{\psi}\left(t ; x, y^{\prime}\right) \tag{4.41}
\end{equation*}
$$

We also introduce the integral operator $\hat{\rho}_{a}(t)$ defined by

$$
\begin{align*}
\hat{\rho}_{a}\left(t ; y, y^{\prime}\right) & =\int d x \psi_{a}(t ; x, y) \overline{\psi_{a}}\left(t ; x, y^{\prime}\right) \\
& =\int d z f(z) V^{m, \omega}(t ; y, z) \int d z^{\prime} \bar{f}\left(z^{\prime}\right) \overline{V^{m, \omega}}\left(t ; y^{\prime}, z^{\prime}\right) \mathcal{I}\left(z, z^{\prime}\right) \tag{4.42}
\end{align*}
$$

where

$$
\begin{equation*}
\mathcal{I}\left(z, z^{\prime}\right) \equiv \int d k\left(W_{+}^{\alpha_{0}, z} g_{\delta}\right)(k) \overline{\left(W_{+}^{\alpha_{0}, z^{\prime}} g_{\delta}\right)}(k)=\left(\left(\Omega_{+}^{\alpha_{0}, z^{\prime}}\right)^{-1} g_{\delta},\left(\Omega_{+}^{\alpha_{0}, z}\right)^{-1} g_{\delta}\right) \tag{4.43}
\end{equation*}
$$

Notice that, from (4.42),(4.43) one has

$$
\begin{equation*}
\hat{\rho}_{a}(t)=V^{m, \omega}(t) \hat{\rho}^{a}(0) V^{m, \omega}(-t) \tag{4.44}
\end{equation*}
$$

where $\hat{\rho}_{0}^{a}$ is defined by the integral kernel

$$
\begin{equation*}
\hat{\rho}^{a}\left(0 ; z, z^{\prime}\right)=f(z) \bar{f}\left(z^{\prime}\right) \mathcal{I}\left(z, z^{\prime}\right) \tag{4.45}
\end{equation*}
$$

It is easily seen that $\mathcal{I}\left(z, z^{\prime}\right)=\overline{\mathcal{I}}\left(z^{\prime}, z\right),\left|\mathcal{I}\left(z, z^{\prime}\right)\right| \leq 1$ and the equality holds only if $z=z^{\prime}$.
Then $\hat{\rho}_{a}(0)$ is a self-adjoint and trace-class operator, with $\operatorname{Tr}\left(\hat{\rho}_{a}(0)\right)=1$; it is also positive since

$$
\begin{align*}
& \left(h, \hat{\rho}_{a}(0) h\right)=\int d z \bar{h}(z) \int d z^{\prime} h\left(z^{\prime}\right) f(z) \bar{f}\left(z^{\prime}\right) \int d k\left(W^{\alpha_{0}, z} g\right)(k) \overline{\left(W^{\alpha_{0}, z^{\prime}} g\right)}(k) \\
& =\int d k\left|\int d z \bar{h}(z) f(y)\left(W^{\alpha_{0}, z} g\right)(k)\right|^{2} \tag{4.46}
\end{align*}
$$

Moreover we have

$$
\begin{equation*}
\operatorname{Tr}\left(\left(\hat{\rho}_{a}(0)\right)^{2}\right)=\int d z d z^{\prime}|f(z)|^{2}\left|f_{\sigma}\left(z^{\prime}\right)\right|^{2}\left|\mathcal{I}\left(z, z^{\prime}\right)\right|^{2}<1 \tag{4.47}
\end{equation*}
$$

We conclude that $\hat{\rho}_{a}(0)$ and its harmonic evolution $\hat{\rho}_{a}(t)$ are density matrices describing mixture states and, by theorem (4.1.1),

$$
\begin{equation*}
\operatorname{Tr}\left(\left|\hat{\rho}(t)-\hat{\rho}_{a}(t)\right|\right)<C \varepsilon^{\frac{1}{2}} \tag{4.48}
\end{equation*}
$$

This means that in our asymptotic regime the motion of the heavy particle is a free evolution. On the other hand the presence of the light particle has a relevant effect, since it produces a transition of the initial state of the heavy particle from $\hat{\rho}_{0}\left(z, z^{\prime}\right)=f(z) \bar{f}\left(z^{\prime}\right)$ to $\hat{\rho}_{a}\left(0 ; z, z^{\prime}\right)$.
We shall see in the next section that this is the origin of the decoherence effect on the heavy particle.

