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Vector calculus in the didactics of mechanics in Italy in the early twentieth century

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Sommario: *Dapprima abbiamo dato alcuni lineamenti storici sul calcolo geometrico e l'algebra geometrica durante la seconda metà del XIX secolo e i primi anni del XX secolo. Dopodiché il nostro obiettivo è stato quello di mettere in luce come in quel periodo in Italia, tenendo presente il contesto internazionale (confronti con Eugène Prouhet (1817-1867), Charles Sturm (1803-1855), Gustav Robert Kirchhoff (1824-1887), Osip Ivanovich Somoff (1815-1876), Paul Appell (1855-1930), Otto Föppl (1854-1924) e altri) il calcolo geometrico e le omografie furono didatticamente utilizzate nei trattati universitari di meccanica e di fisica matematica. Nella nostra rassegna storica, abbiamo tenuto presenti e analizzati i testi di Ottaviano Fabrizio Mossotti (1791-1863), Gian Antonio Maggi (1856-1937), Pietro Burgatti (1868-1938) e quelli relativi alla tradizione didattica peaniana (Filiberto Castellano, Cesare Burali Forti, Roberto Marcolongo, Tommaso Boggio). Il paragrafo quattro è dedicato alla descrizione della teoria sintetica delle omografie nell'insegnamento in Italia della meccanica classica; questa teoria è considerata come un ampliamento del* sistema minimo. *Le nostre considerazioni intorno al testo di Tullio Levi-Civita e Ugo Amaldi sono presentate nel quinto paragrafo. Risulta evidente dalla nostra analisi il ruolo cruciale del sistema minimo. Dal quale vengono sviluppati concetti come le omografie e introdotti altri operatori. Giovanni Giorgi (1871-1950), fisico e matematico molto critico nei confronti delle tecniche proposte dai vettorialisti, dice che i metodi di Burali-Forti e Marcolongo sono "troppo compatti e di difficile lettura". Comunque, sempre secondo il Giorgi uno dei meriti riconosciuti ai vettorialisti italiani fu quello di aver studiato appunto approfonditamente la teoria delle omografie vettoriali.*

Abstract: *Firstly, we have provided a historical outline of the geometrical calculus and geometrical algebra during the second part of XIX century and the first years of XX century. Subsequently, our aim has been to highlight through this period in Italy, and in the international context (comparison with Eugène Prouhet (1817-1867), Charles Sturm (1803-1855), Gustav Robert Kirchhoff (1824-1887), Osip Ivanovich Somoff (1815-1876), Paul Appell (1855-1930), Otto Föppl (1854-1924) and others) the didactic use of geometrical calculus (and of the homographies) in the university treatises of mechanics and of mathematical physics. In this historical overview, we have analyzed the textbooks of Ottaviano Fabrizio Mossotti (1791-1863), Gian Antonio Maggi (1856-1937), Pietro Burgatti (1868-1938) and those of the Peanian tradition (Filiberto Castellano, Cesare Burali-Forti, Roberto Marcolongo, Tommaso Boggio). Section Four is devoted to the description of homography synthetic theory in teaching classical mechanics in the Italian vector school: homography as an extension of the* minimum system. *Our considerations about the Tullio Levi-Civita and Ugo Amaldi textbook are presented in Section Five. The essential role of the* minimum system *is evident in our previous analyses. From this, one can develop other concepts (such as the homographies) and introduce new operators. Giovanni Giorgi (1871-1950), a physicist and mathematician emblematic as critic on the techniques proposed by the vectorialists, states Burali-Forti and Marcolongo's methods are "too compact and difficult to read". However, according to Giorgi, one of the merits of their work was the depth of the theory of vector homographies.*

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1. – Historical preliminaries

Before presenting the topic of this article, which is devoted to the didactical use of the *vector calculus* (and its development) in university treatises of rational mechanics and mathematical physics between the nineteenth and twentieth centuries, it is right to recall the *systems* of *geometric calculus* that since 1844, have been the prelude to what we will propose.

1.1 – *Hamilton's quaternions*

William Rowan Hamilton's (1805-1865) "invention" of *quaternions* had notable resonance not only in the English scientific world, despite its strong originality. In his twenties and thirties, Hamilton began taking an interest in algebra, particularly in complex numbers, with his friend John T. Graves (1806-1870). Peter Guthrie Tait (1831-1901) was also interested in the study of quaternions in order to utilize these techniques for applications in physics. ([2])

Quaternions (also called hyper-complex numbers) constitutes the development – and algebraic extension – of the complex numbers of the theory of oriented segments. Although this last notion had already been studied by August Ferdinand Möbius (1790-1868) and Giusto Bellavitis (1803-1880), an exact formulation of the notion of vector, as an oriented segment, is given by Hamilton in 1844. Subsequently, this was presented and studied by Hamilton in his fundamental works: *Lectures on Quaternions* (1853) and *Elements of Quaternions* (1866). Möbius had already introduced oriented segments in his essential work *Der barycentrische Calcul* (1827). Although with complex numbers (which algebraically constitute a field) the rotations in the plane could be represented, in turn, by quaternions (which constitutes a skew field), it is possible to study the rotations in 3D space. In *Elements of Quaternions* Hamilton presents the notion of the *vector* so:

> A right line AB, considered as having not only *length*, but also *direction*, is said to be a VECTOR. Its initial point A is said to be its origin; and the final point B is said to be its term. A vector AB is conceived to be (or to construct) the difference of two extreme points; or precisely, to be the result of the subtraction of its own origin from its own term; and in conformity with this conception, it is also denoted by the symbol $B - A$ [...]

And then:

> Two vectors are said to be EQUAL to each other, or the equation $AB = CD$, or $B - A = D - C$, is said to hold good, when (and only when) the origin and term of the one can be brought to coincide respectively with the corresponding points of the other by transports (or by translations) without rotation. ([3])

The following section will now address the other general systems.

1.2 – *Grassmann-Peano calculus*

It should be remembered that Hermann Günther Grassmann (1809-1877) published his *Ausdehnungslehre* in 1844, a work full of philosophical reflections, written in a language that had little to do with the mathematical mindset. In Germany, this work made no impression on the mathematical world, and when in 1862, Grassmann published a second edition in which he dedicated considerable space to geometrical interpretations and applications, it was, yet again, no more successful than the first. However, in Italy, Bellavitis read his work and he began an exchange of letters with Grassmann. Grassmann's work was also appreciated by Luigi Cremona and in particular by Giuseppe Peano (1858-1932) and his disciples. In this context, the work of Domenico Chelini (1802-1878) should also be remembered as a first form of vector calculus. He was contemporary with Bellavitis who proposed the *equipollences calculus*, a calculus with oriented segments through which he established a geometrical interpretation of complex numbers. Grassmann's fundamental idea is proposed in the following definition:

> [Def. 5]: Every expression derived by a system of unities (which have not only the absolute unity, that is, the real number 1) utilizing [real] numbers, named numbers of the derivation of the unities [...], is called extensive magnitude [estensive Grösse]. For instance, the polynomial:
>
> $$\sum a_i e_i$$

([2]) See for instance [MC AULAY 1893] and [SINÈGRE 1995].

([3]) Cfr. [KUIPERS 1999].

where a_i are real numbers and e_i form a system of unities, is an extensive magnitude. A magnitude is called numerical if the system is constituted only by the absolute unity.

The previous expression is termed *geometrical formation* by Peano.

Grassmann introduces the algebraic operations (addition, products) among extensive magnitudes. Actually, Grassmann proposes an abstract and general theory about the magnitudes. Peano held the post of lecturer in "Geometrical applications of infinitesimal calculus" at the University of Turin and he was well aware of the problems regarding geometric calculus, so that, when in 1887 he published his lectures in a book entitled *Applicazioni geometriche del calcolo infinitesimale*, he had in mind Bellavitis, Möbius, Hamilton and Grassmann. But it was in 1888 that Peano published the basic work on these topics: *Calcolo geometrico secondo l'*Ausdehnungslehre *di H. Grassmann preceduto dalle operazioni della logica deduttiva*, a crucial work also for the history of logic. Peano, in this volume of 1888, presents Grassmann's ideas in an original way. Among the followers of Peano, who devoted himself to the studies of geometric calculus, was Cesare Burali-Forti (1861-1931); but Filiberto Castellano (1860-1919), Tommaso Boggio (1877-1963) and Mario Pieri (1860-1904) also took an interest in the subject. According to Peano and to Burali-Forti, the co-ordinates method constitutes a numerical intermediation for studying the geometrical objects and their properties, while geometrical calculus proposes absoluteness and conciseness and the approach through it, is immediate and direct to study geometrical problems. However, this calculus does not exclude the use of co-ordinates. Alongside Peano, we call the geometrical formations, which have points as e_i (and only points), as systems of unities. It is of the first kind (or degree), of the second kind if the unities e_i are segments, of the third kind if the unities e_i are triangles and finally of the fourth kind if the unities e_i are tetrahedrons. We will in general denote a geometrical formation by F_q, where q expresses the kind of formation considered.

