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Learning mathematics in a museum

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Sommario: *I musei specificamente dedicati alla matematica esistono dall'inizio del ventunesimo secolo, e si vanno oggi consolidando. La nostra tesi principale è che ci sono idee e competenze matematiche significative e preziose che si possono apprendere e sviluppare all'interno di un contesto espositivo. Per sostenere questa tesi, inquadriamo i musei di matematica nell'ambito della teoria dell'educazione non formale, analizziamo il linguaggio museografico nelle sue applicazioni alla matematica e proponiamo un elenco di concetti matematici particolarmente adatti a essere sviluppati in una mostra. Descriviamo poi alcuni importanti musei matematici in tutto il mondo e la loro offerta museografica. Passiamo infine in rassegna le mostre di IMAGINARY.*

Abstract: *Museums specifically devoted to mathematics have existed since the beginning of the 21st century, and therefore it is a domain consolidating nowadays. Our main thesis is that there are significant and valuable mathematical ideas and skills that can be learned and developed within an exhibition context. To support this thesis, we frame math museums within the non-formal education theory, we analyze the museographic language when applied to mathematics, and then we list a series of mathematical concepts that are especially well suited to be developed in an exhibition. Next, we describe some notable math museums worldwide and their museographic offer, and finally, we review the trajectory of IMAGINARY's exhibitions.*

Introduction

This article is an opinionated introduction to non-formal education in the context of mathematics museums. Non-formal education is the education that occurs outside the school system, by the learner's initiative and adapted to their experience and interests. This education is nevertheless organized, has learning objectives, and is produced by professionals using specific techniques and materials. The museographic language (used in museums and exhibitions) is one of the formats that non-formal education can take, and it has been used successfully in the domain of mathematics during the last quarter of a century in math museums worldwide.

In the first section, we try to clarify definitions and review the main characteristics of non-formal education in the context of mathematics. In the next section, we review what museography is as an expressive language for non-formal education and for communicating mathematics to the public and students. While this discussion is not entirely new in the pedagogical and museographic communities, it may be less known to mathematicians and researchers working on mathematical communication. Next, we focus on aspects of mathematics that can be genuinely learned in a museographic context, discussing their value and the unique opportunities they can offer in a non-formal education framework. In the last two sections, we discuss how these ideas have been implemented practically in museums and exhibitions. In one section, we enumerate a few representative math museums around the world, which take different forms and approaches. In the last section, we give a detailed account of the evolution of the IMAGINARY association exhibitions. We will hint at some of the common steps that most math museums have taken to establish themselves as a reference in their area, and we hope that IMAGINARY's journey can be of inspiration to

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other projects aiming to enter the maturing math museums community.

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Non-formal education, learning beyond the school

For more than two centuries, societies all over the world have strived to achieve universal education. This goal has served different purposes: from nation-building objectives – by preserving the values that are considered important – to competitive development needs – by making schools factories of future workers – to genuinely altruistic ideals – by creating well-informed and critical citizens. In any case, the school institution has been the primary system that society has established to achieve that goal: to transmit knowledge, values, and education from one generation to another. The school system is what we call the formal education system.

Today, however, we live in the age of information. While before this age, the sources of information and access to knowledge were essentially two: books and expert teachers, with today's technology, information is abundant, ubiquitous, instantaneous, and virtually free. Of course, information is not knowledge. Humans still need to digest information in an ordered manner, structure the ideas in the brain, learn to be critical, apply concepts, develop skills, etc. The noise and useless information obstruct the emergence of ideas, and the huge stream of stimuli compete for the attention and time of children and adults. Thus, the always precious resource of knowledge and savvy, expert teachers, remain a scarce asset. This offers alternative means to acquire skills and knowledge, other than the school.

To be precise, authorities (1) and academics (2) agree on distinguishing different types of education:

Formal education: the hierarchically structured, chronologically graded education system from primary through to tertiary, higher education institutions.

Informal education: the process whereby every individual acquires attitudes, values, skills, and knowledge from daily experiences, such as from family, friends, groups, the media, and other influences and factors in the person's environment.

Non-formal education: the organized educational activities outside the established formal system that are intended to serve an identifiable learning clientele with identifiable learning objectives.

In all three settings, a person can experience a learning process, acquiring new skills or knowledge. The formal setting is well-established, and the informal setting is not systematic by nature. In non-formal education, individuals learn outside of school, on their own initiative, adapted to their background and interests, and it continues throughout life. We are interested in exploring the non-formal education setting, in particular, in the context of mathematical education.

Differentiating aspects of non-formal education

In terms of **planning**, the school education is carefully planned. School must be formal and generalistic, each course aims to give a broad and solid base to prepare students for the next steps in their education. It must be systematic and exhaustive in its methods, so successful students develop the desired knowledge and skills. In the case of formal mathematics education, one subject must be mastered before the next one is built on top, and no important block should be skipped. Non-formal

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(1) *Youth, education and action to the new century and beyond: report prepared by UNESCO*, World Conference of Ministers Responsible for Youth, Lisbon, 1998.

https://unesdoc.unesco.org/ark:/48223/pf0000113316

(2) Alan Rogers, *Non-Formal Education. Flexible Schooling or Participatory Education?* CERC Studies in Comparative Education, Volume 15 (2005), ISBN: 978-0-387-24636-9.

https://doi.org/10.1007/0-387-28693-4

education, on the other hand, is not planned but open to the public as broadly as possible. It aims to provide learning opportunities for the interested public, adapting to their background and their interests. The enter barrier must be low, and content must be made attractive. There are no evaluations or exams since all the knowledge and skills learned won't be demanded in a following course or activity. What is learned will nurture the unique and personal "knowledge graph" in the mind of the participant, leading to a different influence in different people.

In terms of **content**, the formal system must select what is worth being taught, and define a curriculum. This is a need since time and resources are finite, but it can also lead to abuse, especially when selecting what content is or is not desirable to teach to future citizens. The content is also chosen in relation to the long-term planning of the school system. The math curriculum (equations, trigonometry, linear algebra...) is selected as what will be needed in the following courses. School can also demand perseverance in training and practice, even if that may be difficult or tedious. In non-formal education, on the contrary, content can be seductive and fun, at the expense of depth. In non-formal education, one can address trending topics (artificial intelligence, climate change...) or exciting connections (mathematics of music, fractal art...) but will necessarily be addressed very shallowly, without the proper foundation and practice that an expert would require. Outreach of current research can also be content of non-formal education, without making such an activity comparable in depth to a scientific conference.

Regarding **timing**, formal education is rigid, subject to the curriculum, and the students must adapt to the allocated time. In non-formal education, the timing is more flexible and participants learn at their own pace.

Regarding **dynamics**, formal education is directed, or hierarchical, from the teacher who has the knowledge to the students who receive it. In a non-formal setting, participants are more heterogeneous and may contribute with different points of view to a collective learning experience. There may be a mediator of the activity, who has the knowledge, but his or her role is as a facilitator of a discovering process. In a formal setting, students generally learn individually and in parallel, while in non-formal settings, the flow of knowledge and information is crossed in collective learning, participants learn from observing other participants, and by teaching by example to others.

In terms of resource **efficiency**, the formal education system is more efficient. When teachers and books were precious resources, and essentially the only means to acquire knowledge, their impact was maximized in the school system. A small group of expert teachers and a supply of books can educate children on a large scale. The non-formal learning is, on the contrary, a system of abundance. Participation in non-formal learning is voluntary, so it will never be anywhere near universal. Non-formal activities can (and should) be attractive and engaging, but a certain predisposition by the public is required since it is not compulsory to participate. Non-formal education has to compete against a large offer of entertainment, sports, and many other activities for the attention of the public, in an increasingly over-stimulating society where the offer of activities largely exceeds anyone's time and attention capabilities.

Bridges between formal and non-formal education

The school education system has been working relatively well for centuries (ignoring for a moment countless reforms, counter-reforms, and pedagogical earthquakes in virtually every country). Some ideas are definitely best taught in school, and we are not advocating for dismantling or yet reforming again the school system.

Non-formal education should work *with* the school but not *for* the school. The first point to recognize is that in non-formal education, participants (be it students or professionals, young or adults) do learn valuable insights into mathematics. We must reject the idea that math communication is about motivating students to learn real mathematics at school or university. Of course, both will be correlated, but the purpose of communicating math cannot be reduced to a marketing campaign for formal education.

Non-formal education cannot fix either problems or deficiencies in the school system. While it will not hurt, and indeed it will be positive to have students in school who spend some of their free time learning mathematics non-formally, the content and most of the skills that can be learned in non-formal settings will be complementary and not overlap with these that can be learned at school. A private tutor is not non-formal education, and certainly, non-formal education cannot be a substitute for formal education. There are, however, many positive synergies between the two educational settings that are worth exploiting.

Schools and universities are still, and probably for a long time will be, the main structure for the education and transmission of knowledge to new generations. But while the formal system is undeniably effective in its purpose, it necessarily misses a focus on a more panoramic, or horizontal, view of mathematics. Questions related to motivation, applications, philosophy of mathematics, current trends, etc., are not always properly addressed in the vertical scheme, and yet, these should also be part of the culture and literacy of well-formed citizens. A non-formal setting, outside the school, can improve this situation by giving such a complementary horizontal view, using all the resources available today. Amongst the resources, the human component, such as math communicators, mediators in exhibitions, and possibly the same teachers and professors who also participate in formal education, must not be forgotten.