### 4.2 Decoherence Effect

Here we discuss an application of formula (4.42),(4.43) to a concrete example of quantum evolution and we give an explicit computation of the decoherence effect.

We shall make further assumptions on the initial state: take $h_{1}, h_{2} \in C_{0}^{\infty}(-1,+1)$ such that $\|f\|=\|g\|=1$; for later use it is convenient to choose $h_{2}$ even. We put

$$
\begin{gather*}
f(y)=\frac{1}{\sqrt{\sigma}}\left[f^{+}(y)+f^{-}(y)\right] \quad f^{ \pm}(y)=h_{1}\left(\frac{y \pm y_{0}}{\sigma}\right) e^{i \frac{p_{0}}{\hbar} y}  \tag{4.49}\\
g(x)=\frac{1}{\sqrt{\delta}} h_{2}\left(\frac{x-x_{0}}{\delta}\right) e^{i \frac{q_{0}}{\hbar} x} \tag{4.50}
\end{gather*}
$$

The heavy particle is assumed to be in a typical superposition state of two spatially separated wave packets, one localized in $y=y_{0}$ with mean value of the momentum $p_{0}$ and the other localized in $y=-y_{0}$ with mean value of the momentum $-p_{0}$.
The parameters $\sigma, \delta$ denote the spreading in position at time zero for the heavy and the light particle respectively.
We define $\beta=\alpha_{0} \delta$ and $\delta^{ \pm}=\left( \pm y_{0}-\sigma, \pm y_{0}+\sigma\right)$.
We shall make the following assumptions on the physical parameters of the system.

$$
\begin{equation*}
r_{0} \in\left(-R_{0}, R_{0}\right), \quad \frac{\delta}{R_{0}-\left|r_{0}\right|} \ll 1, \quad \frac{\sigma}{\delta} \ll 1 \tag{4.51}
\end{equation*}
$$

i.e. we consider the light particle localized in the region between the two wave packets describing the heavy particle and well separated from each wave packet. Moreover the spreading in position of the wave packets is much smaller than the spreading of the light particle. Notice that (4.51) obviously imply $\frac{\sigma}{R_{0}} \ll 1$.
Using assumptions (4.51) we can give an estimate of the basic object $\mathcal{I}\left(z, z^{\prime}\right)$ for $z \in \Delta^{ \pm}$and $z^{\prime} \in \Delta^{ \pm}$and then we can find a more suitable expression for the reduced density matrix of the heavy particle.

Lemma 4.2.1 Assume (4.51). Then

$$
\begin{align*}
& \sup _{z, z^{\prime} \in \Delta^{ \pm}}\left|\mathcal{I}\left(z, z^{\prime}\right)-1\right|<C_{4}\left(\frac{\sigma}{\delta}+\frac{\delta}{R_{0}-\left|r_{0}\right|}\right)  \tag{4.52}\\
& \sup _{z \in \Delta^{+}, z^{\prime} \in \Delta^{-}}\left|\mathcal{I}\left(z, z^{\prime}\right)-\Lambda\right|=\sup _{z \in \Delta^{-}, z^{\prime} \in \Delta^{+}}\left|\mathcal{I}\left(z, z^{\prime}\right)-\bar{\Lambda}\right|<C_{5} \frac{\delta}{R_{0}-\left|r_{0}\right|} \tag{4.53}
\end{align*}
$$

where

$$
\begin{align*}
& \Lambda=\int d k\left|\tilde{g}_{\delta}(k)\right|^{2} \mathcal{T}_{\alpha_{0}}(k)=\int d k\left|\tilde{g}\left(k-k_{0} \delta\right)\right|^{2} \mathcal{T}_{\beta}(k)  \tag{4.54}\\
& k_{0}=\frac{q_{0}}{\hbar}  \tag{4.55}\\
& \mathcal{T}_{\gamma}(k)=-\frac{i k}{\gamma-i k} \quad \gamma>0 \tag{4.56}
\end{align*}
$$

and $C_{4}>0, C_{5}>0$ are explicitly given as functions of $\beta, g$.