Between two geometrical formations we can establish the operation of algebraic addition, which complies with the rules of the algebra of polynomials. However, conceptually the more important operation is the alternated product, which is introduced by Peano and by Burali-Forti:

Suppose we have two geometrical formations in 3D, F_r and F_s, in that case, the alternated product is a product which complies with the rules of algebra of polynomials, but without changing the order of the letters which denote points. If $r+s \leq 4$ the product is said *progressive* (and expresses the geometrical operation of *projection*). If $r+s > 4$ the product is said *regressive* (and represents the geometrical operation of *section*). When considering the 2D plane case, the definition must be replaced by $r+s \leq 3$ and $r+s > 3$ respectively. The alternated product is not commutative. For instance, a segment is represented by the product AB, if A and B are two points that determine the segment. However, a segment can be represented by the formation of the first kind $B-A$. We also have that: $AB = -BA$ from which $AA = 0$. (4)

Peano and Burali-Forti are able to show, by means of their geometrical calculus, theorems of plane projective geometry. Even Bellavitis has proposed the applications of his equipollence calculus to elementary geometry and to projective geometry. Peano and Burali-Forti do not utilize figures as they consider the figures as heuristic representations. Instead, they consider only chains of expressions-identities of their calculus, and afterwards they interpret the last expression (thesis of the theorem) geometrically. Hence a figure is a "model" of the thesis of the theorem.

(4) Some particular progressive products, equalized to 0, have interesting geometrical interpretations:
$ABCD = 0$: the points A, B, C, D are coplanar;
$ABC = 0$: the points A, B, C are in a straight line;
$AB = 0$: the points A and B coincide.
If α is a plane and A a point, the expression:
$A\alpha = 0$ means that A lies on the plane α.
If a, b, c are straight lines:
$ab = 0$ means that the straight line a and b lie on the same plane: or they are parallel or they have a point in common.
$abc = 0$ means that three straight lines a, b, c have a point in common.

1.3 – *Other systems*

A historical recapitulation, albeit brief, on the first forms of vector calculus both in physics and in geometry, is an undertaking that goes beyond this article. However, we must remember some famous proposals. Physicists needed a concept that was easily linked to co-ordinates. James Clerk Maxwell (1831-1879) highlighted the scalar and cross products starting from the quaternions. Meanwhile Arthur Cayley (1821-1895), as early as 1845 had generalized the quaternions by introducing the *octonions*. Hamilton also generalizes the notion of quaternion, introducing the notion of *biquaternion* as a quaternion with complex coefficients. Cayley, however, was inspired by complex numbers. According to Cayley (1845 and 1847) and John Graves (1843) an *octonion* is a number $\alpha = A + A'\varepsilon$ where A and A' are real quaternions, that is, quaternions with real coefficients, and ε is a symbol such that $\varepsilon^2 = -1$. Therefore we have eight *unities*, so:

$$1 = i_0, i_1, i_2, i_3, \varepsilon, i_1\varepsilon, i_2\varepsilon, i_3\varepsilon.$$

If $\alpha = A + A'\varepsilon$ and $\beta = B + B'\varepsilon$ are two octonions, their product is given by

$$\alpha \cdot \beta = \left(A + A'\varepsilon\right) \cdot \left(B + B'\varepsilon\right) = \\ = (AB - B_c{'}A') + \left(B'A + A'B_c\right)\varepsilon$$

where B_c' and B_c are respectively conjugate quaternions of B' and of B. The product between the previous unities is anti-commutative (i.e. $u \cdot v = -(v \cdot u)$) and in particular non-associative. From an algebraic point of view, the set of octonions, for the usual operations $+$ and $\cdot$, is an anti-commutative and non-associative ring. In this period, it is interesting to note the existence of a theorem, which restricts the possibility of constructing finite dimension vector spaces on an associative skew field. Ferdinand Georg Frobenius (1849-1917) shows that only the real, complex numbers and the real quaternions are the associative skew field which, when regarding the additive operation, are finite dimension (respectively 1, 2, 4) vector spaces on the real numbers. If we introduce for the product the following property (weak associativity law):

$$(a \cdot a) \cdot b = a \cdot (a \cdot b) \text{ and}$$
$$(b \cdot a) \cdot a = b \cdot (a \cdot a) \text{ for every } a \text{ and } b$$

we obtain an alternating skew field. Then one has shown that only the skew field of Cayley's real octonions is an alternating skew field on the real numbers, which determines a finite dimension vector space (from the additive point of view).

Furthering Hamilton's work, William Kingdon Clifford (1845–1879) introduced a new kind of *biquaternion*, which also proved useful for applications to physics. Clifford opines that the "vectors of Hamilton are quantities having magnitude and direction, but no particular position", yet he posits that in many cases it is still fundamental to consider the position. Clifford writes: ([5])

> The name *vector* may be conveniently associated with a velocity of *translation*, as the simplest type of the quantity denoted by it. In analogy with this, I propose to use the name *rotor* (short for *rotator*) to mean a quantity having magnitude, direction, and position, of which the simplest type is a *rotation* velocity about a certain axis. A rotor will be geometrically represented by a length proportional to its magnitude measured upon its axis in a certain sense. [...] a vector may move anywise parallel to itself, but a rotor *only* in its own line. The *addition* of rotors will proceed by the rules which govern the composition of forces and rotations. [...] While the *sum* of any number of vectors is always a vector, it will only happen in special cases that the sum of several rotors is a rotor.

The notion of rotor is intuitively associated to the motion of a screw. So, a rigid body's velocity (as a screw) is represented by the combination of a rotation-velocity ω about a given axis and a translation-velocity v along the same axis. This matched velocity is called *twist-velocity* and the sum of two or more rotors is called *motor*. Finally, Clifford denotes by *biquaternion* the ratio of two rotors. A biquaternion is an operation which converts a motor into another motor. In another article of 1878, ([6]) Clifford proposes a general structure (so-called *Clifford algebra*).

Those who renewed the approach to vector analysis in 3D were Josiah Willard Gibbs ([7]) (1839-1903) and Oliver Heaviside ([8]) (1850-1925).

([5]) See [CLIFFORD 1870], p. 182.
([6]) See [CLIFFORD 1878].
([7]) [GIBBS 1881].
([8]) [HEAVISIDE 1893].

1.4 – *Toward the minimum system*

Still in the context of the Grassmann-Peano trend, Burali-Forti and Roberto Marcolongo (1862-1943) developed their studies regarding the vector calculus. Marcolongo was not, in the strict sense, a student of Peano, but with Burali-Forti he had a systematic scientific collaboration. We must consider the following important works of these mathematicians: *Elementi di calcolo vettoriale con numerose applicazioni alla geometria, alla meccanica e alla fisica-matematica* (1st edition 1909, second enriched ed. 1921); *Omografie vettoriali* (1st edition 1909); *Analyse vectorielle générale*, I. *Transformations linéaires* (ed. 1912), II. *Applications à la Mécanique et à la Physique* (ed. 1913). In the book of 1909, they present the vector calculus as a structure of *vector space* with the operations of scalar product and cross product. This one is the *minimum system*, while "we have the *general system* when the geometrical formations are introduced and we use the alternative product" (see Burali-Forti C., "Elementi di calcolo vettoriale", *Enciclopedia delle Matematiche Elementari* [vol. II, part II]). In conclusion, the minimal system was consolidated with the addition of the well-known operators of vector analysis as well. Of course, from the *general system,* we can derive the minimum system where notions or operators of *gradient*, *rotor*, *divergence* are introduced. It is relevant to note that into the minimum system it is possible to give the notion of quaternion. Moreover, quaternions calculus either alone or together with the minimum system, cannot give the general geometric calculus (general system).

2. – Vector calculus and mechanics according to a didactic point of view in Italy, from the end of the nineteenth century

We will now turn to analysing some important texts of mechanics and mathematical physics from the second half of the nineteenth century. We observe that the reference to some forms of oriented segments calculus either does not exist, or it is only implicitly there. The text by Ottaviano Fabrizio Mossotti (9) (1791-1863), *Lezioni elementari di Fisica Matematica, date nell'Università di Corfù nell'anno scolastico 1840-41* (Firenze, 1843) is the first testimony to this. He presents the parallelogram rule with interesting historical references in the 1st Note of the seventh lecture, referring to forces, that is in dynamics. For instance, Eugène Prouhet (1817-1867), disciple of Charles Sturm (1803-1855) on behalf of his professor, publishes the *Cours de Mécanique* held at École Polytechnique. In the 1861 edition, where the subdivision into Statics, Dynamics and Hydrostatics, and Hydrodynamics is respected, the parallelogram rule is recalled (referring in particular to the composition of forces). The treatment develops in purely analytical terms, considering the components (of the oriented segments). It is interesting to note that for candidates who wish to enter Ecole Polytechnique, a more concise volume entitled *Cours de Mécanique* was published later by a professor of mathematics, Ch. Michel, at the Doual lycée. This book, which is divided into Kinematics of the point, Dynamics of the material point and Statics, also proceeds with an analytical approach.

