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A Current Approach to Morse and Novikov Theories

Dottorato in Matematica, Roma «La Sapienza» (2002).

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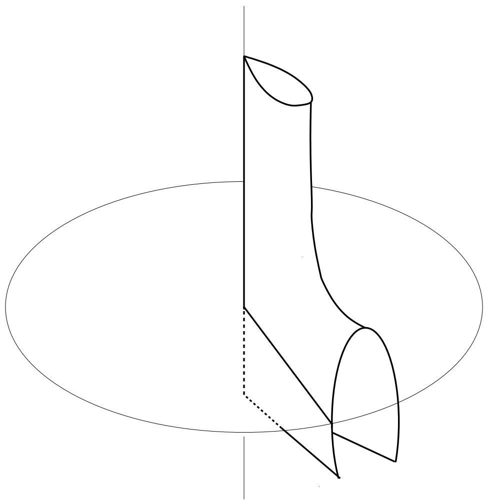
Università degli Studi di Roma "La Sapienza"

Facoltà di Scienze Matematiche, Fisiche e Naturali

Tesi di Dottorato

A Current Approach to Morse and Novikov Theories



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Introduction

The present work might be pictured in several different perspectives, according to which part of its is considered a target, which one a tool and which an application. The results obtained are, in fact, linked to each other in a quite surprisingly strict way. The best choice to introduce and motivate them is thus probably to describe how everything has been worked through.

The starting point is the fundamental article [13] by Harvey and Lawson, where the main results of Morse Theory on compact manifolds are retrieved in a new perspective. In particular, they construct a chain homotopy deforming the deRham complex (of forms or of integral currents) into a finite dimensional complex of currents "the $\mathcal{S}$-complex", isomorphic to the so-called "Morse Complex", giving a new, direct proof to the well-known fact that the latter computes the singular homology of the underlying manifold. The deformation of an object, e.g. a form α, is the asymptotic limit of the pullbacks $\lim_t \phi_t^* (\alpha)$ under the gradient flow of the function (convergence is intended in the sense of currents).

A very important concept in [13] is the "finite volume technique". This technique works when one is able to bound the volume of a certain manifold called the "graph of the flow". In the case of the gradient flow of a Morse function, Harvey and Lawson assumed a technical hypothesis called "tameness" which allowed them to apply a blow up argument and desingularize the graph of the flow, hence bounding its volume. The same hypothesis is also assumed by Laudenbach in [17], dealing with the (a posteriori similar) problem of bounding the volume of stable manifolds in the same framework. Laudenbach too uses a blow up argument and describes the singularities of the stable manifolds as "conical", conjecturing that the volume bound might be proved even without the "tameness" hypothesis, by "desingularizing the cone construction". We recollect these and other preliminaries (in particular the Boundary Value theory for systems of ODE's) in the first chapter.

The second chapter is devoted to find bounds for the volume of the flow and of the stable manifolds of a certain class of flows with non-degenerate singularities. The tameness hypothesis is removed, and the concept of "horned stratification" is introduced to describe the singularities arisen. This model turns out to be very useful and handy: horned stratified subsets define integral currents, can be used as cycles for a homology theory over the integers, and at the same time have nice intersection properties. In the proofs, the tool that replaces the blow up (or "cone") construction is the Boundary Value theory.

In chapter three we apply the local results of the second chapter to obtain global informations. The finite volume technique is extended to a non-compact setting (a certain "weakly proper" condition is assumed in order to compensate lack of compactedness) leading to a "non compact" Morse theory. The target is always to relate the analytic properties of a Morse function to some topological invariant of the underlying manifold. Though, of course, in a noncompact setting the results are different than in the compact case, one might try to obtain them in somewhat the same fashion. We use the dynamics of the flow to define an analogous of the "Morse complex", encoding the topological information. The "forward $\mathcal{S}$-complex" is thus introduced, and the invariants involved are described as groups of cohomology with "forward supports".

Finally, the standard trick of inverting the time leads to a "forward/backward duality", which restricts to Poincaré-deRham duality if the potential function is bounded. Here the proofs involve some tools by functional analysis (in analogy with the proofs of deRham or Serre dualities), since the spaces might be infinite dimensional.

Once constructed a noncompact Morse theory, one looks for interesting examples to fit in such a frame. The observation that flows arising in Novikov theory are weakly proper suggests the attempt to relate the theories. This is done in chapter four.

In classical Novikov theory, cf. [22], one consider a closed 1-form with nondegenerate critical points and looks for relations between the dynamics of a gradient flow for the form and some topological invariant of the underlying manifold and/or of the 1-form. Novikov, ref. cit, considered a covering of the manifold where the form is pulled back to an exact function, which results of Morse type. He then introduced the "Novikov ring" and the "Novikov Complex". The latter is made up of finitely generated modules over the Novikov ring, generators being in one-to-one correspondence with the critical points of the 1-form. This is the key to obtain the Novikov inequalities, in quite a similar way as in Morse theory one obtains Morse inequalities. We construct a modified and arguably more natural version of Morse-Novikov theory, where the Novikov Ring is replaced by a new "Forward Laurent ring". The theory results clarified in several ways. Geometrically, the Novikov complex is described as a subcomplex of the complex of currents, using the constructions in chapter three. Topologically, the invariants are described in a new and concise way as "compact forward cohomology". A new duality called "Lambda duality" is provided for the invariants obtained, as an application of the "forward-backward duality" proved in chapter three.

In chapter five, the previous results are extended to the case of functions and forms with "Bott" singularities (i.e. singularities uniformly distributed along regular submanifolds). The case of functions with Bott singularities on a compact manifold using the approach by finite volume technique had been considered by Latschev in his PhD thesis [18], which we generalize (and largely use!). Also, we construct a Novikov theory for forms with Bott singularities in analogy with the approach in chapter four. Possible applications and perspectives are discussed in a last section.

Finally, two appendixes are added to recollect results about currents and stratifications.

Acknowledgements

In the first place, I'd heartily like to thank my parents.

I'd also like to express deep gratitude to my thesis advisor, prof. Reese F. Harvey, and to my tutor, prof. Alessandro Silva, for their teachings, their interest in my work and their constant encouragement during these years.

I feel then pleased and honored to recall prof. Brendan Hasset, Blaine Lawson and Piero Negrini for many useful suggestions, and prof. Hari Bercovici and Nicola Teleman, whose lessons deeply influenced my taste of mathematics.

Finally, a warm greeting goes to the staff, faculty and students in Math Department at Rice University and to my dear friends Francesco P., Riccardo, Daniela, Luigi, Chun, Huey-ling, Simon, Adriano and Francesco E., thanking the latters for some help with this thesis too.

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Chapter 1

Preliminaries

In this chapter we present a brief review of the main framework we'll work within and the main tool used: Morse Theory and the Finite Volume technique.

A remark: we will usually don't bother about the class of differentiability of our objects and will just work in the $\mathcal{C}^\infty$ category, but of course analogous results might be stated for less regular conditions.

1.1 Morse Functions and The Morse Complex

In the sequel, X will be a smooth manifold of dimension m, not necessarily compact. Let $f : X \to \mathbb{R}$ be a smooth function. A point $p \in X$ is called **critical** for f if $df|_p = 0$, i.e. all first partial derivatives of f vanish in p. It then makes sense to consider the symmetric bilinear form $H_p(f) : T_pM \times T_pM \to \mathbb{R}$ whose expression in a coordinate system $x_1, \ldots, x_n$ is given by $\frac{\partial^2 f}{\partial x^i \partial x^j}$. This form is called the **Hessian** of f in p and the critical point p is called **nondegenerate** if and only if $H_p f$ is nondegenerate. In this case, if the Hessian has signature $(k, n-k)$, say, then k is called the **index** of f in p and denoted by $\#(p)$. The Morse Lemma states that any function with a non degenerate critical point of index k admits (locally near p), an expression in suitable coordinates $x_1, \ldots, x_n$ as a quadratic form:

$$f(x) = f(p) + x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2$$

Definition 1.1.1 *A function whose critical points are all nondegenerate is called a **Morse function**.*

It is well known that the set of Morse function is open and dense in the set of $\mathcal{C}^\infty$ functions on a manifold.

One might define nondegenerate singularities also for vector fields, but it's better to use a slightly less general concept. Suppose that a vector field

V has an isolated singularity in a point p and that for coordinates $x_1, \cdots, x_n$ near p, the expression

$$V(x) = Ax + b(x)$$

holds, A being a constant $n \times n$ matrix (**linearization matrix**) and b a smooth function, vanishing and singular in p. The isolated critical point p is called **hyperbolic** if the matrix A has no purely imaginary eigenvalue (i.e. all eigenvalues have nonvanishing real part, called *characteristic exponent*). The number of negative characteristic exponents of the matrix A is called the index of the critical point p for V. Of course, the singularities of a vector field V which is the gradient of a Morse function f with respect to any Riemannian metric are hyperbolic and the two notion of index agree for $V = -\nabla f$.

If V is a complete vector field, $(\phi_t)_{t\in\mathbb{R}}$ is the corresponding flow and p is a hyperbolic singular point of index k for V, one can introduce the **Stable Manifold** S_p and the **Unstable Manifold** U_p in p:

$$S_p = \{x \in X | \lim_{t\to+\infty} \phi_t(x) = p\}$$
$$U_p = \{x \in X | \lim_{t\to-\infty} \phi_t(x) = p\}$$

Theorem 1.1.2 (Hadamard) *The stable and unstable manifolds at p are smooth immersed submanifolds of X of dim respectively $k = \#(p)$ and $n-k$, transversal to each other.*

The previous theorem is usually called "Stable manifold theorem", and a proof will be provided later in this chapter.

Definition 1.1.3 *A complete vector field with isolated hyperbolic singularities (and its flow) is said to be* ***Smale*** *if and only if for any two critical points p and q, the Stable manifold at p and the Unstable manifold at q intersect transversally (of course the intersection might be empty).*

We next describe the Morse complex, firsts introduced by E. Witten in [31].

Consider a compact oriented manifold X and a Morse function $f : X \to \mathbb{R}$ with a Smale gradient. The group of k-cycles C_k is the free abelian group generated by the critical points of index k, i.e. $C_k \approx \mathbb{Z}^{r_k}$ if there are r_k critical points of index k. The differentials $\delta_k : C_k \to C_{k-1}$ are defined as

$$\delta_k(p) = \sum_{i=1}^{r_{k-1}} c_i q_i$$

where c_i are integers and the q_i's runs through the critical points of index $k-1$. The constants c_i are computed counting *with orientations* the number of flow lines connecting p and q_i (cf. [1] for details).

Theorem 1.1.4 (Morse-...-Floer) *The Morse Complex is a complex (i.e. $\delta^2 = 0$) and has homology isomorphic to the singular homology of X.*

We will prove this statement later. It's interesting to point out that the famous strong Morse inequalities (cf. for example [21]) are a direct algebraic consequence of this fact but we will never use them, so we won't describe them explicitly.

1.2 Finite Volume Flows

The theory of finite volume flows was introduced by Harvey and Lawson in [13]. It is fundamental for our presentation and hence we give a short and intuitive account here, referring to the original paper for details.

Let X be a compact oriented manifold of dimension n and $\phi_t : X \to X$ the flow generated by a vector field V. For some $t > 0$, consider the following subsets:

$$P_t = \{(\phi_t(x), x) | x \in X\} \qquad \textit{and} \qquad T_t = \{(\phi_s(x), x) | x \in X,\, 0 \leq s \leq t\}$$

Of course the P_t is a smooth oriented submanifold, being the (inverted) graph of the diffeomorphism ϕ_t, whereas $T_t = \Phi([0,t] \times X)$ where $\Phi(x,t) = (\phi_t(x), x)$. The map $\Phi : \mathbb{R} \times X \to X \times X$ is an immersion exactly on $\mathbb{R} \times (X - Z)$, where Z is the set of critical points for V. Supposed fixed a Riemannian metric on X:

Definition 1.2.1 *The flow ϕ is called a **finite volume flow** if $\mathbb{R}^+ \times X - Z$ has finite volume with respect to the metric induce by the immersion Φ.*

*If X is not compact, ϕ is called a **locally finite volume flow** if for any compact set K in X, the volume of $\mathbb{R}^+ \times K - Z$ is finite with respect to the metric induce by the restriction of Φ.*

Note that this concept is independent by the choice of the metric and if there are no periodic orbits it's equivalent to ask that the immersed submanifold $\Phi(\mathbb{R}^+ \times X - Z)$ has finite $n+1$-dimensional volume. In the case of periodic orbits, one has to count the volume with "multiplicity" (but so far there is no known flow of finite volume admitting a nontrivial periodic orbit on a compact manifold).

Let's see the applications in the case of a compact X.

Denoting by Δ the diagonal in $X \times X$, observe that Δ and P_t define currents by integration and that $T_t = \Phi_t * ([0,t] \times X)$ defines a current by pushforward, whose boundary is $dT_t = \Delta - P_t$. One then obtains:

Theorem 1.2.2 *If ϕ is a finite volume flow, then both the limits*

$$P = \lim_{t\to+\infty} P_t \qquad \textit{and} \qquad T = \lim_{t\to+\infty} T_t$$

exists as currents and by taking the boundary of T one obtains the following equation of currents on $X \times X$:

$$dT = \Delta - P \tag{1.1}$$

Remark Since the current $T = \Phi_*((0,+\infty)\times X)$ and $(\phi_t(x), x) = (y, \phi_t(y))$ for $y = \phi_{-t}(x)$, it follows that

$$T^* = \Phi_*((-\infty, 0)\times X)$$

is also a well defined current corresponding to the pushforward of T under the flip $(x, y) \to (y, x)$ on $X \times X$.

Using the kernel calculus for currents (an account of which is presented in the appendix), and denoting by bold letters the operators corresponding to the kernels currents, one obtains:

Corollary 1.2.3 *The following limits of operators hold:*

$$\mathbf{T} = \lim_{t\to+\infty} \mathbf{T_t} \qquad \textit{and} \qquad \mathbf{P} = \lim_{t\to+\infty} \mathbf{P_t} \tag{1.2}$$

as well as the equation

$$d\circ \mathbf{T} + \mathbf{T}\circ d = \mathbf{I} - \mathbf{P} \tag{1.3}$$

Remark Note that $\mathbf{T} : \mathcal{E}^k(X) \to \mathcal{D}'^{k-1}(X)$ and $\mathbf{P} : \mathcal{E}^k(X) \to \mathcal{D}'^{k}(X)$; here $\mathcal{E}$ denotes the smooth forms and $\mathcal{D}'$ the currents.

By deRham theory, the operator $\mathbf{I} : \mathcal{E}^k(X) \to \mathcal{D}'^{k}(X)$ induces an isomorphism on real cohomology. Because of the chain homotopy 1.3, so does $\mathbf{P}$:

Corollary 1.2.4 *The map induced in cohomology*

$$\mathbf{P} : H^k(X, \mathbb{R}) \to H^k(X, \mathbb{R})$$

is an isomorphism.

Of course the finite volume technique gains interest and power when there is an explicit description for the kernels T and P and a nice description for the corresponding operators.

If X is not compact, the situation is more complicated and locally finite volumes flows are not always so important as in the compact case. The current equation 1.1 still makes sense, even if the condition on finite volume is only required locally. The corresponding operator equation 1.3 continues to hold, though it is not any longer true that $\mathbf{I}$ induces an isomorphism in cohomology, in general.

To get additional informations out of the operator equations, we need to find a setting where $\mathbf{I}$ is an interesting map. This will be the case in our treatment of noncompact Morse theory, where the operators will be suitably extended.

1.3 The Boundary Value Technique

We are interested in the local properties of solutions of ODE systems on $\mathbb{R}^s \times \mathbb{R}^u$ of the form:

$$\begin{cases} \dot{x} = L^- x + f(x,y) \\ \dot{y} = L^+ y + g(x,y) \end{cases} \tag{1.4}$$

Setting

$$F = (f,g) : \mathbb{R}^s \times \mathbb{R}^u \to \mathbb{R}^s \times \mathbb{R}^u$$

it is assumed that:

$$F(0,0) = 0 \text{ and } dF(0,0) = 0$$

It is also assumed that L^- (respectively L^+) is a constant matrix whose eigenvalues have strictly negative (respectively strictly positive) real parts and that F is smooth.

According to the Grobman-Hartman theorem, the system is *topologically* conjugated to the linear part, but no such description is available, in general, in the smooth category. Nevertheless the behaviour is "dominated" by the linearization: basically, the flow contracts the set of x directions and expands the y's.

The usual way by which one describes the geometry of a dynamical system is the Initial Value problem (abridged *I.V.* problem): give an initial datum, and look for solutions as curves starting at that point for a fixed time. This approach is very intuitive, but does not work properly in terms of stability: a slight modification of the datum can change the picture of solutions dramatically. The phenomenum is critical in presence of singularities; nevertheless, at least for hyperbolic singularities, one can describe the geometry of the flow in a "stable" way using the Boundary Value technique.

We next give an account of this theory, quoting the book [27] and the appendix in the paper [1] as references. For what we know, the BV technique was first introduced by Shilnikov in the late '60 (cf. the introduction in [27]). Nevertheless, Shilnikov is never mentioned in the paper [1], where the authors refer to Smale and Floer as implicit sources.

For reasons that will be clear later, it's better to consider a slightly more general dynamical system than (1.4). The non linear term are supposed small near 0, instead of vanishing, the system non-autonomous and depending on extra parameters:

$$\begin{cases} \dot{x} = L^- x + f(t, x, y, \theta) \\ \dot{y} = L^+ y + g(t, x, y, \theta) \end{cases} \tag{1.5}$$

The matrices L^- and L^+ are assumed to be in Jordan form, the real parts of the eigenvalues (i.e. the characteristic exponents) of L^- to be strictly negative, say $-\lambda_s \leq \cdots \leq -\lambda_1 < 0$, and those of L^+ to be strictly positive, say $0 < \mu_1 \leq \cdots \leq \mu_u$. The Jordan form hypothesis implies that for any fixed value $0 < \alpha < \lambda_1, \mu_1$, the following estimates hold:

$$\left\| e^{tL^-} \right\| \leq e^{-\alpha t} \qquad , \qquad \left\| e^{-tL^+} \right\| \leq e^{-\alpha t} \qquad \text{for } t \geq 0$$

In the sequel the symbol $|x, y|$ will always mean the max between $|x|$ and $|y|$. Now put:

$$F = (f, g) : \mathbb{R} \times \mathbb{R}^s \times \mathbb{R}^u \times W \to \mathbb{R}^s \times \mathbb{R}^u$$

the set W (an open set in some $\mathbb{R}^i$) being the domain of the parameters θ. The "non linear term" F is supposed to vanish at the origin and its spatial derivatives to be uniformly bounded. In other words, there exists constants δ^k such that:

$$\begin{gathered} F(t, 0, 0, \theta) = 0 \\ \delta^k_\varepsilon \stackrel{def}{=} \sup_{|x,y| \leq \varepsilon} \sum_{|m| \leq k} \left| \frac{\partial^k F}{\partial (x, y)^m} \right| \leq \delta^k < +\infty \end{gathered} \tag{1.6}$$

We will later ask the first derivatives to be bounded by a small constant, i.e. δ^1 to be small enough.

Lemma 1.3.1 *In the previous hypotheses, the following inequality holds*

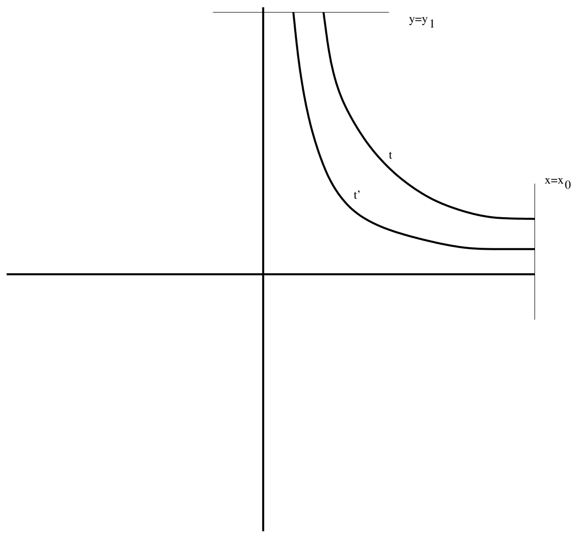
$$|F(t, x, y, \theta)| \leq \delta^1_{|x,y|} |x, y| \tag{1.7}$$

Proof. By the mean value theorem:

$$|F(t, x, y, \theta)| = |F(t, x, y, \theta) - F(t, 0, 0, \theta)| \leq \delta^1_{|x,y|} |x, y| \Box$$

Definition 1.3.2 *We say that the* ***Boundary Value problem*** *(abridged B.V. problem) with data* $(x_0, y_1, \tau) \in \mathbb{R}^n \times [0, +\infty)$ *is solvable for the system* (1.5) *if there exists a solution* $(x^*(t), y^*(t))$ *defined on* $[0, \tau]$ *and satisfying:*

$$(x^*(0), y^*(\tau)) = (x_0, y_1)$$



We can now state the following existence and uniqueness theorem:

Theorem 1.3.3 (Shilnikov) *Suppose the estimate $\delta^1 < \alpha$ holds. Then the Boundary Value problem for the system 1.5 is solvable for any data (x_0, y_1, τ). The solution is unique, it depends smoothly on $(t, x_0, y_1, \tau, \theta)$ and satisfies:*

$$|x^*(t), y^*(t)| \leq 2|x_0, y_1| \quad \textit{for any } t \in [0, \tau]$$

Notations *In the same way as for an Initial Value problem, a BV problem is denoted by:*

$$\begin{cases} \dot{x} = L^- x + f(t, x, y, \theta) \\ \dot{y} = L^+ y + g(t, x, y, \theta) \\ x^*(0) = x_0, \ y^*(\tau) = y_1 \end{cases} \tag{1.8}$$

The solution at time t to the Initial Value problem starting in (x_0, y_0) at $t = 0$ will be denoted by

$$(x(t, x_0, y_0), y(t, x_0, y_0))$$

whereas the solution at time t to the Boundary Value problem with data (x_0, y_0, τ) will be denoted by

$$(x^*(t, x_0, y_1, \tau), y^*(t, x_0, y_1, \tau))$$

The ***"end point"*** (x_1^*, y_0^*) *for the BV solution is defined as*

$$x_1^*(x_0, y_1, \tau) = x^*(\tau, x_0, y_1, \tau) \tag{1.9}$$

$$y_0^*(x_0, y_1, \tau) = y^*(0, x_0, y_1, \tau) \tag{1.10}$$

The following relations hold, by definition:

$$\begin{aligned} x\left(t,x_0,y_0\right) &= x^*\left(t,x_0,y(t,x_0,y_0),t\right) \\ y\left(t,x_0,y_0\right) &= y^*\left(t,x_0,y\left(t,x_0,y_0\right),t\right) \\ x^*\left(t,x_0,y_1,\tau\right) &= x\left(t,x_0,y_0^*\left(x_0,y_1,\tau\right)\right) \\ y^*\left(t,x_0,y_1,\tau\right) &= y\left(t,x_0,y_0^*\left(x_0,y_1,\tau\right)\right) \end{aligned} \tag{1.11}$$

One of the major consequences of the previous theorem is that one can "invert" the unstable variables y in describing the flow of solutions: for any fixed time τ, one can define the two manifolds

$$\begin{aligned} W_\tau^1 &= \{(x_0,y_0,t,x\left(t,x_0,y_0\right),y\left(t,x_0,y_0\right)) \mid (x_0,y_0)\in\mathbb{R}^n,\ 0\le t\le\tau\} \\ &= \{(x_0,y^*(t,x_0,y_1,\tau),t,x^*(t,x_0,y_1,\tau),y_1) \mid (x_0,y_1)\in\mathbb{R}^n,\ 0\le t\le\tau\}\subset\mathbb{R}^{2n+1} \end{aligned}$$

$$\begin{aligned} W_\tau^2 &= \{(x_0,y_0,\tau,x\left(\tau,x_0,y_0\right),y\left(\tau,x_0,y_0\right)) \mid (x_0,y_0)\in\mathbb{R}^n\} \\ &= \{(x_0,y_0^*\left(x_0,y_1,\tau\right),\tau,x_1^*\left(\tau,x_0,y_1,\right),y_1) \mid (x_0,y_1)\in\mathbb{R}^n\}\subset\mathbb{R}^{2n} \end{aligned}$$

The identities above express the fact that the submanifold W_τ^1 can be smoothly parametrized as a graph over both the variables (x_0,y_0,t) and (x_0,t,y_1). Analogously, W_τ^2 can be smoothly graphed over both (x_0,y_0) and (x_0,y_1). As a consequence of the formula for the derivatives in the implicit function theorem, then:

Corollary 1.3.4 *The following matrices are invertible and*

$$\begin{aligned} \left(\frac{\partial y^*}{\partial y_1}|_{(t,x_0,y_1,\tau)}\right)^{-1} &= \frac{\partial y}{\partial y_0}|_{\left(t,x_0,y_0^*(x_0,y_1,\tau)\right)} \\ \left(\frac{\partial y_0^*}{\partial y_1}|_{(x_0,y_1,\tau)}\right)^{-1} &= \frac{\partial y}{\partial y_0}|_{\left(\tau,x_0,y_0^*(x_0,y_1,\tau)\right)} \end{aligned}$$

Proof of the theorem. We'll first prove existence and uniqueness of B.V. solutions, then their regular dependence on (t,x_0,y_1,θ). At this point corollary 1.3.4 would also be proved, since it does not involve any regularity in τ. Finally, we'll use the corollary to prove the regular dependence on τ, completing the proof.

As it should not be surprising, the BV system can be translated in a system of integral equations:

$$\begin{cases} x(t) = e^{tL^-}x_0 + \int_0^t e^{(t-s)L^-}f(s,x(s),y(s),\theta)ds \\ y(t) = e^{-(\tau-t)L^+}y_1 - \int_t^\tau e^{(t-s)L^+}g(s,x(s),y(s),\theta)ds \end{cases} \tag{1.12}$$

Fact 1.1 *Any continuous curve, solution of the equations (1.12) is necessarily smooth in* t *and is a solution to the* BV *problem with data* (x_0, y_1, τ). *The viceversa is also true.*

The proof of the Fact above is the same as for the standard integral formulation of an Initial Value problem. Solving the previous equations is thus equivalent to solve the Boundary Value problem.

The right hand side of equations (1.12) defines an operator on the space of continuous curves. Denote by V_τ the Banach space $\mathcal{C}^0\left([0,\tau], \mathbb{R}^n\right)$ endowed with the sup norm $\| \; \|_\infty$, and define $T : V_\tau \times \mathbb{R}^n \times W \to V_\tau$ by:

$$(X(t), x_0, y_1, \theta) \to \begin{cases} T^x\left(X\right)(t) = e^{tL^-}x_0 + \int_0^t e^{(t-s)L^-} f(s, x(s), y(s), \theta) ds \\ T^y\left(X\right)(t) = e^{-(\tau-t)L^+}y_1 - \int_t^\tau e^{(t-s)L^+} g(s, x(s), y(s), \theta) ds \end{cases}$$

The operator T can be considered as a nonlinear operator on V_τ depending on "spatial" parameters x_0, y_1 and on "nonspatial" parameters θ. Sometimes we will just use the notations $T(X)$ for $T(X, x_0, y_1, \theta)$ if the parameters are understood fixed.

Claim 1.1 *T is continuous (on $V_\tau \times \mathbb{R}^n \times W$).*

This is quite obvious, and can be directly checked by estimating its variations.

Claim 1.2 *T is C^∞ in the V_τ arguments.*

Proof. The differentials d^kT in the V_τ arguments are the linear operators

$$d^kT_X\left(X'_1, \ldots, X'_k\right)(t) = \begin{cases} \int_0^t e^{(t-s)L^-} d^k f|_{(s,X(s),\theta)}\left(X'_1(s), \ldots, X'_k(s)\right) ds \\ \int_t^\tau e^{(t-s)L^+} d^k g|_{(s,X(s),\theta)}\left(X'_1(s), \ldots, X'_k(s)\right) ds \end{cases}$$

where $X, X'_i \in V_\tau$, and $d^k f, d^k g$ denote the differentials in the spatial directions. This can be proved by induction on k, for shortness we just prove the Taylor formula.

$$\left|\left(T(X+X')-T\left(X\right)-dT_XX'-\ldots-d^kT_X\left(X',\ldots,X'\right)\right)(t)\right|$$

$$\leq \sup\begin{cases}\left|\int_0^t e^{(t-s)L^-}f(s,X+X')-f(s,X)-\sum_{i=1}^k d^if|_{(s,X(s),\theta)}(X',\ldots,X')ds\right| \\ \left|\int_t^\tau e^{(t-s)L^+}g(s,X+X')-g(s,X)-\sum_{i=1}^k d^ig|_{(s,X(s),\theta)}(X',\ldots,X')ds\right|\end{cases}$$

$$\leq \sup\begin{cases}\int_0^t e^{-\alpha(t-s)}|d^{k+1}f|_{(s,Y(s),\theta)}\left(X',\ldots,X'\right)|ds \\ \int_t^\tau e^{\alpha(t-s)}|d^{k+1}g|_{(s,Y(s),\theta)}\left(X',\ldots,X'\right)|ds\end{cases}$$

$$\leq \sup\begin{cases}\int_0^t e^{-\alpha(t-s)}\delta^{k+1}_{\|X\|+\|X'\|}\frac{\|X'\|^{k+1}}{k+1!}ds \\ \int_t^\tau e^{\alpha(t-s)}\delta^{k+1}_{\|X\|+\|X'\|}\frac{\|X'\|^{k+1}}{k+1!}ds\end{cases} \leq \frac{\delta^{k+1}_{\|X\|+\|X'\|}}{\alpha(k+1!)}\|X'\|^{k+1}$$

which provides a uniform estimate. We point out that the rate of convergence does not depend on τ $\square$

Claim 1.3 *T is a contraction on V_τ. In fact $\|dT\| < \dfrac{\delta^1}{\alpha}$ (< 1 by hypothesis).*

Proof. We can easily check dT to be a contraction:

$$\left\|dT_XX'\left(t\right)\right\| = \sup_{[0,\tau]}\begin{cases}\left|\int_0^t e^{(t-s)L^-}df|_{(s,X(s),\theta)}\left(X'(s)\right)ds\right| \\ \left|\int_t^\tau e^{(t-s)L^+}dg|_{(s,X(s),\theta)}\left(X'(s)\right)ds\right|\end{cases}$$

$$\sup_{[0,\tau]}\begin{cases}\int_0^t e^{-\alpha(t-s)}\delta^1_{\|X\|}|X'(s)|ds \\ \int_t^\tau e^{(t-s)\alpha}\delta^1_{\|X\|}|X'(s)|ds\end{cases} \leq \frac{\delta^1_{\|X\|}}{\alpha}\left\|X'\right\|$$

and therefore $\|dT_X\| \leq \dfrac{\delta}{\alpha}$ $\square$

By claim 1 and 2, according to the contraction lemma in Banach spaces, for any choice of spatial data (x_0, y_1), there exists a unique fixed point of the operator T. This solves the existence and uniqueness issue for the boundary value problem with data (x_0, y_1, τ).

Claim 1.4 *For any fixed (x_0, y_1, θ), if $\|X\|_\infty \leq 2|x_0, y_1|$, then $\|T(X)\| \leq 2|x_0, y_1|$ too.*

Proof. Suppose $X(t) = (x(t), y(t))$ satisfies $|x(t), y(t)| \leq 2\,|x_0, y_1|$. Then, recalling that T^x and T^y denote the components of T, and putting, for brevity, $\delta = \delta^1_{|x_0,y_1|}$, the estimate 1.7 implies:

$$|T^x(X), T^y(X)(t)| = \sup \begin{cases} |T^x(x(t), y(t))| \\ |T^y(x(t), y(t))| \end{cases}$$

$$\leq \max \begin{cases} e^{-\alpha t}|x_0| + \int_0^t e^{-\alpha(t-s)}|f(s, x(s), y(s))|ds \\ e^{-\alpha(\tau-t)}|y_1| + \int_t^\tau e^{\alpha(t-s)}|g(s, x(s), y(s))|ds \end{cases}$$

$$\leq \max \begin{cases} |x_0| + \delta \int_0^t e^{-\alpha(t-s)}\,|x(s), y(s)|\,ds \\ \leq |y_1| + \delta \int_t^\tau e^{\alpha(t-s)}|x(s), y(s)|ds \end{cases}$$

$$\leq \max \begin{cases} |x_0| + |x_0, y_1|\,\delta\frac{1}{\alpha} \\ |y_1| + |x_0, y_1|\,\delta\frac{1}{\alpha} \end{cases} \leq |x_0, y_1|$$

since $\delta^1 \leq \alpha$ by hypothesis $\square$

Claim 4 proves the estimate on the size of the solution in the statement of the theorem, because of the successive approximations principle (recall that the solutions are the fixed points of the contraction T).