## Proof

First we note that for $z \in \Delta^{-}$

$$
\begin{equation*}
\frac{1}{\sqrt{\delta}}\left(W_{+}^{\alpha_{0}, z} g_{\delta}\right)\left(\frac{k}{\delta}\right)=e^{i\left(k_{0} \delta-k\right) \frac{x_{0}}{\delta}} \tilde{g}\left(k-k_{0} \delta\right)+\mathcal{R}_{\beta}(k) e^{i\left(k_{0} \delta+|k|\right) \frac{x_{0}}{\delta}-i(k+|k|) \frac{z}{\delta}} \tilde{g}\left(|k|+k_{0} \delta\right) \tag{4.57}
\end{equation*}
$$

and for $z \in \Delta^{+}$

$$
\begin{equation*}
\frac{1}{\sqrt{\delta}}\left(W_{+}^{\alpha_{0}, z} g_{\delta}\right)\left(\frac{k}{\delta}\right)=e^{i\left(k_{0} \delta-k\right) \frac{x_{0}}{\delta}} \tilde{g}\left(k-k_{0} \delta\right)+\mathcal{R}_{\beta}(k) e^{i\left(k_{0} \delta-|k|\right) \frac{x_{0}}{\delta}-i(k-|k|) \frac{z}{\delta}} \tilde{g}\left(|k|-k_{0} \delta\right) \tag{4.58}
\end{equation*}
$$

where we have used the fact that

$$
\begin{equation*}
\tilde{g}_{\delta}(k)=\sqrt{\delta} \tilde{g}\left(k \delta-k_{0} \delta\right) e^{-i\left(k-k_{0}\right) r_{0}} \tag{4.59}
\end{equation*}
$$

Then for $z, z^{\prime} \in \Delta^{-}$we have

$$
\begin{align*}
& \mathcal{I}\left(z, z^{\prime}\right)=1+\int d k\left|\tilde{g}\left(|k|+k_{0} \delta\right)\right|^{2}\left|\mathcal{R}_{\beta}(k)\right|^{2} e^{-i(k+|k|) \frac{z-z^{\prime}}{\delta}} \\
& +\int d k \overline{\tilde{g}}\left(k-k_{0} \delta\right) \tilde{g}\left(|k|+k_{0} \delta\right) \mathcal{R}_{\beta}(k) e^{i(k+|k|) \frac{r_{0}-z}{\delta}} \\
& +\int d k \overline{\tilde{g}}\left(|k|+k_{0} \delta\right) \tilde{g}\left(k-k_{0} \delta\right) \overline{\mathcal{R}_{\beta}}(k) e^{-i(k+|k|) \frac{r_{0}-z^{\prime}}{\delta}} \\
& =1+\int_{0}^{\infty} d k\left|\tilde{g}\left(k+k_{0} \delta\right)\right|^{2}\left|\mathcal{R}_{\beta}(k)\right|^{2}\left(e^{-2 i k \frac{z-z^{\prime}}{\delta}}-1\right) \\
& +\int_{0}^{\infty} d k \overline{\tilde{g}}\left(k-k_{0} \delta\right) \tilde{g}\left(k+k_{0} \delta\right) \mathcal{R}_{\beta}(k) e^{2 i k \frac{x_{0}-z}{\delta}} \\
& +\int_{0}^{\infty} d k \overline{\tilde{g}}\left(k+k_{0} \delta\right) \tilde{g}\left(k-k_{0} \delta\right) \overline{\mathcal{R}_{\beta}}(k) e^{-2 i k \frac{x_{0}-z^{\prime}}{\delta}} \\
& \equiv 1+a_{1}+a_{2}+a_{3} \tag{4.60}
\end{align*}
$$

where we have used the identity $\mathcal{R}_{\beta}+\overline{\mathcal{R}_{\beta}}+2\left|\mathcal{R}_{\beta}\right|^{2}=0$ and the fact that $\tilde{g}$ is even.
Using (4.51) we easily estimate $a_{1}$

$$
\begin{align*}
& \left|a_{1}\right| \leq 2 \frac{\left|z-z^{\prime}\right|}{\delta} \int_{0}^{\infty} d k\left|\tilde{g}\left(k+k_{0} \delta\right)\right|^{2} k\left|\mathcal{R}_{\beta}(k)\right|^{2} \leq 4 \frac{\sigma}{\delta} \int_{0}^{\infty} d k\left|\tilde{g}\left(k+k_{0} \delta\right)\right|^{2} k\left|\mathcal{R}_{\beta}(k)\right|^{2} \\
& \leq 2 \beta\|g\|^{2} \frac{\sigma}{\delta} \tag{4.61}
\end{align*}
$$