In this context, although the vector nature of various mechanical quantities had already emerged during the nineteenth century, the explicit use of vectors in mechanics is a process that began to take place during the second half of the century, as shown, for example, by the texts of mechanics by Gustav Robert Kirchhoff (1824-1887) (10) and Osip Ivanovich Somoff (1815-1876). (11)

This process tends to present a clear evolution at the end of the nineteenth century, as mainly attested by one of the most relevant treatises on mechanics of the period, the *Traité de Mécanique Rationnelle* (12) by Paul É. Appell (1855-1930), whose first edition is in 1893. In this text, in fact, a vital role is assigned to vector calculus, to which the entire 'Chapitre I' of the 'Première Partie' is devoted, entitled 'Théorie des vecteurs'; a vector calculus essentially based on the French geometrical tradition

(9) [Mossotti 1845].
(10) [Kirchhoff 1876].
(11) [Somoff 1878-1879].
(12) [Appell 1893].

in mechanics, related to the works by A.-L. Cauchy, M. Chasles and L. Poinsot.

Conversely, the end of the nineteenth century presents some of the first forms of application to mechanics of Gibbs' and Heaviside's vector calculus, a step that is usually associated with the publication in 1897 from the first volume of *Vorlesungen über technische Mechanik* (13) by August Otto Föppl (1854-1924). This is based on Heaviside's theory that Föppl had already examined in classical electrodynamic a few years earlier.

During this period in Italy, (14) a significant university textbook was undoubtedly that of Gian Antonio Maggi (1856-1937), which he wrote in 1894. (15) Maggi's treatise is culturally important. The title is *Principii della Teoria matematica del movimento dei corpi. Corso di Meccanica Razionale*. (16) Maggi begins with an extensive 'Preface and fundamental Preliminaries'. The organization of the volume is divided into the first part, which is dedicated to Kinematics and Dynamics and a second part which concerns the movement of free and constrained rigid bodies, pressure, and variable bodies (which also concerns fluids). In the 'Preliminaries' the notions of co-ordinate axes are introduced, as well as the notion of vector, applied vector and direction cosines. However, the notion of cross product is not explicitly provided. Thus, when he defines the moment M of an applied vector AB with regard to a point or pole P, Maggi describes this product by referring to well-known rules that he expresses through words. Additionally, for the scalar product, Maggi refers to analytical expressions. As an example, we report how he defines the work:

$$\int(Xdx + Ydy + Zdz) = \int_{t_0}^{t}\left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right)dt =$$

$$= \int_{s_0}^{s}\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right)ds = \int_{s_0}^{s} R\cos(Rs)ds$$

where (X, Y, Z) are the components of the force R, s denotes the measure of the trajectory arc of the considered point between its position (x, y, z) at time t and an assigned origin.

Hence, the co-ordinate method was systematically applied by many physicists, as well as by analytical geometers (we remember Enrico D'Ovidio's (1843-1933) work, *Geometria analitica* of 1896, which is influenced by German mathematicians such as Plücker, Hesse, Clebsch, Lindmann, Baltzer and Bobillier). Beyond the scientific usefulness, the co-ordinate method is didactically rather complicated.

In this context, we should also consider the *Lezioni di Meccanica Razionale* (17) held by Tullio Levi-Civita (1873-1941) at the University of Padua, starting from the academic year 1896/1897, which began to be published in lithographed form. Levi-Civita had graduated at the University of Padua with Gregorio Ricci-Curbastro with a thesis on the application of tensor methods for the study of differential invariants and had also been a student of Giuseppe Veronese, particularly in relation to the first systematic considerations on non-archimedean fields.

From the point of view of vector calculus, the most significant aspect of Levi-Civita's initial lectures, consists of a section expressly devoted to this theory. In these early editions, the notion of vector is considered in terms of an oriented segment, expressed by the symbol (R), and vector calculus essentially has a purely analytical character, based on the expression of the components of a vector as a

(13) [Föppl 1897].

(14) Several indications of the development of Italian treatises on Rational Mechanics between the 19th and 20th centuries can be found in [Cirillo, Maschio, Ruggeri, Saccomandi 2013].

(15) Maggi had a degree in Physics and in Mathematics at Pavia University in 1877-78. During the 1881, he studied in Germany, with Gustav Kirchhoff. Back in Italy, he was an assistant lecturer of Physics in Pavia and among his disciples were Carlo Somigliana and Giulio Vivanti. In addition, Maggi taught as full professor of pure mechanics at the Messina, Pisa, Milan universities. His most important works, in fact, concern on the one hand, physical mechanics, the analytical mechanics of material systems in particular, with the discovery, among other things, of the first forms of the equations of motion for non-holonomic systems, today called Maggi's equations, deduced some years before the works of Paul É. Appell on these mechanical systems. On the other hand, there are notable studies on the theory of electromagnetism, on optics and wave propagation. His other vital important contributions concern the theory of elasticity and the theory of potential.

(16) [Maggi 1896].

(17) [Levi-Civita 1896-1897].

function of the co-ordinates of its extremes, once introduced an orthogonal reference system. On this basis, the length R of the vector (R), the expression of the cosinus of the angle between two vectors (R_1), (R_2), and the orthogonal projection (R_g) of a vector (R) along a given line g are then introduced. The only operation explicitly introduced is the geometric sum or resultant $(R_1)+(R_2)+...+(R_n)$ of several vectors, which is defined by considering the extremes of a polygonal having as sides vectors "equivalent" to those assigned.

The theory then includes the introduction of the moment of a vector with respect to a point and an axis. It should be noted that the definition of the momentum of a vector $(R) = AB$ with respect to a certain point P is given here in vector terms – as that "vector $(\Gamma) = PQ$ with origin in P, of length equal to the product of R by the distance of P from the line AB, and directed normally to the plane PAB so that, for an observer with his feet at P and his head towards Q, the direction of the rotation of PA towards PB is that of the balls of the clock" – but without associating to this definition any kind of operation between vectors. The vector (Γ) is then obtained through the analytical determination of its components L, M, N as a function of those X, Y, Z of (R) and of the co-ordinates of the points A and P. In a similar way, that is, without making any reference to vector operations, the 'invariant trinomial' $LX+MY+NZ$ is considered, with L, M, N being the components of the resulting moment of a set of vectors (R_1), (R_2), ..., (R_n) of geometric sum the vector (X, Y, Z) with respect to a point P.

Finally, the 'vector theory', which is presented in these initial editions of Levi-Civita's courses on rational mechanics, is completed by the consideration of a further set of classical topics, such as the theory of couples and systems of parallel vectors.

It is interesting to note that there seems to be a strong analogy between this 'theory of vectors' considered by Levi-Civita and the considerations present in the analogous section of Appell's *Traité*. In particular, this concerns various parts for which it seems possible to speak of a resumption of many topics; a resumption that also concerns the use of parentheses to indicate vector quantities, which is the notation adopted in a systematic way in the 1893 edition of Appell's text.

3. – Teaching mechanics in Peanian tradition and in the Italian vector school

Peano's school was also of significant importance to the study of mechanics, through the application of vector calculus to this discipline. In particular, the publication in 1894 of the *Lezioni di Meccanica Razionale* (18) by Filiberto Castellano, who had graduated from the University of Turin in 1881, where he was Peano's assistant to the chair of Infinitesimal Analysis from 1890 to 1892. The *Lezioni di Meccanica Razionale* were derived from the courses Castellano held as a free lecturer from 1892, when at the Military Academy of Turin. In this text, there is an initial attempt to apply the basic concepts of geometric calculus to the formulation of the principles of mechanics solely in a synthetic form, i.e. without a primary use of Cartesian methods. (19)

The text in effect, includes a short first paragraph entitled "Geometric Formations – Vectors", based on the terminology used in Peano's lectures on infinitesimal analysis, which refers to the theory of formations of the first kind and to the related notions of vector, bivector and trivector. The course of mechanics by Castellano makes systematic use of this terminology. As Peano states, in his review of this text

> [...] A. develops the entire mechanics with considerable simplicity compared to the previous treatises, in which the constant use of co-ordinates is made. Suppose three axes naturally occur in a mechanical problem. In that case it will be convenient to decompose the various vectors according to these three axes, and the vector method will coincide with that of the co-ordinates. In any other case, the method followed by our Author is simpler than the ordinary one. This simplicity is shown by the fact that he writes one equation instead of three, and that the one equation is shorter than any of these three. However, the real advantage is much more important than this simplification in writing and it lies in the fact that one always reasons about the entities that appear in the problem, without needing to introduce into the reasoning entities that are extraneous to the question. (20)

(18) [Castellano 1894].

(19) The importance of the role of 1894 Castellano text in the development of treatises in Rational Mechanics in Italy is also clearly highlighted in [Cirillo, Maschio, Ruggeri, Saccomandi 2013], p. 4.

(20) [Peano 1895], p. 14.

