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Permutation Statistics and Polynomials on Coxeter Groups

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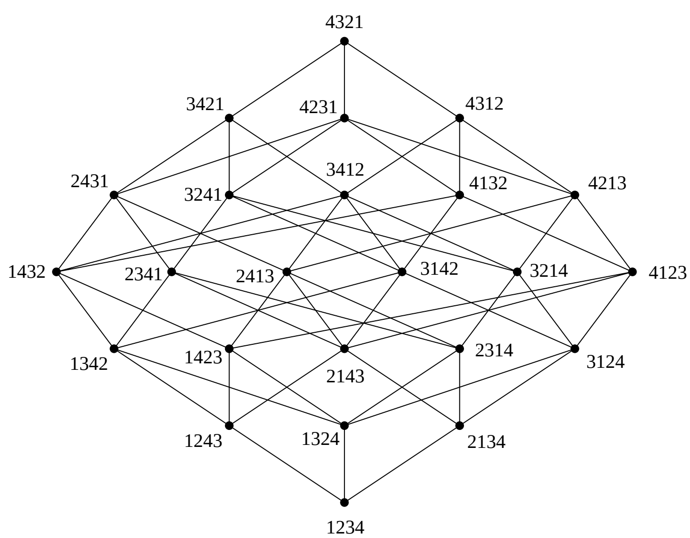

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Riccardo Biagioli

Introduction

The birth of permutation statistics is traditionally attributed to Euler [**29**]. However, it was not until MacMahon's extensive study [**45**] that it became an established discipline of mathematics, and it still took a long time before it developed into the vast field that it is today. In the last thirty years much progress has been made, both in discovering and studying new statistics, and in extending these to arbitrary words with repeated letters and to others groups.

Coxeter groups are a class of groups, defined by Coxeter [**26**] in a certain way by generators and relations, that arise in a multitude of ways in several areas of mathematics such as algebra, geometry, singularity theory as well as in physics. Moreover, their definition allows them to be handled in a combinatorial way. This approach is often referred to as "combinatorics of Coxeter groups", and is one of the prominent parts of what is usually called "algebraic combinatorics". Without any doubt, the most fundamental example of a Coxeter group is the symmetric group S_n. We frequently return to S_n in order to illustrate various general concepts and constructions.

MacMahon considered four different statistics for a permutation σ in S_n. The number of descents $des(\sigma)$, the number of excedances $exc(\sigma)$, the number of inversions $inv(\sigma)$ and the major index $maj(\sigma)$. Given a permutation $\sigma = \sigma_1 \cdots \sigma_n$, we say that $(i, j) \in [n] \times [n]$ is an inversion of σ if $i < j$ and $\sigma_i > \sigma_j$, that $i \in [n-1]$ is a descent if $\sigma_i > \sigma_{i+1}$ and that $i \in [n]$ is a excedance if $\sigma_i > i$. The major index is the sum of all the descents. MacMahon showed, algebraically, that exc is *equidistributed* with des and that inv is equidistributed with maj. Since then any statistic equidistributed with des is said to be *Eulerian*, and any statistic equidistributed with inv is said to be *Mahonian*. Note that most of the permutation statistics found in the literature fall into one of these two categories, and curiously new Mahonian statistics appear much more frequently then Eulerian ones.

From the Coxeter group point of view, these statistics have some fundamental algebraic interpretations. The number of inversions $inv(\sigma)$ is the length of σ, namely the minimum length of an expression of σ in terms of generators, denoted by $\ell(\sigma)$, whereas $des(\sigma)$ is the number of generators s_i such that $\ell(\sigma s_i) < \ell(\sigma)$. Moreover, the symmetric group naturally acts on the polynomial ring in different ways and many mathematicians have studied the rings of the invariants. Surprisingly some of the statistics previously defined, such as major index, a priori completely disconnected from this field, appear with an important role in the representation theory of these actions. Of course these considerations can be made also for other Coxeter groups.

In the first part of this thesis we study the interplay between the combinatorial and algebraic aspects of these statistics on Coxeter groups. In particular we define several new Mahonian and Eulerian statistics on the Coxeter groups of type B and D extending many of the existing ones on the symmetric group. Note that the problem of extending the concept of major index to B_n was open for many years and despite the fact that several "major index" statistics have been defined for B_n no generalization of MacMahon's result has been found until the recent paper [**3**]. First of all we analyze the combinatorial properties of these new statistics and prove some important combinatorial identities that they satisfy, and then we look at their meaning in representation theory. We focus our attention mostly on type D and see that the results obtained extend nicely those for the symmetric group.

In Chapter 1 we briefly give some basic preliminaries about Coxeter groups that are needed in the rest of the work. Moreover, we describe with particular attention the principal source of reference and examples, namely the symmetric group.

The flavour of Chapter 2 is mostly combinatorial. After giving some combinatorial descriptions of the Coxeter groups of types B and D we introduce three new statistics on D_n, $dmaj$, $ddes$ and $fmaj_D$. We show that two of these are Mahonian and that a pair of them gives a generalization to D_n of the well known Carlitz's identity on the Euler-Mahonian distribution of the descent number and major index over S_n [**19**], solving a problem first posed by Foata [**2**].
The results in this chapter are part of paper [**8**] that will appear in Advances in Applied Mathematics.

In Chapter 3 we study the natural and tensor action of W and W^t, respectively, on the polynomial ring $\mathbf{C}[x_1, \dots, x_n]^{\otimes t}$. Denote by DIA and TIA the corresponding invariant algebras. Let $\mathcal{Z}_W(\bar{q})$ be the quotient of the Hilbert series of DIA and TIA. We show that if $W = D_n$ then the series $\mathcal{Z}_{D_n}(\bar{q})$ is actually a polynomial with nonnegative integer coefficients, which admits a nice expression in terms of a new "major index" $Dmaj$. This statistic is Mahonian and is an analogue of major index but, surprisingly, is different from both $dmaj$ and $fmaj_D$. A similar result holds also for S_n and B_n where the polynomial $\mathcal{Z}_W(\bar{q})$ is given by an explicit formula using the major index and the flag-major index (see [**37**], [**3**]). Our proof allows us to give a new and more direct proof also of the result for B_n. Moreover we introduce an Eulerian statistic $Ddes$ that together with $Dmaj$ gives a second generalization of Carlitz's identity.

In Chapter 4 we study the interplay between representation theory of the Coxeter groups of type D and the statistics $Dmaj$ and $Ddes$. The set of elements in a Weyl group having a fixed descent set carries a natural representation of the group, called a descent representation. Descent representations of Weyl groups were first introduced by Solomon (see [**50**]). We construct an analogue of descent representations for the Coxeter groups of type D using the coinvariant algebra as a representation space. The construction of a basis for the coinvariant algebra is important for many applications, and has been approached from different viewpoints. Garsia and Stanton presented a descent basis for a finite dimensional quotient of the Stanley-Reisner ring arising from a finite Weyl group (see [**38**]). For type A, unlike for other types, this quotient is isomorphic to the coinvariant algebra and in this case the basis elements are monomials, indexed by $\sigma \in S_n$, of degree $maj(\sigma)$. Moreover the coinvariant algebra has a natural grading induced from that of $\mathbf{P}_n$ by total degree, and we denote by R_k^W its k-th homogeneous component. In the case of S_n, a well known theorem due to Kraskiewicz-Weyman expresses the multiplicity of the irreducible S_n-representations in R_k^W in terms of the statistic maj. Unfortunately, the Garsia-Stanton approach does not work for other Weyl

groups. For type B these problems are solved by Adin, Brenti and Roichman in the recent paper [**1**]. They provide a descent basis of $R^*(B)$ and an extension of the construction of Solomon's descent representations (see [**50**]) for this type. Here we show how to extend these results to the Weyl groups of type D. We construct an analogue of the descent basis for the coinvariant algebra of type D via a Straightening Lemma. The basis elements, indexed by $\gamma \in D_n$, are monomials of degree $Dmaj(\gamma)$. This new basis lead to the definition of a family of D_n-modules R_{S_1,S_2}, whose elements are even-signed permutations having S_1 and S_2 as "descent set" and "negative set", respectively, and for this reason we call them negative-descent representations. They are analogous but different from Solomon descent representations and Kazhdan-Lusztig representations (see [**43**]). We decompose R_k^W into a direct sums of these R_{S_1,S_2}. Finally, we introduce the concept of D-standard Young bitableaux and by extending the definition of $Dmaj$ on them we give an explicit decomposition into irreducibles of these negative-descent representations, refining a theorem of Stembridge [**54**]. This algebraic setting is then applied to obtain new multivariate combinatorial identities.
The results in Chapters 3 and 4 have been obtained jointly with Fabrizio Caselli.

The second part of this thesis is devoted to the study of a family of polynomials which are generating functions for the dimensions of the Ext-groups between generalized Verma modules. There is one set of generalized Verma modules for each pair (g, p), up to conjugacy, consisting of a semisimple Lie algebra g and a parabolic subalgebra p. If W is the Weyl group of g and $W(m)$ is the Weyl group of a Levi subalgebra of $p = m \oplus u$, then the set of generalized Verma modules attached to (g, p) is indexed by the coset space W^m.

The problem of computing the u-cohomology of irreducible highest weight modules is completely solved by the Kazhdan-Lusztig conjectures which have been proved by Brylinski and Kashiwara in [**17**]. Note that Vogan showed that these conjectures are equivalent to the formula

$$P_{x,y}(q) = \sum_{k \geq 0} q^k \dim_{\mathbf{C}}(\mathrm{Ext}^{\ell(y)-\ell(x)-2k}(M_x, L_y)),$$

where for any $x, y \in W$ with $x \leq y$, $P_{x,y}(q)$ is the Kazdhan-Lusztig polynomial of the pair x, y, and for any $x \in W$ M_x is the Verma module associated to x and L_x is its unique irreducible quotient.

On the other hand, the "Extension Problem", namely the problem of computing the u-cohomology of the (g, p)-generalized Verma modules, is equivalent to compute the dimensions of spaces of extensions between generalized Verma modules. It is as yet unsolved in general, and there are not even conjectures for the form of the answer. If g and p form an indecomposable Hermitian symmetric pair then Shelton [**48**] proves a recursion formula for the polynomials defined by

$$E_{u,v}(q) = \sum_{k \geq 0} (-1)^{\ell(v)-\ell(u)-k} q^k \dim_{\mathbf{C}}(\mathrm{Ext}^k(N_u, N_v)),$$

where N_u and N_v are the generalized Verma modules attached to (g, p) corresponding to u and v in W^m.

In Chapter 5 we solve these recursion relations, in the case when (g, p) corresponds to a classical Hermitian symmetric pair. There are five possibilities for (g, p) when g is classic and our main results are explicit product formulas for the

E-polynomials. Moreover these formulas can be stated in two different ways, one in terms of Weyl group elements and one in terms of partitions. Finally, the formulas imply that the E-polynomials are combinatorial invariants, i.e. that they depend only on the interval defined by u and v in W^m, as an abstract poset.

The material in this chapter is part of the paper [**7**] accepted for publication in Transactions of the American Mathematical Society.

CHAPTER 1

Coxeter groups

Coxeter groups are a class of groups defined by generators and relations. They arise in several fields of mathematics. In this chapter we restrict our attention of the definitions, notation and results that are needed in the rest of this work.

1.1. Definition

We let $\mathbf{P} := \{1,2,3,\ldots\}$, $\mathbf{N} := \mathbf{P} \cup \{0\}$, and $\mathbf{Z}$ be the set of integers; for $a \in \mathbf{N}$ we let $[a] := \{1, 2, \ldots, a\}$ (where $[0] := \emptyset$) and $[\pm n] := [-n, n] \setminus \{0\}$. The cardinality of a set A will be denoted by $|A|$.

Let S be a set. A matrix $m : S \times S \to \mathbf{P} \cup \{\infty\}$ is called a *Coxeter matrix* if it satisfies

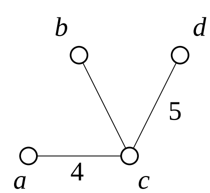
$$m(s,t) = m(t,s), \text{for all} s,t \in S;$$

$$m(s,t) = 1 \Leftrightarrow s = t.$$

Equivalently, m can be represented by a *Coxeter graph* whose node set is S and whose edges are the unordered pairs s, t such that $m(s,t) \geq 3$. The edges with $m(s,t) \geq 4$ are labeled by that numbers.
For example let $S := \{a, b, c, d\}$ with Coxeter matrix

$$m := \begin{pmatrix} 1 & 2 & 4 & 2 \\ 2 & 1 & 3 & 2 \\ 4 & 3 & 1 & 5 \\ 2 & 2 & 5 & 1 \end{pmatrix}$$

then the Coxeter diagram is



A Coxeter matrix m determines a group W with the presentation

$$(1.1) \qquad \begin{cases} \text{Generators:} & S \\ \text{Relations:} & (st)^{m(s,t)} = e, \text{ for all } s,t \in S \ : \ m(s,t) \neq \infty, \end{cases}$$

where "e" denote the identity element of the group. Since $m(s,s) = 1$ we have that

$$s^2 = e, \quad \text{for all } s \in S,$$

and so S is a set of involutions. In particular, $m(s,t) = 2$ if and only if s and t commute; this in the Coxeter graph means that s and t are not joined by an edge.

If a group W has a presentation such as (1.1) then the pair (W,S) is called a *Coxeter system*. The group W is the *Coxeter group*. The cardinality of the set of *Coxeter generators* S is called the *rank* of (W,S). We will investigate only on groups with finite rank. The system is *irreducible* if its Coxeter graph is connected.

The following two statements are equivalent and make explicit what it means for W to be determined by m via the presentation (1.1):

1. If G is a group and $f : G \to G$ is a map such that
$$(f(s)f(t))^{m(s,t)} = e,$$
for all $s,t \in S$ with $m(s,t) \neq \infty$, then there is a unique extension of f to a group homomorphism $f : W \to G$.
2. W is the quotient F_S/N, where F_S is the free group generated by S and N is the normal subgroup generated by $\{(st)^{m(s,t)} \;:\; s,t \in S,\ m(s,t) \neq \infty\}$.

The finite irreducible Coxeter groups have been classified (see e.g., [**42**]). We are mostly interested in the Coxeter groups of type A, B and D: we analyze them in detail in this and next chapter, see Table 1 at the end of this chapter.

1.2. Length function

In this section we define one of the most important statistics on Coxeter groups, the length.

Let (W,S) be a Coxeter group. Each element $w \in W$ can be written as a product of generators $s_i \in S$,
$$w = s_1 s_2 \cdots s_r.$$
The *length* of w is
$$\ell(w) := \min\{r \in \mathbf{N} : w = s_1 \ldots s_r \text{ for some } s_1, \ldots, s_r \in S\}. \tag{1.2}$$
If $r = \ell(w)$ then the word $s_1 \cdots s_r$ is called a *reduced expression* for w. Here some basic facts of the length function.
For all $u, w \in W$

i) $\ell(w) = 1 \Leftrightarrow w \in S$;
ii) $\ell(w) = \ell(w^{-1})$;
iii) $\ell(uw) \equiv \ell(u) + \ell(w)$;
iv) $\ell(sw) = \ell(w) \pm 1, for\ \ s \in S$.

Now we are able to define the polynomial
$$W(q) := \sum_{w \in W} q^{\ell(w)},$$
called the *Poincaré polynomial* of W. It has homological and geometrical interpretations and has also a nice factorization as we will see in more detail later.

We let
$$T := \{wsw^{-1} \;:\; s \in S,\ w \in W\}. \tag{1.3}$$
The elements of T are called *reflections*. The definition shows that $S \subseteq T$, and the elements of S are usually called *simple reflections*. We make the following definitions:
$$T_L(w) := \{t \in T \;:\; \ell(tw) < \ell(w)\},$$

and

$$T_R(w) := \{t \in T \ : \ \ell(wt) < \ell(w)\}.$$

$T_L(w)$ is called the set of *left associated reflections* to w, and similarly for $T_R(w)$. We have the following characterization of the length function

$$|T_L(w)| = \ell(w).$$

Now we introduce two of the basic objects of this thesis, the descent sets. We will study these issues in more details in the rest of the work. We define the *left descent set* of w by

$$D_L(w) := T_L(w) \cap S,$$

and, similarly, the *right descent set* of w by,

$$D_R(w) := T_R(w) \cap S. \tag{1.4}$$

The next is one of the fundamental concept in the combinatorial theory of Coxeter groups. We say that a Coxeter system has the Exchange Property if it's satisfies as follows.

Exchange Property. Let $w = s_1 \cdots s_r$ be a reduced expression and $s \in S$. If $\ell(sw) < \ell(w)$ then $sw = s_1 \cdots \hat{s_i} \cdots s_r$ for some $i \in [r]$.

THEOREM 1.1. *Let W be a group and S a set of generators of order* 2. *Then the following are equivalent:*

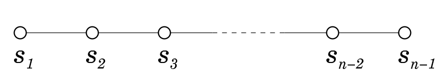
i) (W, S) *is a Coxeter group;*
ii) (W, S) *has the Exchange Property.*

1.3. Symmetric group

In this section we analyze the symmetric group from a combinatorial point of view. We give some notation and definitions that we use in the rest of the work and finally we give a combinatorial proof of the known fact that the symmetric group is a Coxeter group, following the setting of the book **[10]**.

S_n is the set of all bijections $\sigma : [n] \to [n]$. If $\sigma \in S_n$ then we write $\sigma = \sigma_1 \ldots \sigma_n$ to mean that $\sigma(i) = \sigma_i$, for $i = 1, \ldots, n$ and we call this the *complete notation* of σ. If $\sigma \in S_n$ then we may also write σ in *disjoint cycle form* and we will usually omit to write the 1-cycles of σ.
For example, if $\sigma = 64175823$ then we also write $\sigma = (2, 4, 7)(1, 6, 8, 3)$. Given $\sigma, \tau \in S_n$ we let $\sigma\tau := \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$.

As a set of generators for S_n we take $S = \{s_1, \ldots, s_{n-1}\}$ where $s_i := (i, i+1)$, for $i = 1, \ldots, n-1$. The effect of multiplying an element $\sigma \in S_n$ by the transposition s_i on the right is that of interchanging the places of $\sigma(i)$ and $\sigma(i+1)$, and on the left is that of interchanging i and $i+1$ in the complete notation of σ.
For example, let $\sigma = 3765214$ then $\sigma(3, 4) = 3756214$ and $(3, 4)\sigma = 4765213$.



The Coxeter graph of S_n

Now we wonder who is the length function for S_n. The reply is given by the following well known result. We need to introduce the following "statistic" on S_n (and in general for each $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbf{Z}^n$). The *inversion number* of σ is

$$inv(\sigma) := |\{(i,j) \in [n] \times [n] \;:\; i < j,\; \sigma(i) > \sigma(j)\}|.$$

Note that

$$inv(\sigma s_i) = \begin{cases} inv(\sigma) + 1, & \text{if } \sigma(i) < \sigma(i+1), \\ inv(\sigma) - 1, & \text{if } \sigma(i) > \sigma(i+1). \end{cases} \tag{1.5}$$

PROPOSITION 1.2. *Let $\sigma \in S_n$. Then*

$$\ell(\sigma) = inv(\sigma).$$

PROOF. Since $inv(e) = \ell(e) = 0$, then (1.5) implies that $inv(\sigma) \leq \ell(\sigma)$. Conversely, we prove the other inequality by induction on $inv(\sigma)$. If $inv(\sigma) = 0$ then $\sigma = e$ and the thesis holds. So let $\sigma \in S_n$ such that $inv(\sigma) = k > 0$. Then there exists $s_i \in S$ such that $inv(\sigma s_i) = k - 1$ (otherwise would imply $\sigma = e$). Hence by induction $inv(\sigma) = inv(\sigma s_i) + 1 \geq \ell(\sigma s_i) + 1 \geq \ell(\sigma)$ and we are done. $\square$

This proposition can be used to obtain direct proofs the following facts.

The Poincaré polynomial for the S_{n+1}, admits a nice expression, namely

$$S_{n+1}(q) = \sum_{\sigma \in S_{n+1}} q^{\ell(\sigma)} = \prod_{i=1}^{n} [i+1]_q, \tag{1.6}$$

where $[i]_q := 1 + q + \cdots + q^{i-1}$.

PROPOSITION 1.3. *Let $\sigma \in S_n$. Then*

$$D_R(\sigma) := \{s_i \in S \;:\; \sigma(i) > \sigma(i+1)\}.$$

The set of the reflections defined in (1.3) takes the following form

$$T = \{(i,j) \in [n] \times [n] \;:\; 1 \leq i < j \leq n\};$$

Its elements are called *transpositions*.

There are several way to prove the following well known theorem. We present this one as in [**10**].

THEOREM 1.4. *(S_n, S) is a Coxeter system of type A_{n-1}.*

PROOF. We show that the pair (S_n, S) has the Exchange Property, and so the thesis follows by Theorem 1.1. The relations $s_i s_j = s_j s_i$ for $|i - j| \geq 2$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ imply that the type is A_{n-1}.
Let $i, i_1, \dots, i_p \in [n-1]$ and suppose that

$$\ell(s_{i_1} \cdots s_{i_p} s_i) < \ell(s_{i_1} \cdots s_{i_p}). \tag{1.7}$$

We want to show that there exists a $j \in [p]$ such that

$$s_{i_1} \cdots s_{i_p} s_i = s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_p}.$$

Let $\sigma := s_{i_1} \cdots s_{i_p}$, $b := \sigma(i)$ and $a := \sigma(i+1)$. By Proposition 1.2 we know that (1.7) means that $b > a$. Hence, there exists $j \in [p]$ such that a is to the left of b in the complete notation of $s_{i_1} \cdots s_{i_{j-1}}$ but a is to the right of b in the complete notation of $s_{i_1} \cdots s_{i_j}$. Hence, the complete notation of $s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_p}$ is the same of $s_{i_1} \cdots s_{i_p}$ except that a and b are interchanged and this implies the thesis. $\square$

1.4. Bruhat order

Here we introduce this partial order structure, that has a remarkable role in the algebraic and combinatorial theory of Coxeter groups. We start with some poset notation and terminology following the book of Stanley [**51**].

1.4.1. Poset notation. Given a poset $(P, \leq)$ and $u, v \in P$ we let $[u, v] := \{z \in P : u \leq z \leq v\}$ and call this an *interval* of P. We say that u and v are comparable if $u \leq v$ or $v \leq u$. A sequence $(u_1, \ldots, u_n) \in P^n$ such that $u_1 \leq \cdots \leq u_n$ is a *chain*. If P has a minimal element, denoted $\hat{0}$, then we call a subset of the form $[\hat{0}, u]$, for $u \in P$, a *lower interval* of P. We say that v *covers* u, denoted by $u \lhd v$ if $|[u, v]| = 2$. The *Hasse graph* of P is the graph having P as vertex set and $\{\{u, v\} \ : \ u \lhd v \text{ or } v \lhd u\}$ as set of edges. Given any $Q \subseteq P$ we will always consider Q as a poset with the partial ordering induced by P and call Q a *subposet* of P. We say that $z \in P$ is *join-irreducible* if it covers at most one element of P. Given two posets P and Q we write $P \cong Q$ to mean that they are isomorphic as posets.

1.4.2. Definition. Let (W, S) be a Coxeter system and T its set of reflections defined in (1.3). We give the following definitions. Let $u, v \in W$. Then

i) $u \xrightarrow{t} v$ means that $u^{-1}v \in T$ and $\ell(u) < \ell(v)$;
ii) $u \to v$ means that $u \xrightarrow{t} v$ for some $t \in T$;
iii) $u \leq v$ means that there exist $v_i \in W$ such that $u = v_1 \to v_2 \to \cdots \to v_k = v$.

The *Bruhat graph* is the directed graph whose nodes are the elements of W and whose edges are given by *ii)*. Note that the Bruhat graph is acyclic.
Bruhat order is the partial order relation on the set W defined by *iii)*.
The following observations are immediate:

i) $u < v \Rightarrow \ell(u) < \ell(v)$;
ii) if $t \in T$, then $u < ut \Leftrightarrow \ell(u) < \ell(ut)$;
iii) e is the minimum of W.

1.4.3. Quotients. Let (W, S) be a Coxeter system. For any $J \subseteq S$ let W_J be the *parabolic* subgroup of W generated by the set J, and

$$W^J := \{w \in W : \ell(sw) > \ell(w) \text{ for all } s \in J\}.$$

Note that W^J is the system of the minimal right coset representatives of W_J. Note that $W^{\emptyset} = W$. Moreover it's immediate to see that an element w belongs to W^J if and only if no reduced expressions for w begins with a letter from J.

The *quotient* W^J, is a poset and it's partially ordered by Bruhat order. If W is finite w_0^J and w_J^0 denote the unique maximal elements in W^J and W_J respectively. Given $u, v \in W^J$, $u \leq v$, we let

$$[u, v]^J := \{z \in W^J : \ u \leq z \leq v\},$$

and consider $[u, v]^J$ as a poset with the partial ordering induced by W^J.

The preceding construction can of course be mirrored. There is a complete system

$${}^J W := \{w \in W \ : \ \ell(ws) > \ell(w) \text{ for all } s \in J\} = (W^J)^{-1}$$

of minimal length representatives of left cosets wW_J. Furthermore, an element w belongs to ${}^JW$ if and only if no reduced expressions for w ends with a letter from J.

The following is well known (see, e.g., [**10**] or [**42**]).

PROPOSITION 1.5. *Let $J \subseteq S$. Then:*

i) Every $w \in W$ has a unique factorization $w = {}^Jw w_J$ such that ${}^Jw \in {}^JW$ and $w_J \in W_J$.
ii) For this factorization $\ell(w) = \ell({}^Jw) + \ell(w_J)$.

For example, in the case of the symmetric group, the parabolic subgroups are often called *Young subgroups* and the right maximal quotient corresponding to $J := S \setminus \{s_i\}$ has this explicit form

$${}^JS_n = \{\sigma \in S_n \ : \ \sigma(1) < \sigma(2) < \ldots \sigma(k) \text{ and } \sigma(k+1) < \sigma(2) < \ldots \sigma(n)\}.$$

We close this section by giving a well known result on the Poincaré polynomial. We've already seen in the case of the symmetric group that it admits the nice formula (1.6). Is something similar true in general ? Surprisingly, the answer is yes, and it is given by the following theorem (see e.g., [**42**]).

THEOREM 1.6. *Let (W, S) be a finite irreducible Coxeter system, and $n = |S|$. Then there exist positive integers $e_1, \ldots, e_n$ such that*

$$W(q) = \prod_{i=1}^{n} [e_i + 1]_q.$$

In particular, $|W| = \prod_{i=1}^{n}(e_i + 1)$ and $|T| = \ell(w_0) = \sum_{i=1}^{n} e_i$.

The integers $e_1, \ldots, e_n$ are called the *exponents* of (W, S) (see Table 1).

Type	Order	$\|T\|$	Exponents
$A_n, \ \ (n \geq 1)$	$(n+1)!$	$\binom{n+1}{2}$	$1, 2, \ldots, n$
$B_n, \ \ (n \geq 2)$	$2^n n!$	n^2	$1, 3, 5, \ldots, 2n-1$
$D_n, \ \ (n \geq 4)$	$2^{n-1}n!$	$n^2 - n$	$1, 3, \ldots, 2n-3, n-1$

TABLE 1. Classical Coxeter groups

1.5. Partitions

Partitions are combinatorial objects that have several relations with Coxeter groups. We use them often in this work, and now we recall some basic notation.

A *partition* λ of a nonnegative integer n is an infinite sequence $(\lambda_1, \lambda_2, \ldots)$ of nonnegative integers with finitely many terms different from 0, where $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_{i=1}^{\infty} \lambda_i = n$. The sum $\sum_i \lambda_i = n$ is called the *size* of λ and we write $\lambda \vdash n$ or $|\lambda| = n$. The number of parts of λ (i.e., the number of non-zero λ_i) is the *length* of λ, denoted $\ell(\lambda)$. The set of all partition is denoted by $\mathcal{P}$ and its subset with $\ell(\lambda) \leq n$ by $\mathcal{P}(n)$. A partition λ is said to be *strict* if $\lambda_1 > \lambda_2 > \ldots$.We denote by $\mathcal{P}_S$ the set of all (integer) strict partitions and $\mathcal{P}_S(n)$ the subset of all the strict partitions with $\ell(\lambda) \leq n$.

Let $\lambda \vdash n$, then we draw a left-justified array of n boxes with λ_i boxes in the i^{th} row and call it the *Young diagram* of λ.

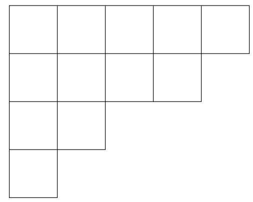


FIGURE 1. The Young diagram of $\lambda = (5, 4, 2, 1)$

The *conjugate partition* $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ of λ is defined by the condition that the Young diagram of λ' is obtained flipping the Young diagram of λ over its main diagonal (from upper left to lower right); equivalently λ'_i is the number of parts of λ that are $\geq i$, for all $i \geq 1$. Note that $\lambda'_1 = \ell(\lambda)$.

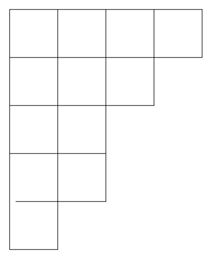


FIGURE 2. The Young diagram of λ'

Given $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}$ we let $d(\lambda)$ be the *length of the Durfee square* of λ,

$$d(\lambda) := \max \{i \in [k] : \lambda_i \geq i\}. \tag{1.8}$$

For any $\mu, \lambda \in \mathcal{P}$ we define $\mu \subseteq \lambda$ if and only if $\mu_i \leq \lambda_i$ for all i. If we identify a partition with its Young diagram, then the partial order $\subseteq$ is given simply by containment of diagrams. It's well known, and not hard to see, that this makes $\mathcal{P}$ into a lattice, usually called *Young's latice* (see e.g., [**52**, §7.2]).

The *dominance order* is a partial order defined on the set of partitions of a fixed nonnegative integer n as follows. If μ and λ are partitions of size n, then we define $\mu \trianglelefteq \lambda$ if

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i$$

for all $i \geq 1$.

CHAPTER 2

Enumerative aspects of Coxeter groups

In this chapter we look at some of basic enumerative aspects of Coxeter groups. Every element possess several significant numerical attributes, called "statistics"; we have already seen, in the previous chapter, the most important example, the length. Here we are interested only in the combinatorial properties of these statistics, without considering their algebraic meaning that will be the main argument of the next two chapters.

We begin by considering the symmetric group. We give some definitions and basic results on permutation statistics on S_n and B_n. Then we introduce and study three new statistics, $dmaj$, $ddes$ and $fmaj_D$ on the even-signed permutation group D_n. We show that $dmaj$ and $fmaj_D$ are Mahonian, and that the pair $(ddes, dmaj)$ gives a generalization of Carlitz's identity.
The results in §2.3 are news, and are part of the paper [**7**] that will appear in Advances in Applied Mathematics.

2.1. Symmetric group

This is a preliminary section in which we collect some basic and well known results on the combinatorics of the symmetric group. This field was influenced mostly by Dominique Foata, and the reader can give a look to [**21, 30, 31, 33**].

2.1.1. Mahonian and Eulerian Statistics. For $\sigma \in S_n$, and in general for any $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbf{Z}^n$, we define a "combinatorial" *descent set* by

$$Des(\sigma) := \{i \in [n-1] \ : \ \sigma(i) > \sigma(i+1)\}.$$

Note that for S_n the map $i \mapsto s_i$ is a bijection between $Des(\sigma)$ and the right descent set $D_R(\sigma)$ defined in (1.4).
As from this object we define the *descent number* of σ by

$$des(\sigma) := |Des(\sigma)|,$$

and the *major index*, first introduced by MacMahon (see [**45**]) by

$$maj(\sigma) := \sum_{i \in Des(\sigma)} i.$$

For example if $\sigma = 325461 \in S_6$ then $Des(\sigma) = \{1, 3, 5\}$, $des(\sigma) = 3$ and $maj(\sigma) = 9$.

The number of permutations in S_n with k descents is denoted $A(n, k+1)$

$$A(n, k+1) = |\{\sigma \in S_n \ : \ des(\sigma) = k\}|.$$

The numbers $\{A(n, k+1) \ : \ n \geq 1, k+1 \in [n]\}$ are called "Eulerian numbers", and it's easy to see that they satisfy the recurrence

$$A(n, k+1) = (k+1)A(n-1, k-1) + (n-k)A(n-1, k),$$

with the initial condition

$$A(1,k+1) := \begin{cases} 1, & \text{for } k=0, \\ 0 & \text{otherwise.} \end{cases}$$

The polynomials

$$A_n(x) := \sum_{\sigma \in S_n} x^{1+des(\sigma)} = \sum_{k=0}^{n-1} A(n,k+1)x^{k+1}$$

are the so-called "Eulerian polynomials" that do not have closed forms for themselves, but do have a nice generating function given by

$$1 + \sum_{n \geq 1} \frac{A_n(x)u^n}{n!} = \frac{1}{1 - \sum_{n \geq 1} \frac{(x-1)^{n-1}u^n}{n!}}.$$

Any permutation statistic whose distribution on S_n is given by n-th Eulerian polynomial $A_n(x)$ is said to be *Eulerian.* One important example of Eulerian statistic is $exc(\sigma)$, the *number of excendances* of a permutation $\sigma \in S_n$

$$exc(\sigma) := |\{i \in [n] \ : \ \sigma(i) > i\}|.$$

The following result is well known.

THEOREM 2.1 (MacMahon). *Let $n \in \mathbf{P}$. Then*

$$\sum_{\sigma \in S_n} q^{maj(\sigma)} = \sum_{\sigma \in S_n} q^{\ell(\sigma)}.$$

It follows that any statistic equidistributed with the length is said to be *Mahonian.* A first proof of the result appeared in [**45**], and other proofs can be found in [**49**] and [**51**]. Here we present the most famous due to Foata, who provide an explicit bijection $\varphi : S_n \longrightarrow S_n$ with the following two properties:

1. $maj(\sigma) = inv(\varphi(\sigma))$;
2. $ides(\sigma) = ides(\varphi(\sigma))$.

where $ides(\sigma) := des(\sigma^{-1})$.
We describe the algorithm to construct the bijection φ. Let $\sigma = \sigma(1)\cdots\sigma(n)$ be a permutation.

i) Define $w_1 := \sigma(1)$; assume that w_k has been defined for some k with $k \in [n-1]$; then
ii) if the last letter of w_k is greater (resp smaller) than $\sigma(k+1)$, split w_k after each letter greater (resp. smaller) than $\sigma(k+1)$; then
iii) in each compartment of w_k determined by the split move the last letter to the beginning; for obtaining w_{k+1} put $\sigma(k+1)$ at the end of the transformed word; replace k by $k+1$;
iv) if $k = n$, then $\varphi(\sigma) = w_k$ if not return to *ii)*.

For example, the image under φ of $\sigma = 749261583$ is obtained as follows:

$$\begin{array}{lcllllllll}
w_1 & = & 7| & & & & & & & \\
w_2 & = & 7| & 4| & & & & & & \\
w_3 & = & 7| & 4| & 9| & & & & & \\
w_4 & = & 7 & 4| & 9 & 2| & & & & \\
w_5 & = & 4| & 7| & 2| & 9| & 6| & & & \\
w_6 & = & 4| & 7 & 2| & 9 & 6 & 1| & & \\
w_7 & = & 4| & 2| & 7| & 1| & 9 & 6| & 5| & \\
w_8 & = & 4| & 2 & 7| & 1 & 6| & 9| & 5| & 8|
\end{array}$$

and $\varphi(\sigma) = w_9 = 472619583$.

2.1.2. Joint-Distribution. It is also interesting the study of pairs of statistics, usually an Eulerian one and a Mahonian one and equidistribution of such "bistatistics".

The most natural joint equidisribution is given by (des, ℓ). There is a simple recursive rule to compute the generating function

$$W(t;q) := \sum_{w \in W} t^{des(w)} q^{\ell(w)},$$

for any Coxeter group W.

THEOREM 2.2. *Let (W, S) be a Coxeter system. Then*

$$W(t;q) = \sum_{J \subseteq S} t^{|J|}(1-t)^{|S \setminus J|} \frac{W(q)}{W_{S\setminus J}(q)}.$$

A proof can be found in [**10**]. Sometimes it's possible to find also simple expression without using the previous theorem. For example, for the symmetric group, we have the following result due to Stanley [**51**]:

$$\sum_{n \geq 0} S_n(t;q) \frac{x^n}{[n]_q!} = \frac{(1-t)exp(x(1-t);q)}{1 - t \cdot exp(x(1-t);q)}$$

in $\mathbf{Z}(q)[t])[[x]]$, where

$$exp(x;q) := \sum_{n \geq 0} \frac{x^n}{[n]_q!},$$

and $S_0(t;q) := S_1(t;q) := 1$.
In the case of S_n there exist a number of enumerative results on the joint distributions (see e.g., [**32**] and [**33**]). Remember that $\ell(\sigma) = inv(\sigma)$ for all $\sigma \in S_n$. The first pair of equidistributed Euler-Mahonian bistatistics was that (des, inv) and $(des, imaj)$, denoted by

$$(des, inv) \approx (des, imaj),$$

where $imaj := maj(\sigma^{-1})$. This is an easy consequence of Foata's bijection.

Of special interest is also the Euler-Mahonian pair (des, maj), whose joint distribution on S_n is given by Carlitz's q-Eulerian polynomials $A_n(t,q)$

$$A_n(t,q) := \sum_{k=0}^{n} A_{n,k}(q) t^k = \sum_{\sigma \in S_n} t^{des(\sigma)} q^{maj(\sigma)},$$

and $A_0(t,q) := 1$. For example, $A_3(t,q) = 1 + 2tq^2 + 2tq + t^2q^3$. Analogously to the Eulerian numbers, the coefficients $A_{n,k}(q)$ satisfy the recurrence

$$A_{n,k}(q) = [k+1]_q A_{n-1,k}(q) + q^k [n-k]_q A_{n-1,k-1}(q),$$

with

$$A_{0,k}(q) := \begin{cases} 1, & \text{for } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The following formula is due to Carlitz's and its proof can be found in [**19**].