Claim 1.5 *The operator $T : V_\tau \times \mathbb{R}^n \times W \to V_\tau$ is C^∞ regular.*

Proof. T is affine in the spatial data (x_0, y_1) and so it depends smoothly on them, together with its derivatives (which do not even depend on x_0 nor on y_1). As for the θ parameters, one can proceed as in claim 2, differentiating under the integral sign both T and its derivatives to get smoothness of the mixed derivatives by induction. The theorem about differentiability after regularity of partial derivatives proves the claim.

We can now prove the smooth dependence of the solutions of the BV problem on the parameters (x_0, y_1, θ), as a consequence of the implicit function theorem in Banach spaces. In fact, claim 3 implies that the derivative of $T - id_{V_\tau} : V_\tau \times \mathbb{R}^n \times W \to V_\tau$ in the V_τ variable is an isomorphism and therefore the zeros of $T - id_{V_\tau}$ (i.e. the fixed points of T) can be smoothly parametrized over (x_0, y_1, θ). By the Fact 1 above, this means that the solutions to the BV problem depend smoothly on (x_0, y_1, θ), completing the proof of the theorem, but for the statement about regularity in τ. We explicitly remark that corollary 1.3.4 is now proven too (cf. what said on the beginning of the present proof).

Claim 1.6 *The solution to the BV problem depends smoothly on τ.*

Proof. A formal problem arises for studying variations of τ in the previous setting, since the Banach space on which T acts changes. It's simpler to check differentiability directly. Recalling the identities 1.3.2 the issue reduces to show that $y^*(0, x_0, y_1, \tau)$ is regular in τ. By the relation:

$$y_1 = y\left(\tau, x_0, y_0^*\left(x_0, y_1, \tau\right)\right)$$

and using corollary 1.3.4 and the implicit function theorem, one gets the desired regularity of $y_0^*(x_0, y_1, \tau)$ on τ (using the well-known regularity of solutions for Initial Value problems). This completes the proof of the last claim and of theorem 1.3.3 □

The previous theorem insured the regularity of solutions to a BV problem with respect to any of its arguments. We will next look for the derivatives. Certainly they solve the "variational" systems of differential equation obtained by formal differentiation of system 1.5. It's then enough to prove uniqueness of the solution to the variational system to identify these derivatives.

Let's consider again the system 1.5 :

$$\begin{cases} \dot{x} = L^- x + f(t, x, y, \theta) \\ \dot{y} = L^+ y + g(t, x, y, \theta) \end{cases} \tag{1.13}$$

with the same hypothesis on the nonlinear terms as in theorem 1.3.3, and let

$$(x^*(t, x_0, y_1, \tau), y^*(t, x_0, y_1, \tau))$$

be the solution of the BV problem with data (x_0, y_1, τ). For any v among $\frac{\partial}{\partial x_0}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial \theta}$, substituting x^*, y^* to x, y in 1.5 and differentiating with respect to v:

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial x^*}{\partial v}\right) = L^- \frac{\partial x^*}{\partial v} + \frac{\partial f}{\partial x}\frac{\partial x^*}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y^*}{\partial v}\left(+\frac{\partial f}{\partial \theta}\right) \\ \frac{\partial}{\partial t}\left(\frac{\partial y^*}{\partial v}\right) = L^+ \frac{\partial y^*}{\partial v} + \frac{\partial g}{\partial x}\frac{\partial x^*}{\partial v} + \frac{\partial g}{\partial y}\frac{\partial y^*}{\partial v}\left(+\frac{\partial f}{\partial \theta}\right) \end{cases} \tag{1.14}$$

where the derivatives of the nonlinear terms are understood evaluated in $(t, x^*(t), y^*(t))$ and the last terms appear only if v is a nonspatial parameter. Denoting x^*, y^* by X, Y, one gets the variational system:

$$\begin{cases} \frac{\partial}{\partial t} X = L^- X + \frac{\partial f}{\partial x} X + \frac{\partial f}{\partial y} Y \left(+\frac{\partial f}{\partial \theta}\right) \stackrel{def}{=} L^- X + F(X, Y) \\ \frac{\partial}{\partial t} Y = L^+ Y + \frac{\partial g}{\partial x} X + \frac{\partial g}{\partial y} Y \left(+\frac{\partial f}{\partial \theta}\right) \stackrel{def}{=} L^+ Y + G(X, Y) \end{cases} \tag{1.15}$$

Observe that 1.15 has the same form as 1.5. Moreover, the estimates on the first derivatives needed for the existence and uniqueness theorem 1.3.3 are exactly the same for the two systems, since $\frac{\partial F, G}{\partial X, Y} = \frac{\partial f, g}{\partial x, y}$. This means that for any BV data (X_0, Y_1, τ), there exists a unique solution to the corresponding BV problem for the system 1.15 . But we already know that $\frac{\partial x^* y^*}{\partial x_0, y_1, \theta}$ solve 1.15, thus it just remains to guess the values of $\frac{\partial x^*}{\partial v}$ in 0 and of $\frac{\partial y^*}{\partial v}$ in τ. Differentiating the equations 1.12 with respect to x_0, y_0 and θ in 0 and τ gives:

$$\frac{\partial x^* y^*}{\partial x_0, y_1, \theta}|_0 = \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

and

$$\frac{\partial x^* y^*}{\partial x_0, y_1, \theta}|_\tau = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & I & 0 \end{array}\right)$$

In particular, for the derivatives along "non spatial" parameters θ, the BV data to choose will just be all 0.

For the successive derivatives, one can proceed by iteration. For example, to find a certain $k+1^{\text{th}}$ derivative of x^*, y^*, one writes the variational system of the system obtained for the k^{th} derivatives. The "nonlinear terms" are sums of terms of two kinds: either linear in the unknowns (having as coefficients the first derivatives of the nonlinear terms of the original system f, g) or not involving the unknowns at all.

Therefore, the estimate on the first derivatives needed in the existence and uniqueness theorem is always fulfilled, as in the above case of first derivatives. This means that any BV problem of a variational system of order $k+1$ has a unique solution. Also, for derivatives of order $k+1 \geq 2$, the variables x_0, y_1 enter the picture as "nonspatial" parameters (the "spatial" ones being the derivatives of order k). Therefore, the desired derivatives are solutions of the BV problem of variational equations with vanishing BV data. Referring to [28] for a more detailed discussion, we just summarize the results in the following:

Corollary 1.3.5 *Under the hypothesis of the existence and uniqueness theorem 1.3.3, the derivatives to the solutions to the BV problem 1.8 with respect to the spatial BV data x_0, y_1 and parameters θ can be found as solutions to the BV problem whose equations are the variational equations and the BV data are given by formal differentiation of the old ones. In particular the BV data are zero for the derivatives in θ or anyway for derivatives of order higher than two.*

Of course the derivatives in t for the solutions are described by the differential equations themselves (of the original system or of the variational ones), so we are left to look for the derivatives in τ. The implicit function theorem allows us to derive them by those in t and in the other variables. Recalling the relation between solutions to the I.V. problem and to the B.V. problem:

$$\begin{aligned} x\left(t, x_0, y_0\right) &= x^*\left(t, x_0, y\left(\tau, x_0, y_0\right), \tau\right) \\ y\left(t, x_0, y_0\right) &= y^*\left(t, x_0, y\left(\tau, x_0, y_0\right), \tau\right) \end{aligned} \tag{1.16}$$

and differentiating the previous w.r.to τ:

$$0 = \frac{\partial x^*}{\partial y_1}|_{(t,x_0,y(\tau,x_0,y_0),\tau)} \frac{\partial y}{\partial t}|_{(\tau,x_0,y_0)} + \frac{\partial x^*}{\partial \tau}|_{(t,x_0,y(\tau,x_0,y_0),\tau)}$$
$$0 = \frac{\partial y^*}{\partial y_1}|_{(t,x_0,y(\tau,x_0,y_0),\tau)} \frac{\partial y}{\partial t}|_{(\tau,x_0,y_0)} + \frac{\partial y^*}{\partial \tau}|_{(t,x_0,y(\tau,x_0,y_0),\tau)}$$

On the other hand, differentiating equations 1.16 in t one obtains:

$$\begin{aligned} \frac{\partial x}{\partial t}|_{(t,x_0,y_0)} &= \frac{\partial x^*}{\partial t}|_{(t,x_0,y(\tau,x_0,y_0),\tau)} \\ \frac{\partial y}{\partial t}|_{(t,x_0,y_0)} &= \frac{\partial y^*}{\partial t}|_{(t,x_0,y(\tau,x_0,y_0),\tau)} \end{aligned}$$

Combining those and using y_1 instead of y_0 as a parameter (replacing the evaluations in a coherent way), one gets the following:

Corollary 1.3.6 *The derivatives of the BV solutions in τ satisfy:*

$$\begin{aligned} \frac{\partial x^*}{\partial \tau}|_{(t,x_0,y_1,\tau)} &= -\frac{\partial x^*}{\partial y_1}|_{(t,x_0,y_1,\tau)} \frac{\partial y^*}{\partial t}|_{(\tau,x_0,y_1,\tau)} \\ \frac{\partial y^*}{\partial \tau}|_{(t,x_0,y_1,\tau)} &= -\frac{\partial y^*}{\partial y_1}|_{(t,x_0,y_1,\tau)} \frac{\partial y^*}{\partial t}|_{(\tau,x_0,y_1,\tau)} \end{aligned} \tag{1.17}$$

Recalling the relations 1.9 and differentiating:

Theorem 1.3.7 *The first derivatives of the endpoint map satisfy:*

$$\frac{\partial x_1^*}{\partial x_0, y_1, \theta}|_{(x_0,y_1,\tau)} = \frac{\partial x^*}{\partial x_0, y_1, \theta}|_{(\tau,x_0,y_1,\tau)} \tag{1.18}$$

$$\frac{\partial y_0^*}{\partial x_0, y_1, \theta}|_{(x_0,y_1,\tau)} = \frac{\partial y^*}{\partial x_0, y_1, \theta}|_{(0,x_0,y_1,\tau)} \tag{1.19}$$

$$\frac{\partial x_1^*}{\partial \tau}|_{(x_0,y_1,\tau)} = \frac{\partial x^*}{\partial t}|_{(\tau,x_0,y_1,\tau)} - \frac{\partial x^*}{\partial y_1}|_{(\tau,x_0,y_1,\tau)} \frac{\partial y^*}{\partial t}|_{(\tau,x_0,y_1,\tau)} \tag{1.20}$$

$$\frac{\partial y_0^*}{\partial \tau}|_{(x_0,y_1,\tau)} = \frac{\partial y^*}{\partial \tau}|_{(0,x_0,y_1,\tau)} \tag{1.21}$$

The k^{th} derivatives of the endpoint map are linear combinations of terms of the form $\dfrac{\partial^h x^*}{\partial^h t, x_0, y_1, \theta}|_{\tau, x_0, y_1, \tau}$ *and* $\dfrac{\partial^h y^*}{\partial^h t, x_0, y_1, \theta}|_{0, x_0, y_1, \tau}$ *with* $h \leq k$*, the coefficients being products of other derivatives of* x^* *and* y^**.*

Remark The previous statement will be relevant in the sequel, since we'll prove some exponential estimates on the distinguished terms above, together with uniform bounds for their "coefficients".

We are now ready to prove an important consequence of the BV technique: the stable manifold theorem. We will restrict the discussion to the case we need, even if similar arguments might be worked out in the general case as well.

Hypothesis From now on, the system 1.5 is supposed to be autonomous (i.e. induced by a vector field).

We start extending the previous results to the case when $\tau = \infty$. Consider in fact the Banach space V of bounded continuous curves defined on the half line, endowed with the sup norm. For $\tau = \infty$ the equations 1.12 become:

$$\begin{cases} x(t) = e^{tL^-}x_0 + \int_0^t e^{(t-s)L^-} f(x(s), y(s))ds \\ y(t) = -\int_t^{+\infty} e^{(t-s)L^+} g(x(s), y(s))ds \end{cases} \tag{1.22}$$

and make still sense because of the sign of the exponentials and the estimates on the nonlinear terms. The analogous of the integral operator T introduced in the proof of theorem 1.3.3 is well defined on V and the Fact and claims in that proof are still valid (but claim 6, which deals with regularity in τ). There is basically nothing new to prove, since all the estimates are already done. As a consequence, for any x_0 (this time this is the only spatial parameter involved), there exists a solution $x^*(t), y^*(t)$ which stays bounded in the ball of radius $2|x_0|$ for all times $t \geq 0$. The map $x_0 \to y^*(0)$ is smooth, and thus its graph S is a smooth submanifold of dimension s.

Lemma 1.3.8 *The manifold* S *in invariant, i.e. the generating vector field is tangent to* S*.*

Equivalently, if $m \in S$ then the trajectory starting on m is contained in S.

Proof. Let $(x(t), y(t))$ be the trajectory trough m: by definition, it solves the integral equations 1.22 with datum x_0. If now $q = (x(t_0), y(t_0))$ is a point on the trajectory, a simple computation proves that $(x(t-t_0), y(t-t_0))$ solves the equations 1.22 with datum $x(t_0)$. That is to say, $q \in S$ $\square$

Lemma 1.3.9 *The integral curve of any solution which stays bounded for any $t \geq 0$ is contained in S.*

Proof. Let $(x(t), y(t))_{t\in\mathbb{R}^+}$ be such a solution. For any $\tau \geq 0$, the restriction $(x(t), y(t))_{t\in[0,\tau]}$ solves the BV problem with data $(x(0), y(\tau), \tau)$ and hence satisfies the BV equations 1.12:

$$\begin{cases} x(t) = e^{tL^-}x(0) + \int_0^t e^{(t-s)L^-} f(s, x(s), y(s), \theta)ds \\ y(t) = e^{-(\tau-t)L^+}y(\tau) - \int_t^\tau e^{(t-s)L^+} g(s, x(s), y(s), \theta)ds \end{cases}$$

for any $t \leq \tau$. The previous equations clearly converge to 1.22 for $\tau \to \infty$. This proves that $(x(0), y(0)) \in S$ and the previous lemma permits to conclude □

Referring to the definition in section 1, the previous lemmata prove:

Proposition 1.3.10 *The manifold S is the stable manifold at the origin for system 1.14. It is a smooth graph over $y = 0$ and is invariant.*

A refined version will be given later.

We now turn back to the initial ("local") system of equations 1.4. In this case, all the information on the size of the nonlinear terms is contained in the fact that their differential vanishes at the origin. The global behavior of solutions might be arbitrarily wild, but since we are interested in local questions, one can use the standard trick of cutting off the vector field away from the origin. This is meaningful provide the solutions under consideration do not leave the region where the system is unmodified. Translating theorem 1.3.3 gives the following.

Theorem 1.3.11 *Suppose $\varepsilon > 0$ is such that the estimate $\delta^1_{2\varepsilon} < \alpha$ holds. Then the BV problem for the system 1.4 is solvable for any data (x_0, y_1, τ) provide $|x_0, y_1| < \varepsilon$. The solution $(x^*(t, x_0, y_1, \tau), y^*(t, x_0, y_1, \tau))$ is unique, it depends smoothly on all its arguments and satisfies $|x^*(t), y^*(t)| < 2|x_0, y_1|$.*

The next refines proposition 1.3.10 (cf. definition in section 1).

Theorem 1.3.12 (Stable Manifold) *Let Ω be a small enough neighborhood of the origin in $\mathbb{R}^n$. Then the stable manifold S at 0 in Ω for system 1.4 is a smooth graph over $y = 0$ (hence a submanifold of dimension s). The unstable manifold is a smooth graph over $x = 0$ and the intersection $S \cap U = \{0\}$ is transversal.*

Proof. We start with the thesis of proposition 1.3.10, which is still valid provide we restrict ourselves to an Ω contained in the ball of radius ε (sufficiently small for the hypotheses of theorem 1.3.11 to hold). We want to show that S is tangent to $y = 0$ in the origin. This is a particular case of the slightly more general fact:

Lemma 1.3.13 *Let R be an invariant submanifold through the origin, which is a smooth graph over $y = 0$. Then R is tangent to $y = 0$ at the origin.*

Proof. Suppose that $R = \{(x, r(x))\}$ for a smooth function $r : \mathbb{R}^s \to \mathbb{R}^u$. Consider than the change of variables

$$\begin{cases} x & \longrightarrow & x \\ y & \longrightarrow & y - r(x) \end{cases}$$

which of course transforms R into the plane $y = 0$. Conjugating the system 1.4 by this diffeomorphism (denoting the new variables by u and v), gives the new system

$$\begin{cases} \dot{u} = L^- u + \dots \\ \dot{v} = dr|_0 L^- u + L^+ v + \dots \end{cases} \tag{1.23}$$

where the dots denote terms which vanishes to the second order at the origin. We already remarked that the plane $v = 0$ is the stable manifold, which is invariant. Imposing $\dot{v} = 0$ on $y = 0$ proves that $dr|_0 = 0$ since otherwise the term $-dr|_0 L^- u$ cannot be killed by terms vanishing at the second order. But $dr|_0 = 0$ just means that R is tangent to $y = 0$ in the origin. This proves that the stable manifold is tangent to $y = 0$ in 0; analogously the unstable manifold is proven tangent to $x = 0$ at 0, completing the proof of the stable manifold theorem $\square$

A first, direct consequence of the previous theorem is that one can choose coordinates such that locally near the origin the stable and unstable manifold are the coordinate planes. The invariance of these submanifolds implies the following.

Corollary 1.3.14 (Straighten Coordinates) *After a smooth change of coordinates, the system 1.4 can be written as*

$$\begin{cases} \dot{x} = L^- x + f(x, y) x \\ \dot{y} = L^+ y + g(x, y) y \end{cases} \tag{1.24}$$

*where f and g are square matrices of smooth functions vanishing in the origin and x, y are column vectors. The new coordinates are called **straighten coordinates** since the local stable and unstable manifolds are given by $y = 0$ and $x = 0$ respectively.*

Straighten coordinates are very useful. For example we can prove estimates on the behavior of solutions in a rather simple way. Let's start recalling the famous Gronwall lemma.

Lemma 1.3.15 (Gronwall) *Suppose that $\phi : [0, +\infty) \to \mathbb{R}$ is continuous and satisfies*

$$\phi(t) \leq a + b \int_0^t \phi(s) ds$$

for $a, b \in \mathbb{R}$, $b \geq 0$. Then the following estimate holds

$$\phi(t) \leq a e^{bt}$$

Consider the system 1.24 in straighten coordinates, and suppose the estimate $\delta^1_{2\varepsilon} < \alpha$ holds, where $\delta^1_{2\varepsilon} = n \max_{|x,y| \leq \varepsilon} \{|f(x,y)|, |g(x,y)|\}$ (this is just the hypothesis of theorem 1.3.3). Let

$$(x^*(t), y^*(t)) = (x^*(t, x_0, y_1, \tau), y^*(t, x_0, y_1, \tau))$$

denote the solution to the BV problem with data (x_0, y_1, τ) and put $\delta = \delta^1_{2\varepsilon}$.

Theorem 1.3.16 *Suppose $|x_0, y_1| \leq \varepsilon$. Then for any $\tau \in [0, +\infty)$ the following inequalities hold:*

$$\begin{cases} |x^*(t, x_0, y_1, \tau)| \leq |x_0| e^{-(\alpha-\delta)t} \\ |y^*(t, x_0, y_1, \tau)| \leq |y_1| e^{(\alpha-\delta)(t-\tau)} \end{cases} \tag{1.25}$$

$$\begin{cases} |x_1^*(x_0, y_1, \tau)| \leq |x_0| e^{-(\alpha-\delta)\tau} \\ |y_0^*(x_0, y_1, \tau)| \leq |y_1| e^{-(\alpha-\delta)\tau} \end{cases} \tag{1.26}$$

Remark The BV formalism allows us not to mind about checking that the solution does not leave a fixed neighborhood of the origin. In fact this is already granted by the existence and uniqueness theorem 1.3.3. Of course similar estimates hold for solutions to I.V. problems, but they are valid as far as the trajectories stay close to the origin.

The previous corollary can be read in a coordinate free way as estimating the distance of trajectories from the stable and unstable manifolds.

Proof. Let's put $v(s) = y^*(\tau - s)$ and rewrite equations 1.12 in our setting:

$$\begin{cases} x(t) = e^{tL^-} x_0 + \int_0^t e^{(t-s)L^-} f(x(s), v(\tau - s)) x(s) ds \\ v(t) = e^{-tL^+} y_1 - \int_t^\tau e^{(t-s)L^+} g(x(s), v(\tau - s)) v(\tau - s) ds \end{cases}$$

Rescale the second integral to get:

$$\begin{cases} x(t) = e^{tL^-} x_0 + \int_0^t e^{(t-s)L^-} f(x(s), v(\tau - s)) x(s) ds \\ v(t) = e^{-tL^+} y_1 - \int_0^t e^{-(t-s)L^+} g(x(\tau - s), v(s)) v(s) ds \end{cases} \tag{1.27}$$

The two equations are formally identical, so we'll just work on the first. Recalling that $\left\| e^{tL^-} \right\| \leq e^{-t\alpha}$:

$$|x(t)| \le e^{-\alpha t}|x(0)| + \int_0^t e^{-(t-s)\alpha}\delta|x(s)|ds$$

Gronwall's lemma applied to $e^{\alpha t}|x(s)|$ gives the desired estimate □

Similar estimates also hold for the derivatives of the solution to the BV problem. It has already been observed that these derivatives can be found as solutions to BV problems of the appropriate variational systems. Unfortunately, the variational system are not any longer "straighten" with respect to their own coordinates (and not even autonomous) and so we cannot just repeat the same argument. Still, there is a common form for all the variational systems we need. In particular any system computing the $(k)^{th}$ derivatives either in t or in the spatial variables x_0, y_1 will have variables X^k, Y^k and an expression like:

$$\begin{cases} \dot{X^k} &= L^- X^k + f(x^*, y^*)X^k + (\cdots)X^{k-1} + \ldots + (\cdots)x \\ \dot{Y^k} &= L^+ Y^k + g(x^*, y^*)Y^k + (\cdots)Y^{k-1} + \ldots + (\cdots)y \end{cases} \tag{1.28}$$

where the symbols X^i stand for terms which are i^{th} derivatives of entries of x and similarly for Y. In this case the coefficients of each addendum X^i is a sum of k-i derivatives of the non linear terms f and g, and therefore they can be estimated using the global bounds δ^h. This permits to use an induction argument in order to estimate X^k and Y^k. There is a warning. The estimates are not any longer valid within the same ball as before but on smaller ones, depending on the order of derivative; quite surprisingly, though, the size of the neighborhood depends only on estimates on the first derivatives of f and g. Before stating the theorem, we need a refined version of the Gronwall lemma.

Lemma 1.3.17 *Suppose that $\phi, a, b : [0, +\infty) \to \mathbb{R}$ are three continuous functions; a is nondecreasing, b nonnegative, and:*

$$\phi(t) \le a(t) + \int_0^t b(s)\phi(s)ds$$

Then the following holds:

$$\phi(t) \le a(t)e^{\int_0^t b(s)ds}$$

With same notations as in the previous theorem 1.3.16 we can state:

Theorem 1.3.18 *Suppose $\varepsilon > 0$ is such that $k\delta^1 < \alpha$. Let $X^*(t), Y^*(t)$ be some k^{th}-order derivatives of (x^*, y^*) in the spatial variables x_0, y_1 and/or* *in the time variables t, τ. The following inequalities hold, for some constant $C_k > 0$:*

$$|X^*(t)| \le C_k e^{-(\alpha - k\delta)t}$$
$$|Y^*(t)| \le C_k e^{(\alpha - k\delta)(t-\tau)}$$

Proof. By corollary 1.3.6, it's enough to mind about the case of derivatives in the spatial or t variables. We can then use the expressions 1.28. As in the proof of theorem 1.3.16 let's put $V(s) = Y(\tau - s)$, so that the integral equations corresponding to 1.28 become:

$$\begin{cases} X(t) = e^{tL^-}X(0) + \int_0^t e^{(t-s)L^-}\left(f(x(s), v(\tau - s))X(s) + \dots\right) ds \\ V(t) = e^{-tL^+}V(0) - \int_0^t e^{-(t-s)L^+}\left(g(x(\tau - s), v(s))V(s) + \dots\right) ds \end{cases} \tag{1.29}$$

where we've already rescaled the second integral and the dots mean terms factoring through less order derivatives of the solutions. Since the two equations are of the same kind, we just give the proof for the first one. By induction, supposing the thesis true for $1, \dots, k-1$, one can find a D_k s.t.

$$|X(t)| \le e^{-\alpha t}|X(0)| + \int_0^t e^{-(t-s)\alpha}\left(\delta|X(s)| + D_k e^{-(\alpha-(k-1)\delta)s}\right) ds$$
$$\le e^{-\alpha t}|X(0)| + D_k/((k-1)\delta)e^{-\alpha + (k-1)\delta t} + \int_0^t e^{-(t-s)\alpha}\delta|X(s)|ds$$

By Gronwall lemma applied to $e^{\alpha t}|X(t)|$ then:

$$e^{\alpha t}|X(t)| \le \left(|X(0)| + \frac{D_k}{(k-1)\delta}e^{(k-1)\delta t}\right)e^{\delta t}$$

from which the thesis follows readily □

By theorem 1.3.7, in the same hypothesis of the previous theorem, it then follows:

Theorem 1.3.19 *Let $V^*(\tau)$ denote some derivative among $\dfrac{\partial^k x_1^*}{\partial^k x_0, y_1, \tau}|_{(x_0, y_1, \tau)}$ or among $\dfrac{\partial^k y_0^*}{\partial x_0, y_1, \tau}|_{(x_0, y_1, \tau)}$. Then the following inequality holds, for some constant $C_k > 0$ not depending on x_0, y_1, τ:*

$$|V^*(\tau)| \le C_k e^{-(\alpha - k\delta)\tau} \tag{1.30}$$

Conclusions We finally draw out the two consequence which will be used in the sequel of the present work. For the sake of clearness we repeat all the hypothesis here.

Consider a vector field on $\mathbb{R}^n$ having the origin as an isolated hyperbolic singularity (cf. definition 1.1) and let $\phi = (\phi_t)_{t\in\mathbb{R}}$ be the flow of solutions. Then there exist "straighten coordinates" x, y near the origin for which the local stable manifold S is given by $y = 0$, the local unstable manifold U by $x = 0$. Moreover, consider the unit ball $B = \{(x, y) | \, |x, y| < 1\}$ and decompose its boundary in the two pieces

$$\partial^+ B = \{|x| = 1, \, |y| \leq 1\} \qquad \partial^- B = \{|x| \leq 1, \, |y| = 1\}$$

We can suppose that $\partial^- B$ and $\partial^+ B$ are transversal to the vector field and that if a point $m \in \partial^+ B$ does not belong to the stable manifold U, then the trajectory starting from m will touch ∂B again in $\partial^- B$. This is granted by corollary 1.3.14 and the estimates in theorem 1.3.16. The construction defines a "first escape" map

$$\varphi : \partial^+ B \backslash S \to \partial^- B \backslash U$$

which is clearly bijective. Since the vector field is not tangent to $\partial^\pm B$ and the flow of solutions is a smooth map, an application of the implicit function theorem proves

Theorem 1.3.20 *The first escape map*

$$\varphi : \partial^+ B \backslash S \to \partial^- B \backslash U$$

is a diffeomorphism.

Consider now the submanifold

$$W = \{\left(\phi_{\frac{t}{1-t}}(m), m, t\right) | m \in \mathbb{R}^n, \, t \in (0, 1)\} \cap B \times B \times (0, 1) \subset \mathbb{R}^{2n+1}$$

Theorem 1.3.21 *The closure* $\overline{W} = (W, \partial W)$ *inside* $B \times B \times \mathbb{R}$ *is a smooth, closed submanifold inside with boundary:*

$$\partial W = U \times S \times \{1\} \cup \Delta \times \{0\}$$

where Δ *denotes the diagonal in* $B \times B$.

Proof. Using the starred notation for the solution to the BV problem, by the existence and uniqueness theorem one can reinterpret W as:

$$\begin{aligned} W &= \left\{ \left(\phi_{\frac{t}{1-t}}(x_0, y_0), x_0, y_0, t\right) \Big| \, |x_0, y_0| < 1, \, |\phi_{\frac{t}{1-t}}(x_0, y_0)| < 1, \, t \in (0, 1)\right\} \\ &= \left\{ \left(x_1^* \left(x_0, y_1, \tfrac{\tau}{1-\tau}\right), y_1, x_0, y_0^* \left(x_0, y_1, \tfrac{\tau}{1-\tau}\right), \tau\right) \Big| \, |x_0, y_1| < 1, \, \tau \in (0, 1)\right\} \end{aligned}$$

The estimates in theorem 1.3.19 now partially prove the statement. In fact so far we just proved that W is $\mathcal{C}^k$ for arbitrarily large k on a neighborhood which depends on k. But W is invariant under the map

$$\rho_{(r,s)} = \big(m', m, t\big) \longrightarrow \big(\phi_r(m'), \phi_s(m), \sigma_{r,s}(t)\big)$$

for any $r < 0 < s$, provide $r + \frac{t}{1-t} = s + \frac{\sigma}{1-\sigma}$.

The value $\sigma_{r,s}(t)$ solving this condition is $\dfrac{(1-t)(s-r)-t}{(1-t)(s-r-1)-t}$ and ρ is hence a diffeomorphism near $t = 1$.

Therefore, if we fix $(m', m, 1) \in \partial W$, the (local near $t = 1$) diffeomorphism $\rho(r, s)$ maps a neighborhood of W near $(m', m, 1)$ to a neighborhood of W near a point which we can suppose arbitrarily close to $(0, 0, 1)$ by choosing the right r, s. Since we know that W is arbitrarily smooth near $(0, 0, 1)$, this concludes the proof $\square$

Chapter 2

Horned Stratifications and Volume Bounds

We here introduce the problem of bounding the volume of the image of a submanifold under a flow. Some pathologies are described as well as the hypotheses (Smale and Weakly Proper) needed to avoid them, and the model for the expected singularities, i.e. the horned stratifications, is studied in detail. The main theorem 2.3.1 is then proved and applied to show that a Weakly Proper, Smale flow has locally finite volume.

2.1 Problem and Counterexamples

Given a flow ϕ on X and a regular submanifold $M \subset X$ of dimension m, we want to study the subset $N \subset X$ obtained by moving M under the flow, i.e. the union of trajectories of ϕ starting on points in M. We then state the following:

Problem To look for conditions on the flow ϕ and on the submanifold M so that N has locally finite $(m+1)$-volume and there is a reasonable description for the boundary of N, in the sense of geometric measure theory.

One soon realizes that it's reasonable to suppose that M contains no fixed point and that the flow has no periodic nor dense orbits in X. It is also clear that singularities appear when N "meets" some fixed point for the flow, and the more involved the fixed point is, the more involved N might result. We thus assume that ϕ is:

- not tangent to M;
- gradient-like with respect to a function on X;
- with isolated hyperbolic singularities.

Still this is not enough, in general. In fact, we will next present two (counter)examples of possible bad behaviors, pointing out what is missed in each case.