For the estimate of $a_{2}$ it is convenient to integrate by parts

$$
\begin{align*}
& \left|a_{2}\right|=\left|\frac{1}{2 i} \frac{\delta}{r_{0}-z} \int_{0}^{\infty} d k \overline{\tilde{g}}\left(k-k_{0} \delta\right) \tilde{g}\left(k+k_{0} \delta\right) \mathcal{R}_{\beta}(k) \frac{d}{d k} e^{2 i k \frac{r_{0}-z}{\delta}}\right| \\
& =\frac{\delta}{2\left|r_{0}-z\right|}\left|\int_{0}^{\infty} d k \frac{d}{d k}\left(\overline{\tilde{g}}\left(k-k_{0} \delta\right) \tilde{g}\left(k+k_{0} \delta\right) \mathcal{R}_{\beta}(k)\right) e^{2 i k \frac{r_{0}-y}{\delta}}-\left|\tilde{g}\left(k_{0} \delta\right)\right|^{2}\right| \\
& \leq \frac{\delta}{R_{0}-\left|r_{0}\right|}\left[\frac{1}{2 \pi}\left(\int d r|g(r)|\right)^{2}+\frac{1}{\beta} \int_{0}^{\infty} d k\left|\overline{\tilde{g}}\left(k-k_{0} \delta\right) \tilde{g}\left(k+k_{0} \delta\right)\right|\right. \\
& \left.+\int_{0}^{\infty} d k\left|\overline{\tilde{g}^{\prime}}\left(k-k_{0} \delta\right) \tilde{g}\left(k+k_{0} \delta\right)\right|+\int_{0}^{\infty} d k\left|\overline{\tilde{g}}\left(k-k_{0} \delta\right) \tilde{g}^{\prime}\left(k+k_{0} \delta\right)\right|\right] \\
& \leq \frac{\delta}{R_{0}-\left|r_{0}\right|}\left[\frac{1}{2 \pi}\left(\int d r|g(r)|\right)^{2}+\frac{1}{\beta}\|g\|^{2}+2\left\|\tilde{g}^{\prime}\right\|\|g\|\right] \tag{4.62}
\end{align*}
$$

The term $a_{3}$ is analyzed exactly in the same way and then we get the estimate (4.52) for $z, z^{\prime} \in \Delta^{-}$. Since in the case $z, z^{\prime} \in \Delta^{+}$the computation is similar we conclude that (4.52) holds.

In order to prove (4.53) we consider the case $z \in \Delta^{+}$and $z^{\prime} \in \Delta^{-}$(the case $z \in \Delta^{-}$and $z^{\prime} \in \Delta^{+}$ can be treated exactly in the same way) and we obtain

$$
\begin{align*}
& \mathcal{I}\left(z, z^{\prime}\right)=1+\int d k \overline{\tilde{g}}\left(|k|+k_{0} \delta\right) \tilde{g}\left(|k|-k_{0} \delta\right)\left|\mathcal{R}_{\beta}(k)\right|^{2} e^{-2 i|k| \frac{r_{0}}{\delta}+i(|k|-k) \frac{z}{\delta}+i(|k|+k) \frac{z^{\prime}}{\delta}} \\
& +\int d k \overline{\tilde{g}}\left(k-k_{0} \delta\right) \tilde{g}\left(|k|-k_{0} \delta\right) \mathcal{R}_{\beta}(k) e^{-i(|k|-k) \frac{r_{0}-z}{\delta}} \\
& +\int d k \overline{\tilde{g}}\left(|k|+k_{0} \delta\right) \tilde{g}\left(k-k_{0} \delta\right) \overline{\mathcal{R}_{\beta}}(k) e^{-i(|k|+k) \frac{r_{0}-z^{\prime}}{\delta}} \\
& =1+\int_{0}^{\infty} d k\left(\left|\tilde{g}\left(k-k_{0} \delta\right)\right|^{2} \mathcal{R}_{\beta}(k)+\left|\tilde{g}\left(k+k_{0} \delta\right)\right|^{2} \overline{\mathcal{R}_{\beta}}(k)\right) \\
& +\int_{0}^{\infty} d k \overline{\tilde{g}}\left(k+k_{0} \delta\right) \tilde{g}\left(k-k_{0} \delta\right)\left|\mathcal{R}_{\beta}(k)\right|^{2} e^{-2 i k \frac{r_{0}-z^{\prime}}{\delta}} \\
& +\int_{0}^{\infty} d k \overline{\tilde{g}}\left(k+k_{0} \delta\right) \tilde{g}\left(k-k_{0} \delta\right) \overline{\mathcal{R}_{\beta}}(k) e^{-2 i k \frac{r_{0}-z^{\prime}}{\delta}} \\
& +\int_{0}^{\infty} d k \overline{\tilde{g}}\left(k+k_{0} \delta\right) \tilde{g}\left(k-k_{0} \delta\right)\left|\mathcal{R}_{\beta}(k)\right|^{2} e^{-2 i k \frac{r_{0}-z}{\delta}} \\
& +\int_{0}^{\infty} d k \overline{\tilde{g}}\left(k+k_{0} \delta\right) \tilde{g}\left(k-k_{0} \delta\right) \mathcal{R}_{\beta}(k) e^{-2 i k \frac{r_{0}-z}{\delta}} \tag{4.63}
\end{align*}
$$

