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Louis Nirenberg in ricordo

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There is no doubt that the work of Louis Nirenberg has influenced many areas of Mathematics.

Nirenberg has tackled relevant issues in pure and applied mathematics by means of a P.D.E. approach, in the spirit of the long standing tradition of the Courant Institute, where he spent the entire academic life.

I was always surprised to see how many colleagues (other than mathematicians) knew about him and his work. Even in the Engineering Faculty where I teach, the same colleagues that were never happy to assign additional credits for mathematics courses, would be extremely impressed by my scientific relation with Louis Nirenberg.

Here I will not present Nirenberg remarkable mathematical results, as I would risk to reproduce (only with less insight) the beautiful review papers already available in literature, as for example by S. Donaldson [D] and by Y.Y. Li [L] in occasion of the Chern Medal award (2010), or by T. Rivière [R] in occasion of the Abel Prize award (2015).

Incidentally, let me add that L. Nirenberg was a gracious and humble receiver of an impressive list of awards, such as the Bôcher Memorial Prize (1959), the Jeffery-Williams Prize (1987), the Steele Prize (1994 and 2014), the National Medal of Science (1995), the Crafoord Prize (1982), etc.

In order to describe the great impact of Nirenberg's work, I have chosen to discuss some of the vast research activity that has flourished out of Nirenberg's original contribution. To keep this note of reasonable length, I will focus only on few examples in Geometrical Analysis, naturally related to well known functional inequalities. We all know how Louis Nirenberg has been so very passionate about *inequalities*.

Starting with his PhD thesis on Weyl's isoperimetric embedding (under the direction of James Stoker), Nirenberg's research activity has been often motivated and inspired by geometrical problems.

Nirenberg Problem: One example of Nirenberg's geometrical insight is described by the following question he posed in 1970:

> *when a (smooth) function K on the 2-sphere S^2 can be realized as the Gauss curvature of some metric on S^2?*

Beside the obvious total integral identity given by the Gauss-Bonnet Theorem (which in particular implies the K must be positive somewhere), there is another less obvious integral identity that K must satisfy. It was derived by Kazdan-Warner in 1974 [KW] from the conformal invariance of the problem, and it forbids K to satisfy suitable monotonicity properties. This fact should be compared with Moser's result [M], who proved in 1973 that any *antipodal* symmetric function on S^2 (positive somewhere) can be realized as the Gauss curvature of a metric which in fact is conformal to the standard metric on S^2.

Actually in [KW], the authors formulated a more general "*assigned Gauss curvature problem*" over Riemann surfaces within a given class of conformal metrics. While, in higher dimension, they asked a similar question about the "*assigned scalar curvature problem*", extending the classical "*Yamabe problem*" where the scalar curvature is required to be a constant.

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In all such conformal invariant problems, one can identify obstructions to solvability as a consequence of some integral identities. It has become commonplace to refer to such identities as the "Pohozaev identities". It is worth to note that local versions of suitable Pohozaev identities have been used also to derive qualitative information.

A PDE approach to Nirenberg problem, or more generally to the assigned Gauss curvature problem, is possible. Indeed, with a fixed background metric (e.g. the standard one on S^2) we see easily how the conformal factor of the desired metric having Gauss curvature K must satisfy a Liouville type equation. The solvability of such class of elliptic equations is evidently rather tricky in account of the obstructions in [KW]. For example, by adopting a variational approach as in [M], one may try to obtain solutions as externals of a Moser-Trudinger- Onofri type inequality with weight K.

Note that, the Moser-Trudinger-Onofri (MTO) inequality enters rather naturally in the "assigned Gauss curvature problem". Actually on S^2, it captures some "limiting" properties of the Sobolev inequality and so it yields to sharp bi-dimensional versions of compact embedding.

However, in presence of a general (non-constant) weight, the MTO-inequality admits no extremals on S^2, and thus minimizing sequences fail to converge (*no compactness*).

So, to attack Nirenberg problem, one must elaborate more sophisticated "min-max" principles, which can be applied successfully only after one overcomes the possible "loss of compactness", as attained for example in [M], (see also [A]).

The compactness issue is common to other conformal invariant problems. It was handled successfully by Sacks-Uhlenbeck [SU] for area minimizing minimal surfaces. The authors in [SU] were the first to observe that compactness may be lost only through the manifestation of "bubbling" phenomena.

It turns out that such a feature is shared by other problems in conformal geometry, including the assigned Gauss curvature problem.

With such information in hand, many authors (too many to mention) were able to obtain interesting contributions towards Nirenberg problem (and the assigned Gauss curvature problem), by means of variational and topological methods and more recently, by flow methods.

We only mention the substantial contributions of A. Chang and P. Yang in [CY1,2], which provide rather sharp results about Nirenberg problem.

Furthermore, starting with the work of Brezis-Merle [BM], a very accurate blow-up analysis is now available for "bubbling" solutions of Liouville type equations on Riemann surfaces. In this way, "concentration-compactness" principles and "mass quantization" properties have been established in various contexts, together with very accurate pointwise estimates about the "bubbling profile".

On such basis, C.C. Chen-C.S. Lin [CL] were able to obtain an explicit Leray-Schauder degree formula for the corresponding Fredholm operator, only in terms of the surface genus.