Mathematics (as many sciences) has an inherent access difficulty due to its verticality. In a domain like art, history, or many humanities it is possible for a non-expert to take a research paper and follow the main lines. This non-expert reader will understand the meaning of the words and the sentences, and while he or she may lack a background in the topic that allows him or her to make deeper connections or read critically, the facts stated in the paper will be at least understood. In mathematics and sciences, it will most often be the case that a non-expert cannot even understand most of the keywords that encode an abstract object, the "basic" properties of which are assumed to be known by the reader. We can say that mathematical knowledge is more "vertical" than that of humanities. This poses a barrier to entry for non-specialists, and it is also a major factor in the education abandonment rate – once a student fails to master the content of a course, it is increasingly difficult for this student to catch up in higher courses.

Communicating mathematics to the general public (and to students) necessarily must break some of this verticality. The non-formal education approach needs to assume no requisites (or almost no requisites) for the learner, who comes from many backgrounds and interests.

This idea of communicating mathematics with non-formal education activities is a relatively young professional field, which can be seen by the fact that even the name of the field is not universally agreed, and often depends on personal preferences, trends and fashions. Although there is no normative definition, each name or designation tries to convey a different nuance that was not covered in other designations. Perhaps the most traditional name has been "math popularization", meaning the mathematics that can be of interest to the general population, some knowledge that can be considered general culture to a more or less cultivated and interested public. The term "recreational mathematics" is usually reserved for games, puzzles, riddles, and other types of playful intellectual challenges that are faced mostly for ludic reasons. More neutral is "math communication", which, strictly speaking, is any message related to the mathematical science that is exchanged between two parts. This aseptic definition could include scientific articles or a math symposium, but it is usually implied that the two parts involved are a professional math side and a non-expert general public side. Often a preferred term in academia is "math outreach", usually referring to recent mathematical research (which is born in universities and academic institutions) that reaches out to the general public, which is outside of that academia. An interview in a newspaper with a researcher who recently proved a particularly hard or famous theorem, including a basic explanation for the general audience, would be math outreach.

The term "math awareness" ([3]) assumes that the general public does not have a clear picture of what a

([3]) See a defense of the term in: Ehrhard Behrends, Nuno Crato, José Francisco Rodrigues (eds), *Raising Public Awareness of Mathematics.* Springer Science & Business Media, 2012, ISBN 3642257100, 9783642257100.

contemporary mathematician does or the implications of mathematics in sciences and technology today. By being made aware of these issues, the public can develop an appreciation for mathematics. A recently coined term is "math engagement," which focuses on the reaction that the public will obtain from the communicative action. This comes from an increasing tendency to evaluate the efficiency of communicative actions in terms of the student's academic results or an increase in the applications to mathematics and STEM careers.

Concerning the evaluation, we already mentioned that non-formal education is less efficient than formal education if measured by the number of individuals reached. It would be a mistake and a deceiving selling point to evaluate non-formal education solely based on the improvement of students' school performance, the increase in the number of applications to STEM careers, statistics on the perception of science and mathematics by the whole population, or the improvement of the whole country or region in international education rankings such as the Programme for International Students Assessment (PISA) ([4]). These indicators can only provide an indirect measure of a successful non-formal education activity. Sure, these indicators will improve with good non-formal education activities, but their primary goal (learning by initiative and discovery, with personal fulfillment) is better evaluated by the participants' necessarily subjective opinions and by the objective learning opportunities that are created.

The PISA ranking and other indicators alike are designed to evaluate the formal education of the students. The competencies evaluated there are those that supposedly students should acquire at school, and therefore any action aimed at improving PISA rankings should start by acting on the formal system: at school. Now, since formal and non-formal education are related, and there are many bridges and synergies between them, it is reasonable to assert that improving non-formal education in mathematics will improve, in the long run, the perception of mathematics, the mathematical culture, and the logical skills of the population, which in turn will likely increase the interest in STEM careers and the overall performance of students in the formal education system.

Precisely because the non-formal education is not meant to be formally evaluated, the PISA tests will not ask "general mathematical culture" questions. For instance, "How did the ancient Greeks discover irrational numbers?", "How are the sound frequencies associated with notes in a musical scale?", "Is the cable of a suspension bridge in the shape of a catenary or a parabola?" "What is the Mandelbrot set?". It does not make sense to evaluate a person's mathematical knowledge using these questions. Maybe, if formulated as a choice question like the suspension bridge shape, it would be a good fit for some TV contests, which are designed to evaluate general culture in a playful context. To give a full answer to these proposed questions would require a deep level of study and understanding of the subject. Yet, most people would agree that having some basic notions to understand the question and the general ideas in the direction of the answer would be a valuable general mathematical culture desirable and accessible to all the population in our modern, knowledge-driven society.

To end this section, let us mention the importance of the didactical and mathematical skills of the professionals conducting non-formal math education. In higher education, personnel often combine duties as professors and researchers, since both need a high level of mathematical skills. However, not all good researchers are good professors or vice-versa. Additional skills in didactics are required for a good teacher. Similarly, a single person can combine working in formal and non-formal education (this is actually a very common situation nowadays, especially secondary school teachers involved in extra-curricular activities), but the methods and didactical skills are not exactly the same in the school system as in non-formal learning. A school teacher will enjoy the privilege of ruling over the use of time and attention of his or her students for a full year, with the reward of seeing them grow intellectually and as persons across time, but the teacher will be bound to a curriculum, to a sometimes arid content or repetitive exercises, and evaluation needs. A non-formal educator, on the other hand, will enjoy the freedom of choosing appealing topics that captivate the

([4]) http://www.oecd.org/pisa

imagination of the participants and spark joyful feelings, but he or she will be bound to a lack of depth in the content, to using attention-catching tricks, and to quick and limited activities. Both sides of that coin can coexist in a single person, or in different people that work together in harmony. A challenge for the professionalization of non-formal maths education (particularly math museums and exhibitions) is to create the possibility of developing a long-term career in the field. This would need at least recognition from the formal education institutions of merits while working on non-formal education, and the sustainability of non-formal education institutions (museums, academies...) that can provide stability to the young professionals aiming to enter the field.

Museography, a language for exploration

Museums have been part of human societies since antiquity. However, historically, museums were associated with preserving rare items and with academic interests. Only in modern times (since the early 20th century) have museums taken a role in educating the public, and much more recently have they taken an entertaining and leisure role for the public. Today, we can find all kinds of museums – from traditional museums of arts and history to a myriad of thematic museums (devoted to a type of food, craft, toys, cinema, a relevant person, etc.). Since the beginning of this century, we can also find museums of mathematics.

Etymologically, a museum is the place where the muses live, the Greek goddesses of arts and sciences that inspired humans in their creative endeavors. Thus, a museum is a place for inspiration, discovery, motivation, and personal transformation. Traditional museums devoted to arts and history clearly put their main effort into the preservation of patrimonial goods and into the research by giving experts access to these artifacts. If we think of science museums, traditionally, that used to mean Natural History museums: fossils, preserved specimens of animals and plants, and thus mostly related to biology or geology. We can also find a few science history museums that preserve the antique instruments of renowned scientists.

The modern conception of a science museum without a patrimonial collection can be traced back to the Exploratorium in San Francisco, USA, which opened in 1969. This institution, founded by the physicist Frank Oppenheimer (brother of the also physicist Robert Oppenheimer known for his involvement in the atomic bomb development), pioneered the idea of a place of discovery and popularization of physical phenomena such as classical experiments in electricity, magnetism, radio waves, light, sound..., but also on current scientific and technic developments of the time like the space race, or, precisely, the atomic energy. The concept was quite innovative at the time. The Exploratorium was open to children and adults alike, and it did not contain any patrimonial artifacts, nothing irreplaceable. The exhibits were built imitating the "magistral demonstration" devices that professors used in university lectures, but in this case, the exhibits were made safe and interactive so as to be manipulated by visitors who explored the topic without the need for any prior knowledge, and often without a guide. To avoid potential friction with the existing community of science (natural history) museums, and to placate those who claimed that such a place was closer to an amusement park than to a museum, the term "Science Center" was coined.

In 1974, the International Council of Museums (ICOM), enlarged their definition of museum to include science centers. The ICOM is an international non-governmental organization linked to UNESCO that maintains a worldwide network of museums and professionals. The issue of a museum's definition is not trivial. In many jurisdictions, an institution that bears the appellation of "museum" must satisfy some legal conditions (for instance, the three classical attributions: preservation of patrimony, supporting research, and education to the public). The current ICOM definition ([5]) (approved in 2022) is the following:

> "A museum is a not-for-profit, permanent institution in the service of society that researches, collects, conserves, interprets and exhibits tangible and

([5]) https://icom.museum/en/resources/standards-guidelines/museum-definition/

intangible heritage. Open to the public, accessible and inclusive, museums foster diversity and sustainability. They operate and communicate ethically, professionally and with the participation of communities, offering varied experiences for education, enjoyment, reflection and knowledge sharing."