The theory of vectors in the "Introduction" of the text by Castellano is based on the following definition: "the segment AB considered in magnitude and direction is called a vector, and is represented by $B-A$". (21) On this basis, the sum of point A with a vector u is introduced and the sum of several vectors u_1, u_2, u_3,..., u_n is defined as the difference between points

$$(A+u_1+u_2+u_3+\ldots+u_n)-A$$

Next, the product of a real number with a vector, the inner product $a\,|\,b$, the outer product or bivector between vectors, trivectors and the index "$|\,B$" of a bivector are also introduced. In the last part of this "Introduction", the notion of 'geometric formations' is presented, a vector as the geometric formation $A-B$ is interpreted and the notion of a derivative in the case of vectors is also presented.

On this basis, these notions of a 'vectorial' nature have multiple applications in Castellano's text.

For example, in the initial section regarding kinematics, after introducing the following symbols

$$P'=\frac{dP}{dt}=\lim_{h=0}\frac{P(t+h)-P(t)}{h}$$

$$P''=\frac{d^2P}{dt^2}=\frac{dP'}{dt}$$

$$\ldots$$

$$P^{(n)}=\frac{d^nP}{dt^n}=\frac{dP^{(n-1)}}{dt}$$

the author introduces the notions of "zero-order acceleration of point P", as the segment that originates in P and whose vector is P'; of the "first-order acceleration of point P" as the segment that originates in P and whose vector is P"; and so on.

In the context of these early didactic applications of vector calculus to rational mechanics in Italy, we should also consider the 1905 first edition of *Meccanica Razionale* (22) by Roberto Marcolongo, followed by a German translation in 1911 and subsequent other Italian editions. The text of 1905 includes a brief introduction to vector calculus, which explicitly refers to geometric calculus, as in the case of Castellano's text. Written a decade later, it is more systematic in its analysis.

This is evident from "Chapter One", which includes a more extended exposition on the theory of vectors in the Peanian style, which explicitly covers the "algebraic" part of the theory concerning the operations on vectors, yet remains substantially in line with Castellano's text. In this case, a vector is "imagined to be represented in magnitude and direction by a segment AB" and denoted by the symbol $B-A$, i.e. as a difference of points, in relation to the basic notion of "sum of a point A and a vector a"

$$B=A+a$$

Moreover, Marcolongo explicitly introduces the sum between vectors, the product of a vector u by a real number m and two other kinds of product: the inner or scalar product of two vectors, defined as "the product of the moduli and cosine of the angle of the vectors"; and the outer product or bivector ab "represented in magnitude and sense by the area of the parallelogram that has the two vectors as adjacent sides". The vector product is obtained by considering the index "$|\,ab$" of a bivector ab, considered as the vector u that is normal to the plane of a, b, of modulus that of ab, and whose direction is such that the trihedron a, b, u is dextrorotatory.

On this basis, various vector properties are then considered, including some notions of vector analysis, and, in particular, the rules of derivation in the case of the scalar product $a\,|\,b$ and the vector product:

$$D(a|b)=a|Db+b|Da$$

$$D(|ab)=|Da.b+|a.Db$$

already present in Castellano's test. A section of "Chapter One" in Marcolongo's book is explicitly devoted to the differential geometry of curves. Its final part presents a reformulation of the classical theory of moments and couples in terms of the operations and symbols introduced. For example, the moment of the vector $P-A$ with respect to point O is defined in this context:

$$M-O=|(A-O)(P-A)$$

(21) [Castellano 1894], p. 2.
(22) [Marcolongo 1905].

Regarding the application of vector notions to rational mechanics, it is interesting, for example, to see how the topic of continuous motions of a plane figure in its plane is developed in "Chapter Five" of Marcolongo's text. Starting from the relation obtained previously on the instantaneous motions of a rigid system

$$\dot{P} = \dot{O} + |\Omega(P - O)$$

and noting that angular velocity Ω is normal to the plane of the figure, Marcolongo obtains

$$|\Omega(P - O) = \omega(P - O)i$$

where $(P - O)i$ constitutes the rotation of the vector $(P - O)$ of 90° and therefore

$$\dot{P} = \dot{O} + \omega(P - O)i$$

In this way, a unique point C is then defined such that

$$0 = \dot{O} + \omega(C - O)i$$

which is the instantaneous center of rotation.

As already stated, the key moment in the history of the Italian vector school regards the contributions of Burali-Forti and Marcolongo to the topic of unification of vector notations, ([23]) starting from the papers published in "Rendiconti del Circolo Matematico di Palermo" in 1907 and 1908 (in connection with the discussion of this topic during the International Congress of Mathematicians in 1908) and soon after, with their books *Elementi di calcolo vettoriale* and *Omografie vettoriali* ([24]) in 1909 and *Analyse vectorielle générale* ([25]) in 1912 and 1913.

On this basis, it is unsurprising that in these books, the question on the application of vector calculus to classical mechanics is important. For example, "Chapter III" of Part 2 of *Elementi di calcolo vettoriale* is entitled "Applicazioni alla Meccanica", treating in vector terms topics such as the velocity and the acceleration of a point, central motions and rigid motions.

([23]) See [FREGUGLIA 1992], [FREGUGLIA, BOCCI 2008], [SALLENT DEL COLOMBO 2010].

([24]) [BURALI-FORTI, MARCOLONGO 1909a, 1909b].

([25]) [BURALI-FORTI, MARCOLONGO 1912, 1913].

Soon after the publication of these texts from the Italian vectorial school, the 1911 German edition of Marcolongo's 1905 *Meccanica* reflects on the use of vector calculus in teaching mechanics. In effect, this translation tends to reproduce the same order of topics from the 1905 Italian edition, but with a shift to the notions and symbolism of the Italian vector school. For example, in the German edition, the preliminary introduction to vector calculus is divided into two chapters, the first dealing with the basic principles of vector calculus and the second devoted explicitly to vector analysis. The structure of the first chapter, entitled *Vektorgeometrie*, initially provides a mechanical and physical explanation regarding the preliminary treatment of vector calculus, but then follows Burali-Forti and Marcolongo's *Elementi di calcolo vettoriale*, in a complete but rather abbreviated form. In particular, it includes all the algorithmic aspects of vector calculus considered in the book, starting from the sum between a point and a vector, and including some mechanical observations on the theory of moments of a vector, or a system of vectors.

It should be noted that in this approach to vector calculus, there is an ongoing consideration of vector identities in terms of the calculus of points. This can be seen directly, as in the following example:

$$D = A + (B - A) + (C - B) + (D - C)$$

or indirectly, for example, concerning the definition of the sum of two vectors in terms of the sum of points and vectors and the difference of points:

$$\boldsymbol{a} + \boldsymbol{b} = \{(O + \boldsymbol{a}) + \boldsymbol{b}\} - O$$

Moreover, in the second chapter of the text, entitled *Vektoranalysis*, there is a systematic introduction to the theory of vector fields expressed in the symbolic language of the Italian vector school. This also includes a short introduction to the differential geometry of curves, and a final segment on the vector expression of Green and Stokes' theorems.

It is interesting to observe that vector calculus, as introduced by Marcolongo in this German edition of his *Meccanica,* plays an active role in the symbolic expression of mechanical concepts. This is due to several factors.

Firstly, by introducing the vector principles of the differential theory of curves, it allows Marcolongo to formulate the theory of kinematics and dynamics of a material point in terms of vector analysis. This is similar to the 1905 edition of Marcolongo's text, but now with the use of the new symbolism.

For example, Marcolongo treats the topic of the vector expression of central motions, characterized in the first place by the relation

$$(P-O)\wedge\ddot{P}=0$$

where O is a given point or, by integrating

$$(P-O)\wedge\dot{P}=\boldsymbol{g}$$

with $\boldsymbol{g}$ a constant vector. If

$$\ddot{P}=-\frac{a^2}{r^3}(P-O)$$

the following equation is obtained

$$\ddot{P}\wedge\boldsymbol{g}=-\frac{a^2}{r^3}(P-O)\wedge\left[(P-O)\wedge\dot{P}\right]=a^2\frac{d}{dt}\frac{(P-O)}{r}$$

which upon integration yields:

$$\frac{\dot{P}\wedge\boldsymbol{g}}{a^2}=\frac{P-O}{r}+\varepsilon\boldsymbol{h}$$

where $\varepsilon\boldsymbol{h}$ is a constant vector and from which it follows that the trajectory is a conic.

In addition, the role that vector calculus plays in the formal expression of the kinematics of rigid bodies is somewhat significant, and is developed for example, in the analysis of instantaneous motions of rigid systems. Similar considerations on the systematic use of vector calculus are developed in the second volume, in particular regarding the dynamics of systems of material points.

Another key moment in the didactic applications of vector calculus to rational mechanics within the Italian vector school is given by the *Lezioni di Meccanica Razionale* by Pietro Burgatti (1868-1938). (26)

His best-known contributions are those on vector calculus and its applications in mechanics. Among his other noted works are: *Elementi di fisica ad uso delle scuole medie* (with Quirino Majorana (1871-1957)), Bologna 1927; *Fondamenti di geometria differenziale* (with Tommaso Boggio and Cesare Burali-Forti), Bologna 1929; *Teoria matematica della elasticità,* Bologna 1931; *Elementi di calcolo vettoriale e omografico,* Milano 1937.