Example 1

Let's consider the linear flow on $\mathbb{R}^3$ given, in cylindric coordinates, by:

$$(\rho, \theta, z) \overset{\phi_t}{\longmapsto} \left(e^{-t}\rho, \theta, e^t z\right)$$

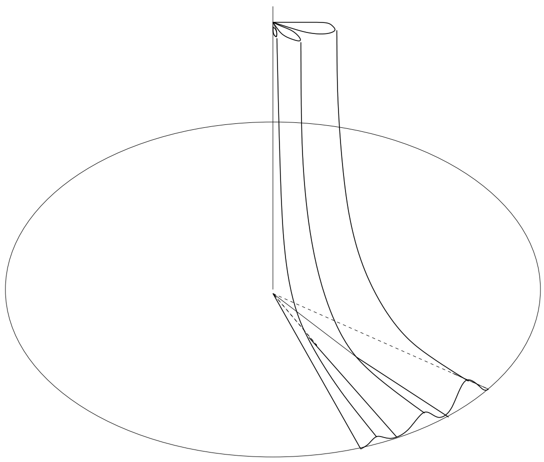
The stable manifold at 0 is obviously $S = \{z = 0\}$ and the unstable is given by $U = \{\rho = 0\}$. Consider a curve γ parametrized by

$$\{(\rho, \theta, z) \mid \rho = 1,\ z = f(\theta)\}$$

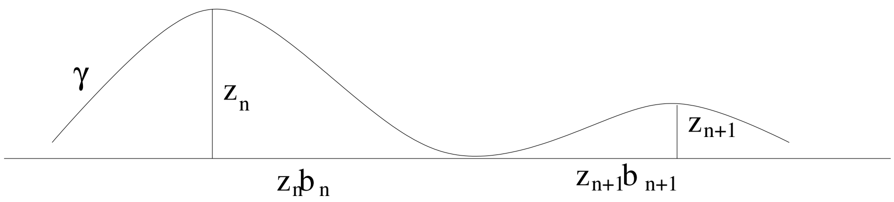
Certainly γ is contained in the cylinder $|\theta| = 1$ and hence never tangent to the flow. We consider the image of γ under the flow, i.e

$$\Gamma = \bigcup_{r>0} \phi_r(\gamma)$$

The origin 0 is in $\bar{\Gamma}$ if and only if $\gamma \cap S \neq \emptyset$, and Γ is smooth (with no boundary) otherwise. It will be proved later that if the intersection $\gamma \cap S$ is transversal and nonempty, then $\bar{\Gamma}$ has finite area (this is the motivation to introduce the Smale hypothesis). We now show that there exists a $\mathcal{C}^1$ curve γ whose image under the flow has unbounded area near the origin.



To find a parametrization f for γ, we'll use Whitney extension theorem. Put $z_n = \frac{1}{n\log^2(n)}$, $b_n = \log(\log(n))$ and $\theta_n = \sum_{i\leq n} a_i b_i$ and observe that $\sum z_n$ and $\sum z_n b_n$ converge but $\sum z_n \log(z_n)$ doesn't.



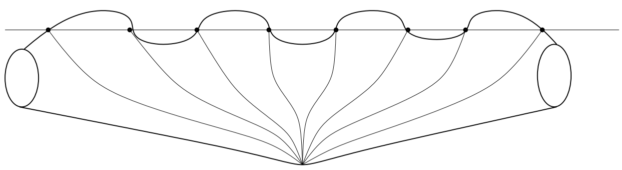
Since the “candidate slopes”

$$\frac{z_n - 0}{\theta_n - \theta_{n-1}} = \frac{1}{b_n} \to 0$$

Whitney’s theorem implies that there exists a $\mathcal{C}^1$ function $f(\theta)$ such that $f(\theta_n) = z_n$ and $f(\frac{1}{2}(\theta_{n+1} - \theta_n)) = 0$. The area of the surface generated by the corresponding curve γ is then bigger than the diverging sum $\sum z_n |\log(z_n)|$, which is the area of the image under the flow of the vertical middle segments.

Example 2

Here the problem is compactedness. Too much mass from distant points accumulates in a small region. The flow is the gradient of the height function on a tubular surface with some wiggles. The manifold M is a discrete sequence of points and N is hence a family of flow lines. Since they accumulate in the lowest point, the volume of N necessarily blows up, as the picture shows.



To avoid the pathological behaviour shown in this example, we’ll next introduce the Weakly Proper hypothesis.

Let ϕ be the flow of a complete vector field V on a manifold X. By a broken flow line of ϕ we mean a piecewise smooth curve whose pieces parametrize flow lines. For points $x, y \in X$ we’ll write $x \preccurlyeq y$ if there is a broken flow line connecting x and y (and $x \prec y$ if in addition $x \neq y$).

Definition 2.1.1 *Let $A, B \subset X$. The set of points lying on a broken flow line starting at a point in A and ending on a point in B is called the* ***shadow of*** *A* ***into*** *B and denoted by $L_B^A \subset X$. In particular, the* ***shadow of*** *A, denoted by $L^A = L_\emptyset^A$ is the set-union of broken flow lines starting at a point in A; similarly one defines L_B, the* ***shadow into*** *B.*

Recall now that the vector field V is said to be **gradient-like** with respect to a function $f : X \to \mathbb{R}$ if and only if $V(f) < 0$ away from the critical points of f. Of course, any gradient vector field is gradient like[1] with respect to his potential function (by analogy, if V is gradient-like with respect to f we also say that f is a weak potential for V). We can now introduce the following:

Definition 2.1.2 *A vector field V is said **Weakly Proper** with respect to f if*

- *V is gradient-like with respect to f*
- *For any compact set K contained in a slab $F = f^{-1}[a, b]$ then the shadows L_F^K and L_K^F are compact.*

Actually, for some vector fields it's enough to restrict the test to the case when K is a point, in fact:

Proposition 2.1.3 *Suppose V is gradient like for f and has isolated singularities. If each broken flow line contained in a slab is relatively compact, then V is weakly proper with respect to f.*

Proof. For any vector field, the closure of an invariant set is an invariant set. If the vector field has isolated singularities, this implies that if a sequence of broken flow lines has a common accumulation point, then it contains a broken flow line in its closure.□

In the general case, sequences of flow lines might accumulate inside critical sets (and so not converge to broken flow lines), and the previous statement doesn't hold. Nevertheless, being isolated is not a necessary condition: a counterexample is provided by Bott singularities (cf. chapter 5), where again the set of broken flow lines is "weakly compact" in the sense of the previous proof.

2.2 Horned Stratifications

We introduce the model of *horned stratifications* for the submanifold arising in the problem discussed in the previous section and have a digression on their features. A review about stratified spaces is provided in the Appendix B.

Definition 2.2.1 *A compact stratified set Y of dimension k will be called **compact horned-stratified** (or **compact h-stratified**) set if:*

[1] We use this convention to be consistent with later choices.

- *The stratification of Y is AB Whitney regular*
- *There exists a submersive map $\pi : M \to Y$ from a k-dimensional compact manifold with corners M to Y.*

*The manifold M will be called a **desingularization** for Y and the map π a **h-projection**. A **locally h-stratified set** $Y \subset X$ is a stratified set which locally coincide with a compact horned stratified set.*

The word "horn" comes from the typical picture of a horned stratified set near the zero dimensional strata. Note that M and Y have same dimension.

Remark. The A-regularity requirement in the previous definition is redundant, being a consequence of the existence of the desingularization. It is not so for B regularity, though.

The horned stratified sets have very nice geometric properties. Let's start with a topological one: it helps understanding the structure of a h-stratified set, but we'll never use it and thus omit the (not difficult) proof.

Lemma 2.2.2 *Let $M \xrightarrow{f} Y$ be a surjective map between two compact Hausdorff spaces. Suppose that $\overset{\circ}{M}$ is connected, $f^{-1}(\overset{\circ}{Y}) = \overset{\circ}{M}$ and $f|_{\overset{\circ}{M}} : \overset{\circ}{M} \to \overset{\circ}{Y}$ is a local homeomorphism. Then $f|_{\overset{\circ}{M}} : \overset{\circ}{M} \to \overset{\circ}{Y}$ is a finite covering (in particular all fibers over interior points have same cardinality).*

The compactdness hypothesis was fundamental in the previous lemma, a proof of which can be found via the open-close trick.

Proposition 2.2.3 *Let $M \xrightarrow{f} Y$ be a desingularization of a compact h-stratification and $f_{ij} : M_i \to Y_j$ the restrictions to the strata. If the dimension of M_i and Y_j coincide, then f_{ij} is a finite covering.*

We next analyze the behavior under intersections.

Proposition 2.2.4 *Let $Y \subset X$ be a h-stratified set and $Z \subset X$ a closed submanifold. Suppose that each stratum of Y is transversal to Z. Then the family of all intersections of strata makes up a h-stratification for $Z \cap Y$.*

Proof. By proposition B.0.6, it remains to prove the desingularization part. Let $f : M \subset N \to X$ be a local desingularization for Y. By hypothesis, the restriction of f to any stratum of M is transversal to Z and thus $M \cap f^{-1}(Z)$ is a compact manifold with boundary. The latter h-projects onto $Y \cap Z$ via f, because of the following linear-algebraic lemma $\square$

Lemma 2.2.5 *Let $V^1 \subset V^2$ and $W^1 \subset W^2$ be finite dimensional vector spaces and let $L : V^2 \to W^2$ be a linear map. Suppose $L(V^1) + W^1 = W^2$ (i.e. the restriction of L to V^1 is transversal to W^1). Then $V^1 + L^{-1}(W^1) = V^2$ and $L : V^1 \cap L^{-1}(W^1) \to V^1 \cap W^1$ is surjective.*

Corollary 2.2.6 *The previous proposition holds if Z is replaced by a closed submanifold with corners, provide all possible intersection are transversal.*

Actually, Z can be replaced by a horned stratified set too, as we next show.

Lemma 2.2.7 *Let $f : X_1 \to X_2$ be a smooth map and $M \xrightarrow{\pi} Y$ be a h-stratified subset of X_1. Then the graph G of the restriction $f|_Y$, i.e.*

$$G(f) = \{(y, f(y))\,|y \in Y\} \subset X_1 \times X_2$$

is naturally a h-stratified set, the desingularization of $G(f)$ being the graph of the map $f \circ \pi$.

Proof. The graph $G(f)$ is the intersection of the graph of f with $Y \times X_2$; apply proposition 2.2.4 □

Proposition 2.2.8 *Let $f : X_1 \to X_2$ be a smooth proper map, and $Y \subset X_2$ a h-stratified space. If f is transversal to Y, then $f^{-1}(Y)$ is h-stratified as well.*

Proof. Let $g : M \subset N \to Y \subset X_2$ be a local the desingularization of Y and let $G_M \subset (X_2 \times N)$ be the (inverted) graph of the restriction of g to M. Consider now, inside $X_1 \times X_2 \times N$, the closed submanifold $graph(f) \times N$ and the submanifold with corners $X_1 \times G_M$. The transversality hypothesis forces those to be transversal, while the proper assumption makes the intersection compact when M is. The projection onto X_1 is then a h-projection onto $f^{-1}(Y)$ □

The previous proposition, combined with corollary 2.2.6 and lemma 2.2.5 yields:

Corollary 2.2.9 *Let $f : X_1 \to X_2$ be a smooth map, $Y \subset X_2$ a h-stratified space and Z a compact manifold with corners. If f restricted to any stratum of Z is transversal to Y, then $f^{-1}(Y) \cap Z$ is horned-stratified too.*

The intersection of transversal horned stratified sets is a horned stratified set:

Proposition 2.2.10 *Let $Y_1, Y_2 \subset X$ be h-stratified sets and suppose that all possible intersections of strata of Y_1 and Y_2 are transversal. Then the family of these intersections is a h-stratification for the set $Y = Y_1 \cap Y_2$.*

Proof. Since the intersection of regular stratified sets is regular (cf. proposition B.0.6), it's enough to find a desingularization for Y; we also assume everything to be compact. Let $f, g : M_i \subset N_i \to Y_i \subset X$ for $i = 1, 2$ be desingularizations (the domains are compact manifold with corners). A manifold with boundary M which desingularizes $Y = Y_1 \cap Y_2$ is the pull-back intersection: $M = \{(p, f(p) = g(q), q) | p \in M_1, q \in M_2\}$. Observe that $M_1 \cap f^{-1}(Y_2)$ is a h-stratified space inside N_1 (and the same property holds with indexes interchanged), then apply the trick of the graph as in the proof of proposition 2.2.8 □

Corollary 2.2.11 *Let $Y_1, Y_2 \subset X$ be h-stratified sets and $y \in Y_1 \cap Y_2$. If the two strata containing y intersect transversally, then all the strata near y intersect transversally and hence, near y, the intersection $Y_1 \cap Y_2$ is a h-stratified set.*

Let's now look at the metric properties of h-stratified sets.

Proposition 2.2.12 *A (locally) h-Stratified set of dimension m has (locally) finite m-dimensional measure.*

In the previous statement, one can take as "m-measure" the Hausdorff measure of any Riemannian metric on the ambient space; this equals the m-dimensional volume of the (union of the) top strata, which is a locally closed submanifold. The volume of the stratification is in fact bounded by a constant times the volume of the desingularization. The constant depends on an upper bound for the first derivatives of the h-projection, as a consequence of the Area formula (cf. [10], theorem 1.2.1).

Remark Suppose that for a compact h-stratified set $M \subset N \xrightarrow{f} Y \subset X$, the desingularizing manifold with corners M is oriented, i.e. its top stratum is oriented, so defining orientations on the codimension 1 strata (the "faces"). In this case, M defines a current of integration $[M]$ on N and there is a well defined pushforward current $f_*[M]$ on X which is obviously supported on Y. Because of possible multiplicities, this might not coincide with the current of integration on Y (assuming Y is oriented) whose existence would be granted by the previous proposition.

Definition 2.2.13 *A current T of dimension k on a manifold X is a* ***finite h-chain (current)****, or finite horned chain, if there exists a finite number of compact h-stratifications $M_i \xrightarrow{\pi_i} Y_i$ (with all M_i oriented) such that $T = \sum_i \pi_{i*}([M_i])$. A* ***(local) h-chain (current)*** *is a current which locally coincides with a finite h-chain.*

Remark The support of a h-chain does not need to be a h-stratified set, but it is a locally finite union of compact h-stratified sets. This extra structure of the support is relevant when dealing with intersections, as we shall see later.

We recall that a **finite chain (current)** is a finite sum of currents defined as pushforward of standard simplexes Δ_k under smooth maps, whereas a **local chain (current)** is a current which locally coincides with a finite chain. This terminology was introduced by deRham in [7]. Clearly the h-chain currents are chain currents and in particular integral currents (i.e. rectifiable with rectifiable boundary, cf. [8] def. 4.1.24), in fact:

Proposition-Definition 2.2.14 *The current boundary dT of a (horned) chain T is again a (horned) chain current. The spaces $\mathcal{C}^*$ of chain currents and $\mathcal{H}^*$ of h-chains are therefore subcomplexes of the complex of integral currents.*

Proof. The proof is trivial for chain currents. As for horned chains, if the desingularization $T = \pi_* M$ holds on a open set U (note that $\pi(M)$ is not supposed contained in U), then the boundary $dT = \pi_*(dM)$. We know that dM is a sum of faces (since M is a manifold with corners), and each face does contribute (after the pushforward) to the boundary dT only if the dimension does not decrease via the projection. The restriction of π to these faces are again h-projections and so dT is a local h-chain □

Observed that the pushforward of a chain current under a smooth proper map is again a chain current, the following is a consequence of corollary 2.2.9 and proposition 2.2.10.

Proposition 2.2.15 *Let $[T]$ be a k-dimensional horned chain current in $\mathbb{R}^n$ and $[S]$ a chain current of dimension l. Suppose that the maps which locally define $[S]$ are transversal to any stratum of the stratified sets on which T is supported. Then $[S]$ and $[T]$ can be "intersected" and $[S] \wedge [T] = [S \cap T]$ is a chain current of dimension $h + k - n$. In particular if $[S]$ is a h-chain too, then $[T \cap S]$ is a h-chain.*

The thesis means that for any test forms γ_n converging to $[S]$ (as currents), and for any β of the correct degree, the numbers $T(\gamma_n \wedge \beta)$ converge, and the limit is $[S \cap T](\beta)$.

Remark We will later prove, as a corollary of Morse theory, that h-chains can be used to compute the integral cohomology of a manifold. This is difficult to prove directly since problems arise with the "Poincaré lemma".

We instead sketch a direct proof of how the local chain currents can be used to compute the integral cohomology of a manifold. This fact is implicit in the work of deRham (cf. [7]).

Proposition-Definition 2.2.16 *The map $U \to \mathcal{C}_k(U)$ associating to each open set U in X^n the abelian group of k-dimensional chain currents on U, with obvious restriction homomorphisms, defines a sheaf, called the* ***sheaf of (germs of) chains of dimension*** *k (or degree $n-k$). This is a subsheaf of the sheaf of germs of currents.*

The only thing to prove is the restriction axiom of a presheaf. If one likes to restrict an elementary chain (i.e. a simplex) Δ on an open set U , he can simply present the restriction $T|_U$ (which is a current) as a locally finite sum of elementary chains in U, by triangulating the simplex in smaller and smaller pieces. Since chains are defined to be currents with an extra local property, the other axioms are simple (being true for the family of currents).

The following is a trivial case of the so called “constancy theorem”, cf. [10].

Proposition 2.2.17 *Let T be a 0-degree d-closed chain on the open and connected set U. Then there exists an integer $c \in \mathbb{Z}$ such that $T = c[U]$.*

Let’s now prove the “Poincaré lemma” for chains.

Proposition 2.2.18 *Let T be a d-closed chain of degree $k > 0$ (i.e. dimension $n-k < n$) on the n-dimensional manifold X. Then, for any $p \in X$ there is a neighborhood U of p and a h-chain S on U of dimension $k-1$ such that $T|_U = dS$.*

Proof (Cone construction). We can suppose U to be a ball in $\mathbb{R}^n$. Let $M = \cup\Delta_i \subset \mathbb{R}^{n'} \xrightarrow{f} T$ be a (local) desingularization, with M a compact manifold with corners (the desingularization is not assumed to be submersive as for h-chains but it just is a sum of smooth simplexes). Consider a point x in U and a point y close to x which is not in the support of T. For any simplex $f : \Delta \to X$ belonging to M, let $C(\Delta) = \{(t, \theta) |\, t \in [0, 1], \theta \in \Delta\}$ and put

$$\widetilde{f} : C(\Delta) \to X \qquad (t, \theta) \to ty - (1-t)f(\theta)$$

Clearly the cones $\widetilde{f}$ define chain currents by pushforward; by $f_*(dM) = dT = 0$ it then follows that $\widetilde{f}_*(C(dM)) = 0$. Near any fixed x there exists a y and a small enough neighborhood V of x and y such that for any of the simplexes Δ_i partially supported on V, the cone over $C(\Delta_i)$ is not identically zero. Summing over all the simplexes Δ_i, one then gets:

$$d\widetilde{f}_*C(M) = \widetilde{f}_*(dC(M)) = \widetilde{f}_*(M - C(dM)) = T$$

the relation $dC(\Delta) = \Delta - C(d\Delta)$ holding because the dimension of the Δ’s is not zero $\Box$

The sheaves of chain currents are acyclic, in fact they are soft sheaves. Recalling that a germ T of chain current on a closed set A is the equivalence class of h-chains defined on a neighborhood of A under the equivalence class determined by coincidence on a (smaller) neighborhood of A:

Lemma 2.2.19 *Let $A \subset X$ be a closed set and T a germ of h-chain current on A. Then there exists a "global" h-chain current R on X which extends T. The sheaf of germs of h-chains of dimension k is hence soft.*

Proof (Slicing). Let U be an open neighborhood of A, and R_0 a chain current on U which extends T. There exist "tubular" neighborhoods $V \Subset U$ of A with $\mathcal{C}^\infty$ boundary ∂V transversal to the simplexes of R_0. Transversality can be attained by a generic perturbation of any given regular neighborhood of A. The intersection $R = R_0 \cap \overline{V}$ is again a chain, providing an extension of T to all of X □

The previous propositions, together with a basic theorem in sheaf cohomology proves:

Theorem 2.2.20 *The sequence of sheaf maps*

$$\mathbb{Z} \to \mathcal{C}^0 \overset{d}{\to} \mathcal{C}^1 \overset{d}{\to} \cdots \overset{d}{\to} \mathcal{C}^{n-1} \overset{d}{\to} \mathcal{C}^n \to 0$$

is an exact resolution of the locally constant sheaf $\mathbb{Z}$, and the sheaves $\mathcal{C}^k$ are acyclic. Hence the cohomology groups of the sheaf $\mathbb{Z}$ can be computed as "deRham" groups

$$H^k(X, \mathbb{Z}) = \frac{\{T \in \mathcal{C}^k \,|\, dT = 0\}}{\{dR \,|\, R \in \mathcal{C}^{k-1}\}}$$

2.3 Movements of Submanifolds

All over the section (X, ϕ, f) will be a Weakly Proper, Smale (abridged WPS) dynamical system.

Recall that "Weakly Proper with respect to f" means that the values of the "potential-like" function f strictly decrease along nontrivial trajectories of the flow and that the shadows of compact sets are compact within each slab $f^{-1}([a, b])$. "Smale" means that the system has isolated hyperbolic singularities whose stable and unstable manifolds intersect transversally.

For each critical point, say p, we'll assumed fixed charts Ω_p in "straighten coordinates" (cf. section 1.3) such that

$$\Omega_p = \{(x, y) \in R^s \times R^u |\, \|x, y\| \leq \varepsilon\}$$

and the stable and unstable manifold are given by

$$S_p = \{\|x\| \leq \varepsilon, \|y\| = 0\} \qquad U_p = \{\|x\| = 0, \|y\| \leq \varepsilon\}$$

The boundary of Ω_p decomposes as $\partial\Omega_p = \partial^+\Omega_p \cup \partial^-\Omega_p$ where

$$\partial^+\Omega_p = \{\|x\| = \varepsilon, \|y\| \leq \varepsilon\} \qquad \partial^-\Omega_p = \{\|x\| \leq \varepsilon, \|y\| = \varepsilon\}$$

Introducing the "**link**" of the stable and unstable manifold:

$$S_p^+ = \{\|x\| = \varepsilon, \|y\| = 0\} = S_p \cap \partial^+\Omega_p$$
$$U_p^- = \{\|x\| = 0, \|y\| = \varepsilon\} = U_p \cap \partial^-\Omega_p$$

the "first escape map" provides a diffeomorphism

$$\Theta_p^p : \partial^+\Omega_p \backslash S_p^+ \to \partial^-\Omega_p \backslash U_p^-$$

We can now state the main theorem of the chapter:

Theorem 2.3.1 *Let (X, ϕ, f) be a Weakly Proper Smale dynamical system, $c \in \mathbb{R}$ a regular value for f and $M \subset f^{-1}(c)$ a smooth (embedded) submanifold of dimension m. Suppose M is transversal to every stable manifold and let $N = \bigcup_{t>0} \phi_t(M)$. Then the closure $\overline{N}$ coincides with the shadow L^M and is a horned stratified subset whose singular strata are unstable manifolds.*

The proof of the theorem is postponed to the end of the section, but we draw out an important consequence:

Corollary 2.3.2 *The unstable manifolds of a WPS dynamical system are horned stratified subsets whose singular strata are unstable manifolds. The same is true for stable manifolds (singular strata being other stable manifolds).*

Proof of the corollary For any critical point p, the unstable manifold U_p is smooth near p, and its intersection with a level set $f = f(p) - \varepsilon$ is a smooth compact submanifold, thus the theorem applies. The standard trick of reversing time proves the dual statement for stable manifolds □

We next prove three local results which will be the bricks in the proof of theorem 2.3.1. They describe what happens during the movement of a submanifold respectively away from the critical points, near the first one, and near the successive ones.

The first result is just a particular case of the existence of linearizing charts for a nonvanishing vector field and we'll omit the proof:

Proposition 2.3.3 *Let $A^- \subset f^{-1}(b)$ be a closed subset and suppose $L^{A^-}_{f^{-1}(a)}$ contains no critical point, for some $a < b$. Then, there exist neighborhoods U of A^- in $f^{-1}(b)$ and V of L^{A^-} in $f^{-1}([a, b])$ and a diffeomorphism $\rho : V \to U \times [0, 1]$ which maps flow lines to "vertical" segments. In particular ρ maps A^- to $A^- \times \{0\}$ and $L^{A^-}_{f^{-1}(a)}$ to $A^- \times [0, 1]$, providing an isotopy between A^- and $A^+ = \rho^{-1}(A^- \times \{1\})$.*

For the second "brick", let p be a critical point and $\Omega \subset \mathbb{R}^n$ a straighten chart centered in p (notations as above).

Proposition 2.3.4 *Suppose $A^+ \subset \partial^+\Omega$ is a compact manifold with corners such that all strata have transversal intersection with S_p. Put*

$$A = \{\phi_t(x) | x \in A^+, t > 0\} \cap \Omega$$

Then the closure $\overline{A}$ in Ω coincides with the shadow $L_{\Omega}^{A^+} = A \cup U_p$ and is a compact horned stratified set. The singular strata of $\overline{A}$ are the shadows of the strata in A^+ and the unstable manifold U_p.

The closure of $A^- \stackrel{def}{=} A \cap \partial^-\Omega$ is compact h-stratified too and contains the link U_p^+ as singular stratum.

Proof. Suppose first $A^+ \subset \partial^+\Omega$ to be a smooth, compact submanifold, transverse to S_p. Since the flow is not tangent to A^+:

Fact 2.1 *The parametrization*

$$(t, m) \in [0, +\infty) \times A^+ \stackrel{\sigma}{\longmapsto} \phi_t(m) \in A$$

is regular, i.e. it is a diffeomorphism.

Consider now the following submanifold of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$:

$$W = \left\{ (x_1, y_1, x_0, y_0, t) \;|\; \begin{array}{c} 0 < t < 1,\ |x_0, y_0, x_1, y_1| < \varepsilon \\ (x_1, y_1) = \phi_{\frac{t}{1-t}}(x_0, y_0) \end{array} \right\}$$

By the BV technique (cf. theorem 1.3.21), the manifold W is smooth in $\Omega \times \Omega \times \mathbb{R}$ with boundary

$$\partial W = \{(0, y_1, x_0, 0, 1) |\ \ |x_0, y_1| < \varepsilon\} \cup \{(m, m, 0) | m \in \Omega\}$$

Define two subsets of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ by:

$$Z \stackrel{def}{=} \mathbb{R}^n \times A^+ \times \mathbb{R}$$

$$(A^1, \partial A^1) \stackrel{def}{=} Z \cap (W, \partial W) = \left\{ (m', m, t) |\, 0 \leq t \leq 1, m \in A^+, m' = \phi_{\frac{t}{1-t}}(m) \right\}$$

It clearly is

$$\partial A^1 = U_p \times (A^+ \cap S_p) \times \{1\} \cup \Delta_{A^+} \times \{0\}$$

The submanifold Z is transverse to ∂W since A^+ is transverse to S_p. Away from ∂W, any intersection of the form $W \cap \mathbb{R}^n \times pt$ is transversal, in particular $W \cap Z$. Therefore, $(A^1, \partial A^1)$ is a smooth compact manifold with boundary and, projecting onto the first factor via

$$\pi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

it's $\pi\left(A^1\right) = A$ and the topological boundary of A is hence $U_p = \pi\left(\partial A^1\right)$.

The restriction of π is clearly a submersion on the singular stratum ∂A^1. On the other hand, the open top stratum of A^1 can be regularly parametrized by the map σ by Fact 2.1. Also, σ factors trough $\pi|_{A^1}$, by the definition of W as a graph and so the restriction of π to the top stratum in A^1 is a diffeomorphism too. Summarizing, the restriction

$$\pi|_{(A^1,\partial A^1)} : (A^1, \partial A^1) \to (A, U_p)$$

is a h-projection over the stratified space $\overline{A} = A \cup U_p$. To show that $\overline{A}$ is horned we are left to prove the AB Whitney regularity conditions for a stratification. It's actually enough to prove B, thanks to the remark after definition 2.2.1. But B-regularity is equivalent to prove that the normal spheres in any small $\mathcal{C}^1$ tubular neighborhood of the singular stratum U_p are transversal to A (see theorem B.0.5 in the appendix). This is trivially true because A is a union of flow lines, which are already transversal to those spheres.

Finally, the intersection $(A^-, U_p^-) = (A, U_p) \cap \partial^- \Omega_p$ is transversal, and so the closure $\overline{A^-}$ is h-stratified too, by proposition 2.2.4.

All the previous arguments still hold with obvious modifications if A^- is replaced by a manifold with corners transversal to the stable manifold (in the sense that any stratum is transversal to the stable manifold). The only really "singular" stratum is now the unstable manifold U_p: the other strata are just the shadows of the strata in A^+, and their singularities are those of a manifold with corners, as one can prove locally using proposition 2.3.3 □

We can now state and prove the third "brick":

Proposition 2.3.5 *The previous proposition 2.3.4 holds word for word by replacing "compact manifold with corners" with "compact h-stratified set".*

Proof. Let $\mathcal{A}^+$ be a compact submanifold with boundary which desingularizes the h-stratified space A^+. It is not restrictive to suppose $\mathcal{A}^+ \subset \mathbb{R}^j \times \partial^+\Omega$ and the h-projection to be the restriction of the projection onto the second factor.

Introduce now the auxiliary flow ψ on $\mathbb{R}^j \times \mathbb{R}^n$ by extending ϕ via a linear contraction in the $\mathbb{R}^j$ components. In other words, $\psi_t(x', x) = (e^{-t}x', \phi_t(x))$ and the following diagram commutes

$$\begin{array}{ccc} \mathbb{R}^j \times \mathbb{R}^n & \xrightarrow{\psi_t} & \mathbb{R}^j \times \mathbb{R}^n \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{R}^n & \xrightarrow{\phi_t} & \mathbb{R}^n \end{array} \qquad (2.1)$$

The origin is a hyperbolic isolated singularity for the flow ψ; its stable manifold is $\mathbb{R}^j \times S_p$ and the unstable manifold is $\{0\} \times U_p$. Clearly the stable manifold "upstairs" is transversal to the desingularization $\mathcal{A}^+ \subset \mathbb{R}^j \times \partial^+\Omega$, which is a compact manifold with corners. The hypothesis of proposition 2.3.4 are fulfilled for the flow ψ by the "incoming link" $\mathcal{A}^+$ and therefore, putting

$$\mathcal{A} = \bigcup_{t>0} \psi_t(\mathcal{A}^+)$$

the closure $\overline{\mathcal{A}}$ (which is the shadow of $\mathcal{A}^+$) is a compact h-stratified set containing the unstable manifold $U_p \times \{0\}$ as singular stratum.

Let's now look to the closure $\overline{A}$ "downstairs". Since the diagram 2.1 commutes, it follows that $A = \pi(\mathcal{A})$ and hence $\overline{A} = L_{\Omega}^{A^+} = A \cup U_p$. The restriction of π on the singular stratum $\{0\} \times U_p$ is a trivial projection over U_p. Besides those, the other strata in $\overline{\mathcal{A}}$ and $\overline{A}$ are respectively the shadows of the strata in $\mathcal{A}^+$ and A^+. It readily comes out that the projection π restricts to be a submersion on any stratum.

To conclude that the $\overline{A}$ is h-stratified, we are left to check B-regularity (A-regularity following by the existence of the h-projection). Besides the stratum U_p, the regularity conditions for the other strata can be checked locally by applying proposition 2.3.3. On the other hand, condition B for U_p is a trivial consequence of the invariance of A (cf. the analogous argument in the proof of proposition 2.3.4).

Finally, the intersection $(A, U_p) \cap \partial^-\Omega$ is transversal and hence a compact h-stratified set too $\square$

We are now ready to prove theorem 2.3.1. Let's describe its idea, which consists of two steps.

First, given any broken flow line γ starting on M, we'll show that γ belongs to $\overline{N}$ and that there is a subset $\Omega(\gamma) \subset \overline{N}$ containing γ which is a h-stratified space and satisfies the thesis of the theorem.