In this modern conception of museum, a new discipline arises: **museography**. It is the practice of developing exhibitions with the goal of transmitting a message to the visitors through the interaction with physical artifacts. Following Fernández ([6]), museography is an expressive and communicative language. Other examples of languages could be literature, cinema, or music, for instance. Each language has its own attributes. In the three examples we just mentioned, we can identify physical support (books, films, and songs/ musical pieces respectively), places where it fits naturally (home, library, cinema theater, concert hall...), or a technique (narrative technique, photography/camera, harmony/composition), just to name a few attributes. Different languages are also suited to different communicative messages. Literature is more fitted to reflexive messages, while music fits messages related to emotions. Borders are permeable, so for instance, a song blends poetry (literature) and music, borrowing techniques from both languages.

In the case of the museography language, its support is exhibitions, modules, and physical artifacts. Its natural place is a museum or an exhibition room, and although it can be a polyvalent space, an exhibition always needs a relatively large surface area hall. There exists a collection of expositive techniques that professionals of exhibitions must develop to make an effective language, including storytelling techniques but also design tricks to attract attention or to lead the intuition of visitors. Furthermore, museography is a tangible language (it relies on the interaction with objects and artifacts), and collective (it is usually visited accompanied by other visitors or with the mediation of a guide). Finally, according to Fernández, a museum (particularly a science museum) must aim to be transformative, that is, the visitor is transformed by his or her visit. This transformation can be in the form of acquired knowledge, but also in the form of a changed mindset, approach, or general impression about the exhibition's subject.

In a science museum, and more particularly in math museums, a double language interaction occurs. Mathematics is often characterized as a language. It is suited for describing natural phenomena (physics, chemistry, statistics...), and more generally, it is a language for describing logical reasonings about abstract objects. A math exhibition has thus a somehow nested language: It uses the exhibition language (which uses objects to tell a story) to tell a story of mathematics (which uses abstractions - numbers/graphs/equations...- to tell how a particular domain works).

A mind-opening experience. What mathematics can be learned in a museum?

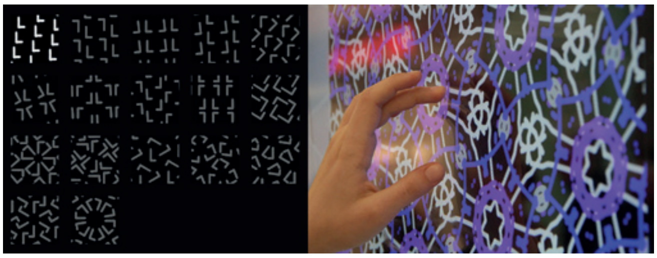
In this section, we will give some examples of concepts, ideas, and topics in mathematics that would be part of a true "mathematical culture" and that are best suited to a non-formal setting, specifically using the museography language. This is a list of examples of mathematics that can be learned in a museum.

Abstraction. A fundamental idea in mathematics is abstraction. Not only as opposed to "concrete", but as the idea of extracting a fundamental principle that governs the object of study. This fundamental principle is often shared with other objects that would be unrelated at first glance. For example, the concept of symmetry, which is widely exploited in math exhibitions, is a perfect example to describe different levels of abstraction and how they can be addressed in and outside of school.

Symmetry arises in repetition patterns, in flowers, animals, and other natural phenomena, in crystals and chemistry, in arts like painting and sculpture, in many musical works, in uncountable games and brain teasers... Many math museums and even amusement parks and other recreational facilities have developed exhibits with a mathematical focus

([6]) Guillermo Fernández Navarro, *El museo de ciencia transformador The transformative science museum* (2018), ISBN: 978-84-09-07652-9.

https://www.elmuseodecienciatransformador.org around mirrors, kaleidoscopes, and other optical games (7). With kaleidoscopes we can generate an infinitude of images that copy an object by reflection on mirrors. However, all these images share the same structure, the same symmetries, which are determined by the positions of the mirrors and not by the object reflected. Some exhibitions use software (8) that allows to generate all the 17 bi-periodic tilings of the plane by translations, rotations, and reflections of any drawing made by the user. Again, the possibilities of the drawing are infinite, but the structure of the symmetries can be only one of 17 possible. There is a beautiful proof that there are only 17 of these symmetry groups using topology, orbifolds, and the Euler characteristic. The elements of this proof are presented to a general audience in *GeCla* (9), by the Portuguese *Atractor* association (see also the DVD *Symmetry – the dynamical view* (10) by this association). This subject is usually not taught in school nor even in the regular university math studies, only in specific graduate courses on geometric topology. Exhibitions usually include modules related to polyhedra, where it is also presented the idea of symmetry.



The program *iOrnaments* allows you to draw tilings using the 17 symmetry groups of the plane.

Symmetry is often presented as a geometric concept, but from a mathematical point of view, it can be described with the more fundamental idea of *group* as an algebraic concept. The concept of group is quite advanced, not because of the difficulty of its definition, but due to the abstraction required to imagine and manipulate this algebraic structure. Groups are usually introduced on a first or second-year course on basic algebra at the university, for math students. After the definition and first examples, one advances through theorems and proofs such as Lagrange theorem for the order of an element, or the classification of abelian groups, and one learns to manipulate and deduce properties on these structures. This is, of course, a formal learning of advanced mathematics, and it can be easily agreed that this is only for people seeking a higher education in mathematics.

The crucial point here is that the idea of a group, a structure of elements that can be combined to produce new elements with very basic properties, is a concept accessible to anybody with a bit of curiosity, almost independently of its mathematical background education. The symmetry structure of a kaleidoscope, a plane tiling, or a polyhedron can be identified as a particular group, named and labeled, and this group governs the symmetries of that particular kaleidoscope or tiling but another kaleidoscope or tiling may be intrinsically different because its group of symmetries is different.

This idea can be conveyed without the need to even use the word "group" or stating its definition (although it is the mediator's task to identify the audience's background and decide how much information to expose). The topic of symmetries of the plane appears in some school textbooks, and chemistry students may have seen the structure of crystals in high school and be familiar with crystallographic classifications. The message in an out-of-school context is that symmetry is not dependent on chemistry, art, or polyhedra, but rather it is an abstract, mathematical framework that explains all of these phenomena.

Mathematical modeling. The idea of modeling is, in some sense, dual to that of abstraction. Creating a model is application, is the practical use of mathematics to solve, or at least to understand, a concrete problem from real life, from outside mathematics. Teachers may find the question of "What is this useful for?" irritating. Mathematics is not only an instrumental science and may be practiced

(7) Such as Museu de Matemàtiques de Catalunya (Barcelona, Spain), Matemilano (Milan, Italy), Le labosaïque (Caen, France), Mathematikum (Giessen, Germany), and many others.

(8) For instance software like *Morenaments*, *GeCla* or *iOrnaments*, see below.

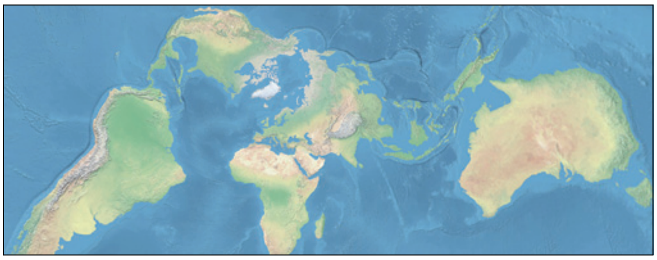
(9) https://www.atractor.pt/mat/GeCla/index-_en.html

(10) https://www.atractor.pt/publicacoes/publs_Atractor-_en.html

just by the quest for knowledge, but "What is this useful for?" is a legitimate question that deserves an argued answer. Actually, applied mathematics is a field on its own and counts as more than half of math production in current research. Of course, one could defend that the whole of physics is applied mathematics, but to fix definitions, applied maths is the use of mathematics to solve problems in fields external to mathematics, usually requiring the collaboration of experts in both mathematics and the field in question.

Learning about applied mathematics and modeling is enriching from the cultural point of view, it gives a sense of interconnection between human knowledge, it is also motivational to learn a subject, and unleashes the freedom of mathematics as a language. Formal education includes physics, chemistry, technology, and other sciences, however, there are plenty of examples of modeling in these fields that cannot be treated in school but are perfect for an out-of-school discussion.

The initiative *Mathematics of Planet Earth* [11] puts the focus on any phenomena related to our planet where mathematics helps to solve or understand. Such phenomena can be physical, natural, human, or any other kind. For instance, cartography is a science that developed from the need to draw maps of regions of the Earth. As a science, it has multiple branches and concerns, one is how to flatten the curved surface of the Earth into a plane, but also others such as how to measure the Earth (geodesy), instruments to make field measures, hundreds of physical constraints, etc. From a mathematical modeling point of view, it triggers the problem of representing a spherical surface onto a plane, what is called a map projection. One can think of multitude of ingenious plane representations of a sphere, drawing the coordinates, projecting imaginary lines on planes, and cylinders... and eventually one needs tools to analyze objectively the different projections, to define and measure distortion, and to evaluate the usefulness of the map for real life problems such as finding routes between locations [12]. Cartography and geodesy were historically a starting point of the mathematical area of differential geometry. Forgetting about some physical constraints is modeling, and developing a theory from this model is abstraction.