Burgatti's treatise (27) (*Lezioni di Meccanica Razionale*, p. 540) presents topics in mechanics which were typically taught in Italian universities. Here, the Introduction is devoted to vector calculus, with the subsequent sections dealing with kinematics, and the examination of statics, where he analyses the role of the constraints and the equilibrium conditions. Finally, he explains the principle of virtual work according to the Lagrange formulation. The Dynamics is explained further in the third part where we find D'Alembert's principle, as well as Lagrange and Hamilton's equations. The mechanics of deformable bodies, the linear elasticity and the fluids, are treated in the fourth and final part of the book. Burgatti concludes, as an appendix, a wealth of historical information regarding the development of mechanics.

In the Preface of 1914 edition he writes:

> [...] I owe a special thanks to prof. R. Marcolongo of the University of Naples, very learned in this matter; I want to thank him publicly. The reader who knows his work (28) will see how much I owe him [...].

(26) [Burgatti 1914] Burgatti was born in Cento (Ferrara) in 1868. As a young man he began the military career. He graduated in mathematics at the University of Rome in 1893, and in 1895 he began his academic career. After qualifying as a university teacher in 1908 he became as professor of Pure Mechanics at the University of Messina. In the same year he was transferred to the same chair at the University of Bologna. Full professor from 1911, he remained in Bologna until his death in 1938. He also taught mechanics and Mathematical Physics at the nearby University of Ferrara. He was member of various academies, as the *Accademia dei Lincei*, Turin Academy of Sciences and that of Bologna; he was also vice-president of the UMI. Burgatti's studies and research focused particularly on mechanics, vector calculus, vector analysis, theory of elasticity, hydrodynamics, differential geometry and astronomy.

(27) We kept in account the article [Freguglia, Graffi 2014].

(28) Burgatti refers to the treaties: [Marcolongo 1905] and its German translation in 1911.

And in the Preface of 1919 edition:

> [...] I thank in particular, prof. Burali-Forti for his precious suggestions, to which I owe the improvement made to the theory of vectors exposed in the introduction [...].

As can be seen, Burgatti's gratitude to the teachings of both Marcolongo and Burali-Forti is explicitly declared. The introduction is organized over fifteen paragraphs, which includes: vector quantities and scalar quantities, vector addition, scalar product, cross product, Cartesian representation of vectors, i as operator, derivative of points and vectors, geometrical applications, scalar field, vector field and potential gradient, applied vectors, applied vector moment, equivalence between applied vector systems, axis momentum, couples and others fundamental notions and the related properties on the vector calculus. Hence Burgatti presents the so-called minimal system derived from the general Grassmann-Peano system. It follows that all the mechanics' concepts and their properties developed in the book are affected by the vector techniques. Some examples are as follows.

If (p. 184, ed. 1919) M is a point of a system and $\boldsymbol{F}$ (as vector) is an applied force at M, the *virtual work* of $\boldsymbol{F}$ is

$$Lav\boldsymbol{F} = F \times \delta M$$

where $\times$ is the scalar product and δM is the virtual displacement. Note the application of δ to a point gives a vector. We have

$$\delta M = \frac{dM}{dt}\delta t$$

At page 291, Burgatti defines the *actual work* so:

$$\textstyle\sum_{s=1}^{n} \boldsymbol{F}_s \times dM_s$$

where dM_s are the actual displacements. As well as, for instance in a compact way he defines (p. 282) the *momentum of inertia forces* so

$$\sum m(M-0) \wedge \frac{d^2M}{dt^2} \times \boldsymbol{a}$$

where m denotes the mass of an infinitesimal particle belonging to a body, M is one of its points, $\boldsymbol{a}$ is a unit vector which determines an axis, O is a point of this axis, $\wedge$ is the cross product and $\times$ is the scalar product. Of course $(M-O)$ is a vector. Then, (29) in respect of the perturbation theory, he suggests some words which somehow anticipate the KAM theory. "Thus, the theory reproduces perfectly the observations; but, moreover, it gives life to them in our thought; it shows to the eyes of our mind, as if we were an immortal audience, year after year, the mechanical evolution of the solar system. Whatever the future will be of this perturbation theory, it will remain as one of the highest creations of the human mind". (30)

If one wants to establish a comparison with classical treatises, the Italian vector school offers a succinct way of explaining the mechanics and the development of related theorems This approach would also allow a fairly smooth transition into the method of co-ordinates from the classical treatises in mechanics.

4. – The use of homography synthetic theory in teaching classical mechanics in the Italian vector school

A specific the use of vector calculus in teaching rational mechanics in the Italian vector school concerns the role played by the theory of homographies. This theory is well presented and analysed in the treatises *Omografie vettoriali* and *Analyse vectorielle générale* and we will present the latter of these approaches below. (31) Burali-Forti and Marcolongo define a *homography* as a linear operator which transforms vectors into vectors, that is, symbolically:

$$\tilde{\omega}\boldsymbol{v} = \boldsymbol{w}$$

where $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors and $\tilde{\omega}$ is a linear operator.

The kinds of homographies are various. For instance, the vector *homotheties* are represented as:

$$\tilde{\omega}_h\boldsymbol{v} = m\boldsymbol{w}$$

(29) See [Freguglia, Graffi 2014], p. 203.

(30) [Burgatti 1919], p. 391.

(31) For a classical modern reading of the theory of homographies in terms of tensor calculus, see [Finzi, Pastori 1961], chap. III.

where m is a real number; the axial homography, so:

$$\tilde{\omega}_a \boldsymbol{x} = \boldsymbol{u} \wedge \boldsymbol{x}$$

where $\wedge$ is the vector product.

By getting the idea from Gibbs, our mathematicians introduce the notion of *dyade*, so:

$$\boldsymbol{H}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{x} = \boldsymbol{u} \times \boldsymbol{x}.\ \boldsymbol{v}$$

where $\times$ is the scalar product and "**.**" is the product vector by the scalar which derives from $\boldsymbol{u} \times \boldsymbol{x}$.
A dyad transforms a vector into a parallel vector. If α is any homography, one has (Fundamental theorem of dyads):

$$\alpha = \boldsymbol{H}(\boldsymbol{i}, \alpha\boldsymbol{i}) + \boldsymbol{H}(\boldsymbol{j}, \alpha\boldsymbol{j}) + \boldsymbol{H}(\boldsymbol{k}, \alpha\boldsymbol{k})$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the *versors*.

In this context, the second volume of *Analyse vectorielle générale* explicitly entitled "Application à la méchanique et à la physique" includes a first chapter solely dedicated to "Moments d'inertie et quantité de mouvement dans les systèmes solides". This presents the first explicit intervention of the theory of homographies, as an application of what was presented in the first volume. For example, in this context, the moment of inertia of a solid body of mass m is introduced by explicitly considering the homography

$$\alpha = -\sum m\{(P-O)\wedge\}^2$$

where P is a point of mass m of the solid body and O is a fixed point, and the summation is relative to all the points of the system. In this way, the moment of inertia with respect to an axis $O\boldsymbol{u}$ is defined as

$$I = \boldsymbol{u} \times \alpha\boldsymbol{u}$$

$\boldsymbol{u}$ being a unit vector.

We can also find the notion of *homography* in "Chap. II" in the section "Meccanica dei corpi deformabili", within Burgatti's *Lezioni di Meccanica Razionale*. Here, Burgatti states:

> [...] To obtain a deformation, starting from a given state of a body, it is sufficient to imagine that each of its points (molecules) P undergoes a displacement $\boldsymbol{s}(P)$, which varies on the various points; but, so that their whole does not represent a rigid body motion. [...] Let us consider a particle containing a given point P, chosen at will, and let $P_1 = P + dP$ be another of its points; then we denote by P' and P'_1 the points corresponding respectively to P and P_1 after the deformation. We easily obtain:
>
> $$P' - P = \boldsymbol{s}(P), P'_1 - P_1 = \boldsymbol{s}(P_1) = \boldsymbol{s}(P + dP)$$
>
> and so, by difference
>
> $$(P'_1 - P') - (P_1 - P) = \boldsymbol{s}(P + dP) - \boldsymbol{s}(P)$$
>
> from which
>
> $$P'_1 - P' = dP + \frac{d\boldsymbol{s}}{dP}dP$$
>
> This formula defines the deformation of the considered particle; since, holding P as fixed and varying P_1, it describes the law of correspondence between the vector elements $P_1 - P = dP$ of the particle in the initial state and those $P'_1 - P'$ of the same particle after deformation. This correspondence depends on the $d\mathbf{s}/dP$ homography; hence this will be called *deformation homography*.