THEOREM 2.3. *Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{A_n(t,q)}{\prod_{i=0}^{n}(1-tq^i)}$$

in $\mathbf{Z}[q][[t]]$.

Denert in 1990 conjectured that the pair (des, maj) was equidistributed with (exc, den), where exc is the number of excedances of a permutation and den is a Mahonian statistic crucially different fron inv. This result was proved first by Foata and Zeilberger [**33**] that called den the *Denert* statistic and after by Han [**40**]. More recently Skandera [**49**] defined a new Eulerian statistic stc such that the pair (stc, inv) is equally distributed with (des, maj) and provide a simple bijective proof of this fact. Note that (des, inv) and (des, maj) are two different families of Euler-Mahonian bistatistics on S_n.

It's possible also to consider pairs of Mahonian statistics. The following results hold.

PROPOSITION 2.4. *The three pairs of statistics*

$$(maj, inv) \approx (imaj, inv) \approx (imaj, maj),$$

have the same distribution on S_n.

PROPOSITION 2.5. *The pair of statistics (maj, inv) is symmetric, namely*

$$(maj, inv) \approx (inv, maj).$$

Proofs of this results can be found in [**32**].

2.2. The Hyperoctahedral group

An increasing number of enumerative results true for S_n have been generalized to the hyperoctahedral group. Several "major index" statistics have been introduced and studied for B_n, (see [**22, 23**], [**46**] and [**53**]), but no generalization of MacMahon's result has been found until the discover of the *flag-major index* in the recent paper [**3**]. After that Foata posed the following question, recently solved in [**2**].

PROBLEM 1 (Foata). Extend the "Euler-Mahonian" bivariate distribution of descent number and major index to B_n.

Here we introduce some basic combinatorial descriptions of the Coxeter groups of type B and then we show the results mentioned above.

2.2.1. Combinatorial description. We denote by B_n the group of all bijections β of the set $[-n, n] \setminus \{0\}$ onto itself such that

$$\beta(-i) = -\beta(i)$$

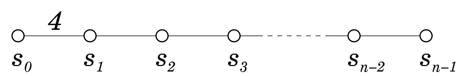
for all $i \in [-n, n] \setminus \{0\}$, with composition as the group operation. This group is usually known as the group of *signed permutations* on $[n]$, or as the *hyperoctahedral group* of rank n. We identify S_n as a subgroup of B_n, and B_n as a subgroup of S_{2n}, in the natural ways. If $\beta \in B_n$ then we write $\beta = [\beta_1, \ldots, \beta_n]$ to mean that $\beta(i) = \beta_i$ for $i = 1, \ldots, n$, and we call this the *window* notation of β. As set of generators for B_n we take $S_B := \{s_1^B, \ldots, s_{n-1}^B, s_0^B\}$ where for $i \in [n-1]$

$$s_i^B := [1, \ldots, i-1, i+1, i, i+2, \ldots, n],$$

and

$$s_0^B := [-1, 2, \ldots, n] = (-1, 1).$$

Note that multiplying an element $\beta \in B_n$ on the right by s_i^B (s_0^B) has the effect of exchanging the values in position i and $i+1$ (respectively, changing the sign of the value in the first position). This makes clear that S_B generates B_n, moreover it's well known that (B_n, S_B) is a Coxeter system of type B (see e.g., [**10**]).



The Coxeter graph of B_n

As for S_n we give an explicit combinatorial description of the length function ℓ of B_n with respect to S_B. For $\beta \in B_n$ we let

$$Neg(\beta) := \{i \in [n] : \beta_i < 0\}, \tag{2.1}$$

$$neg(\beta) = N_1(\beta) := |Neg(\beta)|,$$

and

$$N_2(\beta) := |\{\{i, j\} \in \binom{[n]}{2} : \beta_i + \beta_j < 0\}|.$$

Note that, if $\beta \in B_n$, then it's not hard to see that

$$N_1(\beta) + N_2(\beta) = -\sum_{i \in Neg(\beta)} \beta(i). \tag{2.2}$$

PROPOSITION 2.6. *Let $\beta \in B_n$. Then*

$$\ell(\beta) = inv(\beta) + N_1(\beta) + N_2(\beta).$$

A consequence of this proposition-definition is this characterization of the right descent set.

PROPOSITION 2.7. *Let $\beta \in B_n$. Then*

$$D_R(\beta) = \{s_i^B \in S_B \ : \ \beta(i) > \beta(i+1)\}, \tag{2.3}$$

where $\beta(0) := 0$.

2.2.2. Flag statistics on B_n. We introduce the so-called *flag-statistics*. They have a very important role in representation theory as showed in next chapter. Here we are only interested in their combinatorial properties.

Following [**3**] we let

$$\tau_i := s_i s_{i-1} \cdots s_0^B.$$

The family $\{\tau_i\}_i$ is a set of generators for B_n and for any $\beta \in B_n$ there exist unique integers $r_0, \dots, r_{n-1}$, with $0 \leq r_i \leq 2i+1$ for $i = 0, \dots, n-1$ such that

$$\beta = \tau_{n-1}^{r_{n-1}} \cdots \tau_2^{r_2} \tau_1^{r_1} \tau_0^{r_0}. \tag{2.4}$$

The *flag-major index* is defined by

$$fmaj(\beta) := \sum_{i=0}^{n-1} r_i. \tag{2.5}$$

The $fmaj$ was the first statistic of type "major" on B_n to be Mahonian, (see [**3**, Theorem 2.2]).

Proposition 2.8. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\beta \in B_n} q^{fmaj(\beta)} = \sum_{\beta \in B_n} q^{\ell_B(\beta)}.$$

If we consider the following order on $\mathbf{Z}$

$$-1 \prec -2 \prec \cdots \prec -n \prec \cdots \prec 0 \prec 1 \prec 2 \prec \cdots \prec n \prec \cdots$$

instead of the usual ordering, then there is the following, (see [**3**, Theorem 3.1]).

Proposition 2.9. *Let $\beta \in B_n$. Then*

$$fmaj(\beta) = 2 \cdot maj(\beta) + N_1(\beta).$$

We will use often this characterization in the rest of this work.

The *flag-descent number* of β is defined by

$$fdes(\beta) := 2 \cdot des(\beta) + \eta_1(\beta), \tag{2.6}$$

where

$$\eta_1(\beta) := \begin{cases} 1, & \text{if } \beta(1) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

For example, if $\beta = [-4, -3, 5, 1, -2] \in B_5$ then $fmaj(\beta) = 2 \cdot 8 + 3 = 19$ and $fdes(\beta) = 2 \cdot 3 + 1 = 7$.

The pair of statistics $(fdes, fmaj)$ gives a generalization of Carlitz's identity (Theorem 2.3) to B_n. More precisely we have the following theorem due to Adin, Brenti and Roichman [**2**] (see [**1**] for a refinement).

Theorem 2.10. *Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\beta \in B_n} t^{fdes(\beta)} q^{fmaj(\beta)}}{(1-t) \prod_{i=1}^n (1 - t^2 q^{2i})}$$

in $\mathbf{Z}[q][[t]]$.

Note that the powers of q in the denominator of the formula are the degrees of the Weyl group B_n (see Table 1 of Chapter 4).

2.2.3. Negative statistics. Now we introduce, following [**2**] another type of statistics called *negatives*. Their nature is more combinatorial respect to the flag-statistics and they are the natural analogues of maj and des for B_n.

For any $\beta \in B_n$ the *negative-descent multiset* is defined by

$$NDes(\beta) := Des(\beta) \biguplus \{-\beta(i) \ : \ i \in Neg(\beta)\}.$$

For example, if $\beta = [-3, 1, -6, 2, -4, -5] \in B_6$ then $Des(\beta) = \{2, 4, 5\}$ and $NDes(\beta) = \{2, 3, 4^2, 5^2, 6\}$. Note that if $\beta \in S_n$ then $NDes(\beta)$ is a set and coincides with the usual descent set of β. Also, note that $NDes(\beta)$ can be defined rather naturally also in purely Coxeter group theoretic terms. In fact, for $i \in [n]$ let $\eta_i \in B_n$ be defined by

$$\eta_i := [1, 2, \ldots, i - i, -i, i + 1, \ldots, n],$$

so $\eta_1 = s_0^B$. Then $\eta_1, \ldots, \eta_n$ are reflections (in the Coxeter group sense, see e.g., [**10**] or [**42**]) of B_n and it is clear that

$$NDes(\beta) := \{i \in [n-1] : \ell(\beta s_i^B) < \ell(\beta)\} \biguplus \{i \in [n] : \ell(\beta^{-1}\eta_i) < \ell(\beta^{-1})\}.$$

These considerations explain why it is natural to think of $NDes(\beta)$ as a "descent set", so the following definitions are natural.
For $\beta \in B_n$ we let

$$ndes(\beta) := |NDes(\beta)|$$

and

$$nmaj(\beta) := \sum_{i \in NDes(\beta)} i.$$

The two most important combinatorial proprieties of these two statistics are that $nmaj$ is Mahonian and that they give a generalization of Carlitz's identity to B_n solving the Foata's problem.

THEOREM 2.11. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\beta \in B_n} t^{ndes(\beta)} q^{nmaj(\beta)} = \sum_{\beta \in B_n} t^{fdes(\beta)} q^{fmaj(\beta)}.$$

2.3. The even-signed permutation group

In this section we give some basic combinatorial descriptions of the Coxeter groups of type D. Then we introduce the analogues of the negative statistics and flag-major index for D_n and we study some of their combinatorial properties. Moreover we find a Carlitz's identity for D_n. The definition of flag-descent number for D_n needs some background and so it will be given at the end of next chapter.

2.3.1. Combinatorial description. We denote by D_n the subgroup of B_n consisting of all the signed permutations having an even number of negative entries in their window notation, more precisely

$$D_n := \{\gamma \in B_n : N_1(\gamma) \equiv 0 \ (\ mod\ 2\ )\}.$$

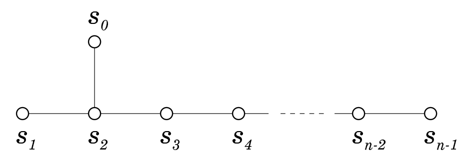
As a set of generators for D_n we take $S_D := \{s_0^D, s_1^D, \ldots, s_{n-1}^D\}$ where for $i \in n-1]$

$$s_i^D := s_i^B,$$

and

$$s_0^D := [-2, -1, 3, \ldots, n] = (1, -2)(2, -1).$$

It's clear that S_D generates D_n. Note that $S_n \subset D_n \subset B_n$. For the rest of the section, if there is no danger of confusion, we will write simply "S" and "s_i" instead "S_D" and "s_i^D", for $i = 0, 1, \ldots, n-1$. It's well known that the pair (D_n, S_D) is a Coxeter system of type D, and this can be proved in similar way as Theorem 1.4 using next Proposition 2.13.



The Coxeter graph of D_n

There is a well known direct combinatorial way to compute the length for $\gamma \in D_n$ (see, e.g., [**10**, §8.2]), namely

Proposition 2.12. *Let $\gamma \in D_n$. Then*

$$\ell(\gamma) = inv(\gamma) - \sum_{i \in Neg(\gamma)} \gamma(i) - N_1(\gamma). \tag{2.7}$$

Equivalently, using the observation (2.2)

$$\ell(\gamma) = inv(\gamma) + N_2(\gamma). \tag{2.8}$$

For example, if $\gamma := [-4, 1, 3, -5, -2, -6] \in D_6$ then $inv(\gamma) = 10$, $des(\gamma) = 2$, $maj(\gamma) = 8$, $N_1(\gamma) = 4$, $N_2(\gamma) = 13$ and $\ell(\gamma) = 23$.

This characterization of the length function allows a simple description of the right descent set of an element of D_n.

Proposition 2.13.

$$D_R(\gamma) = \{s_i \in S \ : \ \gamma(i) > \gamma(i+1)\},$$

where $\gamma(0) := -\gamma(2)$ *and* $\gamma(n+1) := 0$.

For example, if $\gamma = [-2, 1, 5, -3, -4, -6]$ then $D_R(\gamma) = \{s_0, s_3, s_4, s_5\}$.

2.3.2. Negative statistics on D_n. The goal of this section is to show that the results of previous one can be generalized to D_n. Toward this end we will use often the following decomposition of D_n. We let

$$T := \{\gamma \in D_n : des(\gamma) = 0\}. \tag{2.9}$$

It is well known, and easy to see, that

$$D_n = \biguplus_{u \in S_n} \{\gamma u : \gamma \in T\}, \tag{2.10}$$

where $\biguplus$ denotes disjoint union. Note that (2.10) is one case of the multiplicative decomposition of a Coxeter group into a parabolic subgroup and its minimal coset representatives (see Proposition 1.5), more precisely T is the right quotient corresponding to the maximal parabolic subgroup generated by $J := S \setminus \{s_0\}$. We are ready to introduce the main object of this section, a new "descent set" for the elements of D_n. This gives rise in a very natural way, to new "major index" and "descent number". We define the *D-negative descent multiset*

$$DDes(\gamma) := Des(\gamma) \biguplus \{-\gamma(i) - 1 \ : \ i \in Neg(\gamma)\} \setminus \{0\}, \tag{2.11}$$

where $Neg(\gamma)$ is the set of positions of negative entries in γ, defined in (2.1).
For example, if $\gamma = [-4, 1, 3, -5, -2, -6] \in D_6$ then $Des(\gamma) = \{3, 5\}$ and $DDes(\gamma) = \{1, 3^2, 4, 5^2\}$.
Note that if $\gamma \in S_n$ then $DDes(\gamma)$ is a set and coincides with the usual descent set of γ. Also, note that $DDes(\gamma)$ can be defined rather naturally also in purely Coxeter group theoretic terms. In fact, for $i \in [n-1]$ let $\xi_i \in D_n$ be defined by

$$\xi_i := [-1, 2, \dots, i, -i-1, i+2, \dots, n].$$

Then $\xi_1, \dots, \xi_{n-1}$ are reflections (in the Coxeter group sense, see e.g., [**10**] or [**42**]) of D_n and it is clear from (2.7) that

$$DDes(\gamma) := \{i \in [n-1] : \ell(\gamma s_i) < \ell(\gamma)\} \biguplus \{i \in [n-1] : \ell(\gamma^{-1}\xi_i) < \ell(\gamma^{-1})\}.$$

These considerations explain why it is natural to think of $DDes(\gamma)$ as a "descent set", so the following definitions are natural.
For $\gamma \in D_n$ we let

$$ddes(\gamma) := |DDes(\gamma)|$$

and

$$dmaj(\gamma) := \sum_{i \in DDes(\gamma)} i.$$

For example if $\gamma = [-4, 1, 3, -5, -2, -6] \in D_6$ then $ddes(\gamma) = 6$, and $dmaj(\gamma) = 21$.
Note that from (2.11) there follows that

$$dmaj(\gamma) = maj(\gamma) - \sum_{i \in Neg(\gamma)} \gamma(i) - N_1(\gamma) = maj(\gamma) + N_2(\gamma). \tag{2.12}$$

This formula is also one the motivations behind our definition of $dmaj(\gamma)$, because of the corresponding formulas (2.7) and (2.8).
Also note that

$$ddes(\gamma) = des(\gamma) + N_1(\gamma) + \epsilon(\gamma), \tag{2.13}$$

where

$$\epsilon(\gamma) := \begin{cases} -1 & \text{if } 1 \notin \gamma([n]) \\ 0 & \text{if } 1 \in \gamma([n]). \end{cases}$$

Our first result shows that $dmaj$ and ℓ are equidistribuited in D_n.

Proposition 2.14. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\gamma \in D_n} q^{dmaj(\gamma)} = \sum_{\gamma \in D_n} q^{\ell(\gamma)}.$$

Proof. Let T be defined by (2.9). It is clear from our definitions that for all $u \in S_n$ and $\sigma \in T$,

$$maj(\sigma u) = maj(u), \qquad inv(\sigma u) = inv(u), \qquad N_2(\sigma u) = N_2(\sigma). \tag{2.14}$$

Therefore, from (2.8), (2.10), (2.12) and the corresponding result for S_n (Theorem 2.1) we conclude that

$$\begin{aligned}
\sum_{\gamma\in D_n} q^{dmaj(\gamma)} &= \sum_{\sigma\in T}\sum_{u\in S_n} q^{dmaj(\sigma u)} \\
&= \sum_{\sigma\in T}\sum_{u\in S_n} q^{maj(\sigma u)+N_2(\sigma u)} \\
&= \sum_{\sigma\in T} q^{N_2(\sigma)} \sum_{u\in S_n} q^{maj(u)} \\
&= \sum_{\sigma\in T} q^{N_2(\sigma)} \sum_{u\in S_n} q^{inv(u)} \\
&= \sum_{\sigma\in T}\sum_{u\in S_n} q^{inv(\sigma u)+N_2(\sigma u)} \\
&= \sum_{\gamma\in D_n} q^{\ell(\gamma)},
\end{aligned}$$

as desired. □

As before, let $T = \{\gamma \in D_n : des(\gamma) = 0\}$ so

$$T = \{\gamma \in D_n : \gamma(1) < \gamma(2) < \ldots < \gamma(n)\}.$$

Therefore, given $\gamma \in T, \gamma \neq e$, there is a unique $k \in [n]$ such that

$$\gamma(k) < 0 < \gamma(k+1).$$

Given $\gamma \in T$ we associate to γ the strict partition

$$\Lambda(\gamma) := (-\gamma(1)-1, -\gamma(2)-1, \ldots, -\gamma(k)-1). \tag{2.15}$$

The following proposition will be treat with more detail in Chapter 5 (Proposition 5.15).

Proposition 2.15. *The map Λ defined by (2.15) is a bijection between T and $\mathcal{P}_S(n-1)$. Furthermore $\gamma \leq \sigma$ in T if and only if $\Lambda(\gamma) \subseteq \Lambda(\sigma)$ and $\ell(\gamma) = |\Lambda(\gamma)|$ for all $\gamma, \sigma \in T$.*

We find it convenient to identify a strict partition $\lambda \in \mathcal{P}_S(n)$ with a subset of $[n]$. In fact we have an inclusion preserving obvious bijection ϕ between $\mathcal{P}_S(n)$ and $\wp(n) := \{S : S \subseteq [n]\}$ given by:

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \stackrel{\phi}{\longleftrightarrow} \{\lambda_1, \lambda_2, \ldots, \lambda_n\}.$$

We begin with the following lemma.

Lemma 2.16. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\sigma\in T} t^{N_1(\sigma)+\epsilon(\sigma)} q^{N_2(\sigma)} = \sum_{S\subseteq[n-1]} t^{|S|} q^{\sum_{i\in S} i} = \prod_{i=1}^{n-1}(1+tq^i).$$

Proof. From (2.8) we have that $N_2(\sigma) = \ell(\sigma)$, for all $\sigma \in T$. By Proposition 2.15 we have $\ell(\sigma) = |\Lambda(\sigma)|$ and by definition of ϕ that $|\Lambda(\sigma)| = \sum_{i\in\phi(\Lambda(\sigma))} i$. Therefore $N_2(\sigma) = \sum_{i\in\phi(\Lambda(\sigma))} i$.
Let $\sigma \in T$. Suppose first that $1 \in \sigma([n])$, then $|\phi(\Lambda(\sigma))| = N_1(\sigma)$. On the other hand, if $1 \notin \sigma([n])$, we have that $|\phi(\Lambda(\sigma))| = N_1(\sigma) - 1$. Hence $|\phi(\Lambda(\sigma))| = N_1(\sigma) + \epsilon(\sigma)$, and if we let $S = \phi(\Lambda(\sigma))$ the result follows. □

We are now ready to prove the main result of this work, namely that the pair of statistics $(ddes, dmaj)$ solves Foata's problem for the group of even-signed permutations D_n.

THEOREM 2.17 (Carlitz's identity for D_n). *Let $n \in \mathbf{P}$. Then*

$$\sum_{r\geq 0}[r+1]_q^n t^r = \frac{\sum_{\gamma\in D_n} t^{ddes(\gamma)}q^{dmaj(\gamma)}}{(1-t)(1-tq^n)\prod_{i=1}^{n-1}(1-t^2q^{2i})} \tag{2.16}$$

in $\mathbf{Z}[q][[t]]$.

PROOF. Let T be defined by (2.9). Then it is clear from our definitions that

$$des(\sigma u) = des(u)\ ,\ N_1(\sigma u) = N_1(\sigma)\ ,\ \epsilon(\sigma u) = \epsilon(\sigma)$$

and

$$\sum_{i\in Neg(\sigma u)} \sigma u(i) = \sum_{i\in Neg(\sigma)} \sigma(i),$$

for all $\sigma \in T$ and $u \in S_n$. Therefore we have from (2.10), (2.12), (2.13), (2.14) and Lemma 2.16 that

$$\begin{aligned}
\sum_{\gamma\in D_n} t^{ddes(\gamma)}q^{dmaj(\gamma)} &= \sum_{\sigma\in T}\sum_{u\in S_n} t^{des(\sigma u)+N_1(\sigma u)+\epsilon(\sigma u)}q^{maj(\sigma u)+N_2(\sigma u)}\\
&= \sum_{\sigma\in T} t^{N_1(\sigma)+\epsilon(\sigma)}q^{N_2(\sigma)}\sum_{u\in S_n} t^{des(u)}q^{maj(u)}\\
&= \prod_{i=1}^{n-1}(1+tq^i)\sum_{u\in S_n} t^{des(u)}q^{maj(u)}
\end{aligned}$$

and the result follows from Theorem 2.3. □

Note that, as in Theorem 2.10 for B_n, the powers of q in the denominator of formula are the Coxeter degrees of D_n (see Table 1).

The following result is the analogue of Proposition 2.5.

PROPOSITION 2.18. *The pair of statistics $(dmaj, \ell)$ is symmetric, namely*

$$(dmaj, \ell) \approx (\ell, dmaj).$$

PROOF. Let T defined as in (2.9), using the decomposition (2.10), the relations (2.8), (2.12), (2.14) and Proposition 2.5 we obtain

$$\begin{aligned}
\sum_{\gamma\in D_n} t^{\ell(\gamma)}q^{dmaj(\gamma)} &= \sum_{\sigma\in T}\sum_{u\in S_n} t^{inv(\sigma u)+N_2(\sigma u)}q^{maj(\sigma u)+N_2(\sigma u)}\\
&= \sum_{\sigma\in T} t^{N_2(\sigma)}q^{N_2(\sigma)}\sum_{u\in S_n} t^{inv(u)}q^{maj(u)}\\
&= \sum_{\sigma\in T} t^{N_2(\sigma)}q^{N_2(\sigma)}\sum_{u\in S_n} t^{maj(u)}q^{inv(u)}\\
&= \sum_{\sigma\in T}\sum_{u\in S_n} t^{maj(\sigma u)+N_2(\sigma u)}q^{inv(\sigma u)+N_2(\sigma u)}\\
&= \sum_{\gamma\in D_n} t^{dmaj(\gamma)}q^{\ell(\gamma)}
\end{aligned}$$

□

2.3.3. The D-Flag Major Index. For $i = 0, \dots, n-1$ we define

$$t_i := s_i s_{i-1} \cdots s_0, \tag{2.17}$$

explicitly for all $i \in [n-1]$

$$t_i = [-1, -i-1, 2, 3, \dots, i, i+2, \dots, n], \tag{2.18}$$

and for $i = 0$

$$t_0 = [-2, -1, 3, \dots, n] = s_0. \tag{2.19}$$

These are Coxeter elements (see e.g., [**42**, §3.16]), in a distinguished flag of parabolic subgroups

$$1 < G_1 < G_2 < \dots < G_n = D_n$$

where $G_i \simeq D_i$ $(i \geq 2)$ is the parabolic subgroup of D_n generated by $s_0, s_1, \dots, s_{i-1}$. The family $\{t_i\}_i$ is a new set of generators for D_n, and we have the following proposition.

PROPOSITION 2.19. *For every $\gamma \in D_n$ there exists a unique representation*

$$\gamma = t_0^{h_{n-1}} t_{n-1}^{k_{n-1}} t_0^{h_{n-2}} t_{n-2}^{k_{n-2}} \cdots t_0^{h_1} t_1^{k_1} \tag{2.20}$$

with $0 \leq h_r \leq 1$, $0 \leq k_r \leq 2r-1$ and

$$k_r \in \{2r-1, r-1\} \text{ if } h_r = 1 \tag{2.21}$$

for all $r = 1, \dots, n-1$.

PROOF. We proceed by induction on n. For $n = 2$ the result is clear, so suppose $n \geq 3$. We define

$$D_{n,*} := \{t_{n-1}^{k_{n-1}} w \ : \ k_{n-1} \in [0, 2n-3]\ ,\ w \in D_{n-1}\},$$

$$D_{n,1} := \{t_0 t_{n-1}^{2n-3} w \ : \ w \in D_{n-1}\},$$

$$D_{n,-1} := \{t_0 t_{n-1}^{n-2} w \ : \ w \in D_{n-1}\}.$$

It is not hard to see that $|D_{n,1}| = |D_{n,-1}| = |D_{n-1}|$ and that $D_{n,1} \cap D_{n,-1} = \emptyset$ as $\gamma(n) = 1$ and $\sigma(n) = -1$, for all $\gamma \in D_{n,1}$ and $\sigma \in D_{n,-1}$.
On the other hand if $t_{n-1}^r w_1 = t_{n-1}^s w_2$ with $w_1, w_2 \in D_{n-1}$ and $r, s \in [0, 2n-3]$, it is easy to see that $r = s$ and $w_1 = w_2$, hence $|D_{n,*}| = (2n-2)|D_{n-1}|$. Moreover the elements $\gamma \in D_{n,*}$ satisfy $\gamma(n) \neq \pm 1$. Therefore we have the following decomposition of D_n

$$D_n = D_{n,*} \biguplus D_{n,1} \biguplus D_{n,-1},$$

and so the result follows by induction. $\square$

Note that the representation (2.20) is not unique if we drop the condition (2.21). For example consider $\gamma = [-2, 4, 1, -3] \in D_4$. Then γ has two different representations of type (2.20), namely, $\gamma = t_3^4 t_0 t_2^3 t_0 t_1$ and $\gamma = t_0 t_3^4$. The representation of Proposition 2.19 is the first one.

Let $\gamma \in D_n$, then we define the *D-flag major index* of γ by

$$fmaj_D(\gamma) := \sum_{i=1}^{n-1} k_i + \sum_{i=1}^{n-1} h_i. \tag{2.22}$$

For $0 \leq m \leq 2n-1$ we define $r_{n,m} \in D_n$ as follows: for $n = 2$,

$$r_{2,m} := \begin{cases} e & \text{if } m = 0 \\ s_1 & \text{if } m = 1 \\ s_1 s_0 & \text{if } m = 2 \\ s_0 & \text{if } m = 3 \end{cases}$$

and for $n > 2$,

$$r_{n,m} := \begin{cases} e & \text{if } m = 0 \\ s_{n-m} s_{n-m+1} \cdots s_{n-1} & \text{if } 0 < m < n \\ s_{m-n+1} s_{m-n} \cdots s_0 s_2 s_3 \cdots s_{n-1} & \text{if } n \leq m < 2n-1 \\ s_0 s_2 s_3 \cdots s_{n-1} & \text{if } m = 2n-1 \end{cases}$$

The set $\{r_{n,m} \ : \ 0 \leq m < 2n\}$ forms a complete set of representatives of minimal length for the left cosets of D_{n-1} in D_n. Moreover this is still valid for every $i \in [3, n]$, namely, $r_{i,m} \in D_i^{J_i}$ for all $m \in [0, 2i-1]$, where $J_i := S \setminus \{s_{n-1}, \ldots, s_{i-1}\}$. Hence we have the following decomposition

$$D_n = D_n^{J_n} D_{n-1}^{J_{n-1}} \cdots D_2.$$

Note that the length of $r_{i,m}$ is $\bar{m}$, where

$$\bar{m} := \begin{cases} m & \text{if } 0 \leq m \leq 2i-2 \\ i-1 & \text{if } m = 2i-1. \end{cases}$$

From $i)$ of Proposition 1.5 we know that each element $\gamma \in D_n$ has a unique representation as a product

$$\gamma = \prod_{k=1}^{n-1} r_{n+1-k, m_{n+1-k}} \tag{2.23}$$

where $0 \leq m_j < 2j$ for all j. From $ii)$ of Proposition 1.5 it follows that

$$\ell(\gamma) = \sum_{j=2}^{n} \bar{m}_j. \tag{2.24}$$

Thanks to the unique representation (2.23) we define a map $\phi : D_n \to D_n$ in the following way:

$$\phi\Big(\prod_{k=1}^{n-1} r_{n+1-k, m_{n+1-k}}\Big) := \prod_{k=1}^{n-1} \phi(r_{n+1-k, m_{n+1-k}}),$$

where for $i \neq 2$,

$$\phi(r_{i,m}) := \begin{cases} t_{i-1}^{m} & \text{if } m < 2i-2 \\ t_0 t_{i-1}^{\bar{m}-1} & \text{if } 2i-2 \leq m \leq 2i-1, \end{cases}$$

and for $i = 2$,

$$\phi(r_{2,m}) := \begin{cases} e & \text{if } m = 0 \\ t_1 & \text{if } m = 1 \\ t_0 t_1 & \text{if } m = 2 \\ t_0 & \text{if } m = 3. \end{cases}$$

The definition of ϕ, together with Proposition 2.19 and (2.23), imply the following result.

PROPOSITION 2.20. *The map* $\phi : D_n \to D_n$ *is a bijection.* □

Now we are ready to state one of the main results of this section, namely that the D-flag major index is equidistributed with the length in D_n.

THEOREM 2.21. *Let* $n \in \mathbf{P}$. *Then*

$$\sum_{\gamma \in D_n} q^{fmaj_D(\gamma)} = \sum_{\gamma \in D_n} q^{\ell(\gamma)}.$$

PROOF. By definition of $\bar{m}$, the map ϕ is a bijection which sends the length function in the D-flag major index. □

Note that the flag major index ($fmaj$) defined on B_n (2.5) does not work on D_n. Namely if we consider $\gamma \in D_n$ as an element of B_n, then $fmaj(\gamma)$ is not equidistributed with length on D_n. For example, if $\gamma = [-2, -1]$ then $fmaj_D(\gamma) = 1$ while $fmaj(\gamma) = 4$, and in D_2 there is no element of length 4.
Note also that $fmaj_D$ restricted to S_n is not the major index and it's not equidistributed with length. It seems to be a new statistic on S_n. It's easy to see that for each $\gamma \in S_n$, $fmaj_D(\gamma)$ is always even and that $fmaj_D(\gamma) \geq maj(\gamma)$. If we let $E_n(q) := \sum_{\gamma \in S_n} q^{fmaj_D(\gamma)}$, for $n \leq 4$ we have $E_1(q) = 1$, $E_2(q) = 1 + q^2$, $E_3(q) = 1 + 3q^2 + q^4 + q^6$ and $E_4(q) = 1 + 5q^2 + 6q^4 + 7q^6 + 3q^8 + q^{10} + q^{12}$.

We finish this section by describing a combinatorial algorithm that allows us to compute the D-flag major index $fmaj_D$, without using the representation of Proposition 2.19.

Let $\sigma = (a_1, \ldots, a_n) \in \mathbf{Z}^n$ and $i \geq 1$. We use this *split-notation*

$$\sigma = [a_1][a_2, \ldots, a_{i+1}][a_{i+2}, \ldots, a_n].$$

Sometimes it will be useful to denote the first part by A and the second by C_i where i represents the number of its elements.
We define the following operations on $\sigma \in \mathbf{Z}^n$:

$$\overset{\rightarrow_i^0}{\sigma} := [-a_2][-a_1, a_3, \ldots, a_{i+1}][a_{i+2}, \ldots, a_n],$$

and

$$\overset{\rightarrow_i^1}{\sigma} := [-a_1][-a_{i+1}, a_2, \ldots, a_i][a_{i+2}, \ldots, a_n].$$

In these cases we will write $\overset{\rightarrow_i^0}{\sigma} = (A^0, C_i^0, [a_{i+2}, \ldots, a_n])$ and $\overset{\rightarrow_i^1}{\sigma} = (\overset{\rightarrow_1}{A}, \overset{\rightarrow_1}{C_i}, [a_{i+2}, \ldots, a_n])$.
Moreover for all $n \in \mathbf{P}$ we define

$$\rightarrow_i^n := \rightarrow_i^1 \circ \cdots \circ \rightarrow_i^1 \quad n\text{-times}. \tag{2.25}$$

Note that for every $\sigma \in \mathbf{Z}^n$ and $i \geq 1$, $\stackrel{\rightarrow^{2i}_{i}}{\sigma} = \sigma$.
For example, if $\gamma \in D_5$, $\gamma = [-2][1,3,-4,5] = (A, C_4)$, then

$$\stackrel{\rightarrow^{0}_{4}}{\gamma} = [-1][2,3,-4,5] = (A^0, C_4^0),$$

$$\stackrel{\rightarrow^{5}_{4}}{\gamma} = [2][5,-1,-3,4] = (\stackrel{\rightarrow^{5}}{A}, \stackrel{\rightarrow^{5}}{C_4}),$$

and

$$\stackrel{\rightarrow^{2}_{3}}{\gamma} = [-2][-3,4,1][5] = (\stackrel{\rightarrow^{2}}{A}, \stackrel{\rightarrow^{2}}{C_3}, [5]).$$

These are the two technical properties that we will use in the algorithm. Fix $i \in [n-1]$, let t_i be as in (5.11),

$$t_i = [-1][-i-1, 2, 3, \ldots, i][i+2, \ldots n].$$

It's easy to see that for all $i \in [n-1]$ we have

$$t_i^2 = t_i t_i = \stackrel{\rightarrow^{1}_{i}}{t_i}, \tag{2.26}$$

and by (2.25) that for $k \in \mathbf{P}$

$$t_i^k = \stackrel{\rightarrow^{k-1}_{i}}{t_i}. \tag{2.27}$$

Now consider $t_{i-1} = [-1][-i, 2, \ldots, i-1][i+1, \ldots, n]$. As before it is not hard to see that

$$t_i t_{i-1} = \stackrel{\rightarrow^{1}_{i-1}}{t_i}. \tag{2.28}$$

Now we are able to state the algorithm to compute the unique representation of γ as in Proposition 2.19, namely

$$\gamma = f_{n-1} \cdots f_1$$

where for all $r \in [n-1]$, $f_r = t_0^{h_r} t_r^{k_r}$ with $h_r \in [0,1]$ and $k_r \in [0, 2r-1]$.
We construct a sequence $e_0, \ldots, e_{n-1}$ of elements of D_n such that

$$\begin{aligned} &i) \quad e_0 = e\ ,\ \ e_{n-1} = \gamma\ ; \\ &ii) \quad e_i = f_{n-1} \cdots f_{n-i}, \ \text{ for all } \ i \in [1, n-1]\ ; \\ &iii) \quad \gamma(j) = e_i(j), \ \text{ for all } \ j > n-i. \end{aligned}$$

From *iii*) there immediately follows that $e_{n-1} = \gamma$.

We need to do $n-1$ steps. From now on to avoid confusion, we put on A an index corresponding to the number of steps. We begin with $e_0 = [1][2, \ldots, n]$. Assume that e_{n-i} has been constructed, and we will construct $e_{n-i+1} = e_{n-i} f_{i-1}$. Then by *iii*),

$$e_{n-i} = (A_{n-i}, C_{i-1}, [\gamma(i+1), \ldots \gamma(n)]).$$

For simplicity, we define $p(i)$ and $p(-i)$ to be the positions of $\gamma(i)$ and $-\gamma(i)$ in C_{i-1} or C_{i-1}^0respectively. There are four cases to consider.

1) $\gamma(i) \in C_{i-1}$

Then we let $k_{i-1} = i-1-p(i)$ and $h_{i-1} = 0$. Hence $f_{i-1} = t_{i-1}^{i-1-p(i)}$.

2) $-\gamma(i) \in C_{i-1}$

Then we let $k_{i-1} = 2i-2-p(-i)$ and $h_{i-1} = 0$. Hence $f_{i-1} = t_{i-1}^{2i-2-p(-i)}$.

3) $\gamma(i) \in A_{n-i}$

Then $-\gamma(i) \in C_{i-1}^0$ and in particular $p(-i) = 1$. We let $k_{i-1} = 2i-3$ and $h_{i-1} = 1$. Hence $f_{i-1} = t_0 t_{i-1}^{2i-3}$.

4) $-\gamma(i) \in A_{n-i}$

Then $\gamma(i) \in C_{i-1}^0$ and $p(i) = 1$. We let $k_{i-1} = i-2$ and $h_{i-1} = 1$. Hence $f_{i-1} = t_0 t_{i-1}^{i-2}$.

We have determined the factor f_{i-1}. By $ii)$ we let $e_{n-i+1} = e_{n-1} f_{i-1}$. From (2.25) and (2.27) it follows that $e_{n-i+1}(i) = \gamma(i)$ and by (2.28) $iii)$ again holds. Therefore

$$e_{n-i+1} = (A_{n-i+1}, C_{i-2}, [\gamma(i), \dots, \gamma(n)]),$$

where in cases 1) and 2),

$$A_{n-i+1} := \overset{\rightarrow k_{i-1}}{A_{n-i}} \ , \ \ C_{i-2} := \overset{\rightarrow k_{i-1}}{C_{i-1}} \setminus [\gamma(i)],$$

while in cases 3) and 4),

$$A_{n-i+1} := \overset{\rightarrow k_{i-1}}{A_{n-i}^0} \ , \ \ C_{i-2} := \overset{\rightarrow k_{i-1}}{C_{i-1}^0} \setminus [\gamma(i)].$$

Observe that in the first step $p(n) = \gamma(n) - 1$ and $p(-n) = -\gamma(n) - 1$. These can be used for the computation of e_1.

