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A Central Result from Newton's *Principia Mathematica*: The Body of Least Resistance

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Sommario: *Su invito di Hugo Beirâo da Veiga ho tenuto il 28 febbraio 2018 due conferenze al Centro de Matemática Computacional e Estocástica, Instituto Superior Técnico, Universidade de Lisboa. Rivolgo a lui i miei ringraziamenti per avermi offerto l'occasione di presentare queste preziose scoperte di Newton. Gli argomenti delle conferenze riguardavano "Risultati fondamentali nei Principia Mathematica di Newton", in particolare "Il corpo di minima resistenza", dove Newton affronta un problema variazionale in un modo davvero sorprendente, e "La forza di attrazione di un corpo sferico", dove egli impiega gli strumenti della geometria greca per calcolare un integrale singolare. Il presente articolo offre una versione estesa della prima conferenza.*

Abstract: *Following an invitation of Hugo Beirâo da Veiga I gave on February, 28, 2018, two lectures at the Centro de Matemática Computacional e Estocástica, Instituto Superior Técnico, Universidade de Lisboa, and I thank him very much for the possibility of presenting theses gems of Newton's inventions. The topics were "Central Results from Newton's Principia Mathematica", namely "The Body of Least Resistance", where Newton treats a variational problem in a truly astonishing way, and "The Force of Attraction of a Spherical Body", where he uses Greek geometry in order to evaluate a singular integral. This paper contains the first lecture.*

1. – Introduction

Newton's Treatise *"Philosophiae Naturalis Principia Mathematica"* is the fundamental work of modern science as it developed in the 17^{th} century. In this work celestial mechanics and other areas of physics are analyzed in a way that became characteristic for the exact sciences. As indicated in the title, mathematics provides the basic tools for the scientific investigation. At about the same time infinitesimal calculus was invented and developed, and this branch of mathematics became the distinctive way to formulate propositions in physics. As it was invented by Newton and Leibniz it is by no means obvious that in the Principia Mathematica Newton proves theorems of celestial mechanics by using methods of Euclidean geometry. From the assumption that all points of a material body attract each other with a force that is inversely proportional to the square of the distances between these points, Newton concludes that the attraction of a homogeneous sphere towards a material point outside of it is inversely proportional to the square of the distance between the point and the center of the sphere. Nowadays the proof of this statement would be given by evaluating a singular integral, and therefore it may be surprising – in any case it is most admirable – that Newton does so using ancient

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geometry. ([1]) And he proceeds in the same way when he investigates the motion of a heavenly body along a parabolic path; the force that acts on the comet is determined by properties of conic sections as they were known in classical geometry.

A comprehensive analysis of Newton's methods is given by N. Guicciardini in his fundamental work [2] "Reading the Principia".

There are however theorems in the Principia that Newton proved using methods of infinitesimal calculus. Of special interest is a minimum problem which in today's terminology belongs to the calculus of variations: a rotationally symmetric body of prescribed base and height is to be determined such that its resistance in a uniform fluid flow becomes minimal. ([2])

In 1690 and in 1696 the problems of the catenary and of the brachistochrone were posed: What is the shape of a chain of uniform density under the influence of gravity? And what is the curve that connects two points A and B in a vertical plane such that a material point moving along it from A to B does so in shortest time? These questions were published in the Acta Eruditorum as a challenge to all mathematicians, and in both cases several solutions were submitted. Consequently these problems are regarded to be the beginning of a new field, the calculus of variations. As Newton published the problem of the body of least resistance several years earlier, we will look at the following questions:

(i) How was Newton's contribution received by his contemporaries, in particular when they investigated the catenary and the brachistochrone?

(ii) How did Newton react to the challenges, in particular to the one from 1696, where the deadline for submitting a solution was extended in order to give mathematicians abroad, who would receive the Acta Eruditorum later, enough time to take part in the competition?

We will deal with these points in §6. We start the presentation of Newton's results in §2 by formulating the problem, in particular the physical assumptions. As the body is assumed to be rotationally symmetric it can be described by a real function $u = u(r)$ of one real variable. The corresponding resistance is then an integral whose integrand depends on r and the derivative of u only: $\mathcal{R}(u) = \int f(r, u'(r))\, dr$. After defining $\mathcal{R}(u)$ Newton considers two examples, a sphere and a circular cylinder; in this case the flow hits the planar disc at its top. These simple examples demonstrate distinctly how Newton argued: rather than calculating two integrals and finding out that the value for the sphere is half that for the cylinder he compares the forces the fluid exerts on related points of the sphere and the cylinder; the forces are constant on the cylinder's top disc, and they form a paraboloid in case of the sphere. Finally a theorem of Archimedes gives that the volume of the paraboloid is half the one of the cylinder which circumscribes it.

In §3 the body of minimal resistance is determined in the class of truncated cones. Such a cone is characterized by a real parameter e and its resistance turns out to be an algebraic function $\mathcal{R} = \mathcal{R}(e)$. Because of this Newton obtains the minimizer by calculating the value of e for which the derivative $\mathcal{R}'(e)$ vanishes.

§4 contains the basic result: the proportion that characterizes the minimizer. In modern terminology it is called a first integral of the Euler equation. The Principia do not contain a derivation of this statement; from Newton's notes to this proposition we take a perturbation (variation) of the minimizing function that can be used in an elementary proof using modern notation.

Any admissible body $\mathcal{B}$ of prescribed base is bounded on the lower end by a disc B_R due to the assumption of symmetry. Newton found the surprising fact that the minimizing configuration is bounded

([1]) The instruction at St. John's College, Anaheim, MD, is based on the original literature rather than modern textbooks. Hence celestial mechanics is studied by an intensive reading of parts of Newton's Principia Mathematica. This is made possible by providing appropriate aids, which in the present case are collected in the book [1] by Dana Densmore. The expanded proofs in this book not only contain a comprehensive description of the arguments; in many cases it is also listed where the statements that are used occur in Euclid's elements.