A map of the Earth under an oblique stereographic projection.

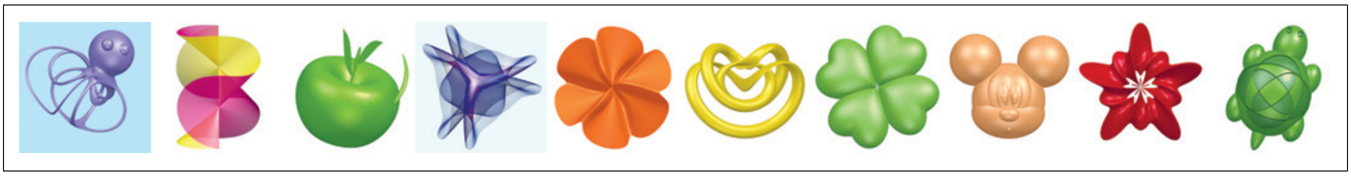
Mathematical modeling is also a perfect framework for introducing some areas of mathematics, such as differential equations. While the theory for solving equations with unknown functions and derivatives (both ordinary or partial) is an advanced topic, it is intuitive and accessible the idea of a function as modeling a physical magnitude, and constraints involving some change rates of this magnitude. At the *Mathematics of Planet Earth*, there are examples such as models of the movement of ice in a glacier, models of the wind and particles suspended in the atmosphere, models of the sea currents and tsunamis, etc. [13]

Modeling of the movement of volcanic ashes in the atmosphere.

[11] Both the name of a research initiative (www.mpe2013.org) as well as a related exhibition managed by IMAGINARY (http://imaginary.org/exhibition/mathematics-of-planet-earth), see below.

[12] See the exhibit *Mappae mundi* https://www.imaginary.org/program/mappae-mundi

[13] *The Future of Glaciers*, *Dune Ash*, and *TsunaMath* respectively, at imaginary.org



Submissions for an international SURFER competition, created by school children.

It is also remarkable that applied mathematics and examples of modeling suit perfectly historical presentations of mathematics. The history of mathematics (old and recent) is shaped by the domains it relates to. Examples of this historical display are the Winton Gallery at the London Science Museum or the calculus sections at Musée des Arts et Métiers in Paris. Both museums use their patrimonial collections to show areas where mathematics has played a central role.

Creativity. A big problem in the formal approach to learning mathematics is the usual lack of freedom to create something using the language that mathematics provides. Mathematics is presented as a solving tool for predefined problems and not as a creation tool. While research into new mathematics can be a creative process, research is far from students and the general public's experience. Thus, it is important to develop tools and activities that allow the creative expression of the user, making use of a mathematical language.

Creativity requires an open end of the activity, there is no creativity if the outcome is pre-defined. Modules based on construction blocks provide a traditional space for creativity, especially for the youngest public. Polyhedral building blocks are common in math museums (commercially available ([14]) or homemade). These blocks allow multiple connections and shapes, but they also have geometric restrictions that make the creation more challenging. Huge, soft blocks are also fun for children, where they can be challenged to build bridges, towers, packings, or other structures.

Drawing is another inherently creative activity. Software to draw symmetric tilings, as mentioned before, mixes the creativity of drawing with the "magical" multiplication of the lines drawn, making pleasant patterns. The *Atractor* association (Portugal) has set inter-school competitions ([15]) using their software *GeCla* (Generation and Classification) for plane tilings, where each contestant has to draw a pattern and guess the symmetry group of other's drawings.

SURFER ([16]) is an interactive program to visualize algebraic surfaces, that is, surfaces in the three-dimensional space defined by a polynomial equation in the three spatial variables x, y, z. This program is part of the IMAGINARY exhibitions, and contains a tutorial where one gets familiar with the equations that one can use as an input, and then one obtains pictures of surfaces as output. The possibilities for input are endless, but to get any meaningful drawing, one needs to develop an intuition on how changes in the formulas affect the surface. Ultimately, this is a tool for using a symbolic, mathematical language in a creative way. SURFER has been used in competitions ([17]), where participants are asked to create the most beautiful, appealing, or interesting image, which is evaluated by a jury.

Mind exercise. Math fairs, exhibitions, and museums tend to have an important part of their collections devoted to games, puzzles, and brain teasers. This makes many of these exhibitions a brain training camp for the visitors, which is by itself a quite positive exercise. Education is not only the transmission of knowledge, it is also training and acquiring competencies. Games and riddles help to develop strategy and problem-solving skills. They also develop reasoning and logical thinking.

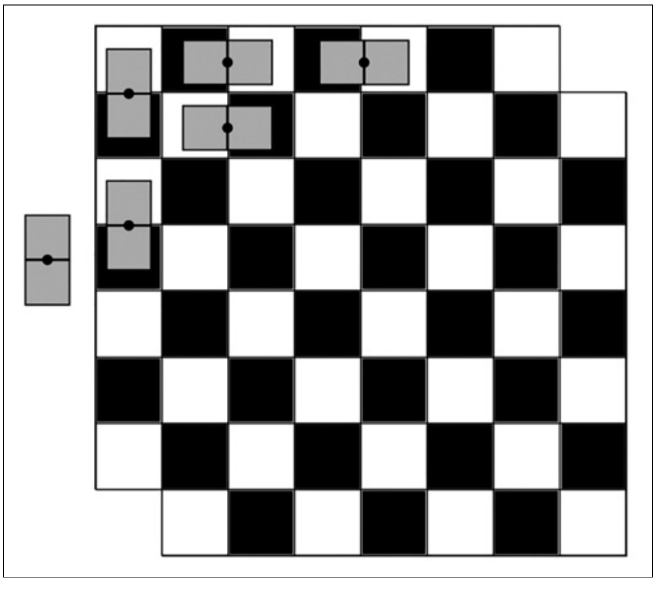
A classical puzzle is to find whether it is possible to cover a chessboard with dominoes. A follow-up then asks if we can remove one square on the corner of the

([14]) Such as *Polydron*, *Zometool*, *MathLink*...

([15]) http://www.atractor.pt/mat/GeCla/Competicao-_en.html

([16]) Available at https://imaginary.org/program/surfer

([17]) https://imaginary.org/search/node/surfer%20competition board and still cover the board with dominoes. Finally, the challenge is to find whether it is possible to do it when we remove two diagonally opposite corner squares. In the first case, the solution is affirmative because we can construct a solution. The second case is impossible because there is an odd number of squares to be covered, and dominoes cover squares in pairs. In the third case, although there is an even number of squares, there is a different number of black and white squares (both removed corners had the same color) and each domino covers one of each color. The puzzle may be more challenging if we draw the original checkerboard just as an 8 by 8 grid with no colors. The three arguments require increasingly complex reasoning. While proving an existence result by construction is quite intuitive, for a public not familiar with rigorous mathematical proofs, such as children (and many adults), it is a challenge to make a logical argument to prove an impossibility result, like that a necessary condition is not met. Going further, one can ask what is a condition necessary and sufficient for the existence of a solution on a chessboard with removed squares.



The dominoes and mutilated chessboard problem.

Puzzles can also play a role in rewarding the effort of the visitor. The visitor who solves a puzzle feels more engaged and satisfied with his, or her, position of 'intellectual advantage'. Solving a puzzle is empowering. In any math exhibition with puzzles, we can observe a fascinating phenomenon: children who manage to solve a puzzle run to show and tell their parents, siblings, or other children. They feel in possession of knowledge that is satisfying to share or to be proud of. A famously difficult puzzle is the Rubik's cube. Anyone able to solve it raises a bit of respect for his, or her, skill. This is mostly due to the fact that the solution cannot be revealed as a simple trick, you are forced to spend a fair amount of time learning the solution algorithm, even with a guide to solve it. In the decade of the 80s, with no Internet and the Rubik's cube as a novelty, the ultimate skill was to solve the cube. In the next decades, new challenges appeared, such as "speedcubing" to solve it in the least possible time. Today, with world speed records close to human limits, attention is shifted to puzzle variations, with different structures, challenging the mathematics and engineering limits. The Rubik's cube is a great mathematical toy, it can lead to learning (and research!) in group theory, to develop logic and visual skills, and it is still a children's toy that anyone can play with ([18]).