We finish this section by illustrating the procedure with an example.

Let $\gamma = [5, 3, -4, 1, -2] \in D_5$. We start from

$$e = e_0 = [1][2,3,4,5] = (A_0, C_4).$$

$1^{st}-$ step $i = 5$, $-\gamma(5) = 2 \in C_4$

We are in case 2) and $p(-5) = 1$, so $k_4 = 7$, $h_4 = 0$ and $f_4 = t_4^7$. It follows that $A_1 = \overset{\rightarrow 7}{A_0} = [-1]$ and $C_3 = \overset{\rightarrow 7}{C_4} \setminus [-2] = [3,4,5]$. Hence,

$$e_1 = [-1][3,4,5][-2].$$

$2^{nd}-$ step $i = 4$, $-\gamma(4) = -1 \in A_1$

We are in case 4) so $k_3 = 2$, $h_3 = 1$ and $f_3 = t_0 t_3^2$. It follows that $A_2 = \overset{\rightarrow 2}{A_1^0} = [-3]$ and $C_2 = \overset{\rightarrow 2}{C_3^0} \setminus [1] = [-4, -5]$. Hence,

$$e_2 = [-3][-4,-5][1,-2].$$

$3^{rd}-$ step $i = 3$, $\gamma(3) = -4 \in C_2$

We are in case 1) and $p(3) = 1$, so $k_2 = 1$, $h_2 = 0$ and $f_2 = t_2$. It follows that $A_3 = \overset{\rightarrow 1}{A_2} = [3]$ and $C_1 = \overset{\rightarrow 1}{C_2} \setminus [-4] = [5]$. Hence,

$$e_3 = [3][5][-4,1,-2].$$

$4^{th}-$ step $i = 2$, $\gamma(2) = 3 \in A_3$

We are in case 3) so $k_1 = 1$, $h_1 = 1$ and $f_1 = t_0 t_1$. It follows that $A_4 = \overset{\rightarrow 1}{A_3^0} = [5]$ and $C_0 = \emptyset$. Hence,

$$e_4 = [5][3,-4,1,-2] = \gamma,$$

and we are done. Finally $\gamma = t_4^7 t_0 t_3^2 t_2 t_0 t_1$ and $fmaj_D(\gamma) = 12$.

CHAPTER 3

Invariant algebras

Let W be a classical Weyl group. Consider the natural, diagonal and tensor, actions of W and W^t, respectively, on the polynomial ring $\mathbf{C}[x_1, \ldots, x_n]^{\otimes t}$ and denote by DIA and TIA the corresponding invariant algebras. Let $\mathcal{Z}_W(\bar{q})$ be the quotient of the Hilbert series of DIA and TIA. This series, for S_n and B_n, is a polynomial which admits a nice expression in terms of maj and $fmaj$, respectively.

In this chapter we analyze the case of D_n. We find an explicit formula for $\mathcal{Z}_{D_n}(\bar{q})$ which implies, in particular, that this series is actually a polynomial with nonnegative integer coefficients (Theorem 3.28). To do that we introduce several new mahonian statistics on B_n and D_n and a new "major index" for D_n called $Dmaj$. Our proof is based on the theory of t-partite partitions introduced by Gordon in [**39**] and further studied by Garsia and Gessel in [**37**]. Using similar ideas, we find a new and simpler proof of Adin-Roichman formula for $\mathcal{Z}_{B_n}(\bar{q})$ (Theorem 3.5). Finally, we define a new descent number $Ddes$ on D_n so that the pair $(Ddes, Dmaj)$ satisfies the Carlitz's identity for D_n.

3.1. Algebraic setting

In this preliminary section we give some tools that are needed in the rest of this chapter.

3.1.1. Notation. In all this chapter we use this linear order on $\mathbf{Z}^n$

$$-1 \prec -2 \prec \ldots \prec -n \prec \ldots \prec 0 \prec 1 \prec 2 \prec \ldots \prec n \prec \ldots$$

instead of the usual ordering. Moreover it's also convenient use this different notation for elements in B_n and D_n, that we call *pair notation.*

For each $\sigma \in S_n$ and $H \subseteq [n]$, we let $(\sigma, H) := [\beta_1, \ldots, \beta_n]$ be the signed permutation defined as follows:

$$\beta_i := \begin{cases} -\sigma_i, & \text{if } i \in H, \\ \sigma_i, & \text{if } i \notin H. \end{cases}$$

Note that in this notation we have

$$(\sigma, H)^{-1} = (\sigma^{-1}, \sigma(H)) \tag{3.1}$$

and

$$(\sigma, H)(\tau, K) = (\sigma\tau, K \triangle \tau^{-1}(H)) \tag{3.2}$$

For example, if $(\sigma, H) = (43512, \{1, 2, 5\}) = [-4, -3, 5, 1, -2] \in B_5$ then $(\sigma, H)^{-1} = (45213, \{2, 3, 4\}) = [4, -5, -2, -1, 3]$ and if $(\tau, K) = (21345, \{2, 5\})$ then $(\sigma, H)(\tau, K) = (34512, \{1\})$.

As for B_n we introduce a *pair* notation for D_n. For each $\sigma \in S_n$ and $K \subseteq [n-1]$ we let $(\sigma, K)_D := [\gamma_1, \ldots, \gamma_n]$ be the unique even-signed permutation γ such that $|\gamma_i| = \sigma_i$ for all $i \in [n]$ and $K \cup \{n\} \supseteq Neg(\gamma) \supseteq K$. More precisely

$$\gamma_i := \begin{cases} -\sigma_i, & \text{if } i \in K, \\ \sigma_i, & \text{if } i \notin K \cup \{n\}, \\ (-1)^{|K|}\sigma_n, & \text{if } i = n. \end{cases}$$

For example $(54312, \{1,3,4\})_D = [-5, 4, -3, -1, -2] \in D_5$. We will usually omit the index D in the pair notation of D_n when there is no risk of confusion with the pair notation of B_n.

3.1.2. Group Actions on Polynomial Rings. Let W be a classical Weyl group, i.e $W = S_n, B_n$ or D_n. There is a natural action of W on the polynomial ring $\mathbf{P}_n := \mathbf{C}[x_1, \ldots, x_n]$, $\varphi : W \to \mathrm{Aut}(\mathbf{P}_n)$ defined on the generators by

$$\varphi(w) : x_i \mapsto \frac{w(i)}{\mid w(i) \mid} x_{|w(i)|},$$

for all $w \in W$ and extended uniquely to an algebra homomorphism. This action gives rise to two actions on the tensor power $\mathbf{P}_n^{\otimes t} := \mathbf{P}_n \otimes \cdots \otimes \mathbf{P}_n$ (t-times): the natural *tensor action* φ_T of $W^t := W \times \cdots \times W$ (t-times), and the *diagonal action* of W on $\mathbf{P}_n^{\otimes t}$, $\varphi_D := \varphi_T \circ d$ defined using the diagonal embedding $d : W \hookrightarrow W^t$, $w \mapsto (w, \ldots, w)$.
The *tensor invariant algebra*

$$\mathrm{TIA} := \{\bar{p} \in \mathbf{P}_n^{\otimes t} : \varphi_T(\bar{w})\bar{p} = \bar{p} \text{ for all } \bar{w} \in W^t\}$$

is a subalgebra of the *diagonal invariant algebra*

$$\mathrm{DIA} := \{\bar{p} \in \mathbf{P}_n^{\otimes t} : \varphi_D(w)\bar{p} = \bar{p} \text{ for all } w \in W\}.$$

These two algebras are naturally multigraded and hence we can consider the corresponding Hilbert series

$$F_D(\bar{q}) := \sum_{n_1, \ldots, n_t} \dim_{\mathbf{C}}(\mathrm{DIA}_{n_1, \ldots, n_t}) q_1^{n_1} \cdots q_t^{n_t},$$

$$F_T(\bar{q}) := \sum_{n_1, \ldots, n_t} \dim_{\mathbf{C}}(\mathrm{TIA}_{n_1, \ldots, n_t}) q_1^{n_1} \cdots q_t^{n_t},$$

where $\mathrm{DIA}_{n_1, \ldots, n_t}$ and $\mathrm{TIA}_{n_1, \ldots, n_t}$ are the homogeneous components of multi-degree $(n_1, \ldots, n_t)$ in DIA and TIA respectively and $\bar{q} = (q_1, \ldots, q_t)$.
We denote the quotient series by

$$\mathcal{Z}_W(\bar{q}) := \frac{F_D(\bar{q})}{F_T(\bar{q})} \in \mathbf{Z}[[\bar{q}]].$$

3.1.3. t-Partite Partitions. In this section we recall the language of t-partite partitions which were originally defined by Gordon [**39**] as well as some results of Garsia and Gessel [**37**] that we use in the rest of this work.

Let $\mathcal{F}_n$ be the set of all functions $f : [n] \to \mathbf{N}$. For $f \in \mathcal{F}_n$ we let

$$|f| := \sum_{i=1}^{n} f(i),$$

and we denote $\mathcal{F}_{n,t} := (\mathcal{F}_n)^t$. Moreover, for $f = (f_1, \dots, f_t) \in \mathcal{F}_{n,t}$, we define

$$\alpha_j(f) := \sum_{i=1}^{t} f_i(j),$$

and we let $\mathcal{F}^e_{n,t} := \{f \in \mathcal{F}_{n,t} \ : \ \alpha_j(f) \equiv 0 \ \text{ for all } j \in [n]\}$ and $\mathcal{F}^o_{n,t} := \{f \in \mathcal{F}_{n,t} \ : \ \alpha_j(f) \equiv 1 \ \text{ for all } j \in [n]\}$.

A *t-partite partition with n parts* is a sequence $f = (f_1, \dots, f_t) \in \mathcal{F}_{n,t}$,

$$f = \begin{pmatrix} f_1(1) & f_1(2) & \dots & f_1(n) \\ f_2(1) & f_2(2) & \dots & f_2(n) \\ \vdots & \vdots & & \vdots \\ f_t(1) & f_t(2) & \dots & f_t(n) \end{pmatrix}$$

satisfying the following condition:
for $i_o \in [t]$ and $j \in [n]$, if $f_i(j) = f_i(j+1)$ for all $i < i_0$, then $f_{i_0}(j) \geq f_{i_0}(j+1)$.
Note, in particular, that for $i_0 = 1$ this implies that

$$f_1(1) \geq f_1(2) \geq \dots \geq f_1(n) \geq 0,$$

so f_1 is a partition with at most n parts.
We denote the set of all the t-partite partitions with n parts by $\mathcal{B}_{n,t}$. In particular, $\mathcal{B}_{n,1}$ is the set of all integer partitions with at most n parts.
For example, if $n = 5$ and $t = 2$, then $f = (f_1, f_2)$ with $f_1 = (4,4,4,3,3)$ and $f_2 = (3,3,2,5,4)$ is a bipartite partition with 5 parts.

Given a permutation $\sigma = \sigma_1 \cdots \sigma_n$ we say that the partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is *σ-compatible* if, for all $i \in [n-1]$,

$$\lambda_i - \lambda_{i+1} \geq \varepsilon_i(\sigma) := \begin{cases} 1, & \text{if } \sigma_i > \sigma_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

We also set $\varepsilon_n(\sigma) := 0$. Clearly, a partition λ is σ-compatible if and only if it is of the form

$$\lambda_i = p_i + p_{i+1} + \cdots + p_n$$

with $p_i \geq \varepsilon_i(\sigma)$ for all i. We let $\mathcal{P}(\sigma)$ be the set of all σ-compatible partitions.
For example, if $\sigma = 15342$ then $\lambda = (6,6,4,4,3) \in \mathcal{P}(\sigma)$.
The following theorems are due to Garsia and Gessel (see [**37**, Theorems 2.1 and 2.2]):

THEOREM 3.1. *The map Ω,*

$$(\sigma, \lambda, \mu) \longmapsto \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \mu_{\sigma_1} & \mu_{\sigma_2} & \dots & \mu_{\sigma_n} \end{pmatrix},$$

is a bijection between $\mathcal{B}_{n,2}$ and the set $\mathcal{P}_{n,2}$ of the triplets (σ, λ, μ), where
i) $\sigma \in S_n$;
ii) $\lambda \in \mathcal{P}(\sigma)$;
iii) $\mu \in \mathcal{P}(\sigma^{-1})$.

THEOREM 3.2. *Let $W = S_n$. Then*

$$\mathcal{Z}_{S_n}(q_1, q_2) = \frac{\sum_{f \in \mathcal{B}_{n,2}} q_1^{|f_1|} q_2^{|f_2|}}{\sum_{g,h \in \mathcal{B}_{n,1}} q_1^{|g|} q_2^{|h|}} = \sum_{\sigma \in S_n} q_1^{maj(\sigma)} q_2^{maj(\sigma^{-1})}.$$

We let

$$\mathcal{B}^e_{n,2} := \{f \in \mathcal{B}_{n,2} \ : \ \alpha_j(f) \equiv 0 \ \text{ for all } j \in [n]\}$$

and

$$\mathcal{B}^o_{n,2} := \{f \in \mathcal{B}_{n,2} \ : \ \alpha_j(f) \equiv 1 \ \text{ for all } j \in [n]\}$$

the sets of all the *even* and *odd* bipartite partitions with n parts, respectively. Moreover we let

$$\mathcal{P}^e_{n,2} := \{(\sigma, \lambda, \mu) \in \mathcal{P}_{n,2} \ : \ \lambda_i + \mu_{\sigma(i)} \equiv 0 \ \text{ for all } i \in [n]\}$$

and

$$\mathcal{P}^o_{n,2} := \{(\sigma, \lambda, \mu) \in \mathcal{P}_{n,2} \ : \ \lambda_i + \mu_{\sigma(i)} \equiv 1 \ \text{ for all } i \in [n]\}.$$

It's clear that, by restriction, the map Ω of Theorem 3.1 gives rise to two bijections $\mathcal{B}^e_{n,2} \leftrightarrow \mathcal{P}^e_{n,2}$ and $\mathcal{B}^o_{n,2} \leftrightarrow \mathcal{P}^o_{n,2}$.

Theorems 3.1 and 3.2 can be extended to the general case ($t > 2$) as follows, (see [**37**, Remark 2.2]).

THEOREM 3.3. *There exists a bijection between $\mathcal{B}_{n,t}$ and the set $\mathcal{P}_{n,t}$ of the $2t$-tuples*

$$(\sigma_1, \dots, \sigma_t, \lambda^{(1)}, \dots, \lambda^{(t)})$$

where $\sigma_i \in S_n$, $\lambda^{(i)} \in \mathcal{P}(\sigma_i)$ for all $i \in [t]$ and $\sigma_t \cdots \sigma_2\sigma_1 = e$. This bijection is given by

$$\Omega(\sigma_1, \dots \sigma_t, \lambda^{(1)}, \dots, \lambda^{(t)}) := \begin{pmatrix} \lambda^{(1)}_1 & \lambda^{(1)}_2 & \cdots & \lambda^{(1)}_n \\ \lambda^{(2)}_{\sigma_1(1)} & \lambda^{(2)}_{\sigma_1(2)} & \cdots & \lambda^{(2)}_{\sigma_1(n)} \\ \vdots & \vdots & & \vdots \\ \lambda^{(t)}_{\sigma_{t-1}\cdots\sigma_1(1)} & \lambda^{(t)}_{\sigma_{t-1}\cdots\sigma_1(2)} & \cdots & \lambda^{(t)}_{\sigma_{t-1}\cdots\sigma_1(n)} \end{pmatrix}.$$

We define $\mathcal{B}^e_{n,t}$, $\mathcal{B}^o_{n,t}$, $\mathcal{P}^e_{n,t}$ and $\mathcal{P}^o_{n,t}$ analogously to the case $t = 2$. Note again that the correspondence Ω restricts to bijections $\mathcal{B}^e_{n,t} \leftrightarrow \mathcal{P}^e_{n,t}$ and $\mathcal{B}^o_{n,t} \leftrightarrow \mathcal{P}^o_{n,t}$.

THEOREM 3.4. *Let $W = S_n$ and $t \in \mathbf{N}$. Then*

$$\mathcal{Z}_{S_n}(\bar{q}) = \sum_{\sigma_1, \dots, \sigma_t} \prod_{i=1}^{t} q_i^{maj(\sigma_i)},$$

where the sum is over all t-tuples $(\sigma_1, \dots, \sigma_t)$ of permutations in S_n such that $\sigma_t\sigma_{t-1} \cdots \sigma_1 = e$.

The following is the corresponding result of Theorem 3.4 for B_n and it is due to Adin and Roichman [**3**].

THEOREM 3.5. *Let $n, t \in \mathbf{N}$. Then*

$$\mathcal{Z}_{B_n}(\bar{q}) = \sum_{\beta_1, \dots, \beta_t \in B_n} \prod_{i=1}^{t} q_i^{fmaj(\beta_i)},$$

where the sum is over all the signed permutations $\beta_1, \dots, \beta_t \in B_n$ such that $\beta_t \cdots \beta_1 = e$.

3.2. New Statistics on B_n and D_n

In this section we introduce some new combinatorial objects and we prove some preliminary results that are used in the proof of the main result of this chapter (Theorem 3.28).

3.2.1. Bijections and Parity Sets. We define a bijection $\varphi_n : 2^{[n]} \to 2^{[n]}$, for every $n \in \mathbf{N}$, in the following inductive way: for $n \geq 1$,

$$\varphi_n(H) := \begin{cases} \mathcal{C}_n \varphi_{n-1}(H), & \text{if } H \subseteq [n-1], \\ \varphi_{n-1}(H \setminus \{n\}), & \text{if } H \not\subseteq [n-1], \end{cases}$$

and $\varphi_0(\emptyset) := \emptyset$.
For example, let $n = 4$ and $H = \{2\}$, then,

$$\begin{aligned} \varphi_4(\{2\}) &= \mathcal{C}_4\varphi_3(\{2\}) = \mathcal{C}_4\mathcal{C}_3\varphi_2(\{2\}) = \mathcal{C}_4\mathcal{C}_3\varphi_1(\emptyset) = \mathcal{C}_4\mathcal{C}_3\mathcal{C}_1\varphi_0(\emptyset) \\ &= \mathcal{C}_4\mathcal{C}_3(\{1\}) = \mathcal{C}_4(\{2,3\}) = \{1,4\}. \end{aligned}$$

There is also a direct way to compute φ_n.

LEMMA 3.6. *Let $n \in \mathbf{N}$ and $H \subseteq [n]$. Then*

$$\varphi_n(H) = \{i \in [n] : \ |[i,n] \setminus H| \equiv 1\}.$$

PROOF. We proceed by induction on n. If $n = 0$ it is trivial, so suppose $n \geq 1$. If $n \in H$ we have

$$\begin{aligned} \varphi_n(H) &= \varphi_{n-1}(H \setminus \{n\}) \\ &= \{i \in [n-1] : |[i, n-1] \setminus (H \setminus \{n\})| \equiv 1\} \\ &= \{i \in [n] : |[i,n] \setminus H| \equiv 1\}. \end{aligned}$$

The case $n \notin H$ is similar and is left to the reader. □

Our goal is to understand the action of a permutation σ on $\varphi_n(H)$. For this it's useful to introduce the following concept. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, 0, 0, \ldots) \in \mathcal{B}_{n,1}$. We define the *parity set* of λ to be

$$H(\lambda) := \{i \in [n] \ : \ \lambda_i - \lambda_{i+1} \equiv 0\}.$$

Let $\sigma \in S_n$ and $H \subseteq [n]$. Let $\lambda \in \mathcal{B}_{n,1}$ be such that $H = H(\lambda)$. Then we define

$$A^\sigma := H(\mu),$$

where μ is any partition in $\mathcal{B}_{n,1}$ such that $\lambda_i + \mu_{\sigma(i)} \equiv 0$ for all $i \in [n]$. Note that the definition of A^σ doesn't depend on λ and μ but only on H and σ.
Observe that the following statements are equivalent:

i) $(H(\lambda))^\sigma = H(\mu)$;
ii) $\lambda_i + \mu_{\sigma(i)} \equiv 0$ for all $i \in [n]$.

For example, suppose $n = 4$ and $\sigma = 4312$. Let $\lambda_i = p_i + \cdots + p_n$ and $\mu_i = r_i + \cdots + r_n$, for $i = 1, \ldots, n$. The condition $\lambda_i + \mu_{\sigma(i)} \equiv 0$ for all $i \in [n]$ is equivalent to the following system of congruences:

$$\begin{cases} p_1 + p_2 + p_3 + p_4 \equiv r_4 \\ p_2 + p_3 + p_4 \equiv r_3 + r_4 \\ p_3 + p_4 \equiv r_1 + r_2 + r_3 + r_4 \\ p_4 \equiv r_2 + r_3 + r_4. \end{cases}$$

If $H = \{1,3\}$ is the parity set of λ then p_1, p_3 are even, and p_2, p_4 are odd. All these conditions force r_3, r_4 to be even and r_1, r_2 to be odd, hence $A^\sigma = \{3,4\}$.

It is also possible to give an explicit direct description of A^σ.

LEMMA 3.7. *Let $n \in \mathbf{N}$, $H \subseteq [n]$ and $\sigma \in S_n$. Then*

$$A^\sigma = \{i \in [n] : |[\sigma^{-1}(i), \sigma^{-1}(i+1)) \setminus H| \equiv 0\},$$

where $\sigma^{-1}(n+1) := n+1$.

PROOF. Let λ be a partition with parity set H and set $p_i := \lambda_i - \lambda_{i+1}$. Let μ be a partition such that $\lambda_i + \mu_{\sigma(i)} \equiv 0$ and set $r_i := \mu_i - \mu_{i+1}$. Then, by definition, $i \in A^\sigma$ if and only if r_i is even. But

$$\begin{aligned} r_i &= \mu_i - \mu_{i+1} \\ &\equiv \lambda_{\sigma^{-1}(i)} - \lambda_{\sigma^{-1}(i+1)} \\ &\equiv \sum_{j \in [\sigma^{-1}(i), \sigma^{-1}(i+1))} p_j \end{aligned}$$

and the result follows. $\square$

We can now prove the main technical result of this section.

LEMMA 3.8. *Let $n \in \mathbf{N}$. Then for all $H \subseteq [n]$ and $\sigma \in S_n$ we have*

$$\sigma\varphi_n(H) = \varphi_n(H^\sigma).$$

PROOF. From Lemma 3.6 we have that

$$i \in \sigma\varphi_n(H) \iff |[\sigma^{-1}(i), n] \setminus H| \equiv 1,$$

and

$$i \in \varphi_n(A^\sigma) \iff |[i, n] \setminus A^\sigma| \equiv 1.$$

The latter condition is equivalent to the following statement: the number of the following congruences

$$\begin{aligned} |[\sigma^{-1}(i), \sigma^{-1}(i+1)) \setminus H| &\equiv 0 \\ |[\sigma^{-1}(i+1), \sigma^{-1}(i+2)) \setminus H| &\equiv 0 \\ &\vdots \\ |[\sigma^{-1}(n), \sigma^{-1}(n+1)) \setminus H| &\equiv 0 \end{aligned}$$

which are not satisfied is congruent to 1. Hence the sum of the members in the left-hand side is congruent to 1. But

$$\sum_{j=i}^{n} |[\sigma^{-1}(j), \sigma^{-1}(j+1)) \setminus H| \equiv |[\sigma^{-1}(i), n+1) \setminus H|$$

and we are done. $\square$

Note that Lemma 3.8 implies that $(\sigma, H) \mapsto H^\sigma$ is a left action of S_n on $2^{[n]}$.

Let $p : 2^{[n]} \to 2^{[n-1]}$ be the following projection of sets

$$H \mapsto \begin{cases} H, & \text{if } n \notin H, \\ \mathcal{C}_n(H), & \text{if } n \in H. \end{cases} \tag{3.3}$$

Let $\sigma \in S_n$, $H \subseteq [n]$ and $\lambda \in \mathcal{B}_{n,1}$ be such that $H(\lambda) = H$. We define

$$\overline{H^\sigma} := H(\mu)$$

where $\mu \in \mathcal{B}_{n,1}$ is such that $\lambda_i + \mu_{\sigma(i)} \equiv 1$ for all $i \in [n]$.
The proof of the following technical lemma is left to the reader.

LEMMA 3.9. *Let $\sigma \in S_n$ and $H \subseteq [n]$. Then*

$$\overline{H^\sigma} = H^\sigma \triangle \{n\} = (H \triangle \{n\})^\sigma.$$

LEMMA 3.10. *Let $\sigma \in S_n$ and $K \subseteq [n-1]$. Then*

$$\varphi_{n-1}(K^\sigma \setminus \{n\}) = p\left(\sigma \varphi_{n-1}(K)\right).$$

PROOF. Suppose $n \notin K^\sigma$. Then, by Lemma 3.8, we have that

$$\varphi_{n-1}(K^\sigma) = \mathcal{C}_n \varphi_n(K^\sigma) = \mathcal{C}_n \sigma \varphi_n(K) = \mathcal{C}_n \sigma \mathcal{C}_n \varphi_{n-1}(K) = \sigma \varphi_{n-1}(K).$$

If $n \in K^\sigma$ we have similarly that

$$\varphi_{n-1}(K^\sigma \setminus \{n\}) = \varphi_n(K^\sigma) = \sigma \varphi_n(K) = \sigma \mathcal{C}_n \varphi_{n-1}(K) = \mathcal{C}_n \sigma \varphi_{n-1}(K)$$

and the result follows. □

3.2.2. Generalization to the Multivariable Case. In this section we generalize the definitions and results given in §3.2.1 to the multivariable case.
Let $n, t \in \mathbf{N}, \sigma_1, \ldots, \sigma_t \in S_n$ and $H_1, \ldots, H_t \subseteq [n]$. Let $\lambda^{(1)}, \ldots, \lambda^{(t)} \in \mathcal{B}_{n,1}$ be such that the parity set $H(\lambda^{(i)}) = H_i$ for all $i \in [t]$. Then we define

$$(H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)} := H(\mu),$$

where the partition $\mu \in \mathcal{B}_{n,1}$ is such that for all $j \in [n]$, $\lambda_j^{(1)} + \lambda_{\sigma_1(j)}^{(2)} + \cdots + \lambda_{\sigma_{t-1} \cdots \sigma_1(j)}^{(t)} + \mu_{\sigma_t \cdots \sigma_1(j)} \equiv 0$. Note that, as for the one-dimensional case, the definition of $(H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)}$ doesn't depend on the $\lambda^{(i)}$'s and μ but only on the H_i's and σ_i's.
Observe that the following conditions are equivalent:

i) $(H(\lambda^{(1)}), \ldots, H(\lambda^{(t)}))^{(\sigma_1, \ldots, \sigma_t)} = H(\mu)$;

ii) $\lambda_j^{(1)} + \lambda_{\sigma_1(j)}^{(2)} + \cdots + \lambda_{\sigma_{t-1} \cdots \sigma_1(j)}^{(t)} + \mu_{\sigma_t \cdots \sigma_1(j)} \equiv 0$.

LEMMA 3.11. *Let $n, t \in \mathbf{N}$, $\sigma_i \in S_n$ and $H_i \subseteq [n]$ for all $i \in [t]$. Then*

$$(H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)} = \mathcal{C}_n^{t+1} \left(H_1^{\sigma_t \cdots \sigma_1} \triangle H_2^{\sigma_t \cdots \sigma_2} \triangle \cdots \triangle H_t^{\sigma_t}\right).$$

PROOF. We sketch the proof in the case $t = 2$, for $t > 2$ it is similar. We have to prove that

$$(H_1, H_2)^{(\sigma_1, \sigma_2)} = \mathcal{C}_n (H_1^{\sigma_2 \sigma_1} \triangle H_2^{\sigma_2}).$$

Let $\lambda^{(1)}, \lambda^{(2)}, \mu \in \mathcal{B}_{n,1}$ be such that for all $i = 1, 2$, $H(\lambda^{(i)}) = H_i$, and for all $j \in [n]$

$$\lambda_j^{(1)} + \lambda_{\sigma_1(j)}^{(2)} \equiv \mu_{\sigma_2 \sigma_1(j)}. \tag{3.4}$$

Let $p_j = \lambda_j^{(1)} - \lambda_{j+1}^{(1)}$, $r_j = \lambda_j^{(2)} - \lambda_{j+1}^{(2)}$ and $s_j = \mu_j - \mu_{j+1}$ for all $j \in [n]$. The condition (3.4) is equivalent to

$$s_j \equiv \sum_{i \in [\sigma_2^{-1}(j), \sigma_2^{-1}(j+1))} r_i + \sum_{i \in [\sigma_1^{-1}\sigma_2^{-1}(j), \sigma_1^{-1}\sigma_2^{-1}(j+1))} p_i,$$

and the thesis follows from Lemma 3.7. □

The next result says that the bijection φ_n is "almost" distributive with respect to the symmetric difference of sets.

LEMMA 3.12. *Let $n \in \mathbf{N}$. Then for all $H_1, \ldots, H_t \subseteq [n]$ we have:*

$$\varphi_n(H_1)\triangle \cdots \triangle \varphi_n(H_t) = \varphi_n \mathcal{C}_n^{t+1}\left(H_1 \triangle \cdots \triangle H_t\right).$$

PROOF. We proceed by induction on t. If $t = 1$ it is trivial, so suppose $t = 2$. In this case we have to prove that

$$\varphi_n(H_1)\triangle \varphi_n(H_2) = \varphi_n \mathcal{C}_n\left(H_1 \triangle H_2\right). \tag{3.5}$$

By Lemma 3.6 the set in the left-hand side is given by the $i \in [n]$ that verify exactly one of the following congruences

$$\begin{array}{c} |[i,n] \setminus H_1| \equiv 1 \\ |[i,n] \setminus H_2| \equiv 1. \end{array}$$

Hence

$$\begin{aligned} \varphi_n(H_1)\triangle\varphi_n(H_2) &= \{i \in [n] : |[i,n] \setminus H_1| + |[i,n] \setminus H_2| \equiv 1\} \\ &= \{i \in [n] : |H_1 \triangle H_2 \cap [i,n]| \equiv 1\}. \end{aligned}$$

The set in the right-hand side is

$$\begin{aligned} \varphi_n \mathcal{C}_n(H_1 \triangle H_2) &= \{i \in [n] : |[i,n] \setminus \mathcal{C}_n(H_1 \triangle H_2)| \equiv 1\} \\ &= \{i \in [n] : |H_1 \triangle H_2 \cap [i,n]| \equiv 1\}. \end{aligned}$$

Now suppose $t > 2$. We have

$$\begin{aligned} \varphi_n(H_1)\triangle \cdots \triangle \varphi_n(H_t) &= \left(\varphi_n \mathcal{C}_n^t(H_1 \triangle \cdots \triangle H_{t-1})\right) \triangle \varphi_n(H_t) \\ &= \varphi_n \mathcal{C}_n \left(\mathcal{C}_n^t(H_1 \triangle \cdots \triangle H_{t-1}) \triangle H_t\right) \\ &= \varphi_n \mathcal{C}_n^{t+1}(H_1 \triangle \cdots \triangle H_t), \end{aligned}$$

where we have used the induction hypothesis and (3.5). $\square$

We can now prove the following generalization of Lemma 3.8.

COROLLARY 3.13. *Let $n \in \mathbf{N}$. Then for all $H_1, \cdots, H_t \in [n]$ and $\sigma_1, \ldots, \sigma_t \in S_n$ we have*

$$\sigma_t \cdots \sigma_1 \varphi_n(H_1) \triangle \sigma_t \cdots \sigma_2 \varphi_n(H_2) \triangle \cdots \triangle \sigma_t \varphi_n(H_t) = \varphi_n \left((H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)}\right).$$

PROOF. By Lemmas 3.11, 3.12 and 3.8 there follows that

$$\begin{aligned} \varphi_n \left((H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)}\right) &= \varphi_n(\mathcal{C}_n^{t+1}\left(H_1^{\sigma_t \cdots \sigma_1} \triangle H_2^{\sigma_t \cdots \sigma_2} \triangle \cdots \triangle H_t^{\sigma_t}\right)) \\ &= \varphi_n(H_1^{\sigma_t \cdots \sigma_1}) \triangle \cdots \triangle \varphi_n(H_t^{\sigma_t}) \\ &= \sigma_t \cdots \sigma_1 \varphi_n(H_1) \triangle \cdots \triangle \sigma_t \varphi_n(H_t). \end{aligned}$$

$\square$

Let $\sigma_1, \ldots, \sigma_t \in S_n$ and $H_1, \ldots, H_t \subseteq [n]$. Moreover, let $\lambda^{(1)}, \ldots, \lambda^{(t)} \in \mathcal{B}_{n,1}$ be such that $H(\lambda^{(i)}) = H_i$ for all $i \in [n]$. Then we define

$$\overline{(H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)}} = H(\mu)$$

where $\mu \in \mathcal{B}_{n,1}$ is such that $\lambda_j^{(1)} + \lambda_{\sigma_1(j)}^{(2)} + \cdots + \lambda_{\sigma_{t-1} \cdots \sigma_1(j)}^{(t)} + \mu_{\sigma_t \cdots \sigma_1(j)} \equiv 1$ for all $j \in [n]$.

The following two results are natural generalizations of Lemmas 3.9 and 3.10 and again we leave the proof of the former to the reader.

LEMMA 3.14. *Let $\sigma_1, \ldots, \sigma_t \in S_n$ and $H_1, \ldots, H_t \subseteq [n]$. Then for all $i \in [t]$ we have:*

$$\begin{aligned}\overline{(H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)}} &= (H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)} \triangle \{n\} \\ &= (H_1, \ldots, H_{i-1}, H_i \triangle \{n\}, H_{i+1}, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)}.\end{aligned}$$

LEMMA 3.15. *Let $\sigma_1, \ldots, \sigma_{t-1} \in S_n$ and $K_1, \ldots, K_{t-1} \subseteq [n-1]$. Then*

$$\varphi_{n-1}\big((K_1, \ldots, K_{t-1})^{(\sigma_1, \ldots, \sigma_{t-1})} \setminus \{n\}\big) = p\left(\sigma_{t-1} \cdots \sigma_1 \varphi_{n-1} K_1 \triangle \cdots \triangle \sigma_{t-1} \varphi_{n-1} K_{t-1}\right).$$

PROOF. If $n \notin (K_1, \ldots, K_{t-1})^{(\sigma_1, \ldots, \sigma_{t-1})}$ we have, by Corollary 3.13,

$$\begin{aligned}\varphi_{n-1}\big((K_1, \ldots, K_{t-1})^{(\sigma_1, \ldots, \sigma_{t-1})}\big) &= \mathcal{C}_n \varphi_n\big((K_1, \ldots, K_{t-1})^{(\sigma_1, \ldots, \sigma_{t-1})}\big) \\ &= \mathcal{C}_n\big(\sigma_{t-1} \cdots \sigma_1 \varphi_n K_1 \triangle \cdots \triangle \sigma_{t-1} \varphi_n K_{t-1}\big) \\ &= \mathcal{C}_n\big(\sigma_{t-1} \cdots \sigma_1 \mathcal{C}_n \varphi_{n-1} K_1 \triangle \cdots \triangle \sigma_{t-1} \mathcal{C}_n \varphi_{n-1} K_{t-1}\big) \\ &= \mathcal{C}_n^t\big(\sigma_{t-1} \cdots \sigma_1 \varphi_{n-1} K_1 \triangle \cdots \triangle \sigma_{t-1} \varphi_{n-1} K_{t-1}\big).\end{aligned}$$

Similarly, if $n \in (K_1, \ldots, K_{t-1})^{(\sigma_1, \ldots, \sigma_{t-1})}$ we have that

$$\varphi_{n-1}(K_1, \ldots, K_{t-1})^{(\sigma_1, \ldots, \sigma_{t-1})} = \mathcal{C}_n^{t+1}\big(\sigma_{t-1} \cdots \sigma_1 \varphi_{n-1} K_1 \triangle \cdots \triangle \sigma_{t-1} \varphi_{n-1} K_{t-1}\big)$$

and the result follows. □

3.2.3. The Statistics *ned* and *Dmaj*. In this section we introduce the fundamental statistics *ned* and *Dmaj* and study some of their basic properties. For every $\beta \in B_n$ we define $\bar{\beta} \in B_{n-1}$ by deleting the last entry of β and scaling the others as follows

$$\bar{\beta}(i) := \begin{cases} \beta(i), & \text{if } |\beta(i)| < |\beta(n)|, \\ \beta(i) - 1, & \text{if } \beta(i) > 0 \text{ and } |\beta(i)| > |\beta(n)|, \\ \beta(i) + 1, & \text{if } \beta(i) < 0 \text{ and } |\beta(i)| > |\beta(n)|. \end{cases}$$

For example, if $\beta = [-4, -3, 5, 1, -2] \in B_5$ then $\bar{\beta} = [-3, -2, 4, 1]$.
We let B_n^+ the set of the signed permutations $\beta \in B_n$ such that $\beta(n) > 0$.

LEMMA 3.16. *Let $\beta \in B_n^+$. Then*

$$maj(-\beta) = maj(\beta) + neg(\beta),$$

where $-\beta := [-\beta(1), \ldots, -\beta(n)]$.