Now if we put $dP = h\boldsymbol{a}$, where $\boldsymbol{a}$ is a unit vector and $h \in \boldsymbol{R}^+$, utilizing calculations Burgatti obtains:

$$\varepsilon = \mathrm{mod}\left(1 + \frac{d\boldsymbol{s}}{dP}\right)\boldsymbol{a} - 1$$

where ε expresses the *linear dilatation coefficient along the direction* $\boldsymbol{a}$ *related to the point* P.
$d\boldsymbol{s}/dP$ is a homography, a vector $\boldsymbol{s}(P)$ being transformed into a vector $d\boldsymbol{s}/dP$. This transformation is determined (32) as:

$$\frac{d\boldsymbol{s}}{dP}\boldsymbol{i} = \frac{\partial\boldsymbol{s}}{\partial x} = \frac{\partial s_x}{\partial x}\boldsymbol{i} + \frac{\partial s_y}{\partial x}\boldsymbol{j} + \frac{\partial s_z}{\partial x}\boldsymbol{k}$$

$$\frac{d\boldsymbol{s}}{dP}\boldsymbol{j} = \frac{\partial\boldsymbol{s}}{\partial y} = \frac{\partial s_x}{\partial y}\boldsymbol{i} + \frac{\partial s_y}{\partial y}\boldsymbol{j} + \frac{\partial s_z}{\partial y}\boldsymbol{k}$$

$$\frac{d\boldsymbol{s}}{dP}\boldsymbol{k} = \frac{\partial\boldsymbol{s}}{\partial z} = \frac{\partial s_x}{\partial z}\boldsymbol{i} + \frac{\partial s_y}{\partial z}\boldsymbol{j} + \frac{\partial s_z}{\partial z}\boldsymbol{k}$$

However, Burgatti did not systematically develop the theory of homographies.

Another early use of the theory of homographies can be found in the second Italian edition of Marcolongo's *Meccanica Razionale*. Here, Marcolongo resumes what had been done in the volume *Omografie vettoriali*, as he devotes paragraphs §7 to §9 of the first chapter, dedicated to vector calculus, and specifically to the basic notions on the theory of

(32) See [LAZZARINO 1929], p. 97.

homographies. These notions are then used within the second volume of this text in respect of the treatment of continuum mechanics.

However, the text on rational mechanics, which surely shows the most significant use of the theory of homographies, is the 1921 volume by Burali-Forti and Boggio *Meccanica Razionale*, (33) a text that considerably introduces this theory. Thus homography α is determined when three non-coplanar vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are assigned, and the vectors $\boldsymbol{u}_1 = \alpha\boldsymbol{u}$, $\boldsymbol{v}_1 = \alpha\boldsymbol{v}$ and $\boldsymbol{w}_1 = \alpha\boldsymbol{w}$ are known by the following table:

$$\alpha = \begin{pmatrix} \boldsymbol{u}_1, & \boldsymbol{v}_1, & \boldsymbol{w}_1 \\ \boldsymbol{u}, & \boldsymbol{v}, & \boldsymbol{w} \end{pmatrix}$$

If even $\boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{w}_1$ are not coplanar, then it is possible to consider the inverse of α so:

$$\alpha^{-1} = \begin{pmatrix} \boldsymbol{u}, & \boldsymbol{v}, & \boldsymbol{w} \\ \boldsymbol{u}_1, & \boldsymbol{v}_1, & \boldsymbol{w}_1 \end{pmatrix}$$

The sum and the product of homographies have, respectively, the following properties:

$$(\alpha + \beta)\boldsymbol{u} = \alpha\boldsymbol{u} + \beta\boldsymbol{u} \text{ and } (\beta\alpha)\boldsymbol{u} = \beta(\alpha\boldsymbol{u})$$

being α and β homographies and $\boldsymbol{u}$ a vector. The homographies have the following laws:

$$\alpha + \beta = \beta + \alpha; \qquad \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$

and in general $\alpha\beta \neq \beta\alpha$

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma; (\alpha\beta)\gamma = \alpha(\beta\gamma)$$

$$(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$$

where α and β are invertible

$$\alpha^n = \alpha\alpha \ldots \alpha(n \; times)$$

As our authors write, the homographies form a linear system. Then the *dyads* are introduced. They are all degenerate and do not form a linear system. If α is a homography and O a fixed point, all points P satisfying the expression

$$(P - 0) \times \alpha(P - 0) = \text{cost}$$

stay on a quadric called the *indicator* of α. Taking into account also the volume of Burali-Forti and Marcolongo from 1909 (2d ed. 1920) a vector homography α, considering an orthogonal reference $O, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, is expressed by the Cartesian way by determining the vectors $\alpha\boldsymbol{i}$, $\alpha\boldsymbol{j}$, $\alpha\boldsymbol{k}$ through the system:

$$(*) \begin{cases} \alpha\boldsymbol{i} = a_{11}\boldsymbol{i} + a_{12}\boldsymbol{j} + a_{13}\boldsymbol{k} \\ \alpha\boldsymbol{j} = a_{21}\boldsymbol{i} + a_{22}\boldsymbol{j} + a_{23}\boldsymbol{k} \\ \alpha\boldsymbol{k} = a_{31}\boldsymbol{i} + a_{32}\boldsymbol{j} + a_{33}\boldsymbol{k} \end{cases}$$

where a_{ij} are real numbers. So, a *dilatation* is a particular homography determined by:

$$\boldsymbol{u} \times \alpha\boldsymbol{v} = \boldsymbol{v} \times \alpha\boldsymbol{u}$$

with $\boldsymbol{u}$ and $\boldsymbol{v}$ any vectors.
If in the system (*) we put $a_{rs} = a_{sr}$ then we get the Cartesian expression of a dilatation. In the theory of vector homographies, operators on homographies play an important role. So, it is called *the vector of* α and we will indicate it by $V\alpha$, the vector defined synthetically by the expression:

$$2(V\alpha) \times \boldsymbol{u} \wedge \boldsymbol{v} = \boldsymbol{v} \times \alpha\boldsymbol{u} - \boldsymbol{u} \times \alpha\boldsymbol{v}$$

$(V\alpha)\wedge$ represents the axial homography of which the vector $(V\alpha)$ is the axis. The related Cartesian expression is:

$$V\alpha = (a_{23} - a_{32})\boldsymbol{i} + (a_{31} - a_{13})\boldsymbol{j} + (a_{12} - a_{21})\boldsymbol{k}$$

where a_{ij} are real numbers.

In turn, we have dilatation of α and $D\alpha$ denotes it, the operator on α establishes a homography so:

$$D\alpha = \alpha(V\alpha)\wedge$$

Finally, we can indicate *conjugate of a* by $K\alpha$, the homography (vector) defined by the expression

$$K\alpha = D\alpha - (V\alpha)\wedge$$

Cartesian expression of $K\alpha$ is:

$$\begin{cases} K\alpha\boldsymbol{i} = a_{11}\boldsymbol{i} + a_{21}\boldsymbol{j} + a_{31}\boldsymbol{k} \\ K\alpha\boldsymbol{j} = a_{12}\boldsymbol{i} + a_{22}\boldsymbol{j} + a_{32}\boldsymbol{k} \\ K\alpha\boldsymbol{k} = a_{13}\boldsymbol{i} + a_{23}\boldsymbol{j} + a_{33}\boldsymbol{k} \end{cases}$$

whose coefficients refer to (*).

A separate paragraph is dedicated by our authors to the invariants I_1, I_2, I_3 of a homography. There

(33) [BURALI-FORTI, BOGGIO 1921]. A subsequent significant use of the theory of homographies in mechanics occurs with the application to the issue of nonlinear elasticity, as mainly highlighted in some work by A. Signorini [SIGNORINI 1943, 1949, 1955, 1960] and [SIGNORINI 1960].

are three kinds, and they are real numbers defined as follows:

$$\begin{cases} \boldsymbol{u}\wedge\boldsymbol{v}\times\boldsymbol{w}\cdot I_1\alpha = \boldsymbol{v}\wedge\boldsymbol{w}\times\alpha\boldsymbol{u}+\boldsymbol{w}\wedge\boldsymbol{u}\times\alpha\boldsymbol{v}+\boldsymbol{u}\wedge\boldsymbol{v}\times\alpha\boldsymbol{w} \\ \boldsymbol{u}\wedge\boldsymbol{v}\times\boldsymbol{w}\cdot I_2\alpha = \alpha\boldsymbol{v}\wedge\alpha\boldsymbol{w}\times\boldsymbol{u}+\alpha\boldsymbol{w}\wedge\alpha\boldsymbol{u}\times\boldsymbol{v}+\alpha\boldsymbol{u}\wedge\alpha\boldsymbol{v}\times\boldsymbol{w} \\ \boldsymbol{u}\wedge\boldsymbol{v}\times\boldsymbol{w}\cdot I_3\alpha = \alpha\boldsymbol{u}\wedge\alpha\boldsymbol{v}\times\alpha\boldsymbol{w} \end{cases}$$

with $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ vectors not parallel to the same plane. These numbers are uniquely determined by α. For example, considering the Cartesian orthogonal system $O, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ we have the following expression:

$$I_1\alpha = \boldsymbol{i}\times\alpha\boldsymbol{i}+\boldsymbol{j}\times\alpha\boldsymbol{j}+\boldsymbol{k}\times\alpha\boldsymbol{k}$$

and so the Cartesian expression of $I_1\alpha$ is given by:

$$I_1\alpha = a_{11}+a_{22}+a_{33}$$

which coincides with the matrix trace of the (*) coefficients.