The ultimate mind exercise is fostering personal investigation. Not only trying to find a solution to a problem, but discovering the problem itself in the given framework. This process can be guided by a mediator, but it is essential to give freedom and time to explore. One of such frameworks is the spherical blackboard ([19]), consisting of a sphere that we can draw on, and possibly a compass and a curved ruler to draw "straight lines" (properly, geodesics). A canvas to draw always sparks creativity, and visitors will try drawing figures, possibly countries resembling the Earth. If faced with the question "can you draw a square?", they will start struggling with the problems of curved space. It is possible to draw a quadrilateral with equal sides, but not equal angles. This information does not need to be revealed as a given knowledge, but can be discovered by

([18]) See for instance *Mathologer*'s channel on YouTube for mathematical content on the cube.

https://www.youtube.com/results?search_query=mathologer+rubik

([19]) True chalk blackboards on a sphere exist, but a modern educational set is the Lénárt Sphere

http://lenartsphere.com/,

which is transparent.

experimentation. Or rather, it can be guessed after several attempts, but not to be sure without a deeper examination. The fact that there are unlimited possibilities to try forces creativity to make a strategy to understand the problem, and there is always the sense that the experience does not end with the answer to the proposed question. The end goal is not to draw a square. What about triangles or pentagons? What about drawing a polyhedron over the sphere...?

Discovering fields. Many fields of knowledge are not present in formal education, just because there is no time to cover everything in school. In particular, many fields of mathematics remain unknown to the population, while in many cases knowing about their existence is enriching and accessible. Of course, mere introductory notions do not substitute proper courses, but some non-formal approaches serve to discover fields that are unknown or little spread. Furthermore, there are often popular culture references that can anchor the attention of the public and serve as a starting point for the presentation.

Fractals are one of these popular culture topics in mathematics. Many people have heard about them, maybe they have heard about the Mandelbrot set and seen psychedelic pictures on the Internet, or heard references in pop songs. This kind of public is naturally motivated, so that is a good anchor point to offer more information on the topic, while at the same time making the learning process entertaining and pleasant.

Fractals link to the notion of infinity, because of the pass to the limit in their construction. The self-similarity property is intuitive and it has support on nature, sometimes more evident (trees, fern leaves, types of broccoli), sometimes more revealing (how long is the coast of Great Britain? ([20])). The idea of the limit of a process can be explored with Peano and Hilbert curves (that fill the plane and space, respectively) or von Koch snowflake. Finite approximations can be constructed physically, and computer simulations can be used to explore them virtually ([21]). One can also explore the notions of fractal dimension, and the iterative process is also the basis for talking about chaos.

Complex numbers and complex dynamics are also essential to understand fractals in depth. Unfortunately, complex numbers is not always a topic granted in non-university education. However, the idea of algebraic operations between points on the plane (or pairs of coordinates, or equivalently complex numbers) is accessible even if the public has no prior knowledge of complex numbers. From this point, one can dive into complex variable functions and complex analysis. The jump is qualitatively huge, and complex analysis is often taught on a second or third year university course in mathematics. However, the idea of the subject can be conveyed to the general public. A complex variable (holomorphic) function happens to be conformal, which means that it preserves angles. This is a notion that can be experienced with the help of appropriate software ([22]), designed to that end. An image (which can be static or can be a video input from a camera) is presented after a holomorphic transformation, and dynamically one can test the angles before and after. One can also see the zeros and poles of meromorphic functions, and play with the complex geometry. The next step is iterating a function (composing it with itself several times) to get to complex dynamics, and from there, to Julia sets and Mandelbrot sets.

This journey on complex analysis and fractals needs most probably a mediator with the ability to stop, advance, and skip parts, but in a good environment, this is a viable exposition for adolescents and the general public. The ideal setting would be an interactive talk of about 20-25 min in a museum or in an after-school activity, using the software and maybe a few objects (sculpture of a Hilbert curve, a piece of broccoli...). A more classical talk is always

([20]) This question is quoted from Mandelbrot. The meaning is that there is no good notion of length, since the smaller and smaller details of the abrupt coast make the length grow to infinity if we use smaller and smaller units of length. Instead, it motivates the idea of fractal dimension.

([21]) See for instance some *Cinderella* applets (see below for *Cinderella*).

https://imaginary.org/program/cinderella-applets.

([22]) See the Conformal Webcam by Christian Mercat.

https://math.univ-lyon1.fr/~mercat/CindyJS/examples/cindygl/22_webcamconf.html an option if the audience is too large, but it is important that the public gets the opportunity to touch and interact with a holomorphic function or with a fractal, changing parameters and discovering details by himself.

Pearls of mathematics. In mathematics, there are endless examples of small stories, a theorem, a conjecture, some elements of a study field; that can be told without context and can be easily understood and always contain a surprising result. These stories, which we will call "pearls", are delightful for casual discussions on mathematics, are engaging, and at the same time, satisfyingly bounded. These pearls are not really isolated stories, and every iteration of the story that adds a detail makes it more enriching. For this nature, this plethora of stories does not fit into formal learning, and it has always been perfect for math vulgarization. Let us see some examples.

The Banach-Tarski paradox is a theorem, stating that it is possible to cut a sphere into five pieces and re-arrange them into two spheres, identical in size to the original one. This is called a paradox because it is counter-intuitive and contradicts the additive property of volume, but at the same time, it is a proven theorem. To understand this theorem, or should we say, to understand the *concepts* that this theorem involves, one has to talk about measure theory and the fact that not all the sets of points that can be defined can also be assigned a volume, the so-called non-measurable sets. Their existence, in turn, depends on accepting the Axiom of Choice, which is also a delicate subject. But for our discussion, the interesting point is that the story can be told in a few words, and one can appeal to the authority of the sentence "it is a mathematical theorem, it is true" to astonish the audience and, hopefully, to spark the curiosity and hunger to understand the concepts behind such a result in more depth ([23]).

Another pearl is Ramanujan's sum of all natural numbers. The result states that

$$1 + 2 + 3 + 4 + 5 + ... = -1/12.$$

This is an astounding statement and every person will refuse to believe it at first sight. But again, it is a proven theorem (in a suitable sense). The explanation is that our familiar concept of sum is originally only defined for two, three, or any finite number of addends. If we want to make infinite sums, we need to extend the definition of addition to deal with an infinite amount of addends. One of Zeno's paradoxes was related to the sum $1 + 1/2 + 1/4 + 1/8 + 1/16 + ... = 2$, and it was astonishing for the ancient Greeks and for all humanity for centuries. Nowadays, it is accepted and used widely in many areas of mathematics and we have a robust and well-known theory of convergent series. Convergent series admit a numerical value while non-convergent series do not. However, it is possible to extend the theory to assign a numerical value to series that are non-convergent in the "classical" convergence theory. In this case, Ramanujan's sum of all natural numbers must have a value of minus one-twelfth. This framework turns out to be very useful in some areas like string theory in physics, and also brings rich and interesting mathematics. This story tells that one can go beyond the rules, and the established theory should be always put to the test. Things discarded or deemed as impossible should always be reconsidered ([24]).

There are many such pearls that can be used to surprise and intrigue the public. The format of these stories is usually short books or articles of math popularization and, more recently, on Internet sites such as YouTube. The mathematics popularization video is really a flourishing domain. The relatively short duration of the videos (rarely more than 30 minutes) and the close and casual style have made this a successful format and the breeding ground for a growing community of math communicators. Lan-

([23]) See for instance the video from Vsauce
(https://www.youtube.com/watch?v=s86-Z-CbaHA).

([24]) See videos from *Numberphile*
(https://www.youtube.com/watch?v=w-I6XTVZXww
https://www.youtube.com/watch?v=0Oazb7IWzbA&t=36s),
from *Mathologer*
(https://www.youtube.com/watch?v=jcKRGpMiVTw)
or from *3brown1blue*
(https://www.youtube.com/watch?v=sD0NjbwqlYw&t=8s).

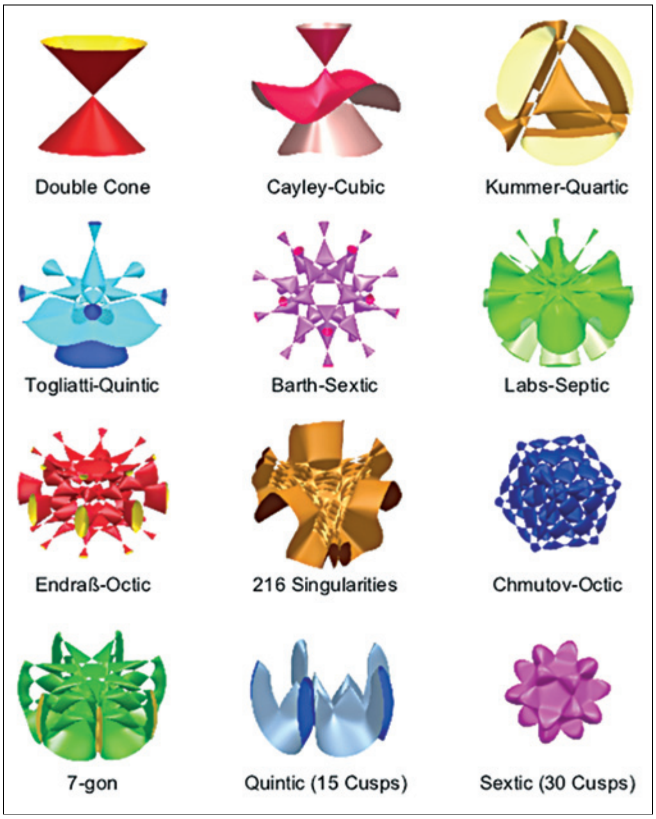
guage and geographical barriers still exist, but it is a lesser problem for younger generations. Topics are re-formulated between authors with different points of view, often acknowledging a "response", and in different languages for different publics. The audio-visual character of the video allows for mathematical visualizations, animations, diagrams, and cartoons, that represent a qualitative improvement over text and picture books. The technique and skills of the authors are in some cases as professional as television productions, without the time and budget constraints of television.