PROOF. We proceed by induction on n. For $n = 1$ it's true, so let $n > 1$. We have three cases to consider:

i) $\beta(n-1) > \beta(n)$

Then $maj(\beta) = maj(\bar{\beta}) + n - 1$,$maj(-\beta) = maj(-\bar{\beta}) + n - 1$ and $neg(\beta) = neg(\bar{\beta})$. Since $\bar{\beta} \in B_{n-1}^+$ by induction we have

$$maj(-\bar{\beta}) = maj(\bar{\beta}) + neg(\bar{\beta}) \tag{3.6}$$

and the thesis follows.

ii) $\beta(n) > \beta(n-1) > 0$

Then $maj(\beta) = maj(\bar{\beta})$, $maj(-\beta) = maj(-\bar{\beta})$ and $neg(\beta) = neg(\bar{\beta})$, and the result follows by (3.6), as above.

iii) $\beta(n-1) < 0$

Then we have $maj(-\beta) = maj(-\bar{\beta}) + n - 1$ and $maj(\beta) = maj(\bar{\beta})$. Since $-\beta \in B^+_{n-1}$ by induction there follows that $maj(\bar{\beta}) = maj(-\bar{\beta}) + neg(-\bar{\beta})$. Hence

$$maj(-\beta) = maj(\bar{\beta}) - neg(-\bar{\beta}) + n - 1 = maj(\beta) - neg(-\bar{\beta}) + n - 1,$$

and the result follows since $neg(-\bar{\beta}) = n - 1 - neg(\beta)$. □

COROLLARY 3.17. *Let $\beta \in B^+_n$. Then*

$$fmaj(-\beta) = fmaj(\beta) + n.$$

PROOF. This follows immediately from $neg(-\beta) = n - neg(\beta)$ and Lemma 3.16. □

The verification of the following observation is left to the reader.

LEMMA 3.18. *Let $\sigma \in S_n$ and $H \subseteq [n]$. Then*

$$\overline{(\sigma, \varphi_n(H))} := \begin{cases} -(\bar{\sigma}, \varphi_{n-1}(H)), & \text{if } n \notin H, \\ (\bar{\sigma}, \varphi_{n-1}(H \setminus \{n\})), & \text{if } n \in H. \end{cases}$$

We now define one of the crucial concept of this work. For $(\sigma, H) \in B_n$ then we let

$$ned_B(\sigma, H) := \sum_{i \in H} 2i\varepsilon_i(\sigma) + \sum_{i \in \mathcal{C}_n(H)} i \tag{3.7}$$

and similarly for $(\sigma, K)_D \in D_n$ we let

$$ned_D(\sigma, K) := \sum_{i \in K} 2i\varepsilon_i(\sigma) + \sum_{i \in \mathcal{C}_{n-1}(K)} i. \tag{3.8}$$

For example, if $\beta = [-2, 4, -3, -1] = (2431, \{1, 3, 4\}) \in B_4$ then $ned_B(\beta) = 2 \cdot 3 + 2 = 8$ and if $\gamma = [2, 4, -3, -1] = (2431, \{3\}) \in D_4$ then $ned_D(\gamma) = 2 \cdot 3 + 1 + 2 = 9$. The main property of ned_B is the following one.

THEOREM 3.19. *For every $(\sigma, H) \in B_n$*

$$ned_B(\sigma, H) = fmaj(\sigma, \varphi_n(H)). \tag{3.9}$$

PROOF. We proceed by induction on n, (3.9) being easy to check for $n = 1$. Let $n > 1$, $H \subseteq [n]$ and $\sigma \in S_n$. We have four cases to consider.

a) $n \notin H$, $n - 1 \in Des(\sigma)$ and $n - 1 \in H$

Then

$$\begin{aligned} ned_B(\sigma, H) &= 2(n-1) + \sum_{i \in H} 2i\varepsilon_i(\bar{\sigma}) + \sum_{\mathcal{C}_{n-1}(H)} i + n \\ &= 3n - 2 + ned_B(\bar{\sigma}, H). \end{aligned}$$

Let' s compute the right-hand side of (3.9). We have $n - 1, n \in \varphi_n(H)$ and $n - 1 \notin \varphi_{n-1}(H)$. From this, Lemma 3.18 and Corollary 3.17, it follows that

$$\begin{aligned} fmaj(\sigma, \varphi_n(H)) &= fmaj(\overline{\sigma, \varphi_n(H)}) + 2(n-1) + 1 \\ &= fmaj(-(\bar{\sigma}, \varphi_{n-1}(H))) + 2n - 1 \\ &= fmaj(\bar{\sigma}, \varphi_{n-1}(H)) + 3n - 2, \end{aligned}$$

so (3.9) follows from our induction hypothesis.

b) $n \notin H$ and either $n - 1 \notin Des(\sigma)$ or $n - 1 \notin H$

Then

$$\begin{aligned} ned_B(\sigma, H) &= \sum_{i \in H} 2i\varepsilon_i(\bar{\sigma}) + \sum_{\mathcal{C}_{n-1}(H)} i + n \\ &= ned_B(\bar{\sigma}, H) + n. \end{aligned}$$

Consider now the right-hand side of (3.9). We have two possibilities.
If $n-1 \notin H$ then $n-1 \notin \varphi_n(H)$, $n \in \varphi_n(H)$ and $n-1 \in \varphi_{n-1}(H)$. By Lemma 3.18 and Corollary 3.17 we obtain

$$\begin{aligned} fmaj(\sigma, \varphi_n(H)) &= fmaj(\overline{\sigma, \varphi_n(H)}) + 2(n-1) + 1 \\ &= fmaj(-(\bar{\sigma}, \varphi_{n-1}(H))) + 2n - 1 \\ &= fmaj(\bar{\sigma}, \varphi_{n-1}(H)) + n. \end{aligned}$$

If $n-1 \notin Des(\sigma)$ and $n-1 \in H$ then $n-1, n \in \varphi_n(H)$ and $n-1 \notin \varphi_{n-1}(H)$. By Lemma 3.18 and Corollary 3.17 we have that

$$\begin{aligned} fmaj(\sigma, \varphi_n(H)) &= fmaj(\overline{\sigma, \varphi_n(H)}) + 1 \\ &= fmaj(-(\bar{\sigma}, \varphi_{n-1}(H))) + 1 \\ &= fmaj(\bar{\sigma}, \varphi_{n-1}(H)) + (n-1) + 1, \end{aligned}$$

and (3.9) follows.

c) $n \in H$, $n-1 \in Des(\sigma)$ and $n-1 \in H$

Then

$$\begin{aligned} ned_B(\sigma, H) &= \sum_{i \in H \setminus \{n\}} 2i\varepsilon_i(\bar{\sigma}) + \sum_{\mathcal{C}_{n-1}(H \setminus \{n\})} i + 2n - 2 \\ &= ned_B(\bar{\sigma}, H \setminus \{n\}) + 2n - 2. \end{aligned}$$

On the other hand, from $n-1, n \notin \varphi_n(H)$ and Lemma 3.18 we have that

$$\begin{aligned} fmaj(\sigma, \varphi_n(H)) &= fmaj(\overline{\sigma, \varphi_{n-1}(H)}) + 2(n-1) \\ &= fmaj(\bar{\sigma}, \varphi_{n-1}(H \setminus \{n\})) + 2n - 2, \end{aligned}$$

and (3.9) again follows.

d) $n \in H$ and either $n-1 \notin Des(\sigma)$ or $n-1 \notin H$

Then

$$\begin{aligned} ned_B(\sigma, H) &= \sum_{i \in H \setminus \{n\}} 2i\varepsilon_i(\bar{\sigma}) + \sum_{\mathcal{C}_{n-1}(H \setminus \{n\})} i \\ &= ned_B(\bar{\sigma}, H \setminus \{n\}). \end{aligned}$$

But $n \notin \varphi_n(H)$ hence by Lemma 3.18 it follows that

$$fmaj(\sigma, \varphi_n(H)) = fmaj(\overline{\sigma, \varphi_n(H)}) = fmaj(\bar{\sigma}, \varphi_{n-1}(H \setminus \{n\})),$$

and this concludes the proof. $\square$

COROLLARY 3.20. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\beta \in B_n} q^{ned_B(\beta)} = \sum_{\beta \in B_n} q^{fmaj(\beta)}.$$

□

The following statistic is fundamental for this work and its definition is naturally suggested by Theorem 3.19. We will show in §3.3 and in §3.4 that it's Mahonian and, moreover, that it plays the same algebraic role for D_n, as maj for S_n and $fmaj$ for B_n, in the Hilbert series of DIA and TIA.
Let $\gamma \in D_n$, we define

$$Dmaj(\gamma) := fmaj([\gamma_1, \ldots, \gamma_{n-1}, |\gamma_n|]).$$

For example, if $\gamma = [-2, 3, -1, -5, -4]$, then $Dmaj(\gamma) = fmaj([-2, 3, -1, -5, 4]) = 2 \cdot 2 + 3 = 7$. Note that $Dmaj((\sigma, K)_D) = fmaj((\sigma, K))$.
The next result follows immediately from Theorem 3.19.

Corollary 3.21. *Let $(\sigma, K) \in D_n$, then*

$$ned_D(\sigma, K) = Dmaj(\sigma, \varphi_{n-1}(K)).$$

□

3.3. The Main Result

In this section we use the combinatorial tools developed in §3.2 to find a closed formula for $\mathcal{Z}_{D_n}(\bar{q})$ in terms of the statistic $Dmaj$.

3.3.1. A Basis for TIA and DIA for D_n. Let $W = D_n$. The tensor invariant algebra TIA is $(\mathbf{P}_n^{D_n})^{\otimes t}$, and $\mathbf{P}_n^{D_n}$ is freely generated (as an algebra) by the $n-1$ elementary symmetric functions $e_j(x_1^2, \ldots, x_n^2)$ for $j \in [n-1]$ and the monomial $x_1 \cdots x_n$ (see, e.g., [**42**, §3]). Hence

$$F_T(\bar{q}) = \prod_{i=1}^{t} \Big(\frac{1}{(1-q_i^n)} \prod_{j=1}^{n-1} \frac{1}{(1-q_i^{2j})} \Big).$$

A linear basis for $\mathbf{P}_n^{\otimes t}$ consists of all tensor monomials

$$\bar{x}^f := \bigotimes_{i=1}^{t} \prod_{j=1}^{n} x_j^{f_i(j)}$$

where $f = (f_1, \ldots, f_t) \in \mathcal{F}_{n,t}$. The canonical projection $\pi : \mathbf{P}_n^{\otimes t} \to \mathrm{DIA}$ is defined by

$$\pi(\bar{p}) := \sum_{\gamma \in D_n} \varphi_D(\gamma)(\bar{p})$$

so that

$$\mathrm{DIA} = < \{\pi(\bar{x}^f) : \ f \in \mathcal{F}_{n,t}\} > .$$

Lemma 3.22. *For $f \in \mathcal{F}_{n,t}$,*

$$\pi(\bar{x}^f) \neq 0 \iff f \in \mathcal{F}_{n,t}^e \cup \mathcal{F}_{n,t}^o,$$

where $\mathcal{F}_{n,t}^e$ and $\mathcal{F}_{n,t}^o$ are defined in §3.1.3

Proof. Let $\delta_i = [-1, 2, 3, \ldots, -i, \ldots, n]$ for $i \in [2, n]$. Note that

$$\varphi_D(\delta_i)(\bar{x}^f) = (-1)^{\alpha_1(f)+\alpha_i(f)} \bar{x}^f.$$

Therefore, if C is any coset in D_n of the subgroup $T_i = \{e, \delta_i\}$, then

$$\sum_{\gamma \in C} \varphi_D(\gamma)(\bar{x}^f) \neq 0$$

if and only if

$$\alpha_1(f) + \alpha_i(f) \equiv 0.$$

Hence we conclude that if $\pi(\bar{x}^f) \neq 0$ then $f \in \mathcal{F}^e_{n,t} \cup \mathcal{F}^o_{n,t}$.
For the converse, let H be the subgroup of all the *generalized identity permutations* $h = (e, B) \in D_n$, (i.e. $|h(i)| = i$ for all $i \in [n]$). We have that $D_n = S_n \ltimes H$, hence every $\gamma \in D_n$ has a unique representation $\gamma = \sigma \cdot h$ with $\sigma \in S_n$ and $h \in H$.
For any $f \in \mathcal{F}^e_{n,t} \cup \mathcal{F}^o_{n,t}$ and for any $h \in H$ we have $\varphi_D(h)(\bar{x}^f) = \bar{x}^f$ hence

$$\sum_{\gamma \in \sigma H} \varphi_D(\gamma)(\bar{x}^f) = |H| \varphi_D(\sigma)(\bar{x}^f),$$

for any $\sigma \in S_n$, and the thesis follows. □

Clearly $\mathcal{B}^e_{n,t} \cup \mathcal{B}^o_{n,t}$ is a complete system of representatives for the orbits of all $f \in \mathcal{F}^e_{n,t} \cup \mathcal{F}^o_{n,t}$, under the action of the symmetric group. Hence we have

PROPOSITION 3.23. *The set*

$$\left\{\pi(\bar{x}^f) : f \in \mathcal{B}^e_{n,t} \cup \mathcal{B}^o_{n,t}\right\}$$

is a homogeneous basis for DIA.

COROLLARY 3.24. *The Hilbert series for* DIA *is*

$$F_D(\bar{q}) = \sum_{f \in \mathcal{B}^e_{n,t} \cup \mathcal{B}^o_{n,t}} q_1^{|f_1|} \cdots q_t^{|f_t|}.$$

3.3.2. The Polynomial $\mathcal{Z}_{D_n}(q_1, q_2)$. We define an involution $\alpha : D_n \to D_n$ by

$$(\sigma, K) \quad \mapsto \quad (\sigma^{-1}, p(\sigma(K))), \tag{3.10}$$

where p is the projection defined in (3.3)
For example, $\alpha(4213, \{1,3\}) = (3241, p(\{1,4\}) = (3241, \{2,3\})$.
We are now ready to state and prove the following

THEOREM 3.25. *Let $n \in \mathbf{N}$. Then*

$$\mathcal{Z}_{D_n}(q_1, q_2) = \sum_{\gamma \in D_n} q_1^{Dmaj(\gamma)} q_2^{Dmaj(\alpha(\gamma))}.$$

PROOF. By Corollary 3.24 and the note below Theorem 3.2 we have that

$$\begin{aligned} F_D(q_1, q_2) &= \sum_{f \in \mathcal{B}^e_{n,2} \cup \mathcal{B}^o_{n,2}} q_1^{|f_1|} q_2^{|f_2|} \\ &= \sum_{(\sigma,\lambda,\mu) \in \mathcal{P}^e_{n,2}} q_1^{|\lambda|} q_2^{|\mu|} + \sum_{(\sigma,\lambda,\mu) \in \mathcal{P}^o_{n,2}} q_1^{|\lambda|} q_2^{|\mu|}. \end{aligned} \tag{3.11}$$

The first part in (3.11) can be rewritten as

$$\sum_{(\sigma,\lambda,\mu) \in \mathcal{P}^e_{n,2}} q_1^{|\lambda|} q_2^{|\mu|} = \sum_{\sigma \in S_n} \sum_{p_i, r_i} q_1^{\sum j p_j} q_2^{\sum j r_j}$$

where the last sum runs through all $p_i, r_i \in \mathbf{N}$, $i \in [n]$, such that $p_i \geq \varepsilon_i(\sigma)$, $r_i \geq \varepsilon_i(\sigma^{-1})$ and $(H(\lambda))^\sigma = H(\mu)$, with $\lambda_i = p_i + \cdots + p_n$ and $\mu_i = r_i + \cdots + r_n$. Now we split the previous sum according to the parity set of λ. Note that p_i is even if and only if $i \in H(\lambda)$, and similarly for μ. Hence we obtain $\sum_{\mathcal{P}^e_{n,2}} q_1^{|\lambda|} q_2^{|\mu|} =$

$$\sum_{\sigma \in S_n} \sum_{H \subseteq [n]} \left(\prod_{i \in H} q_1^{2i\varepsilon_i(\sigma)} \prod_{i \in \mathcal{C}_n(H)} q_1^i \prod_{i \in H^\sigma} q_2^{2i\varepsilon_i(\sigma^{-1})} \prod_{i \in \mathcal{C}_n(H^\sigma)} q_2^i \sum_{\pi_i, \rho_i \in \mathbf{N}} q_1^{2\sum j\pi_j} q_2^{2\sum j\rho_j} \right)$$

(3.12)

$$= \prod_{i=1}^{2} \prod_{j=1}^{n} \frac{1}{(1 - q_i^{2j})} \sum_{\sigma \in S_n} \sum_{H \subseteq [n]} \left(\prod_{i \in H} q_1^{2i\varepsilon_i(\sigma)} \prod_{i \in \mathcal{C}_n(H)} q_1^i \prod_{i \in H^\sigma} q_2^{2i\varepsilon_i(\sigma^{-1})} \prod_{i \in \mathcal{C}_n(H^\sigma)} q_2^i \right)$$

where $p_i = 2\pi_i + 2\varepsilon_i(\sigma)$ for $i \in H$, $p_i = 2\pi_i + 1$ for $i \in \mathcal{C}_n(H)$, $r_i = 2\rho_i + 2\varepsilon_i(\sigma^{-1})$ for $i \in A^\sigma$ and $r_i = 2\rho_i + 1$ for $i \in \mathcal{C}_n(A^\sigma)$.

Analogously we can evaluate the second part of (3.11), substituting A^σ with $\overline{A^\sigma}$, obtaining $\sum_{(\sigma,\lambda,\mu) \in \mathcal{P}^o_{n,2}} q_1^{|\lambda|} q_2^{|\mu|} =$

(3.13)

$$\prod_{i=1}^{2} \prod_{j=1}^{n} \frac{1}{(1 - q_i^{2j})} \sum_{\sigma \in S_n} \sum_{H \subseteq [n]} \left(\prod_{i \in H} q_1^{2i\varepsilon_i(\sigma)} \prod_{i \in \mathcal{C}_n(H)} q_1^i \prod_{i \in \overline{H^\sigma}} q_2^{2i\varepsilon_i(\sigma^{-1})} \prod_{i \in \mathcal{C}_n(\overline{H^\sigma})} q_2^i \right).$$

Hence by (3.11), (3.12) and (3.13) we have that

$$\begin{aligned}
F_D(q_1, q_2) \prod_{i=1}^{2} \prod_{j=1}^{n} (1 - q_i^{2j}) &= \sum_{\sigma} \sum_{H \subseteq [n]} \prod_{i \in H} q_1^{2i\varepsilon_i(\sigma)} \prod_{i \in \mathcal{C}_n(H)} q_1^i \cdot \\
&\left(\prod_{i \in H^\sigma} q_2^{2i\varepsilon_i(\sigma^{-1})} \prod_{i \in \mathcal{C}_n(H^\sigma)} q_2^i + \prod_{i \in \overline{H^\sigma}} q_2^{2i\varepsilon_i(\sigma^{-1})} \prod_{i \in \mathcal{C}_n(\overline{H^\sigma})} q_2^i \right) \\
&= \sum_{\sigma} \sum_{K \subseteq [n-1]} \sum_{H \in \{K, K \cup \{n\}\}} \prod_{i \in H} q_1^{2i\varepsilon_i(\sigma)} \prod_{i \in \mathcal{C}_n(H)} q_1^i \cdot \\
&\left(\prod_{i \in H^\sigma} q_2^{2i\varepsilon_i(\sigma^{-1})} \prod_{i \in \mathcal{C}_n(H^\sigma)} q_2^i + \prod_{i \in H^\sigma \triangle \{n\}} q_2^{2i\varepsilon_i(\sigma^{-1})} \prod_{i \in \mathcal{C}_n(H^\sigma \triangle \{n\})} q_2^i \right) \\
&= \sum_{\sigma} \sum_{K \subseteq [n-1]} (1 + q_1^n)(1 + q_2^n) \prod_{i \in K \cup \{n\}} q_1^{2i\varepsilon_i(\sigma)} \prod_{i \in \mathcal{C}_n(K \cup \{n\})} q_1^i \cdot \\
&\qquad \prod_{i \in K^\sigma \cup \{n\}} q_2^{2i\varepsilon_i(\sigma^{-1})} \prod_{i \in \mathcal{C}_n(K^\sigma \cup \{n\})} q_2^i
\end{aligned}$$

where we have used the fact that $\varepsilon_n(\sigma) = 0$ for all $\sigma \in S_n$, and Lemma 3.9. Applying Corollary 3.21 and Lemma 3.10 it follows that

$$\begin{aligned}
\mathcal{Z}_{D_n}(q_1, q_2) &= \sum_{\sigma} \sum_{K \subseteq [n-1]} q_1^{ned_D(\sigma, K)} q_2^{ned_D(\sigma^{-1}, (K^\sigma \setminus \{n\}))} \\
&= \sum_{\sigma} \sum_{K \subseteq [n-1]} q_1^{Dmaj(\sigma, \varphi_{n-1}(K))} q_2^{Dmaj(\sigma^{-1}, \varphi_{n-1}(K^\sigma \setminus \{n\}))} \\
&= \sum_{\sigma} \sum_{K \subseteq [n-1]} q_1^{Dmaj(\sigma, \varphi_{n-1}(K))} q_2^{Dmaj(\sigma^{-1}, p\sigma\varphi_{n-1}(K))} \\
&= \sum_{\gamma \in D_n} q_1^{Dmaj(\gamma)} q_2^{Dmaj(\alpha(\gamma))},
\end{aligned}$$

as desired. □

We denote by ι the inversion in D_n so that $\iota(\gamma) := \gamma^{-1}$. The next lemma says that it is possible to "substitute" α with ι in Theorem 3.25.

LEMMA 3.26. *α and ι are conjugate in $S(D_n)$.*

PROOF. It is well known that two elements of a symmetric group are conjugate if and only if they have the same cycle type. Since both α and ι are involutions it is enough to show that they have the same number of fixed points. For this it is sufficient to show that, if we set, for $\sigma \in S_n$,

$$i_\sigma = \left| \left\{ K \in 2^{[n-1]} : (\sigma, K)^{-1} = (\sigma, K) \right\} \right|$$

and

$$a_\sigma = \left| \left\{ K \in 2^{[n-1]} : \alpha(\sigma, K) = (\sigma, K) \right\} \right|,$$

then $i_\sigma = a_\sigma$, for all $\sigma \in S_n$. It is clear that $i_\sigma = a_\sigma = 0$ if σ is not an involution in S_n. On the other hand if σ is an involution with some fixed point then we have $i_\sigma = a_\sigma = 2^{c_1(\sigma)+c_2(\sigma)-1}$ while if σ has no fixed point then $a_\sigma = i_\sigma = 2^{c_2(\sigma)}$, where $c_i(\sigma)$ is the number of cycles of length i of σ. □

COROLLARY 3.27. *There exists a function $M : D_n \to \mathbf{N}$, equidistributed with length, such that*

$$\mathcal{Z}_{D_n}(q_1, q_2) = \sum_{\gamma \in D_n} q_1^{M(\gamma)} q_2^{M(\gamma^{-1})}.$$

PROOF. By Lemma 3.26 we know that there exists $\psi \in S(D_n)$ such that $\alpha\psi = \psi\iota$. Then the function $M := Dmaj \circ \psi$ realizes the above formula for $\mathcal{Z}_{D_n}(q_1, q_2)$. It follows immediately from next Proposition 3.35 that this M is equidistributed with length on D_n. □

3.3.3. The Polynomial $\mathcal{Z}_{D_n}(\bar{q})$. In this section we provide an explicit simple formula for the polynomial $\mathcal{Z}_{D_n}(\bar{q})$ in terms of $Dmaj$.
We denote by $\alpha : D_n^{t-1} \to D_n$ the map

$$((\sigma_1, K_1), \ldots, (\sigma_{t-1}, K_{t-1})) \quad \mapsto \quad \left((\sigma_{t-1} \cdots \sigma_1)^{-1}, p\left(\sigma_{t-1} \cdots \sigma_1 K_1 \triangle \cdots \triangle \sigma_{t-1} K_{t-1}\right)\right).$$

For example,

$$\begin{aligned}
\alpha\left((4231, \{1,3\}), (2143, \{3\})\right) &= (2413, p(3142(\{1,3\}) \triangle 2143(\{3\})) \\
&= (2413, p(\{3\})) = (2413, \{3\}).
\end{aligned}$$

Note that this is consistent with the definition of α given in (3.10).

THEOREM 3.28. *Let $n \in \mathbf{N}$. Then*

$$\mathcal{Z}_{D_n}(\bar{q}) = \sum_{\gamma_1,\ldots,\gamma_t \in D_n} \prod_{i=1}^{t} q_i^{Dmaj(\gamma_i)},$$

where the sum runs through all $\gamma_1,\ldots,\gamma_t \in D_n$ such that $\gamma_t = \alpha(\gamma_1,\ldots,\gamma_{t-1})$.

PROOF. The proof is similar to that of Theorem 3.25, and hence we will not go through all the details. By Corollary 3.24 we have that

$$F_D(\bar{q}) = \sum_{(f_1,\ldots,f_t)\in \mathcal{B}^e_{n,t}\cup \mathcal{B}^o_{n,t}} \prod_{i=1}^{t} q_i^{|f_i|}. \tag{3.14}$$

Let's consider the sum in (3.14) restricted to $\mathcal{B}^e_{n,t}$. By the note first Theorem 3.4 we have that

(3.15)
$$\sum_{(f_1,\ldots f_t)\in \mathcal{B}^e_{n,t}} \prod_{i=1}^{t} q_i^{|f_i|} = \sum_{(\sigma_1,\ldots,\sigma_t,\lambda^{(1)},\ldots,\lambda^{(t)})\in \mathcal{P}^e_{n,t}} \prod_{i=1}^{t} q_i^{|\lambda_i|} = \sum_{\sigma_t\cdots\sigma_1=e} \sum_{p_j^{(i)}} \prod_{i=1}^{t} q_i^{\sum k p_k^{(i)}},$$

where the last sum is over all $p_j^{(i)} \in \mathbf{N}$, for $i \in [n]$ and $j \in [t]$, such that $p_j^{(i)} \geq \varepsilon_j(\sigma_i)$ and $(H(\lambda^{(1)}),\ldots,H(\lambda^{(t-1)}))^{(\sigma_1,\ldots,\sigma_{t-1})} = H(\lambda^{(t)})$,with $\lambda_j^{(i)} = p_j^{(i)} + \cdots + p_n^{(i)}$. We proceed in a similar way for the sum in (3.14) over $\mathcal{B}^o_{n,t}$. If we split these sums according to the parity sets of the $\lambda^{(i)}$'s for $i \in [t-1]$ we obtain, by Lemma 3.14, that

$$F_D(\bar{q}) \prod_{i=1}^{n}\prod_{j=1}^{t}(1-q_j^{2i}) = \sum_{\sigma_1,\ldots,\sigma_t} \sum_{H_1,\ldots,H_t} \prod_{j=1}^{t-1} \prod_{h\in H_j} q_j^{2h\varepsilon_h(\sigma_j)} \prod_{h\in \mathcal{C}_n(H_j)} q_j^h \cdot \tag{3.16}$$

$$\left(\prod_{h\in H_t} q_t^{2h\varepsilon_h(\sigma_t)} \prod_{h\in \mathcal{C}_n(H_t)} q_t^h + \prod_{h\in H_t\triangle\{n\}} q_t^{2h\varepsilon_h(\sigma_t)} \prod_{h\in \mathcal{C}_n(H_t\triangle\{n\})} q_t^h \right),$$

where the sums run through all $\sigma_1,\ldots,\sigma_t \in S_n$ such that $\sigma_t = (\sigma_{t-1}\cdots\sigma_1)^{-1}$ and all $H_1,\ldots,H_t \subseteq [n]$ such that $H_t = (H_1,\ldots,H_{t-1})^{(\sigma_1,\ldots,\sigma_{t-1})}$. Now using the fact that $\varepsilon_n(\sigma) = 0$ for all $\sigma \in S_n$, and Lemma 3.14 we obtain that

$$F_D(\bar{q}) \prod_{i=1}^{n}\prod_{j=1}^{t}(1-q_j^{2i}) = \sum_{\sigma_1,\ldots,\sigma_t} \sum_{K_1,\ldots,K_t} \prod_{i=1}^{t}(1+q_i^n) \prod_{k\in K_i\cup\{n\}} q_k^{2k\varepsilon_k(\sigma_i)} \prod_{k\in \mathcal{C}_n(K_i\cup\{n\})} q_k^i$$

where the second sum runs over all $K_1,\ldots,K_t \subseteq [n-1]$ such that $K_t = (K_1,\ldots,K_{t-1})^{(\sigma_1,\ldots,\sigma_{t-1})}$ and hence, by Corollary 3.21, Lemma 3.15 and the definition of α we conclude that

$$\begin{aligned}
\mathcal{Z}_{D_n}(\bar{q}) &= \sum_{\sigma_1,\ldots,\sigma_t} \sum_{K_1,\ldots,K_t} \prod_{k \in K_i \cup \{n\}} q_k^{2k\varepsilon_k(\sigma_i)} \prod_{k \in \mathcal{C}_n(K_i \cup \{n\})} q_k^i \\
&= \sum_{\sigma_1,\ldots,\sigma_t} \sum_{K_1,\ldots,K_t} \prod_{i=1}^{t} q_i^{ned_D(\sigma_i, K_i \setminus \{n\})} \\
&= \sum_{\sigma_i} \sum_{K_i} \prod_{i=1}^{t-1} q_i^{Dmaj(\sigma_i, \varphi_{n-1}(K_i)} q_t^{Dmaj(\sigma_t, \varphi_{n-1}\left((K_1,\ldots,K_{t-1})^{(\sigma_1,\ldots,\sigma_{t-1})} \setminus \{n\}\right)} \\
&= \sum_{\sigma_i} \sum_{K_i} \prod_{i=1}^{t-1} q_i^{Dmaj(\sigma_i, \varphi_{n-1}K_i)} q_t^{Dmaj(\sigma_t, p(\sigma_{t-1}\cdots\sigma_1\varphi_{n-1}K_1 \triangle \cdots \triangle \sigma_1 \varphi_{n-1} K_{t-1}))} \\
&= \sum_{\gamma_1,\ldots,\gamma_{t-1} \in D_n} \prod_{i=1}^{t-1} q_i^{Dmaj(\gamma_i)} q_t^{Dmaj(\alpha(\gamma_1,\ldots,\gamma_{t-1}))}.
\end{aligned}$$

$\square$

3.3.4. The case n odd. If n is odd the formula appearing in Theorem 3.28 can be slightly improved. In particular we define one more statistic, $Dmaj^o$, that allows us to obtain a formula for $\mathcal{Z}_{D_n}(\bar{q})$ similar to the corresponding ones for S_n and B_n appearing in Theorem 3.4 and Theorem 3.5. Consider the set $S_n \times 2^{[n-1]}$ with the binary operation

$$(\sigma, H) * (\tau, K) := \left(\sigma\tau, p(K \triangle \tau^{-1}(H))\right).$$

PROPOSITION 3.29. *Let $n > 1$. Then $\Delta_n = \left(S_n \times 2^{[n-1]}, *\right)$ is a group.*

PROOF. The operation is clearly well-defined, the identity element is $(e, \emptyset)$ and inversion is given by $(\sigma, H)^{-1} = (\sigma^{-1}, p\sigma(H))$. We check the associativity property

$$\begin{aligned}
(\sigma, H) * ((\tau, K) * (\upsilon, L)) &= (\sigma, H) * \left(\tau\upsilon, p(L \triangle \upsilon^{-1}(K))\right) \\
&= \left(\sigma\tau\upsilon, p(p(L \triangle \upsilon^{-1}(K)) \triangle \upsilon^{-1}\tau^{-1}(H))\right) \\
&= \left(\sigma\tau\upsilon, p(L \triangle \upsilon^{-1}(K) \triangle \upsilon^{-1}\tau^{-1}(H))\right)
\end{aligned}$$

and

$$\begin{aligned}
((\sigma, H) * (\tau, K)) * (\upsilon, L) &= \left(\sigma\tau, p(K \triangle \tau^{-1}(H))\right) * (\upsilon, L) \\
&= \left(\sigma\tau\upsilon, p(L \triangle \upsilon^{-1} p(K \triangle \tau^{-1}(H)))\right) \\
&= \left(\sigma\tau\upsilon, p(L \triangle \upsilon^{-1}(K) \triangle \upsilon^{-1}\tau^{-1}(H))\right),
\end{aligned}$$

where we have used the distributivity of υ^{-1} with respect to the symmetric difference and the fact that $p(p(H) \triangle K) = p(H \triangle K)$ for all $H, K \subseteq [n]$. $\square$

THEOREM 3.30. *Δ_n is isomorphic to D_n if and only if n is odd.*

PROOF. It's not difficult to see that, if n is odd, the map $\Phi : D_n \to \Delta_n$ defined by

$$\gamma \mapsto (|\gamma|, p(Neg(\gamma))) \tag{3.17}$$

is an isomorphism, where $|\gamma| := (|\gamma_1|, \ldots, |\gamma_n|)$. Now suppose that n is even and let $\varphi : D_n \to \Delta_n$ be a group homomorphism. Let $(\sigma_i, K_i) = \varphi(s_i)$, for $i = 0, \ldots, n-1$, be the images of the Coxeter generators of D_n. Then the Coxeter relations for D_n force the permutations $\sigma_0, \ldots, \sigma_{n-1}$ to have the same sign and the sets $K_0, \dots, K_{n-1}$ to have all the same parity. These conditions imply that the set $\{(\sigma_i, K_i) : i = 0, \dots, n-1\}$ cannot generate Δ_n. $\square$

Let $n \in \mathbf{N}$ be odd. Then we let

$$Dmaj^o := Dmaj \circ \Phi,$$

where we identify Δ_n with D_n through the pair notation and Φ is defined as in (3.17).
For example $Dmaj^o([3, -1, 5, 2, -4]) = Dmaj(31524, \{1, 3, 4\}) = 2 \cdot 5 + 3 = 13$.

Corollary 3.31. *Let* $n \in \mathbf{N}$. *Then*

$$\mathcal{Z}_{D_{2n+1}}(\bar{q}) = \sum_{\gamma_1, \dots, \gamma_t} \prod_{i=1}^{t} q_i^{Dmaj^o(\gamma_i)}.$$

where the sum is over all $\gamma_1, \dots, \gamma_t \in D_{2n+1}$ *such that* $\gamma_t \cdots \gamma_1 = e$.

Proof. It is an immediate consequence of the proof of Theorem 3.30 that

$$\alpha(\Phi(\gamma_1), \dots, \Phi(\gamma_2)) = \Phi(\gamma_t \cdots \gamma_1)^{-1}$$

and the thesis follows from Theorem 3.28. $\square$

Theorem 3.30 implies that, if n is even, there is no $\Phi \in S(D_n)$ such that $\alpha(\Phi(\gamma_1), \dots, \Phi(\gamma_2)) = \Phi(\gamma_t \cdots \gamma_1)^{-1}$ that would imply the corresponding result of Corollary 3.31. Nevertheless, we know that this result holds for $t = 2$ (Corollary 3.27) but we haven't been able to define a nice statistic, $Dmaj^e$, that works in this case, or to understand if it exists for $t > 2$. We therefore propose the following

Problem 2. Let $n \in \mathbf{N}$ be even. Is there a statistic $Dmaj^e : D_n \to \mathbf{N}$, necessarily equidistributed with length on D_n, such that

$$\mathcal{Z}_{D_n}(\bar{q}) = \sum_{\gamma_1, \dots, \gamma_t} \prod_{i=1}^{t} q_i^{Dmaj^e(\gamma_i)}$$

with $\gamma_t \cdots \gamma_1 = e$?

3.4. Applications to Weyl groups of type B

In this section we show how the ideas developed for the Weyl groups of type D can be used to give a new and simpler proof of the closed formula for $\mathcal{Z}_{B_n}(\bar{q})$ appearing in Theorem 3.5 which was discovered by Adin and Roichman [**3**] using different methods.

3.4.1. A Basis for TIA and DIA for B_n. Let $W = B_n$. The tensor invariant algebra TIA is clearly equal to $(\mathbf{P}_n^{B_n})^{\otimes t}$. It is well known, (see, e.g.,[**42**, §3]), that $\mathbf{P}_n^{B_n}$ is freely generated (as an algebra) by the n elementary symmetric functions in the squares of the indeterminates, $x_1^2, \dots, x_n^2$,

$$e_j(x_1^2, \dots, x_n^2) := \sum_{1 \le i_1 < \dots < i_j \le n} x_{i_1}^2 \cdots x_{i_j}^2$$

for $j \in [n]$. Hence

$$F_T(\bar{q}) = \prod_{i=1}^{t} \prod_{j=1}^{n} \frac{1}{(1 - q_i^{2j})}.$$

For all $\bar{x}^f \in \mathbf{P}_n^{\otimes t}$ let

$$\pi(\bar{x}^f) := \sum_{\beta \in B_n} \varphi_D(\beta)(\bar{x}^f)$$

be the corresponding invariant element in DIA. In [**3**] the authors prove that

$$\pi(\bar{x}^f) \neq 0 \Longleftrightarrow f \in \mathcal{F}^e_{n,t},$$

which implies the following

LEMMA 3.32. *The set*

$$\left\{\pi(\bar{x}^f) : f \in \mathcal{B}^e_{n,t}\right\}$$

is a homogeneous basis for DIA.

COROLLARY 3.33. *The Hilbert series for* DIA *is*

$$F_D(\bar{q}) = \sum_{f \in \mathcal{B}^e_{n,t}} q_1^{|f_1|} \cdots q_t^{|f_t|}.$$

Note that we choose a different parametrization of the basis of DIA with respect to [**3**] and we use this one to compute the generating function $F_D(\bar{q})$.

3.4.2. The polynomial $\mathcal{Z}_{B_n}(\bar{q})$. We provide a new proof of Theorem 3.5 using the statistic *ned* introduced in (3.7).

THEOREM 3.34. *Let $n, t \in \mathbf{N}$. Then*

$$\mathcal{Z}_{B_n}(\bar{q}) = \sum_{\beta_1, \ldots, \beta_t \in B_n} \prod_{i=1}^{t} q_i^{fmaj(\beta_i)},$$

where the sum is over all the signed permutation $\beta_1, \ldots, \beta_t \in B_n$ such that $\beta_t \cdots \beta_1 = e$.