([2]) This problem is treated in Section VII: "The motion of fluids and the resistance made to projected bodies" of Book II: "The motion of bodies" in Vol. I of the Principia, cf. in particular Proposition XXXIV and the following Scholium, see [9], pp. 331-334, and [8], pp. 471-475.

by a similar disc at the upper end, too. Its radius is determined by the fact that the lateral surface meets this disc at an angle of 135°. Also for this result we give a proof in modern notation in §5, and we point out that this property of the solution can be regarded as a transversality or a free-boundary condition, depending on the type of representation we choose; such properties were studied systematically in the calculus of variations centuries later.

2. – The resistance of a rigid body in a uniform flow: Newton's assumptions. Examples

We first derive a formula for the resistance that a rotationally symmetric body experiences in a flow which is parallel to its axis of rotation. We write Newton's assumptions and arguments first in modern notation in order to make them easier accessible. (3) If a particle hits the body in some Point P with force $\mathbf{e} = PQ$, then only the normal component $e_n = \mathbf{e} \cdot \mathbf{n} = PR_1$, where $\mathbf{n} = PR$ is the unit normal to the lateral boundary at P, contributes to the resistance. We may assume $||\mathbf{e}|| = 1$, then $e_n = \cos\alpha$, where α is the angle between the normal and the direction of the flow. The vector $\mathbf{e}_\mathrm{n} := \mathrm{e}_\mathrm{n}\mathbf{n}$ has one component, R_1Q_1, that is orthogonal to $\mathbf{e}$; in the end it does not contribute to the resistance because due to the symmetry of the body there is a point P', opposite to P, where the corresponding component is of the same size but has opposite direction. Hence only the projection of $\mathbf{e}_n$ in direction of $\mathbf{e}$ must be taken into account, and that is $\cos^2\alpha \cdot \mathbf{e}$. In order to determine the resistance of some part of the lateral boundary, e.g. the graph of $u = u(r)$ for r in some interval (r_1, r_2), Newton assumes the medium to consist of equal particles that are at equal distances from each other; furthermore these particles do not interact with each other. If ds is the line element of the curve $\mathcal{C}$, which generates the lateral boundary, then the number of particles that hit ds equals $\cos\alpha\, ds$. As the curve is

(3) As in the original figures the axis of rotation points in horizontal direction, and we consider graphs of functions $u = u(r)$ that are defined on an interval of the vertical axis.

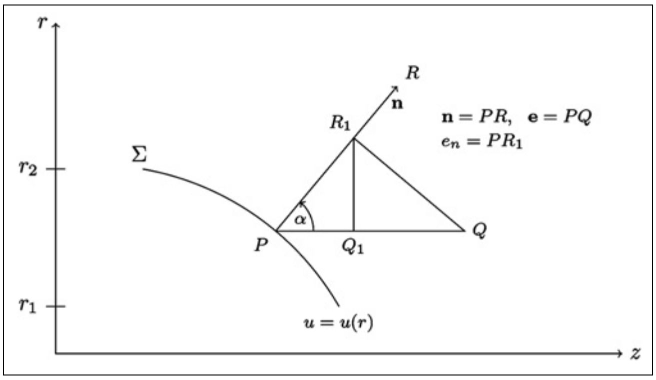


FIGURA 2.1 – Resistance due to the flow.

the graph of some function $u = u(r)$, we eventually get for the resistance

$$\mathcal{R} = \mathcal{R}(u) = \int_0^{2\pi} \int_{r_1}^{r_2} \cos^2\alpha \cdot \cos\alpha \sqrt{1 + |u'(r)|^2}\, r\, dr\, d\alpha.$$

With $\cos\alpha = \dfrac{1}{\sqrt{1 + |u'(r)|^2}}$ this gives

$$\mathcal{R}(u) = \int_{r_1}^{r_2} \frac{r}{1 + |u'(r)|^2}\, dr. \tag{2.1}$$

We will specify r_1 and r_2 when we look at the examples. The assumptions about the medium and its interaction with the rigid body are stated in Proposition XXXIV; then the resistance of a sphere and a circular cylinder are calculated. "In a rare medium consisting of equal particles freely disposed at equal distances from each other, a globe and a cylinder described on equal diameters move with equal velocities in the direction of the axis of the cylinder, the resistance of the globe will be half as great as that of the cylinder." (4)

We obtain this result from (2.1) immediately. In the set-up used there the sphere is generated by $u_1(r) = \sqrt{R^2 - r^2}, 0 \le r \le R, R = h$, and this gives

$$\mathcal{R}(u_1) = \frac{\pi}{2} R^2.$$

For the cylinder of height R the resistance equals the area of its top which is a disc of radius R, and hence we get

$$\mathcal{R} = \pi R^2.$$

(4) [9], p. 331.

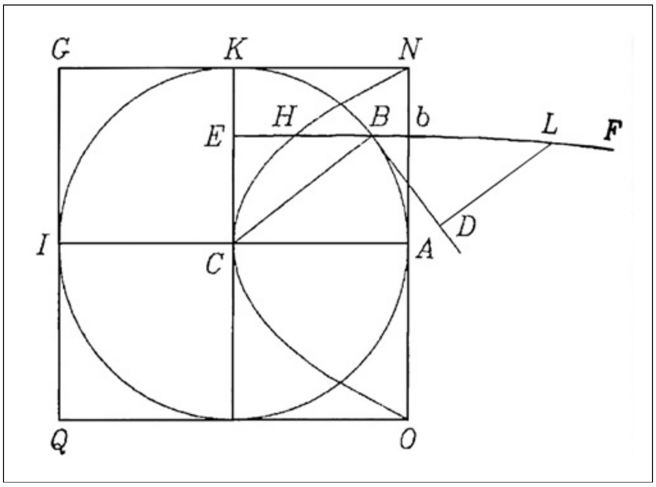


FIGURA 2.2 – Sphere and Circular Cylinder.