Math communication in short formats, be it videos or more traditional articles, is a huge challenge in didactics. Time, attention, and background constraints push the capacity for synthesis, visual resources, and appeal of the author. The "pearls" are often the finest introductions to many topics, coated in a mixture of entertainment and mathematical insight. In that regard, the educational value is out of doubt, and these resources should be exploited by teachers and math educators.

Edge of knowledge. Mathematics is not a finished science. There are still and always questions where we don't know the answer, unsolved problems, and entire domains to be discovered. This is difficult to transmit to the public, since the advances are not apparent in society as it is the case of many technologies (communications, medical research...), nor can be easily described in layman's terms as in physics (we discovered a new particle using an accelerator laboratory). Mathematics has for a long time been a rare case where communication of current research was deemed as a futile effort. It is the duty of researchers to publish and communicate their advances to their mathematician colleagues, but it is also their duty, or at least the duty of the research community, to communicate also with the non-specialists. Fortunately, this tendency has changed in the last decade with many projects around the world.

The link between research and the public can be done by the researchers themselves or by an increasing number of professional math communicators.

IMAGINARY's exhibition contains many exhibits connected to current research topics in mathematics. For instance, the aforementioned program SURFER visualizes algebraic surfaces. In addition to allowing users to creatively explore surfaces they create, the program also provides an introduction to the theory of singularities. Singularities are points where an algebraic surface fails to be smooth, such as cusps or edges. These points are rare to find, and hence it is an interesting field of study. For polynomials of low degree (up to 6), there are theorems that find the maximum number of singular points for a surface of the given degree. SURFER provides visualization of these surfaces, that are named after the mathematicians who discovered them, in many cases in recent years. For higher degrees (7 and up), there is not yet a definitive result for this problem of finding the maximum possible singularities. These are unsolved problems and current research. Of course, one needs more than SURFER to discover a new theorem, but this exhibit gives a close sense of what these new theorems would be, and it sparks the imagination to feel that anyone could write a polynomial that sets a new world record surface on singularities.



Algebraic surfaces with record of singularities, in SURFER.

Other examples of communication of current research would include almost any exhibit on applied mathematics addressing hot research topics such as artificial intelligence, climate change, quantum computing, etc. which will easily engage to communicate research done in the last decade. Just to cite some examples, *Google's A.I. experiments* ([25]) and *QuBobs* ([26]) are educational activities prepared by researchers in these fields.

Some museums of mathematics

In order to illustrate how this museographic language can be materialized in practice, we would like to describe some remarkable examples of mathematics museums and exhibitions worldwide. Although any list we made here would be incomplete, we have intentionally selected some projects that take very different approaches to their vision of what is a mathematics museum. In past studies, we compiled a list and map of math museums worldwide ([27]), which includes more than 50 institutions that fit an inclusive definition of a math museum (any space with a permanent public exhibition, devoted mainly to mathematics). We also prepared a report ([28]) with highlighted exhibits from math museums around the world. The reader interested in deeper comparisons of math museums and exhibits may find these two references relevant.

Mathematikum (Giessen, Germany)

Opened in 2002, Mathematikum is one of the first math museums in the world, and it has had an undeniable influence on other math museums, both in Germany and abroad. Their museographic approach includes many puzzles but also manipulative exhibits that display some mathematical phenomenon to be discovered and observed. It definitely opts for physical objects instead of computer-based exhibits, and an interaction driven from a visitor's point of view, with many big exhibits that involve using the full body, and many mechanical artifacts. All exhibits are designed in-house and built and repaired by their own workshop. Mathematikum sells exhibits as part of its business, and its exhibits can be found in many other math museums. Other museums that have followed their inspiration or that share a similar museographic approach would include the Museum of Mathematics of Catalonia (Spain), the Garden of Archimedes (Italy, temporarily closed), the Fermat museum (France), and many others.

Winton Gallery of the London Science Museum (London, United Kingdom)

The London Science Museum is foremost a patrimonial museum, preserving and displaying thousands of artifacts from Britain's rich scientific and technological tradition. The part of the museum devoted to mathematics is named the Winton Gallery, and its exhibition is composed of historical artifacts (protected behind glass, not interactive exhibits). The exhibition escapes from the traditional division in areas of mathematics (geometry, arithmetic, calculus...) and instead it focuses on human stories where mathematics has played a significant role. The gallery is divided into six areas: War and peace, Trade and travel, Form and beauty, Maps and models, Life and death, and Money. Each area tells a story around an iconic historical artifact. As an example, there is one original MONIAC (a wordplay between Money and ENIAC, one of the first computers). It is a hydraulic demonstration device created by the economist Bill Philips in 1949 to illustrate the money flow in the economy. Aside from the anecdotal technical details of the water-powered machine, the story behind is that for decades economists were taught that money flow can be modeled by the motion of an incompressible fluid, which sparks a debate on the benefits and limitations of that thinking paradigm.

Other museums in the history of science and technology category would be the Museum of Arts and Crafts (Musée des Arts et Métiers) in Paris or the German Museum of Technic (Deutsches Technikmuseum) in Berlin and Munich. These museums

([25]) https://experiments.withgoogle.com/

([26]) https://qubobs.irif.fr/portfolio/

([27]) https://www.mathcom.wiki/index.php?title=Math_Museums

([28]) https://www.mathcom.wiki/index.php?title=File:2020-06-01-best-maths-exhibits-report.pdf are rare (they only exist in a few places with rich scientific traditions) and not easily reproducible since they conform to the traditional preservation role of museums. It is even more rare that such a museum devotes a section to mathematics.

National Museum of Mathematics, MoMath (New York, United States)

Opened in 2012 in the center of Manhattan, MoMath offers a unique experience of interactive math exhibits, developed with the support of mathematicians, often aiming for the outreach of recent research. The exhibits often use high technology (touchscreens, custom user interfaces, robotics, computer vision, AI systems...), and are produced by professional agencies that yield high production quality. As examples of exhibits, we can mention the *Square-wheeled tricycles* (which ride smoothly around a circular path with the floor shaped suitably); the *Ring/Wall of fire*, where several lasers sweep a plane, and some geometric translucent pieces can be moved across, revealing cross-sections of the shape; or the *Human tree*, where a camera recognizes the image of a visitor and creates a fractal by inserting a copy of the body into the body's arms. While this kind of production and the location of the museum itself hint that this museum has a sizeable budget, MoMath manages to excite visitors with a stunning experience that can compete with the rest of the cultural offer in the city, and sets it apart from other math museums. MoMath also enjoys a good support from the mathematical community at large (researchers, recreational mathematicians, math communicators...), and has set itself as a reference. MoMath organizes two bi-annual conferences in alternating years: MOVES for recreational mathematics and MATRIX (the last few editions, co-organized with IMAGINARY) for math museums and math communication.

Maison Poincaré (Paris, France)

This recent math museum (opened in 2023) is also quite unique. Located in the extended premises of the Institut Henri Poincaré, a research center with almost a century of history, it aims to make mathematical research visible to the public, while highlighting the human side of mathematicians. This is not a museum planned for young children, the discourse is officially intended for students at lycée age (15-18 years old) and adults.

The museum has several spaces, but only a couple are devoted to strictly mathematical content (modeling of acoustics, chaos, fluids...). Other spaces, however, put the focus on stories about mathematicians. The museum has many biographies and portraits of famous mathematicians (with an absolute parity of male-female), and messages of "how a mathematician thinks", "how a mathematician shares his/her findings", "how mathematicians have lectures", "how mathematicians are awarded"... Maison Poincaré is a museum created by researchers, and researchers are a substantial part of the intended public and also part of the exhibition. The building also hosts lectures and conferences, and visiting mathematicians are supposed to walk casually around the museum on their way to lectures. The visiting public will see "real math formulas in chalk over blackboard" as part of their visit experience. This is also a quite unique museum amongst mathematics museums, trying to attract the public in the city with the most museums in the world.

IMAGINARY exhibitions

We will finish this article by describing our museographic journey at IMAGINARY, which is the author's affiliation. IMAGINARY is not a classical museum, but a non-profit association whose main activity is the creation of exhibitions to communicate modern mathematics and sciences. IMAGINARY does not have its own physical museum or interaction space, and thus it relies on its partners or clients (museums, public agencies, private foundations, educators, schools...) to achieve this interaction with the public. Nevertheless, IMAGINARY "speaks" the language of museography, and it embraces the non-formal education paradigm. Furthermore, it is fully committed to the open-source philosophy and the collaboration within the community.