PROOF. By Corollary 3.33, (3.15) and (3.16) in the proof of Theorem 3.28 we easily obtain that

$$\mathcal{Z}_{B_n}(\bar{q}) = \sum_{\sigma_1, \ldots, \sigma_t} \sum_{H_1, \ldots, H_t} \prod_{i=1}^{t} \left(\prod_{h \in H_i} q_i^{2h\varepsilon_h(\sigma_i)} \prod_{h \in \mathcal{C}_n(H_i)} q_i^h \right),$$

where the sums run through all $\sigma_1, \ldots, \sigma_t \in S_n$ and $H_1, \ldots, H_t \subseteq [n]$ such that $\sigma_t = (\sigma_{t-1} \cdots \sigma_1)^{-1}$ and $H_t = (H_1, \ldots, H_{t-1})^{(\sigma_1, \ldots, \sigma_{t-1})}$. By Theorem 3.19 and Corollary 3.13, we conclude that

$$\begin{aligned}
\mathcal{Z}_{B_n}(\bar{q}) &= \sum_{\sigma_1, \ldots, \sigma_t} \sum_{H_1, \ldots, H_t} \prod_{i=1}^{t} q_i^{ned_B(\sigma_i, (H_i))} \\
&= \sum_{\sigma_1, \ldots, \sigma_t} \sum_{H_1, \ldots, H_t} \prod_{i=1}^{t} q_i^{fmaj(\sigma_i, \varphi_n(H_i))} \\
&= \sum_{\sigma_i} \sum_{H_i} \prod_{i=1}^{t-1} q_i^{fmaj(\sigma_i, \varphi_n(H_i))} q_t^{fmaj(\sigma_t, \sigma_{t-1} \ldots \sigma_1 \varphi_n(H_1) \triangle \cdots \triangle \sigma_{t-1} \varphi_n(H_{t-1}))} \\
&= \sum_{\beta_t \cdots \beta_1 = e} \prod_{i=1}^{t} q_i^{fmaj(\beta_i)},
\end{aligned}$$

since

$$((\sigma_{t-1}, H_{t-1}) \cdots (\sigma_1, H_1))^{-1} = ((\sigma_{t-1} \cdots \sigma_1)^{-1}, \sigma_{t-1} \cdots \sigma_1(H_1) \triangle \cdots \triangle \sigma_{t-1}(H_{t-1})).$$

□

3.5. Combinatorial Properties of *Dmaj*

In this section we show that $Dmaj$ is a Mahonian statistic. Moreover we introduce two new "descent numbers" on D_n, $Ddes$ and $fmaj_D$ so that the pairs $(Ddes, Dmaj)$ and $(fdes_D, fmaj_D)$ give two different generalizations of Carlitz's identity to D_n. Note that the *flag-descent number* $fdes_D$ is the Eulerian statistic pre-announced in §2.3.

With the notation of §2.2, it is not hard to see that for each $n \geq 2$ there is the following decomposition of B_n^+,

$$B_n^+ = \bigcup_c \bigcup_\xi \left(\{\tau_{n-1}^c \xi\} \biguplus \{\tau_{n-1}^c \tau_{n-2}^{n-1} \xi\}\right),$$

where $c = 0, \ldots, n-1$ and $\xi \in B_{n-1}^+$.

Proposition 3.35. *Let* $n \in \mathbf{P}$. *Then*

$$\sum_{\gamma \in D_n} q^{Dmaj(\gamma)} = \sum_{\gamma \in D_n} q^{\ell(\gamma)}.$$

Proof. We define a map $\Psi : D_n \to B_n^+$ as follows:

$$\Psi(\prod_{i=1}^{n-1} t_0^{h_{n-i}} t_{n-i}^{k_{n-i}}) := \prod_{i=1}^{n-1} \Psi(t_0^{h_{n-i}} t_{n-i}^{k_{n-i}}),$$

where

$$\begin{array}{rcll} \Psi(t_0 t_{n-i}^{2n-2i-1}) & := & \tau_{n-i}^{n-i} \tau_{n-i-1}^{n-i}; & \\ \Psi(t_0 t_{n-i}^{n-i-1}) & := & \tau_{n-i-1}^{n-i}; & \\ \Psi(t_{n-i}^{k_{n-i}}) & := & \tau_{n-i}^{k_{k-i}}, & \text{if } k_{n-1} \leq n-i; \\ \Psi(t_{n-i}^{k_{n-i}}) & := & \tau_{n-i}^{k_{n-i}-n+i} \tau_{n-i-1}^{n-i}, & \text{if } k_{n-1} > n-i. \end{array}$$

It's easy to see that the map Ψ is a bijection that sends $fmaj_D$ to $fmaj$. The thesis follows from the equidistribution of the D-flag major index and the definition of $Dmaj$. □

For $\beta \in B_n$, by Corollary 3.17 we know that

$$fmaj(-\beta) = fmaj(\beta) + n. \tag{3.18}$$

From definition (2.6), it is not hard to prove that for $\beta \in B_n^+$

$$fdes(-\beta) = fdes(\beta) + 1. \tag{3.19}$$

For $\gamma \in D_n$ we define the *D-descent number* by

$$Ddes(\gamma) := fdes([\gamma_1, \ldots, \gamma_{n-1}, |\gamma_n|]).$$

For example, if $\gamma = [-2, -1, 4, 5, -6, -3]$ then $Ddes(\gamma) = fdes([-2, -1, 4, 5, -6, 3]) = 2 \cdot 2 + 1 = 5$.

THEOREM 3.36. *Let $n \in \mathbf{P}$. Then*

$$\sum_{r\geq 0}[r+1]_q^n t^r = \frac{\sum_{\gamma\in D_n} t^{Ddes(\gamma)}q^{Dmaj(\gamma)}}{(1-t)(1-tq^n)\prod_{i=1}^{n-1}(1-t^2q^{2i})}$$

in $\mathbf{Z}[q][[t]]$.

PROOF. From (3.18) and (3.19) we have that

$$\begin{aligned}
\sum_{\beta\in B_n} t^{fdes(\beta)}q^{fmaj(\beta)} &= \sum_{\beta\in B_n^+} t^{fdes(\beta)}q^{fmaj(\beta)} + t^{fdes(-\beta)}q^{fmaj(-\beta)}\\
&= \sum_{\beta\in B_n^+} t^{fdes(\beta)}q^{fmaj(\beta)} + t^{fdes(\beta)+1}q^{fmaj(\beta)+n}\\
&= (1+tq^n)\sum_{\beta\in B_n^+} t^{fdes(\beta)}q^{fmaj(\beta)}\\
&= (1+tq^n)\sum_{\gamma\in D_n} t^{Ddes(\gamma)}q^{Dmaj(\gamma)}.
\end{aligned}$$

Now the result follows easily from Theorem 2.10. □

Finally, we define the following D-flag descent number on D_n,

$$fdes_D(\gamma) := fdes(\Psi(\gamma)),$$

where Ψ has been defined in the proof of Proposition 3.35. Then from Theorem 3.36 and the definition of Ψ it is easy to see that the two pairs of statistics $(fdes_D, fmaj_D)$ and $(Ddes, Dmaj)$ are equidistributed in D_n, and hence we may conclude that

COROLLARY 3.37. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\gamma\in D_n} t^{ddes(\gamma)}q^{dmaj_D(\gamma)} = \sum_{\gamma\in D_n} t^{fdes_D(\gamma)}q^{fmaj_D(\gamma)} = \sum_{\gamma\in D_n} t^{Ddes(\gamma)}q^{Dmaj(\gamma)}.$$

□

Finally, the case $t = 1$ and Proposition 3.36 imply the following result.

COROLLARY 3.38. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\gamma\in D_n} q^{\ell(\gamma)} = \sum_{\gamma\in D_n} q^{dmaj(\gamma)} = \sum_{\gamma\in D_n} q^{fmaj_D(\gamma)} = \sum_{\gamma\in D_n} q^{Dmaj(\gamma)}.$$

□

CHAPTER 4

Coinvariant algebra

In this chapter we use the statistics $Dmaj$ and $Ddes$ defined in Chapter 3 to obtain new results in the representation theory of the Weyl groups of type D. We construct a family of representations for Weyl groups of type D, called negative-descent representations, using the coinvariant algebra as representation space. A monomial descent basis the coinvariant algebra of type D is constructed using a new Straightening Lemma. The basis elements, indexed by $\gamma \in D_n$, have degree $Dmaj(\gamma)$. Extending the statistic $Dmaj$ to D-standard Young bitableaux we give a refinement of a theorem of Stembridge on the decomposition of the negative-descent representations into irreducible components. Finally, using some new multivariate statistics, introduced to prove the previous results, we derive some combinatorial identities on Weyl groups of type D by comparing suitable Hilbert series. One of these identities is then used to prove Carlitz's identity for D_n in a very direct way.

4.1. Combinatorial representation theory

We introduce the basic definitions of *D-standard bitableaux* and $Dmaj$ on D-bitableaux, that allow us to prove the main result of this chapter (Theorem 4.21).

4.1.1. Tableaux. Let $\lambda \vdash n$ be a partition, a *Young tableau* of shape λ is obtained by inserting the integers $1, 2, \ldots, n$ as *entries* in the cells of the Young diagram of shape λ allowing no repetitions. A *standard Young tableau* of shape λ is a Young tableau whose entries increase along rows and down columns. We denote by $SYT(\lambda)$ the set of all standard Young tableaux.
For example the tableau T in Figure 1 belongs to $SYT(5, 3, 2, 1)$.

$$T := \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 8 & 10 \\ \hline 2 & 6 & 7 \\ \cline{1-3} 4 & 11 \\ \cline{1-2} 9 \\ \cline{1-1} \end{array}$$

FIGURE 1

A *descent* in a standard Young tableau T is an entry i such that $i+1$ is strictly below i. We denote the set of descents in T by $Des(T)$. The *major index* of a tableau T is

$$maj(T) := \sum_{i \in Des(T)} i.$$

In the example above $Des(T) = \{1, 3, 5, 8, 10\}$ and so $maj(T) = 27$.

A *semistandard Young tableau* of shape λ ($SSYT(\lambda)$) is obtained by inserting non-negative integers as entries in the boxes of the Young diagram of shape λ, so that the entries weakly increase along rows and strictly increase down columns. For $T \in SSYT(\lambda)$ the *content vector* is

$$cont(T) := (m_0, m_1, \dots),$$

where $m_i :=\mid \{\text{cells in } T \text{ with entry } i\} \mid$, for $i \geq 0$.

A *bipartition* of a non-negative integer n is an ordered pair $(\lambda^{(1)}, \lambda^{(2)})$ of partitions such that $|\lambda^{(1)}| + |\lambda^{(2)}| = n$ denoted by $(\lambda^{(1)}, \lambda^{(2)}) \vdash n$. The *Young diagram* of shape $(\lambda^{(1)}, \lambda^{(2)})$ is obtained by the union of the Young diagrams of shape $\lambda^{(1)}$ and $\lambda^{(2)}$ by positioning the second to the south-west of the first. We define a *standard Young bitableau* $T = (T_1, T_2)$ of shape $(\lambda^{(1)}, \lambda^{(2)}) \vdash n$ by inserting the integers $1, 2, \dots, n$ in the corresponding Young diagram increasing along rows and columns. We denote the set of all standard Young bitableaux of shape $(\lambda^{(1)}, \lambda^{(2)})$ by $SYT(\lambda^{(1)}, \lambda^{(2)})$. We define $Des(T)$ and $maj(T)$ as above and the *negative number* $neg(T)$ as the number of entries of T_2. We define the *flag-major index* of a bitableau by

$$fmaj(T) := 2 \cdot maj(T) + neg(T).$$

Given two partitions $\lambda^{(1)}, \lambda^{(2)}$ such that $|\lambda^{(1)}| + |\lambda^{(2)}| = n$, we define a *D-standard (Young) bitableau* $T = (T_1, T_2)$ of type $\{\lambda^{(1)}, \lambda^{(2)}\}$ as a standard Young bitableau of shape $(\lambda^{(1)}, \lambda^{(2)})$ or $(\lambda^{(2)}, \lambda^{(1)})$ with the condition that $n \in T_1$. We let $Des(T)$, $maj(T)$ and $neg(T)$ as above and we define *D-major index* of a D-standard bitableau, the restriction of $fmaj$, denoted by

$$Dmaj(T) := 2 \cdot maj(T) + neg(T).$$

For example T and S in Figure 2 are two D-standard bitableau of type $\{(3,1), (2,2,1)\}$ and we have $Dmaj(T) = 2 \cdot 15 + 5 = 35$ and $Dmaj(S) = 2 \cdot 13 + 4 = 30$.

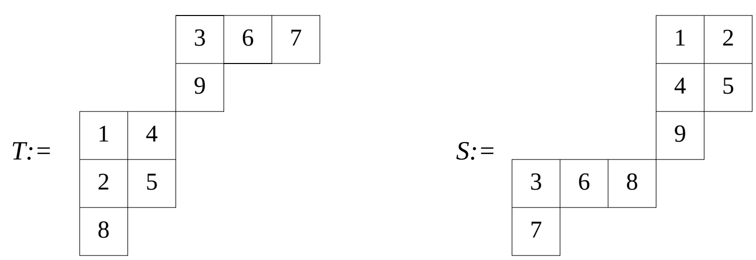


FIGURE 2

We denote by $DSYT\{\lambda^{(1)}, \lambda^{(2)}\}$ the set of all D-standard tableaux of type $\{\lambda^{(1)}, \lambda^{(2)}\}$.

4.1.2. Irreducible representations. The ring of the S_n-invariants of $\mathbf{P}_n$, $\mathbf{P}_n^{S_n}$ is the ring of the symmetric polynomials. We recall some of its bases.

For $k \in \mathbf{N}$ we let

$$e_k(\bar{x}) := \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k} \tag{4.1}$$

where $\bar{x} := (x_1, x_2, \dots, x_n)$. For any partition $\lambda = (\lambda_1, \dots, \lambda_r)$ we denote the *elementary symmetric polynomials* by $e_\lambda := e_{\lambda_1} \cdots e_{\lambda_r}$.

For $k \in \mathbf{N}$ we let

$$p_k(\bar{x}) := \sum_{i=1}^{n} x_i^k.$$

For any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ we denote the *power sum symmetric polynomials* by $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_r}$.

The *Schur symmetric functions* s_λ can be quickly defined using tableaux

$$s_\lambda(\bar{x}) := \sum_{T \in SSYT(\lambda)} \bar{x}^{cont(\lambda)}.$$

The irreducible representations, as well as the conjugacy classes, of the symmetric group S_n are known to be indexed by partitions of n. If λ and α are two partitions of n, we denote by χ_α^λ the value assumed by the character of the irreducible representation indexed by λ on the conjugacy class of cycle type α. The following theorem is due to Frobenius and we refer the reader to [**52**, Corollary 7.17.4] for its proof.

Theorem 4.1 (Frobenius Formula). *Let α be a partition of n. Then*

$$p_\alpha(\bar{x}) = \sum_{\lambda \vdash n} \chi_\alpha^\lambda s_\lambda(\bar{x}).$$

In the case of B_n, the conjugacy classes and the irreducible characters are parametrized by ordered pairs of partitions such that the total sum of their parts is equal to n. The pair (α, β) corresponding to a conjugacy class describes the sign cycle type for any element in the class and we use the convention that α corresponds to the positive and β to the negative cycles. For two partitions α, β and two sets of independent variables $\bar{x}$ and $\bar{y}$ we define

$$p_{\alpha,\beta}(\bar{x}, \bar{y}) := \prod_i (p_{\alpha_i}(\bar{x}) + p_{\beta_i}(\bar{y})) \cdot \prod_i (p_{\alpha_i}(\bar{x}) - p_{\beta_i}(\bar{y})),$$

and we let $\chi_{\alpha,\beta}^{\lambda,\mu}$ be the value of the irreducible character of B_n indexed by (λ, μ) on the conjugacy class of type (α, β). The analogue of Theorem 4.1 is the following (see [**44**, p. 178])

Theorem 4.2. *For any bipartition (α, β) of n*

$$p_{\alpha,\beta}(\bar{x}, \bar{y}) = \sum_{\lambda,\mu} \chi_{\alpha,\beta}^{\lambda,\mu} s_\lambda(\bar{x}) s_\mu(\bar{y}),$$

where the sum runs through all ordered pairs of partitions (λ, μ) of total size $|\lambda| + |\mu| = |\alpha| + |\beta| = n$.

Since D_n is a subgroup of index 2 of the Weyl group B_n, the intersection of a conjugacy class of B_n with D_n is either empty or a single class or splits up into two classes in D_n. This also leads to a parametrization of the classes of D_n by pairs of partitions (α, β) as before, but where $\ell(\beta)$ is even and where there are two classes of this type if β is empty and all parts of α are even. In the latter case these two classes in D_n are usually labeled by $(\alpha, +)$ and $(\alpha, -)$. On the other hand the restrictions of an irreducible character of B_n to D_n is either irreducible or splits up into two irreducible components. Let (λ, μ) be a pair of partitions with total size n. If $\lambda \neq \mu$ then the restrictions of the irreducible characters of B_n labeled by (λ, μ) and (μ, λ) are irreducible and equal. If $\lambda = \mu$ then the restriction of the character labeled by (λ, λ) splits into two irreducible components which we denote by $(\lambda,\lambda)^+$ and $(\lambda,\lambda)^-$. Note that this can only happen if n is even. Hence we may label all irreducible characters of D_n by $(\lambda, \mu)^\epsilon$ where λ and μ are two partitions such that $|\lambda| + |\mu| = n$, $\lambda \preceq \mu$ in some total order $\prec$ on the set of all integer partitions, and ϵ is equal to $\prec$ if $\lambda \neq \mu$ and ϵ is equal to $+$ or $-$ if $\lambda = \mu$.

We refer to the following theorem as the Frobenius formula for D_n; it's certainly known, however, for lack of an adequate reference, we give a proof here.

THEOREM 4.3 (Frobenius Formula for D_n). *For any bipartition $(\alpha, \beta) \vdash n$ with $\ell(\beta) \equiv 0$ we have*

$$\begin{aligned} p_{\alpha,\beta}(\bar{x}, \bar{y}) &= \sum_{\lambda,\mu} \chi_{\alpha,\beta}^{(\lambda,\mu)^{\prec}} \left(s_\lambda(\bar{x}) s_\mu(\bar{y}) + s_\mu(\bar{x}) s_\lambda(\bar{y})\right) \\ &+ \sum_{\lambda:|\lambda|=\frac{n}{2}} \left(\chi_{\alpha,\beta}^{(\lambda,\lambda)^+} + \chi_{\alpha,\beta}^{(\lambda,\lambda)^-}\right) s_\lambda(\bar{x}) s_\lambda(\bar{y}) \end{aligned}$$

where the first sum is taken over all $(\lambda, \mu) \vdash n$ such that $\lambda \prec \mu$.

PROOF. It is a consequence of Theorem 4.2 observing that if $\lambda \prec \mu$ then $\chi_{\alpha,\beta}^{\lambda,\mu} = \chi_{\alpha,\beta}^{\mu,\lambda} = \chi_{\alpha,\beta}^{(\lambda,\mu)^{\prec}}$ and if $|\lambda| = \frac{n}{2}$ then $\chi_{\alpha,\beta}^{\lambda,\lambda} = \chi_{\alpha,\beta}^{(\lambda,\lambda)^+} + \chi_{\alpha,\beta}^{(\lambda,\lambda)^-}$. □

4.1.3. Coinvariant algebra. Consider the natural action φ of a classical Weyl group W (with $W = A_{n-1}, B_n, D_n$) on $\mathbf{P}_n$ defined as in §3.1.2 on the generators by

$$\varphi(w) : x_i \mapsto \frac{w(i)}{|w(i)|} x_{|w(i)|},$$

for all $w \in W$ and extended uniquely to an algebra homomorphism. The algebra of invariants $\mathbf{P}_n^W$ is generated by n homogeneous algebraically independent elements of positive degree together with 1, (see, e.g., [**42**, § 3.5]). A set of algebraically independent homogeneous generators of $\mathbf{P}_n^W$ is called a set of *basic invariants*. The set of basic invariants is not uniquely determined, however their degrees $d_1, \ldots, d_n$ turn out to be independent of the choice of the generators and are called the *degrees* of W (see Table 1).

For example, in the case of S_n, the n-elementary symmetric polynomials $e_i(x_1, \ldots, x_n)$ $i \in [n]$, defined in (4.1), are a set of basic invariants of $\mathbf{P}_n^{S_n}$.

Let I_n^W be the ideal generated by a set of basic invariants of W. The quotient

$$R^*(W) := \mathbf{P}_n / I_n^W$$

is called the *coinvariant algebra* of W and it is well known that it has dimension $|W|$ as a $\mathbf{C}$-vector space. Moreover, W acts naturally as a group of linear operators on this space and it can be shown that this representation of W is isomorphic to the *regular representation* (see e.g., [**41**, § II.3] or [**42**, § 3.6]). All these properties naturally lead to the problem of finding a "nice" basis for $R^*(W)$.

4.1.4. The Garsia-Stanton descent basis and its extension. A basis for the coinvariant algebra of type A has been found by Garsia and Stanton [**38**] using the theory of Stanley-Reisner rings. For $\sigma \in S_n$ they define

$$a_\sigma := \prod_{j \in Des(\sigma)} (x_{\sigma(1)} \cdots x_{\sigma(j)}).$$

Type	degrees
$A_n, \quad (n \geq 1)$	$1, \ldots, n$
$B_n, \quad (n \geq 2)$	$2, 4, , \ldots, 2n$
$D_n, \quad (n \geq 4)$	$2, 4, \ldots, 2n-2, n$

TABLE 1. Degrees of basic invariants

If we denote the number of descents in σ from position i on by

$$d_i(\sigma) := |\{j \in Des(\sigma) \ : \ j \geq i\}|,$$

it's immediate to see that $a_\sigma := \prod_{i=1}^n x_{\sigma(i)}^{d_i(\sigma)}$. They show that the set $\{a_\sigma + I_n^{S_n} \ : \ \sigma \in S_n\}$ is a basis of $R^*(S_n)$, called the *descent basis*. Note that the representatives a_σ of this basis are actually monomials with $deg(a_\sigma) = maj(\sigma)$.

In a recent work [**1**] Adin, Brenti and Roichman defined a monomial basis for the coinvariant algebra of type B that seems to be the right generalization of the Garsia-Stanton basis for type A. We need to introduce the following statistic. For $\beta \in B_n$ we let

$$f_i(\beta) := 2 \cdot d_i(\beta) + \eta_i(\beta),$$

where

$$\eta_i(\beta) := \begin{cases} 1, & \text{if } \beta(i) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

To any $\beta \in B_n$ we associate the monomial

$$b_\beta := \prod_{i=1}^n x_{|\beta(i)|}^{f_i(\beta)}.$$

Then the set $\{b_\beta + I_n^{B_n} \ : \ \beta \in B_n\}$ is a basis of $R^*(B_n)$.

The coinvariant algebra has a natural grading induced from the grading of $\mathbf{P}_n$ by total degree and we denote by R_k its k-th homogeneous component, so that

$$R^*(W) = \bigoplus_{k \geq 0} R_k.$$

In the case of the symmetric group, the major index on tableaux plays a crucial role in the decomposition of R_k into irreducible representations, and we have the following theorem due to Kraskiewicz-Weyman, (see [**47**, Theorem 8.8] for a proof).

THEOREM 4.4 (Kraskiewicz-Weyman). *In type A, for $0 \leq k \leq \binom{n}{2}$, the representation R_k is isomorphic to the direct sum $\oplus m_{k,\lambda} S^\lambda$, where λ runs through all partitions of n, S^λ is the corresponding irreducible S_n-representation, and*

$$m_{k,\lambda} = | \{T \in SYT(\lambda) \ : \ maj(T) = k\} | .$$

The following are the corresponding results for B_n and D_n of Theorem 4.4 and are due to Stembridge [**54**]. Here we state them in our terminology.

THEOREM 4.5 (Stembridge). *In type B, for $0 \leq k \leq n^2$, the representation R_k^B is isomorphic to the direct sum $\oplus m_{k,(\lambda,\mu)} S^{(\lambda,\mu)}$, where $S^{(\lambda,\mu)}$ is the irreducible representation of B_n corresponding to $(\lambda, \mu) \vdash n$ and*

$$m_{k,(\lambda,\mu)} = | \{T \in SYT(\lambda, \mu) \ : \ fmaj(T) = k\} | .$$

The following is the analogous theorem for D_n.

THEOREM 4.6. *In type D, for* $0 \le k \le n^2 - n$, *the representation* R_k^D *is isomorphic to the direct sum* $\oplus m_{k,(\lambda,\mu)^\epsilon} S^{(\lambda,\mu)^\epsilon}$, *where* $S^{(\lambda,\mu)^\epsilon}$ *is the irreducible representation of* D_n *labelled as in §4.1.2, and*

$$m_{k,(\lambda,\mu)^\epsilon} := \mid \{T \in DSYT\{\lambda,\mu\} \, : \, Dmaj(T) = k\} \mid .$$

4.2. Negative-Descent Representations for D_n

We first define a family of monomials, indexed by D_n, and we prove that it is a basis for the coinvariant algebra of type D, that we call the (even-signed) descent basis. To this end we present a straightening lemma for expanding an arbitrary monomial in $\mathbf{P}_n$ in terms of the descent basis with coefficients in $\mathbf{P}_n^{D_n}$. This algorithm is a generalization of the one presented in [**1**] for types A and B and allows us to define a family of negative-descent representations of D_n.

4.2.1. The descent basis for $R^*(D_n)$. Let $\gamma \in D_n$, for $i \in [n-1]$, we define

$$\delta_i(\gamma) := \mid \{j \in Des(|\gamma|_n) \, : \, j \ge i\} \mid ,$$

$$\eta_i(\gamma) := \begin{cases} 1, & \text{if } \gamma(i) < 0; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h_i(\gamma) := 2\delta_i(\gamma) + \eta_i(\gamma).$$

Note that

$$\sum_{i=1}^{n-1} h_i(\gamma) = Dmaj(\gamma) \tag{4.2}$$

and that

$$h_1(\gamma) = Ddes(\gamma).$$

DEFINITION. *For any* $\gamma \in D_n$, *we define*

$$c_\gamma := \prod_{i=1}^{n-1} x_{|\gamma(i)|}^{h_i(\gamma)}.$$

For example, let $\gamma := [6,-4,-2,3,-5,-1] \in D_6$, then $(h_1(\gamma),\dots,h_5(\gamma)) = (6,5,3,2,1)$ and $c_\gamma = x_2^3 x_3^2 x_4^5 x_5 x_6^6$.

The goal of this section is to prove that the set $\{c_\gamma + I_n^D \, : \, \gamma \in D_n\}$ is a linear basis for the coinvariant algebra of type D. We call it the *even-signed descent basis* and c_γ a *descent basis element.* We denote by

$$f_i(x_1,\dots,x_n) := \begin{cases} e_i(x_1^2,\dots,x_n^2), & \text{for } i \in [n-1]; \\ x_1 \cdots x_n, & \text{for } i = n. \end{cases}$$

Note that $\{f_i\}_{i=1}^n$ is a set of basic invariants for $\mathbf{P}_n^{D_n}$. Moreover, for any partition $\lambda = (\lambda_1,\dots,\lambda_t)$ with $\lambda_1 \le n$, we define $f_\lambda := f_{\lambda_1} \cdots f_{\lambda_t}$. For the moment we restrict our attention on the quotient $S := \mathbf{P}_n/(f_n)$. We naturally identify a monomial that hasn't all the variables with its corresponding class in S. We start by associating to any monomial $M \in S$ an even-signed permutation $\gamma(M)$. Let M be a non-zero monomial in S, $M = \prod_{i=1}^n x_i^{p_i}$ (note that $p_i = 0$ for some $i \ge 1$).

We define $\gamma = \gamma(M) \in D_n$ as the unique even-signed permutation such that, for $i \in [n-1]$,

i) $p_{|\gamma(i)|} \geq p_{|\gamma(i+1)|}$;
ii) $p_{|\gamma(i)|} = p_{|\gamma(i+1)|} \Longrightarrow |\gamma(i)| < |\gamma(i+1)|$;
iii) $p_{|\gamma(i)|} \equiv 0 \Longleftrightarrow \gamma(i) > 0$.

Note that the last condition determines also the sign of $\gamma(n)$.

We call $\gamma(M) \in D_n$ the *index even-permutation* and

$$\lambda(M) := (p_{|\gamma(1)|}, \ldots, p_{|\gamma(n)|}),$$

the *exponent partition* of M.

LEMMA 4.7. *Let $M = \prod_{i=1}^n x_i^{p_i} \in S$ and $\gamma := \gamma(M)$. The sequence $(p_{|\gamma(i)|} - h_i(\gamma))$, $i = 1, \ldots, n-1$, of exponents in M/c_γ consists of nonnegative even integers and is weakly decreasing.*

PROOF. We have, by definition, $p_{|\gamma(i)|} \equiv 0 \Leftrightarrow \eta_i(\gamma) = 0$ and hence our sequence consists of even integers. Now it is easy to check that $p_{|\gamma(n-1)|} - h_{n-1}(\gamma) \geq 0$ so it remains to show that the sequence is weakly decreasing. If $p_{|\gamma(i)|} = p_{|\gamma(i+1)|}$ we also have $h_i(\gamma) = h_{i+1}(\gamma)$. If $p_{|\gamma(i)|} > p_{|\gamma(i+1)|}$ and $p_{|\gamma(i)|} \equiv p_{|\gamma(i+1)|}$ then $p_{|\gamma(i)|} \geq p_{|\gamma(i+1)|} + 2$ and $h_i(\gamma) \leq h_{i+1}(\gamma) + 2$ and the result follows. Finally, if $p_{|\gamma(i)|} > p_{|\gamma(i+1)|}$ and $p_{|\gamma(i)|} \not\equiv p_{|\gamma(i+1)|}$ then $h_i(\gamma) = h_{i+1}(\gamma) + 1$ and we are done. $\square$

We denote by $\mu(M)$ the partition conjugate to $\left(\frac{p_{|\gamma(i)|} - h_i(\gamma)}{2}\right)_{i=1}^{n-1}$, where $\gamma = \gamma(M)$ (note that $\mu(M)_1 < n$) and we let $X_k^\gamma := x_{|\gamma(1)|}^2 \cdots x_{|\gamma(k)|}^2$, for all $k \in [n-1]$. For any monomial $M \in S$ we have

$$M = c_\gamma X_{\mu(1)}^\gamma X_{\mu(2)}^\gamma \cdots X_{\mu(r)}^\gamma \tag{4.3}$$

where $\gamma = \gamma(M)$, $\mu = \mu(M)$ and $r := \ell(\mu)$.

For example, for $n = 4$, if $M = x_1^6 x_2^2 x_3$, then $\gamma(M) = [1, 2, -3, -4]$, $c_\gamma = x_1^2 x_2^2 x_3$, $\mu(M) = (1, 1)$ and $M = c_\gamma \cdot x_1^2 \cdot x_1^2$.

We now define a partial order on the monomials of the same total degree in S.

DEFINITION. *Let M and M' monomials in S with the same total degree and such that the exponents of x_i in M and M' have the same parity for every $i \in [n]$. Then we write $M' < M$ if one of the following holds*

1. $\lambda(M') \lhd \lambda(M)$, *or*
2. $\lambda(M') = \lambda(M)$ *and* $inv(|\gamma(M')|_n) > inv(|\gamma(M)|_n)$

From now on denote

$$M^{(k)} := M \cdot X_k^\gamma,$$

where $\gamma = \gamma(M)$.

LEMMA 4.8. *Let $M \in S$ be a monomial and $k \in [n-1]$. Then*

$$M \cdot f_k = M^{(k)} + \sum_{M' < M^{(k)}} \epsilon_{M,M'} M',$$

where $\epsilon_{M,M'} \in \{0, 1\}$.

Proof. Let $\lambda(M) = (\lambda_1, \dots, \lambda_n)$, with $\lambda_n = 0$ be the exponent partition of M, then $\lambda(M^{(k)}) = (\lambda_1 + 2, \lambda_2 + 2, \dots, \lambda_k + 2, \lambda_{k+1}, \dots, \lambda_n)$. It's clear that $M^{(k)}$ is a terms appearing in the expansion of the polynomial $M \cdot f_k$ and that, if M' is another term appearing in $M \cdot f_k$ then $\lambda(M') \trianglelefteq \lambda(M^{(k)})$. If $\lambda(M^{(k)}) \neq \lambda(M')$ we are done so we can suppose $\lambda(M^{(k)}) = \lambda(M')$. This can happen only if $\gamma(M)(j) = \gamma(M')(j)$ for all $j \notin J$, where $J := \{j \in [n] : \lambda_j(M) = \lambda_k(M)\}$. Since $|\gamma(M)|_n$ is monotone increasing in J it follows that $inv(|\gamma(M)|_n) > inv(|\gamma(M')_n|)$ and the proof is complete. □

By the description of $\lambda(M^{(k)})$ given in the proof of Lemma 4.8 we clearly have that, for all $k \in [n-1]$,

$$M' < M \Longrightarrow M'^{(k)} < M^{(k)}. \tag{4.4}$$

Moreover, by the definition of $\gamma(M)$, we have that

$$\gamma(M) = \gamma(M^{(k)}). \tag{4.5}$$

Now let N be a monomial, $\gamma = \gamma(N)$ and $\mu(N) = (\mu_1, \dots, \mu_r)$. By Lemma 4.8 we have that

$$N \cdot f_{\mu_1} = N X^{\gamma}_{\mu_1} + \sum_{M' < N X^{\gamma}_{\mu_1}} \varepsilon_{N,M'} \, M'.$$

By (4.4) and (4.5) we obtain that

$$\begin{aligned} N \cdot f_{\mu_1} f_{\mu_2} &= N X^{\gamma}_{\mu_1} f_{\mu_2} + \sum_{M' < N X^{\gamma}_{\mu_1}} \varepsilon_{N,M'} \, M' f_{\mu_2} \\ &= N X^{\gamma}_{\mu_1} X^{\gamma}_{\mu_2} + \sum_{M'' < N X^{\gamma}_{\mu_1} X^{\gamma}_{\mu_2}} a_{N,M''} \, M'' \end{aligned}$$

where $a_{N,M''}$ are suitable integers. Now let M be any monomial in S. By iterating this procedure and by taking $N := c_{\gamma(M)}$, from $\gamma(c_{\gamma(M)}) = \gamma(M)$ and (4.3) we have that

$$M = c_{\gamma(M)} f_{\mu(M)} + \sum_{M' < M} a_{M',M} M'. \tag{4.6}$$

Lemma 4.9 (Straightening Lemma). *Let M be a monomial in S. Then M admits the following expression*

$$M = f_{\mu(M)} \cdot c_{\gamma(M)} + \sum_{M' < M} n_{M',M} f_{\mu(M')} \cdot c_{\gamma(M')},$$

where $n_{M,M'}$ are integers.

Proof. It follows immediately by iterating (4.6). □

For example, let $n = 4$ and $M = x_1^4 x_2^3 x_4^4$. We have $\lambda(M) = (4, 4, 3)$, $\gamma(M) = [1, 4, -2, -3]$, $(h_1, h_2, h_3) = (2, 2, 1)$, $c_\gamma = x_1^2 x_2 x_4^2$ and $\mu(M) = (3)$. Then

$$M = c_{\gamma(M)} f_3$$

in S.

Now we are ready to state and prove the main result of this section.

THEOREM 4.10. *The set*

$$\{c_\gamma + I_n^D \ : \gamma \in D_n\}$$

is a basis for $R^*(D_n)$.

PROOF. By the Straightening Lemma 4.9 the set $\{c_\gamma + I_n^D \ : \ \gamma \in D_n\}$ spans $R^*(D_n)$ as a $\mathbf{C}$-vector space and it has the right dimension as stated in §2.2 □

4.2.2. Homogeneous components in $R^*(D_n)$.

PROPOSITION 4.11. *Let* $\gamma, \xi \in D_n$. *Then*

$$\xi \cdot c_\gamma = \sum_{\{u \in D_n \,:\, \lambda(c_u) \trianglelefteq \lambda(c_\gamma)\}} n_u c_u + p,$$

where $n_u \in \mathbf{Z}$ *and* $p \in I_n^D$.

PROOF. Apply the Straightening Lemma (Lemma 4.9) to $M = \xi(c_\gamma)$. Note that $f_{\mu(M')} \notin I_n^D$ if and only if $\mu(M') = \emptyset$. Hence, if M' gives a non-zero contribute in this expansion of M we have $M' = c_{\gamma(M')}$. If we let $u = \gamma(M')$, then

$$\lambda(c_u) = \lambda(M') \trianglelefteq \lambda(M) = \lambda(\xi \cdot c_\gamma) = \lambda(c_\gamma).$$

□

By Proposition 4.11 we have that

$$J_{\lambda,D}^{\trianglelefteq} := \mathrm{span}_{\mathbf{C}}\{c_\gamma + I_n^D \mid \gamma \in D_n,\, \lambda(c_\gamma) \trianglelefteq \lambda\}$$

and

$$J_{\lambda,D}^{\triangleleft} := \mathrm{span}_{\mathbf{C}}\{c_\gamma + I_n^D \mid \gamma \in D_n,\, \lambda(c_\gamma) \triangleleft \lambda\}$$

are two submodules of R_k^D where $k = |\lambda|$. Their quotient is still a D_n-module denoted by

$$R_\lambda := \frac{J_{\lambda,D}^{\trianglelefteq}}{J_{\lambda,D}^{\triangleleft}}.$$

For any $S \subseteq [n-1]$ we define the partition

$$\lambda(S) := (\lambda_1, \ldots, \lambda_{n-1}),$$

where $\lambda_i := |S \cap [i, n-1]|$. For $S_1, S_2 \subseteq [n-1]$, we define the vector

$$\lambda_{S_1,S_2} := 2 \cdot \lambda_{S_1} + \mathbf{1}_{S_2},$$

where $\mathbf{1}_{S_2} \in \{0,1\}^{n-1}$ is the characteristic vector of S_2.