Second, we'll show that $\overline{N}$ can be stratified arranging a locally finite family of these h-stratified subsets in such a way that the top strata don't overlap. This would complete the proof.

More precisely, in the first step we define $\Omega(\gamma)$ by cutting the shadow L^M along γ into "**pipes**" and "**pants**". The pipes are parts not containing any critical point, and can be described using proposition 2.3.3. The pants are parts containing a single critical point and propositions 2.3.4 and 2.3.5 will be invoked.

Each pipe and pant has two distinguished components in the boundary: the "**incoming link**" and the "**outgoing link**". For example, in the notations of proposition 2.3.3, the shadow $L^{A^-} \cap f^{-1}[a, b]$ is a pipe, its incoming link is $A^- \approx A^- \times 0$ and its outgoing link is $L^{A^-} \cap f^{-1}(a) \approx A^- \times 1$. On the other hand, in the terminology of proposition 2.3.4, the set $\overline{A}$ is a pant, A^+ is its incoming link and $A^- \cup U^-$ its outgoing link. Note that new singularities arise only within pants.

Proof of the theorem. Step 1.

Choose a broken flow line γ starting on M and let $p_1, \ldots, p_r$ be the critical points on γ so that the flow line can be decomposed in

$$\gamma = (\gamma_0, \gamma_1, ..., \gamma_k) \qquad p_i \xrightarrow{\gamma_i} p_{i+1}$$

connecting the points $x_0 \in M$, $p_1, ..., p_k$ (x_0 is not critical but the p_i's are); observe that each γ_i is contained in $S_{p_{i+1}}$ (and in U_{p_i} if $i \neq 0$). For each p_i it is assumed chosen a chart Ω_{p_i} in straighten coordinates and the intersections of its boundary with the flow line γ are denoted by:

$$x_i = \gamma \cap \partial^+\Omega_{p_i} \qquad y_i = \gamma \cap \partial^-\Omega_{p_i}$$

We next show how to cut off the pipes along the flow lines γ_i and the pants near the critical points p_i.

Let's start with the first pipe, along γ_0, recalling that M is a closed submanifold of the regular level set $f^{-1}(c)$, and any stable manifold is transversal to M.

Choose a closed ball B inside $f^{-1}(c)$, centered in x_0 and small enough so that B "flows" inside Ω_{p_1}, i.e. so that the all the flow lines starting on B meet Ω_{p_1} without breaks. Since the stable manifold S_{p_1} is closed near x_0 (the flow line γ not being broken up to p_1), S_{p_1} intersect transversally $M \cap \partial B$, provide B is small enough.

Moreover, since the flow has no fixed points on the shadow $L^{M\cap B}_{\partial^-\Omega_{p_1}}$, proposition 2.3.3 provides a diffeomorphism $L^{M\cap B}_{\partial^-\Omega_{p_1}} \approx M \cap B \times [0,1]$ obtained by trivializing the trajectories.

This shadow is the first pipe: it is a smooth manifold with corners, the incoming link is $M \cap B$ and the outgoing link A_1^+ (corresponding to $M \cap B \times \{1\}$ via the trivializing diffeomorphism) is a submanifold with boundary contained in $\partial^+\Omega_{p_1}$ and transversal to S_{p_1}.

The outgoing link A_1^+ of the pipe above coincides with the incoming link of the next pant, near p_1, which we denote by A_1. This pant is just the shadow of A_1^+ into Ω_{p_1}, i.e. $A_1 = L^{A_1^+}_{\Omega_{p_1}}$.

Since A_1^+ is a submanifold with boundary transversal to S_{p_1}, we can apply proposition 2.3.4 to desingularize the pant A_1 and to find its outgoing

link: it follows that A_1 is a h-stratified set containing U_{p_1} as singular stratum. The outgoing link $A_1^- = A_1 \cap \partial^- \Omega_{p_1}$ is h-stratified too and contains the link $U_{p_1}^-$ of the unstable manifold as singular stratum.
Differing from the connection pipe-pant, the incoming link for the successive pipe will not be all the outgoing link A_1^- of the present pant, but just a small portion of it, cut near γ.

We now proceed by finite induction to construct the pipes and pants along γ. Suppose constructed the pipes along $\gamma_1, \ldots, \gamma_{i-2}$ and the pants near $p_1, \ldots, p_{i-1}$ (this implies that the shadow of M has been desingularized near γ up to p_{i-1}). The only inductive hypothesis we need to keep track of is that the outgoing link A_{i-1}^- of the last pant is a h-stratified set containing the link of the unstable manifold $U_{p_{i-1}}^- = U_{p_{i-1}} \cap \partial^- \Omega_{p_1}$ as singular stratum.

The next critical point is p_i, joined to p_{i-1} by the unbroken flow line γ_{i-1}. The stable manifold S_{p_i} contains γ_{i-1} and is closed near $x_{i-1} = \gamma \cap \partial^- \Omega_{p_{i-1}}$. If B is a small ball in $\partial^- \Omega_{p_{i-1}}$ centered in x_{i-1}, then $(B, \partial B)$ and A_{i-1}^- intersect transversally; in particular their intersection is a h-stratification and contains a neighborhood of x_{i-1} inside $U_{p_{i-1}}^-$. This last fact and corollary 2.2.11 imply that, if B is small enough, then all the strata in $(B, \partial B) \cap A_{i-1}^-$ are transversal to S_{p_i}, since $U_{p_{i-1}}^-$ is. Finally, by possibly further shrinking B, we can also assume that $B \cap A_{i-1}^-$ "flows" into Ω_{p_i} without meeting other critical points.

Choosing such a ball B, the next pipe can be defined as the shadow $L_{\partial^+ \Omega_{p_i}}^{B \cap A_{i-1}^-}$, which is h-stratified as a trivial consequence of proposition 2.3.3. The outgoing link $A_i^+ = L_{\partial^+ \Omega_{p_i}}^{B \cap A_{i-1}^-} \cap \partial^+ \Omega_{p_i}$ is a h-stratified space transversal to the stable manifold S_{p_i}. Note that A_i^+ will be the incoming link for the next pant, as always at the connection pipe-pant.

The pipe near p_i, denoted by A_i, is the shadow $L_{\Omega_{p_i}}^{A_i^+}$. To complete step one, we apply proposition 2.3.5 to the incoming link A_i^+. It comes out that the pant A_i is h-stratified and contains the unstable manifold U_{p_i} as singular stratum. Moreover, the intersection with $\partial^- \Omega_{p_i}$ is transversal and hence the outgoing link of the pant A_i is a h-stratified space containing the link $U_{p_i}^-$ of the unstable manifold as singular stratum. This completes the inductive step and proves Step 1.

Step 2.

So far, we desingularized the shadow L^M along each compact broken flow line γ; it remains to prove that these desingularizations can be arranged into a locally finite family without overlappings on the top dimensional strata.

Fix any compact set K and any $b < c$ (recall $M \subset f^{-1}(c)$). Because of the Weakly Proper hypothesis, the shadow $L^K_{f^{-1}(b)}$ is compact, and hence it contains a finite number of critical points.

The set K can be covered by a finite family of small balls B_j in M whose top dimensional strata do not overlap (these balls are compact manifold with corners of the same dimension of M). Also, for any j we can assume that either B_j is disjoint by all the stable manifolds of the critical points in $L^K_{f^{-1}(b)}$ or there is one of these critical points, say p, such that B_i flows into $\partial^+\Omega_p$ without meeting any other critical point.

In the first case, the shadow $L^B_{f^{-1}(b)}$ is a pipe. In the second case, the shadow L^B can be decomposed in a pipe up to Ω_p, a pant near p and thereafter we can decompose the outgoing link (which is a compact stratified set contained in $\partial^-\Omega_{p_i}$) cutting it by a finite number of balls, as done for M near K. Proceeding by finite induction using Step 1, one constructs the desired stratification, completing the proof of Step 2.

Summarizing, we proved that the closure of $N = \bigcup_{t>0} \phi_t(M)$ is just the shadow L^M, it is a stratified set whose singular strata are unstable manifolds and we showed, by passing to a refinement, that the stratification is horned; the proof of the theorem is complete $\Box$

As a byproduct of the previous proof, we can describe the current boundary on N.

Suppose X is oriented. For any critical point p choose an arbitrary orientation to the stable manifold U_p so that an orientation is induced on S_p by requiring that $< X >=< U_p > + < S_p >$. Suppose finally M is oriented too and orient N by $< N >=< M > + < L >$, where $< L >$ is the direction of the flow. Observe that U_p, S_p, M and N all define horned chain currents. In particular, if M has dimension k, then the current $[N]$ has dimension $k+1$ and, since the singular strata of N are unstable manifolds, $d[N]$ is a sum with integer coefficients of unstable manifolds of critical points of index $n-k$ (by dimension restrictions). We now sketch how to compute this coefficients, referring to [13] for more details.

The Smale and Weakly Proper hypothesis imply that the set of points in M joined to a critical point p of index $n-k$ is finite, the intersection $S_p \cap M$ being discrete and compact. Hence, for any p of index $n-k$ there is a finite number of flow lines γ from M to p.

For each of these γ, the previous proof implies that the h-projection desingularizing dN is a diffeomorphism onto the stratum U_p. This means that the contribution of "γ" to the boundary of N is just $\pm[U_p]$, the correct sign being determined by the orientations.

Corollary 2.3.6 *In the notations of the previous theorem, suppose X, M and all the stable and unstable manifolds are oriented (in such a way that $< X >=< S_p > + < U_p >$ at each critical point p). Then $[M]$ and $[N]$ are horned chain currents and*

$$\partial [N] = [M] - \sum c_q [U_q]$$

The sum goes trough all the critical points q of index $n - dimM$ and for any such q, the coefficient c_q "counts with orientation" the number of flow lines joining M and q.

Of course a similar statement holds for backward shadows, by replacing unstable manifolds by stable manifolds. In particular, for any critical point p, the current boundary of the stable manifold S_p is given by:

$$\partial [S_p] = \sum c_q^p [S_q]$$

Important Remark If the manifold is compact, the coefficients c_q^p in the previous formula are exactly the same we described in the construction of the Morse Complex in section 1.1. This implies that replacing a critical point by its stable manifold provides a "geometric" realization of the Morse Complex as a subcomplex of the complex of currents (interpreting the boundary in the Morse complex as the boundary of currents).

This embedding of the Morse Complex suggests a way to define a generalized Morse complex in a noncompact setting. This will be the subject of the next chapter.

2.4 Volume bound for the Flow and the Fundamental Morse Equation

The aim of this section is to use theorem 2.3.1 to find a bound for the volume of a Weakly Proper, Smale flow.

We need to prove that the submanifold $T \subset X \times X$ defined in section 1.2 (the graph of the flow) has locally finite volume. We actually show much more: T is a horned stratified set and its current boundary will be described.

Consider the height function h on the circle S^1 and put:

$$H : S^1 \times X \times X \to \mathbb{R}$$
$$H(\theta, x, y) = h(\theta) - f(x) + f(y)$$

The flow Ψ on $S^1 \times X \times X$ given by:

$$\Psi_t(s,x,y) = \begin{cases} \left(e^{-t}s, \phi_t(x), \phi_{-t}(y)\right) & \text{for s coordinate near S} \\ \left(e^{t}s, \phi_t(x), \phi_{-t}(y)\right) & \text{for s coordinate near N} \end{cases}$$

is gradient like with respect to H (here S and N are the South and North pole of S^1).

Lemma 2.4.1 *The dynamical system $(S^1 \times X \times X, \Psi, H)$ is Weakly Proper and Smale.*

Proof. The critical points of Ψ are of the kind (S,p,q) and (N,p,q), for p and q critical points of ϕ, and the corresponding stable and unstable manifolds at such points are:

$$\begin{array}{ll} S_{(S,p,q)} = S^1 \times S_p \times U_q & S_{(N,p,q)} = \{N\} \times S_p \times U_q \\ U_{(S,p,q)} = \{S\} \times U_p \times S_q & U_{(N,p,q)} = S^1 \times U_p \times S_q \end{array}$$

Clearly all possible intersection are transversal, thus insuring the Smale hypothesis. It remains to prove the Weakly Proper condition. By contradiction, let γ be a non compact flow line for Ψ contained in a slab $H^{-1}[a,b]$; it is not restrictive to choose γ unbroken. Since the first coordinate is bounded, the lack of compactedness comes from the x or y variable. But the contribute by x and y to H are of the same sign (either for positive times or for negative ones) and so the weakly proper hypothesis for ϕ on X implies the same hypothesis for Ψ $\square$

Let now $1 \in S^1$ be the point whose coordinate in the chart containing the south pole is 1 and consider the subset

$$\mathcal{T} = \{(e^{-t}, \phi_t(x), \phi_{-t}(x)) \,|\, t \geq 0,\, x \in X\} \subset S^1 \times X \times X$$

Observe that

$$\mathcal{T} = \bigcup_{t \geq 0} \Psi_t(\{1\} \times \Delta)$$

and that $\{1\} \times \Delta$ is contained in the level set of $1/2$, which we can suppose to be regular for H (if not, start from another point in S^1).

Lemma 2.4.2 *The submanifold $\{1\} \times \Delta$ is transversal to all the stable manifolds.*

Proof. First observe that $\mathcal{T}$ does not intersect any stable manifold of the critical points (N,p,q). The remaining stable manifolds are of the form $S^1 \times S_p \times U_q$ and transversality for $S_p \times U_q \cap \Delta$ is exactly the same then transversality of $S_p \cap U_q$ in X, the Smale hypothesis proves the lemma $\square$

Proposition 2.4.3 *The closure of the submanifold $\mathcal{T}$ is a horned stratified set with singular strata $\{1\} \times \Delta$ and $U_{(S,p,q)}$ for any critical points $q \preccurlyeq p$. The boundary of the corresponding chain current $[\mathcal{T}]$ is:*

$$\partial[\mathcal{T}] = \{1\} \times \Delta - \sum [U_{(S,p,p)}] \tag{2.2}$$

Proof. The previous lemma and theorem 2.3.1 proves the first statement. on the other hand, the boundary value technique implies that, near each (S,p,p), the set $\mathcal{T}$ is a manifold with boundary given by $\{1\} \times \Delta \cup \{S\} \times U_p \times S_q$ (cf. theorem 1.3.21), so the equation in the statement holds near points (S,p,p) (recall S is just the south pole in the circle S^1).

The current $\mathcal{T}$ is a horned chain and its boundary is thus a sum of singular strata. By dimension restrictions, the only admissible ones are exactly $\{1\} \times \Delta$ and $U_{(S,p,p)}$, thanks to corollary 2.3.6. Since the equation holds near (S,p,p), it then holds everywhere. □

We can now state the main theorem:

Theorem 2.4.4 *A Weakly Proper Smale flow has locally finite volume. In fact the closure of the "total graph" T is a horned stratified set whose singular strata are Δ and $U_p \times S_q$ for any critical points $q \preccurlyeq p$.*

*The following "**Fundamental Morse Equation**" of h-chain currents holds in $X \times X$:*

$$dT = \Delta - P \tag{FME}$$

where (denoting by Cr the set of critical points)

$$P = \sum_{p \in Cr} [U_p \times S_p] \tag{2.3}$$

Proof. Continuing with the previous notations, the projection onto the second factor $\pi : S^1 \times X \times X \to X \times X$ maps $\mathcal{T}$ onto the "total flow" T. The projection is a diffeomorphism on the top dimensional stratum of $\mathcal{T}$, and a trivial submersion on the singular strata $\{1\} \times \Delta$ and $U_{(S,p,q)} = \{S\} \times U_p \times S_q$ (here S is just the south pole in the circle and $q \preccurlyeq p$).

Since π is proper and $\mathcal{T}$ is h-stratified, to prove that $\overline{T}$ is h-stratified too it remains to show the A-B regularity. By the remark after definition 2.2.1 it's enough to prove "B". This is trivial along the stratum Δ (the singularity is smooth here). Along the other strata $U_p \times S_q$, the argument is similar to the corresponding one in the proof of theorem 2.3.1, by using the fact that T is made up of flow lines for the (auxiliary) flow (ϕ_t, ϕ_{-t}) on $X \times X$. Note that Δ is not contained in a regular level set for this auxiliary flow, that's why we needed to introduce $\mathcal{T}$.

It's thus proved that T is horned stratified too and the corresponding h-chain current is just $T = \pi_* \mathcal{T}$. The equations in the statement then follow by pushforwarding the relation 2.2 □

We end this section by applying the “locally finite volume” technique to the previous equation. Using the kernel calculus, let

$$\mathbf{T}, \mathbf{I}, \mathbf{P} : \mathcal{E}^*_{cpt}(Y) \quad \longrightarrow \quad \mathcal{D}'^*(Y)$$

be the operators whose kernels are T, Δ and P (recall $\mathcal{E}$ denotes smooth forms and $\mathcal{D}'$ currents).

Proposition 2.4.5 *For any Weakly Proper, Smale dynamical system (X, ϕ), and any test form α on X, the limit and the integral:*

$$\mathbf{P}(\alpha) = \lim_{t \to +\infty} \phi_t^*(\alpha) = \sum_{p \in Cr} \left(\int_{U_p} \alpha \right) [S_p] \quad , \quad \mathbf{T}(\alpha) = \int_{\mathbb{R}^+} \phi_t^*(\alpha) \tag{2.4}$$

converge as currents and the following equation holds:

$$d(\mathbf{T}(\alpha)) + \mathbf{T}(d\alpha) = \alpha - \mathbf{P}(\alpha) \tag{2.5}$$

Proof. The only nontrivial fact is the integral expression for $\mathbf{T}$. Since the current $T = \Phi_*(\mathbb{R}^+ \times X)$, where $\Phi(t, x) = (\phi_t(x), x)$, for any forms α and β on X it is:

$$T(\alpha \otimes \beta) = \int_{\mathbb{R}^+ \times X} \Phi^*(\alpha \wedge \beta) = \int_{\mathbb{R}^+ \times X} \phi_t^*(\alpha) \wedge \beta = \int_X \left(\int_{\mathbb{R}^+} \phi_t^*(\alpha) \right) \wedge \beta$$

and this proves the expression 2.4 for $\mathbf{T}$. Note that the current $\mathbf{T}(\alpha)$ is clearly smooth away from the stable manifolds of critical points in the backward shadow of the support of α □

Remark As observed in section 1.2 about the finite volume technique, equation 2.5 is not so relevant if X is not compact. In fact, it provides a chain homotopy between $\mathbf{I}$ and $\mathbf{P}$ but $\mathbf{I}$ does not induce any algebraically interesting map in topology, hence $\mathbf{P}$ doesn't as well. Nevertheless, one can extend the operators involved to act on spaces where this equation regains power. This will be done in the next section by introducing the “forward supports”.

Chapter 3

Non Compact Morse Theory

In this chapter, starting from the Fundamental Morse Equation proved in the last chapter, we'll derive an equation of operators called the Morse Chain Homotopy. As the name suggests, the new equation provides a chain homotopy between a complex of forms with some restriction on the support and a non compact analogous of the Morse complex. A noncompact Morse theory is established, relating the dynamics of a Smale, Weakly Proper flow to the spaces of cohomology with "forward supports" of the manifold. Those are often infinite dimensional and endowed with a natural locally convex topology. The standard trick of inverting the flow leads to a forward-backward duality.

All over the chapter we assume that X is an oriented n-dimensional manifold, endowed with a Weakly Proper, Smale dynamical system (ϕ, f). An arbitrary orientation is assumed chosen for the unstable manifold of each critical point, inducing an orientation on the stable manifolds in such a way that $< X >=< S_p > + < U_p >$ at each critical point p.

3.1 Forward Supports and the Morse Chain Homotopy

We here study the operators $\mathbf{P}$ and $\mathbf{T}$ introduced in the last chapter. The structure and position of the stable and unstable manifolds and the expressions 2.4 suggests that $\mathbf{P}$ and $\mathbf{T}$ extend to forms which are supported in $f^{-1}[a, +\infty)$ for some $a \in \mathbb{R}$ and it's quite obvious that the range of the operator consists of currents supported in the same way. This justifies the following definition.

Definition 3.1.1 *A closed set $A \subset X$ is a **compact/forward** set (abbreviated c/f set) with respect to the function f if both*

- $A \cap f^{-1}([b, c])$ is compact for any $b \leq c \in \mathbb{R}$ (i.e. A is **slab compact**)

- $A \subset f^{-1}([a, +\infty))$ for some constant $a \in \mathbb{R}$ (i.e. A is **forward**).

One can defined ***backward*** *and* ***compact/backward*** *sets in a similar manner. Of course, a closed set is forward (resp. backward) if and only if it has compact intersection with all the compact/backward (resp. compact/forward) sets.*

Remark (stability) *Suppose f_0 and f_1 are two weakly proper functions on X whose difference is bounded (say by the constant $c \geq 0$). Then f_0 and f_1 determine the same family of compact/forward sets.*

Since $f_1^{-1}([-\infty, a)) \subset f_0^{-1}([-\infty, a+c))$ for any $a \in \mathbb{R}$, if A is c/f with respect to f_0 then A is also c/f with respect to f_1. Actually, both the notions forward set and slab compact (cf definition of c/f set) are the same for f_0 and f_1.

The subscript $\uparrow$ will denote the family of forward sets while $c\uparrow$ or c/f will denote the compact/forward ones; analogously for backward sets. For example, $\mathcal{E}^*_{\uparrow}(X) = \Gamma_{\uparrow}(X, \mathcal{E}^*)$ and $\mathcal{E}^*_{c\uparrow}(X) = \Gamma_{c\uparrow}(X, \mathcal{E}^*)$ denote the space of smooth forms with forward and compact/backward support, respectively.

Clearly either of the forward, backward, c/f and c/b family of sets is a paracompactifying family for X in the terminology of Godement ([G]). It makes thus sense to compute (sheaf) cohomology with supports in such families of sets. For example, one can compute the cohomology of X with real coefficients (i.e. the cohomology of the locally constant sheaf $\mathbb{R}$) and forward supports via deRham theorem, using either smooth forms or currents as:

$$H^k_{\uparrow}(X, \mathbb{R}) \approx \frac{\{\omega \in \mathcal{E}^k_{\uparrow}(X) \mid d\omega = 0\}}{\{d\eta \mid \eta \in \mathcal{E}^{k-1}_{\uparrow}(X)\}} \approx \frac{\{\theta \in \mathcal{D}'^k_{\uparrow}(X) \mid d\theta = 0\}}{\{d\rho \mid \rho \in \mathcal{D}'^{k-1}_{\uparrow}(X)\}}$$

Note that the stable manifolds have compact/forward support, whereas the unstable manifolds are compact/backward ("compact" because of the Weakly Proper hypothesis).

As one expects, the following duality hold:

Proposition 3.1.2 *The currents with compact/forward support are the continuous linear functionals on the space of forms with backward support, and similarly the forward-supported currents are functionals over the compact/backward forms:*

$$\left(\mathcal{E}^k_{\downarrow}(X)\right)' = \mathcal{D}'^k_{c\uparrow}(X) \qquad \left(\mathcal{E}^k_{c\downarrow}(X)\right)' = \mathcal{D}'^k_{\uparrow}(X)$$

Proof. We just prove the first statement. For $j = 1, 2 \ldots$ the closed sets $B_j = f^{-1}(-\infty, j]$ form an exhaustive sequence of backward sets. Now each $\mathcal{E}^*_{B_j}(X)$ (the space of forms with support contained in B_j) is a closed subspace of the Frechet space $\mathcal{E}^*(X)$. Therefore

$$\mathcal{E}^*_{\downarrow}(X) = \varinjlim_{j \to +\infty} \mathcal{E}^*_{B_j}(X)$$

is the strict inductive limit of Frechet spaces (in particular, an LF-space).

Since the inclusion map $\mathcal{E}^k_{cpt}(X) \hookrightarrow \mathcal{E}^k_{\downarrow}(X)$ is continuous with dense range, the adjoint map $\mathcal{E}^k_{\downarrow}(X)' \hookrightarrow \mathcal{D}'^k(X)$ is similar (i.e. 1-1, continuous, with dense range). In particular each continuous linear functional on $\mathcal{E}^k_{\downarrow}(X)$ is a current. Now if $T \in \mathcal{E}^k_{\downarrow}(X)' \subset \mathcal{D}'^k(X)$ and $\mathrm{spt}(T) \cap B_j$ is not compact (for some j), there exist a sequence of pairwise disjoint open balls $U_n \subset B_j$ whose centers don't accumulate. Pick $\varphi_n \in \mathcal{E}^k_{cpt}(U_n)$ with $T(\varphi_n) = 1$. Since $\varphi = \sum \varphi_n \in \mathcal{E}^k_{\downarrow}(X)$ but $T(\varphi)$ is not defined this is a contradiction, proving that spt$T$ is compact forward.

Conversely, if $T \in \mathcal{D}'^k_{c\uparrow}(X)$ has c/f support then $T(\varphi)$ is defined for all $\varphi \in \mathcal{E}^k_{\downarrow}(X)$, since $\mathrm{spt}(T) \cap \mathrm{spt}(\varphi)$ is compact. Also, the maps $T : \mathcal{E}^k_{B_j}(X) \to \mathbb{R}$ are continuous. That is, $\mathcal{D}'^k_{c\uparrow}(X) \subset \mathcal{E}^k_{\downarrow}(X)'$ $\Box$

We now turn back to the kernel currents T, Δ, P and their associated operators $\mathbf{T}$, $\mathbf{I}$ and $\mathbf{P}$. Let's introduce a terminology which will only appear in the next lemma: a set $A \times B \subset X \times X$ will be said "compact/forward-backward" (abridged cf-b) if A is compact forward and B is backward in X and analogously one defines "forward-compact/backward" product sets.

Lemma 3.1.3 *The support of T, Δ and P has compact intersection with any product subset of $X \times X$ which is compact/forward-backward or forward-compact/backward and hence the three currents act on any form with such support.*

Proof. It suffices to prove the statement for T, since the supports of P and Δ are contained in spt$T$. Let A be forward and B compact/backward in X. Choose a sequence $(x_n, y_n) \in (A \times B) \cap \mathrm{spt}(T)$ and suppose that (x_n, y_n) doesn't accumulate. The points $x_n \geq y_n$ lie on the same (broken) flow line. There are two cases:

- If the y_n don't accumulate, then necessarily $f(y_n)$ is not bounded below (since B is c/b) and therefore also $f(x_n) \leq f(y_n)$ is unbounded below, which contradicts x_n to be a forward set.
- If the y_n accumulate in y, it's $f(x_n) \leq f(y) + \epsilon$ and hence the points x_n lie in some slab $f^{-1}[-a, f(y) + \epsilon]$, since A is forward. The weakly proper hypothesis then implies that the points x_n necessarily accumulate since the y_n do. An analogous argument proves the c/f-c case $\Box$

By the very construction of the operator associated to a kernel current, the previous lemma, together with proposition 2.4.5 proves the next theorem:

Theorem 3.1.4 *The operators* $\mathbf{T}$, $\mathbf{I}$ *and* $\mathbf{P}$ *associated to the kernel currents* T, Δ *and* P, *which a priori are operators from*

$$\mathcal{E}^*_{cpt}(X) \quad \longrightarrow \quad \mathcal{D}'^*(X)$$

extend to continuous operators from

$$\mathcal{E}^*_{\uparrow}(X) \quad \longrightarrow \quad \mathcal{D}'^*_{\uparrow}(X)$$

and satisfy the following equation, which we call ***Morse Chain Homotopy****:*

$$d \circ \mathbf{T} + \mathbf{T} \circ d = \mathbf{I} - \mathbf{P} \tag{3.1}$$

In particular, for any forwardly supported form φ*:*

$$\mathbf{P}(\varphi) = \lim_{t \to +\infty} \phi_t^*(\varphi) = \sum_{p \in Cr} \left(\int_{U_p} \varphi \right) [S_p] \tag{3.2}$$

The statement remains true by replacing "forward" with "compact/forward" everywhere.

We also get the important:

Corollary 3.1.5 *The operators* $\mathbf{T}$, $\mathbf{I}$ *and* $\mathbf{P}$ *act on a local chain current* K *on* X *with forward support (resp. c/f) provide* K *is transversal to any unstable manifold. The result is a local chain current with forward (resp. c/f) support. The equation of chain currents*

$$d(\mathbf{T}K) + \mathbf{T}(dK) = K - \mathbf{P}(K) \tag{3.3}$$

holds as well as the expression:

$$\mathbf{P}(K) = \lim_{t \to +\infty} \phi_t^*(K) = \sum_{p \in Cr} ([U_p] \bullet K)\, [S_p] \tag{3.4}$$

where the coefficients in the previous sum are integers and vanish if $dimK \neq dimS_p$. *In particular,* $\mathbf{P}$ *acts as the identity on any stable manifold* S_p. *All the statements remains true by replacing "chain" with "horned chain".*

Observe that, by definition, it's $([U_p] \bullet K) = [U_p \cap K](1) \in \mathbb{Z}$.

Proof. Note first that for any point x, the intersection $\{x\} \times X \cap \Delta$ is transversal. Also, for any point (y, x) in the interior of T, with $y = \phi_t(x)$, using the automorphism (ϕ_t, id) on $X \times X$ it's not difficult to prove that the intersection $\{x\} \times X \cap T$ is transversal. Finally, given a submanifold $K \subset X$ and two critical points p, q, the intersection $K \times X \cap U_q \times S_p$ is transversal if and only if K is transversal to U_q. Since all the singular strata in $\overline{T}$ are of this form, the first statement follows from proposition 2.2.15.

Now, suppose that the dimension of K is k and α is a smooth form of degree k on X. By lemma 3.1.3, if the support of K is forward in X, then $T \cap K \times X$ has compact intersection with any set of the form $X \times B$ provide B is compact/backward. The relation $[T \cap (K \times X)](1(x) \wedge \alpha(y)) = [T](K(x) \wedge \alpha(y)) = [\mathbf{T}(K)](\alpha)$ proves that the operator $\mathbf{T}$ extends to K and $\mathbf{T}(K)$ is a chain current with forward support. Exactly the same argument works for P and Δ and for the cases when K has compact/forward support (obtaining a chain with compact forward support) or is horned (obtaining horned chains).

The last statement holds since for any pair of critical points (p, q) it's $[U_p] \bullet [S_q] = [U_p \cap S_q](1) = \delta^{pq}$ □

Remark The chain $T(K)$ is the "backward shadow" of K, cf. definition 2.1.1.

3.2 Realization of the Morse Complex

Let's start with the definition:

Definition 3.2.1 *The **(compact/forward) $\mathcal{S}$-complex over** $\mathbb{Z}$, denoted by ${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(\phi)$ is the subcomplex of $\mathcal{D}'^*_{c\uparrow}(X)$ (the complex of currents with c/f support) consisting of those currents of the form*

$$\sum_{p \in F} a_p\, [S_p] \quad \text{where F is a c/f set of critical points and } a_p \in \mathbb{Z}$$

The boundary $d : {}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(\phi) \to {}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(\phi)$ is the current boundary.

*Similarly we define ${}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(\phi)$ (the **$\mathcal{S}$-complex over** $\mathbb{R}$) and $\mathcal{S}^*_{\uparrow}(\phi)$ (the **forward $\mathcal{S}$-complex**).*

Note that each element of ${}_{\mathbb{Z}}\mathcal{S}^*_{\uparrow}(f)$ is the sum of a locally finite family of stable manifolds and it is a horned chain (cf. corollary 2.3.6). We will sometime skip the explicit refence to the flow ϕ and just write ${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}$. The following is a direct consequence of corollary 3.1.5.

Lemma 3.2.2 *The operators $\mathbf{T}$ and $\mathbf{P}$ extend to act on ${}_{\mathbb{R}}\mathcal{S}^*_{\uparrow}$. In particular $\mathbf{P}$ acts as the identity on ${}_{\mathbb{R}}\mathcal{S}^*_{\uparrow}$.*

We can now state the main theorem of the chapter:

Theorem 3.2.3 *The maps*

$$\mathbf{P} : \mathcal{E}^*_{c\uparrow}(X) \longrightarrow {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(\phi) \quad \text{and the inclusion} \quad \mathbf{J} : {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(\phi) \hookrightarrow \mathcal{D}'^*_{c\uparrow}(X)$$

induces isomorphisms in cohomology:

$$H^p_{c\uparrow}(X, \mathbb{R}) \approx H^p\left({}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(\phi)\right)$$

The statement also holds by replacing compact/forward with forward everywhere.