IMAGINARY originated in 2007 within the Mathematical Research Institute Oberwolfach (MFO) to celebrate the German Year of Mathematics 2008 by creating an interactive traveling

Impressions from IMAGINARY exhibitions in Taiwan and Turkey.

exhibition, *IMAGINARY: Through the Eyes of Mathematics* (29), to display what contemporary mathematics looks like. Many German and international mathematicians contributed exhibits, including the celebrated program SURFER, on which the user can select or write as input any polynomial expression F in three variables x, y, z, and the program automatically draws as output the surface given by the zeroset $F(x, y, z) = 0$. Another successful exhibits were *Morenaments* (30), for which one could draw on a touchscreen and the program copied the pattern tiling the plane in any of the 17 flat crystallographic groups; or *Cinderella* (31), a collection of small interactive apps showing a visual effect in topics as simulation, chaos or symmetries in a playful way.

The original exhibition was a success, and after traveling around Germany, it started traveling internationally under an open license, so everybody could copy the content and organize an exhibition locally. For instance, universities organized SURFER school workshops and competitions, on which students learned and played to create intricate formulas that produce nice visualizations on the SURFER program, or added programs or visualizations from local researchers. Thanks to this open approach, the original and other subsequent IMAGINARY exhibitions have reached public in more than 70 countries, with an estimated 2+ million people having seen at some point IMAGINARY content in real life (not online).

In 2013, IMAGINARY expanded its team, opened its open online platform to host exhibits, and created its second big exhibition, *Mathematics of Planet Earth* (MPE) (32), also from international contributors via a contribution call and competition co-organized by UNESCO. The MPE exhibition included exhibits on cartography, modeling of alpine glacier melting, dispersion of volcano ashes, tsunamis, sundials, or analysis of electrical grid infrastructure, to name a few. This remains the second most-spread exhibition of IMAGINARY, with 32 installations in 17 countries. Four science museums have included exhibits from MPE into their collections.

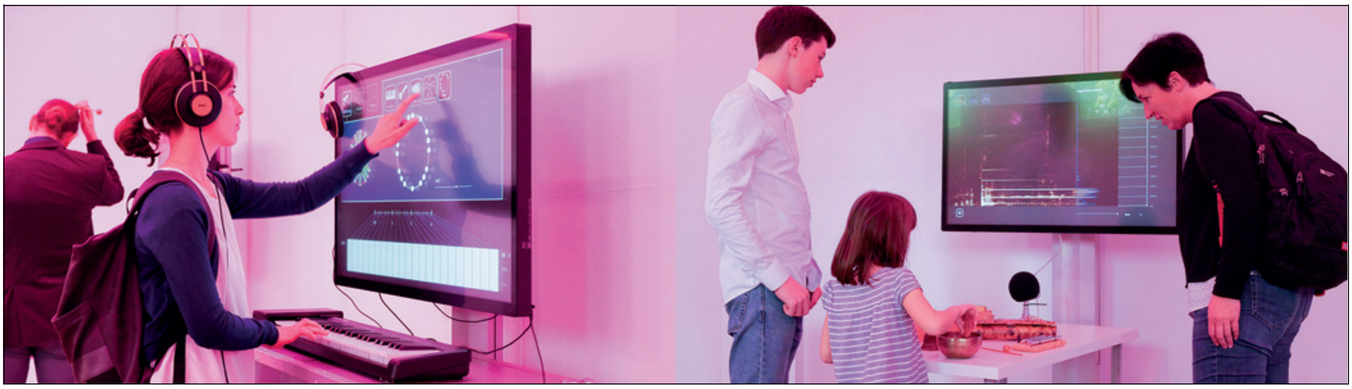
By 2016, the project had grown too big from its original purpose that it was appropriate to become a non-profit organization independent from MFO, headquartered in Berlin, although the institute remains a shareholder of IMAGINARY.

(29) https://www.imaginary.org/exhibition/imaginary-through-the-eyes-of-mathematics

(30) https://www.imaginary.org/program/morenaments

(31) The name "Cinderella" refers to a collection of apps (https://www.imaginary.org/program/cinderella-applets) and to the software used to develop them, by Jürgen Richter-Gebert and the Technical University Munich team. IMAGINARY has a long-time collaboration history with the *Cinderella* team.

(32) https://www.imaginary.org/exhibition/mathematics-of-planet-earth



Impressions from *La La Lab* in Heidelberg, Germany.

In 2019, we inaugurated the exhibition *La La Lab: the Mathematics of Music* ([33]), which represented a turning point in IMAGINARY's museographic approach. While our previous exhibitions relied exclusively on the contributors who decided to participate, leaving a relatively small choice of the content to feature in the exhibition, with *La La Lab* we could decide the topics and the story to be told beforehand, and then we looked for researchers or artists who could help us build such an exhibition. A significant part of the exhibits was created in-house to fit exactly our storytelling. In *La La Lab*, we wanted to tell a story in which music is constructed from several accumulative building blocks, such as acoustics, scales and tuning, harmony, rhythm, composition, and expressiveness. Each of these building blocks uses some specific mathematical tools, and so mathematics supports music in a fashion similar to how it supports physics. Therefore, we ensured that at least one exhibit was addressing each of these topics. We had a sound synthesizer and analyzer with Fourier analysis, a scale exhibit exploring tunings, a *Tonnetz* exhibit portraying geometrical visualizations of harmony, and *Con Espressione!*, an AI-assisted performer that reacts to the hand indications of the user as if it gave hints to a pianist on how to make a piece more expressive.

We made sure that we portrayed real nice-sounding musical pieces, some current research, and techniques such as artificial intelligence, and we took care to have truly artistic exhibits. We tried to balance the more insightful and complex exhibits with more playful and light ones, and also the computer-based exhibits with several hands-on interactives. With more than 20 exhibits (including many computers and big format touchscreens) and taking more than 500 square meters, it had a significantly higher budget than our previous exhibitions. The exhibition was supported by the Klaus Tschira Foundation. *La La Lab* has been displayed in Heidelberg, Belgrade, Salzburg, Strasbourg, and other cities. An independent copy made by a school in Traunstein (Germany) has toured some high schools in Bavaria ([34]).

In 2020 we were expecting to inaugurate our fourth major exhibition, *I AM A.I.: Explaining Artificial Intelligence* ([35]), supported by the Carl Zeiss Foundation, but unfortunately, the pandemic altered everyone's plans. Instead, we launched a virtual exhibition ([36]) featuring five exhibits designed to engage visitors through physical actions such as making music, playing, investigating, reading, and crafting. Additionally to the explanatory texts, we offered a virtual interactive tour that combined these exhibits on A.I. with a pre-recorded video, providing a guided experience visitors could follow at their own pace. Exhibits on that website format included *Neural Numbers*, which explains how a neural network can be trained to recognize hand-written digits, *Gradient Descent*, an arcade-like video game that mimics the mathematical opti-

([33]) https://lalalab.imaginary.org/

([34]) https://lalalab.akg-ts.de/

([35]) https://www.imaginary.org/exhibition/i-am-ai-explaining-artificial-intelligence

([36]) https://www.i-am.ai/

Impressions from *I AM A.I.* at Phaeno Science Center in Wolfsburg, Germany, photographer Janina Snatzke.

mization process that helps train a neural network, or *Piano Genie*, an example of how A.I. can help to generate music from a user without requiring the user to understand and apply harmony and composition rules.

When the full exhibition finally opened, in 2022, it also included bigger production exhibits such as *Reinforcement Learning*, about this technique in Machine Learning, the *Turing board game*, to reflect on the meaning of understanding a problem vs solving a problem, or the *Ethics of Autonomous Vehicles*, which tries to spark reflection on the implications of shifting to a society increasingly reliant on artificial intelligence systems. From a museographic perspective, we deepened our path started with *La La Lab* to begin with a central story to tell, identify key topics that must be part of that story, and build our exhibition around the story. We had outsourced furniture production for the first time, a visual language, design code, and other production aspects that made us closer to commercial quality agencies. On the other hand, no commercial exhibit manufacturer has the type of mathematical perspective or content we have. We displayed *I AM A.I. in* some museums and science centers such as Phaeno Science Center in Wolfsburg (Germany).

The *I AM A.I.* exhibition also triggered a series of projects related to communicating A.I. Besides the exhibition and the website, we launched *Explaining A.I.* (2020), a series of online workshops during pandemic through the Goethe Institute network; *AI Explorables for Schools* (2022) (37), a massive open online course (MOOC) in the KI-campus platform; *MusKI* (2023) (38), an explorable web book on history and techniques of electronic music and artificial intelligence; and the ongoing Erasmus+ collaboration project *A.I. Exhibits (2024)* for museums and *AI suitcases* (2024), each one exploring one core concept of A.I. With these projects, we exploit our acquired expertise in communicating A.I., and we simultaneously expand our range of expositive techniques, by establishing contact with schools and other museums or increasing our online offer.