For notation convenience, we introduce this new descent set

$$Des_n(\gamma) := Des(|\gamma|_n)$$

and we let

$$Neg_n(\gamma) := Neg(\gamma) \cap [n-1].$$

REMARK 4.12. For any $\gamma \in D_n$ we have

$$\lambda(c_\gamma) = \lambda_{S_1,S_2},$$

where $S_1 = Des_n(\gamma)$ and $S_2 = Neg_n(\gamma)$.

The next proposition tells us when the representation R_λ is nontrivial.

PROPOSITION 4.13. *The following three conditions on* $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ *are equivalent:*

i) $R_\lambda \neq 0$.
ii) $\lambda = \lambda(c_\gamma)$, *for some* $\gamma \in D_n$.
iii) $\lambda = \lambda_{S_1,S_2}$ *for some* $S_1, S_2 \in [n-1]$ *and* $\lambda_i - \lambda_{i+1} \in \{0, 1, 2\}$ *for all* $i = 1, \dots, n-1$, *where* $\lambda_n := 0$.

PROOF. $R_\lambda \neq 0$ if and only if there exists $\gamma \in D_n$ such that $\lambda(c_\gamma) = \lambda$ and by Remark 4.12, $\lambda(c_\gamma) = \lambda_{S_1,S_2}$ with $S_1 = Des_n(\gamma)$ and $S_2 = Neg_n(\gamma)$. On the other hand given λ_{S_1,S_2} with $\lambda_i - \lambda_{i+1} \in \{0,1,2\}$ and S_1, $S_2 \subseteq [n-1]$ there exists a $\gamma \in D_n$ with $Des_n(\gamma) = S_1$ and $Neg_n(\gamma) = S_2$, and we are done. $\square$

From now we denote $R_{S_1,S_2} := R_{\lambda_{S_1,S_2}}$.

PROPOSITION 4.14. *For any* S_1, $S_2 \subseteq [n-1]$, *the set*

$$\{\bar{c}_\gamma \ : \ \gamma \in D_n, \ Des_n(\gamma) = S_1 \text{ and } Neg_n(\gamma) = S_2\},$$

where $\bar{c}_\gamma$ *is the image of* c_γ *in the quotient* R_λ, *is a basis of* R_{S_1,S_2}.

PROOF. Let $\gamma \neq \gamma'$ two even-signed permutations such that $\lambda(c_\gamma) = \lambda(c_{\gamma'})$. By Remark 4.12 follows that $\lambda_{S_1,S_2} = \lambda_{S_1',S_2'}$, where S_1, S_1' and S_2, S_2' are Des_n and Neg_n of γ and γ', respectively. Hence $S_1 = S_1'$, $S_2 = S_2'$ and $inv(|\gamma|_n) = inv(|\gamma'|_n)$ so that the monomials c_γ and $c_{\gamma'}$ are incomparable. $\square$

By the previous proposition it is natural to call the D_n-module R_{S_1,S_2} a *negative-descent representation*.

THEOREM 4.15. *For every* $0 \leq k \leq n^2 - n$,

$$R_k^D \cong \bigoplus_{S_1,S_2} R_{S_1,S_2}$$

as D_n*- modules, where the sum is over all* $S_1, S_2 \in [n-1]$ *such that*

$$2 \cdot \sum_{i \in S_1} i + |S_2| = k.$$

PROOF. Note that by (4.2), $c_\gamma + I_n^D \in R_k^D$ if and only if $Dmaj(\gamma) = k$. By Theorem 4.10 we have that $\{c_\gamma + I_n^D \ : \ Dmaj(\gamma) = k\}$ is a basis for R_k^D and so by Corollary 4.14, $R_k^D \cong \oplus_{S_1,S_2} R_{S_1,S_2}$, as $\mathbf{C}$-vector spaces, where $S_1, S_2 \subseteq [n-1]$ with $\sum_{i\in S_1} i + |S_2| = k$. By Maschke's theorem, if V is a finite dimensional G-module for a finite group G (over a field of characteristic zero) and $W \subseteq V$ is a G-submodule, then $V \cong W \oplus (V/W)$ as G-module. Apply this theorem to the poset

$$\{J^{\trianglelefteq}_{\lambda_{S_1,S_2},D} \ : \ S_1, S_2 \subseteq [n-1], \ 2 \cdot \sum_{i \in S_1} i + |S_2| = k\}$$

ordered by dominance order on the partitions λ_{S_1,S_2}. The result follows by induction on this poset and by the definition of λ_{S_1,S_2}. $\square$

4.2.3. Irreducible components of the descent representations. In this section we prove a simple combinatorial way to compute the multiplicities of the irreducible representations of D_n in R_{S_1,S_2}. This result is a refinement of Stembridge's Theorem (Theorem 4.6).

Define $\mathbf{C}[[p_1, p_1p_2, \ldots]]$ to be the ring of formal series in countably many variables $p_1, p_1p_2, \ldots, p_1p_2\cdots p_k, \ldots$; a linear basis for it consists of the monomials $p^\lambda := p_1^{\lambda_1}\cdots p_n^{\lambda_n}$ for all partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$.

Let $\iota : \mathbf{C}[[p_1, p_1p_2, \ldots]] \to \mathbf{C}[[q_1, q_1q_2, \ldots]]$ be the map defined on the generators by

$$\iota(p^\lambda) := q^{\lambda'},$$

and extended by linearity. Note that ι is not a ring homomorphism.

For any standard Young bitableau $T = (T_1, T_2)$ of shape $(\lambda^{(1)}, \lambda^{(2)})$, following [**1**], we define for $i \in [n]$,

$$d_i(T) := |\{j \geq i \ : \ j \in Des(T)\},$$

$$\epsilon_i(T) := \begin{cases} 1, & \text{if } i \in Neg(T); \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_i(T) := 2 \cdot d_i(T) + \epsilon_i(T). \tag{4.7}$$

The following lemma is known and we refer the reader to [**1**] for its proof.

Lemma 4.16. *If $(\lambda^{(1)}, \lambda^{(2)}) \vdash n$. Then*

$$\iota[s_{\lambda^{(1)}}(1, p_1p_2, p_1p_2p_3p_4, \ldots) \cdot s_{\lambda^{(2)}}(p_1, p_1p_2p_3, \ldots)] =$$
$$= \frac{\sum_{T \in SYT(\lambda^{(1)},\lambda^{(2)})} \prod_{i=1}^n q_i^{f_i(T)}}{\prod_{i=1}^n (1 - q_1^2q_2^2\cdots q_i^2)}.$$

Let $T = (T_1, T_2)$ be a Young standard bitableau. We denote by $\hat{T} := (T_2, T_1)$ the bitableau obtained by swapping the two tableaux in T and we call it the *transpose bitableau* of T.

The following technical lemma is fundamental in the proof of the main result of this section (Theorem 4.21).

Lemma 4.17. *Let $T = (T_1, T_2)$ be a Young standard bitableau of total size n such that $n \in T_1$. Then*

$$f_i(T) = f_i(\hat{T}) + 1$$

for all $i = 1, \ldots, n$.

Proof. We proceed by backwards induction on i. If $i = n$ it's obvious. So suppose $i < n$. We have four cases to consider.

a) $i \in Des(T)$, and $i \in T_2$

This implies that $i + 1 \in T_2$ and so $d_i(T) = d_{i+1}(T) + 1$ and $\epsilon_i(T) = \epsilon_{i+1}(T)$. In $\hat{T}$ we have again $d_i(\hat{T}) = d_{i+1}(\hat{T}) + 1$ and $\epsilon_i(\hat{T}) = \epsilon_{i+1}(\hat{T})$. Hence, by induction,

$$\begin{aligned} f_i(T) &= 2d_i(T) + \epsilon_i(T) = 2d_{i+1}(T) + \epsilon_{i+1}(T) + 2 \\ &= f_{i+1}(T) + 2 = f_{i+1}(\hat{T}) + 3 \\ &= f_i(\hat{T}) + 1. \end{aligned}$$

b) $i \in Des(T)$, and $i \in T_1$

If $i+1 \in T_1$, then all the conditions are as in case a) and the result follows by induction.
Otherwise, if $i+1 \in T_2$, then $d_i(T) = d_{i+1}(T) + 1$ $\epsilon_i(T) = \epsilon_{i+1}(T) - 1$, $d_i(\hat{T}) = d_{i+1}(\hat{T})$ and $\epsilon_i(\hat{T}) = \epsilon_{i+1}(\hat{T}) + 1$. Hence, by induction,

$$\begin{aligned} f_i(T) &= 2d_{i+1} + 2 + \epsilon_{i+1} - 1 = f_{i+1}(T) + 1 \\ &= f_{i+1}(\hat{T}) + 2 = 2d_i(\bar{T}) + \epsilon_i(\bar{T}) + 1 \\ &= f_i(\hat{T}) + 1. \end{aligned}$$

c) $i \notin Des(T)$, and $i \in T_1$

This implies that $i+1 \in T_1$ then $d_i(T) = d_{i+1}(T)$ $\epsilon_i(T) = \epsilon_{i+1}(T)$ and the same relation for $\bar{T}$ and so the thesis easily follows.

d) $i \notin Des(T)$, and $i \in T_2$

There are two possibilities: $i+1 \in T_1$ or T_2. In both cases the thesis easily follows by induction. □

For an element $\gamma \in D_n$ let the (graded) trace of its action on the polynomial ring $\mathbf{P}_n$ be

$$Tr_{\mathbf{P}_n}(\gamma) := \sum_m \langle \gamma \cdot m, m \rangle \cdot \bar{q}^{\lambda(m)},$$

where the sum is over all monomials $m \in \mathbf{P}_n$, $\lambda(m)$ is the exponent partition of m, and the inner product is such that the set of all monomials is an orthonormal basis for $\mathbf{P}_n$. Note that $\langle \gamma \cdot m, m \rangle \in \{0, \pm 1\}$.

The following lemma is the restriction to D_n of Claim 5.12 in [**1**].

LEMMA 4.18. *If $\gamma \in D_n$ is of cycle type (α, β) then the trace of its action on $\boldsymbol{P}_n$ is*

$$Tr_{\boldsymbol{P}_n}(\gamma) = \iota[p_{\alpha,\beta}(\bar{x}, \bar{y})],$$

where $\bar{x} := (1, p_1p_2, p_1p_2p_3p_4, \ldots)$ and $\bar{y} := (p_1, p_1p_2p_3, \ldots)$.

The coinvariant algebra $R^*(D_n)$ has the descent basis $\{c_\gamma + I_n^D \ : \ \gamma \in D_n\}$. So similarly to $\mathbf{P}_n$ we let

$$Tr_{\mathbf{P}_n/I_n^D}(\gamma) := \sum_{\gamma \in D_n} \langle \gamma \cdot (c_\gamma + I_n^D), c_\gamma + I_n^D \rangle \cdot \bar{q}^{\lambda(c_\gamma)}, \tag{4.8}$$

where the inner product is such that the descent basis is orthonornal.

LEMMA 4.19. *Let $n \in \boldsymbol{P}$. Then*

$$Tr_{\boldsymbol{P}_n}(\gamma) = Tr_{\boldsymbol{P}_n/I_n^D}(\gamma) \cdot \sum_{t \geq 0} \bar{q}^t \sum_{\ell(\lambda) \leq n-1} \bar{q}^{2\lambda}.$$

PROOF. We consider the following basis of homogeneous polynomials $\{f_{\mu_D} c_\gamma \ : \ \gamma \in D_n \ , \ \mu_D = ((n)^t, \bar{\mu}_D)\}$ instead of the monomial basis. The trace $Tr_{\mathbf{P}_n}$ is not changed. But for the Straightening Lemma we have that the maximal monomial m in $f_{\mu_D} c_\gamma$ has exponent partition $\lambda(m) = \lambda(c_\gamma) + \mu_D(m)'$, where the sum of partitions is componentwise. By (4.8) the thesis follows. □

REMARK 4.20. It's clear that

$$\sum_{t\geq 0}(q_1\cdots q_n)^t = \frac{1}{1-q_1\cdots q_n}$$

and

$$\sum_{\ell(\lambda)\leq n-1}(q_1\cdots q_n)^{2\lambda} = \prod_{i=1}^{n-1}\frac{1}{1-q_1^2\cdots q_i^2}.$$

We are now ready to state and prove the main result of this section.

THEOREM 4.21. *For any pair of subsets $S_1, S_2 \subseteq [n-1]$, and a bipartition of n $(\lambda,\mu)\vdash n$, the multiplicity of the irreducible D_n-representation corresponding to $(\lambda,\mu)^\epsilon$ in R_{S_1,S_2} is*

$$m_{S_1,S_2,(\lambda,\mu)^\epsilon} := |\{T\in DSYT\{\lambda,\mu\} \ : \ Des(T)=S_1 \text{ and } Neg(T)=S_2\}|.$$

PROOF. By Lemma 4.19 and Remark 4.20 we have

$$Tr_{R_n^D}(\gamma)\cdot\prod_{i=1}^{n-1}\frac{1}{(1-q_1^2\cdots q_i^2)(1-q_1\cdots q_n)} = Tr_{\mathbf{P}_n}(\gamma). \tag{4.9}$$

By Lemma 4.18 we have

$$Tr_{\mathbf{P}_n}(\gamma) = \iota[p_{\alpha,\beta}(\bar{x},\bar{y})] \tag{4.10}$$

that, by Frobenius Formula for type D, is equal to

$$\iota\left[\sum_{\lambda,\mu}\chi_{\alpha,\beta}^{(\lambda,\mu)^\prec}\left(s_\lambda(\bar{x})s_\mu(\bar{y})+s_\mu(\bar{x})s_\lambda(\bar{y})\right)+\sum_{\lambda:|\lambda|=\frac{n}{2}}\left(\chi_{\alpha,\beta}^{(\lambda,\lambda)^+}+\chi_{\alpha,\beta}^{(\lambda,\lambda)^-}\right)s_\lambda(\bar{x})s_\lambda(\bar{y})\right].$$

Hence applying the linearity of ι and Lemma 4.16 we obtain that $R.H.S.$ of (4.10) is equal to

$$\begin{aligned} = \quad & \frac{1}{\prod_{i=1}^n(1-q_1^2\cdots q_n^2)}\Big\{\sum_{\lambda,\mu}\chi_{\alpha,\beta}^{(\lambda,\mu)^\prec}\Big(\sum_{T\in SYT(\lambda,\mu)}\prod_{i=1}^n q_i^{f_i(T)}+\sum_{T\in SYT(\mu,\lambda)}\prod_{i=1}^n q_i^{f_i(T)}\Big) \\ + \quad & \sum_{\lambda:|\lambda|=\frac{n}{2}}\left(\chi_{\alpha,\beta}^{(\lambda,\lambda)^+}+\chi_{\alpha,\beta}^{(\lambda,\lambda)^-}\right)\sum_{T\in SYT(\lambda,\lambda)}\prod_{i=1}^n q_i^{f_i(T)}\Big\}. \end{aligned}$$

Splitting each sum on the SYT in two pieces depending if n is in T_1 or in T_2 we have

$$\begin{aligned}
\iota[p_{\alpha,\beta}(\bar{x},\bar{y})] &= \frac{1}{\prod_{i=1}^{n}(1-q_1^2\cdots q_n^2)}\Big\{\sum_{\lambda,\mu}\chi_{\alpha,\beta}^{(\lambda,\mu)^{\prec}}\cdot \\
&\Big(\sum_{(T_1,T_2)\in SYT(\lambda,\mu)\; n\in T_1}\prod_{i=1}^{n} q_i^{f_i(T)}+\sum_{(T_1,T_2)\in SYT(\lambda,\mu)\; n\in T_2}\prod_{i=1}^{n} q_i^{f_i(T)} \\
&+\sum_{(T_1,T_2)\in SYT(\mu,\lambda)\; n\in T_1}\prod_{i=1}^{n} q_i^{f_i(T)}+\sum_{(T_1,T_2)\in SYT(\mu,\lambda)\; n\in T_2}\prod_{i=1}^{n} q_i^{f_i(T)}\Big) \\
&+\sum_{\lambda:|\lambda|=\frac{n}{2}}(\chi_{\alpha,\beta}^{(\lambda,\lambda)^{+}}+\chi_{\alpha,\beta}^{(\lambda,\lambda)^{-}})(\sum_{(T_1,T_2)\in SYT(\lambda,\lambda),\; n\in T_1}\prod_{i=1}^{n} q_i^{f_i(T)} \\
&+\sum_{(T_1,T_2)\in SYT(\lambda,\lambda),\; n\in T_2}\prod_{i=1}^{n} q_i^{f_i(T)})\Big\}
\end{aligned}$$

Thanks to Lemma 4.17

$$\begin{aligned}
\iota[p_{\alpha,\beta}(\bar{x},\bar{y})] &= \frac{1}{\prod_{i=1}^{n}(1-q_1^2\cdots q_n^2)}\cdot\{\sum_{\lambda,\mu}\chi_{\alpha,\beta}^{(\lambda,\mu)^{\prec}}(1+q_1\cdots q_n)\cdot \\
&\cdot\;(\sum_{(T_1,T_2)\in SYT(\lambda,\mu)\; n\in T_1}\prod_{i=1}^{n} q_i^{f_i(T)}+\sum_{(T_1,T_2)\in SYT(\mu,\lambda)\; n\in T_1}\prod_{i=1}^{n} q_i^{f_i(T)}) \\
&+\sum_{\lambda:|\lambda|=\frac{n}{2}}(\chi_{\alpha,\beta}^{(\lambda,\lambda)^{+}}+\chi_{\alpha,\beta}^{(\lambda,\lambda)^{-}})\sum_{(T_1,T_2)\in SYT(\lambda,\lambda),\; n\in T_1}\prod_{i=1}^{n} q_i^{f_i(T)}(1+q_1\cdots q_n)\}.
\end{aligned}$$

Hence by (4.9) and the definition of D-standard bitableaux we have

$$\begin{aligned}
Tr_{R_n^D}(\gamma) &= \sum_{\lambda,\mu}\chi_{\alpha,\beta}^{(\lambda,\mu)^{\prec}}\sum_{T\in DSYT\{\lambda,\mu\}}\prod_{i=1}^{n-1} q_i^{f_i(T)} \\
&+\sum_{\lambda:|\lambda|=\frac{n}{2}}(\chi_{\alpha,\beta}^{(\lambda,\lambda)^{+}}+\chi_{\alpha,\beta}^{(\lambda,\lambda)^{-}})\sum_{T\in D-SYT\{\lambda,\lambda\}}\prod_{i=1}^{n-1} q_i^{f_i(T)}.
\end{aligned}$$

We conclude that the graded multiplicity of the irreducible D_n-representation corresponding to $(\lambda,\mu)^{\epsilon}$, in $R^*(D_n)$, is

$$\sum_{T\in DSYT\{\lambda,\mu\}}\prod_{i=1}^{n-1} q_i^{f_i(T)}=\sum_{T\in DSYT\{\lambda,\mu\}}\bar{q}^{\lambda_{Des(T),Neg(T)}},$$

and so we are done. $\square$

Theorem 4.6 can be easily obtained from this, by observing that for any $T\in DSYT\{\lambda,\mu\}$, $\sum_{i=1}^{n-1} f_i(T)=Dmaj(T)$.

4.3. Combinatorial Identities

In this section we compute the Hilbert series of polynomial ring $\mathbf{P}_n$ with respect to multi-degree rearranged into a weakly decreasing sequence. Moreover we will see Corollary 3.37 as subcase of Theorem 4.25.

For any partition $\lambda = (\lambda_1, \dots, \lambda_n)$ with at most n parts we let for every $j \geq 0$

$$m_j(\lambda) := |\{i \in [n] \ : \ \lambda_i = j\}|,$$

and

$$\binom{n}{\bar{m}(\lambda)} := \binom{n}{m_0(\lambda), m_1(\lambda), \dots},$$

the multinomial coefficient.

The following theorem can be found in [**1**][Theorem 6.2] and give an explicit formula for the Hilbert series of the polynomial ring $\mathbf{P}_n$ with respect to weakly decreasing multi-degree.

THEOREM 4.22. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\ell(\lambda) \leq n} \binom{n}{\bar{m}(\lambda)} \prod_{i=1}^{n} q_i^{\lambda_i} = \frac{\sum_{\sigma \in S_n} \prod_{i=1}^{n} q_i^{d_i(\sigma)}}{\prod_{i=1}^{n}(1 - q_1 \cdots q_i)},$$

in $\mathbf{Z}[[q_1, \dots, q_n]]$.

The Hilbert series of $\mathbf{P}_n$ can be computed by considering the even-signed descent basis for the coinvariant algebra of type D and applying the Straightening Lemma for this type.

It's easy to see that the map $\mathbf{P}_n \to D_n \times \mathcal{P}(n)$ given by

$$m \mapsto (\gamma(m), \mu_D(m)'), \tag{4.11}$$

is a bijection.

THEOREM 4.23. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\ell(\lambda) \leq n} \binom{n}{\bar{m}(\lambda)} \prod_{i=1}^{n} q_i^{\lambda_i} = \frac{\sum_{\gamma \in D_n} \prod_{i=1}^{n-1} q_i^{2\delta_i(\gamma)+\eta_i(\gamma)}}{\prod_{i=1}^{n-1}(1 - q_1^2 \cdots q_i^2)(1 - q_1 \cdots q_n)},$$

in $\mathbf{Z}[[q_1, \dots, q_n]]$.

PROOF. The L.H.S. of the theorem is the Hilbert series of polynomial ring by exponent partition. So using the Straightening Lemma for type D, the bijection 4.11 and Remark 4.20 we have

$$\sum_{m \in \mathbf{P}_n} \bar{q}^{\lambda(m)} = \sum_{m \in \mathbf{P}_n} \bar{q}^{\lambda(c_{\gamma(m)}) + \mu_D(m)'} = \sum_{\gamma \in D_n} \bar{q}^{\lambda(c_\gamma)} \cdot \sum_{t \geq 0} \bar{q}^t \cdot \sum_{\mu \in \mathcal{P}(n-1)} \bar{q}^{2\mu}.$$

By Remark 4.12 the thesis follows. $\square$

Now we compute the Hilbert series in a different way using the following observation. The D-negative multiset defined in (2.11) can be written also in this form

$$DDes(\gamma) = Des(\gamma) \biguplus \{Neg(\gamma^{-1})\} \setminus \{n\}.$$

Now we define

$$\bar{n}_i(\gamma) := |\{j \geq i \ : \ j \in Neg(\gamma) \cap [n-1]\}|.$$

Hence we have that

$$ddes(\gamma) = d_1(\gamma) + \bar{n}_1(\gamma). \tag{4.12}$$

THEOREM 4.24. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\ell(\lambda)\leq n} \binom{n}{\bar{m}(\lambda)} \prod_{i=1}^{n} q_i^{\lambda_i} = \frac{\sum_{\gamma\in D_n}\prod_{i=1}^{n-1} q_i^{d_i(\gamma)+\bar{n}_i(\gamma^{-1})}}{\prod_{i=1}^{n-1}(1-q_1^2\cdots q_i^2)(1-q_1\cdots q_n)},$$

in $\mathbf{Z}[[q_1,\ldots,q_n]]$.

PROOF. Let consider the usual decomposition $D_n = \biguplus_{u\in S_n}\{\sigma u : \sigma \in T\}$, where $T = \{\sigma \in D_n : des(\sigma) = 0\}$. From definitions is clear that for every $\sigma \in T$, $u \in S_n$ and $i \in [n-1]$

$$d_i(\sigma u) = d_i(u) \text{ and } \bar{n}_i(u^{-1}\sigma^{-1}) = \bar{n}_i(\sigma^{-1}).$$

Therefore

$$\begin{aligned}
\sum_{\gamma\in D_n}\prod_{i=1}^{n-1} q_i^{d_i(\gamma)+\bar{n}_i(\gamma^{-1})} &= \sum_{u\in S_n}\sum_{\sigma\in T}\prod_{i=1}^{n-1} q_i^{d_i(\sigma u)+\bar{n}_i((\sigma u)^{-1})} \\
&= \sum_{u\in S_n}\sum_{\sigma\in T}\prod_{i=1}^{n-1} q_i^{d_i(u)+\bar{n}_i((\sigma)^{-1})} \\
&= \sum_{u\in S_n}\prod_{i=1}^{n-1} q_i^{d_i(u)}\cdot\sum_{\sigma\in T}\prod_{i=1}^{n-1} q_i^{\bar{n}_i((\sigma)^{-1})}.
\end{aligned}$$

An element $\sigma \in T$ is uniquely determined by the set $Neg(\sigma^{-1}) \cap [n-1]$. Hence

$$\sum_{\sigma\in T}\prod_{i=1}^{n-1} q_i^{\bar{n}_i(\sigma^{-1})} = \prod_{i=1}^{n-1}(1+q_1\cdots q_i).$$

The thesis follows by Theorem 4.22 observing that $d_n(u) = 0$. □

The following identity easily follows by Theorems 4.23 and 4.24.

THEOREM 4.25. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\gamma\in D_n}\prod_{i=1}^{n-1} q_i^{d_i(\gamma)+\bar{n}_i(\gamma^{-1})} = \sum_{\gamma\in D_n}\prod_{i=1}^{n-1} q_i^{2\delta_i(\gamma)+\eta_i(\gamma)}.$$

□

Using Theorem 4.25 we prove that the two pair of statistics $(ddes, dmaj)$ and $(Ddes, Dmaj)$ have the same distributed on D_n, as already seen in Corollary 3.37

$$\sum_{\gamma\in D_n} t^{ddes(\gamma)}q^{dmaj(\gamma)} = \sum_{\gamma\in D_n} t^{Ddes(\gamma)}q^{Dmaj(\gamma)}.$$

PROOF OF COROLLARY 3.37. In Theorem 4.25 replace q_1 with qt and q_i with q for $i \in [2, n-1]$ and apply the identities

$$d_1(\gamma)+\bar{n}_1(\gamma) = ddes(\gamma), \quad \sum_{i=1}^{n-1}(d_i(\gamma)+\bar{n}_i(\gamma^{-1})) = dmaj(\gamma),$$

and

$$2\cdot\delta_1(\gamma)+\eta_1(\gamma) = Ddes(\gamma), \quad \sum_{i=1}^{n-1}(2\cdot\delta_i+\eta_i) = Dmaj(\gamma).$$

□

CHAPTER 5

Hermitian symmetric pairs

In this chapter we give explicit combinatorial product formulas for the polynomials, defined by Shelton, encoding the dimensions of the spaces of extensions of (g,p)-generalized Verma modules, in the cases when (g,p) corresponds to an indecomposable classic Hermitian symmetric pair. The formulas imply that these dimensions are combinatorial invariants. We discuss also how these polynomials are related to the parabolic R-polynomials introduced by Deodhar.
The results in this chapter are part of a paper [**7**] that will appear in Transactions of the American Mathematical Society.

5.1. Shelton polynomials

For a Lie algebra g we let $U(g)$ its universal enveloping algebra. Let g be a semisimple complex Lie algebra with Cartan subalgebra h, root system $\Delta \subset h^*$ and positive root system Δ^+ with respect to a fixed basis π. Let $b = h \oplus n$ be the Borel subalgebra with $n = \sum_{\alpha>0} g_\alpha$, $\bar{n} = \sum_{\alpha>0} g_{-\alpha}$ and let $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$. Let W be the Weyl group of Δ.

Fix a parabolic subalgebra $p = m \oplus u$ of g which contains b and has nilradical u. Assume that h is contained in m and put $\Delta(m)$ its root system and $\Delta^+(m) = \Delta \cup \Delta(m)$. Let $W(m)$ be the parabolic subgroup of W generated by the reflections corresponding to roots in $\Delta(m)$ and W^m the quotient, i.e. the set of minimal length coset representatives of $W(m)$ in W.

Fix $\lambda \in h^*$ and consider a one-dimensional $b = h \oplus n$-module $\mathbf{C}_{\lambda-\rho}$, such that $(H+N)(z) := (\lambda-\rho)(H)z$ for all $H \in h$, $N \in n$ and $z \in \mathbf{C}$. The module

$$M(\lambda) := U(g) \bigotimes_{U(b)} \mathbf{C}_{\lambda-\rho},$$

is called the *Verma module,* associated with b, h, π with highest weight $\lambda - \rho$.

If λ is $\Delta^+(m)$ dominant integral, let $F(m,\lambda)$ be the irreducible finite-dimensional m-module with highest weight λ. Letting u acts by zero this become a p-module. The module

$$N(\lambda) := U(g) \bigotimes_{U(p)} F(m, \lambda-\rho),$$

is the (g,p)-*generalized Verma module* with highest weight $\lambda - \rho$. Note that the quotient W^m parametrizes the set of all (g,p)- generalized Verma modules. More precisely to each $w \in W^m$ we may associate the generalized Verma module $N_w = N(w_J^0 w\lambda)$.

We let $\mathcal{O}(g,p)$ be the category of all g-modules M which are:

i) finitely generated over $U(g)$;
ii) $U(p)$-locally finite;

iii) completely reducible as m-modules.

In the case $p = b$ this is the category $\mathcal{O}$ introduced by Bernstein-Gelfand-Gelfand (see, e.g., [**11**]). In this work all Ext^* groups are computed in the category $\mathcal{O}(g, p)$.

The following result is due to Shelton, and we refer the reader to [**48**] for its proof. He defines for any u and v in W^m, a polynomial $E_{u,v}(q)$ by:

$$E_{u,v}(q) = \sum_{k \geq 0} (-1)^{\ell(v)-\ell(u)-k} q^k \dim_{\mathbf{C}}(\mathrm{Ext}^k(N_u, N_v)), \tag{5.1}$$

and proves that,

THEOREM 5.1. *Suppose that (g, p) corresponds to an indecomposable Hermitian symmetric pair. Then for all $u, v \in W^m$:*

i) $E_{u,v}(q) = 0$ *if* $u \not\leq v$;

ii) $E_{u,u}(q) = 1$;

iii) *if* $u < v$ *and* $s \notin D_R(u)$ *with* $us \in W^m$ *then*

$$E_{u,v}(q) = \begin{cases} E_{us,vs}(q) & \text{if } s \notin D_R(v) \text{ and } vs \in W^m; \\ (q-1)E_{us,v}(q) & \text{if } s \in D_R(v) \text{ and } us \not\leq vs; \\ (q-q^{-1})E_{us,v}(q) + E_{us,vs}(q) & \text{if } u \leq us \leq vs \leq v; \\ qE_{us,v}(q) & \text{if } vs \notin W^m. \end{cases}$$

In this paper we solve these recurrence relations in the cases when (g, p) is an indecomposable classic Hermitian symmetric pair. Our main results are explicit product formulas for these polynomials. Moreover, these formulas imply that the polynomials are combinatorial invariants.

We designate a pair by the types of the root systems $\Delta(g)$ and $\Delta(m)$ (see Table below) and with a change of notation, consistent with those of previous chapters, we denote the corresponding Weyl groups by W and W_J and the quotient by W^J, where J is the suitable subset of the set of generators.

(g, p)	g	$[m, m]$
$SU(r, s)$	A_N	$A_{r-1} \times A_{s-1}$
$SO(2n-1, 2)$	B_n	B_{n-1}
$Sp(2n, \mathbf{R})$	C_n	A_{n-1}
$SO(2n-2, 2)$	D_n	D_{n-1}
$SO^*(2n)$	D_n	A_{n-1}

The indecomposable classic Hermitian Symmetric pairs

5.2. The case $(A_{n-1}, A_{i-1} \times A_{n-i-1})$

In this section, we give an explicit product formula for the polynomials $E_{u,v}(q)$ in the case of the pair $(A_{n-1}, A_{i-1} \times A_{n-i-1})$. We give two different formulations of this result, one in terms of permutations and one in terms of partitions. Throughout this section we fix $n \in \mathbf{P}$ and $i \in [n-1]$, and we let $W := S_n$, $s_i := (i, i+1)$ for $i = 1, \ldots, n-1$, $S := \{s_1, \ldots, s_{n-1}\}$, and $J := S \setminus \{s_i\}$.

We identify a partition λ with its diagram,

$$\{(i, j) \in \mathbf{P}^2 : \ 1 \leq i \leq k, \ 1 \leq j \leq \lambda_i\}.$$

If we replace the dots (i, j) by juxtaposed squares, we obtain the Young diagram of λ, rotated counterclockwise by $\frac{\pi}{4}$ radians with respect to the usual anglophone convention, as in Chapter 1. So, for example, the diagram of $(5, 4, 2, 1)$ is illustrated in Figure 1.

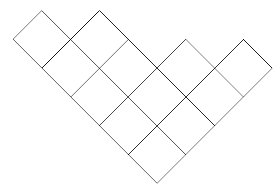


FIGURE 1

Given $n \in \mathbf{P}$ and $i \in [n-1]$ we let

$$\mathcal{P}(n, i) := \{\mu \in \mathcal{P} : \ \mu \subseteq (n-i)^i\}.$$

Let $\mu, \lambda \in \mathcal{P}$, $\mu \subseteq \lambda$. We then call $\lambda \setminus \mu$ a *skew partition*. Note that, in poset theoretic language, partitions (respectively, skew-partitions) are the finite order ideals (respectively, finite convex subsets) of $\mathbf{P}^2$. Given two skew partitions $\rho, \nu \subset \mathbf{P}^2$ we write $\rho \approx \nu$ if ρ is a translate of ν. Given a skew partition $\nu \subseteq \mathbf{P}^2$ its *conjugate* is

$$\nu' := \{(j, i) \in \mathbf{P}^2 \ : \ (i, j) \in \nu\}.$$

For symmetric groups, the parabolic subgroups of S_n, are called *Young subgroups*. In the case of maximal parabolic subgroups (i.e. $J = S \setminus \{s_i\}$) their quotients take this particularly simple form:

$$W^J = \{w \in W : w^{-1}(1) < \ldots < w^{-1}(i) \text{ and } w^{-1}(i+1) < \ldots < w^{-1}(n)\}.$$

Given $v \in W^J$ we associate to v the partition

$$\Lambda(v) := (v^{-1}(i) - i, \ldots, v^{-1}(2) - 2, v^{-1}(1) - 1). \tag{5.2}$$

The following is well known (see, [**14**, Proposition 2.8]).

PROPOSITION 5.2. *The map Λ defined by (5.2) is a bijection between W^J and $\mathcal{P}(n, i)$. Furthermore $u \leq v$ in W^J if and only if $\Lambda(u) \subseteq \Lambda(v)$, and $\ell(v) = |\Lambda(v)|$ for all $u, v \in W^J$.*

We find it sometimes convenient to identify a partition $\lambda \in \mathcal{P}(n, i)$ with a lattice path, with $(1, 1)$ and $(1, -1)$ steps. This path is the union of the lower boundary of the diagram of the skew-partition $(n-i)^i \setminus \lambda$ and the upper boundary of the partition λ. Note that it starts at $(0, 0)$ and ends at $(n, 2i - n)$ (equivalently, it has n steps and exactly i are $(1, 1)$-steps). We call a $(1, 1)$-step (respectively, $(1, -1)$-step) an *up-step* (respectively, *down-step*). Given $j \in [n-1]$ we say that λ has a *peak* at j if the j-th step of λ is up and its $(j+1)$-th step is down.
For example, if we take the partition of Figure 1, $\lambda = (5, 4, 2, 1) \in \mathcal{P}(9, 4)$ then the associated path is the one shown in Figure 2 and it has peaks at $1, 3, 6$, and 8.
Note that this identification between partitions and paths depends on n and i. For example, the same partition $(5, 4, 2, 1)$ corresponds to the path in Figure 3 if

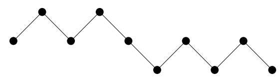


FIGURE 2. The lattice path corresponding to $(5, 4, 2, 1)$ if $n = 9$ and $i = 4$

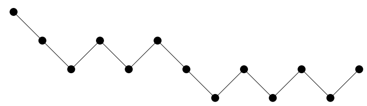


FIGURE 3. The lattice path corresponding to $(5, 4, 2, 1)$ if $n = 12$ and $i = 5$

$n = 12$ and $i = 5$. Since n and i are fixed throughout this section, this will cause no confusion.

The following elementary lemma is known, (see, [**14**, Lemma 2.9]).

LEMMA 5.3. *Let $v \in W^J$, and $j \in [n-1]$. Then $s_j \in D_R(v)$ if and only if $\Lambda(v)$ has a peak at $n - j$.*

Note that the k-th step of $\Lambda(v)$ is an up step if and only if

$$k \in \{n + 1 - v^{-1}(i), n + 1 - v^{-1}(i - 1), \ldots, n + 1 - v^{-1}(1)\}.$$

Let $u, v \in W^J$, $u \leq v$. For $j \in [n]$ we let, following [**14**],

$$a_j(u, v) := |\{r \in u^{-1}([i]) : \ r < j\}| - |\{r \in v^{-1}([i]) : \ r < j\}|. \tag{5.3}$$

For example, if $n = 9$, $i = 4$, $u = 123564789$, and $v = 516278394$ then

$$(a_1(u, v), \ldots, a_9(u, v)) = (0, 1, 1, 2, 1, 1, 2, 1, 1). \tag{5.4}$$

Note that it follows easily from Proposition 5.2 that $a_j(u, v) \geq 0$ for $j = 1, \ldots, n$ if and only if $u \leq v$, and that $a_j(u, v) > 0$ if $j \in v^{-1}([i]) \setminus u^{-1}([i])$ and $u \leq v$. Also note that, if $u \in W^J$ and $j \in [n]$, then

$$|\{r \in u^{-1}([i]) : r < j\}| = \begin{cases} u(j) - 1, & \text{if } j \in u^{-1}([i]), \\ j + i - u(j), & \text{if } j \notin u^{-1}([i]). \end{cases}$$

This may be used to obtain a more explicit formula for $a_j(u, v)$, if desired.

We can now state and prove the main result of this section.