The estimate of the last four terms of (4.63) proceeds exactly as the estimate of $a_{2}$ in (4.62). On the other hand

$$
\begin{align*}
& 1+\int_{0}^{\infty} d k\left(\left|\tilde{g}\left(k-k_{0} \delta\right)\right|^{2} \mathcal{R}_{\beta}(k)+\left|\tilde{g}\left(k+k_{0} \delta\right)\right|^{2} \overline{\mathcal{R}_{\beta}}(k)\right) \\
& =\int d k\left|\tilde{g}\left(k-k_{0} \delta\right)\right|^{2}\left(\frac{-i k}{\beta-i k}\right) \tag{4.64}
\end{align*}
$$

and this concludes the proof of the proposition.

Proposition (4.2.1) allows us to find a further approximate form for the reduced density matrix.

Theorem 4.2.2 Under the assumptions (4.51) and for any $t \geq 0$ we have

$$
\begin{equation*}
\left[\operatorname{Tr}\left(\left(\hat{\rho}^{a}(t)-\hat{\rho}^{f}(t)\right)^{2}\right)\right]^{1 / 2}<C_{6}\left(\frac{\sigma}{\delta}+\frac{\delta}{R_{0}-\left|r_{0}\right|}\right) \tag{4.65}
\end{equation*}
$$

where

$$
\begin{align*}
& \hat{\rho}^{f}(t)=V^{m, \omega}(t) \hat{\rho}_{0}^{f} V^{m, \omega}(-t)  \tag{4.66}\\
& \hat{\rho}_{0}^{f}\left(z, z^{\prime}\right)=\frac{1}{2} f^{+}(z) \overline{f^{+}}\left(z^{\prime}\right)+\frac{1}{2} f^{-}(z) \overline{f^{-}}\left(z^{\prime}\right)+\frac{\Lambda}{2} f^{+}(z) \overline{f^{-}}\left(z^{\prime}\right)+\frac{\bar{\Lambda}}{2} f^{-}(z) \overline{f^{+}}\left(z^{\prime}\right) \tag{4.67}
\end{align*}
$$

and $C_{6}>0$ is explicitely given as function of $\beta, g$.

## Proof

$$
\begin{align*}
& \operatorname{Tr}\left(\left(\hat{\rho}^{a}(t)-\hat{\rho}^{f}(t)\right)^{2}\right)=\operatorname{Tr}\left(\left(\hat{\rho}_{0}^{a}-\hat{\rho}_{0}^{f}\right)^{2}\right) \\
& \left.=\frac{1}{4} \int d y d z^{\prime} \right\rvert\, f^{+}(z) \overline{f^{+}}\left(z^{\prime}\right)\left(\mathcal{I}\left(z, z^{\prime}\right)-1\right)+f^{-}(z) \overline{f^{-}}\left(z^{\prime}\right)\left(\mathcal{I}\left(z, z^{\prime}\right)-1\right) \\
& +f^{+}(z) \overline{f^{-}}\left(z^{\prime}\right)\left(\mathcal{I}\left(z, z^{\prime}\right)-\Lambda\right)+\left.f^{-}(z) \overline{f^{+}}\left(z^{\prime}\right)\left(\mathcal{I}\left(z, z^{\prime}\right)-\bar{\Lambda}\right)\right|^{2} \\
& \leq \sup _{z, z^{\prime} \in \Delta^{+}}\left|\mathcal{I}\left(z, z^{\prime}\right)-1\right|^{2}+\sup _{z, z^{\prime} \in \Delta^{-}}\left|\mathcal{I}\left(z, z^{\prime}\right)-1\right|^{2} \\
& +\sup _{z \in \Delta^{+}, z^{\prime} \in \Delta^{-}}\left|\mathcal{I}\left(z, z^{\prime}\right)-\Lambda\right|^{2}+\sup _{z \in \Delta^{-}, z^{\prime} \in \Delta^{+}}\left|\mathcal{I}\left(z, z^{\prime}\right)-\bar{\Lambda}\right|^{2} \tag{4.68}
\end{align*}
$$

Using lemma (4.2.1) we conclude the proof.