Proof. Since $\mathbf{J} \circ \mathbf{P}$ is homotopic to $\mathbf{I}$ and the latter induces an isomorphism in cohomology, we just need to show that $\mathbf{P}$ is surjective. This means that if $S \in {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}$ is closed than there is a closed $\varphi \in \mathcal{E}^*_{c\uparrow}(X)$ such that $P(\varphi) = S$. This fact is a little technical, though not very difficult and the proof is a refinement of an argument in [13]. Suppose $S \in {}_{\mathbb{R}}\mathcal{S}^k_{c\uparrow}$ is closed and $f(S) \geq 0$. The idea is to consider slabs $f^{-1}([0,n])$ and find a sequence of forms φ_n with compact forward support on X such that:

- φ_n is closed
- $\mathrm{spt}(\varphi_n) \subset f^{-1}([n, +\infty[)$ and it is compact/forward
- $\mathbf{P}\left(\sum_{i=1}^n (\varphi_i)\right) - S$ vanishes on $f^{-1}([0,n])$

In this case $\varphi = \sum \varphi_n$ would define a closed form with c/f support and $P(\varphi) = S$.

Recall $k = dimS$ and let R_{k+1} be the set of critical points of index $k+1$ contained in the slab $f^{-1}([0,1])$. The unstable manifolds of these critical points don't accumulate and are disjoint from $\mathrm{spt}(S)$. We can therefore choose a neighborood A of $\mathrm{spt}S$ with empty intersection with the unstable manifolds of critical points in R_{k+1}. Of course we can (and do) also suppose $\overline{A}$ to be compact forward since S is. Consider now the family of closed sets well contained in A (i.e. the closed B such that there exists C open for which $B \subset C \subset \overline{C} \subset A$). By deRham theory, one can compute cohomology of A with such supports using either smooth forms or currents. Therefore there exists a closed smooth form φ_1 and a current θ such that both have support well contained in A and satisfy:

$$S - \varphi_1 = d\theta$$

The form φ_1 and the current θ extend to X respectively to a closed smooth form and a current, both with compact forward support contained in $f^{-1}([0, +\infty[)$. We now want to prove that $P(\varphi_i) - S$ vanishes on $f^{-1}([0,1])$. Observe that for any critical point q of order k in A, the corresponing unstable manifold U_q is closed (and a closed current) in A, since the boundary of an unstable manifold is another unstable manifold and we already cut out those of the fitting dimension. Also, the family of unstable manifolds of critical points of order k is discrete in A, since contained in a slab. Therefore, within A, one can present any of these U_q as a limit of smooth closed forms $u_q^\varepsilon \rightharpoonup U_q$ (even if such forms might not extend to closed forms on all of X). Now, the operator $\mathbf{P}$ acts on $S - \varphi_1$ and hence on $d\theta$. Moreover, since there is no unstable manifold of dimension $n-k-1 = deg(\theta)$, it is:

$$[\mathbf{P}(d\theta)]|_{f^{-1}[0,1]} = \sum_{0\leq f(p)\leq 1} ([U_p](d\theta))\,[S_p] = \sum_{0\leq f(p)\leq 1} \lim_{\varepsilon} \left([u_p^\varepsilon](d\theta)\right)[S_p] = 0$$

where the last equality holds since the forms u^ε are closed on the support of $d\theta$. Replacing S by $S - \mathbf{P}(\sum_{i=1}^{r} \varphi_i)$, the same argument proves the r^{th} inductive step. The proof is similar in the forward case □

As for integer coefficients, one can use the $\mathcal{S}$-complex over $\mathbb{Z}$ to compute $H^*_{c\uparrow}(X, \mathbb{Z})$.

Theorem 3.2.4 *The maps*

$$\mathbf{P} : \mathcal{C}^*_{c\uparrow}(X) \dashrightarrow {}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(\phi) \quad \textit{and the inclusion} \quad \mathbf{J} : {}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(\phi) \hookrightarrow \mathcal{C}^*_{c\uparrow}(X)$$

induce isomorphisms in cohomology

$$H^p\left({}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}\right) \approx H^p_{c\uparrow}(X, \mathbb{Z})$$

The $\mathcal{S}$-complex over $\mathbb{Z}$ thus computes integral cohomology with forward supports. Consequently, if ${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}$ is a finitely generated group, then so is $H^*_{c\uparrow}(X, \mathbb{Z})$ and the Morse inequalities can be derived (the strong inequalities over $\mathbb{Z}$) as in the classic case.

Remark The dashed map in the previous statement is of course defined only on chains transversal to the unstable manifolds.

Proof. By lemma 3.2.2, and since the operator $\mathbf{P}$ commutes with the boundary and has range in $\mathcal{S}$, it readily follows that it's enough to prove that in each class $[K] \in H^p(\mathcal{C}^*)$ there is a representative K' in the domain of both the operators $\mathbf{P}$ and $\mathbf{T}$ since in that case the Morse chain homotopy 3.3 applies.

It is thus necessary, given a chain current K, to find a chain K' which is transversal to all the unstable manifolds and is homologous to K. Since the unstable manifolds make up an A-B regular stratification of X, the maps transversal to all the unstable manifolds are an open and dense set among the possible ones (cf. proposition B.0.10). The result then follows by pushforward of a 1-parameter deformation. □

3.3 Forward-Backward Duality

So far we considered the forward semigroup of the flow ϕ; in this section we'll use the backward semigroup to obtain a duality.

Denote by $\mathcal{U}^*_\downarrow$ the subcomplex made up of all the currents of the form $\sum_{p\in B} a_p [U_p]$ where B is a backward set of critical points. These currents are backward supported, of course.

Since all the hypotheses were symmetric in time, the result so far established for the asymptotic limits of the flow ϕ_t can be as well obtained for the limits at $-\infty$, using the flow with reversed time ϕ_{-t}. In particular, what has been proven for the $\mathcal{S}$ complex holds for the complex $\mathcal{U}$ with obvious modifications.

Flowing backwards in time produces hence a projection operator from $\mathcal{E}^*_\downarrow(X)$ to $\mathcal{U}^*_\downarrow(\phi)$ which is chain homotopic to the identity. Thus

$$H^p_\downarrow(X,\mathbb{R}) \approx H^p\left(\mathcal{E}^*_\downarrow(X)\right) \approx H^p\left(\mathcal{U}^*_\downarrow(\phi)\right)$$

Recall (cf. proposition 3.1.2) that if $B_j = f^{-1}((-\infty, j])$:

$$\mathcal{E}^*_\downarrow(X) = \varinjlim_{j\to+\infty} \mathcal{E}^*_{B_j}(X)$$

and the complexes $\mathcal{E}^*_\downarrow(X)$ and $\mathcal{D}'^*_{c\uparrow}(X)$ are dual complexes. Once we know that $d : \mathcal{E}^{p-1}_\downarrow(X) \longrightarrow \mathcal{E}^p_\downarrow(X)$ and $d : {\mathcal{D}'_{c\uparrow}}^{q-1}(X) \longrightarrow {\mathcal{D}'_{c\uparrow}}^{q}(X)$ have closed range, an elementary Hahn-Banach argument would establish the duality

$$\left(H^p\left(\mathcal{E}^*_\downarrow(X)\right)\right)' \approx H^{n-p}\left(\mathcal{D}'^*_{c\uparrow}(X)\right)$$

This same argument establishes DeRham duality and Serre duality under the closed range hypothesis.

Now, $\mathcal{U}^*_\downarrow(\phi) = \varinjlim_{j\to+\infty} \mathcal{U}^*_{B_j}(X)$ where $\mathcal{U}^*_{B_j}(X)$ is a closed subspace of $\mathcal{D}'^*_{B_j}(X)$. Equivalently, $\mathcal{U}^*_{B_j}(X)$ can be defined as the infinite product $\prod\limits_{p\in B_j\cap Cr(\phi)} \mathbb{R}$. All linear subspaces of an infinite product of $\mathbb{R}$'s are closed subspaces. Therefore $d : \mathcal{U}^*_{B_j}(X) \longrightarrow \mathcal{U}^*_{B_j}(X)$ automatically has closed range. This proves that $d : \mathcal{U}^*_\downarrow(\phi) \longrightarrow \mathcal{U}^*_\downarrow(\phi)$ has closed range. The isomorphisms $H^p\left(\mathcal{E}^*_\downarrow(X)\right) = H^p\left(\mathcal{U}^*_\downarrow(\phi)\right)$ and $H^p\left(\mathcal{E}^*_\downarrow(X)\right) = H^p\left(\mathcal{D}'^*_\downarrow(X)\right)$, along with the chain homotopy $\mathbf{P}$ and the inclusion $\mathbf{I}$ implies that if $\varphi \in \mathcal{E}_\downarrow(X)$ is d-closed and is mapped by $\mathbf{P}$ to an exact current, then φ is exact in $\mathcal{E}_\downarrow(X)$. This proves that

$$\mathbf{P}^{-1}(\mathrm{im}(d)) \cap \ker(d) = \mathrm{im}(d)$$

and hence, by the continuity of the projection $\mathbf{P} : \mathcal{E}^*_\downarrow(X) \longrightarrow \mathcal{U}^*_\downarrow(\phi)$, it follows that $d : \mathcal{E}^{p-1}_\downarrow(X) \longrightarrow \mathcal{E}^p_\downarrow(X)$ has closed range. Note that the previous isomorphism $H^*\left(\mathcal{E}^*_{\downarrow}(X)\right) \approx H^*\left(\mathcal{U}^*_{\downarrow}(\phi)\right)$ is an isomorphism of locally convex linear topological spaces.

The situation is similar for compact/forward cohomology. The families of compact/forward sets and backward sets are characterized by the fact that intersections are compact. Choose a countable exhaustive sequence F_j of c/f sets for X. Again,

$$\mathcal{E}^*_{c\uparrow}(X) = \varinjlim_{j\to+\infty} \mathcal{E}^*_{F_j}(X)$$

is the strict inductive limit of Frechet spaces, and

$$\mathcal{E}^*_{c\uparrow}(X)' = \mathcal{D}'^*_{\downarrow}(X)$$

As before, using the Morse Chain Homotopy and the fact that $d : \mathcal{S}^*_{c\uparrow}(X) \longrightarrow \mathcal{U}^*_{c\uparrow}(X)$ has closed range, it follows that $d : \mathcal{D}'^{\,q-1}_{c\uparrow}(X) \to \mathcal{D}'^{\,q}_{c\uparrow}(X)$ has closed range. We just proved:

Theorem 3.3.1 (Duality) *The spaces*

$$H^p_{\downarrow}(X,\mathbb{R}) \qquad \text{and} \qquad H^{n-p}_{c\uparrow}(X,\mathbb{R})$$

are dual locally convex linear topological vector spaces.

The duality can be simply interpreted via the representatives in the $\mathcal{S}$ and $\mathcal{U}$ complexes since $\mathcal{U}^*_{\downarrow}$ and $\mathcal{S}^*_{c\uparrow}$ are in "perfect" duality.

First $(U_p, S_q) = \delta_{pq}$ for all $p, q \in Cr(\phi)$. Second, if $U = \sum_{p\in B} a_p [U_p]$ and $S = \sum_{q\in F} b_q [S_q]$,where B is a backward set of critical points and F is a c/f set of critical points, then $B \cap F$ is finite and $(U,S) = \sum_{p\in B\cap F} a_p b_p$. Using integers coefficients this yields the following:

Corollary 3.3.2 *The groups $H^p_{\downarrow}(X,\mathbb{Z})$ and $H^{n-p}_{c\uparrow}(X,\mathbb{Z})$ are dual.*

Remarks

a) If f is a proper exhaustion function, then backward sets are compact sets and compact/forward ones are closed sets. Consequently the forward backward duality is just ordinary DeRham duality between $H^p\left(\mathcal{E}^*_{cpt}(X)\right)$ and $H^{n-p}\left(\mathcal{D}'^*(X)\right)$.

b) If f is a bounded function then all closed sets are backward and compact/forward is the same as compact. Again this duality reduces to the other case of ordinary DeRham duality between $H^p\left(\mathcal{E}^*(X)\right)$ and $H^{n-p}\left(\mathcal{D}'^*_{cpt}(X)\right)$.

Chapter 4

Novikov Theory Revisited

Morse Novikov theory is a variation of non compact Morse theory, governed by the addition of the action of a certain (Novikov) ring on subsets of the ambient manifold. This action commutes with the flow and the complexes of forms and currents are endowed with a natural structure of modules over the Novikov ring. We will first consider the special case of "cyclic coverings", and later deal with the general case, where there is less compatibility between the algebraic structure and the dynamical system.

4.1 Cyclic Coverings (Circle Valued Morse theory)

Suppose X to be compact and $g : X \longrightarrow \mathbb{R}/\mathbb{Z}$ to be a function whose singularities are those of a Morse function (the domain does not influence the definition of "nondegenerate"), i.e. a **circle valued Morse function**. Let now $\sigma : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ be the quotient map, $\rho : Y \to X$ the covering map induced by pulling back σ and $f : Y \to \mathbb{R}$ the corresponding lifting of g.

$$\begin{array}{ccc} Y & \xrightarrow{f} & \mathbb{R} \\ \downarrow\rho & & \downarrow\sigma \\ X & \xrightarrow{g} & \mathbb{R}/\mathbb{Z} \end{array}$$

The group of deck transformations of the covering $Y \to X$ is the integers $\mathbb{Z} = \langle t \rangle$, where $t : Y \longrightarrow Y$ is a diffeomorphism. The equivariance $f(ty) = f(y) + 1$ relates the covering group and the function f, which is clearly Morse.

For a simpler exposition, we'll just consider gradient flows (instead of gradient-like ones); this fact will be later stressed in the notation of the $\mathcal{S}$-complex.

Choose a Riemannian metric on X and let ϕ be the gradient flow of g (i.e. the vector field obtained by raising indexes to the one form dg).

By pulling back via ρ, the flow ϕ can be lifted to a flow ψ on Y, which is the gradient of the Morse function f (for the pullback metric). We assume ψ to be Smale (this is a generic condition).

The function f is clearly proper (hence the flow ψ is automatically Weakly Proper). The critical points upstairs are just the preimages of the critical points downstairs. The main difference between the two dynamical systems is that upstairs there are no closed orbits (nor closed broken flow lines) whereas downstairs there might be some. Actually any flow line that has no finite limit point downstairs lifts to a closed curve (necessarily tending to ∞ in Y).

Consider the **group rings** of the covering

$$\mathbb{R}\left[t, t^{-1}\right] \quad \text{and} \quad \mathbb{Z}\left[t, t^{-1}\right]$$

i.e. the the rings of Laurent polynomials in t.

Definition 4.1.1 *The* ***(Laurent) Novikov rings***

$$\Lambda_{\mathbb{R}} = \mathbb{R}\left[[t]\right]\left[t^{-1}\right] \quad \text{and} \quad \Lambda_{\mathbb{Z}} = \mathbb{Z}\left[[t]\right]\left[t^{-1}\right]$$

are the rings of formal Laurent series with finite principal parts. In particular $\Lambda_{\mathbb{R}}$ is actually a field. Moreover $\Lambda_{\mathbb{R}}$ is a $\mathbb{R}\left[t, t^{-1}\right]$-module and $\Lambda_{\mathbb{Z}}$ is a $\mathbb{Z}\left[t, t^{-1}\right]$-module.

By the geometric series trick it readily follows:

Algebraic Fact 1 $\Lambda_{\mathbb{R}}$ is a field.

Compact/forward sets can be defined algebraically, as a simple consequence of the interaction of the deck map t and of f:

Lemma 4.1.2 *A closed set $A \subset Y$ is* a compact/forward set if and only if there exists a compact set $K \subset Y$ and an integer $N \in \mathbb{Z}$ such that $A \subset \bigcup_{n \geq N} t^n(K)$.

Let's now reconsider the complexes of forms and currents $\mathcal{E}^*_{c\uparrow}(Y)$, $\mathcal{D}'^*_{c\uparrow}(Y)$, $\mathcal{C}^*_{c\uparrow}(Y)$, and $\mathcal{S}^*_{c\uparrow}(f)$ defined in the previous chapter. We remark that in this setting, the words "forward" and "compact/forward" have the same meaning (the Morse function being proper), nevertheless we mantain the terminology "compact/forward" to be consistent with the following section, dealing with a more general case.

Since the covering map t commutes with the flow ψ, the previous lemma implies that the action of t by pushforward is a self map of all the previous complexes. This induces actions of the group rings and Novikov rings. The operators

$$\begin{aligned}\mathbf{T} &: \mathcal{E}^*_{c\uparrow}(Y) \longrightarrow \mathcal{D}'^*_{c\uparrow}(Y)\\ \mathbf{P} &: \mathcal{E}^*_{c\uparrow}(Y) \longrightarrow {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f) \subset \mathcal{D}'^*_{c\uparrow}(Y)\end{aligned}$$

commute with the action of t by pushforward, and hence they are $\Lambda_{\mathbb{R}}$-linear maps.

Theorem 4.1.3 *The map of $\Lambda_{\mathbb{R}}$-complexes*

$$\mathbf{P} : \mathcal{E}^*_{c\uparrow}(Y) \longrightarrow {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f)$$

induces an isomorphism of finite dimensional $\Lambda_{\mathbb{R}}$-vector spaces

$$H^i_{c\uparrow}(Y,\mathbb{R}) \approx H^i\left({}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f)\right)$$

Moreover, $\dim_{\Lambda_{\mathbb{R}}} {}_{\mathbb{R}}\mathcal{S}^k_{c\uparrow} = \#(critical\ points\ of\ index\ k\ for\ g)$ *is finite.*

Proof. The first statement is a direct consequence of theorem 3.2.3 and of Fact 1. In addition, any choice of a lifting $p \in Cr(g) \longmapsto \bar{p} \in Cr(f)$ for the set of critical points downstairs provides a (finite) $\Lambda_{\mathbb{R}}$ basis for ${}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f)$, consisting of the stable manifolds $S_{\bar{p}} \in {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}$ at those (lifted) points $\square$

The inequalities of Morse type between the dimensions (over $\Lambda_{\mathbb{R}}$) of ${}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f)$ and of $H^i_{c\uparrow}(Y,\mathbb{R})$ are an algebraic consequence of this theorem.

As for the theory with integer coefficients, a key result is:

Algebraic Fact 2 $\Lambda_{\mathbb{Z}}$ is a principal ideal domain.

Now, observe that the inclusion map ${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f) \hookrightarrow \mathcal{C}^*_{c\uparrow}(Y)$ commutes with the action (as pushforward) of t and the complexes involved are complexes of $\mathbb{Z}\left[t,t^{-1}\right]$-modules as well as of $\Lambda_{\mathbb{Z}}$-modules. Theorem 3.2.4 then implies:

Theorem 4.1.4 *The inclusion map of $\Lambda_{\mathbb{Z}}$-complexes*

$${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f) \hookrightarrow \mathcal{C}^*_{\uparrow}(Y)$$

induces an isomorphism of $\Lambda_{\mathbb{Z}}$-modules

$$H^i\left({}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f)\right) \approx H^i_{c\uparrow}(Y,\mathbb{Z})$$

Moreover, ${}_{\mathbb{Z}}\mathcal{S}^k_{c\uparrow}$ is finitely generated, with one generator in ${}_{\mathbb{Z}}\mathcal{S}^k_{c\uparrow}$ for every critical point of g of index k (downstairs).

Because of Fact 2, the Novikov inequalities over the integers are again an algebraic consequence of this theorem, exactly as in Morse theory.

Next we compare $H^*_{cpt}(Y,\mathbb{Z})$ and $H^*_{c\uparrow}(Y,\mathbb{Z})$, in analogy with the result in [25]. The sheaf cohomology groups $H^p_{cpt}(Y,\mathbb{Z})$ are standard topological invariants of Y (isomorphic to $H_{n-p}(Y,\mathbb{Z})$, i.e. homology). Altough the constructions of $\mathbf{T}$, $\mathbf{P}$ and the $\mathcal{S}$-complex depend on f, the cohomology with c/f supports $H^*_{c\uparrow}(Y,\mathbb{Z})$ only depends on the covering translation t. Consequently, to compute the c/f supported cohomology we may replace f by a new f which is the lift to Y of a (single valued!) Morse function $g : X \to \mathbb{R}$.

The isomorphism of Λ-modules $H^p\left({}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f)\right) \approx H^p_{c\uparrow}(Y,\mathbb{Z})$ remains valid for the new f.

But with the new f, each stable manifold S_p is relatively compact in Y, therefore $[S_p]$ has compact support and its boundary consists of a finite sum of other stable manifolds. In particular the space ${}_{\mathbb{Z}}\mathcal{S}^*_{cpt}(f)$ (made up of finite sums of stable manifolds) is closed under taking boundary, i.e. it is a complex. Moreover, the operator $\mathbf{P}$ maps $\mathcal{E}^*_{cpt}(Y)$ to ${}_{\mathbb{R}}\mathcal{S}^*_{cpt}(f)$ and the operator $\mathbf{T}$ is a chain homotopy between $\mathbf{P}$ and the identity $\mathbf{I} : \mathcal{E}^*_{cpt}(Y) \longrightarrow \mathcal{D}'^{\,*}_{cpt}(Y)$.

Consequently, there are isomorphisms of real vector spaces and abelian groups:

$$H^p_{cpt}(Y,\mathbb{R}) \approx H^p\left(\mathcal{E}^*_{cpt}(Y)\right) \approx H^p\left({}_{\mathbb{R}}\mathcal{S}^*_{cpt}(f)\right)$$

$$H^p_{cpt}(Y,\mathbb{Z}) \approx H^p\left(\mathcal{C}^*_{cpt}(Y)\right) \approx H^p\left({}_{\mathbb{Z}}\mathcal{S}^*_{cpt}(f)\right)$$

The (covering) group ring $\mathbb{Z}[\pi] = \mathbb{Z}\left[t,t^{-1}\right]$ of Laurent polynomials acts on ${}_{\mathbb{Z}}\mathcal{S}^*_{cpt}(f)$ and $\mathcal{C}^*_{cpt}(Y)$. Therefore ${}_{\mathbb{Z}}\mathcal{S}^*_{cpt}(f) \underset{\mathbb{Z}[\pi]}{\otimes} \Lambda_{\mathbb{Z}}$ and $\mathcal{C}^*_{cpt}(f) \underset{\mathbb{Z}[\pi]}{\otimes} \Lambda_{\mathbb{Z}}$ are complexes of $\Lambda_{\mathbb{Z}}$ modules and there are isomorphisms of $\Lambda_{\mathbb{Z}}$ modules:

$${}_{\mathbb{Z}}\mathcal{S}^*_{cpt}(f) \underset{\mathbb{Z}[\pi]}{\otimes} \Lambda_{\mathbb{Z}} = {}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f) \quad \text{and} \quad \mathcal{C}^*_{cpt}(f) \underset{\mathbb{Z}[\pi]}{\otimes} \Lambda_{\mathbb{Z}} = \mathcal{C}^*_{c\uparrow}(f)$$

We now need:

Algebraic Fact 3 $\Lambda_{\mathbb{Z}}$ is flat over $\mathbb{Z}[\pi] = \mathbb{Z}\left[t,t^{-1}\right]$.

By taking homology of the complexes, this implies:

Theorem 4.1.5 *As finitely generated $\Lambda_{\mathbb{Z}}$-modules:*

$$H^*_{c\uparrow}(Y,\mathbb{Z}) \approx H^*_{cpt}(Y,\Lambda_{\mathbb{Z}})$$

where, by definition, $H^*_{cpt}(Y,\Lambda_{\mathbb{Z}}) = H^*_{cpt}(Y,\mathbb{Z}) \underset{\mathbb{Z}[\pi]}{\otimes} \Lambda_{\mathbb{Z}}$.

4.2 General Case

Let's now look at the general case of Novikov theory. We start with a closed one form ω with nondegenerate singularities on the compact manifold X. We'll call such forms "**Novikov**" forms. We recall that the irrationality degree of ω is the minimal number $k-1$ such that all the periods of ω are rationally dependent by the periods $\chi_1, \ldots, \chi_k$. It corresponds to the smallest number of generators of ω by forms in $H^1(X, \mathbb{Z})$.

Let $k-1$ be the irrationality index of ω and $\chi = (\chi_1, .., \chi_k)$ denote its generating periods. Let $\rho : Y \to X$ be a minimal covering such that ω pulls back to an exact form, say df, with $f : Y \longrightarrow \mathbb{R}$. The group π of deck translations of (Y, ρ) is a free abelian group with k generators, say $t_1, ..., t_k$ (i.e. $\pi \approx \mathbb{Z}^k$) and the **group rings** (over $\mathbb{R}$ and $\mathbb{Z}$) are the Laurent polynomials rings:

$$\mathbb{R}[\pi] = \mathbb{R}\left[t_1, .., t_k, t_1^{-1}, .., t_k^{-1}\right] \quad \text{and} \quad \mathbb{Z}[\pi] = \mathbb{Z}\left[t_1, .., t_k, t_1^{-1}, .., t_k^{-1}\right]$$

The equivariance relations $f(t_i(y)) = f(y) + \chi_i$ hold for any $i = 1, .., k$.

If $k = 1$ the covering is cyclic and the one form ω can be seen as the differential of a circular valued function, which was the case in the previous section.

Using the covering map ρ as it has been done for cyclic coverings, the form ω lifts to a Morse function f on Y. Note that f is not proper.

We choose a metric Y which is the lift of a metric on X and for which the gradient flow ψ of f is Smale. The gradient flow of ω on X is denoted by ϕ.

The critical points of f on Y are just the preimages of the critical points of ω on X. Of course ψ has no closed orbits (nor broken closed orbits).

Lemma 4.2.1 *The lifted flow ψ is weakly proper.*

Proof. Suppose $\bar{\gamma} : [0, +\infty[\to Y$ is a forward flow-half line of ψ which is not relatively compact in Y (i.e. $\bar{\gamma}$ does not converge to a critical point); we just need to show that f is unbounded on $\bar{\gamma}$. Consider the projected curve $\gamma = \rho(\bar{\gamma})$, which is a forward flow-half line for ϕ. Since $\rho^*(\omega) = df$, and

$$\lim_{s \to +\infty} [f(\bar{\gamma}(s)) - f(\bar{\gamma}(0))] = \lim_{s \to +\infty} \int_0^s \frac{d}{dt} f(\bar{\gamma}(t))\, dt = \int_{\bar{\gamma}} df = \int_{\gamma} \omega$$

we are left to prove that $\int_\gamma \omega = +\infty$. Observe that γ cannot converge to a critical point for ϕ in X, otherwise $\bar{\gamma}$ would also converge to a critical point for ψ in Y. Moreover, for any open set $D \subset X$ containing all the critical points, there exists a constant $c > 0$, determined by $\|\omega\|$ on $X \backslash D$, such that for any piece of an integral curve α contained in $X \backslash D$, the estimate

$\int_\alpha \omega = \int |\dot{\alpha}|^2 \, dt > c \int |\dot{\alpha}| \, dt$ holds. Since γ doesn't converge to a critical point, we can choose D so that γ has unbounded lenght in $X \backslash D$ $\Box$

Remark *The previous lemma (and proof) holds for any covering where ω pullbacks to an exact form, in particular the lifted flow is weakly proper for the universal covering.*

Now we modify the classic Novikov Theory (for $k > 1$) by introducing a new ring. Let χ also denote the linear functional on $\mathbb{R}^k$ defined by $\chi(v) = \chi \cdot v$, where χ is the vector of periods.

Definition 4.2.2 *A subset $F \subset \mathbb{Z}^k$ is said:*

1) ***slab compact*** if F intersect each slab $\chi^{-1}([a, b])$ is compact (i.e. finite),
2) ***forward*** if $F \subset \chi^{-1}([a, +\infty))$ for some $a \in \mathbb{R}$,
3) ***compact/forward*** *or c/f if F is both slab compact and forward.*

First, consider formal Laurent series $\alpha = \sum a_n t^n$, where $t = (t_1, .., t_k)$ and $n = (n_1, .., n_k)$. The **support** of α, denoted by $|\alpha|$, consists of all $n \in \mathbb{Z}^k$ such that $a_n \neq 0$.

Definition 4.2.3 *The* ***forward (Laurent) ring*** *Λ consists of all formal Laurent series $\alpha = \sum a_n t^n$ with compact/forward support and with integer coefficients $a_n \in \mathbb{Z}$. The alternative notations $\Lambda = \Lambda_{cf} = \Lambda(\mathbb{Z})$ will also be used.*

Note 1 The support of α is c/f if and only if $|\alpha| \cap \chi^{-1}((-\infty, a])$ is finite for all $a \in \mathbb{R}$.

Note 2 Consequently, given $\alpha, \beta \in \Lambda$, the Cauchy product $\gamma = \alpha\beta$ is defined by the finite sums $c_n = \sum\limits_{p+q=n} a_p b_q$, and γ has c/f support.

Notations. The **degree** of a monomial term t^n is defined to be $\chi(n)$. The set of degrees of all the non zero monomial terms in the expansion of $\alpha \in \Lambda$ is the image $\chi(|\alpha|)$ of the support $|\alpha|$ under the map $\chi : \mathbb{Z}^k \to \mathbb{R}$. This set $DEGS(\alpha) \overset{def}{=} \chi(|\alpha|)$ is a discrete (possibly finite) subset of $\mathbb{R}$, bounded below. Consequently, each $\alpha \in \Lambda$ has a unique expansion $\alpha = \sum\limits_{j=0}^{N \leq \infty} a_j t^{A_j}$ with each a_j non zero and $\deg t^{A_j} < \deg t^{A_{j+1}}$. The **degree** of α is defined to be the degree of the **leading term** $a_0 t^{A_0}$.
Define $l : \Lambda \to \mathbb{Z}$ by taking the leading coefficient, namely $l(\alpha) = a_0$ (and $l(0) = 0$). Note that l is not a ring homomorphism, though it is a homomorphism for the multiplicative group of Λ.

Lemma 4.2.4 *The map l sends ideals to ideals and an element $\alpha \in \Lambda$ is a unit if and only if $l(\alpha)$ is a unit.*

Proof. Let I be an ideal in Λ, and $m = l(\mu)$, $n = l(\nu)$ for some $\mu, \nu \in \Lambda$ that we can suppose to be of degree zero. If $m+n \neq 0$ then $m+n = l(\mu+\nu)$. Moreover, $-m = l(-\mu)$ and therefore $l(I)$ is a subgroup of I. Since l is a homomorphism for the multiplicative group of Λ, it follows that $l(I)$ is an ideal in Λ. As for the second statement, the nontrivial part is the "only if". We can assume $\alpha = 1 - \beta$ with $\deg(\beta) > 0$. Since $\deg\left(\beta^k\right) = k \deg(\beta)$, the geometric series in β provides the inverse for α in $\Lambda(\mathbb{Z})$ □

As a consequence we obtain the analogous of the Algebraic Fact 1:

Corollary 4.2.5 *The ring $\Lambda(\mathbb{R})$ obtained by considering coefficients in $\mathbb{R}$ is a field.*

The proof of the second algebraic fact is more involved:

Lemma 4.2.6 *The forward ring $\Lambda(\mathbb{Z})$ is a principal ideal domain.*

Proof. Suppose I is an ideal of Λ. Since $\mathbb{Z}$ is a p.i.d., $l(I) = \mathbb{Z}a$ for some integer $a \in \mathbb{Z}$. Choose an element $\alpha = a + \sum_{j=0}^{\infty} a_j t^{A_j}$ in the ideal I with degree zero and leading coefficient $l(\alpha) = a$. Given $\gamma \in I$, we will inductively define $\beta = \sum_{j=0}^{\infty} \beta_j \in \Lambda(\mathbb{Z})$ so that $\gamma = \beta\alpha$, proving that $I = \Lambda\alpha$.

