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RECENSIONE

Review of Eugenio Calabi's Collected Works

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In 2021, Springer has published a beautiful volume collecting all of Eugenio Calabi's mathematical papers. This is very opportune in view of his 100th birthday, May 11th 2023. This book, very substantial in depth and length (as it sums to 843 pages), represents an invaluable tool in the hands of all geometers and geometric analysts of any age, to understand the roots of many still very active nowadays on which Calabi has given foundational contributions.

The volume, edited by J.-P. Bourguignon, X.X. Chen and S. Donaldson, contains also an extremely precious section (of more than a hundred pages) with detailed commentaries on various "chapters" of Calabi's research by some of the most distinguished scholars in their fields, such as H.B. Lawson, M. Berger, J.-P. Bourguignon, C. LeBrun, X.X. Chen and S. Donaldson.

Calabi's research has shaped a vast territory in between Algebraic, Differential and Complex Geometry, with crucial excursions in into the PDE aspects of these subjects. As I will only comment on few topics (the ones more familiar to the reviewer), he has set the ground for entire subjects, many of them still full of open problems and active research, and in some notable cases its depth and importance has been fully recognized only after some time (notable examples are his work on extremal Kähler metrics and on viscosity solutions to PDEs). Yet, in few other cases, Calabi's research has been immediately recognized recognised as central by the global mathematical community, as for example his work on the existence of Ricci flat metrics and on the rigidity of arithmetic groups (with E. Vesentini), which are the grounds for two research topics that led to two Fields Medals being awarded to S.T. Yau in 1982 and G. Margulis in 1978.

Before entering into some of mathematical aspects of Calabi's work, some comments are due in describing his relationship with the Italian academic/mathematical community. Calabi was born in Milan and aged fifteen when his family had to escape to France for one year and then to the United States due to the infamous racist laws promulgated during one of the darkest moments of our history. While Eugenio was immediately admitted to M.I.T. to study chemical engineering, and then built his mathematical career overseas, at Louisiana State, Caltech, the Institute for Advanced Study, Minnesota and Pennsylvania, most of his family reestablished in Italy in 1946. Particularly well known to the Italian public has been the civic and political activities of one of Eugenio's older sisters, Tullia Zevi Calabi.

Eugenio has been keeping a strong relationship with the Italian mathematical community ever since, often visiting Italian universities and collaborating with many Italian mathematicians of his generation. He had long and strong links to the Scuola Normale and another notable occasion has been his visit to a PDE meeting organised by Fichera and Courant in 1954 in Trieste where, as he and Louis Nirenberg recalled later, various discussions on the famous *a priori* estimates needed to prove Calabi's Conjectures took place (they were fully established only in 1976 by Yau). A famous meeting was organised in

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Pisa by P. De Bartolomeis, F. Tricerri and E. Vesentini to honour Eugenio's seventieth birthday and its proceedings have been published in the prestigious series "Symposia Mathematica" by Cambridge University Press.

At a personal level, I also recall a beautiful meeting organised in Mondello (Sicily) in 2003 to celebrate Eugenio's eightieth birthday where, in between spectacular talks by prominent mathematicians of various fields, many of us much younger had the opportunity to get to know him and to hear directly from him the origin of some of his most profound ideas.

There is no doubt Calabi's name will be linked forever to the existence of Kähler Ricci-flat metrics, a central problem both in Mathematics, in our attempts to describe Nature as a whole via String Theory, and likely to be the only mathematical problem with a Broadway Show dedicated to...

Let me recall that a Kähler manifold M is a complex manifold (we will denote by J the associated endomorphism of the tangent bundle with $J^2 = -Id$) with a Riemannian metric g such that J is a parallel isometry (or equivalently the associated 2-form $\omega(X,Y) := g(JX,Y)$ is closed, hence defining a real-valued cohomology class $[\omega]$). Two foundational results are needed to set up any meaningful question about these spaces: the first one is a classics of Hodge Theory known as the $\partial\bar{\partial}-$Lemma:

PROPOSITION 1. – *On a compact Kähler manifold* (M, ω_0) *any smooth Kähler form* ω *representing* $[\omega_0]$ *is given by* $\omega_\phi = \omega_0 + i\partial\bar{\partial}\phi$, *where* $\phi \in C^\infty(M)$ *is uniquely determined up to a constant.*

This result tells us that the space of Kähler metrics representing a fixed cohomology class can be parametrised by the open cone H_ω of $C^\infty(M)/\mathbb{R}$ of Kähler potentials ϕ for which $\omega_\phi(\cdot, J\cdot) > 0$.