THEOREM 5.4. *Let $u, v \in W^J$, $u \leq v$. Then*

$$E_{u,v}(q) = q^{\ell(v) - \ell(u)} \prod_{j \in v^{-1}([i]) \setminus u^{-1}([i])} (1 - q^{-2a_j(u,v)+1}). \tag{5.5}$$

PROOF. Let, for brevity, $D_i(u, v) := v^{-1}([i]) \setminus u^{-1}([i])$. We proceed by induction on $\ell(w_0^J) - \ell(u)$. If $\ell(w_0^J) - \ell(u) = 0$, we have $w_0^J = v = u$ and the result is trivially true. So suppose that $\ell(w_0^J) - \ell(u) > 0$ and let $s = (k, k + 1)$ be such that $s \notin D_R(u)$ and $us \in W^J$. Note that, since $u \in W^J$, this implies that $k \in u^{-1}([i])$ and $k + 1 \notin u^{-1}([i])$. We have four cases to consider.

a) $s \notin D_R(v)$, and $vs \in W^J$

Since $v \in W^J$, this implies that $k \in v^{-1}([i])$ and $k+1 \notin v^{-1}([i])$, moreover $us, vs \in W^J$ and so, $D_i(us, vs) = D_i(u,v)$ and $a_j(us,vs) = a_j(u,v)$ for all $j \in [n]$ since $(us)^{-1}([i])$ is obtained from $u^{-1}([i])$ by replacing k by $k+1$, and similarly for v. Hence, by Theorem 5.1 and our induction hypothesis,

$$\begin{aligned} E_{u,v}(q) &= E_{us,vs} \\ &= q^{\ell(vs)-\ell(us)} \prod_{j \in D_i(us,vs)} (1 - q^{-2a_j(us,vs)+1}) \\ &= q^{\ell(v)-\ell(u)} \prod_{j \in D_i(u,v)} (1 - q^{-2a_j(u,v)+1}), \end{aligned}$$

as desired.

b) $s \in D_R(v)$, and $us \not\leq vs$

Since $v \in W^J$ this implies that $k \notin v^{-1}([i])$, $k+1 \in v^{-1}([i])$, and $vs \in W^J$. So we have that $D_i(us, v) = D_i(u,v) \setminus \{k+1\}$, and

$$a_j(us, v) = a_j(u,v) = a_j(us, vs) \tag{5.6}$$

for all $j \in [n] \setminus \{k+1\}$, and

$$a_{k+1}(us, vs) = a_{k+1}(u,v) - 2. \tag{5.7}$$

Since $us \not\leq vs$ by (5.7) and the note before the statement of this theorem, we conclude that $a_{k+1}(u,v) = 1$. Hence, by Theorem 5.1 and the induction hypothesis,

$$\begin{aligned} E_{u,v}(q) &= (q-1)E_{us,v}(q) \\ &= (q-1)q^{\ell(v)-\ell(u)-1} \prod_{j \in D_i(u,v)\setminus\{k+1\}} (1 - q^{-2a_j(us,v)+1}) \\ &= \frac{(q-1)q^{\ell(v)-\ell(u)-1}}{(1 - q^{-2a_{k+1}(u,v)+1})} \prod_{j \in D_i(u,v)} (1 - q^{-2a_j(u,v)+1}) \\ &= q^{\ell(v)-\ell(u)} \prod_{j \in D_i(u,v)} (1 - q^{-2a_j(u,v)+1}), \end{aligned}$$

as desired.

c) $s \in D_R(v)$, and $us \leq vs$

Since $v \in W^J$ this implies that $k \notin v^{-1}([i])$, $k+1 \in v^{-1}([i])$, and $vs \in W^J$. So we have that

$$D_i(us, v) = D_i(u,v) \setminus \{k+1\},$$

$$D_i(us, vs) = (D_i(u,v) \setminus \{k+1\}) \cup \{k\},$$

$a_j(us,v) = a_j(us,vs) = a_j(u,v)$ for all $j \in [n]\backslash\{k+1\}$, $a_{k+1}(us,vs) = a_{k+1}(u,v)-2$ and $a_{k+1}(u,v) = a_k(u,v)+1$. Hence, by Theorem 5.1 and our induction hypothesis,

$$\begin{aligned}
E_{u,v}(q) &= (q-q^{-1})E_{us,v}(q) + E_{us,vs}(q) \\
&= (q-q^{-1})q^{\ell(v)-\ell(u)-1} \prod_{j\in D_i(u,v)\backslash\{k+1\}} (1-q^{-2a_j(us,v)+1}) \\
&\quad + q^{\ell(v)-\ell(u)-2} \prod_{j\in (D_i(u,v)\backslash\{k+1\})\cup\{k\}} (1-q^{-2a_j(us,vs)+1}) \\
&= (q-q^{-1})q^{\ell(v)-\ell(u)-1} \prod_{j\in D_i(u,v)\backslash\{k+1\}} (1-q^{-2a_j(u,v)+1}) \\
&\quad + q^{\ell(v)-\ell(u)-2}(1-q^{-2a_k(u,v)+1}) \prod_{j\in D_i(u,v)\backslash\{k+1\}} (1-q^{-2a_j(u,v)+1}) \\
&= q^{\ell(v)-\ell(u)-2}\frac{(q^2-q^{-2a_k(u,v)+1})}{(1-q^{-2a_{k+1}(u,v)+1})} \prod_{j\in D_i(u,v)} (1-q^{-2a_j(u,v)+1}) \\
&= q^{\ell(v)-\ell(u)} \prod_{j\in D_i(u,v)} (1-q^{-2a_j(u,v)+1}),
\end{aligned}$$

and the result again follows.

d) $vs \notin W^J$

Then $s \notin D_R(v)$ and we have two cases. In the first one, we have $k, k+1 \in v^{-1}([i])$ and this implies that $D_i(us,v) = (D_i(u,v)\setminus\{k+1\})\cup\{k\}$ and $a_j(us,v) = a_j(u,v)$ for $j \in [n]\setminus\{k+1\}$, $a_{k+1}(u,v) = a_k(us,v)$. Hence, by Theorem 5.1 and the induction hypothesis,

$$\begin{aligned}
E_{u,v}(q) &= qE_{us,v}(q) \\
&= q^{\ell(v)-\ell(u)} \prod_{j\in (D_i(u,v)\backslash\{k+1\})\cup\{k\}} (1-q^{-2a_j(us,v)+1}) \\
&= q^{\ell(v)-\ell(u)}\frac{(1-q^{-2a_k(us,v)+1})}{(1-q^{-2a_{k+1}(u,v)+1})} \prod_{j\in D_i(u,v)} (1-q^{-2a_j(u,v)+1}) \\
&= q^{\ell(v)-\ell(u)} \prod_{j\in D_i(u,v)} (1-q^{-2a_j(u,v)+1}).
\end{aligned}$$

In the second case we have, $k, k+1 \notin v^{-1}([i])$, hence $D_i(us,v) = D_i(u,v)$ and $a_j(us,v) = a_j(u,v)$ for $j \in [n]\setminus\{k+1\}$. Hence, by Theorem 5.1 and induction hypothesis,

$$\begin{aligned}
E_{u,v}(q) &= qE_{us,v}(q) \\
&= q^{\ell(v)-\ell(u)} \prod_{j\in D_i(u,v)} (1-q^{-2a_j(us,v)+1}) \\
&= q^{\ell(v)-\ell(u)} \prod_{j\in D_i(u,v)} (1-q^{-2a_j(u,v)+1}),
\end{aligned}$$

and the result again follows.

This completes the induction step and hence the proof. □

Because W^J is isomorphic, as a poset, to a lower interval in Young's lattice (by Proposition 5.2), it is natural to rephrase Theorem 5.4 in the language of partitions rather than in that of permutations.

Let $\mu, \lambda \in \mathcal{P}(n, i)$, with $\mu \subseteq \lambda$. Think of μ and λ as paths as explained above. Then, by Proposition 5.2, the path λ lies (weakly) above the path μ. Let $1 \leq j \leq n$ and consider the j-th step of λ (from the left). Following [**14**], we say that such a step is *allowable* with respect to μ if it is an up-step and the j-th step of μ is a down-step.
For example, if $n = 9$, $i = 4$, $\lambda = (5, 4, 2, 1)$, and $\mu = (2, 0, 0, 0)$ then the j-th step of λ is allowable with respect to μ exactly if $j \in \{1, 3, 6\}$ (see Figure 4).

Now let $\tilde{a}_j(\mu, \lambda)$ be the vertical distance (divided by two, since it is always even) between the (right end of the) j-th step of λ and the (right end of the) j-th step of μ. We then have the following result, and we refer the reader to [**14**, Proposition 3.3] for its proof.

PROPOSITION 5.5. *Let $u, v \in W^J$, $u \leq v$. Then*

$$a_j(u, v) = \tilde{a}_{n+1-j}(\Lambda(u), \Lambda(v))$$

for $j = 1, \dots, n$. Furthermore $n + 1 - j \in v^{-1}([i]) \setminus u^{-1}([i])$ if and only if the j-th step of $\Lambda(v)$ is allowable with respect to $\Lambda(u)$.

We can now rephrase Theorem 5.4 in terms of partitions.

COROLLARY 5.6. *Let $u, v \in W^J$, $u \leq v$. Then*

$$E_{u,v}(q) = q^{|\lambda \setminus \mu|} \prod_j (1 - q^{-2\tilde{a}_j(\mu,\lambda)+1}) \tag{5.8}$$

where $\mu = \Lambda(u)$, $\lambda = \Lambda(v)$ and j runs over all the allowable steps of λ with respect to μ. In particular, $E_{u,v}(q)$ depends only on $\Lambda(v) \setminus \Lambda(u)$. $\square$

In the case of a lower interval, the formula (5.8) takes up a particularly simple form. The proof of the next result is analogous to the one of [**14**, Corollary 3.5] and we leave it to the reader.

COROLLARY 5.7. *Let $v \in W^J$. Then*

$$E_{e,v}(q) = q^{|\mu|} \prod_{j=1}^{d(\mu)} (1 - q^{-2j+1}),$$

where $\mu = \Lambda(v)$ and $d(\mu)$ is the length of the Durfee square of μ. $\square$

We close this section with an example, to illustrate Theorem 5.4 and Corollary 5.6. Let $u = 123564789$ and $v = 516278394$ in $S_9^{S \setminus \{(4,5)\}}$, we have $D_4(u, v) = \{4, 7, 9\}$. From (5.4) and (5.5), it follows that

$$E_{u,v}(q) = q^{10}(1 - q^{-1})(1 - q^{-3})(1 - q^{-3}). \tag{5.9}$$

Observe that we have $\Lambda(v) = (5, 4, 2, 1) = \lambda$ and $\Lambda(u) = (2, 0, 0, 0) = \mu$. The diagram of the skew-partition $\Lambda(v) \setminus \Lambda(u)$ is drawn in Figure 4. The allowable-steps are indicated by arrows and $\tilde{a}_1(\mu, \lambda) = 1$, $\tilde{a}_3(\mu, \lambda) = 2$, $\tilde{a}_6(\mu, \lambda) = 2$.

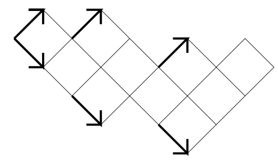


FIGURE 4

5.3. The case (C_n, A_{n-1})

A partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is *strict* if $\lambda_1 > \lambda_2 > \ldots > \lambda_k$. We denote by $\mathcal{P}_s$ the set of all (integer) strict partitions. Let

$$H := \{(i,j) \in \mathbf{P}^2 : i \leq j\}$$

with the ordering induced by the product ordering on $\mathbf{P}^2$. We call the finite order ideals of H *shifted partitions*. Denote by $\mathcal{I}$ the set of all finite order ideals of H. Note that $\mathcal{I}$ is partially ordered by set inclusion. It is well known that this makes $\mathcal{I}$ into a distributive lattice. We identify a shifted partition with its diagram

$$\{(i,j) \in \mathbf{P}^2 : 1 \leq i \leq k, i \leq j \leq \lambda_i - 1 + i\},$$

and we draw it rotated counterclockwise by $\frac{\pi}{4}$ radians. So for example the diagram of $(7, 6, 5, 4, 2)$ is illustrated in Figure 5.

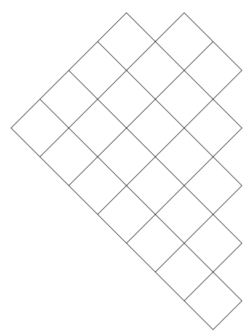


FIGURE 5

Let

$$\tilde{\mathcal{P}} := \{\lambda \in \mathcal{P} : \lambda \supseteq \ell(\lambda)^{\ell(\lambda)}\}.$$

Note that there are inclusion preserving bijections between strict partitions, shifted partitions and partitions in $\tilde{\mathcal{P}}$, given by:

$$(\lambda_1, \lambda_2, \ldots, \lambda_k) \leftrightarrow \{(i,j) : 1 \leq i \leq k, i \leq j \leq \lambda_i - 1 + i\} \leftrightarrow (\lambda_1, \lambda_2 + 1, \ldots, \lambda_k + k - 1).$$

For this reason, we will freely identify these objects. Note, however, that only the bijection between strict partitions and shifted partitions preserves size. In fact, if $\lambda, \mu, \in \tilde{\mathcal{P}}$ and $\nu \in \mathcal{P}$, with $\mu \subseteq \nu \subseteq \lambda$, then it does not necessarily follow that $\nu \in \tilde{\mathcal{P}}$. Therefore the subposet

$$\{\nu \in \mathcal{P} : \mu \subseteq \nu \subseteq \lambda\}$$

of $\mathcal{P}$ is not isomorphic to the subposet

$$\{\nu \in \mathcal{I} : \mu \subseteq \nu \subseteq \lambda\}$$

of $\mathcal{I}$.

Our purpose in this section is to obtain an explicit product formula for the E-polynomials in the case of the pair (C_n, A_{n-1}). For the rest of this section, we fix $n \in \mathbf{P}$ and we let $W := B_n$, $s_i := (i, i+1)(-i-1, -i)$ for $i = 1, \dots, n-1$, $s_0 := (1, -1) = s_0^B$, $S := \{s_0, s_1, \dots, s_{n-1}\}$ and $J := S \setminus \{s_0\}$.

From Proposition 2.19 we have that

$$W^J = \{v \in W : v^{-1}(1) < v^{-1}(2) < \dots < v^{-1}(n)\}.$$

Therefore, given $v \neq e$, $v \in W^J$, there is a unique $k \in [n]$ (in fact, $k = N_1(v)$) such that

$$v^{-1}(k) < 0 < v^{-1}(k+1) \tag{5.10}$$

and we associate to v the shifted partition

$$\Lambda_B(v) := (-v^{-1}(1), -v^{-1}(2), \dots, -v^{-1}(k)). \tag{5.11}$$

Let

$$\tilde{\mathcal{I}}(n) := \{\lambda \in \mathcal{I} :\ \lambda \subseteq (n, n-1, \dots, 2, 1)\},$$

the following is known:

Proposition 5.8. *The map Λ_B defined by (5.11) is a bijection between W^J and $\tilde{\mathcal{I}}(n)$. Furthermore $u \leq v$ in W^J if and only if $\Lambda_B(u) \subseteq \Lambda_B(v)$ and $\ell(v) = |\Lambda_B(v)|$ for all $u, v \in W^J$.*

As before it is convenient to identify a shifted partition $\lambda \in \tilde{\mathcal{I}}(n)$ with a lattice path with $(1,1)$ and $(1,-1)$ steps starting at $(0,0)$ and having n steps. We have the obvious bijection between the peaks of λ as a path and the upper peaks of λ as a partition. Note that, as in Proposition 5.2, this bijection depends on n, but for us n is fixed and so there is no confusion. For example, the partition $\lambda = (7, 6, 5, 4, 2)$ corresponds to the path in Figure 6.

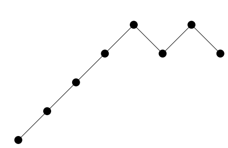


Figure 6

Lemma 5.9. *Let $v \in W^J$, and $j \in [n-1]$. Then $s_j \in D_R(v)$ if and only if $\Lambda_B(v)$ has a peak at $n - j$. Furthermore, $s_o \in D_R(v)$ if and only if the last step of $\Lambda_B(v)$ is up.*

This result can be proved in a way similar to Lemma 5.3 (see also Lemma 5.16) and is due to Brenti [**15**]. Note that i-th step of $\Lambda_B(v)$ is an up step if and only if

$$i \in \{n+1+v^{-1}(1), n+1+v^{-1}(2), \dots, n+1+v^{-1}(k)\}. \tag{5.12}$$

PROPOSITION 5.10. *Let $v \in W^J$, and $i \in [n]$. Then the i-th step (from the left) of $\Lambda_B(v)$ (seen as a path) is an up-step if and only if $v(n+1-i) < 0$.*

PROOF. We know that the i-th step of $v \in W^J$ is an up step if and only if

$$n+1-i \in \{-v^{-1}(1), -v^{-1}(2), \ldots, -v^{-1}(k)\}. \tag{5.13}$$

But this, by the definition of k, happens if and only if $v(n+1-i) < 0$, as desired. $\square$

We are now ready to prove the main theorem of this section, which gives an explicit product formula for the polynomials $E_{u,v}(q)$ in the case of the symmetric pair (C_n, A_{n-1}). As in the preceding section we give two different formulations of this result, one in terms of signed permutations and one in terms of shifted partitions.

Let $u, v \in W^J$, $u \leq v$. For $j \in [n]$ let, following [**15**],

$$b_j(u,v) := |\{r \geq j : \ v(r) < 0\}| - |\{r \geq j : \ u(r) < 0\}|. \tag{5.14}$$

For example, if $u = [4, 5, -3, -2, 6, 7, -1]$ and $v = [6, -5, 7, -4, -3, -2, -1]$, then

$$(b_1(u,v), \ldots, b_7(u,v)) = (2, 2, 1, 2, 2, 1, 0). \tag{5.15}$$

Note that it follows from Proposition 5.8 that $b_j(u,v) \geq 0$ for $j = 1, \ldots, n$ if and only if $u \leq v$. Also, if $u \leq v$, then $b_j(u,v) > 0$ when $v(j) < 0 < u(j)$. We let

$$N(u,v) := \{r \in [n] : \ u(r)v(r) < 0\}, \tag{5.16}$$

and

$$D(u,v) := \{r \in N(u,v) : \ (-1)^{b_r(u,v)} < 0\}. \tag{5.17}$$

THEOREM 5.11. *Let $u, v \in W^J$, $u \leq v$. Then*

$$E_{u,v}(q) = q^{\ell(v)-\ell(u)} \prod_{j \in D(u,v)} (1 - q^{-\tilde{b}_j(u,v)}), \tag{5.18}$$

where

$$\tilde{b}_j(u,v) := \begin{cases} 2b_j(u,v) - 1 & \text{if } u(j) > 0, \\ 2b_j(u,v) + 1 & \text{if } u(j) < 0. \end{cases} \tag{5.19}$$

PROOF. We proceed by induction on $\ell(w_0^J) - \ell(u)$. If $\ell(w_0^J) - \ell(u) = 0$, we have $w_0^J = v = u$ and the result is trivially true. So suppose that $\ell(w_0^J) - \ell(u) > 0$ and let s be such that $s \notin D_R(u)$ and $us \in W^J$.
Suppose first that $s = (-i-1, -i)(i, i+1)$ for some $i \in [n-1]$, then $u(i) < u(i+1)$; note that, since $u \in W^J$, this implies that $u(i) < 0 < u(i+1)$. We have four cases to consider.

a) $s \notin D_R(v)$, and $vs \in W^J$

Since $v \in W^J$, this implies that $v(i) < 0 < v(i+1)$. Moreover $us, vs \in W^J$ and so, $N(us, vs) = N(u,v)$ and $b_j(us, vs) = b_j(u,v)$ for all $j \in [n]$, so $D(us, vs) = D(u,v)$ and $\tilde{b}_j(us, vs) = \tilde{b}_j(u,v)$ for all $j \in [n] \setminus \{i, i+1\}$. Hence, by Theorem 5.1 and our induction hypothesis,

$$\begin{aligned} E_{u,v}(q) &= E_{us,vs}(q) \\ &= q^{\ell(vs)-\ell(us)} \prod_{j\in D(us,vs)} (1-q^{-\tilde{b}_j(us,vs)}) \\ &= q^{\ell(v)-\ell(u)} \prod_{j\in D(u,v)} (1-q^{-\tilde{b}_j(u,v)}), \end{aligned}$$

as desired.

b) $s \in D(v)$, and $us \not\leq vs$

Then $v(i) > 0 > v(i+1)$, therefore $N(us,v) = N(u,v) \setminus \{i, i+1\}$, $b_j(us,v) = b_j(u,v) = b_j(us,vs)$ for all $j \in [n] \setminus \{i+1\}$, and

$$b_i(u,v) = b_{i+1}(u,v) - 1. \tag{5.20}$$

Since $us \not\leq vs$, it follows from the note before the statement of the theorem that $b_{i+1}(u,v) > 0 > b_{i+1}(us,vs)$. Also $b_{i+1}(u,v) - b_{i+1}(us,vs) = 2$, so $b_{i+1}(u,v) = 1$. This and (5.20) imply that $i \notin D(u,v)$ and $i+1 \in D(u,v)$. It follows that $D(us,v) = D(u,v) \setminus \{i+1\}$, $\tilde{b}_j(us,v) = \tilde{b}_j(u,v)$ for all $j \in [n] \setminus \{i, i+1\}$, and that $\tilde{b}_{i+1}(u,v) = 2b_{i+1}(u,v) - 1 = 1$. Hence, by Theorem 5.1 and the induction hypothesis,

$$\begin{aligned} E_{u,v}(q) &= (q-1)E_{us,v}(q) \\ &= (q-1)q^{\ell(v)-\ell(u)-1} \prod_{j\in D(u,v)\setminus\{i+1\}} (1-q^{-\tilde{b}_j(us,v)}) \\ &= \frac{(q-1)q^{\ell(v)-\ell(u)-1}}{(1-q^{-\tilde{b}_{i+1}(u,v)})} \prod_{j\in D(u,v)} (1-q^{-\tilde{b}_j(u,v)}) \\ &= q^{\ell(v)-\ell(u)} \prod_{j\in D(u,v)} (1-q^{-\tilde{b}_j(u,v)}). \end{aligned}$$

c) $s \in D_R(v)$, and $us \leq vs$

Then, as above, $v(i) > 0 > v(i+1)$, $N(us,v) = N(u,v) \setminus \{i, i+1\}$, $b_j(us,v) = b_j(u,v)$ for all $j \in [n] \setminus \{i+1\}$, and

$$b_i(u,v) = b_{i+1}(u,v) - 1. \tag{5.21}$$

It follows that $D(us,v) = D(u,v) \setminus \{i, i+1\}$ and $\tilde{b}_j(us,v) = \tilde{b}_j(u,v)$, for all $j \in [n] \setminus \{i, i+1\}$. On the other hand we have: $N(us,vs) = N(u,v)$ and $b_j(us,vs) = b_j(u,v)$ for all $j \in [n] \setminus \{i+1\}$, and $b_{i+1}(us,vs) = b_{i+1}(u,v) - 2$. It follows that $D(us,vs) = D(u,v)$ and $\tilde{b}_j(us,vs) = \tilde{b}_j(u,v)$ for all $j \in [n] \setminus \{i, i+1\}$. Hence, by Theorem 5.1 and the induction hypothesis,

$$\begin{aligned} E_{u,v}(q) &= (q-q^{-1})E_{us,v}(q) + E_{us,vs}(q) \\ &= (q-q^{-1})q^{\ell(v)-\ell(u)-1} \prod_{j\in D(u,v)\setminus\{i,i+1\}} (1-q^{-\tilde{b}_j(us,v)}) \\ &\quad + q^{\ell(v)-\ell(u)-2} \prod_{j\in D(u,v)} (1-q^{-\tilde{b}_j(us,vs)}). \end{aligned} \tag{5.22}$$

From (5.21) we have that $i \in D(u,v)$ if and only if $i+1 \notin D(u,v)$, so we have two cases. If $i \in D(u,v)$, then $D(us,v) = D(u,v) \setminus \{i\}$ and $\tilde{b}_i(us,vs) = 2b_i(us,vs) - 1 = 2b_i(u,v) - 1 = \tilde{b}_i(u,v) - 2$ because $u(i) < 0 < u(i+1)$. Hence, by (5.22),

$$\begin{aligned}
E_{u,v}(q) &= (q^2-1)q^{\ell(v)-\ell(u)-2} \prod_{j \in D(u,v)\setminus\{i\}} (1-q^{-\tilde{b}_j(u,v)}) \\
&\quad + q^{\ell(v)-\ell(u)-2}(1-q^{-\tilde{b}_i(us,vs)}) \prod_{j \in D(u,v)\setminus\{i\}} (1-q^{-\tilde{b}_j(u,v)}) \\
&= q^{\ell(v)-\ell(u)-2} \frac{(q^2-q^{-\tilde{b}_i(us,vs)})}{(1-q^{-\tilde{b}_i(u,v)})} \prod_{j \in D(u,v)} (1-q^{-\tilde{b}_j(u,v)}) \\
&= q^{\ell(v)-\ell(u)} \prod_{j \in D(u,v)} (1-q^{-\tilde{b}_j(u,v)}).
\end{aligned}$$

If $i+1 \in D(u,v)$, then $D(us,v) = D(u,v) \setminus \{i+1\}$ and $\tilde{b}_{i+1}(us,vs) = 2b_{i+1}(us,vs) + 1 = (2b_{i+1}(u,v) - 1) - 2 = \tilde{b}_{i+1}(u,v) - 2$. Hence, by (5.22),

$$\begin{aligned}
E_{u,v}(q) &= (q^2-1)q^{\ell(v)-\ell(u)-2} \prod_{j \in D(u,v)\setminus\{i+1\}} (1-q^{-\tilde{b}_j(u,v)}) \\
&\quad + q^{\ell(v)-\ell(u)-2}(1-q^{-\tilde{b}_{i+1}(us,vs)}) \prod_{j \in D(u,v)\setminus\{i+1\}} (1-q^{-\tilde{b}_j(u,v)}) \\
&= q^{\ell(v)-\ell(u)-2} \frac{(q^2-q^{-\tilde{b}_{i+1}(us,vs)})}{(1-q^{-\tilde{b}_{i+1}(u,v)})} \prod_{j \in D(u,v)} (1-q^{-\tilde{b}_j(u,v)}) \\
&= q^{\ell(v)-\ell(u)} \prod_{j \in D(u,v)} (1-q^{-\tilde{b}_j(u,v)}),
\end{aligned}$$

and the result follows also in this case.

d) $vs \notin W^J$

Then $s \notin D_R(v)$ and we have two cases. In the first one, we have $v(i) < v(i+1) < 0$ and this implies that $N(us,v) = (N(u,v) \setminus \{i+1\}) \cup \{i\}$, $b_j(us,v) = b_j(u,v)$ for all $j \in [n] \setminus \{i+1\}$, $b_i(us,v) = b_{i+1}(u,v)$. It follows that

$$D(us,v) \setminus \{i\} = D(u,v) \setminus \{i+1\},$$

$i \in D(us,v)$ if and only if $i+1 \in D(u,v)$, $\tilde{b}_j(us,v) = \tilde{b}_j(u,v)$ for all $j \in [n]\setminus\{i,i+1\}$ and $\tilde{b}_i(us,v) = \tilde{b}_{i+1}(u,v)$. Hence, by Theorem 5.1 and the induction hypothesis, if $i \notin D(us,v)$, $D(us,v) = D(u,v)$ and the thesis easily follows; otherwise if $i \in D(us,v)$ we have,

$$\begin{aligned}
E_{u,v}(q) &= qE_{us,v}(q) \\
&= qq^{\ell(v)-\ell(u)-1} \prod_{j \in D(us,v)} (1-q^{-\tilde{b}_j(us,v)}) \\
&= q^{\ell(v)-\ell(u)} \frac{(1-q^{-\tilde{b}_i(us,v)})}{(1-q^{-\tilde{b}_{i+1}(u,v)})} \prod_{j \in D(u,v)} (1-q^{-\tilde{b}_j(u,v)}) \\
&= q^{\ell(v)-\ell(u)} \prod_{j \in D(u,v)} (1-q^{-\tilde{b}_j(u,v)}),
\end{aligned}$$

because $\tilde{b}_i(us, v) = \tilde{b}_{i+1}(u, v)$.

In the second, we have $0 < v(i) < v(i+1)$ and this implies that $N(us, v) = (N(u, v) \setminus \{i\}) \cup \{i+1\}$, $b_j(us, v) = b_j(u, v)$ for all $j \in [n] \setminus \{i+1\}$, $b_{i+1}(us, v) = b_i(u, v)$. It follows that

$$D(us, v) \setminus \{i+1\} = D(u, v) \setminus \{i\},$$

$i+1 \in D(us, v)$ if and only if $i \in D(u, v)$, $\tilde{b}_j(us, v) = \tilde{b}_j(u, v)$ for all $j \in [n] \setminus \{i, i+1\}$ and $\tilde{b}_{i+1}(us, v) = \tilde{b}_i(u, v)$. Hence, by Theorem 5.1 and the induction hypothesis, if $i+1 \notin D(us, v)$, $D(us, v) = D(u, v)$ and the thesis easily follows; otherwise if $i+1 \in D(us, v)$ we have,

$$\begin{aligned} E_{u,v}(q) &= qE_{us,v}(q) \\ &= q^{\ell(v)-\ell(u)} \prod_{j \in D(us,v)} (1 - q^{-\tilde{b}_j(us,v)}) \\ &= q^{\ell(v)-\ell(u)} \frac{(1 - q^{-\tilde{b}_{i+1}(us,v)})}{(1 - q^{-\tilde{b}_i(u,v)})} \prod_{j \in D(u,v)} (1 - q^{-\tilde{b}_j(u,v)}), \end{aligned}$$

since $\tilde{b}_{i+1}(us, v) = \tilde{b}_i(u, v)$, the result again follows.

Suppose now that $s = (-1, 1) = s_0$. Then $u(1) > 0$, and we observe that $us, vs \in W^J$. We therefore have three cases to consider.

1) $s \notin D_R(v)$

Then $vs \in W^J$ and $v(1) > 0$. Hence $N(us, vs) = N(u, v)$, $b_j(us, vs) = b_j(u, v)$ for all $j \in [n]$ and so $D(us, vs) = D(u, v)$, $1 \notin D(u, v)$, $\tilde{b}_j(us, vs) = \tilde{b}_j(u, v)$, for all $j \in [n] \setminus \{1\}$ and the result follows from Theorem 5.1 and the induction hypothesis.

2) $s \in D_R(v)$, and $us \not\leq vs$

Then $v(1) < 0$ so $N(us, v) = N(u, v) \setminus \{1\}$, $b_j(us, v) = b_j(u, v)$ for all $j \in [n] \setminus \{1\}$. Therefore $D(us, v) = D(u, v) \setminus \{1\}$ and $\tilde{b}_j(us, v) = \tilde{b}_j(u, v)$ for all $j \in [n] \setminus \{1\}$. Also, $u < v$ and $us \not\leq vs$, so by the remark before the statement of the theorem, $b_1(u, v) = 1$ and hence $\tilde{b}_1(u, v) = 1$. Hence,

$$\begin{aligned} E_{u,v}(q) &= (q-1)E_{us,v}(q) \\ &= (q-1)q^{\ell(v)-\ell(u)-1} \prod_{j \in D(u,v) \setminus \{1\}} (1 - q^{-\tilde{b}_j(u,v)}) \\ &= q^{\ell(v)-\ell(u)} \prod_{j \in D(u,v)} (1 - q^{-\tilde{b}_j(u,v)}). \end{aligned}$$

3) $s \in D_R(v)$, and $us \leq vs$

Then $v(1) < 0$, $N(us, vs) = N(u, v)$, $N(us, v) = N(u, v) \setminus \{1\}$, $b_j(us, vs) = b_j(u, v) = b_j(us, v)$ for all $j \in [n] \setminus \{1\}$, and $b_1(us, vs) = b_1(u, v) - 2 = b_1(us, v) - 1$. It follows that

$$D(us, v) = D(u, v) \setminus \{1\} = D(us, vs) \setminus \{1\}$$

and $\tilde{b}_j(us, vs) = \tilde{b}_j(u, v) = \tilde{b}_j(us, v)$ for all $j \in [n] \setminus \{1\}$. Hence,

$$\begin{aligned} E_{u,v}(q) &= (q - q^{-1}) E_{us,v}(q) + E_{us,vs}(q) \\ &= (q - q^{-1}) q^{\ell(v)-\ell(u)-1} \prod_{j \in D(u,v) \setminus \{1\}} (1 - q^{-\tilde{b}_j(us,v)}) \\ &\quad + q^{\ell(v)-\ell(u)-2} \prod_{j \in D(us,vs)} (1 - q^{-\tilde{b}_j(us,vs)}). \end{aligned} \tag{5.23}$$

Now we have two cases. If $1 \in D(u, v)$, then $1 \in D(us, vs)$ and $\tilde{b}_1(us, vs) = \tilde{b}_1(u, v) - 2$, so from (5.23),

$$\begin{aligned} E_{u,v}(q) &= (q^2 - 1) q^{\ell(v)-\ell(u)-2} \prod_{j \in D(u,v) \setminus \{1\}} (1 - q^{-\tilde{b}_j(u,v)}) \\ &\quad + q^{\ell(v)-\ell(u)-2} (1 - q^{-\tilde{b}_1(us,vs)}) \prod_{j \in D(u,v) \setminus \{1\}} (1 - q^{-\tilde{b}_j(u,v)}) \\ &= q^{\ell(v)-\ell(u)} \frac{(1 - q^{-\tilde{b}_1(us,vs)-2})}{(1 - q^{-\tilde{b}_1(u,v)})} \prod_{j \in D(u,v)} (1 - q^{-\tilde{b}_j(u,v)}). \end{aligned}$$

If $1 \notin D(u, v)$, then $1 \notin D(us, vs)$ and from (5.23) we have that

$$\begin{aligned} E_{u,v}(q) &= (q^2 - 1) q^{\ell(v)-\ell(u)-2} \prod_{j \in D(u,v)} (1 - q^{-\tilde{b}_j(u,v)}) \\ &\quad + q^{\ell(v)-\ell(u)-2} \prod_{j \in D(u,v)} (1 - q^{-\tilde{b}_j(u,v)}) \\ &= q^{\ell(v)-\ell(u)} \prod_{j \in D(u,v)} (1 - q^{-\tilde{b}_j(u,v)}) \end{aligned}$$

and the result follows. This completes the induction step and hence the proof. $\square$

As in the previous section it is natural to rephrase Theorem 5.11 in the language of shifted partitions. Let $\mu, \lambda \in \tilde{\mathcal{I}}(n)$, with $\mu \subseteq \lambda$. We think of μ and λ as paths as explained at the beginning of this section. Then, by Proposition 5.8, the path λ lies (weakly) above the path μ. Let $j \in [n]$ and consider the j-th step of λ (from the left). Following **[15]** we say that such a step is *B-allowable* with respect to μ if the j-th step of μ is not parallel to it and $\tilde{a}_j(\mu, \lambda)$ is odd.
For example, if $\mu = (7, 4, 3, 0, 0, 0, 0)$, and $\lambda = (7, 6, 5, 4, 2, 0, 0)$ then the j-th step of λ is B-allowable with respect to μ exactly if $j \in \{2, 5\}$ (see Figure 7).

PROPOSITION 5.12. *Let $u, v \in W^J$, $u \leq v$. Then*

$$b_j(u, v) = \tilde{a}_{n+1-j}(\Lambda_B(u), \Lambda_B(v))$$

for $i = 1, \ldots, n$. Furthermore $n + 1 - j \in D(u, v)$ if and only if the j-th step of $\Lambda_B(v)$ is B-allowable with respect to $\Lambda_B(u)$.

This result can be proved in a way similar to Proposition 5.5 and is due Brenti **[15]**.
We can now rephrase Theorem 5.11 in terms of shifted partitions.

COROLLARY 5.13. *Let $u, v \in W^J$, $u \leq v$. Then*

$$E_{u,v}(q) = q^{|\lambda \setminus \mu|} \prod_j (1 - q^{-\bar{a}_j(\mu,\lambda)}) \tag{5.24}$$

where $\mu = \Lambda_B(u)$, $\lambda = \Lambda_B(v)$, j runs over all the B-allowable steps of λ with respect to μ, and

$$\bar{a}_j(\mu, \lambda) := \begin{cases} 2\tilde{a}_j(\mu,\lambda) - 1 & \text{if the } j\text{-th step of } \mu \text{ is down,} \\ 2\tilde{a}_j(\mu,\lambda) + 1 & \text{if the } j\text{-th step of } \mu \text{ is up.} \end{cases}$$

In particular, $E_{u,v}(q)$ depends only on $\Lambda_B(v) \setminus \Lambda_B(u)$. □

In the case of a lower interval, the formulas (5.18) and (5.24) take up a particularly simple form.

COROLLARY 5.14. *Let $v \in W^J$. Then*

$$E_{e,v}(q) = q^{\ell(v)} \prod_{j=1}^{\lfloor \frac{N_1(v)+1}{2} \rfloor} (1 - q^{-4j+3}) = q^{|\lambda|} \prod_{j=1}^{\lfloor \frac{\ell(\lambda)+1}{2} \rfloor} (1 - q^{-4j+3}),$$

where $\lambda = \Lambda_B(v)$.