In the third chapter, our authors introduce the notion of *isomerism* and afterwards, that of *rotor*. A (vector) isomerism is a (vector) homography that preserves the modulus of the vector, i.e.:

$$(\alpha\boldsymbol{u})^2 = \boldsymbol{u}^2$$

The isomeries that have the third invariant equal to $+1$ are the rotors. The rotor is defined so:

$$R(\varphi,\boldsymbol{u}) = \cos\varphi + (1-\cos\varphi)H(\boldsymbol{u},\boldsymbol{u}) + \operatorname{sen}\varphi.\boldsymbol{u}\wedge$$

where $\boldsymbol{u}$ is an unit vector and $H(\boldsymbol{u},\boldsymbol{u})$ is a particular dyad. Therefore, a rotor $R(\varphi,\boldsymbol{u})$ is a homography that expresses a φ radians rotation around $\boldsymbol{u}$, giving any vector φ radians rotation around $\boldsymbol{u}$. At this point, of course, a comparison with Hamiltonian quaternions arises.

Following Burali-Forti and Boggio's introduction to the principles of the theory of homographies, they are then used in a systematic way within the text.

In particular, this concerns the topic of finite motions, and instantaneous and continuous motions, which are developed in "Chap.II" and "Chap.III" respectively. The section relating to the moments of inertia can be found in "Chap.III" in Part Two of the volume. Other uses of the theory of homographies are also included in the last chapter of the text, "Chap.XI", entitled "Miscellaneous Applications".

Concerning the kinematics of finite motions of rigid bodies, the authors directly define a "finite motion" in terms of homographies. This is done in the following way. Given two points A and B and a rotor $\alpha = R(\varphi,\boldsymbol{u})$, the operator λ

$$\lambda = \left(\frac{B}{A},\ \alpha\right) = \left(\frac{B}{A},\ R(\varphi,\boldsymbol{u})\right)$$

associates to the generic point $P = A + (P - A)$ the point

$$\lambda P = B + \alpha(P-A) = B + R(\varphi,\boldsymbol{u})(P-A)$$

On this basis, a "finite motion" is defined by the authors as the operator λ applied to the points of a figure F to obtain a figure F'. Hence, in the following paragraph, based on the properties of homographies, the classification of finite motions is developed, considering the particular cases of translations for $\alpha = 1$, of rotations for $B = A$; and the case of helicoidal motions.

Regarding the part concerning the moments of inertia, Boggio and Burali-Forti's volume also presents a radical approach. After introducing of the basic definition related to the moment of inertia, the discussion, in fact, immediately shifts to how these notions can be formulated in terms of homographies, systematically expanding what was already seen in previous works.

More specifically, the authors define the moment of inertia of a material system $S = (m_i, P_i)$ with respect to a line u as

$$I = \sum m_i r_i^2 = \sum m_i[(P_i - O)\wedge\boldsymbol{u}]^2$$

O being a point, $\boldsymbol{u}$ a vector and u the line through O and parallel to the vector $\boldsymbol{u}$; and indicating the radius of inertia of S with respect to the line u, the number R such that

$$I = mR^2$$

with $m = \sum m_i$. On this basis, the "homography of inertia" of S with respect to the point O is introduced as

$$\eta = -\sum m_i[(P_i - O)\wedge]^2$$

which can also be put in the form

$$\eta = \sum m_i\left[(P_i - O)^2 - H(P_i - O, P_i - O)\right]^2$$

In this way – in analogy with the earlier applications of homographies to classical mechanics – the moment of inertia of S with respect to the axis $O\boldsymbol{u}$, being

$\boldsymbol{u}$ a unit vector, is expressed as

$$(**) \qquad I = \boldsymbol{u} \times \eta \boldsymbol{u}$$

with $\boldsymbol{u}^2 = 1$.

The importance of this homography in Boggio and Burali-Forti's text is also related to the fact that the entire theory of moments of inertia are expressed according to this concept. For example, the following results are present in the text:

1. If G is the center of mass of the system S and O is a generic point, there is a definite relation between the homographies of inertia η_O and η_G relative to the points O and G expressed, in terms of the theory of homographies, by the relation:

$$\eta_O = \eta_G - m[(G-O)\wedge]^2 = \\ = \eta_G + m(G-O)^2 - mH(G-O, G-O)$$

2. the calculation of moments of inertia in various specific cases, through that of the respective homographies of inertia.

As already said, the final part of the volume by Boggio and Burali-Forti includes other applications of the theory of homographies.

An interesting example in this sense is the question of the motion of a free "point-mass" (m, P) in a resistant medium in which the motion presents a resistance, acting opposite to that of the velocity P' of the point P, which is subject simultaneously to a central force of center O and to a force parallel to a given unit vector $\boldsymbol{k}$.

Then, the forces acting on the point-mass (m, P) are $[P, -mh_1P']$, $[P, mh_2(P-O)]$, $[P, mh_3\boldsymbol{k}]$, where $h_1 \geq 0$ is a finite function of t, P, P' and h_2, h_3 are finite functions of t and P.

The differential equation of motion is then:

$$P'' = -h_1P' + h_2(P-O) + h_3\boldsymbol{k}$$

and operating with the axial homography $P'\wedge$, the authors obtain the expression

$$P' \wedge P'' = h_2P' \wedge (P-O) + h_3P' \wedge \boldsymbol{k}$$

which can be used to obtain the expression of P'''

$$P''' = \left(h_1^2 - h_1' + h_2\right)P' - \\ - \left(h_2' - h_1h_2\right)(P-O) + \left(h_3' - h_1h_3\right)\boldsymbol{k}$$

5. – Vector calculus in Levi-Civita's and Amaldi's *Lezioni di Meccanica Razionale*

The publication in 1923 of the first volume of *Lezioni di Meccanica Razionale* (34) by Levi-Civita and Amaldi can be considered the result of a multifaceted process. It was preceded, between 1920 and 1922, by some lithographed editions of these *Lezioni*, with Levi-Civita and Amaldi as co-authors. (35) These lithographed editions are very similar to the definitive edition and shed light on the gestation process of the *Lezioni di Meccanica Razionale*. This work, in effect, was the fruit of a collaboration that arose during 1918, on the basis of different didactic experiences in mechanics. On the one hand, in relation to Levi-Civita's courses in rational mechanics which, as mentioned, began to be taught and published in lithographed form starting from the academic year 1896/1897. On the other hand, during the second decade of the twentieth century, Amaldi too had begun to hold courses in rational mechanics first in Modena and then in Padua at the R. Scuola d'Applicazione per gli Ingegneri, whose lithographed editions also began to be published. These lectures were explicitly written "on the basis of the Course of Rational Mechanics of Prof. T. Levi-Civita" and can therefore be seen as a sort of rereading by Amaldi of Levi-Civita's courses.

Despite this, in Amaldi's lectures, it is possible to notice a different attitude towards vector calculus. Unlike Levi-Civita's courses, he does not dedicate a chapter explicitly to this, but instead, vector calculus takes an 'integrated role', which is introduced progressively throughout the course, gradually highlighting the vector nature of the mechanical notions that he introduces. In this way, for example, the lectures by Amaldi do not present a preliminary auton-

(34) [Levi-Civita, Amaldi 1923]. The *Lezioni di Meccanica Razionale* by Levi-Civita and Amaldi were republished in 2013 in an edition, [Levi-Civita, Amaldi 2013], which includes a volume of Supplements contributed by some distinguished Rational Mechanics experts. In particular, regarding the peculiarity of these *Lezioni* in the Italian and international context of treatises in Rational Mechanics, see [Cirillo, Maschio, Ruggeri, Saccomandi 2013].

(35) We kept in account the article [Dell'Aglio 2013].

omous treatment of vector algebra: in particular, the scalar product and the vector product are considered only when these notions are essential in the treatment, in particular, of the kinematics of rigid bodies.