The second critical observation, due to Kähler himself, is a surprisingly simple global relationship between the volume form and the Ricci curvature (which is more elegantly expressed in terms of the *Ricci* form $\rho_\omega(X,Y) := Ric_g(JX,Y)$, bearing in mind that $\frac{1}{n!}\,\omega^n$ is nothing but the Riemannian volume form):

PROPOSITION 2. – *On a Kähler manifold of complex dimension* n, *given two Kähler metrics* ω_0 *and* ω_1

$$\rho_{\omega_1} = \rho_{\omega_0} - i\partial\bar{\partial}\left(\log\frac{\omega_1^n}{\omega_0^n}\right).$$

These two facts clearly suggest that any question about prescribing the volume form and the Ricci curvature (and hence in turn the scalar curvature) of a manifold boils down to solving a non-linear scalar PDE on H_ω of Monge-Ampère type.

The series of short papers and announcements written by Calabi between 1954 and 1957 are all reproduced in the volume (finally!) and account for Eugenio's first contributions to these problems where the modern reader can almost physically feel the intellectual efforts and the human hopes he had put on this program. Eventually, he had to leave the critical existence problem about solving the corresponding complex Monge-Ampère equations open (hence passed into history as "Calabi's Conjectures", waiting for the solution by Yau in 1976). Yet, what he left on the ground has shaped research in Complex Differential Geometry (and will certainly continue so for years to come) and have become a general paradigma for many problems in Geometric Analysis:

- Definition on H_ω of a "natural" structure of infinite dimensional Riemannian manifold;
- Proof of the uniqueness of solutions to the relevant Monge-Ampère equations by a careful use of the maximum principle;
- Clarification on how the prescribed volume Conjecture implies the Ricci-flat Conjecture;
- Use of the continuity method to solve the corresponding equations; proof of non-emptyness and openness;
- Definition of the functional

$$C(\phi) = \int_M Scal^2_{\omega_\phi} dVol_{\omega_\phi}\ ;$$

- Geometric interpretation of its critical points and their relationship with the space of holomorphic vector fields.

All this, and in fact even more, in the pages $181 - 198$ of the present volume (and reading them in order, the footnote on page 197 of the present volume of the 1957 "Lefschetz" paper acquires an even more special taste...), is the set up for a variety of lines of research which have become central themes in Geometry and Geometric Analysis and to which Calabi himself has devoted a large part of his subsequent study:

- Study the (Riemannian) geometry of the space H_ω, in terms of curvature, geodesics and its metric completions;
- Study real and complex Monge-Ampère equations arising from Geometry (see e.g. Calabi's work on Affine Real Differential Geometry);
- Study critical points of quadratic functionals such as C from a Morse-theoretic point of view. This theme was expanded by Calabi himself at the beginning of the eighties in two famous papers on "Extremal metrics", whose existence has been and is one of the main problems in the subject.

Having in a single volume all of Calabi's work, enriched by detailed comments by distinguished experts, gives also the possibility to realise how the efforts Calabi made to prove his Conjectures have indeed produced outstanding ideas and results also in a priori unrelated (or not obviously related) areas. Of particular interest are his attempts to find the missing "a priori estimates" in similar but hopefully simpler situations. In this sense, witnessing the birth of the idea of a "viscosity solution" to a non-linear PDE in Calabi's work on the real Monge-Ampère equation with applications to Bernstein's Theorems in Affine Geometry is of tremendous interest.

Calabi-type Conjectures have been formulated in various different situations and are still objects of study and important research (for example by Tosatti-Weinkove and others in the Hermitian case, or by Verbitsky and others in the hypercomplex setting).

All these themes and results originating from Calabi's most famous work are not the first ones he tackled. In fact the first problem Calabi studied in Complex Geometry is the following one: are *all* Kähler metrics induced as pullbacks via holomorphic maps of the constant curvature metrics on complex space forms?

This very natural problem is clearly very reminiscent of the similar general problem for Riemannian manifolds (not necessarily complex and Kähler) which was under study at Princeton in the same years and was to become the celebrated Nash Isometric Embedding Theorem (as the answer is well known to be indeed positive).