IMAGINARY's fifth major exhibition was the *10-Minute Museum on the Mathematics of Climate Crisis* (2022) (39), a new format for us consisting of a self-contained, free-standing, portable, physical station that brings the exhibition format to a very compact space, intended for a very short-time interaction. This 10-minute museum is a four-sided column occupying one square meter footprint, with several interactive exhibits displayed on its sides. It is intended to be installed in lobbies, entrance halls, shopping malls, train stations... or museums. From a design point of view, it offers a tight integration of the exhibits with a well-defined story, albeit giving the visitor the possibility to explore the exhibits in any order. It features a central eye-catching piece – a polyhedral pixelated globe, evoking the digital models that abstract features of our planet – a use of the three-dimensional space (holes, perpendicular surfaces), and several screens playing animations

(37) https://ki-campus.org/courses/ai-explorables-schools

(38) https://www.muski.io/

(39) https://climatecrisis.imaginary.org/

Impressions of the *10-Minute-Museum* at Jacob-und-Wilhelm-Grimm-Zentrum in Berlin, Germany.

and interactive apps that drive the visitor's attention. From the point of view of content, the story focuses on the concept of mathematical models, applied to climate sciences. Individual exhibit modules include a touchscreen simulation of the Navier-Stokes equations for fluid motion, followed by describing global ocean-atmosphere circulation models that can be used for weather prediction, but can also reveal long-term trends like changes in the gulf stream currents or the polar jet streams. Next, a module describes the Earth's energy balance and the Greenhouse effect by using box models of stocks and flows governed by some differential equations whose parameters can be changed by the visitor. A model is a tool to explore dynamics, is the motto. Finally, some physical pieces, like a tipping point marble metaphor and some sea shells showing the effects of ocean acidification, make the hands-on part of the mini-museum. The *Climate Crisis* project was also supported by the Klaus Tschira Foundation. Thanks to its portable format, it has traveled (and still does) to many venues and cities around Germany. We have used this experience to enter some science center circuits.

Since 2021 IMAGINARY has established a fruitful relationship with the Futurium museum in Berlin, a science center oriented to technological and societal challenges for the future of humanity. We have developed three exhibits for Futurium. The first one is the *Future Mobility Simulator* (2021) ([40]), which combines a scientific urban mobility simulation with elements from computer games. Visitors can build their own city scenarios of mobility, combining residential and commercial areas, parks, and roads with Lego bricks. What at first looks like just a blocky city model is brought to life: you can see how roads connect, how traffic flows and how CO_2 emissions change. The exhibit allows to playfully simulate how urban planning decisions, new laws and technological advances change mobility in a city. Technically, a camera recognizes the position and type of Lego tiles and feeds information to a computer, which runs a simulation. A light projector on top casts images of cars moving, and color-codes the plain-white tiles according to the area they represent in the city, and optionally also to air pollution or other values resulting from the simulation. A screen displays scores and other information about how successful the current arrangement is, and offers to switch between mobility scenarios like adding autonomous vehicles or speed restrictions in the city. This exhibit is inspired by the simulation software *CityScope* ([41]) created by researchers at MIT MediaLab from the Massachusetts Institute of Technology. This represented a qualitative jump in the type of physical interfaces we had built up to the moment, which posed no small technical challenges, but on the upside, it was our first exhibit designed for and sold to a major science center, meeting quality, safety, and design criteria from commercial agencies.

([40]) https://futurium.de/en/future-mobility-simulator

([41]) https://www.media.mit.edu/projects/cityscope/overview/

Impressions of the *Future Mobility Simulator* at Futurium in Berlin, Germany, photographer David von Becker.

The second exhibit at Futurium is *Citizen Quest* (2023) ([42]), which tackles the question: What does the future of democracy look like? The multiplayer game is set in a virtual city where players engage in democratic decision-making processes. Through navigating diverse opinions and stakeholders, players strive to reach a consensus on complex issues. The game explores future-oriented concepts of digital democracy, such as citizens' assemblies and algorithmic decision systems. With this exhibit we developed a type of meta exhibit (a game exhibit engine) that can easily be adapted to other content, stories and topics. By changing the image design and the texts of the players, we have been able to reuse the platform in other social topics, like freedom and data ([43]). This is the exhibit that we have most copies and variations, and the most versatile so far.

The third and most recent exhibit on display at Futurium is *RRRRR* (2024) ([44]). In this multiplayer cooperative simulation game, the main aim is to learn about five key strategies of the Circular Economy: Reduce, Repair, Reuse, Refurbish, and Recycle. The goal is to establish a functional circular economy model centered around a particular item: smartphones. The exhibit revolves around a dynamic simulation depicting smartphone production and usage lifecycle in a screen-filling graphic-novel-like animation, accessible from all sides of a table. Visitors can engage with the simulation through a game mechanic based on worker placement. This mechanic allows them to modify various model parameters by triggering specific actions and events. By participating in this interactive experience, visitors can directly shape the simulation's development and influence its stakeholders. These engagements provide valuable insights into how the principles of reducing, repairing, reusing, refurbishing, and recycling impact the economy in real-world scenarios. On this exhibit, we leveraged previous experience with stocks and flows modeling (Earth energy balance and greenhouse effect from our climate mini-museum) to define a plausible model based on recent research literature, albeit the exhibit is only suitable for didactical purposes, and not for real decision making. On the technical side, we explored the technology of physical tokens for worker placement. These tokens resemble transparent hockey pucks, which are positioned on the table-sized horizontal touchscreen specially designed to be sensitive to these tokens but not to finger touch.

We have covered the major exhibitions and exhibits developed and used by IMAGINARY. Our portfolio includes other exhibition-oriented projects, notably several collaborations with European partners through the EU's Erasmus+ scheme (*MathSpaces*, *Mathina*, *Significant Mathematics for Early Mathematicians (SMEM)*, and the ongoing *AI exhibits* and *Quantum arcade*),

([42]) https://www.imaginary.org/event/new-exhibit-citizen-quest-opening-futurium

([43]) https://www.imaginary.org/event/citizen-quest-on-board-of-the-ms-wissenschaft

([44]) https://www.imaginary.org/event/new-exhibit-rrrrr-about-circular-economy-at-futurium-berlin as well as quite a few other non-exhibition-oriented projects.

It is relevant to mention that IMAGINARY participates in several dissemination projects that support communities. For instance, we develop the website and workshops for the International Day of Mathematics ([45]) (a global celebration of math for schools and students), we collaborate in the outreach of the MaRDI ([46]) project (spreading the Research Data culture to the math researchers community), and we co-organize the MATRIX $\times$ IMAGINARY conferences ([47]) for math museums and math communicators. All these projects have in common that a) aim to disseminate ideas to a large heterogeneous audience and b) IMAGINARY is in a position to use its experience in exhibitions and non-formal education to address these audiences.

As we have seen, IMAGINARY started and keeps close to the academic world, it focuses on communicating modern mathematics, and its main target public is largely grown students and adults (except for a few projects like the Erasmus+ collaborations). Through partnerships with experts, IMAGINARY often takes core models or engines that researchers provide as prototypes, and IMAGINARY's team transforms them into fully fledged exhibits, with attention to user interaction, message, didactical considerations, design, and coherent storytelling across an integrated exhibition. Because IMAGINARY places itself as an agent between researchers, museums, the general public, educators, and school children, it relies on open-source licenses to spread and re-use content, and in a strong community network to articulate the whole system and reach the end public.

Conclusion

Mathematics communication is already passing from an emergent field to a consolidated domain. In particular, many mathematics museums that started as enthusiastic projects but without any future security, are now well-established institutions that have proven to be sustainable and appealing to the public. Furthermore, the number of these institutions is growing (50+ worldwide), and this tendency will likely continue globally, with an open community of math museums and math communicators already existing.

IMAGINARY has placed itself in a particular niche of communicating modern mathematics and its connections with many diverse fields, staying close to the research community. IMAGINARY relies on interactions and partnerships with the community to spread its content, which is offered under open licenses.

Understanding museography as a language is crucial to achieve better exhibitions. While other museums and science centers have developed museographic skills for decades, and their techniques and contents are more or less stable, many math museums are still pioneering and experimenting with novel expositive techniques and contents. It will be a task for museographers and pedagogues to analyze the development and performance of the math museums.

Math museums and other math communication initiatives are already interacting with the school education system, with some of these becoming references on mathematical extra-curricular offers in their respective regions. The non-formal education theory is the appropriate pedagogical framework to embed math museums and other math communication activities, to analyze their goals, their performance, and their development from a theoretical perspective. On the practical side, placing math museums and math communication in a functional role within education and society will help secure funding, support, and security for the professional development of their members.

The role of math communication as a non-formal mode of education and its connection to the formal education provided in schools and universities are still being researched and developed. We hope for a synergy between the two sides of the same coin to achieve a better education and a more enlightened society.

([45]) https://www.idm314.org

([46]) https://www.mardi4nfdi.de

([47]) https://matrix.imaginary.org

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Daniel Ramos (Zaragoza, Spain, 1983) is a mathematician working on math communication. He obtained a PhD in Mathematics in 2014 from the Autonomous University of Barcelona (Spain) and held postdoc positions in Montpellier (France) and Lisbon (Portugal). In 2018 he started working on math communication at IMAGINARY as Chief Content Officer, curating exhibitions like La La Lab: The Mathematics of Music*. He develops mathematical content, prototypes software apps, conducts workshops, and writes diverse texts in many IMAGINARY projects. Occasionally, he also combines his work at IMAGINARY with other freelance math communication for math institutes, such as the Centre de Recerca Matemàtica in Barcelona. He is based in Luxembourg.*