On the contrary, the various editions of Levi-Civita and Amaldi's lectures tend to present an explicit preliminary formulation of vector calculus, to which the first chapter is devoted, like in the earlier courses by Levi-Civita in rational mechanics. ([36]) In particular, in these editions of Levi-Civita and Amaldi's lectures, it is possible to note a consideration of vector quantities that shows analogies to the approach of the Italian vector school. For example, in these texts, the sum of a point and a vector plays a central role. In the lithographed edition of 1922, we read:

> Still in accordance to the operational definition of the vector, point B is said to be the sum of point A and the vector $\underline{v}$ and it is written:
>
> (1) $$B = A + \underline{v}$$
>
> in which the sign $=$ means that the two symbols B and $A + \underline{v}$ represent the same point. ([37])

On this basis, a vector can be written as a difference of points:

> By considering the common rules of algebraic calculus valid for the equalities of type (1), we can also write
>
> (2) $$B - A = \underline{v}$$
>
> meaning that the two symbols $\underline{v}$ and B–A (to be intended as "B minus A") represent the same vector. This way, the vector appears as a 'difference of points' [...]. This new use of the sign $=$ does not need to be justified since, by definition, in this case it simply denotes a logical identity and hence, without any doubt, the usual equalities rules apply. ([38])

Of course, the consideration of a vector in terms of a difference of points $B - A$ constitutes an element that can be seen as a clear form of adherence to the conceptions of a synthetic nature of the Italian vector school. Indeed, in these editions of Levi-Civita and Amaldi's *Lezioni* there is an explicit introduction to all elementary vector operations, with detailed formulation of their basic properties, in a way similar to the *Elementi di calcolo vettoriale* by Burali-Forti and Marcolongo. It should be noted that the symbols used here for the scalar product and the vector product coincide with those used by the Italian vector school, in line with what happens in the lithographed editions of Levi-Civita's lectures after 1910. ([39])

Moreover, also in these lectures by Levi-Civita and Amaldi, the notion of sum of a point and a vector plays a foundational role because it is on this basis that the sum of vectors is introduced exactly as in the texts of the vectorialists. Given a point O and n vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$, it is possible to obtain the points

$$A_1 = O + \boldsymbol{v}_1, A_2 = A_1 + \boldsymbol{v}_2, \ldots, A_n = A_{n-1} + \boldsymbol{v}_n$$

and this leads to the construction of the polygonal $OA_1A_2 \ldots A_n$, whose sides 'represent' the assigned vectors. Under these assumptions, the sum of vectors is expressed by the difference of points

$$A_n - O = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \ldots + \boldsymbol{v}_n$$

In this context, it is not by chance that there are explicit references to Grassmann's work in the edition of *Lezioni di Meccanica Razionale* by Levi-Civita and Amaldi published in 1923, as the origin of some of the concepts and symbols they introduce. Similarly, many aspects of Levi-Civita's and Amaldi's theory of vectors are illustrated by making explicit use of the consideration of vectors as differences of points. For example, the decomposition of a vector $\boldsymbol{v} = B - A$ is formulated in this context by considering points $A_1, A_2, \ldots, A_{n-1}$ such that:

$$B - A = (A_1 - A) + (A_2 - A_1) + \ldots + (B - A_{n-1})$$

an expression that, in the case of the decomposition of a vector with respect to three non-coplanar directions, leads to the following expression:

$$B - A = (B_1 - A) + (B_2 - A) + (B_3 - A)$$

([36]) Indeed, some of the features in the various editions of Levi-Civita's and Amaldi's *Lezioni* are already present in the lithographed edition of Levi-Civita's lectures on 'vector theory' from 1910.

([37]) [Levi-Civita, Amaldi 1922], p. 7.

([38]) Idem.

([39]) A fact that is fully confirmed by Levi-Civita's interest for the contributions of the vectorialists on the question of the unification of vector symbolisms: cf. [Sallent Del Colombo 2010].

where B_1, B_2, B_3 are three points identified on the straight lines r_1, r_2, r_3 passing through the point A and having those prefixed directions.

It is important to emphasize that the process of algebraization which characterizes the 'theory of vectors' in the first volume of Levi-Civita's and Amaldi's *Lezioni di Meccanica Razionale,* tends, in part, to distance it from the use of vector calculus in other texts in rational mechanics, as in the case of Appell's *Traité*. (40)

It should be noted, on the other hand, that the various editions of Levi-Civita's and Amaldi's lectures similarly represent the topics contained in Levi-Civita's previous 'theories of vectors': in particular, giving space, as usual, to the consideration of the theory of moments and of couples, always in the form in which these subjects appear in Appell's *Traité*. The key point is that all the arguments in the second part of this 'theory of vectors' are now reformulated by using the new notations adopted in the Italian vector school. For example, the way to vary the resultant moment of a system of vectors of resultant $\boldsymbol{R}$ when the center of reduction P varies, is now expressed as:

$$\boldsymbol{M}' = \boldsymbol{M} + (P - P') \wedge \boldsymbol{R}$$

P' being the new center of reduction. Similarly, the proof of the constancy of the invariant trinomial is now obtained in a synthetic form by applying in the previous expression the scalar product by $\boldsymbol{R}$:

$$\boldsymbol{M}' \times \boldsymbol{R} = \boldsymbol{M} \times \boldsymbol{R} + [(P - P') \wedge \boldsymbol{R}] \times \boldsymbol{R}$$

and obtaining the result for the definition of the vector product.
In this context, the absence in the part concerning 'theory of vectors' of explicit references to other techniques should be noted – and in particular, of the theory of homographies, which, as mentioned above, play a significant role in the texts of the vectorialists in rational mechanics.

One mention of the theory of homographies in the text by Levi-Civita and Amaldi can be found in the second volume, first published in 1926, within "Chap. 4", entitled "Caratteristiche dinamiche e cinematiche dei sistemi". In particular, with respect to vectors $\boldsymbol{\omega}$ and $\boldsymbol{K}$, which express the angular velocity and the angular momentum of a system of material points, the authors affirmed:

> The correspondence of the two vectors $\boldsymbol{\omega}$ and $\boldsymbol{K}$, which we have already analyzed from a geometric point of view [...], is a first example of those biunivocal correspondence among (variable) vectors that, considering a set of three axis as reference, are defined by expressing the components of one of the two vectors as a homogeneous linear function of the components of the other one. These are the so-called vectorial homographies, following the name coined for them by Burali-Forti, who with Marcolongo developed the theory thereof. We will not spend any more time on this theory [...]. (41)

In addition, the correspondence between $\boldsymbol{\omega}$ and $\boldsymbol{K}$ is referred to as the "(vector) homography of inertia of the solid respect to its point O". On this basis, the concept "homography (vector) of inertia" occurs in several subsequent points of the second volume of Levi-Civita's and Amaldi's treatise, such as in "Chap.7" in relation to "Dynamic and structural conditions of plane motions" and in "Chap. 8".

It can therefore be said that within the *Lezioni di Meccanica Razionale* by Levi-Civita and Amaldi, the theory of homographies finds very limited space. In effect, this theory is totally absent as a general theoretical tool in the initial part concerning the 'theory of vectors' and it is present, with the aim of expressing notions and laws of a mechanical nature, only in relation to the notion of "homography of inertia", which feature of the second volume of the *Lezioni di Meccanica Razionale*. This attitude of reluctance toward a systematic use of the theory of homographies, can of course, be related to the discussion between Levi-Civita and some exponents of the Italian vectorial school in the same period; even

(40) Cfr. [APPELL 1909], pp. 4-5. In fact, the other editions of Appell's *Traité* during the first two decades of the twentieth century show changes from the point of view of vector calculus only in the third edition of 1909, but in a completely different direction, because it relates to the consideration of a strictly geometric classification of vectors. A more modern exposition of the principles of vector calculus in Appell's work occurs with the publication of the volume [APPELL 1921].

(41) [LEVI-CIVITA, AMALDI 1926], p. 294.

in respect of the expressive role of this theory even in relation to tensor calculus. ([42])

6. – Some conclusions

We have seen in our paper how, in the context of Italian treatises between the end of the nineteenth century and the early twentieth century, the exposition of the topics of classical mechanics, from a purely didactic point of view, may depend substantially on the type of approach chosen for vector calculus and the way it is presented.

The essential role of the minimum system is evident from our previous analyses and considerations. From this, one can develop other concepts (homographies included) and other operators. Note that the use of co-ordinates is an interpretation of the vector calculus. This is useful from a didactic point of view and for the applications.

On the other hand, we have seen how the systematic use of the theory of homographies, such as that developed by Burali-Forti and Boggio in their treatise, represents a choice that does not find unanimous consensus.

To explain how he arrived at the minimum system theoretically, Burali-Forti wrote a well-articulated article on the *Enciclopedia delle Matematiche Elementari* (vol. II, part II published in 1938). Later, in 1947, again in the same *Enciclopedia* (vol. III, part I) Giovanni Giorgi (1871-1950), a physicist and mathematician emblematic as critic on the techniques proposed by the vectorialists, says something noteworthy. He mentioned, resizing the importance of the contribution to vector calculus given by Burali-Forti and Marcolongo, that the minimum system constitutes only "an adaptation of Gibbs' system, with some modifications of form and nomenclature, with a more marked independence from the Hamiltonian system, and from any complete system, and with notable extensions on linear operations on vectors treatment". While one of the relevant points asserted by the supporters of vector calculus was that the use of co-ordinates presents complexity and prolixity of calculation, Giorgi instead argues Burali-Forti and Marcolongo's methods are "too compact and difficult to read". However, Giorgi does give merit to the two Italian vectorialists due to the depth of their study into the theory of vector homographies. Nonetheless, it was precisely this treatment that was the catalyst to the criticisms to the vectorialist approach. The alternative approach proposed by Giorgi, was the one of compact matrix (linear systems theory). ([43]) Despite Giorgi's views, the vectorialists also took Cartesian aspects into account, as seen in section 4.

([42]) On the question of the use of the theory of homographies, also in connection with general relativity, cf. [Bernardini 2005], [Guerraggio, Nastasi 2005], [Nastasi, Tazzioli 2004].

([43]) But the approach to the classical mechanics according to the methods of the Geometric Algebra is always topical (see for instance [Hestenes 2002]).

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