Now we present Newton's proof. The figure shows the cross-section GNOQ of a circular cylinder and the one of a sphere, IKA. We may assume that a particle that hits the sphere in B and the cylinder in b does so with force BL. Let $F(B)$ and $F(b)$ denote the forces that act perpendicularly on the sphere in B and on the circular cylinder in b, resp. Then

$$F(B) : F(b) = DL : BL,$$

because the tangential component of BL does not contribute to the resistance. Since we are interested in the ratio of the forces we may assume that $|BC| = |BL|$, and therefore we get

$$F(B) : F(b) = BE : BC.$$

Next, the force that moves the globe in direction of FB relates to the force that moves it in direction of BC as BE to BC. Hence, joining the two ratios we get that the force which moves the globe in direction of AC is to the force on the cylinder like BE^2 to BC^2. Now we sum up these quantities. We define the point H on the line EF by

$$bH = \frac{BE^2}{BC} = BC \cdot \frac{BE^2}{BC^2} = AC \cdot cos\,\alpha,$$

with α as in Fig. 2.1. If we define H in this way for every b between O and N we get a parabola. To see this, let $x = EC$ and $y = EC$ be the coordinates of H. Then by definition of x we have $bH = AC - x$; with $BE^2 = BC^2 - CE^2 = AC^2 - y^2$ we obtain

$$AC - x = bH = \frac{BE^2}{BC} = \frac{1}{BC} \cdot (AC^2 - y^2),$$

hence,

$$AC^2 - AC \cdot x = AC^2 - y^2,$$

and this means

$$y^2 = AC \cdot x,$$

and AC is the latus rectum of the parabola.

If we now rotate the parabola $OCNH$ that is generated by the segments bH about the axis CA, we get a paraboloid; taking bE instead of bH and proceeding in the same way we get the circular cylinder that circumscribes the paraboloid. According to a theorem of Archimedes the volume of the cylinder is twice that of the paraboloid.

3. – The truncated cone of minimal resistance

In the Scholium to Proposition XXXIV Newton describes another class of problems in which the one of least resistance is to be determined. "As if upon the circular base $CEBH$ from the center O, with radius OC, and the altitude OD, one would construct a frustum $CBGF$ of a cone, which should meet with less resistance than any other frustum constructed with the same base and altitude, and

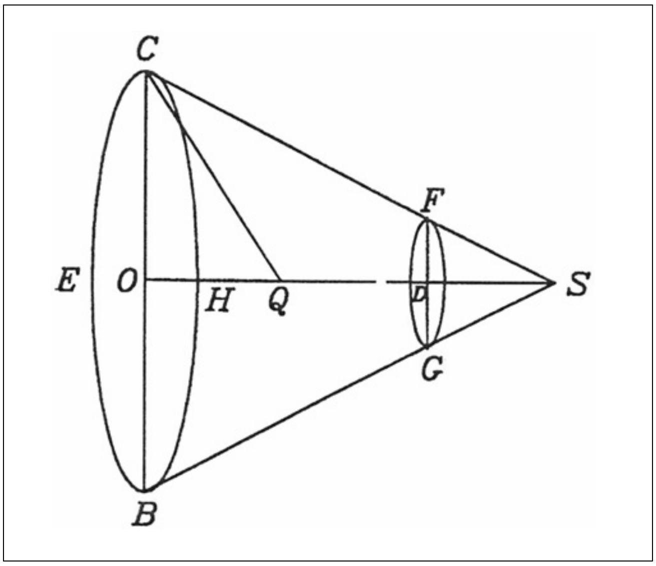


FIGURA 3.1 – Truncated Cone.