In addition, it has been possible to extend Nirenberg problem to surfaces with conical singularities, a problem surprisingly connected to the construction of self-dual gauge field vortices [T].

Also in such "singular" situation the sphere enters as a "critical" case. However, by considering minimal immersion problems with "assigned" second fundamental form, interesting new "critical' cases occur also for surfaces with positive genus.

By now, the analytical machinery used in the study of Nirenberg problem is so well tuned and powerful to crack much more involved conformal invariant problems. We only mention the case of Toda-type system, of interest in algebraic geometry and in non-abelian gauge field theory, see [LYZ] and [T]; or the case of fully nonlinear conformal invariant problem of Monge- Ampère type, see [CGY].

Elliptic equations involving the critical Sobolev exponent

Another of Nirenberg's successes has been his work with H. Brezis, concerning the Dirichlet problem for semi-linear elliptic equations involving the critical Sobolev exponent. A problem evidently prompted by the work of Aubin [A] on the Yamabe problem.

According to MathScinet the Brezis-Nirenberg [BN] paper has received more than 1450 citations.

Again, the problem admits a variational formulation, now in terms of extremals for the Sobolev inequality in dimension 3 or higher. Again, we have obstructions to solvability (in star-shaped domains) as expressed by a (well known) Pohozaev identity and manifested through "bubbling" phenomena.

Brezis-Nirenberg showed that it suffices a linear perturbation to restore solvability. Actually, the non-existence situation seems rather "unstable" against the more robust role of a "rich topology" of the domain. Indeed, after the seminal work of Bahri-Coron [BC] and Bahri [B], a vast literature is now available about existence and multiplicity results for elliptic equations involving the critical Sobolev exponent in different contexts. This is paired by a rather precise analysis of the bubbling behavior of (approximate) solutions.

In the geometrical context, an even deeper investigation has led to striking existence and compactness results for solutions of the Yamabe problem. The pioneering work of Aubin [A] has been completed by an impressive bulk of original papers starting with the excellent contributions of Schoen, see [S1,2].

While, the more general assigned scalar curvature problem posed by Kazdan and Warner is still very actively investigated, especially for the n-dimensional sphere, where the problem remains as puzzling as ever.

The Caffarelli-Kohn-Nirenberg inequality

Finally, I wish to conclude with another class of popular problems originated by Nirenberg work. It concerns extremals for the weighted Caffarelli-Kohn-Nirenberg (CKN) inequality [CKN1, 2]. Such inequality was introduced by the authors as interpolation between the Hardy and the Sobolev inequality, in order to control the singular set of solutions for the Navier-Stokes equations, a *millennium* problem.

Typically, as seen above, extremals of inequalities can be viewed as minimizers of certain energies and, as such, typically inherit the energy's symmetries.

The CKN-inequality are radially symmetric and Nirenberg has significantly contributed to enhance and popularize one of the most powerful tool to handle such situation, namely: the "*moving plane method*" of Alexandroff (see [GNN] and [BN]). It is based upon the "*maximum principle*", one of Nirenberg's favorite subject.

However, in the context of CKN-inequality, such methods apply successfully only within certain range of parameters. Indeed, it was shown by Catrina-Wang [CW] that there are cases where a symmetry breaking phenomenon does occur.

Naturally, one wants to understand what triggers such phenomenon also in connection with the best constant involved in CKN-inequality.

However, for extremals of the CKN-inequality, a complete description of the phenomenon of "*symmetry*" versus *"symmetry breaking"* turned out to be a very subtle issue to tackle, even though it was not particularly difficult to formulate a reasonable conjecture.

Still, many authors have tried different methods to match spectral information with a "global" bifurcation analysis. We mention, the *entropy method* borrowed from Kinetic and Information theory, *the Carré du Champ* method successfully applied in Markov processes and various other methods involving flows and corresponding monotonicity formulae, mass transport as well as nonlinear techniques to attain uniqueness ad hence symmetry.

All such methods (reviewed in [DEL1]) have allowed to gain a remarkable insight about extremals for the CKN-inequality as well as for more general functional-inequalities.

But to obtain a full understanding of the symmetry breaking phenomenon and eventually confirm the conjecture, finally Dolbeault-Esteban-Loss [DEL2] introduced a method based on a nonlinear diffusion flow which (by unifying several of the previous approaches) helped to untangle the uniqueness/rigidity issue. More interestingly, nonlinear diffusion flows seem very prone for generalization in other directions.

On a personal note

Learning and discussing mathematics with Louis Nirenberg has forged my style as a mathematician. I enjoyed the collaborative and supporting atmosphere that Louis Nirenberg created among his students and collaborators. Never a trace of competition, only the sheer enthusiasm to learn from each other.

I like to believe that Louis' teaching helped me to acquire a certain taste for a "good problem" or a "nice proof". But what I certainly learnt is to always mind and never underestimate mere technical efforts. Forever, I shall prize Louis great ability to carry out involved technical arguments with elegance till the last detail.

In those moments when nothing seemed to work, Louis would say that a good mathematician can hope to have at most two original ideas in a year. Translated into the current lingo, this means at most two publications per year. Perhaps it would be appropriate to reaffirm this principle and pass it on to the younger generations.

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