PROOF. We know from Theorem 5.11 that

$$E_{e,v}(q) = q^{\ell(v)} \prod_{j \in D(e,v)} (1 - q^{-\tilde{b}_j(e,v)}). \tag{5.25}$$

Clearly, $N(e,v) = \{r \in [n] : v(r) < 0\}$, $\tilde{b}_j(e,v) = 2b_j(e,v) - 1$ for all $j \in [n]$. But, $b_j(e,v) = |\{r \geq j : v(r) < 0\}|$ for $j \in [n]$, and $D(e,v) = \{r \in N(e,v) : b_r(e,v) \text{ is odd}\}$. Hence,

$$E_{e,v}(q) = q^{\ell(v)} \prod_{j=1}^{\lfloor \frac{N_1(v)+1}{2} \rfloor} (1 - q^{-4j+3}),$$

as desired. From the definition we have that $\ell(\Lambda_B(v)) = N_1(v)$, so the second equation follows. □

We close this section with an example. Let $n = 7$, $u = [4, 5, -3, -2, 6, 7, -1]$, $v = [6, -5, 7, -4, -3, -2, -1] \in B_7^J$. We have $N(u,v) = \{2, 3, 5, 6\}$ and $D(u,v) = \{3, 6\}$. Since $u(3) < 0$ and $u(6) > 0$, from (5.19) and (5.18) it follows that

$$E_{u,v}(q) = q^{10}(1 - q^{-3})(1 - q^{-1}).$$

Observe that $\Lambda_B(u) = (7, 4, 3, 0, 0, 0, 0) = \mu$ and $\Lambda_B(v) = (7, 6, 5, 4, 2, 0, 0) = \lambda$. The paths corresponding to $\Lambda_B(v)$ and $\Lambda_B(u)$ are drawn in Figure 7, and the B-allowable steps are indicated by arrows. We have that $\bar{a}_2(\mu, \lambda) = 1$ and $\bar{a}_5(\mu, \lambda) = 3$.

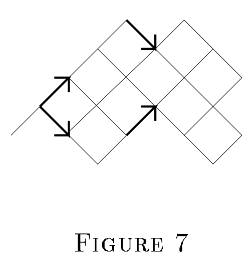


FIGURE 7

5.4. The case (D_n, A_{n-1})

In this section we study the E-polynomials in the case of the pair (D_n, A_{n-1}). For the rest of this section, we fix $n \in \mathbf{P}$ and we let $W := D_n$, $s_i := (i, i+1)(-i-1, -i)$ for $i = 1, \dots, n-1$, $s_0 := (1, -2)(-1, 2)$, $S := \{s_0, s_1, \dots, s_{n-1}\}$ and $J := S \setminus \{s_0\}$.

We have that $W^J = \{v \in W : v^{-1}(1) < v^{-1}(2) < \dots < v^{-1}(n)\}$ and for every $v \in W^J$ $v \neq e$, there is a unique $k \in [n]$ such that

$$v^{-1}(k) < 0 < v^{-1}(k+1) \tag{5.26}$$

and we associate to v the shifted partition

$$\Lambda_D(v) := (-v^{-1}(1) - 1, -v^{-1}(2) - 1, \dots, -v^{-1}(k) - 1). \tag{5.27}$$

It's not so hard to see that,

Proposition 5.15. *The map Λ_D defined by (5.27) is a bijection between W^J and $\tilde{\mathcal{I}}(n-1)$. Furthermore $u \leq v$ in W^J if and only if $\Lambda_D(u) \subseteq \Lambda_D(v)$ and $\ell(v) = |\Lambda_D(v)|$ for all $u, v \in W^J$.* □

Let $\mathcal{A}$ be the subset of $\tilde{\mathcal{I}}(n)$ consisting of all the shifted partition with an even number of entries different to zero. More precisely,

$$\mathcal{A} := \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \tilde{\mathcal{I}}(n) \ : \ k \text{ is even}\},$$

we call this set the *even shifted partitions*. Observe that $\Lambda_B(W^J) = \mathcal{A}$. Since, for $u, v \in W^J$, $u \leq v$ in B_n^J if and only if $u \leq v$ in D_n^J, we have an inclusion preserving bijection between shifted partition in $\tilde{\mathcal{I}}(n-1)$ and even shifted partitions in $\tilde{\mathcal{I}}(n)$. Thanks to this bijection we can identify $\lambda \in \tilde{\mathcal{I}}(n-1)$ with the lattice path associated to $\lambda \in \tilde{\mathcal{I}}(n)$ as explained after Proposition 5.8. We observe that this lattice path starts at $(0,0)$, ends after n steps, and has an even number of up steps.
For example, let $v = [-4, 5, -3, 6, -2, -1] \in D_6^J$, then $\Lambda_D(v) = (5, 4, 2, 0)$, $\Lambda_B(v) = (6, 5, 3, 1)$ and the lattice path associate to v is drawn in Figure 8.

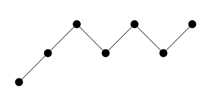


Figure 8

Lemma 5.16. *Let $v \in W^J$, and $j \in [n-1]$. Then $s_j \in D_R(v)$ if and only if $\Lambda_D(v)$ has a peak at $n-j$. Furthermore, $s_o \in D_R(v)$ if and only if the last two steps of $\Lambda_D(v)$ are up.*

Proof. Let k defined by (5.26) and $j \in [n-1]$, we have that $s_j \in D_R(v)$ if and only if $v(j) > v(j+1)$. Since $v \in W^J$ this happens if and only if $v(j) > 0 > v(j+1)$. Equivalently, this happens if and only if $j \in v^{-1}([n])$ and $j+1 \notin v^{-1}([n])$. But $\Lambda_D(v)$ (as a path) has a peak at $n-j$ if and only if its $(n-j)$-th step is up and its $(n-j+1)$-th step is down. But the i-th step of $\Lambda_D(v)$ is an up step if and only if

$$i \in \{n+1+v^{-1}(1), n+1+v^{-1}(2), \dots, n+1+v^{-1}(k)\}. \tag{5.28}$$

Therefore $\Lambda_D(v)$ has a peak at $n-j$ if and only if

$$-j = v^{-1}(i) + 1 \tag{5.29}$$

for some $i \in [k]$, and

$$-j \neq v^{-1}(i) \tag{5.30}$$

for all $i \in [k]$. Equivalently, if and only if $j \notin v^{-1}([-k,-1])$ and $j+1 \in v^{-1}([-k,-1])$, but by the definition of k, $i \notin v^{-1}([-k,-1])$ if and only if $i \notin v^{-1}([-n,-1])$, which is if and only if $i \in v^{-1}([n])$, for all $i \in [n]$. The result follows.
Now let $s_o \in D_R(v)$, this happens if and only if $v(1)+v(2) < 0$, this implies that $v(1) < v(2) < 0$. Follows that $v^{-1}(k) = -1$ and $v^{-1}(k-1) = -2$ and so that the n-th and the $n-1$-th steps of $\Lambda_D(v)$ are up. $\square$

The following is exactly the analogue of Proposition 5.10 for permutation in D_n^J.

PROPOSITION 5.17. *Let $v \in W^J$, and $i \in [n]$. Then the i-th step (from the left) of $\Lambda_D(v)$ (seen as a path) is an up-step if and only if $v(n+1-i) < 0$.* $\square$

Now we are ready to prove the analogue of Theorem 5.11 for the permutations in $(S_n^D)^J$. The formula is exactly the same but observe that the polynomials are not always the same in fact the function length is different.

THEOREM 5.18. *Let $u, v \in W^J$, $u \leq v$. Then*

$$E_{u,v}(q) = q^{\ell(v)-\ell(u)} \prod_{j \in D(u,v)} (1 - q^{-\tilde{b}_j(u,v)}), \tag{5.31}$$

where $\tilde{b}_j(u,v)$ is defined as in Theorem 5.11.

PROOF. We proceed by induction on $\ell(w_0^J) - \ell(u)$. If $\ell(w_0^J) - \ell(u) = 0$, we have $w_0^J = v = u$ and the result is trivially true. So suppose that $\ell(w_0^J) - \ell(u) > 0$ and let s be such that $s \notin D_R(u)$ and $us \in W^J$. If $s = (-i-1,-i)(i,i+1)$ for some $i \in [n-1]$, then the proof is exactly the same as for Theorem 5.11. So suppose that $s = (-1,2)(1,-2) = s_0$, then we have $u(1)+u(2) > 0$. We observe that $us \in W^J$ implies that $0 < u(1) < u(2)$. Moreover we have that $b_1(u,v)$ is even for every $u, v \in W^J$ and so $1 \notin D(u,v)$. We have four cases to consider.

a) $s \notin D_R(v)$, and $vs \in W^J$

We have $0 < v(1) < v(2)$, then $N(us,vs) = N(u,v)$ and $b_j(us,vs) = b_j(u,v)$ for all $j \in [n]$ and so $D(us,vs) = D(u,v)$ and $\tilde{b}_j(us,vs) = \tilde{b}_j(u,v)$ for all $j \in [n] \setminus \{1,2\}$. Hence, by Theorem 5.1 and our induction hypothesis,

$$\begin{aligned} E_{u,v}(q) &= E_{us,vs}(q) \\ &= q^{\ell(v)-\ell(u)} \prod_{j \in D(u,v)} (1 - q^{-\tilde{b}_j(u,v)}). \end{aligned}$$

b) $s \in D_R(v)$, and $us \not\leq vs$

Hence $v(1)+v(2) < 0$, and $v(1) < v(2) < 0$. We have that $N(us,v) = N(u,v) \setminus \{1,2\}$, $b_j(us,v) = b_j(u,v)$ for all $j \in [n] \setminus \{1,2\}$. Reasoning as in Theorem 5.11 b), $b_1(u,v) > b_2(u,v) \geq 0$, so $b_1(u,v) \geq 2$; $b_2(us,vs) > b_1(us,vs)$ so $b_1(us,vs) \leq -2$ and $b_1(u,v) - b_1(us,vs) = 4$ so $b_1(u,v) = 2$ and $b_2(u,v) = 1$. So $D(us,v) = D(u,v) \setminus \{2\}$ and $\tilde{b}_j(us,v) = \tilde{b}_j(u,v)$ for all $j \in [n] \setminus \{1,2\}$ and $\tilde{b}_2(u,v) = 1$. Hence, by Theorem 5.1 and the induction hypothesis,

$$\begin{aligned} E_{u,v}(q) &= (q-1)E_{us,v}(q) \\ &= (q-1)q^{\ell(v)-\ell(u)-1} \prod_{j \in D(u,v)\setminus\{2\}} (1-q^{-\tilde{b}_j(u,v)}) \\ &= q^{\ell(v)-\ell(u)} \prod_{j \in D(u,v)} (1-q^{-\tilde{b}_j(u,v)}). \end{aligned}$$

c) $s \in D_R(v)$, and $us \leq vs$

We have $v(1)+v(2) < 0$, and $v(1) < v(2) < 0$. So $N(us,vs) = N(u,v)$, $N(us,v) = N(u,v) \setminus \{1,2\}$, $b_j(us,vs) = b_j(u,v) = b_j(us,v)$ for all $j \in [n] \setminus \{1,2\}$, and $b_2(us,vs) = b_2(u,v) - 2$. It follows that

$$D(us,v) = D(u,v) \setminus \{2\}, \quad D(us,vs) = D(u,v),$$

$\tilde{b}_j(us,vs) = \tilde{b}_j(u,v) = \tilde{b}_j(us,v)$ for all $j \in [n] \setminus \{1,2\}$, and $\tilde{b}_2(us,vs) = \tilde{b}_2(u,v) - 2$. Hence, by Theorem 5.1 and our induction hypothesis,

$$\begin{aligned} E_{u,v}(q) &= (q-q^{-1})E_{us,v}(q) + E_{us,vs}(q) \\ &= (q-q^{-1})q^{\ell(v)-\ell(u)-1} \prod_{j \in D(u,v)\setminus\{2\}} (1-q^{-\tilde{b}_j(u,v)}) + \\ &+ q^{\ell(v)-\ell(u)-2}(1-q^{-\tilde{b}_2(us,vs)}) \prod_{j \in D(u,v)\setminus\{2\}} (1-q^{-\tilde{b}_j(u,v)}) \\ &= q^{\ell(v)-\ell(u)} \frac{(1-q^{-\tilde{b}_2(us,vs)-2})}{(1-q^{-\tilde{b}_2(u,v)})} \prod_{j \in D(u,v)} (1-q^{-\tilde{b}_j(u,v)}), \end{aligned}$$

and the result follows.

d) $vs \notin W^J$

We have two cases. Suppose that $v(1) > 0$ and $v(2) < 0$. We have $N(us,v) = (N(u,v) \cup \{1\}) \setminus \{2\}$, $b_j(u,v) = b_j(us,v)$ for all $j \in [n] \setminus \{1,2\}$ and,

$$b_1(u,v) = b_2(u,v), \tag{5.32}$$

$$b_1(us,v) = b_1(u,v) - 2. \tag{5.33}$$

Since $b_1(u,v)$ is even, (5.32) implies that $2 \notin D(u,v)$, and (5.33) implies that $1 \notin D(us,v)$. It follows that $D(u,v) = D(us,v)$ and $\tilde{b}_j(us,v) = \tilde{b}_j(u,v)$ for all $j \in [n] \setminus \{1,2\}$, so the thesis follows immediately by induction.

Suppose now that $v(1) < 0$ and $v(2) > 0$. Then $N(us,v) = (N(u,v) \cup \{2\}) \setminus \{1\}$, $b_j(u,v) = b_j(us,v)$ for all $[n] \setminus \{1,2\}$ and

$$b_1(u,v) = b_2(us,v) + 2. \tag{5.34}$$

Since $b_1(u,v)$ is even, (5.34) implies that $2 \notin D(us,v)$. It follows that $D(u,v) = D(us,v)$ and $\tilde{b}_j(us,v) = \tilde{b}_j(u,v)$ for all $j \in [n] \setminus \{1,2\}$, so the thesis follows immediately by induction. This completes the proof.

□

As in the previous cases it is natural to rephrase Theorem 5.18 in the language of shifted partitions. Let $\lambda, \mu \in \mathcal{A}$, with $\lambda \subseteq \mu$. We think μ and λ as paths as explained in previous section. So using Proposition 5.12 we have that

COROLLARY 5.19. *Let $u, v \in W^J$, $u \leq v$, then*

$$E_{u,v}(q) = q^{|\lambda \setminus \mu|} \prod_j (1 - q^{-\bar{a}_j(\mu,\lambda)}) \tag{5.35}$$

where $\mu = \Lambda_D(u)$, $\lambda = \Lambda_D(v)$, j runs over all the B-allowable steps of λ with respect to μ, and $\bar{a}_j(u,v)$ is defined as in Corollary 5.13. In particular, $E_{u,v}(q)$ depends only on $\Lambda_D(v) \setminus \Lambda_D(u)$. □

For lower intervals we obtain the following

COROLLARY 5.20. *Let $v \in W^J$. Then*

$$E_{e,v}(q) = q^{\ell(v)} \prod_{j=1}^{\frac{N_1(v)}{2}} (1 - q^{-4j+3}) = q^{|\lambda|} \prod_{j=1}^{\frac{\ell(\lambda)}{2}} (1 - q^{-4j+3}),$$

where $\lambda = \Lambda_D(v)$.

PROOF. The results follows immediately from Corollary 5.14 , observing that $N_1(v)$ is even for every $v \in W^J$. □

5.5. The cases (B_n, B_{n-1}) and (D_n, D_{n-1})

In this section we analyze the E-polynomials in the cases (B_n, B_{n-1}) and (D_n, D_{n-1}). We start with the first one. Hence $W = B_n$ but now we let $J := S \setminus \{s_{n-1}\}$ so that $W_J = B_{n-1}$. It follows that the quotient W^J is a totally ordered set, more precisely is the chain

$$W^J = \{e, s_{n-1}, \dots, s_{n-1}s_{n-2}\dots s_1 s_0, s_{n-1}\dots s_1 s_0 s_1, \dots, s_{n-1}s_{n-2}\dots s_1 s_0 s_1 \dots s_{n-1}\}.$$

PROPOSITION 5.21. *Let (W, S) be a Coxeter system, $J \subset S$ and $u, v \in W^J$ such that $u \leq v$. If $[u, v]^J$ is a chain, then*

$$E_{u,v}(q) = q^{\ell(v)-\ell(u)}(1 - q^{-1}).$$

PROOF. We proceed by induction on $\ell(w_0^J) - \ell(u)$. If $\ell(w_0^J) - \ell(u) = 0$ the result is trivially true. So suppose that $\ell(w_0^J) - \ell(u) > 0$ and let $s \notin D_R(u)$ and $us \in W^J$. We have four cases to consider.

a) $s \notin D_R(v)$, and $vs \in W^J$

Then the result follows immediately by induction.

b) $s \in D_R(v)$, and $us \not\leq vs$

We know that $[u, v]^J$ is a chain, so $us \not\leq vs$ implies that $us = v$. It follows that

$$E_{u,v}(q) = (q - 1)E_{us,v}(q) = q(1 - q^{-1}).$$

c) $s \in D_R(v)$, and $us \leq vs$

We have

$$\begin{aligned} E_{u,v}(q) &= (q - q^{-1})q^{\ell(v)-\ell(u)-1}(1 - q^{-1}) + q^{\ell(v)-\ell(u)-2}(1 - q^{-1}) \\ &= q^{\ell(v)-\ell(u)-2}(1 - q^{-1})q^2 = q^{\ell(v)-\ell(u)}(1 - q^{-1}). \end{aligned}$$

d) $vs \notin W^J$

Then the result follows immediately by induction. □

By the comment preceding the proposition this settles the case (B_n, B_{n-1}).

Let's examine the case (D_n, D_{n-1}). Hence $W = D_n$, $S = \{s_0, \ldots, s_{n-1}\}$ and $s_0 = (-1, 2)(1, -2)$. Now we let $J := S \setminus \{s_{n-1}\}$. The quotient can be written in this form

$$W^J = \{w \in W : w^{-1}(-2) < w^{-1}(1) < \ldots < w^{-1}(n-1)\},$$

and its Bruhat order is drawn in Figure 9.

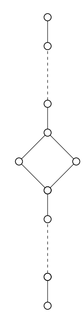


FIGURE 9

Moreover we know the unique reduced expression of each $w \in W^J$, in fact the $n-1$ elements in the chain on the bottom are $e, s_{n-1}, \ldots, s_{n-1} \cdots s_2$, the $n-1$ elements in the chain on the top are $s_{n-1} \cdots s_2 s_0 s_1, \ldots, s_{n-1} \cdots s_2 s_0 s_1 s_2 \cdots s_{n-1}$, and the remaining two elements are exactly $s_{n-1} \cdots s_2 s_0$ and $s_{n-1} \cdots s_2 s_1$. So every $i \in [2, n-1]$ identifies two elements in the quotient, $u_i := s_{n-1} \cdots s_i$ in the chain on the bottom, and $v_i := s_{n-1} \cdots s_2 s_0 s_1 \cdots s_i$ in the chain on the top. Moreover we define $u_n := e$.
Note that for all $i \in [2, n]$ and and for all $j \in [1, n-1]$ we have $\ell(u_i) = n - i$, and $\ell(v_j) = n + j - 1$.

We call (u, v) a *equidistant pair* if there exists $i \in [2, n]$ such that $u = u_i$ and $v = v_{i-1}$.
The verification of the following observation is left to the reader.

LEMMA 5.22. *Let $u, v \in W^J$. If (u, v) is an equidistant pair then $u^{-1}(j) = v^{-1}(j)$ for all $j \in [2, n-1]$.* $\square$

PROPOSITION 5.23. *Let $u, v \in W^J$. Then*

$$E_{u,v}(q) = \begin{cases} (q^{\ell(v)-\ell(u)} - q)(1 - q^{-1}) & \textit{if } (u, v) \textit{ is equidistant} \\ q^{\ell(v)-\ell(u)}(1 - q^{-1}) & \textit{otherwise.} \end{cases}$$

PROOF. We have three cases to consider.

1) $(u, v) = (u_i, v_j)$, with $i \in [2, n]$, and $j \in [1, n-1] \setminus \{i-1\}$

If $j \geq i$, then $u_i = s_{n-1} \cdots s_i$ and $v_j = s_{n-1} \cdots s_2 s_0 s_1 \cdots s_j$. Hence $s_{i-1} \notin D_R(u_i)$ and $u_i s_{i-1} = u_{i-1} \in W^J$, while $v_j s_{i-1} \notin W^J$ so we have $E_{u_i, v_j}(q) = q E_{u_{i-1}, v_j}(q)$. We have $i-2$ steps like this to do, and one more step for $s = s_0$, so at the end we have

$$E_{u_i, v_j}(q) = q^{i-2} E_{u_2, v_j}(q) = q^{i-1} E_{u_2 s_0, v_j}(q). \tag{5.36}$$

Now $[u_2s_0, v_j]^J$ is a chain with j steps and so by Proposition 5.21 it follows $E_{u_2s_0,v_j}(q) = q^{j-1}(q-1)$ and so from (5.36) we are done.
If $j < i-1$, after $(i-j-2)$ steps in the diagram we have

$$E_{u_i,v_j}(q) = q^{i-j-2}E_{u_{j+2},v_j}(q). \tag{5.37}$$

The next step is for $s = s_{j+1}$; $s_{j+1} \notin D_R(u_{j+2})$ and $s_{j+1} \notin D_R(v_j)$, so by the first part of the proof we have

$$E_{u_{j+2},v_j}(q) = E_{u_{j+1},v_{j+1}}(q) = q^{2j+1}(1-q^{-1}).$$

and the result follows from (5.37).

2) (u, v) is an equidistant pair

We proceed by induction on i. If $i = 2$, then $u_2 = s_{n-1}\cdots s_2$ and $v_1 = s_{n-1}\cdots s_2s_0s_1$. Hence $s_0 \notin D_R(u_2)$, $s_0 \in D_R(v_1)$ and $u_2s_0 \not\leq v_1s_0$, so $E_{u_2,v_1}(q) = (q-1)E_{u_2s_0,v_1}(q)$. In the second step, $s := s_1$ we have $u_2s_0s_1 \not\leq v_1s_1$ so $E_{u_2,v_1}(q) = (q-1)^2E_{u_2s_0s_1,v_1}(q)$ $= (q-1)^2$, since $v_1 = u_2s_0s_1$.
So suppose $i > 2$. Then $s_{i-1} \notin D_R(u_i)$, $u_is_{i-1} = u_{i-1} \in W^J$ and $v_{i-1}s_{i-1} = v_{i-2} \in W^J$ so since $u_{i-1} \leq v_{i-2}$ we have

$$E_{u_i,v_{i-1}}(q) = (q-q^{-1})E_{u_{i-1},v_{i-1}}(q) + E_{u_{i-1},v_{i-2}}(q).$$

By case 1), $E_{u_{i-1},v_{i-1}}(q) = q^{\ell(v_{i-1})-\ell(u_{i-1})}(1-q^{-1})$ and by induction, $E_{u_{i-1},v_{i-2}}(q) = q^{\ell(v_{i-2})-\ell(u_{i-1})}(1-q^{-1})$. The result follows.

3) $[u, v]^J$ is a chain

The result follows by Proposition 5.21. This completes the proof. □

5.6. Consequences and further remarks

In this section we derive some consequences of our results. We start by proving that the E-polynomials are combinatorial invariants, i.e. that they depend only on the poset $[u, v]^J$. To do this we need a purely order theoretic result on skew partitions, that was first proved in [**14**, Lemma 5.5].

LEMMA 5.24. *Let ρ, ν be two connected skew partitions that are isomorphic as posets. Then either $\rho \approx \nu$ or $\rho \approx \nu'$.*

We can now prove the main result of this section.

COROLLARY 5.25. *Let $J \subset S$, as in §5.2, §5.3, §5.4, and $u, v \in W^J$, $x, y \in W^J$ be such that $[u, v]^J \cong [x, y]^J$. Then*

$$E_{u,v}(q) = E_{x,y}(q).$$

PROOF. Now we prove this result in the case when $W = S_n$. By Proposition 5.2 we have that $[u, v]^J$ is isomorphic, as a poset, to the interval $[\Lambda(u), \Lambda(v)]$ in Young's lattice. But it follows immediately from the definitions and well known results in the theory of partially ordered sets (see, e.g., [**51**, §3.4]) that the subposet of join-irreducibles of $[\Lambda(u), \Lambda(v)]$ is isomorphic to $\Lambda(v) \setminus \Lambda(u)$, where the skew partition $\Lambda(v) \setminus \Lambda(u)$ is seen as a poset. Therefore, since $[u, v]^J \cong [x, y]^J$, we conclude that $\Lambda(v) \setminus \Lambda(u) \cong \Lambda(y) \setminus \Lambda(x)$ (as poset), and the result follows from Lemma 5.24 and Corollary 5.6.
Similarly, we can prove the result for the other cases, but we need to replace Proposition 5.2, Corollary 5.6 and Λ for (C_n, A_{n-1}) with Proposition 5.8, Corollary 5.13 and Λ_B, and for (D_n, A_{n-1}) with Proposition 5.15, Corollary 5.19 and Λ_D, respectively. □

Note that in the case when $W = S_n$ the proof of Corollary 5.25 applies whenever $[u,v]^K \cong [x,y]^H$ with $K, H \subset S$, $|K| = |H| = |S| - 1$.

We conclude this section by discussing the connections mentioned at the end of the introduction. In [**36**], Gabber and Joseph define for every $u, v \in W$ a polynomial

$$R'_{u,v}(q) = \sum_{k \geq 0} (-1)^{\ell(v)-\ell(u)-k} q^k \dim(\mathrm{Ext}^k(M_u, M_v))$$

and they conjectured (although this is not explicitly stated) that

$$R'_{u,v}(q) = R_{u,v}(q).$$

This conjecture is not true (see [**12**]), but the R' and the R-polynomials are not so different. In fact, Carlin shows that the R'-polynomials are monic of degree $\ell(v) - \ell(u)$ (see [**19**, Theorem 3.8]) as are the R-polynomials, and proves that the conjecture is true in two cases: when $\ell(v) - \ell(u) \leq 3$ (see [**19**, Proposition 3.13]) and when (u, v) is a Coxeter pair ([**19**, Proposition 3.11]). The E-polynomials play the same role as the R'-polynomials in the generalized case, so it is natural to wonder about the analogous question, i.e. if $E_{u,v}(q) = R^J_{u,v}(q)$, where $R^J_{u,v}(q)$ are the parabolic R-polynomials (see e.g., [**27**]). This question also has a negative answer. In fact, for example, let $v = [3, 4, 1, 2, 5] \in S_5^{S\setminus\{(2,3)\}}$, then we have that $R^J_{e,v}(q) = q^4(1 - q^{-1})(1 - q^{-2})$, while $E_{e,v}(q) = q^4(1 - q^{-1})(1 - q^{-3})$.
However, we can prove the analoguous results of Carlin, for generalized Verma modules. The first two results are very simple, and their proofs are immediate from Theorems 5.4, 5.11 and 5.18.

COROLLARY 5.26. *Let $u, v \in W^J$, $u \leq v$. Then $E_{u,v}(q)$ is a monic polynomial of degree $\ell(v) - \ell(u)$.* □

COROLLARY 5.27. *Let $u, v \in W^J$. If $u \leq v$ then*

$$\dim(\mathrm{Ext}^{\ell(v)-\ell(u)}(N_u, N_v)) = 1.$$

□

In [**14**] Brenti finds explicit formulas for the maximal parabolic R-polynomials of the symmetric group and, in [**15**] for the group of signed permutations, when $J = S \setminus \{s_0\}$. He proves the following

THEOREM 5.28. *Let $u, v \in S_n^J$, $u \leq v$. Then*

$$R^J_{u,v}(q) = q^{|\Lambda(v)\setminus\Lambda(u)|} \prod_j (1 - q^{-\tilde{a}_j(\Lambda(u),\Lambda(v))})$$

where j runs over the allowable steps of $\Lambda(v)$ with respect to $\Lambda(u)$.

THEOREM 5.29. *Let $u, v \in (S_n^B)^J$, $u \leq v$. Then*

$$R^J_{u,v}(q) = q^{|\Lambda_B(v)\setminus\Lambda_B(u)|} \prod_j (1 - q^{-\bar{b}_j(\Lambda_B(u),\Lambda_B(v))})$$

where j runs over the B-allowable steps of $\Lambda_B(v)$ with respect to $\Lambda_B(u)$ and

$$\bar{b}_j(\Lambda_B(u), \Lambda_B(v)) := \begin{cases} \tilde{a}_j(\Lambda_B(u), \Lambda_B(v)) & \textit{if the } j\textit{-th step of } \Lambda_B(u) \textit{ is down,} \\ \tilde{a}_j(\Lambda_B(u), \Lambda_B(v)) + 1 & \textit{if the } j\textit{-th step of } \Lambda_B(u) \textit{ is up.} \end{cases}$$

It's not hard to see that the formula in Theorem 5.29 can be extended to the case $W = D_n$ replacing Λ_B by Λ_D.

Our and Brenti's formulas are very similar, and it is easy to see that the following results hold. The statements and proofs are given for $W = S_n$, but the results are also true for B_n and D_n. We simply need to replace Λ by Λ_B and Λ_D, respectively.

We say that a skew partition is a *border strip* (also called a *ribbon*), if it contains no 2×2 square of cells.

PROPOSITION 5.30. *Let $u, v \in W^J$ be such that the skew partition $\Lambda(v) \setminus \Lambda(u)$ is a border strip. Then*

$$E_{u,v}(q) = R^J_{u,v}(q).$$

□

So, in particular, we obtain the analog of Proposition 3.13 of [**19**].

COROLLARY 5.31. *Let $u, v \in W^J$. If $\ell(v) - \ell(u) \leq 3$ then*

$$\dim(\mathrm{Ext}^k(N_u, N_v)) = r^J_k(u, v)$$

where $r^J_k(u, v)$ is the absolute value of the coefficient of q^k in $R^J_{u,v}(q)$. □

Let $u, v \in W^J$, we call (u, v) a *generalized Coxeter pair* if $r^J_1(u, v) = \ell(v) - \ell(u)$. The next result is the analogous of Proposition 3.11 of [**19**].

PROPOSITION 5.32. *Let $u, v \in W^J$. If (u, v) is a generalized Coxeter pair then*

$$\dim(\mathrm{Ext}^k(N_u, N_v)) = r^J_k(u, v) = \binom{n}{k},$$

for $k = 0, \ldots, n$, where $n = \ell(v) - \ell(u)$.

PROOF. If (u, v) is a generalized Coxeter pair then the only possibility is that $R^J_{u,v}(q) = q^n(1 - q^{-1})^n$. This means that the skew partition $\Lambda(v) \setminus \Lambda(u)$ is a border strip, and so the result follows by Proposition 5.30. □

It seems that the situation is exactly analogous, hence we are led to think that, as for the generalized Verma modules, there should exist a recursion formula also for the ordinary ones.

Bibliography

[1] R. M. Adin, F. Brenti and Y. Roichman, *Descent representations and multivariate statistics*, math.CO/0112073.

[2] R. M. Adin, F. Brenti and Y. Roichman, *Descent numbers and Major indices for the Hyperoctahedral group*, Adv. Applied Math., **27** (2001), 210-224.

[3] R. M. Adin and Y. Roichman, *The Flag Major index and group actions on polynomial rings*, Europ. J. Combinatorics, **22** (2001), 431-446.

[4] E. E. Allen, *The diagonal monomials and basis for the diagonally symmetric polynomials*, J. Alg. Combin., **3** (1994), 5-16.

[5] E. E. Allen, *New bases for the decomposition of the graded left regular representation of the reflection groups of type B_n and D_n*, J. Algebra, **173** (1995), 122-143.

[6] H. Barcelo and A.Ram, *Combinatorial representation theory*, New Perspectives in Geometric Combinatorics, MSRI Publications, Vol. 38, 1999.

[7] R. Biagioli, *Closed product formulas for extensions of generalized Verma modules*, Trans. Amer. Math. Soc., to appear.

[8] R. Biagioli, *Major and descent statistics for the even-signed permutation group*, Adv. in Appl. Math., to appear.

[9] R. Biagioli and F. Caselli, *Invariant algebras and major indices for classical Weyl groups*, preprint, Roma 2002, pp. 35.

[10] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, G.T.M., Springer-Verlag, Berlin, to appear.

[11] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, *On a category of g-modules*, Funct. Anal. Appl., **10** (1976), 1-8.

[12] B. Boe, *A Counterexample to the Gabber-Joseph Conjecture*, Kazhdan-Lusztig Theory and related topics, Contemp. Math., **139**, 1-2, Amer. Math. Soc., Providence, RI, 1992.

[13] B. Boe, T.J. Enright, B. Shelton, *Determination of the interwining operators for holomorphically induced representations of Hermitian symmetric pairs*, Pacific J. Math., **131** (1988), 39-50.

[14] F. Brenti, *Kazhdan-Lusztig and R-polynomials, Young's lattice, and Dyck partitions*, Pacific J. Math., to appear.

[15] F. Brenti, personal communication, December 2000.

[16] F. Brenti, *q-Eulerian polynomials arising from Coxeter groups*, Europ. J. Combinarorics, **15** (1994), 417-441.

[17] J. Brylinski, M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math., **64** (1981), 387-410.

[18] K.J. Carlin, *Extensions of Verma modules*, Trans. Amer. Math. Soc., **294** (1986), 29-43.

[19] L. Carlitz, *A combinatorial property of q-Eulerian numbers*, Amer. Math. Montly, **82** (1975), 51-54.

[20] L. Casian, D. Collingwood, *The Kazhdan-Lusztig conjecture for generalized Verma modules*, Math. Z., **195** (1987), 581-600.

[21] R. J. Clarke and D. Foata, *Eulerian calculus. I. Univariable statistics*, Europ. J. Combinarorics, **15** (1994), 345-362.

[22] R. J. Clarke and D. Foata, *Eulerian calculus. II. An extension of Han's fundamental trasformation*, Europ. J. Combinarorics, **16** (1995), 221-252.

[23] R. J. Clarke and D. Foata, *Eulerian calculus. III. The ubiquitous Cauchy formula*, Europ. J. Combinarorics, **16** (1995), 329-355.

[24] D.H. Collingwood, *The n-Homology of Harish-Chandra modules: generalizing a theorem of Kostant*, Math. Ann., **272** (1985), 161-187.

[25] D.H. Collingwood, B. Shelton, *A duality theorem for extensions on induced Highest Weight modules*, Pacific. J. Math., **146** (1990), 227-237.

[26] H.S.M. Coxeter, *Discrete groups generated by reflections*, Ann. of Math., **35** (1934), 588-621.

[27] V. Deodhar, *On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan-Lusztig polynomials*, J. Algebra, **111** (1987), 483-506.

[28] T.J. Enright, B. Shelton, *Categories of highest weight modules: applications to Hermitian symmetric spaces*, Mem. Amer. Math. Soc., **367** (1987).

[29] L. Euler, *Institutiones calculi differentialis*, in Opera Omnia, Series Prima, vol. X, Verlag von B.G. Teubner, Leipzig, 1913.

[30] D. Foata, *On the Netto inversion number of a sequence*, Proc. Amer. Math. Soc., **19** (1968), 236-240.

[31] D. Foata, *Distributions eulériennes et mahoniennes sur le groupes des permutations*, in Higher Combinatorics, 27-49, D. Reidel, Amsterdam 1997 (Proc. NATO Adv. Study Inst.).

[32] D. Foata and M.P. Schützenberger, *Major index and inversion number of permutations*, Math. Nachr., **83** (1978), 143-159.

[33] D. Foata and D. Zeilberger, *Denert's permutation statistic is indeed Euler-Mahonian*, Studies in Appl. Math., **83** (1990), 31-59.

[34] W. Fulton, *Young tableaux*, London Math. Soc. student texts 35, Cambridge Univ. Press, New York 1997.

[35] W. Fulton and J. Harris, *Representation theory*, G.T.M. 129, Springer-Verlag, New York 1991.

[36] O. Gabber, A. Joseph, *Towards the Kazdhan-Lusztig conjecture*, Ann. Sci. École. Norm. Sup., **14** (1981), 261-302.

[37] A. Garsia and I. Gessel, *Permutation statistics and partitions*, Adv. Math., **31** (1979), 288-305.

[38] A. Garsia and D. Stanton, *Group actions of Stanley-Reisner rings and invariant of permutation groups*, Adv. Math., **51** (1984), 107-201.

[39] B. Gordon, *Two theorems on multipartite partitions*, J. London Math. Soc., **38** (1963), 459-464.

[40] G.-H. Han, *Une transformation fondamentale sur les réarrangements de mots*, Adv. in Math., **105** (1994), 26-41.

[41] H. L. Hiller, *Geometry of Coxeter groups*, Res. Notes in Math. 54, Pitman, Boston, 1982.

[42] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Stud. Adv. Math., no.29, Cambridge Univ. Press, Cambridge, 1990.

[43] D. Kazhdan, G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math., **53** (1979), 165-184.

[44] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second edition, Oxford Math. Monographs, Oxford Univ. Press, Oxford 1995.

[45] P. A. MacMahon, *Combinatory analysis*, vol.1, Cambridge Univ. Press, London, 1915.

[46] V. Reiner, *Signed Permutation Statistics*, Europ. J. Combinatorics, **14** (1993), 553-567.

[47] C. Reutenauer, *Free Lie algebras,* London Mathematical Soc. Monographs, New series 7, Oxford Univ. Press, 1993.

[48] B. Shelton, *Extensions between generalized Verma modules: The Hermitian symmetric case*, Math. Z., **197** (1988), 305-318.

[49] M. Skandera, *An Eulerian partner for inversions*, Sém. Lothar. Combin., **46**, B46d (2001), pp. 19.

[50] L. Solomon, *The order of the finite Chevalley groups*, J. Algebra, **3** (1966), 376-393.

[51] R. P. Stanley, *Enumerative combinatorics* , vol.1, Wadsworth and Brooks/Cole, Monterey, CA, 1986.

[52] R. P. Stanley, *Enumerative combinatorics* , vol.2, Cambridge Stud. Adv. Math., no.62, Cambridge Univ. Press, Cambridge, 1999.

[53] E. Steingrimsson, *Permutation statistics of indexed permutations*, Europ. J. Combinarorics, **15** (1994), 187-205.

[54] J. Stembridge, *On the eigenvalues of representations of reflection groups and wreath products,* Pacific J. Math., **140** (1989), 359-396.