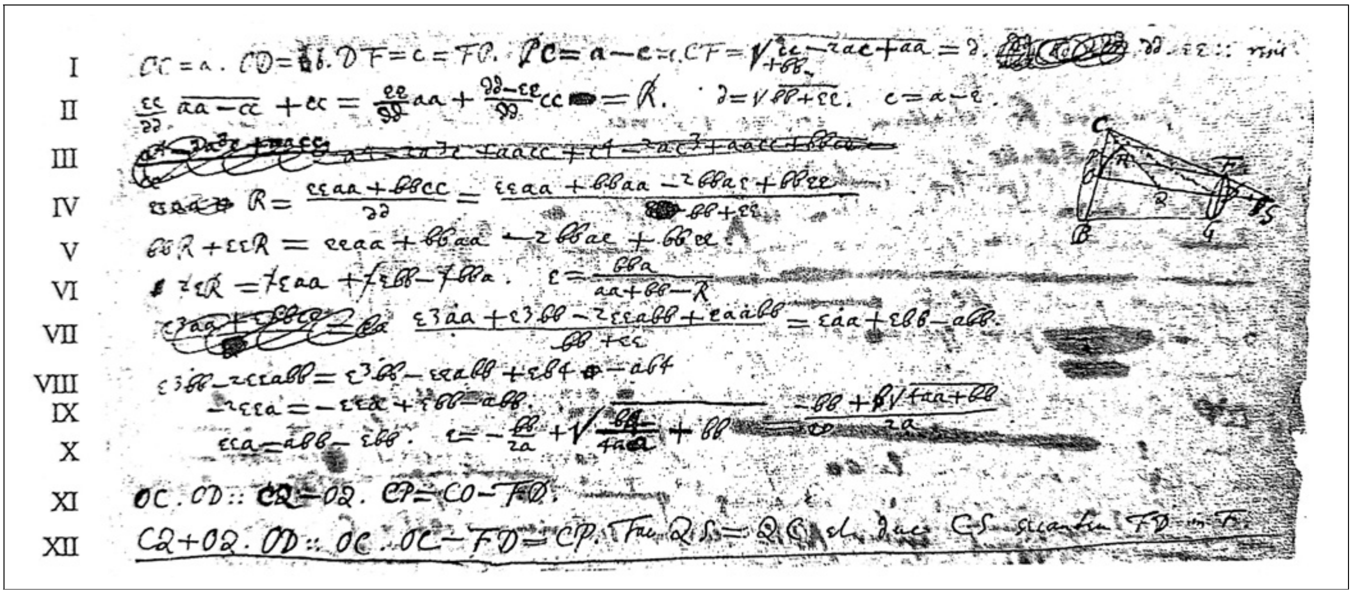


FIGURA 3.2 – Newton's Notes.

going forward towards D in the direction of its axis: bisect the altitude OD in Q and produce OQ to S so that QS may be equal to QC, and S will be the vertex of the cone whose frustum is sought." (5)

At first sight this example may appear somewhat artificial. We will see later that in the general case where the line CF that generates the lateral boundary of the cone is replaced by some curve the minimizing configuration contains also such a disc at the upper end. A proof that the cone which is characterized by choosing its vertex in this way is not given in the Principia. Newton's notes from 1685 however contain a proof that uses calculus in the same way we would do it nowadays.

These notes contain a drawing of the truncated cone where all points are denoted as in Fig. 3.1 which is taken from the Principia; there is, however, an additional point P on the segment OC with $|OP| = |DF|$. This point P or, equivalently, the length of CP serves as parameter in the family of truncated cones, for it determines the disc on the top as well as the lateral boundary of the body.

Newton sets (6) $a = OC, b = OD$, and $e = CP$; a and b are the data of our problem, and the resistance $R = R(e)$ is given as follows: The contribution of the upper disc of radius FD equals its area which is $\pi \cdot (a-e)^2$. The resistance of the lateral surface is $\pi \cdot [a^2 - (a-e)^2] \cdot \dfrac{e^2}{b^2+e^2}$; here the first factor is the area of the projection of the lateral boundary in the direction of the flow, and this gives the area of the annulus with radii OC and CP. The second one is $sin^2\,\theta$, where θ is the angle $\angle OSF$; there holds $\theta = \dfrac{\pi}{2} - \alpha$ with α being the angle between the normal to the line CF and the direction of the flow, cf. the previous example. With $\sin\theta = \dfrac{CP}{CF}$ and $CF^2 = CP^2 + PF^2$ we obtain $\sin^2\theta = \dfrac{e^2}{b^2+e^2}$ which finally gives

$$(3.1)\quad R(e) = \pi \cdot (a^2 - e^2) + \pi\,[a - (a-e)^2] \cdot \frac{e^2}{b^2+e^2}.$$

In Newton's notes this reads

$$R = \frac{aaee + bbaa - 2bbae + bbee}{bb + ee},$$

cf. line (IV) in Fig. 4. This is equivalent to

$$bbR + eeR = aaee + bbaa - 2bbae + bbee,$$

cf. line (V); differentiating with respect to e and setting $R'(e) = 0$ Newton gets

$$\not{2}eR = \not{2}eaa + \not{2}bb - \not{2}bba,$$

(5) [9], p. 333.

(6) [11], pp. 456-480, in particular plate III, after p. 462. The first line (I) starts with "$OC = a.CD = b.DF = c = FP$" The last expression should read "$DF = c = FP$, which is clear from the next equation "$PC = a - c = e$".

and inserting $aa + \dfrac{eebb + bbaa - 2bbae + bbaa}{bb + ee}$ for R he eventually obtains a quadratic quation for e:

$$\text{(X)} \qquad eea = abb - ebb.$$

In the formulas leading to (X) terms that cancel each other are indicated by a dot as in $e\dot{a}a$. The solution to the quadratic equation (X) reads

$$\text{(X)} \quad e = -\frac{bb}{2a} + \sqrt{\frac{bb}{4aa} + bb} = \frac{-bb + b\sqrt{bb + 4aa}}{2a}.$$

From (X) it is clear that R is minimal at the critical point e in (XI), and we may regard the problem to be solved. Newton however transforms the result such that the cone frustum of minimal resistance can be geometrically constructed in an elementary way: S is determined by $CQ = QS$, as was stated in the scholium. He arrives at this statement in two steps:

(XI) $OC : OD = CQ - OQ : CP = CQ - OQ : CO - FD$ (7)

This follows from (X) by writing it in the form

$$\text{(3.2)} \qquad \frac{a}{b} = \frac{\sqrt{\frac{1}{4}b^2 + a^2} - \frac{b}{2}}{e};$$

now it contains $\frac{b}{2} = OQ = QD$. The next relation

$$\text{(XII)} \quad CQ + OQ : OD = OC : OC - FD = OC : CP$$

follows from (3.2), if we multiply it by $\sqrt{\frac{1}{4}b^2 + a^2} + \frac{b}{2}$; this gives

$$\frac{a}{b}\left(\sqrt{\frac{1}{4}b^2 + a^2} + \frac{b}{2}\right) = \frac{\frac{1}{4}b^2 + a^2 - \frac{1}{4}b^2}{e}$$

or

$$\frac{\sqrt{\frac{1}{4}b^2 + a^2} + \frac{b}{2}}{b} = \frac{a}{e},$$

which is (XII). On the other hand we have from the intercept theorem

$$OC : CP = OS : PF = OS : OD = OQ + QS : OD,$$

which is the same relation as (XII) with QS insted of CQ; hence these quantities must be equal. Newton concludes this calculation with the statement how the point F is to be constructed: fac $QS = QC$ et duc CS secantem FD in F, which means: make $QS = QC$ and draw the line CS that cuts FD in F.