From theorems (4.2.2) and (4.1.1) we conclude that the reduced density matrix for the heavy particle in the position representation can be approximated by the density matrix

$$
\begin{align*}
& \hat{\rho}^{f}\left(t, y, y^{\prime}\right)=\frac{1}{2}\left(V^{m, \omega}(t) f^{+}\right)(y)\left(V^{m, \omega}(-t) \overline{f^{+}}\right)\left(y^{\prime}\right)+\frac{1}{2}\left(V^{m, \omega}(t) f^{-}\right)(y)\left(V^{m, \omega}(-t) \overline{f^{-}}\right)\left(R^{\prime}\right) \\
& +\frac{\Lambda}{2}\left(V^{m, \omega}(t) f^{+}\right)(y)\left(V^{m, \omega}(-t) \overline{f^{-}}\right)\left(y^{\prime}\right)+\frac{\bar{\Lambda}}{2}\left(V^{m, \omega}(t) f^{-}\right)(y)\left(V^{m, \omega}(-t) \overline{f^{+}}\right)\left(y^{\prime}\right) \tag{4.69}
\end{align*}
$$

with an explicit control of the error.
If the interaction with the light particle is switched off, i.e. for $\alpha_{0}=0$, we have $\Lambda=1$ and then (4.69) reduces to the pure state corresponding to the coherent superposition of the free evolution of the two wave packets $f^{ \pm}$.
On the other hand, if $\alpha_{0}>0$ one easily sees that $0<|\Lambda|<1$ and then (4.69) is a mixed state for which the interference terms are reduced by the factor $\Lambda$ and this is the typical manifestation of the (partial) decoherence effect induced by the light particle on the heavy one.
The relevant parameter $\Lambda$ (see (4.56)) is defined in terms of the probability distribution of the momentum of the light particle at time zero $\left|g_{\delta}(k)\right|^{2}$ and of the transmission coefficient $\mathcal{T}_{\alpha_{0}}(k)$
associated to the hamiltonian $H_{\alpha_{0}, 0}$ defined by the laplacian perturbed by a point interaction of strength $\alpha_{0}$ placed at the origin,(see e.g. [AGH-KH]).
Then the decoherence effect is emphasized if the fraction of transmitted wave for the light particle is small.
In fact for $k_{0} \gg \alpha_{0}$ the light particle is completely transmitted and $\Lambda \rightarrow 1$, i.e. there is no decoherence effect.
On the other hand, for $k_{0} \ll \alpha_{0}$ the decoherence effect is present since $\Lambda \rightarrow \int d k|\tilde{g}(k)|^{2}\left|\mathcal{T}_{\beta}(k)\right|^{2}$. In such case one can say that the decoherence effect is proportional to the reflection probability for a particle described by the hamiltonian $H_{\beta, 0}$ which at time zero is in the state $g$.

## Appendix A

## The Green's function of the free system

Here we derive the expression of the integral kernel of the resolvent of $H_{0}$ used in the paper; let $d=1,2,3$ be the dimension of the space

$$
\begin{align*}
& G^{\lambda}\left(x, y ; x^{\prime}, y^{\prime}\right)=\left(H_{0}+\lambda\right)^{-1}\left(x, y ; x^{\prime} y^{\prime}\right)=\frac{1}{(2 \pi)^{\frac{d}{2}}} \frac{1}{\hbar \omega} \frac{1}{\pi^{\frac{d N}{2}}}\left(\frac{m \omega}{\hbar}\right)^{\frac{d N}{2}}\left(\frac{M \omega}{\hbar}\right)^{\frac{d}{2}} \\
& \quad \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+\frac{d N}{2}-1} \frac{1}{\left(1-\nu^{2}\right)^{\frac{d N}{2}}} \frac{1}{\left(\ln \frac{1}{\nu}\right)^{\frac{d}{2}}} e^{-\frac{1}{2} \frac{1-\nu}{1+\nu} \frac{m \omega}{\hbar}\left(y^{2}+y^{\prime 2}\right)} e^{-\frac{M \omega}{\hbar} \frac{1}{2 \ln \frac{1}{\nu}}\left(x-x^{\prime}\right)^{2}} e^{-\frac{m \omega}{\hbar} \frac{\nu}{1-\nu^{2}}\left(y-y^{\prime}\right)^{2}} \tag{A.1}
\end{align*}
$$