Define $\gamma_0 = \gamma$ and, given $\gamma_k \in I$, define the monomial $\beta_k = b_k t^{B_k}$ as the leading term of γ_k divided by a. Since $l(I) = \mathbb{Z}a$, the coefficient $b_k \in \mathbb{Z}$. Now define

$$\gamma_{k+1} = \gamma_k - \beta_k\alpha = \gamma - (\beta_0 + \beta_1 + .. + \beta_k)\,\alpha$$

as the error in the factorization. Thus $\deg\beta_k = \deg\gamma_k < \deg\gamma_{k+1}$. Put $z_k = \deg\beta_k$: if no γ_k vanishes, it remains to show that $\lim_{n\to\infty} z_n = +\infty$. Note that $z_k = \min DEGS(\gamma_k)$, the set of degrees of terms in γ_k. Let

$$\{y_1, y_2, ...\} = \left\{\deg t^{A_1}, \deg t^{A_2}, ...\right\} = DEGS(\alpha) \setminus \{0\}$$
$$\{x_0, x_1, ...\} = \left\{\deg t^{C_0}, \deg t^{C_1}, ...\right\} = DEGS(\gamma) = DEGS(\gamma_0)$$

Now

$$\gamma_1 = \gamma_0 - \beta_0\alpha = C_1 t^{C_1} + C_2 t^{C_2} + ... - b_0 a_1 t^{C_0 + A_1} - b_0 a_2 t^{C_0 + A_2} - ...$$

and $z_0 = x_0 = \deg t^{C_0}$. Therefore,

$$DEGS(\gamma_1) \subset (DEGS(\gamma_0) \sim z_0) \cup \{z_0 + y_1, z_0 + y_2, ...\}$$

Similarly,

$$DEGS\,(\gamma_{k+1}) \subset (DEGS\,(\gamma_k) \sim z_k) \cup \{z_k + y_1, z_k + y_2, ...\}$$

Consequently, the union of all the sets $DEGS\,(\gamma_k)$ is contained in the set D of real numbers of the form $x_j + y_{i_1} + ... + y_{i_k}$. Since both the set of x_j's and the set of y_i's are discrete and bounded below, the set D is also discrete and bounded below. Therefore $\lim_{k\to\infty} z_k = \infty$ $\square$

Exactly as in the cyclic covering case, $compact/forward$ sets based on f can be defined algebraically in terms of the covering group $\mathbb{Z}^k$.

Lemma 4.2.7 *A closed set $A \subset Y$ is a compact/forward set if and only if there exists a compact set $K \subset Y$ and a c/f set F in the lattice $\mathbb{Z}^k$ such that A is contained in the union of the sets $t^n\,(K)$ over $n \in F$.*

Theorem 4.2.8 *Each of the theorems 4.1.3, 4.1.4, 4.1.5 stated in the last section holds for $k > 1$ if one substitutes the ring Λ_{cf} for Λ.*

The statements and their proofs are identical, (the third algebraic fact will be proved in the next section as corollary 4.3.4), and they will not be repeated here. We just point out an important remark.

Remark (Topological Stability) Any two Novikov forms in the same cohomology class in $H^1\,(X, \mathbb{R})$ define the same c/f sets on Y. In fact they differ by the differential of a bounded function (since X is compact). Therefore their liftings to Y differ by a bounded function. The stability remark in the section on Morse theory applies.

4.3 Comparison with Novikov Theory

Finally, we compare the previous results with Novikov theory ($k > 1$). In order to be concise, we will restrict to integer coefficients. It is convenient to define the Novikov ring in terms of supports.

Definition 4.3.1 *A subset $F \subset \mathbb{Z}^k$ is a* ***cone-forward*** *(or* ***Novikov forward****, abbreviated n/f) set in the lattice with respect to χ if there exist $a \in \mathbb{R}$ and $\varepsilon > 0$ such that $F \subset \chi^{-1}([a, +\infty))$ and ("****stability****") this remains true for all χ^* with $|\chi - \chi^*| < \varepsilon$. The* ***Novikov ring*** *Λ_{nf} consists of all formal Laurent series $\alpha = \sum_{n\in F} a_n t^n$ with integer coefficients whose support $F = |\alpha|$ is a cone-forward set in the lattice $\mathbb{Z}^k$.*

Note that any cone-forward set is compact/forward, so that the Novikov ring Λ_{nf} is a subring of the ring Λ_{cf}. Again, the geometric power series argument shows that the Novikov ring $\Lambda_{nf}\,(\mathbb{R})$ over $\mathbb{R}$ is a field.

Pajithnov in [26] attributes to J. Sikorav the following result; we provided a new proof for completeness.

Proposition 4.3.2 *The Novikov ring Λ_{nf} is a p.i.d.*

Proof. Given an ideal $I \subset \Lambda_{nf}(\mathbb{Z})$, let $\bar{I}$ be the ideal in $\Lambda_{cf}(\mathbb{Z})$ generated by I. Pick $\alpha \in I$, such that the ideal $l(I) \subset \mathbb{Z}$ is generated by $l(\alpha)$. Then $\bar{I} = \Lambda_{cf}(\mathbb{Z})\,\alpha$. In particular, if $\gamma \in I$ then $\beta = \gamma\alpha^{-1} \in \Lambda_{cf}(\mathbb{Z})$, but $\gamma\alpha^{-1}$ also belongs to $\Lambda_{nf}(\mathbb{R})$. Finally note $\Lambda_{nf}(\mathbb{Z}) = \Lambda_{nf}(\mathbb{R}) \cap \Lambda_{cf}(\mathbb{Z})$ $\square$

Proposition 4.3.3 *The Novikov ring Λ_{nf} is a flat algebra over the Laurent Polynomials ring $L = Z\left[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}\right]$*

Proof. Let Λ_{nf}^+ be the subring of the Novikov ring Λ_{nf} made up of the elements of positive degree, and L^+ the subring of L made up of polynomials of positive degree (according to our definition of degree), so that Λ_{nf}^+ is an algebra over L^+. We now observe that Λ_{nf} can be presented as $\Lambda_{nf} = L \otimes_{L^+} \Lambda_{nf}^+$. Since "extending the scalars" preserves flatness (cf. [B], I.2.7, Cor. 2), we are left to prove that Λ_{nf}^+ is flat over L^+.

Let's restrict to two variables, but the idea can be simply generalized to the general case. For any $\alpha \in [0, \pi]$, let Λ_α^+ be the subring of Λ_{nf}^+ made up of series whose support in $\mathbb{Z}^2$ is contained in a cone of angle α centred in the origin and symmetric with respect to the vector χ. In particular $\Lambda_0^+ = \mathbb{Z}$ (since the origin is the only point of the line with direction χ and integer coordinates) and $\Lambda_\pi^+ = \Lambda_{nf}^+$. Let finally $L_\alpha^+ = L \cap \Lambda_\alpha^+$ and observe that $\Lambda_{nf}^+ = \varinjlim \Lambda_\alpha^+$, and $L^+ = \varinjlim L_\alpha^+$where the direct limit is taken for $\alpha \nearrow \pi$.

We will next prove that any Λ_α^+ is a completion of L_α^+. Since the set of points with integer coordinates is discrete in the angular regions described above, the degree of elements in L_α^+ (and in Λ_α^+) is a discrete (hence well ordered) set in $[0, +\infty)$, say $\{a_0 = 0, a_1, \ldots, a_k, \ldots\}$. Also, the sets

$$I_k^\alpha = \{\lambda \in L_\alpha^+ \mid \deg(\lambda) \geq a_k\}$$

are ideals for L_α^+ and the quotients

$$L_\alpha^+ / I_k^\alpha \approx \{\lambda \in L_\alpha^+ \mid \deg(\lambda) < a_k\} (\operatorname{mod} I_k^\alpha)$$

"are" the polynomials in L_α^+ of degree less than a_k. It is the clear that $\Lambda_\alpha^+ = \varinjlim L_\alpha^+ / I_k^\alpha$, i.e. Λ_α^+ is a completion of L_α^+. Since a completion of a ring is always flat over that ring (cf. [Ma]), the previous proves that any Λ_α^+ is flat over L_α^+.

Recall that $\Lambda_{nf}^+ = \varinjlim \Lambda_\alpha^+$, and $L^+ = \varinjlim L_\alpha^+$, and since ,roughly speaking, the direct limit preserves flatness (cf. [5] I.2.7, Prop. 9), it follows that Λ_{nf}^+

is flat over L^+, completing the proof of the lemma $\square$

As a consequence, we get the analogous of Algebraic Fact three for the compact/forward ring.

Corollary 4.3.4 *The compact/forward ring Λ_{cf} is a flat algebra over the Laurent Polynomials ring L.*

Proof Any torsion free module over a p.i.d. is flat (cf. [20]), and the compact/forward ring Λ_{cf} is torsion free over the Novikov ring Λ_{nf}, so Λ_{cf} is flat over Λ_{nf}. We just proved Λ_{nf} is flat over the Laurent polynomials L and since flatness has the transitive property we are done $\square$

Definition 4.3.5 *A closed subset $A \subset Y$ is a **Novikov-forward** set (abbreviated n/f-set) if there exists a compact set $K \subset Y$ and a n/f set F in the lattice $\mathbb{Z}^k$ such that $A \subset \bigcup_{n \in F} t^n (K)$*

The Novikov-forward sets are compact/forward with respect to f , since $f(t^n(y)) = \chi \cdot n + f(y)$. The converse is not always true if $k > 1$ because the lattice contains compact/forward sets which are not Novikov/forward.

Clearly, each covering translation t_i acts on the various complexes of forms and current with support in Novikov/forward-sets and the different actions commute (since the t_i's do). This allows one to define actions of the group ring and Novikov ring "by linearity" on those complexes; the supports are in fact preserved by the action of the Novikov ring Λ. One can also define an $\mathcal{S}$-complex ${}_{\mathbb{Z}}\mathcal{S}^*_{nf}$ with supports in n/f-sets and all the previous arguments carry over substituting n/f-sets for compact/forward sets with respect to f.

Theorem 4.3.6 *Each of the Theorems 4.1.3, 4.1.5, 4.1.5 stated for cyclic covers holds for the ($k > 1$) Novikov case if one substitutes n/f-supports for compact/forward supports.*

Again, the statements and their proofs are identical and will not be repeated here. These three theorems in the n/f case and the c/f are directly related as follows. Since $\Lambda_{cf}(\mathbb{R})$ is a field, $\Lambda_{cf}(\mathbb{Z})$ is a torsion free module over the ring $\Lambda = \Lambda_{nf}(\mathbb{Z})$. Since the ring $\Lambda = \Lambda_{nf}(\mathbb{Z})$ is a p.i.d., this implies that $\Lambda_{cf}(\mathbb{Z})$ is a flat Λ-module. Therefore:

$$ {}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow} \approx {}_{\mathbb{Z}}\mathcal{S}^*_{nf} \otimes_\Lambda \Lambda_{cf}(\mathbb{Z}) $$

$$ H^*_{c\uparrow}(Y,\mathbb{Z}) \approx H^*\left({}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}\right) \approx H^*\left({}_{\mathbb{Z}}\mathcal{S}^*_{nf}\right) \otimes_\Lambda \Lambda_{cf} \approx H^*_{nf}(Y,\mathbb{Z}) \otimes_{\Lambda_{nf}} \Lambda_{cf} $$

In particular the Novikov numbers and inequalities are the same over Λ_{nf} or Λ_{cf}.

The larger ring $\Lambda_{cf}(\mathbb{Z})$ is geometrically more natural in the context of non-compact Morse Theory, while the smaller ring $\Lambda_{nf}(\mathbb{Z})$ has other advantages. Of course they coincide in the cyclic ($k=1$) case.

4.4 Lambda Duality

Let $\Lambda_{\mathbb{Z}}$ denote either the compact/forward or the Novikov ring, and let $\Lambda_{\mathbb{R}}$ denote the corresponding field. First, consider the field $\Lambda = \Lambda_{\mathbb{R}}$, subscripts are understood. Each critical point $x_0 \in X$, of index h, determines a Λ-line in $\mathcal{S}^h_{c\uparrow}$ denoted $\mathcal{S}^h_{x_0}$ which equals $\Lambda[S_{y_0}]$ where y_0 is any fixed critical point above x_0. Each $S \in \mathcal{S}^h_{x_0}$ uniquely determines $\lambda \in \Lambda$ for which $S = \lambda \cdot [S_{y_0}]$. Similarly, each critical point $x_0 \in X$, of index h determines a Λ-line in $\mathcal{U}^{n-h}_{c\downarrow}$, denoted $\mathcal{U}^{n-h}_{x_0}$, which consists of all sums $U = \sum\limits_{n\in|\mu|} b_n U_{t^{-n}y_0} = \mu U_{y_0}$, where $\mu = \sum\limits_{n\in|\mu|} b_n t^n \in \Lambda$. As before, $U \in \mathcal{U}^{n-h}_{x_0}$ uniquely determines $\mu \in \Lambda$ with $U = \mu U_{y_0}$.

Note. The same Λ acts on $\mathcal{S}$ and $\mathcal{U}$; necessarily, then, the variable t acts by pullback on $\mathcal{U}$ and in particular the line $\mathcal{U}^{n-h}_{x_0} \subset \mathcal{U}^{n-h}_{n\downarrow} \subset \mathcal{U}^{n-h}_{c\downarrow}$.

Definition 4.4.1 *Given $U \in \mathcal{U}^{n-h}_{x_0}$ and $S \in \mathcal{S}^h_{x_0}$ define the Λ -pairing of U and S by*

$$(U,S)_\Lambda = \lambda\mu$$

where $U = \mu U_{y_0}$ and $S = \lambda S_{y_0}$ determine λ and μ.

Lemma 4.4.2 *The pairing $(\ ,\ )_\Lambda$ is Λ-bilinear and independent of the choice of the critical point $y_0 \in Y$ above $x_0 \in Cr(\omega) \subset X$.*

Proof. Suppose y_0' is another choice of critical point above x_0. Then $S_{y_0'} = S_{t^n y_0} = t^n S_{y_0}$ while $U_{y_0'} = U_{t^n y_0} = t^{-n} U_{y_0}$ for the same $n \in \mathbb{Z}$ so that $\lambda = \lambda' t^n$ and $\mu = \mu' t^{-n}$ and $\lambda\mu = \lambda'\mu'$ □

Note that $\mathcal{S}^h = \sum\limits_{x\in Cr(\omega, Ind\, h)} \mathcal{S}^h_x$ and $\mathcal{U}^{n-h} = \sum\limits_{x\in Cr(\omega, Ind\, h)} \mathcal{U}^{n-h}_x$. Given $S \in \mathcal{S}^h$ and $U \in \mathcal{U}^{n-h}$ the Λ-bilinear pairing obviously extends to $(S,U)_\Lambda$.

Remark If $(S \bullet U)$ denotes the current intersection of S with U paired with 1 (i.e. the number of points in the intersection), then $(S \bullet U) \in \mathbb{Z}$ is the leading coefficient of $(S,U)_\Lambda \in \Lambda$.

Theorem 4.4.3 *The Λ-vector spaces $\mathcal{S}^h_{c\uparrow}$ and $\mathcal{U}^{n-h}_{c\downarrow}$ are finite dimensional dual vector spaces under the pairing $(\ ,\ )_\Lambda$. Consequently, the Λ-vector spaces $H^h\left(\mathcal{S}^*_{c\uparrow}\right)$ and $H^{n-h}\left(\mathcal{U}^*_{c\uparrow}\right)$ are Λ-dual finite dimensional vector spaces.*

The proof is immediate.

The integer case produces two finitely generated complexes of ${}_{\mathbb{Z}}\Lambda$-bilinear pairing; and with bases S_i, U_i so that $(S_i, U_j) = \delta_{ij}$. It follows algebraically that Λ-Poincaré duality holds over $\mathbb{Z}$.

Chapter 5

Functions and 1-forms with Bott Singularities

5.1 The Boundary Value Technique

We now present a generalization of the theory discussed in section 1.3, adapting it to the case of Bott singularities. The arguments are often very similar, so the proof are sometimes omitted or sketched. Again, no part of this section is completely original, and we refer to the sources quoted before as references.

Definition 5.1.1 *Let V be a vector field on a manifold X of dimension n and $C \subset X$ a regular (hence closed) submanifold of dimension c. Suppose $V|_C = 0$. The singular points on C are called of "**Bott**" type or "**nondegenerate in the normal direction**" if for any point $p \in C$ there exists local coordinates u, v near p such that $C = \{u = 0\}$ and the local system determined by V has the form*

$$\begin{cases} \dot{u} = Au + a(u, v) \\ \dot{v} = b(u, v) \end{cases}$$

where A is a square matrix of order $n - c$ with no purely imaginary invariant and a and b are vector valued functions which vanish on C together with their differentials.
*The number $\lambda_C = (\lambda_p)$ of negative characteristic exponents of the matrix A is called the **index** of C (or of any $p \in C$). We also put, following [18], $\lambda_C^* = n - c - \lambda_C$.*

*A function $f : X \to \mathbb{R}$ or a one form $\varphi \in \mathcal{E}^1(X)$ is called "**Morse-Bott**" if for some (and hence all) Riemannian metric, the singularities of the corresponding gradient vector field are of Bott type, distributed on compact critical manifolds.*

Consider the following system on $\mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c$:

$$\begin{cases} \dot{x} = L^- x + f(t, x, y, z, \theta) \\ \dot{y} = L^+ y + g(t, x, y, z, \theta) \\ \dot{z} = h(t, x, y, z, \theta) \end{cases} \tag{5.1}$$

where the matrices L^- and L^+ are in Jordan form, the real parts of the eigenvalues of L^- are strictly negative, say $-\lambda_s \leq \cdots \leq -\lambda_1 < 0$, and those of L^+ strictly positive, say $0 < \mu_1 \leq \cdots \leq \mu_u$.

In the sequel the symbol $|x, y, z|$ will always mean the max among $|x|, |y|, |z|$. Now set:

$$F = (f, g, h) : \mathbb{R} \times \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c \times W \to \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c$$

We assume that F vanishes on the **critical manifold** $C = 0 \times \mathbb{R}^c$ and the spatial derivatives of the non linear terms are uniformly bounded, i.e. there exists δ^k such that:

$$F(t, 0, 0, z, \theta) = 0$$

$$\delta^k_\varepsilon \overset{def}{=} \sup_{|x,y| \leq \varepsilon} \sum_{|m|=k} \left| \frac{\partial^k F}{\partial (x,y,z)^m} \right| \leq \delta^k < +\infty \tag{5.2}$$

Note that the bounds δ^k_ε are taken over tubular neighborhoods of the critical manifold.

Lemma 5.1.2 *In the previous hypotheses, the following inequalities hold:*

$$|F(t, x, y, z, \theta)| \leq \delta^1_{|x,y|} |x, y| \tag{5.3}$$

$$|\frac{\partial^k F}{\partial z^k}(t, x, y, z, \theta)| \leq \delta^{k+1}_{|x,y|} |x, y| \tag{5.4}$$

Proof. By $F(t, 0, 0, z, \theta) = 0$ it follows $\frac{\partial^k F}{\partial z^k}(t, 0, 0, z, \theta) = 0$; then apply the mean value theorem □

Definition 5.1.3 *We say that the **Boundary Value problem** (abridged B.V. problem) with data $(x_0, y_1, z_1, \tau) \in \mathbb{R}^n \times [0, +\infty)$ is solvable for the system 5.1 if there exists a solution $(x^*(t), y^*(t), z^*(t))$ defined on $[0, \tau]$ and satisfying:*

$$(x^*(0), y^*(\tau), z^*(\tau)) = (x_0, y_1, z_1)$$

Theorem 5.1.4 *Suppose the estimate $2\delta^1 < \alpha$ hold. Then the Boundary Value problem for the system 5.1 is solvable for any data (x_0, y_1, z_1, τ). The solution is unique, it depends smoothly on $(t, x_0, y_1, z_1, \tau, \theta)$ and satisfies:*

$$|x^*(t), y^*(t)| \leq 2|x_0, y_1| \quad \textit{for any } t \in [0, \tau] \tag{5.5}$$

Notations *In the same way as for an Initial Value problem, a BV problem is denoted by:*

$$\begin{cases} \dot{x} = L^{-}x + f\left(t, x, y, z, \theta\right) \\ \dot{y} = L^{+}y + g\left(t, x, y, z, \theta\right) \\ \dot{z} = h(t, x, y, z, \theta) \\ x^{*}(0) = x_0,\ y^{*}(\tau) = y_1,\ z^{*}(\tau) = z_1 \end{cases} \tag{5.6}$$

*The solutions to Initial and Boundary Value problems will be denoted as in section 1.3; we just recall the "**end point map**" (x_1^*, y_0^*, z_0^*) for the BV solution, defined as:*

$$x_1^*\left(x_0, y_1, z_1, \tau\right) = x^*\left(\tau, x_0, y_1, z_1, \tau\right) \tag{5.7}$$
$$y_0^*\left(x_0, y_1, z_1, \tau\right) = y^*\left(0, x_0, y_1, z_1, \tau\right) \tag{5.8}$$
$$z_0^*\left(x_0, y_1, z_1, \tau\right) = z^*\left(0, x_0, y_1, z_1, \tau\right) \tag{5.9}$$

It is useful to join the "unstable variables" y, z under a single coordinate $w \overset{def}{=} (y, z)$. For example, the discussion after theorem 1.3.3 works word for word by replacing y with w and therefore proving:

Corollary 5.1.5 *The following matrices are invertible and*

$$\left(\frac{\partial w^*}{\partial w_1}|_{(t,x_0,w_1,\tau)}\right)^{-1} = \frac{\partial w}{\partial w_0}|_{\left(t,x_0,w_0^*(x_0,w_1,\tau)\right)}$$
$$\left(\frac{\partial w_0^*}{\partial w_1}|_{(x_0,w_1,\tau)}\right)^{-1} = \frac{\partial w}{\partial w_0}|_{\left(\tau,x_0,w_0^*(x_0,w_1,\tau)\right)}$$

Proof of the theorem. This time the system of integral equations is:

$$\begin{cases} x(t) = e^{tL^-}x_0 + \int_0^t e^{(t-s)L^-}f(s, x(s), y(s), z\left(s\right), \theta)ds \\ y(t) = e^{-(\tau-t)L^+}y_1 - \int_t^\tau e^{(t-s)L^+}g(s, x(s), y(s), z\left(s\right), \theta)ds \\ z\left(t\right) = z_1 - \int_t^\tau h(s, x(s), y(s), z\left(s\right), \theta)ds \end{cases} \tag{5.10}$$

Fact 5.1 *Any continuous curve, solution of the equations (5.10) is necessarily smooth in t and is a solution to the BV problem with data (x_0, y_1, z_1, τ). The viceversa is also true.*

The right hand side of the equations (5.10) defines an operator on the space of continuous curves: its fixed points are our target. In order to make it a contraction, we consider a weighted norm for the curves.

Choose any γ such that $= \delta^1 < \gamma < \alpha/2$ and denote by ${}_\gamma V_\tau$ the Banach space $\mathcal{C}^0\left([0,\tau], \mathbb{R}^n\right)$ of continuous curves endowed with the γ-weighted norm

$$\|X = (x(t), y(t), z(t))\|_\gamma = \sup_{[0,\tau]}(|x(s), y(s), z(s)|e^{\gamma s})$$

Define $T : {}_\gamma V_\tau \times \mathbb{R}^n \times W \to {}_\gamma V_\tau$ by acting on $(X(t), x_0, y_1, z_1, \theta)$ as:

$$\begin{cases} T^x(X)(t) = e^{tL^-}x_0 + \int_0^t e^{(t-s)L^-} f(s, x(s), y(s), z(s), \theta) ds \\ T^y(X)(t) = e^{-(\tau-t)L^+}y_1 - \int_t^\tau e^{(t-s)L^+} g(s, x(s), y(s), z(s), \theta) ds \\ T^z(X)(t) = z_1 - \int_t^\tau h(s, x(s), y(s), z(s), \theta) ds \end{cases}$$

Claim 5.1 *T is continuous (on ${}_\gamma V_\tau \times \mathbb{R}^n \times W$).*

As before, this can be simply checked directly.

Claim 5.2 *T is C^∞ in the ${}_\gamma V_\tau$ arguments.*

Proof. The differentials d^kT in the V arguments are the linear operators

$$d^kT_X\left(X_1', \ldots, X_k'\right)(t) = \begin{cases} \int_0^t e^{(t-s)L^-} d^k f|_{(s,X(s),\theta)}\left(X_1'(s), \ldots, X_k'(s)\right) ds \\ \int_t^\tau e^{(t-s)L^+} d^k g|_{(s,X(s),\theta)}\left(X_1'(s), \ldots, X_k'(s)\right) ds \\ \int_t^\tau e^{(t-s)L^+} d^k h|_{(s,X(s),\theta)}\left(X_1'(s), \ldots, X_k'(s)\right) ds \end{cases}$$

where $X, X_i' \in V$, and $d^k f \ldots$ are the differentials in the spatial directions. In fact let's just check Taylor's formula:

$$\psi(t) = \left|\left(T(X+X') - T(X) - dT_X X' - \ldots - dT_X^k \left(X', \ldots, X'\right)\right)(t)\right| \leq$$

$$\leq \sup \begin{cases} \int_0^t e^{-\alpha(t-s)} |f(s, X+X') - f(s,X) - \sum_{i=1}^k d^i f|_{(s,X(s),\theta)}(X', \ldots, X')\,|ds \\ \int_t^\tau e^{\alpha(t-s)} |g(s, X+X') - g(s,X) - \sum_{i=1}^k d^i g|_{(s,X(s),\theta)}(X', \ldots, X')\,|ds \\ \int_t^\tau |h(s, X+X') - h(s,X) - \sum_{i=1}^k d^i h|_{(s,X(s),\theta)}(X', \ldots, X')\,|ds \end{cases}$$

$$\leq \sup \begin{cases} \int_0^t e^{-\alpha(t-s)} \delta^{k+1}_{\|X\|+\|X'\|} \frac{|X'(s)|^{k+1}}{k+1!} ds \\ \int_t^\tau e^{\alpha(t-s)} \delta^{k+1}_{\|X\|+\|X'\|} \frac{|X'(s)|^{k+1}}{k+1!} ds \\ \int_t^\tau \delta^{k+1}_{\|X\|+\|X'\|} \frac{|X'(s)|^{k+1}}{k+1!} ds \end{cases}$$

$$\leq \frac{\delta^{k+1}_{\|X\|+\|X'\|}}{k+1!} \|X'\|_\gamma^{k+1} \sup \begin{cases} \int_0^t e^{-\alpha t + (\alpha - (k+1)\gamma)s} ds \\ \int_t^\tau e^{\alpha t - (\alpha + (k+1)\gamma)s} ds \\ \int_t^\tau e^{-(k+1)\gamma s} ds \end{cases}$$

Hence we get

$$\psi(t) e^{\gamma t} \leq \frac{\delta^{k+1}_{\|X\|+\|X'\|}}{\min\{(k+1)\gamma,\ |(k+1)\gamma - \alpha|\}(k+1!)} \|X'\|_\gamma^{k+1}$$

which provides an estimate (uniform in τ) for the $\mathcal{C}^\infty$ smoothness of T.

Claim 5.3 *T is a contraction on ${}_\gamma V_\tau$. In fact $\|dT\|_\gamma < \dfrac{\delta^1}{\gamma}$ (which is < 1 by hypothesis).*

Proof. We can easily check dT to be a contraction (here δ means δ^1):

$$\left\|dT_X X'(t)\right\|_\gamma = \sup_{[0,\tau]} e^{\gamma t} \begin{cases} \left|\int_0^t e^{(t-s)L^-} df|_{(s,X(s))} X'(s)\, ds\right| \\ \left|\int_t^\tau e^{(t-s)L^+} dg|_{(s,X(s))} X'(s)\, ds\right| \\ \left|\int_t^\tau dh|_{(s,X(s))} X'(s)\, ds\right| \end{cases}$$

$$\leq \sup_{[0,\tau]} e^{\gamma t} \begin{cases} \int_0^t e^{-\alpha(t-s)} \delta_{\|X\|} |X'(s)|\, ds \\ \int_t^\tau e^{(t-s)L^+} \delta_{\|X\|} |X'(s)|\, ds \\ \int_t^\tau \delta_{\|X\|} |X'(s)|\, ds \end{cases}$$

$$\leq \delta_{\|X\|} \left\|X'\right\|_\gamma \sup_{[0,\tau]} e^{\gamma t} \begin{cases} \int_0^t e^{-\alpha(t-s)} e^{-\gamma s} ds \\ \int_t^\tau e^{\alpha(t-s)} e^{-\gamma s} ds \\ \int_t^\tau e^{-\gamma s} ds \end{cases} \leq \frac{\delta_{\|X\|}}{\gamma} \left\|X'\right\|_\gamma$$

And therefore $\|dT_X\|_\gamma \leq \dfrac{\delta}{\gamma} < 1$ □

Remark The estimate $2\delta^1 < \alpha$ assumed as hypothesis differs from the similar estimate $\delta^1 < \alpha$ assumed in the case of isolated singularities. In the last inequality proven, in fact, we needed at the same time $\delta < \gamma$ and $\delta < \alpha - \gamma$, and hence $\delta^1 < \alpha$ was not sufficient.

Because of the previous claim, the contraction lemma provides the existence of unique fixed points for the operator T for any choice of data (x_0, y_1, z_1, τ), solving the existence and uniqueness issue for the BV problem.

Claim 5.4 *Fix (x_0, y_1, z_1, θ) and a curve $X(t) = (x(t), y(t), z(t))$. If the inequality $|x(t), y(t)| \leq 2|x_0, y_1|$ holds, then $|T^x(X)(t), T^y(X)(t)| \leq 2|x_0, y_1|$ for the first components of $T(X)$ too.*

Proof. Exactly as Claim 1.4; the γ norm is not involved here.

Claim 5.5 *The operator $T : V_\tau \times \mathbb{R}^n \times W \to V_\tau$ is C^∞ regular.*

Proof. T is affine in the spatial data (x_0, y_1, z_1) and the parameters θ can be considered as new variables adding them to "z" via the constant equation $\dot{\theta} = 0$. Since T is continuous, the theorem about differentiability after regularity of partial derivatives proves the claim.

By now, the proof of smooth dependence of solutions on the variables $(x_0.y_1, z_1, \theta)$ goes on exactly as in the case of nondegenerate singularities, relying upon the implicit function theorem in Banach spaces. The same arguments as before (replacing the variables y, z by w and using corollary 5.1.5) works word for word in the present case, proving the smooth dependence of the solution to a BV problem on τ and completing the proof of the existence and uniqueness theorem □

The derivatives of the solutions to a BV problem certainly satisfy the corresponding variational differential systems. Moreover, as in the case of isolated singularities, the variational systems are in the same form as system 5.1. The proof given for corollary 1.3.5 then shows:

Corollary 5.1.6 *Under the hypothesis of the existence and uniqueness theorem 5.1.4, the derivatives to the BV problem 5.6 with respect to the spatial BV data x_0, y_1, z_1 and parameters θ can be found as solutions to the BV problem whose equations are the variational equations and the new BV data are given by formal differentiation of the old ones. In particular the BV data are zero for the derivatives in θ or anyway for derivatives of second order and higher.*

The derivatives in t for the solutions are described by the differential equations themselves (of the original system or of the variational ones), whereas for the τ variable, the discussion in section 1.3 goes through word for word if one here substitutes $w = (y, z)$ to the variables y there.