4. – Newton's Proportion for the Body of Least Resistance

In the Scholium to Proposition XXXIV Newton states the following proportion which characterizes the body of least resistance:

$$\text{(4.1)} \qquad MN : GR = GB^3 : (4BR \times GB^2)$$

> *"If the figure DNFG to be such a curve, that if, from any point thereof, as N, the perpendicular NM be let fall on the axis AB, and from the given point G there be drawn the right line GR parallel to a right line touching the figure in N, and cutting the axis produced in R,* **MN** *becomes to* **GR as GB³** *to* **4BR x GB²**, *the solid described by the revolution of this figure about its axis AB, moving in the before-mentioned rare medium from A towards B, will be less resisted than any other circular solid whatsoever, described of the same length and breadth."* (8)

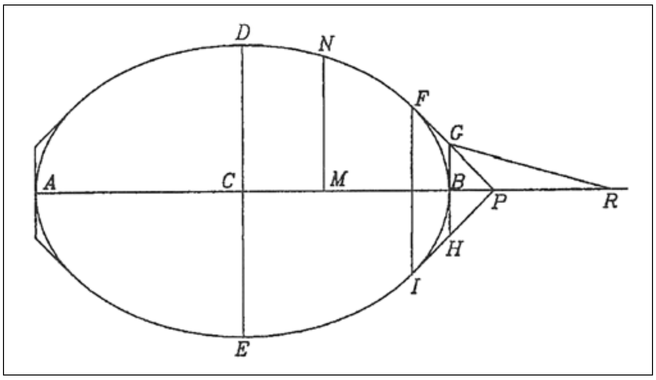


FIGURA 4.1 – Body of Least Resistance.

In modern notation we can describe the curve $DNFG$ by the function $u = u(r)$; then MN corresponds to the variable r, and the point N of the curve

(7) In Newton's notation the equality sign after CP refers only to this quantity.

(8) [9], pp. 333-334.

has coordinates $(r, u(r))$ with $u(r) = CM$, which means that the curve $DNFG$ is a graph above the interval CD on the r-axis. We now transform (4.1) into a differential equation for the function $u = u(r)$. Let GR be parallel to the tangent in N, and let θ be the angle at G between GB and GR, which means

$$BR : GR = \sin\theta, \quad GB : GR = \cos\theta. \tag{4.2}$$

GB is considered to be a constant c, because we are interested in local perturbations of the curve in interior points. This gives $GR = \dfrac{c}{\cos\theta}$, and with (4.2) we can write (4.1) in the form

$$\frac{r}{\frac{c}{\cos\theta}} = \frac{1}{4}\frac{1}{\sin\theta}\cdot\frac{1}{\cos^2\theta.}$$

We multiply by $\dfrac{\sin\theta}{\cos^2\theta}$ and get

$$\frac{r\sin\theta}{\cos\theta} = \frac{c}{4}\frac{1}{\cos^4\theta} = \frac{c}{4}\left(\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}\right)^2 =$$

$$= \frac{c}{4}\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)^2.$$

This equation can be written in terms of the derivative $u'(r) = \dfrac{\sin\theta}{\cos\theta}$ in the form

$$\frac{r\,u'(r)}{(1+|u'(r)|^2)^2} = \frac{c}{4}. \tag{4.3}$$

From (2.1) we see that the resistance, formulated in terms of the function $u = u(r)$, is an integral whose integrand does not depend on u explicitely: $f(r, u, p) = \dfrac{r}{1+p^2}$. Then the Euler equation reduces to $f_p = const$, which is (4.3).

The Scholium in the Principia does not contain a hint how Newton derived condition (4.1).

Goldstine calls this differential equation "the heart of Newton's contribution", ([9]) and he gives a detailed interpretation of Newton's notes from 1685 and 1694. ([10]) Rather than describing the comments of Goldstine, Whiteside, and other historians we give an elementary argument in modern notation that is based on Newton's idea: the minimizing arc is compared with another one that is constructed by moving a piece of it in direction of the flow.

We consider the integral

$$\mathcal{I}(u) = \int_a^b f(x, u'(x))\,dx,$$

and let $u_0 = u_0(x)$ be a minimizer; to define a variation $v = v(x)$ we fix $x' \in (a, b)$ and set

$$v(x) = \begin{cases} u_0(a) + \frac{u_0(a+h)+\varepsilon-u_0(a)}{h}\cdot(x-a) & \text{for } x\in[a,a+h] \\ u_0(x)+\varepsilon & \text{for } x\in[a+h,x'-h] \\ u_0(x')+\frac{u_0(x')-u_0(x'-h)-\varepsilon}{h}\cdot(x-x') & \text{for } x\in[x'-h,x'] \\ u_0(x) & \text{for } x\in[x',b] \end{cases}$$

$v(x)$ differs from u_0 by a constant term ε in $[a+h, x'-h]$, and therefore $v'(x)$ equals $u_0'(x)$ in this interval. Hence we get

$$\begin{aligned} \mathcal{I}(v)-\mathcal{I}(u_0) = & \int_a^{a+h} f(x, v'(x))\,dx + \int_{x'-h}^{x'} f(x, v'(x))\,dx \\ & - \int_a^{a+h} f(x, u_0'(x))\,dx - \int_{x'-h}^{x'} f(x, u_0'(x))\,dx \end{aligned} \tag{4.4}$$

for all $h > 0$ and $|\varepsilon| < 1$.