Starting from the Mehler kernel and performing the Laplace transform with respect to time one finds the Green's function for the harmonic oscillator in $\mathbb{R}^{n}$.

$$
G_{\mathrm{osc}}^{\lambda}\left(y, y^{\prime}\right)=\left(H_{\mathrm{osc}}+\lambda\right)^{-1}\left(y, y^{\prime}\right)=\int_{0}^{+\infty} d t \frac{e^{-\frac{\lambda}{\hbar} t}}{\hbar}\left(\frac{m \omega}{\pi \hbar\left(1-e^{-2 \omega t}\right)}\right)^{\frac{n}{2}} e^{-\frac{m \omega}{\hbar} \frac{y^{2}+y^{\prime 2}}{2 \tanh (\omega t)}+\frac{m \omega}{\hbar} \frac{y \cdot y^{\prime}}{\sinh (\omega t)}} e^{-t \frac{n \omega}{2}}
$$

With the change of variable $\nu=e^{-\omega t}$ one obtains:

$$
\begin{align*}
G_{\mathrm{osc}}^{\lambda}\left(y, y^{\prime}\right)=\left(H_{\mathrm{osc}}+\lambda\right)^{-1}\left(y, y^{\prime}\right)=\left(\frac{m \omega}{\hbar}\right)^{\frac{n}{2}} & \frac{1}{\hbar \omega} \frac{1}{\pi^{\frac{n}{2}}} e^{-\frac{1}{2} \frac{m \omega}{\hbar}\left(y^{2}+y^{\prime 2}\right)} \\
& \int_{0}^{1} d \nu \nu^{\frac{\lambda}{\hbar \omega}+\frac{n}{2}-1} \frac{1}{\left(1-\nu^{2}\right)^{\frac{n}{2}}} e^{\frac{m \omega}{\hbar} \frac{2 \nu y \cdot y^{\prime}-\nu^{2}\left(y^{2}+y^{\prime 2}\right)}{\left(1-\nu^{2}\right)}} \tag{A.2}
\end{align*}
$$

Note that formula A. 2 can be also derived using Hermite functions (see e.g. [BC]). To compute $\left(H_{0}+\lambda\right)^{-1}$, we solve the following equation with respect to $f$ for a given $g$.

$$
\left[\left(-\frac{\hbar^{2}}{2 M} \Delta_{x}-\frac{\hbar^{2}}{2 m} \Delta_{y}+\frac{1}{2} m \omega^{2} y^{2}+\lambda\right) f\right](x, y)=g(x, y)
$$

We have put $y^{0}=0$ for simplicity. Taking the Fourier transform with respect to the $x$ variable we find

$$
\begin{aligned}
\mathcal{F} g(k, y) & =\int d y^{\prime} G_{\text {osc }}^{\lambda+\frac{\hbar^{2} k^{2}}{2 M}}\left(y, y^{\prime}\right) \mathcal{F} f\left(k, y^{\prime}\right) \\
& =\frac{1}{(2 \pi)^{\frac{d}{2}}} \int d x^{\prime} d y^{\prime} e^{-\imath k \cdot x^{\prime}} G_{\text {osc }}^{\lambda+\frac{\hbar^{2} k^{2}}{2 M}}\left(y, y^{\prime}\right) f\left(x^{\prime}, y^{\prime}\right)
\end{aligned}
$$

and then

$$
\left(H_{0}+\lambda\right)^{-1}\left(x, y ; x^{\prime}, y^{\prime}\right)=\frac{1}{(2 \pi)^{d}} \int d k e^{\imath k \cdot\left(x-x^{\prime}\right)} G_{\mathrm{osc}}^{\lambda+\frac{\hbar^{2} k^{2}}{2 M}}\left(y, y^{\prime}\right)
$$

Substituting (A.2) in this equation we finally get (A.1).