Corollary 5.1.7 *The derivatives of the BV solutions in τ satisfy:*

$$\begin{aligned} \frac{\partial x^*}{\partial \tau}|_{(t,x_0,w_1,\tau)} &= -\frac{\partial x^*}{\partial w_1}|_{(t,x_0,w_1,\tau)} \frac{\partial w^*}{\partial t}|_{(\tau,x_0,w_1,\tau)} \\ \frac{\partial w^*}{\partial \tau}|_{(t,x_0,w_1,\tau)} &= -\frac{\partial w^*}{\partial w_1}|_{(t,x_0,w_1,\tau)} \frac{\partial w^*}{\partial t}|_{(\tau,x_0,w_1,\tau)} \end{aligned} \tag{5.11}$$

Theorem 5.1.8 *The first derivatives of the endpoint map satisfy:*

$$\frac{\partial x_1^*}{\partial x_0, w_1, \theta}|_{(x_0,w_1,\tau)} = \frac{\partial x^*}{\partial x_0, w_1, \theta}|_{(\tau,x_0,w_1,\tau)} \tag{5.12}$$

$$\frac{\partial w_0^*}{\partial x_0, w_1, \theta}|_{(x_0,w_1,\tau)} = \frac{\partial w^*}{\partial x_0, w_1, \theta}|_{(0,x_0,w_1,\tau)} \tag{5.13}$$

$$\frac{\partial x_1^*}{\partial \tau}|_{(x_0,w_1,\tau)} = \frac{\partial x^*}{\partial t}|_{(\tau,x_0,w_1,\tau)} - \frac{\partial x^*}{\partial w_1}|_{(\tau,x_0,w_1,\tau)} \frac{\partial w^*}{\partial t}|_{(\tau,x_0,w_1,\tau)} \tag{5.14}$$

$$\frac{\partial w_0^*}{\partial \tau}|_{(x_0,w_1,\tau)} = \frac{\partial w^*}{\partial \tau}|_{(0,x_0,w_1,\tau)} \tag{5.15}$$

The k^{th} derivatives of the endpoint map are linear combinations of derivatives $\dfrac{\partial^h x^*}{\partial^h t, x_0, w_1, \theta}|_{\tau,x_0,w_1,\tau}$ and $\dfrac{\partial^h w^*}{\partial^h t, x_0, w_1, \theta}|_{0,x_0,w_1,\tau}$ with $0 \leq h \leq k$, the coefficients being products of other derivatives of x^* and w^*.

Hypothesis From now on, the differential system is supposed to be local and autonomous, i.e. of the form:

$$\begin{cases} \dot{x} = L^- x + f(x, y, z) \\ \dot{y} = L^+ y + g(x, y, z) \\ \dot{z} = h(x, y, z) \end{cases} \tag{5.16}$$

The non linear terms $F = (f, g, h)$ are just defined locally near the origin, and satisfy $F(0, 0, z) = 0$ and $dF|_{(0,0,z)} = 0$.

In the new hypothesis, there exists a small enough neighborhood of the origin $U \subset \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c$ where the estimate $2\delta^1 \leq \alpha$ on the first derivatives holds. Modifying the system 5.16 away from U by a smooth cut off, the existence and uniqueness theorem 5.1.4 applies, and its thesis then holds on U for system 5.16 as well, thanks to the bound given by relation 5.5. We can now prove:

Theorem 5.1.9 (Stable Bundle) *For any point z in $\mathbb{R}^c \cap U$ there passes a unique invariant manifold S_z of dimension s called "stable manifold for z" which is the union of trajectories in U tending to z. The union $S = \cup S_z$ is a $\mathcal{C}^\infty$ bundle over the critical manifold $\mathbb{R}^c$ called the "stable bundle" and it is made up of all the trajectories which stay bounded in a neighborhood of 0 for all times.*

Proof. Using the cut off trick,it's sufficient to assume the nonlinear terms to be globally defined and the bound on their first derivatives to hold on all of $\mathbb{R}^n$.

Fix now a $z_1 \in C$ and translate the system $\dot{X} = F(X)$ so that $z_1 = 0$. The new system is just $\dot{X} = F(X - (0, 0, z_1))$, which we can consider as a system depending on the "nonspatial" parameter z_1. Consider now the Banach space V of continuous curves defined on the half line and bounded in the γ-weighted norm, for some $2\delta < \gamma < \alpha$.

The choice $z_1 = 0$ allows the integral operator T associated to 5.16 to operate on V for $\tau = \infty$. In fact T is now given by:

$$\begin{cases} T^x\left(X\right)(t) = e^{tL^-}x_0 + \int_0^t e^{(t-s)L^-}f(s, x(s), y(s), z(s), z_1)ds \\ T^y\left(X\right)(t) = -\int_t^{+\infty} e^{(t-s)L^+}g(s, x(s), y(s), z(s), z_1)ds \\ T^z\left(X\right)(t) = -\int_t^{+\infty} h(s, x(s), y(s), z(s), z_1)ds \end{cases}$$

Observe that z_1 is considered as one of the old "θ" parameters. All the estimates in the proof of theorem 5.1.4 still work for the new T, since they never involved τ explicitly. Therefore, for any fixed z_1 and for any x_0 there is a solution $x^*(t), y^*(t), z^*(t)$, defined on the half-line, which converge exponentially (faster than $e^{-\gamma t}$) to the point $(0, 0, z_1)$. One then gets a smooth "endpoint" map $(x_0, z_1) \to (y_0^*(x_0, z_1), z_0^*(x_0, z_1))$. Note that $(y_0^*(0, z_1), z_0^*(0, z_1)) = (0, z_1)$. Therefore, the graph S of the endpoint map is a smooth manifold of dimension $s + c$ and is the disjoint union of the submanifolds S_{z_1}, graphs of the restrictions of the endpoint map to $z = z_1$. Of course, S is the stable bundle and its fibers S_{z_1} the unstable manifolds. In fact, similarly to the case of isolated singularities, one proves that S_{z_1} is invariant by the flow and contains any solution which is bounded in the γ-norm after translation in $(0, 0, z_1)$, i.e. any solution which tend to $(0, 0, z_1)$ faster than $e^{-\gamma t}$ (for any $2\delta < \gamma < \alpha$).

The fact that all bounded small solutions actually belong to the stable bundle can be seen using the estimates we'll later prove □

Remark Repeating all the discussion by considering z as a stable coordinate (i.e. treating z as the x coordinates instead that as the y ones) provides analogous results. In particular there exists the "unstable bundle" of the critical manifold C whose fibers are the "unstable manifolds" for the points in C.

Using an analogous of lemma 1.3.13, it is not difficult to check the transversalities needed to state the following:

Corollary 5.1.10 (Straighten Coordinates) *After a smooth change of coordinates, the system 5.16 can be written as*

$$\begin{cases} \dot{x} = L^{-}x + f(x,y,z)x \\ \dot{y} = L^{+}v + g(x,y,z)y \\ \dot{z} = h(x,y,z)xy \end{cases} \tag{5.17}$$

*where f, g, h are square matrices of smooth functions vanishing on the critical manifold $\mathbb{R}^c$ and $h(x,y,z)xy$ stands for a vector whose components are sums of terms each of which factors through $x_i y_j$, for some i, j. The new coordinates are called **straighten coordinates** since the stable and unstable manifolds in z_0 are given by $y = 0, z = z_0$ and $x = 0, z = z_0$ respectively.*

In straighten coordinates, useful estimates hold on the size of the solution. Again, the "not leaving the neighborhood" issue (usually needed for this kind of estimates) is simple to deal with, for solutions to Boundary Value problems, since these are confined by construction.

Theorem 5.1.11 *Suppose $|x_0, y_1| \leq \varepsilon$. Then for any $\tau \in [0, +\infty)$ the following inequality hold, for some constant C_0 independent on τ:*

$$\begin{cases} |x^*(t, x_0, y_1, \tau)| \leq |x_0| e^{-(\alpha-\delta)t} \\ |y^*(t, x_0, y_1, \tau)| \leq |y_1| e^{(\alpha-\delta)(t-\tau)} \\ |z^*(t, x_0, y_1, \tau) - z_1| \leq C_0 \tau e^{-\delta\tau} \end{cases} \tag{5.18}$$

$$\begin{cases} |x_1^*(x_0, y_1, \tau)| \leq |x_0| e^{-(\alpha-\delta)\tau} \\ |y_0^*(x_0, y_1, \tau)| \leq |y_1| e^{-(\alpha-\delta)\tau} \\ |z_0^*(x_0, y_1, \tau) - z_1| \leq C_0 \tau e^{-\delta\tau} \end{cases} \tag{5.19}$$

Proof. The Gronwall lemma, applied exactly as in the proof of theorem 1.3.16 proves the estimates on the x, y variables, since nothing formally changes. As for z, observe that

$$z(t) - z_1 = -\int_t^\tau h\left(x(s), y(s), z(s)\right) x(s) y(s) ds$$

so that the thesis follows, bounding h by δ $\square$

As for the derivatives of the solutions, we observe that the variational systems for derivatives of order k in the x, y, z or t variables have the form:

$$\begin{cases} \dot{X^k} = L^- X^k + f(x^*, y^*, z^*)X^k + (\cdots)X^{k-1} + \ldots + (\cdots)x \\ \dot{Y^k} = L^+ Y^k + g(x^*, y^*, z^*)Y^k + (\cdots)Y^{k-1} + \ldots + (\cdots)y \\ \dot{Z^k} = \sum(\cdots)(X^i Y^{h-i}) \end{cases} \tag{5.20}$$

This means that each summand in the first equation factors trough some lower order derivatives of x (and similarly in the second), whereas in the third equation all addenda factor through products of derivatives of x and y (of course $0 \leq h \leq k$ in the formula).

Theorem 5.1.12 *Suppose $\varepsilon > 0$ is such that $k\delta < \alpha$. Let $X^*(t), Y^*(t)$ be some k^{th}-order derivatives of (x^*, y^*) in the spatial variables x_0, y_1 and/or in the time variables t, τ. The following inequalities hold, for some constant $C_k > 0$:*

$$|X^*(t)| \leq C_k e^{-(\alpha - k\delta)t}$$

$$|Y^*(t)| \leq C_k e^{(\alpha - k\delta)(t-\tau)}$$

$$|Z^*(t)| \leq C_k \tau e^{-(\alpha - k\delta)(\tau)}$$

Proof. The proof of the first two inequalities is exactly the same as in theorem 1.3.18 since the expressions are formally the same. It remains to deal with Z^*. Since Z^* solves

$$Z(t) - Z_1 = -\int_t^\tau \sum(\cdots(s))(X^i(s)Y^{h-i}(s))ds$$

The dots are bounded by a uniform constant depending on the estimates on the derivatives f, g, h, whereas $|X^i(s)Y^{h-i}(s)| \leq Ce^{(\alpha - k\delta)\tau}$ where C depends on the constants coming from the estimates of X^i and Y^{h-i} and on the number of terms involved. Integrating gives the thesis $\square$

By theorem 5.1.8, in the same hypothesis of the previous theorem, it then follows (again w stands for (y, z)):

Theorem 5.1.13 *Let $V^*(\tau)$ denote some derivative among $\frac{\partial^k x_1^*}{\partial^k x_0, w_1, \tau}|_{(x_0, w_1, \tau)}$ or among $\frac{\partial^k w_0^*}{\partial x_0, w_1, \tau}|_{(x_0, w_1, \tau)}$. Then the following inequality holds, for some constant $C_k > 0$, not depending on x_0, w_1, τ :*

$$|V^*(\tau)| \leq (1 + \tau)C_k e^{-(\alpha - k\delta)\tau}$$

Conclusions We finally draw out the consequences which will be used later. For the sake of clearness we repeat all the hypothesis here.

Consider a vector field on $\mathbb{R}^n = \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c$ having $\mathbb{R}^c$ as critical submanifold of Bott singularities. Let $\phi = (\phi_t)_{t\in\mathbb{R}}$ be the (local) flow of solutions near a critical point, which we assume to be the origin. Then there exists "straighten coordinates" x, y, z near the origin for which $S = \{y = 0\}$ is the stable bundle and $S_{z_1} = \{y = 0, z = z_1\}$ the stable manifolds, $U = \{x = 0\}$ the unstable bundle and $U_{z_1} = \{x = 0, z = z_1\}$ the unstable manifolds.

Consider then the unit cylinder $B = \{(x, y, z)|\,|x, y, z| < 1\,\}$ and decompose its boundary ∂B in the pieces:

$$\partial^+ B = \{|x| = 1,\, |y, z| \leq 1,\} \qquad \partial^- B = \{|x, z| \leq 1,\, |y| = 1\}$$
$$\partial^c B = \{|z| = 1,\, |x, y| \leq 1,\}$$

We can suppose that $\partial^- B$ and $\partial^+ B$ are transversal to the vector field (at least for small z) and that if a point $m \in \partial^- B$ does not belong to the stable bundle S and is distant from $\partial^c B$, then the trajectory starting from m will touch ∂B again in $\partial^+ B$. This is granted by corollary 5.1.10 and the estimates in theorem 5.1.11. The construction defines a "first escape" map

$$\varphi : \partial^+ B \backslash S \to \partial^- B \backslash U$$

Since the vector field is not tangent to $\partial^\pm B$ and the flow of solutions is a smooth map, an application of the implicit function theorem proves

Theorem 5.1.14 *The first escape map $\varphi : \partial^- B\backslash S \to \partial^+ B\backslash U$ is a diffeomorphism onto its image.*

Now, consider the submanifold

$$W = \{\left(\phi_{\frac{t}{1-t}}(m), m, t\right) |m \in \mathbb{R}^n,\, t \in (0, 1)\} \cap B \times B \times (0, 1) \subset \mathbb{R}^{2n+1}$$

Theorem 5.1.15 *The closure $\overline{W} = (W, \partial W)$ is a smooth, closed submanifold in $\times B \times B \times \mathbb{R}$ with boundary*

$$\partial W = U \times_C S \times \{1\} \cup \Delta \times \{0\}$$

where

$$U \times_C S = \{(0,y,z,x,0,z) \in \mathbb{R}^n \times \mathbb{R}^n\}$$

is the fiber product of the stable and unstable bundles, which is a smooth submanifold of dimension n.

Proof. Using the starred notation for the solution to the BV problem, by the existence and uniqueness theorem one can reintrepret W as:

$$W = \Big\{ \Big(\phi_{\frac{t}{1-t}}(x_0, y_0, z_0), x_0, y_0, z_0, t \Big) \,\Big|\, (x_0, y_0, z_0) \in \mathbb{R}^n \Big\} \cap B \times B \times (0,1)$$

$$= \Big\{ \big(x_1^*(x_0, y_1, z_1, \tfrac{\tau}{1-\tau}), y_1, z_1, x_0, y_0^*(x_0, y_1, z_1, \tfrac{\tau}{1-\tau}), z_0^*, \tau \big) \;\Big|\; \begin{matrix} |x_0, y_1, z_1| < 1 \\ \tau \in (0,1) \end{matrix} \Big\}$$

so that the estimates in theorem 5.1.13, combined with the same invariance argument given for theorem 1.3.21 permit to conclude □

5.2 Vector Fields and One Forms with Bott Singularities

All over this section X will denote an oriented (not necessarily compact) n-dimensional manifold, V a complete vector field and ϕ the corresponding flow. The vector field is supposed to have Bott singularities distributed along compact oriented (critical) submanifolds.

Using the notations U_C (resp. S_C) for the unstable bundle (resp. stable bundle) of the critical manifold C and U_p (resp. S_p) for the unstable manifold (resp. stable) of the critical point p, we introduce the following:

Definition 5.2.1 *The flow ϕ satisfies the* ***generalized Smale condition*** *if all the intersections between stable bundles and unstable manifolds and between stable manifolds and unstable bundles are transversal. In other words, for any critical points and critical manifolds $p \in C_p$ and $q \in C_q$, the intersections $S_{C_p} \cap U_q$ and $S_p \cap U_{C_q}$ are transversal.*

We assume the flow ϕ to satisfy the generalize Smale condition and the vector field V to be Weakly Proper with respect to some smooth function $f : X \to \mathbb{R}$. Note that for a vector field with Bott singularities it's enough to check the Weakly Proper hypothesis just on points instead than on compact sets, cf. proposition 2.1.3.

Under these hypotheses, any critical manifold is contained in a level set of f. Two critical manifolds C and C' cannot intersect and if there is a (possibly broken) flow line connecting C and C' then the inequalities $\lambda_C < \lambda_{C'}$ and $\lambda^*_{C'} < \lambda^*_C$ hold between the corresponding indexes.

Remark Contrary to the case of Morse functions, it is not true that any function f with Bott singularities always admits a metric such that the gradient vector field ∇f is generalized Smale, cf. [18] for an example.

We will next show how to generalize the results of chapters 2 and 3 to flows of vector fields with Bott singularities, starting with the following:

Definition 5.2.2 *A compact stratified set Y of dimension k will be called **compact generalized horned-stratified** (or **compact gh-stratified**) set if:*

- *The stratification of Y is AB Whitney regular;*
- *There exists a k-dimensional compact manifold with corners M, a refinement of the stratification of M to an AB regular stratification which does not modify the interior of M, and a submersive stratified map $\pi : M \to Y$.*

*The manifold M with the refined stratification is called a **desingularization** for Y and the map π a **gh-projection**.*

*A **locally gh-stratified set** $Y \subset X$ is a stratified set which locally coincide with a compact generalized horned stratified set.*

Remark. The A-regularity requirement for Y in the previous definition is granted by the existence of the desingularization $M \to Y$. Observe, moreover, that the refinement of the stratification on M is not supposed to be locally finite. It is often needed, for example, to consider a two dimensional face as stratified by one dimensional submanifolds.

The arguments in section 2.2 work in the present setting by replacing h-stratified with gh-stratified and giving to "desingularizing" the new meaning of definition 5.2.2. The only point to keep in mind is the double nature of the desingularization: as manifold with corners and as a stratified set. As an example, we prove the following:

Proposition 5.2.3 *Let $Y \subset X$ be a gh-stratified set and $Z \subset X$ a closed submanifold. Suppose that each stratum of Y is transversal to Z. Then the family of all intersections of strata makes up a gh-stratification for $Z \cap Y$.*

Proof. It's enough to find a desingularization for $Z \cap Y$. Let $\pi : M \subset N \to X$ be a local desingularization for Y. By hypothesis, the restriction of π to any stratum of M (both as a manifold with corners and as a stratified space, after the refinement) is transversal to Z and thus $M \cap f^{-1}(Z)$ is endowed with both the structures of compact manifold with boundary and AB regular stratification (by refinement of the corners). The restriction of π is a gh-projection, because of lemma 2.2.5 $\square$

Referring to section 2.2 for the proofs, that basically work word for word as observed above, we just state:

Proposition 5.2.4 *Let $Y_1, Y_2 \subset X$ be gh-stratified sets and suppose that all possible intersections of strata of Y_1 and Y_2 are transversal. Then the family of these intersections is a gh-stratification for $Y = Y_1 \cap Y_2$.*

Corollary 5.2.5 *Let $Y_1, Y_2 \subset X$ be gh-stratified sets and $y \in Y_1 \cap Y_2$. If the two strata containing y intersect transversally, then all the strata near y intersect transversally and hence, near y, the intersection $Y_1 \cap Y_2$ is a gh-stratified set.*

Remark In the case of h-stratifications, the local desingularizations define "h-chain" currents by pushforward under some orientability assumption. For gh-stratifications one might still define "gh-chain currents" by pushforward (assuming the desingularizations are oriented maps), but such a family would not be close under the boundary operation and so we'll not pursue this aim. Nevertheless the pushforward of a gh-projection is a quite special chain current, in fact (cf. proposition 2.2.15):

Proposition 5.2.6 *Let $[T]$ be a k-dimensional chain current in $\mathbb{R}^n$ locally defined as the pushforward $\pi_*(M)$ under the gh-projection $\pi : M \to T$ of the gh-stratified space $\overline{T}$. If $[S]$ is chain current of dimension l and the simplexes which locally define S are transversal to the strata of T, then $[S]$ and $[T]$ can be "intersected" and $[S] \wedge [T] = [S \cap T]$ is a chain current of dimension $h + k - n$.*

Keeping the notations introduced in section 2.3, we can now state the following generalization of theorem 2.3.1:

Theorem 5.2.7 *Let $(X, \phi, f : X \to \mathbb{R})$ be a Weakly Proper generalized Smale dynamical system, $c \in \mathbb{R}$ a regular value for f and $M \subset f^{-1}(c)$ a smooth (embedded) submanifold of dimension m. Suppose M is transversal to every stable manifold and let $N = \bigcup_{t>0} \phi_t(M)$. Then the closure $\overline{N}$ coincides with the shadow L^M and it is a gh-stratified subset of dimension $m+1$ whose singular strata are unstable manifolds.*

As in the case of isolated singularities, one also gets the following:

Corollary 5.2.8 *The unstable bundle U_C of a critical manifold C (resp. the unstable manifold U_p of a critical point p) is a generalized horned stratified subset whose singular strata are the unstable manifolds in the shadow of C (resp. p). Similarly for stable bundles and manifolds.*

The proof given for the analogous theorem 2.3.1 is still valid for the previous theorem, but its preliminary results (propositions 2.3.4 and 2.3.5) need to be modified to fit in the new frame. We thus just prove the two adequate extensions.

Suppose p is a critical point belonging to the critical manifold C. We fix a chart Ω in "straighten coordinates" x, y, z centred in p (cf. the previous section). The boundary of Ω decomposes as $\partial\Omega = \partial^+\Omega \cup \partial^-\Omega \cup \partial^c\Omega$, where

$$\begin{aligned}\partial^+\Omega &= \{\|x\| = \varepsilon, \|y, z\| \le \varepsilon\}\\ \partial^-\Omega &= \{\|x, z\| \le \varepsilon, \|y\| = \varepsilon\}\\ \partial^c\Omega &= \{\|x, y\| \le \varepsilon, \|z\| = \varepsilon\}\end{aligned}$$

Note that the third component of boundary is not necessarily invariant by the flow. Define the "**links**" of (un)stable bundles and manifolds as:

$$\begin{aligned}S^+ &= S \cap \partial^+\Omega = \{\|x\| = 1,\, y = 0\} \quad , \quad S_z^+ = S_z \cap \partial^+\Omega\\ U^- &= U \cap \partial^-\Omega = \{\|y\| = 1,\, x = 0\} \quad , \quad U_z^- = U_z \cap \partial^-\Omega\end{aligned}$$

We can now prove the analogous of the "second brick" in the proof of theorem 2.3.1 :

Proposition 5.2.9 *Suppose $A^+ \subset \partial^+\Omega$ is a compact manifold with corners such that all strata have transversal intersection with S. Put*

$$A = \{\phi_t(x) | x \in A^+, t > 0\} \cap \Omega$$

Then the closure $\overline{A}$ coincides with the shadow $L_\Omega^{A^+}$ and is a compact generalized horned stratified set. Its singular strata (besides the shadows of the strata in A^+) are unstable manifolds.

The closure of $A^- \overset{def}{=} A \cap \partial^-\Omega$ is gh-stratified too. Its singular strata are links of unstable manifolds (and of the shadows of the strata in A^+).

Remark The shadow of A^+ does not touch $\partial^c\Omega$ at all provide Ω is small enough and A^+ is well disjoint by $\partial^c\Omega$, thanks to the estimates in theorem 5.1.11.

Proof. We restrict to the case when $A^+ \subset \partial^+\Omega$ is a smooth compact submanifold, transverse to the stable bundle S. Since the flow is not tangent to A^+, the parametrization

$$(t, m) \in [0, +\infty) \times A^+ \overset{\sigma}{\longmapsto} \phi_t(m) \in A$$

is a diffeomorphism. Consider now the following submanifold of $\mathbb{R}^{2n} \times \mathbb{R}$:

$$W = \{\left(\phi_{\frac{t}{1-t}}(m), m, t\right) | m \in \mathbb{R}^n,\, t \in (0, 1)\} \cap \Omega \times \Omega \times \mathbb{R} \subset \mathbb{R}^{2n+1}$$

By the BV technique (cf. theorem 5.1.15), within $\Omega \times \Omega \times \mathbb{R}$, the manifold W is smooth, with boundary

$$\partial W = U \times_C S \times \{1\} \cup \Delta \times \{0\}$$

Define two subsets of $\mathbb{R}^{2n} \times \mathbb{R}$ by:

$$Z \stackrel{def}{=} \mathbb{R}^n \times A^+ \times \mathbb{R}$$
$$(A^1, \partial A^1) \stackrel{def}{=} Z \cap (W, \partial W) = \left\{ (m', m, t) \,|\, 0 \leq t \leq 1, m \in A^+, m' = \phi_{\frac{t}{1-t}}(m) \right\}$$

The submanifold Z is transverse to ∂W since A^+ is transverse to S. Besides, transversality of $W \cap Z$ elsewhere is readily proved. Therefore, $(A^1, \partial A^1)$ is a smooth compact manifold with boundary:

$$\partial A^1 = U \times_C \left(A^+ \cap S\right) \cup \Delta_{A^+} \times \{0\}$$

where

$$U \times_C \left(A^+ \cap S\right) \stackrel{def}{=} \bigcup_{(x,0,z) \in A^+ \cap S} U_z \times \{(x, 0, z)\} \subset \mathbb{R}^n \times \mathbb{R}^n$$

It is actually the previous decomposition for $U \times_C (A^+ \cap S)$ that suggests how to refine some of the strata in ∂A^1 into a family of (possibly lower dimensional) strata, in the spirit of definition 5.2.2. From now on, the component of boundary $U \times_C (A^+ \cap S) \subset \partial A^1$ is not any longer to be considered a submanifold of the same dimension of A^+ but as a union of strata of dimension λ^*_C (they are unstable manifolds).

Projecting onto the first factor via

$$\pi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

it is $\pi\left(A^1\right) = A$ and the topological boundary of A is :

$$\bigcup_{(x,0,z) \in S \cup A^+} U_z \subset \mathbb{R}^n$$

so that $\overline{A}$ is exactly the shadow of A^+ in Ω. Moreover, the projection π is clearly a diffeomorphism on the top stratum, and a trivial projection on the singular strata (after the refinement), thus:

$$\pi|_{(A^1, \partial A^1)} : (A^1, \partial A^1) \to \overline{A} = L_\Omega^{A^+}$$

is a gh-projection. As for the "regularity" of the stratified space $\overline{A}$, the A-regularity condition follows from being submersed by the gh-projection (cf. the remark after definition 5.2.2). The B-regularity follows by the invariance of A under the flow, as observed in the proof of proposition 2.3.4.

Finally, the intersection $\overline{A^-} = \overline{A} \cap \partial^- \Omega$ is transversal, and hence gh stratified too, thanks to proposition 5.2.3 $\square$

Next, we state the analogous of the "third brick" (cf. proposition 2.3.5):

Proposition 5.2.10 *The previous proposition 5.2.9 holds word for word by replacing "compact manifold with corners" with "compact gh-stratified set".*

Proof. Let $\mathcal{A}^+$ be a compact submanifold with boundary which desingularizes the gh-stratified space A^+. It is not restrictive to suppose $\mathcal{A}^+ \subset \mathbb{R}^j \times \partial^+\Omega$ and the h-projection to be the restriction of the projection onto the second factor.

The auxiliary flow ψ on $\mathbb{R}^j \times \mathbb{R}^n$given by $\psi_t(x', x) = (e^{-t}x', \phi_t(x))$ on $\mathbb{R}^j \times \mathbb{R}^n$ extends ϕ by a linear contraction and the following diagram commutes

$$\begin{array}{ccc} \mathbb{R}^j \times \mathbb{R}^n & \xrightarrow{\psi_t} & \mathbb{R}^j \times \mathbb{R}^n \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{R}^n & \xrightarrow{\phi_t} & \mathbb{R}^n \end{array} \tag{5.21}$$

Note that C is a critical manifold of Bott singularities for ψ; the stable bundle is $\mathbb{R}^j \times S$ and the unstable bundle is $\{0\} \times U$. Clearly the stable bundle "upstairs" is transversal to the desingularization $\mathcal{A}^+ \subset \mathbb{R}^j \times \partial^+\Omega$, which is a compact manifold with corners. The hypotheses of proposition 5.2.9 are thus fulfilled for the flow ψ by the "incoming link" $\mathcal{A}^+$ and therefore the shadow $\mathcal{A}$ of $\mathcal{A}^+$ into $\mathbb{R}^j \times \Omega$ is a gh-stratified set having unstable manifolds $\{0\} \times U_z$ as singular strata (besides the shadows of the strata in $\mathcal{A}^+$).

Let's now look to the closure $\overline{A}$ "downstairs". Since the diagram 2.1 commutes, it follows that $A = \pi(\mathcal{A})$ and hence:

$$\overline{A} = L_\Omega^{A^+} = A \cup \bigcup_{(x,0,z) \in A^+ \cup S} U_z$$

We now prove $\overline{A}$ is gh-stratified. Denote by $\overline{\mathcal{A}^1}$ the desingularization of $\overline{\mathcal{A}}$ provided by proposition 5.2.9. Then $\overline{\mathcal{A}^1} = (\mathcal{A}^1, \partial\mathcal{A}^1)$ is a compact manifold with corners ($\partial\mathcal{A}^1$ being the family of its "singular" strata). After the refinement of some of the singular strata as

$$\partial\mathcal{A}^1 = \bigcup_{\cdots} U_z$$

the projection $\pi_1 : (\mathcal{A}^1, \partial\mathcal{A}^1) \to (\mathcal{A}, \partial\mathcal{A})$ is a gh projection (i.e. a stratified submersive map), and in particular a trivial submersion on the singular strata of the form U_z. Note that the symbol U_z makes sense both on the "first floor" $\mathbb{R}^n$, "second floor" $\mathbb{R}^j \times \mathbb{R}^n$ and "third floor" $\mathbb{R}^{j'} \times \mathbb{R}^j \times \mathbb{R}^n$, since the critical manifolds are of the form C, $\{0\} \times C$ and $\{0\} \times C$. The composition of projections $(\mathcal{A}^1, \partial\mathcal{A}^1) \to (\mathcal{A}, \partial\mathcal{A}) \to (A, \partial A)$ is then clearly a gh-projection being a diffeomorphism on the top stratum and a submersion on the singular strata.

The A-B regularity conditions for the stratification $\overline{A}$ follow, as before, by the existence of the desingularization and by the invariance of the set A under the flow. Therefore, $\overline{A}$ is gh-stratified. Tranversality of $\overline{A^-} = \overline{A} \cap \partial^- \Omega$ implies that $\overline{A^-}$ is gh-stratified too $\square$

As already observed, the previous two propositions, combined with the same arguments in section 2.3 prove theorem 5.2.7. As an application, we can prove the following generalization of a theorem in [18] (cf. definition 1.2.1):

Theorem 5.2.11 *Let (X, ϕ, f) be a Weakly Proper, generalized Smale dynamical system. Then the closure of the "total graph" $T \subset X \times X$ is a generalized horned stratified space whose singular strata are the diagonal Δ and the product submanifolds $U_p \times S_q$ for $q \preccurlyeq p$. In particular, the flow ϕ has locally finite volume.*

If, moreover, X is oriented and any critical manifold $C \in Cr(\phi)$ and its stable and unstable bundle are oriented (inducing orientations on the stable and unstable manifolds) in such a way that $< X >=< U_p > + < C > + < S_p >$ for any critical point p, then the following ***fundamental Morse-Bott equation*** *holds:*

$$dT = \Delta - P \tag{5.22}$$

where (denoting by Cr the family of critical manifolds)

$$P = \sum_{C \in Cr} U_C \times_C S_C \tag{5.23}$$

We recall that $q \preccurlyeq p$ means that there is a (possibly broken) flow line connecting q and p.

Proof. Using the auxiliary flow on $S^1 \times X \times X$ constructed in section 2.4 (which now is generalized Smale), the statement concerning the gh-stratified structure of $\overline{T}$ follows as there. It is not so for equation 5.22 though, because the boundary of the current defined by a gh-stratified space is not so simple as that of a h-chain. We restrict to sketch the steps of the proof of equations 5.22 and 5.23, referring to [18] and [13] for more details.

1) The current $dT - \Delta$ is supported inside $\mathcal{P} = \cup_{q \preccurlyeq p} U_p \times S_q$, for p and q critical points.

2) The set $\mathcal{P}_0 = \cup_{q \prec p} U_p \times S_q$ has Hausdorff dimension not bigger than $n-2$.

3) Equations 5.22 and 5.23 hold in a neighborhood of (p,p) for any critical point p, as a consequence of the BV technique (cf. the similar proof for non degenerate singularities).

4) Equation 5.22 clearly holds locally on $X \times X - \mathcal{P}$ (meaning that Δ is one component of the current boundary of T, since P is certainly supported

inside $\mathcal{P}$). Using the diffeomorphisms $(\phi_t, \phi_{t'})$ of $X \times X$ and the invariance of T under them, point 3) proves that the two equations hold on $X \times X - \mathcal{P}_0$.