In $[a, a+h]$ and in $[x'-h, x']$ v' is constant; the integrals can be evaluated by the mean value theorem, such that we get

$$\int_a^{a+h} f(x, v'(x))\,dx + \int_{x'-h}^{x'} f(x, v'(x))\,dx =$$
$$= hf(\overline{x}, v'\,(a)) + hf(\tilde{x}, v'(x'))$$

with $\overline{x} \in [a, a+h]$ and $\tilde{x} \in [x'-h, x']$.

As $\mathcal{I}(v)$ is minimal for $\varepsilon = 0$, we get from (4.4)

$$\frac{d}{d\varepsilon}\left\{h\cdot f\left(\overline{x}, \frac{u_0(a+h)+\varepsilon-u_0(a)}{h}\right)+ \right.$$
$$\left. + h\cdot f\left(\tilde{x}, \frac{u_0(x')-[u_0(x'-h)+\varepsilon]}{h}\right)\right\}\Bigg|_{\varepsilon=0} = 0,$$

which leads to

$$0 = h\cdot f_p\left(\overline{x}, \frac{u_0(a+h)-u_0(a)}{h}\right)\cdot\frac{1}{h} +$$
$$+ h\cdot f_p\left(\tilde{x}, \frac{u_0(x')-u(x'-h)}{h}\right)\cdot\left(\frac{-1}{h}\right),$$

([9]) [5], p.15, Footnote 28.

([10]) [6], pp.456-480. The ones from 1894 were made for D. Gregory who had asked Newton to explain (4.1) to him.

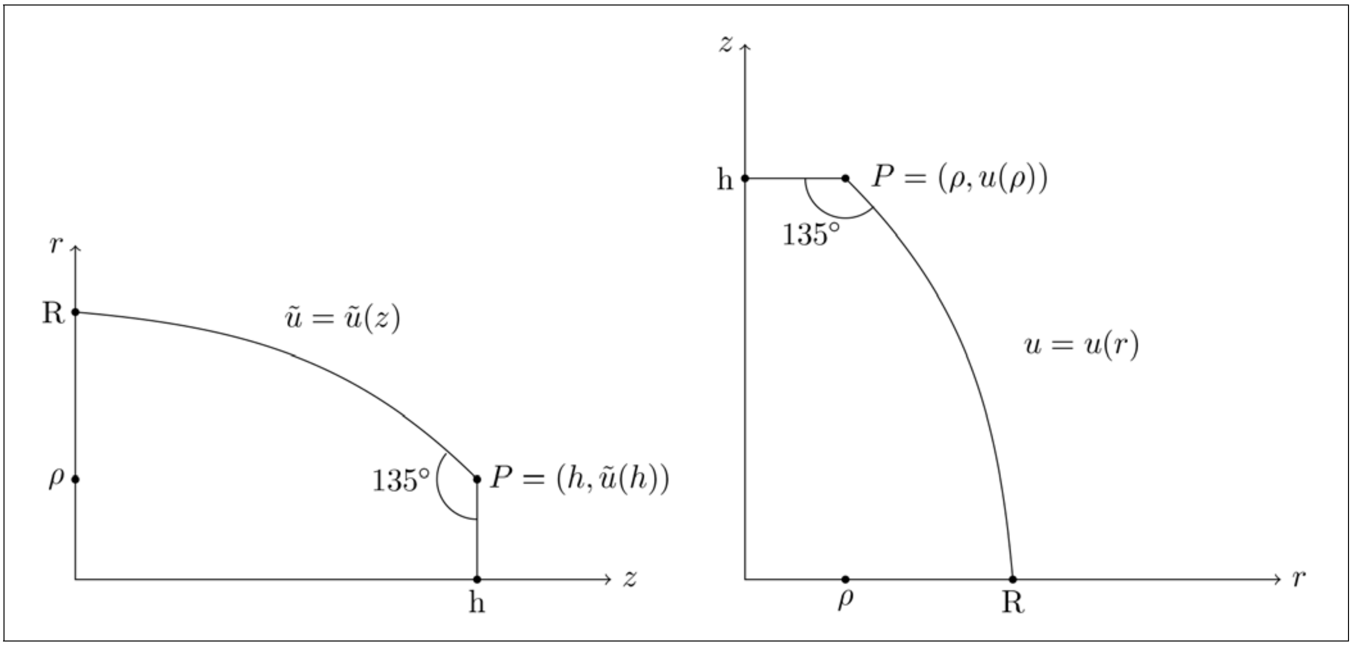


FIGURA 5.1 – a) Free-Boundary and b) Transversality Condition.

and with $h \to 0$ we eventually get

$$f_p(a, u'(a) = f_p(x', u_0(x')).$$

As x' is any point in (a, b), this means that

$$f_p(x', u_0(x')) = \text{const} \quad \forall \text{x}' \in (\text{a}, \text{b}).$$

5. – The Body of Least Resistance – a Free-Boundary Problem

In the Scholium to Proposition XXXIV Newton states that the body of minimal resistance is bounded at its upper end by a disc B_ρ that is parallel to the circular base of the body and that the lateral surface meets B_ρ at an angle of 135°. In modern terminology this statement turns out to be a transversality or a free-boundary condition.