It is really quite remarkable that Calabi's work (which appeared in fact even earlier than Nash's) proves that, in the complex case, there are indeed local obstructions (while in the real case there are none, but the main subtleties come from globalising the local embeddings)! Calabi's 1953 paper (his PhD thesis) is a real gem. It is still today the best account on the problem and it is full of ideas that quite surprisingly took a long time to be recognised by the experts (as Marcel Berger recalls in his 1996 paper included in the present volume "*this wonderful text remains quite unknown and almost unused*") with very few exceptions (most notably by S.S. Chern, B. Smyth and S. Kobayashi). It was only much later, with the work of M. Umehara and D. Hulin, that Calabi's work regained its proper position. Possibly the critical idea at the basis of his paper is the one of *Diastasis*, which can be loosely translated as "taking apart" or "creating distances". Recall that a potential for the Kähler form ω is a real-valued function Φ defined on a open set $U \subset M$ satisfying $\omega = \bar{\partial}\partial\Phi$. A potential is not unique: it is defined up to the sum with the real part of a holomorphic function. Therefore, when ω is real analytic, a potential Φ can be complex analytically continued to an open neighbourhood $V \subset U \times \bar{U}$ of the diagonal. Denote this extension by $\Phi(x, \bar{y})$. It is holomorphic in x and antiholomorphic in y and, with this notation, $\Phi(x) = \Phi(x, \bar{x})$.

The *diastasis function* is the Kähler potential D_p around p defined by

$$D_p(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Among all the potentials the diastasis is characterised by the fact that, in every coordinate system (Z) centered at p,

$$D_p(Z, \bar{Z}) = \sum_{|j|,|k| \geq 0} a_{jk} Z^j \bar{Z}^k,$$

with $a_{j0} = a_{0j} = 0$ for all multi-indices j and D_p satisfies the following key property:

PROPOSITION 3. – *Let* $\varphi : (M, g) \to (N, G)$ *be a holomorphic and isometric embedding between*

Kähler manifolds and suppose that G is real analytic. Then g is real analytic and for every point $p \in M$

$$\varphi(D_p) = D_{\varphi(p)},$$

where D_p *(resp.* $D_{\varphi(p)}$*) is the distasis of g relative to p (resp. of G relative to* φ(p)).

Calabi has found some quite simple and elegant local obstructions to the local holomorphic isometric problem expressed in terms of D_p, and in fact in terms of the coefficients a_{jk} of its expansion above. Yet, the problem itself is far from being understood (a recent account can be found in the Springer book by A. Loi and M. Zedda). In particular let me mention a very natural and frustrating open problem which connects various themes mentioned in this review:

Problem 1. – *If* $\varphi : (M, g) \to (\mathbb{C}P^N, \omega_{FS})$ *is a holomorphic and isometric embedding of an extremal metric (or Kähler-Einstein, or constant scalar curvature...), then* (M, g) *is a homogeneous space.*

Besides the naturality of these problems other very deep motivations for these questions come by mixing Calabi's work with some much more recent studies on the Bergman kernel. To give an idea to the reader of the depth and importance of these developments let us take one step back.

Let g be a Kähler metric on a compact complex manifold M such that the Kähler form ω associated to g is integral, i.e. there exists a holomorphic line bundle L over M, whose first Chern class equals the second de Rham cohomology class of ω, i.e. $c_1(L) = [\omega]_{dR}$. By Kodaira's theory, this is equivalent to saying that M is a projective algebraic manifold and L a positive (or ample) line bundle over M. In algebraic-geometric terms L is said to be a *polarisation* of M, g a *polarised* metric and the pair (M, L) a *polarised manifold*. Fix a polarisation L over M. In this case there exists a Hermitian metric h on L, defined up to the multiplication with a positive constant, such that its Ricci curvature $Ric(h)$ equals ω ($Ric(h)$ is the two-form on M whose local expression is given by $Ric(h) = -\frac{i}{2}\partial\bar{\partial}\log h(\sigma(x), \sigma(x))$, for a trivialising holomorphic section $\sigma : U \to L \setminus \{0\}$). Let $s_0, \dots, s_N$ be an orthonormal basis of $H^0(L)$ (the space of global holomorphic sections of L) with respect to the scalar product

$$\langle s, t\rangle = \int_M h(s(x), t(x))\frac{\omega^n(x)}{n!},\ \ s, t \in H^0(L).$$

Consider the non-negative smooth function B_g on M (called the *Bergman Kernel*) given by:

$$B_g(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x)).$$

As suggested by the notation, this function depends only on the Kähler metric and not on the chosen orthonormal basis chosen.

The metrics on M obtained by pull-backs of the Fubini-Study form ω_{FS} via the Kodaira map (evaluation of holomorphic sections) corresponding to an L^2-orthonormal basis, were studied by G. Tian and called *Bergman metrics*. Among them those (if any) for which B_g is constant were called *critical* by W. Zhang and *balanced* by S. Donaldson (not to be confused with another very important class of metrics on hermitian manifolds called balanced too by Paul Gauduchon more than 20 years before these ones, which were actually first studied by Bourguignon-Li-Yau to estimate eigenvalues of the Laplace operator on Riemann surfaces).