5) Applying the support theorem for flat currents (cf. appendix A) completes the proof $\square$

The kernel calculus (cf. appendix A) translates the Fundamental Morse-Bott Equation into an equations of operators from smooth forms to currents; to better describe those operators it's worth to first consider a special example of kernel.

Let $F \to B$ be a fiber bundle with F and B compact oriented manifolds. Suppose the fibers to be oriented compact manifold with corners. The current pushforward π_{F*} in this case is just the integration along the fibers and sends smooth forms on F to smooth forms on B (decreasing the degree by the dimension of the fiber). In particular, the pullback π_F^* (by duality) acts on the space of currents, so that both operators act indifferently on forms and currents. For example, the pullback a point mass in B is the corresponding fiber, $\pi_F^*([b]) = [F_b]$.

The Stokes theorem implies the following formulas (α and β might be either forms or currents):

$$d\pi_*(\alpha) = \pi_*(d\alpha) + \pi_*^{\partial}(\alpha) \qquad d\pi^*(\beta) = \pi^*(d\beta) + \pi^{\partial *}(\beta) \tag{5.24}$$

where π_*^{∂} is the pushforward of the boundary bundle $\partial F \to B$.

Lemma 5.2.12 *Let $F \to B$ and $G \to B$ two bundles as above and $F \times_B G \subset F \times G$ be the fiber product. Then the operator associated to the current $P = F \times_B G$ in $F \times G$ is given by:*

$$\mathbf{P} : \mathcal{E}^*(F) \to \mathcal{D}'^*(G) \qquad \alpha \to \pi_G^*(\pi_{F*}(\alpha))$$

Proof. Let π_1 and π_2 be the projections of $F \times G$ onto its factors and let $\alpha \in \mathcal{E}^*(F)$, $\beta \in \mathcal{E}^*(G)$. Then, by definition:

$$\begin{aligned}(\mathbf{P}\alpha)(\beta) = P\left(\pi_1^*(\alpha) \wedge \pi_2^*(\beta)\right) &= \int_{F \times_B G} \left(\pi_1^*(\alpha) \wedge \pi_2^*(\beta)\right) \\ &= \int_B \pi_{F*}(\alpha) \wedge \pi_{G*}(\beta) = \int_G \pi_G^*(\pi_{F*}\alpha) \wedge \beta \qquad \square\end{aligned}$$

Return now to the Fundamental Morse-Bott Equation and look at the current $P = \sum_{C \in Cr} U_C \times_C S_C$. It follows by corollary 5.2.8 that for any critical manifold C the unstable bundle admits a desingularization $\widetilde{U}_C \overset{\pi_{\widetilde{U}_C}}{\to} \overline{U_C}$ with compact fibers from a manifold with corners $\widetilde{U}_C$. A similar desingularization $\widetilde{S}_C \overset{\pi_{\widetilde{S}_C}}{\to} \overline{S_C}$ is provided for the stable bundle.

Observe that there is no smooth map $\overline{U_C} \to C$, but just smooth submersions on their interior:

$$U_C \overset{\pi_U}{\to} C \overset{\pi_S}{\leftarrow} S_C$$

Nevertheless, one can construct the diagram:

$$\begin{array}{cccccc}
C & & \leftarrow & \widetilde{S_C} & & \\
 & & & \downarrow & & \\
\uparrow & & & \overline{S_C} & \rightarrow & X \\
 & & & & & \\
\widetilde{U_C} & \rightarrow & \overline{U_C} & & & \\
 & & \downarrow & & & \\
 & & X & & &
\end{array} \tag{5.25}$$

We can now describe the operator $\mathbf{P}_C$ associated to the kernel current $U_C \times_C S_C \in \mathcal{D}'^n(X \times X)$, recalling that both U_C, S_C and $U_C \times S_C$ are defined as pushforwards of the given desingularizations (the currents are denoted by the same symbol as their own support).

Let $\alpha \in \mathcal{E}^*_{cpt}(X)$. Pull it to $\widetilde{\alpha}$ via the composition $\widetilde{U_C} \to \overline{U_C} \to X$. Then, apply lemma 5.2.12 to get the form $\pi^*_{S_C}(\pi_{U_C *}(\alpha))$ on $\widetilde{S_C}$. Finally, push this (smooth) current into X via the composition $\widetilde{S_C} \to \overline{S_C} \to X$. Of course the result is a current supported on $\overline{S_C}$, which will be denoted by $\mathbf{P}_C(\alpha) = \pi^*_{S_C}(\pi_{U_C *}(\alpha))$ with a slight abuse of notation. The same "chasing" of diagram 5.25 can be used to define the action of $\mathbf{P}_C$ over a local chain current K, provide K is transversal to all the unstable manifolds of points in C: intersect K with $\overline{U_C}$, pushforward to C, pullback to S_C, pushforward again into X. Again, we put $\mathbf{P}_C(K) = \pi^*_{S_C}(\pi_{U_C *}(K))$, cf. [18] for more details.

The following formula is a direct consequence of the relations 5.24 and holds when θ is either a smooth form or a chain current transversal to all unstable manifolds:

$$d(\mathbf{P}_C\theta) = \mathbf{P}_C(d\theta) + \pi^*_{S_C}(\pi_{\partial U_C *}(\theta)) + \pi^*_{\partial S_C}(\pi_{U_C *}(\theta)) \tag{5.26}$$

Remark It is not difficult to prove (cf. [19]) that for any smooth form $\theta \in \mathcal{E}^*(C)$ there exists a form $\eta \in \mathcal{E}^*(X)$, with η supported on a neighborhood of C, such that $\pi^*_{S_C}(\theta) = \pi^*_{\partial S_C}(\pi_{U_C *}(\eta)) = \mathbf{P}_C(\eta)$. The same is true for transversal chain currents.

So far we described the behaviour of each $\mathbf{P}_C$. Let's now pass to $\mathbf{P} = \sum \mathbf{P_C}$. The discussion about forward and compact/forward supports already done in the case of isolated singularities still works in the present framework, since it depends on the weakly proper assumption for the flow. As before, the operator $\mathbf{P}$ extends to act on any smooth form α with forward (resp c/f) support, and $\mathbf{P}(\alpha)$ is then a forwardly supported current (resp. c/f supported). The same is true when applying $\mathbf{P}$ to chain currents. The application of kernel calculus to equations 5.22 and 5.23 then allows us to generalize theorem 3.1.4 as follows:

Theorem 5.2.13 *The operators* $\mathbf{T}$, $\mathbf{I}$ *and* $\mathbf{P}$ *associated to the kernel currents* T, Δ *and* P, *which a priori are operators from*

$$\mathcal{E}^*_{cpt}(X) \quad \longrightarrow \quad \mathcal{D}'^*(X)$$

extend to continuous operators from

$$\mathcal{E}^*_{\uparrow}(X) \quad \longrightarrow \quad \mathcal{D}'^*_{\uparrow}(X)$$

and satisfy the following equation, called ***Morse-Bott Chain Homotopy****:*

$$d \circ \mathbf{T} + \mathbf{T} \circ d = \mathbf{I} - \mathbf{P} \tag{5.27}$$

In particular, the operator $\mathbf{P}$ *satisfies:*

$$\mathbf{P} = \lim_{t \to +\infty} \phi_t^* = \sum_{C \in Cr} \pi^*_{S_C} \circ \pi_{U_C *} \tag{5.28}$$

Moreover, the operators $\mathbf{T}$, $\mathbf{I}$ *and* $\mathbf{P}$ *act on any local chain current* K *on* X *with forward support provide* K *is transversal to all the unstable manifolds. In this case,* $\mathbf{T}(K)$ *and* $\mathbf{P}(K)$ *are forward chain currents and the previous expression for* $\mathbf{P}$ *still holds.*

All the statements remain true by replacing "forward" with "compact/forward" everywhere.

Corollary 5.2.14 *The operator* $\mathbf{P}$ *commutes with the boundary. Moreover, if* $\mathbf{P}$ *acts on the current* θ *then it acts on* $\mathbf{P}(\theta)$ *too and* $\mathbf{P}(\mathbf{P}(\theta)) = \mathbf{P}(\theta)$.

Following [18], we next define the $\mathcal{S}$-complex for the flow ϕ, as a generalization of the $\mathcal{S}$-complex studied in chapter 3. As before, this will be a geometric realization of the Morse-Bott complex (cf. [1], [9]).

Definition 5.2.15 *The* ***(compact/forward)*** $\mathcal{S}$***-complex over*** $\mathbb{R}$, *denoted by* ${}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(\phi)$, *is the subcomplex of* $\mathcal{D}'^*_{c\uparrow}(X)$ *(the complex of currents with c/f support) consisting of those currents of the form*

$$\sum \pi^*_{S_C}(\varphi_C)$$

where the sum is taken over a family of critical manifolds whose union is a compact forward sets and each $\varphi_C \in \mathcal{E}^*(C)$ *is a smooth form on the critical* *(and compact) manifold C. The* ***(compact/forward) $\mathcal{S}$-complex over $\mathbb{Z}$*** *is analogously defined by requiring φ_C to be a chain current on C. Similarly we define $\mathcal{S}^*_{\uparrow}(\phi)$ (the* ***forward $\mathcal{S}$-complex****).*

The boundary d is the current boundary in all the cases.

The names "over the reals" and "over the integers" are chosen just because the corresponding complex computes topological invariants with such coefficients.

Remark As a consequence of the remark before theorem 5.2.13, the compact-forward ${}_{\mathbb{R}}\mathcal{S}$-complex coincides with the image $\mathbf{P}\left(\mathcal{E}^*_{c\uparrow}(X)\right)$, whereas the ${}_{\mathbb{Z}}\mathcal{S}$-complex is the image of $\mathbf{P}$ over compact forwardly supported chain currents transverse to all the unstable manifolds. The closure of these complexes under the boundary is then granted by corollary 5.2.14.

Using corollary 5.2.14, a proof along the lines of theorem 3.2.3 readily yields:

Theorem 5.2.16 *The maps*

$$\mathbf{P} : \mathcal{E}^*_{c\uparrow}(X) \longrightarrow {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(\phi) \quad \text{and the inclusion} \quad \mathbf{J} : \ {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(\phi) \hookrightarrow \mathcal{D}'^*_{c\uparrow}(X)$$

induces isomorphisms in real cohomology:

$$H^p_{c\uparrow}(X, \mathbb{R}) \approx H^p\left({}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(\phi)\right)$$

Similarly, the maps

$$\mathbf{P} : \mathcal{C}^*_{c\uparrow}(X) \dashrightarrow \ {}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(\phi) \quad \text{and the inclusion} \quad \mathbf{J} : \ {}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(\phi) \hookrightarrow \mathcal{C}^*_{c\uparrow}(X)$$

induces isomorphisms:

$$H^p\left({}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}\right) \approx H^p_{c\uparrow}(X, \mathbb{Z})$$

The statement also holds by replacing compact/forward with forward everywhere.

Remark It is worth to stress the advantage of using the finite volume technique in the previous theorem. On one side, one realizes geometrically both the cycles and the boundary of the "Morse-Bott complex" and secondly, one has a quick and intuitive proof of the result, relating it to the dynamics of the flow (cf. for comparison the exposition in [1], dealing just with real coefficients).

We finally point out that the $\mathcal{S}$-complex is filtered by $\emptyset = \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_n = \mathcal{S}_{\uparrow}(\phi)$ where $\mathcal{F}_i$ is defined to be the subset of the forward $\mathcal{S}$-complex made up of sums of elements supported on stable bundles of index less or equal to i. This observation goes back to Bott [4].

The spectral sequence associated to the previous filtration has $E_1^{k,j}$ terms generated by the cohomology spaces $H^j(C)$, where C runs through the critical manifolds of index k (cf. [18], or [1] for an approach using a "formal" Morse Bott complex). Many informations can then be drawn out of the previous spectral sequence. We just mention two of them, referring to [18] for proofs and further results.

First, if there are a finite number of critical manifolds one obtains the analogous of Morse Bott inequalities usually stated for compact manifolds, relating the Betti numbers and the indexes of the critical manifolds to the Betti numbers of X.

The second application we quote is the following:

Theorem 5.2.17 *Suppose the currents $P_C = U_C \times_C S_C$ are closed for any critical manifold C. Then*

$$H^k_{c\uparrow}(X, \mathbb{Z}) \approx \sum_{c\uparrow} H^{k-\lambda_c}(C, \mathbb{Z})$$

where the symbol $\sum_{c\uparrow}$ means that the sum runs through families of critical manifolds C whose union is a compact forward set.

Of course a similar statement holds for forward supports.

Next, we extend the approach to Novikov theory developed in chapter 4 to the case of Novikov forms with Bott singularities. We consider the general case at once.

Suppose ω is a closed one form with Bott singularities (a "Novikov Bott form") on the compact manifold X and let ϕ be the flow of the gradient of ω. Let k be the irrationality degree of ω, $\chi \in \mathbb{R}^k$ the vector of periods and $Y \xrightarrow{\pi} X$ the minimal covering trivializing ω, endowed with the action of $\mathbb{Z}^k =< t_1, \ldots, t_k >$, where $t_i : Y \to Y$ are covering diffeomorphisms. Let f be a potential for $\pi^*(\omega)$ (hence a Morse-Bott function) and ψ the gradient of f for the pullback metric; the equivariance relations $f(t_i(y)) = f(y) + \chi_i$ hold.

Exactly as proved in lemma 4.2.1, the flow ψ is weakly proper with respect to f. Moreover, the critical manifolds upstairs are clearly compact. We assume the flow ψ to be generalized Smale (this is not a generic condition for the Riemannian metric).

Considering the complex $\mathcal{S}^*_{c\uparrow}(\psi)$ over the reals or the integers, it is trivial to check that the Forward Laurent Ring Λ acts on it (by letting the covering diffeomorphisms t_i act by pushforward) and that the operators $\mathbf{P}, \mathbf{T}$ are Λ-linear maps. The only difference is that the $\mathcal{S}$-complex is not any longer finitely generated over Λ.

Away from the finiteness of generators, though, the other results carry over:

Theorem 5.2.18 *The map of $\Lambda_{\mathbb{R}}$ complexes:*

$$\mathbf{P} : \mathcal{E}^*_{c\uparrow}(Y) \longrightarrow {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f)$$

induces an isomorphism of $\Lambda_{\mathbb{R}}$ vector spaces:

$$H^i_{c\uparrow}(Y, \mathbb{R}) \approx H^i\left({}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f)\right)$$

The inclusion map of $\Lambda_{\mathbb{Z}}$-complexes

$${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f) \hookrightarrow \mathcal{C}^*_{\uparrow}(Y)$$

induces an isomorphism of $\Lambda_{\mathbb{Z}}$-modules

$$H^i\left({}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f)\right) \approx H^i_{c\uparrow}(Y, \mathbb{Z})$$

The statements also holds by replacing Λ with the Novikov Ring.

As a corollary of the theorem, one might obtain the analogous of Morse Bott inequalities in the present setting, since the filtration of the $\mathcal{S}$-complex presented for Morse Bott theory is preserved by the action of the forward Laurent Ring.

The compatibility with the action of the Forward Laurent Ring also permits to extend theorem 5.2.17 to:

Theorem 5.2.19 *Suppose the currents $P_C = S_C \times_C U_C$ on $Y \times Y$ are closed for any critical manifold C. Then the following (not finitely generated) Λ modules are isomorphic:*

$$H^k_{c\uparrow}(X, \mathbb{Z}) \approx \sum_{c\uparrow} H^{k-\lambda_c}(C', \mathbb{Z})$$

The sums runs trough family of critical manifolds (all diffeomorphic) C' whose union is compact forward on Y.

5.3 Applications and Perspectives

In this section we will discuss some generalizations of the theory developed so far and some ideas and open questions deserving further developments.

We like to start by mentioning a list of possible extensions and applications which have not been dealt with, so far, for the sake of brevity and are now presented for completeness. They have been studied by Harvey and Lawson and by Latschev using the finite volume approach: it's thus reasonable to expect them (and their counterparts in Novikov theory) to work with obvious modifications for weakly proper flows with isolated or Bott hyperbolic singularities. Once more we address the reader to [18] and [13] for an adequate explanation of the following.

- Other coefficients.

  We just used real or integer coefficients, assuming all the manifold we met to be oriented. If the objects under consideration carry R-orientations, R being one of the $\mathbb{Z}_p$ for p prime, the theory of chains mod-p (cf. Federer [8], 4.4.6) permits to obtain analogous results for cohomology groups with $\mathbb{Z}_p$ coefficients. Analogously, one can consider cohomology with coefficients in flat bundles.

- Equivariant cohomology.

  If G is a compact group acting on X, and the action commutes with a flow on X, then the operators $\mathbf{T}$ and $\mathbf{P}$ can act on the spaces of equivariant differential forms and equivariant currents. An equivariant $\mathcal{S}$-complex can be defined too and the Morse chain homotopy still provides chain homotopies between the three complexes.

- Cohomology operations.

  Using the flow to construct a deformation of the triple diagonal $\Delta \subset X \times X \times X$ into a current P (similarly to the construction of the Fundamental Morse equation), one can obtain a chain homotopy between the wedge product of two closed forms $\alpha \wedge \beta$ and a sum of currents given by intersection of stable and unstable manifolds, thus realizing the cup product within the $\mathcal{S}$ and $\mathcal{U}$ complexes. Generalizations to other cohomology operations are possible by this kind of technique.

We next like to point out three problems related to the approach presented in this thesis.

The first one concerns the limit of the pullbacks operators ϕ_t^* of a flow ϕ on a manifold X.

For generic gradient-like flows, we proved that the sequence of pullback operators ϕ_t^* converges; we actually didn't spend much time to describe the best convergence one could get, but restricted to consider the "weak star" convergence. The proof we gave shows that the operators converge with respect to the "flat" topology, but we won't discuss it further now either.

In the case of the gradient flow ϕ of a closed one form ω (i.e. in Novikov theory) we introduced a covering Y of X and discussed the gradient flow ψ of the Morse function f such that df lifted ω. It is interesting, though, to ask whether the limit of ϕ_t^* converges in some weaker sense, for example as Cesaro (or "ergodic") limit.

More generally, we can address the following question: is it true that for a generic flow ϕ the pullback operators acting on test forms converge in some sense? Are there special family of forms or currents over which they converge? Of course, if we suppose that the flow ϕ preserves a measure μ, then the operators ϕ_t^* converge over the functions in $L^1(\mu)$ (by Birkhoff's theorem). This is the case, for example, of Hamiltonian gradients, but still no convergence is granted on higher degree differential forms.

Coming back to Novikov theory, and supposing to be able to get some convergence for the pullbacks ϕ_t "downstairs", is there then a relation with the operator P, limit of the the pullbacks ψ_t "upstairs"?

The second application we are interested in is to study the relationships between the dynamics of a flow and the topology of a manifold when the flow arises by a geometric construction.

The example we have in mind, which partially motivated our interest in Novikov theory, is that of locally conformal Kähler manifolds (abridged lcK). These are Hermitian manifolds whose metric is locally conformal to some (local) Kahler metric. If the underlying manifold X is simply connected, then the existence of a lcK metric would imply the existence of a Kähler metric on X. This existence is not granted otherwise, and there are several counterexamples, the most elementary probably being the Hopf manifold $S^1 \times S^3$. Clearly, there are some basic topological obstructions on compact complex manifolds to admit Kähler metrics. It is a conjecture of I. Vaisman that if those topological obstructions vanish, then a locally conformal Kähler manifold admits a Kähler metric.

The conjecture has been proved in the case ω is parallel (in particular never vanishing), and lcK manifold with parallel Lee form are called Vaisman manifolds.

Note that if g is the lcK metric, then the differentials df of the local conformal factors f (defined by the fact that $e^f g$ are local Kähler metrics) glue up to define a closed one form $\omega = df$ called the Lee form of the lcK manifold. The cohomology class of ω vanishes if and only if the underlying manifold admits a Kahler metric.

It would be interesting to try to relate our "finite volume" approach to this geometric problem, which certainly deserves further study.

Appendix A

Currents and Kernel Calculus

In this appendix we briefly review and fix notations about currents and Kernel Calculus, referring to [7], [8] and [10] as comprehensive references. The papers [14] and [15] are useful short references for Kernel Calculus and applications of Currents to cohomology. In the following X is assumed to be an oriented (not necessarily compact) manifold of dimension n.

The space $\mathcal{D}'^k(X)$ is the space of currents of degree k (also called of dimension $n-k$) on X; they are the functionals over the space $\mathcal{E}^{n-k}_{cpt}(X)$ of ("test") smooth forms with compact support. The topology on $\mathcal{E}^k_{cpt}(X)$ is the Whitney topology: in particular, a sequence of forms α_n converges to 0 if and only if there exists a compact set K such that the supports of all the α_n are contained in K and for any chart U, the coefficients of the forms α_n and all their derivatives converge uniformly to 0 on compact subsets of U. The topology on the space of currents $\mathcal{D}'^k(X)$ is the weak-star topology, so that a sequence of currents T_n converges to 0 if and only if $T_n(\alpha)$ converges to 0 for any test form α.

Any smooth form $\alpha \in \mathcal{E}^k(X)$ defines a current R_α of degree k by partial integration : $R_\alpha(\beta) = \int_X \alpha \wedge \beta$ for any test form β of degree $n-k$. Usually R_α is still denoted by α.

Any locally closed oriented submanifold T of dimension k and of locally finite k volume defines a current $[T]$ of dimension k by integration: $[T](\alpha) = \int_T(\alpha)$ for any test form α of degree k.

The boundary ∂ of a current R of degree k is a current of degree $k+1$ defined as the adjoint to the exterior differential on forms, i.e. for any form α of degree $n-k-1$ it is: $\partial R(\alpha) = R(d\alpha)$. The operator $d = (-1)^{k+1}\partial$ on $\mathcal{D}'^k(X)$ is called the "differential" (or sometimes the boundary too); in this thesis we only used the latter. The pushforward π_* defined by a smooth proper map $\pi : X \to Y$ is the adjoint operator of the pull back on forms. It preserves dimensions.

Currents on a product space are called "kernels". If Y has dimension n, each kernel $K \in \mathcal{D}'^{m+n-k}(Y \times X)$ determines an operator from test forms to currents

$$\mathbf{K} : \mathcal{E}^h_{cpt}(Y) \to \mathcal{D}'^{k-h}(Y \times X)$$

for any $0 \leq h \leq k$ defined by the relation:

$$(\mathbf{K}(\alpha))(\beta) = K\left(\pi_Y^*(\alpha) \wedge \pi_X^*(\beta)\right)$$

where α is a test form on Y, β is a test form on X and π_Y, π_X are the projections of the product.

For example, the following correspondences kernel-operators hold:

diagonal $\Delta \subset X \times X$	identity operator $\mathbf{I} : \mathcal{E}^*_{cpt}(X) \to \mathcal{D}'^*(X)$
(inverted) graph of a map $f : X \to Y$ $G_f = \{(f(x), x) \mid x \in X\} \subset Y \times X$	pullback operator $f^* : \mathcal{E}^*_{cpt}(Y) \to \mathcal{D}'^*(X)$
$K, \qquad dK$	$\mathbf{K}, \qquad d \circ \mathbf{K} + \mathbf{K} \circ d$

We explicitly remark that the last relation implies that a closed current defines an operator which commutes with the boundary d.

We finally quote a theorem. It is actually never really used, but in the sketch of proof for theorem 5.2.11. A more general statement involving "flat currents" can be found in [8], 4.1.20.

Theorem A.0.1 (Flat support theorem) *Let dR be the boundary of a current of integration over a locally closed, locally finite volume submanifold R of dimension k. If the Hausdorff $k-1$ measure of spt(dR) is zero, then $dR = 0$.*

Appendix B

Stratifications

We here collect some results about stratifications, referring to [12] for the statements not otherwise motivated.

Definition B.0.2 *Let $Y \subset X$ be a closed set in a manifold. A **stratification** of Y is a family of disjoint sets $\left\{Y_i = Y_i^k \,|\, i \in I_k,\, 0 \le k \le m\right\}$ called **strata** with the following properties:*

- *Any Y_i^k is a locally closed submanifold of X of dimension k;*
- *For any $i \in I_k$, $j \in I_h$, if $Y_i \cap \overline{Y_j} \neq \emptyset$ then $h < k$ and $Y_i \subset \overline{Y_j}$.*

*The number m is called the **dimension** of Y (provide I_m is nonempty) and the stratification is said **locally finite** if the family of strata is a locally finite family (in the ambient space topology). If two stratifications $Y = \cup_{i=1}^m Y_i \subset X$ and $Y' = \cup_{j=1}^{m'} Y_j \subset X'$ are given, a **stratified map** $f : Y \to Y'$ is the restriction to Y of a smooth map $\overline{f} : X \to X'$ with the property that for any stratum Y_i there exists a stratum Y_j' such that $\overline{f}(Y_i) \subset Y_j'$. A stratified map $f : Y' \to Y$ is called **submersive** if any of the restrictions to a stratum $f|_{Y_i} : Y_i \to Y_j'$ is a submersion (i.e. it is surjective and has differential surjective at each point).*

Remark Though the same subset Y might be stratified in several different ways, one sometimes identifies a stratified set just by its support Y. A special example of stratification is that of a submanifold with boundary. In this case there is a unique natural choice for the strata. More generally, any submanifold with corners will be assumed stratified in the natural way.

It is sometime useful to “refine” a given stratification.

Definition B.0.3 *A **refinement** of a stratification $Y = \cup Y_i^k$ is a new stratification $Y = \cup Y_j'^h$ such that for any $Y_j'^h$ there exists an Y_i^k such that $h \le k$ and $Y_j'^h \subset Y_i^k$. In a refinement any stratum is (possibly) replaced by* *some smaller strata (in size and dimension). The refinement will be called* ***locally finite*** *if each stratum is replaced by a locally finite (in the ambient space topology) family of strata.*

Remark One can have a locally finite refinement of a not locally finite stratification, since the definition involves the single strata. Nevertheless, any locally finite refinement of a locally finite stratification is a locally finite stratification.

Definition B.0.4 *A stratification $Y = \cup Y_i^k \subset \mathbb{R}^n$ is said to be* ***AB (Whitney) regular*** *if for any two strata Y_1, Y_2 of Y, such that $Y_1 \cap \overline{Y_2} \neq \emptyset$ and for any sequences $y_n^1 \in Y_1$ and $y_n^2 \in Y_2$, both converging to $y \in Y_1$ the following holds:*

- *A. If the tangent planes $T_{y_n^2} Y_2$ converge to some limiting plane τ, then $T_y Y_1 \subset \tau$*
- *B. if in addition the normalized vectors $[y_n^1, y_n^2]$ converge to a vector l, then $l \in \tau$.*

In the previous definition we supposed the stratification to be embedded in an euclidean space. This is not relevant, since the property defined happens to be a $\mathcal{C}^1$ invariant, as we next show (cf. [30]).

Let Y_1, Y_2 be two submanifolds in $\mathbb{R}^n$ and suppose $Y_1 \subset \overline{Y_2}$ and $y \in Y_1$. Let (U, φ) be any "straightening" chart (U, φ) near y in whose coordinates $Y_1 = V$ is a $k - plane$:

$$\varphi : (U, U \cap Y_1, y) \to \left(\mathbb{R}^n, \mathbb{R}^k \times 0^{n-k}, 0\right)$$

Let r be the retraction onto the stratum:

$$r = \varphi^{-1} \circ \pi_V \circ \varphi : U \to U \cap Y_1$$

and ρ the distance function:

$$\rho = \rho_V \circ \varphi : U \to \mathbb{R}^+$$

where π_V is the orthogonal projection onto V and ρ_V is the square distance from V in $\mathbb{R}^n$ (if $V = \{x_{k+1} = \cdots = x_n = 0\}$ then $\rho_V(x) = x_{k+1}^2 + \cdots + x_n^2$).

Theorem B.0.5 *The previous condition A on (Y_1, Y_2) is equivalent to the following. For any straightening $\mathcal{C}^1$ chart such that Y_1 is a k-plane, and any $y \in Y_1$ there exists a neighborhood $V \subset U$ of y such that $r|_{V \cap Y_2}$ is a submersion. Furthermore, the map $(r, \rho)\,|_{V \cap Y_2}$ is a submersion if and only if (Y_1, Y_2) is B regular.*

Remark Of course the regularity of the maps r, ρ is granted by construction (being restriction of smooth functions to submanifolds), and one needs only to check the surjectivities. Therefore, condition A is equivalent to ask that for any y in Y_1, the n-k plane $V_y^\perp$ orthogonal to y has transversal and nonempty intersection with Y_2. Condition B requires that the spheres of any small radius $S(r) \subset V_y^\perp$ have nonempty and transversal intersection with Y_2.

Proposition B.0.6 *Let $Y_1, Y_2 \subset X$ be two (AB-regular) stratified sets and suppose that each stratum of Y_1 is transversal to any stratum of Y_2. Then the family of all intersections of strata makes up a (AB-regular) stratification for $Y = Y_1 \cap Y_2$.*

Corollary B.0.7 *Let $Y_1, Y_2 \subset X$ be two (AB-regular) stratified sets and $y \in Y_1 \cap Y_2$ be a point of transversal intersection (i.e. the two strata containing y are transversal). Then Y_1 and Y_2 are transversal in a neighborhood of y (and the intersection is AB-regular).*

Suppose Y is a (AB regular) stratified set and $y \in Y$ belongs to the stratum Y^y. For any Riemannian metric on X let r be the distance function from y, $B_\varepsilon(y)$ the corresponding ε-ball and $\partial B_\varepsilon(y)$ the sphere.

Proposition B.0.8 *If Y is AB Whitney then for ε small enough the sphere $\partial B_\varepsilon(y)$ is transversal to Y. Moreover, if N is a smooth submanifold of X which is transversal to all the strata in Y, $N \cap Y^y = \{y\}$ and $dim(Y^y) + dim(N) = dim(X)$, then the sphere $\partial B_\varepsilon(y)$ is also transversal to N and hence to $N \cap Y$.*

The intersection $N \cap Y \cap B_\varepsilon(y)$ is called the **normal slice** through y along N and $N \cap Y \cap \partial B_\varepsilon(y)$ is called the **normal link** through y along N. They are again AB Whitney stratified in a natural way.

Remark These spaces are very important since for ε small enough the topological type of the pair (normal slice, normal link) is indipendent by y , ε, N, and the Riemannian metric chosen to define the distance function. This is a deep result, consequence of the so called "Thom first isotopy lemma"(it has not been used in this thesis).

Definition B.0.9 *Let $Y_1 \subset X_1$ and $Y_2 \subset X_2$ two stratified spaces and $f : X_1 \to X_2$ a smooth map. The restriction $f|_{Y_1}$ is **transversal** to Y_2 if for any point $y \in Y_1$ it is $df|_y(T_y Y_1^y) + T_{f(y)} Y_2^{f(y)} = T_{f(y)} X_2$, where Y_1^y denotes the stratum of Y containing y.*

In particular, if $f|_{X_1}$ is transversal to Y_2 one just says that f is transversal to Y_2. If $Y_1, Y_2 \subset X$ are stratified sets, they are transversal if and only if the restriction to Y_1 of the identity map of X is transversal to Y_2.

Proposition B.0.10 *Let $f : X_1 \to X_2$ be a smooth map, and $Y_1 \subset X_1$ and $Y_2 \subset X_2$ two stratified spaces. If f is transversal to Y_2, then $f^{-1}(Y_2)$ is a stratified space, which is AB Whitney if Y_2 is. If $f|_{Y_1}$ is transversal to Y_2 then $f^{-1}(Y_2) \cap Y_1$ is a stratified space, which is AB Whitney if both Y_1 and Y_2 are. For fixed Y_1 and Y_2, the set $\mathcal{S}$ of maps $f : X_1 \to X_2$ such that the restriction $f|_{Y_1}$ is transversal to Y_2 is open and dense in the set of smooth functions (with the strong "Whitney" topology), provide Y_1 and Y_2 are A-Whitney regular.*

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