If the lateral surface Σ is described as before by a real function $u = u(r)$ then the minimizer to

$$\mathcal{R}(\rho, u) = \pi\rho^2 + 2\pi \int_\rho^R \frac{r}{1 + |u'(r)|^2}$$

is a function that is constant on some interval $[0, \rho]$. In Fig. 5.1.b it is depicted in the usual way, i.e with the independent variable r on the horizontal axis. Now it is clear that at some point P of the graph of u the minimizing curve is not differentiable; the horizontal part of the graph meets the curved one at an angle of 135°. Variational problems of this type have been studied very extensively. This transversality condition satisfied by u at some interior point ρ of $[0, R]$ is the content of Hamilton's formula. ([11])

If we write the lateral boundary as a graph of some function $r = \tilde{u}(z)$ that is defined on the interval $[0, h]$, cf. Fig. 5.1.a, then the resistance reads

$$\mathcal{R}(\tilde{u}) = \pi\,\tilde{u}^2(h) + \int_0^h \frac{\tilde{u}(z)|\tilde{u}'(z)|^2\tilde{u}'(z)}{1 + |\tilde{u}'(z)|^2}\, dz,$$

for $\tilde{u}$: $[0, h] \to [0, \infty)$ with boundary conditions $\tilde{u}(0) = R$ and $\tilde{u}(h) \geq 0$. Now the point P has coordinates $(h, \tilde{u}(h))$; because h is a boundary point and $\tilde{u}$ satisfies in h an inequality, the condition on the angle at P is in modern notation a free-boundary condition.

As mentioned in §3 Newton studies the resistance of truncated cones, figures that are also bounded by a disc at their tops. In that case, how-

([11]) P. Funk gives a systematic treatment of this type of variational problem, cf. [4], pp. 60-82; in the appendix he discusses also Newton's example, cf. pp. 616-621.

ever, the angle depends on the data R and h, whereas here it is a local property that gives the same value for any R and h. Only in the limiting case $h \to 0$ the angle in the frustum of the cone tends to 135°, as D.T. Whiteside points out. ([12]) When we consider how the calculus of variations developed we must regard the fact that Newton asked for such a property of the minimizer to be a truly spectacular aspect of his investigation. As in the previous chapter we will not present Newton's notes nor the literature concerning them. We derive the condition on the angle by a variation of the integral $\mathcal{R}(\rho, u)$ with respect to ρ; as we know the corresponding Euler equation already the argument is rather elementary.

Let (ρ, u) be a minimizer and set

$$u_\varepsilon(r) = \begin{cases} h & \text{for } r \in [0, \rho+\varepsilon] \\ u(r) + \varepsilon\phi(r) & \text{for } r \in [\rho+\varepsilon, R] \end{cases}$$

with $\phi(R) = 0$ and $u_\varepsilon(\rho+\varepsilon) \equiv u(\rho+\varepsilon) + \phi(\rho+\varepsilon) = h$. Clearly $\phi(\rho+\varepsilon)$ tends to $u'(\rho)$, as $\varepsilon \to 0$. A standard calculation then shows that

$$\frac{1}{\varepsilon}\left[\mathcal{R}(\rho+\varepsilon, u_\varepsilon) - \mathcal{R}(\rho, u)\right] \to \text{const} \cdot \left[1 - \frac{1+3\,|u'(\rho)|^2}{(1+|u'(\rho)|^2)^2}\right],$$

as $\varepsilon \to 0$, and the last quantity vanishes for $|u'(\rho)| = 1$.

6. – Newton's Problem and the Beginning of the Calculus of Variations

In 1696 Johann Bernoulli posed the problem "of finding the curve joining two points in a vertical plane along which a frictionless bead will descend in the least possible time". ([13]) A similar challenge had been published in 1690 by Jacob Bernoulli. He formulated the problem of the catenary, i.e. to determine the shape of a chain of constant mass density and given length under the influence of gravity. Besides Bernoulli also Huygens and Leibniz submitted solutions; these contributions, and the vivid interaction of the mathematicians who presented them, is described in Truesdell's introduction to Euler's work on mechanics. ([14]) As Newton was not involved we shall not go into details here; we wish to point out, however, that there are manuscripts of Newton's, ([15]) published much later than Truesdell's work appeared, that show what Newton did in connection with this problem around 1680.

Besides publishing the problem in the Acta Eruditorum Johann Bernoulli informed various colleagues about it directly; he wrote in particular to Leibniz on June, 14^{th}, 1696, informing him about his result. All were invited to submit an answer, and if no one would do so by the end of the year, Bernoulli promised to publish his own demonstration. Leibniz was very enthousiastic about this problem, and in an article he speculated who would be able to solve it. Among the mathematicians he mentioned was of course also Newton; he could do it, Leibniz wrote, provided he would take the trouble. ([16])

A very vivid exchange of ideas evolved, and in order to let also mathematicians from other countries join the competition, where usually the issues of Acta Eruditorum were available rather late, the daedline was postponed to easter of 1697. For details of this contest and in particular for the influence this problem had we refer to Goldstine's book. ([17]) Here we are interested in two questions concerning the role of Newton: (i) Did the mathematicians who worked on the minimum problem of Bernoulli know about Newton's contribution in the Principia? (ii) How did Newton respond to the challenge of Johann Bernoulli?