A beautiful picture arises as one considers the behaviour of Bergman and balanced metrics as a natural parameter $m > 0$ is introduced and one considers the previous notions for the whole sequence of line bundles L^m, equipped with a Hermitian metric h_m such that $Ric(h_m)$ equals $m\,\omega$ and $s_0, \dots, s_{d_m}$, $d_m + 1 = \dim H^0(L^m)$ an orthonormal basis of $H^0(L^m)$ with respect to the L^2-scalar product $\langle s, t\rangle_m$ as above.

Take m suficiently large such that, for each point $x \in M$, there exists $s \in H^0(L^m)$ non-vanishing at x and consider the holomorphic map of M into the complex projective space $\mathbb{C}P^{d_m}$ given by:

$$\varphi_m : x \mapsto [s_0(x) : \dots : s_{d_m}(x)].$$

One can prove that

$$\varphi_m^*\omega_{FS} = m\,\omega + \frac{i}{2}\partial\bar{\partial}\log B_{mg},$$

giving a simple and crucial bridge between Calabi's holomorphic isometric embedding problem and the classical theory of Bergman kernels.

In this respect, the following results form the ground of our present understanding:

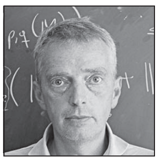
- (Tian-Ruan) any polarised metric on compact complex manifold is the C^∞-limit of (normalised) Bergman metrics;
- (Donaldson) Let (L, h) be a polarisation of a compact Kähler manifold (M, ω) of constant scalar curvature. If $Aut(M, L)/\mathbb{C}^*$, the group of biholomorphisms of M which lift to holomorphic bundles maps $L \to L$ modulo the trivial automorphism group $\mathbb{C}^*$, is trivial, then, for all sufficiently large integers m, there exists a unique balanced metric $\tilde{g}_m$ on M, with polarisation L^m, such that its associated Kähler form $\tilde{\omega}_m$ is cohomologous to $m\,\omega$ and $\frac{1}{m}\,\tilde{\omega}_m$ C^∞ converges to ω. Moreover if $\tilde{g}_m$ is a sequence of balanced metrics on M with $\tilde{\omega}_m \in c_1(L^m)$ such that $\frac{1}{m}\,\tilde{g}_m$ C^∞ converges to a metric g, then g has constant scalar curvature.

Despite many years of work by many authors, many foundational questions about the mutual relationship between these notions and geometric characterisations of Kähler manifolds with special properties of their Bergman kernels in the same spirit as for the isometric problem remain open and no doubt Calabi's original approach via the diastasis function can still provide very useful insight.

Calabi's contributions to Geometry and Geometric Analysis span various other topics and became of fundamental importance for later developments, and as for themes above, it is very interesting to have all his (commented) work in a single volume to see unifying principles appearing under different shapes in different problems. For example:

- Calabi's idea to build a sort of "holomorphic Frenet frame" to an immersed Riemann surface and to connect it to the general theory of minimal surfaces in spheres;
- his use of "symmetries" (in a very generalised sense) to reduce general PDEs to more tractable ODEs, so to produce explicit examples of special metrics in various contexts, an idea he refined to the point of becoming known as Calabi's Ansatz or Moment Map construction;
- the study of Hamiltonians on Riemann surfaces and the definition of two central notions in symplectic topology now called *Flux homomorphism* and *Calabi Invariant*.

In summary, I consider the volume under review an indispensable addition to the library of any student and researcher in Geometry (of any type) and Geometric Analysis. Thanks to having for the first time all Calabi's works together (many of them are very hard to find nowadays) and having beautiful introductions and explanations by top experts, any reader can learn not just the origin of much of today's research, but also find many less known ideas, attempts and problems that are still waiting to be fully understood.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Nato e laureato a Genova, Claudio Arezzo ha conseguito il Dottorato a Warwick e Torino. Dopo aver tenuto posizioni a Stanford e al MIT, è stato Professore Associato (2000-2004) e Professore Ordinario (dal 2004) di Geometria presso l'Università di Parma. Dal 2010 è membro permanente dell'Abdus Salam International Center for Theoretical Physics di Trieste, in cui, dal 2018, dirige la Sezione di Matematica. Autore di circa 50 pubblicazioni scientifiche e co-organizzatore di più di 30 scuole e convegni in tutto il mondo, la sua ricerca verte su vari temi di Geometria Differenziale reale e complessa, Analisi Geometrica e Fisica Matematica.