## Appendix B

## Unitary Group for the Harmonic Oscillator perturbed by a Point Interaction

In this appendix we give an integral representation for the unitary group of the one dimensional hamiltonian

$$
H_{\alpha}=H_{0}+\alpha \delta(y)=-\frac{\hbar^{2}}{2 m} \frac{d^{2}}{d y^{2}}+\frac{1}{2} m \omega^{2} y^{2}+\alpha \delta(y)
$$

In [FI] it was proved that

$$
\begin{equation*}
\left(H_{\alpha}+\lambda\right)^{-1}\left(y, y^{\prime}\right)=G^{\lambda}\left(y, y^{\prime}\right)-\frac{1}{\frac{1}{\alpha}+G^{\lambda}(0,0)} G^{\lambda}(y, 0) G^{\lambda}\left(y^{\prime}, 0\right) \tag{B.1}
\end{equation*}
$$

In the previous formula $G^{\lambda}\left(y, y^{\prime}\right)$ is the integral kernel of $\left(H_{0}+\lambda\right)^{-1}$. Equation (B.1) is equivalent to:

$$
\begin{gathered}
\mathscr{D}\left(H_{\alpha}\right)=\left\{u \in L^{2}(\mathbb{R}) \text { s.t. } u=\varphi^{\lambda}+G^{\lambda} q, \varphi^{\lambda} \in \mathscr{D}\left(H_{0}\right), \alpha u(0)=-q\right\} \\
\left(H_{\alpha}+\lambda\right) u=\left(H_{0}+\lambda\right) \varphi^{\lambda}
\end{gathered}
$$

The domain of $H_{\alpha}$ can be written also in the equivalent way:

$$
\begin{align*}
& \mathscr{D}\left(H_{\alpha}\right)=\left\{u \in L^{2}(\mathbb{R}) \text { s.t. } x^{2} u \in L^{2}(\mathbb{R}), u \in H^{2}\left(\mathbb{R}^{+}\right) \cap H^{2}\left(\mathbb{R}^{-}\right) \cap H^{1}(\mathbb{R})\right. \\
&\left.\frac{\hbar^{2}}{2 m}\left(u^{\prime}\left(0^{+}\right)-u^{\prime}\left(0^{-}\right)\right)=\alpha u(0)\right\} \tag{B.2}
\end{align*}
$$

Now we shall prove that, for a regular initial datum $\psi_{0} \in \mathscr{D}\left(H_{\alpha}\right)$,

$$
\begin{equation*}
\psi(t ; y)=\left(e^{-i \frac{t}{\hbar} H_{0}} \psi_{0}\right)(y)+\frac{i}{\hbar} \int_{0}^{t} d s e^{-i \frac{t-s}{\hbar} H_{0}}(y, 0) q(s) \tag{B.3}
\end{equation*}
$$

is the solution of the time dependent Schrodinger equation.
The function $q(t)$ is solution of

$$
\begin{equation*}
q(t)+\alpha \frac{i}{\hbar} \int_{0}^{t} d s\left[\frac{m \omega}{\pi \hbar\left(1-e^{-2 i \omega(t-s)}\right)}\right]^{\frac{1}{2}} q(s)+\alpha\left(e^{-i \frac{t}{\hbar} H_{0}} \psi_{0}\right)(0)=0 \tag{B.4}
\end{equation*}
$$

If we project equation (B.3) on the harmonic oscillator eigenvalue basis we obtain

$$
\begin{equation*}
\psi(t ; n)=e^{-i n \omega t} \psi_{0}(n) \frac{i}{\hbar} \int_{0}^{t-s} d s e^{-i n \omega(t-s)} q(s) \tag{B.5}
\end{equation*}
$$

Integrating by parts we get

$$
\begin{array}{r}
\psi(t ; n)=e^{-i n \omega t} \varphi_{0}^{\lambda}(n)+\frac{i}{\hbar} \frac{\lambda}{n \hbar \omega+\lambda} \int_{0}^{t} d s e^{-i n \omega(t-s)} q(s)-\frac{1}{n \hbar \omega+\lambda} \int_{0}^{t} d s e^{-i n \omega(t-s)} \stackrel{\circ}{q}(s)+ \\
+\frac{1}{n \hbar \omega+\lambda} q(t) \tag{B.6}
\end{array}
$$

If $\psi_{0}$ is a regular function then the first three terms on the r.h.s.of (B.6) belong to $\mathscr{D}\left(H_{0}\right)$ and then they have continuous first derivative; the only discontuities of the first derivative of $\psi(t ; y)$ come from the last term of (B.6) which is proportional to $G^{\lambda}(y, 0)$.
It is easy to show that $\frac{d}{d y} G^{\lambda}(y, 0) \simeq-\frac{m}{\hbar^{2}} \operatorname{sgm}(y)$ near the origin. Then using (B.2), $\psi(t)$ belongs to $\mathscr{D}\left(H_{\alpha}\right)$ if and only if (B.4) holds. Taking the time derivative of (B.3), and using (B.6) one shows that $i \hbar \frac{d \psi}{d t}(t)=H_{\alpha} \psi(t)$ holds.

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