Up to 1696, which means during the first ten years after the Principia appeared, Newton's variatonal problem was not referred to in the literature. It is not known how Newton's result was received, except for two examples. The Collected Works of Christiaan Huygens, published by the Dutch Society of Sciences in 22 volumes between 1888 and 1950, contain not only his papers and letters but also

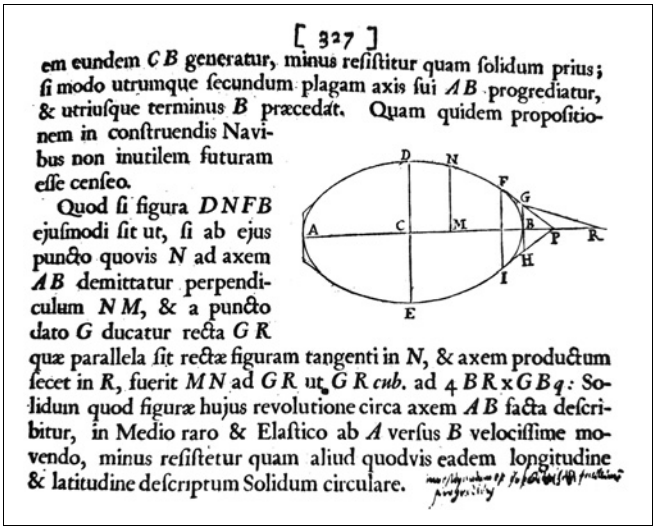
([12]) [11], p. 462.
([13]) [5], p. 30.

([14]) [14], §10: "The contest to find the catenary (1690)", pp. 64-75.
([15]) [10], pp. 520-524.
([16]) [6], p. 334: "si operam hanc in se reciperet".
([17]) [5], pp. 7-29.

notes that he made when reading publications of others. And here, "as we might have expected", as Whiteside writes, (18) we see that Huygens read this part of the Principia between the 22^{nd} and the 25^{th} of April, 1691; he studied Newton's problem and gave proofs of the propopsitions that were stated without any proof by Newton. (19) Besides the "Incomparable Huygens", as Leibniz called him, we know only of Leibniz himself that he read Newton's analysis of the minimum problem. His copy of (the first edition of) the Principia, on various pages richly annotated by his comments, was sold in 1926 by the University of Göttingen, because they owned two copies of the first edition and estimated the proper one, that contained no handwritten comments in the margins, to be more valuable. It was bought by Martin Bodmer, Geneva, and is now in the Bibliotheca Bodmeriana. In 1969 Emil A. Fellmann found out that the notes were written by Leibniz. Now we know that he read the propositions on the body of least resistance, because there is a comment by him. It consists of five words and says that one will very easily (!) make progress if one investigates by – and then the central word "isoperimetricis" is crossed out and written over by another word that is read differently by Fellmann and by Whiteside. (20) But they agree that the remark by Leibniz is not anything clear or substantial.

As long as notes and remarks of other mathematicians are not found we must consider Huygens to be the only one who studied Newton's statements successfully.

Finally we ask whether Newton learned about Bernoulli's challenge, and if so, how he reacted to it. According to his niece, Catherine Bardon, who lived in his household, Newton, after coming home from his work in the mint, would first look whether some book or paper was sent to him before he would eat. On January, 29^{th}, $16\frac{96}{97}$, Bernoulli's problem had arrived, and he started to work on it immediately; by 4 in the morning he had solved it, and on January 30^{th} his solution was sent for publication in the


[327]

em eundem *CB* generatur, minus resistitur quam solidum prius; si modo utrumque secundum plagam axis sui *AB* progrediatur, & utriusque terminus *B* præcedat. Quam quidem propositionem in construendis Navibus non inutilem futuram esse censeo.

Quod si figura *DNFB* ejusmodi sit ut, si ab ejus puncto quovis *N* ad axem *AB* demittatur perpendiculum *NM*, & a puncto dato *G* ducatur recta *GR* quæ parallela sit rectæ figuram tangenti in *N*, & axem productum secet in *R*, fuerit *MN* ad *GR* ut *GR cub.* ad 4 *BR*x*GBq*: Solidum quod figuræ hujus revolutione circa axem *AB* facta describitur, in Medio raro & Elastico ab *A* versus *B* velocissime movendo, minus resistetur quam aliud quodvis eadem longitudine & latitudine descriptum Solidum circulare.

FIGURA 6.1 – Annotation by Leibniz.

Proceedings of the Royal Society (21): It appeared there – anonymously – in the form of a letter to Charles Montagu, chancellor of the Royal Society. When Newton was asked by David Gregory, a scotsman, about the body of least resistance, he sent him detailed calculations that were made just for him. But towards Bernoulli and other mathematicians from the continent Newton reacted differently. "I do not love ... To be dunned and teezed by forreigners about Mathematical things", he wrote in a letter to Flamsteed. (22) The letter to Montagu was also published in May 1697 in the Acta Eruditorum, together with the other solutions. But in the index of the volume of that year the author is listed as "Newton, Isaac". (23) Obviously not only Johann Bernoulli, who claimed to have recognized the lion by its claw, tanquam ex ungue leonem, but also the other mathematicians would not imagine that someone else in England had been able to solve this problem.

(18) [11], p. 466, footnote 25.
(19) [3], pp. 335-341.
(20) [7], pp. 84-86; [10], l.c.

(21) [13], pp. 220-229; [11], pp.72-79.
(22) [13], p. 296. In this letter Newton wrote that he did not want to be cited in a paper Flamsteed was about to publish, and when disapproving this Newton added the foreigners who ask mathematical questions. The correspondence contains also Flamsteed's remarks to these statements as well his answer; obviously it was not easy to deal with Newton.
(23) The library of the University of Halle offers a copy of this volume for download.

Acknoledgement. The author thanks Cambridge University Press for granting the license to reproduce Newton's notes in Fig. 3.2. The copy of Newton's Principia that contains the annotations by Leibniz is now owned by The Martin Bodmer Foundation, Cologny (Geneva). They make the entire book available online, and the author thanks them for permitting to reproduce page [327] in Fig. 6.1.

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