
Matematica, Cultura e Società

RIVISTA DELL'UNIONE MATEMATICA ITALIANA

SERENA DIPIERRO, ENRICO VALDINOCI

Alessio Figalli's contribution to nonlocal minimal surfaces

Matematica, Cultura e Società. Rivista dell'Unione Matematica Italiana, Serie 1, Vol. 4
(2019), n.2, p. 119–124.

Unione Matematica Italiana

http://www.bdim.eu/item?id=RUMI_2019_1_4_2_119_0

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

*Articolo digitalizzato nel quadro del programma
bdim (Biblioteca Digitale Italiana di Matematica)*

SIMAI & UMI

<http://www.bdim.eu/>

Alessio Figalli’s contributions to nonlocal minimal surfaces

SERENA DIPIERRO - ENRICO VALDINOCI

University of Western Australia

E-mail: serena.dipierro@uwa.edu.au, enrico.valdinoci@uwa.edu.au

Sommario: *Nella sua intensa carriera, Alessio ha prodotto un numero straordinario di risultati eccezionali riguardanti una varietà impressionante di temi di ricerca. Tra le molteplici direzioni intraprese, quella riguardante le superfici minime nonlocali ha proposto Alessio come un leader mondiale per i suoi contributi stimolanti e innovativi.*

Abstract: *Alessio has produced in his very intense career an extraordinary number of outstanding results in an impressive variety of topics. Among the multifold research lines in which he acted as a trailblazer, the one focused on nonlocal minimal surfaces offered an excellent opportunity for Alessio to pioneer some of the first settlements in a brand new subject of investigation and pave the way to a broad spectrum of future research.*

Alessio has produced in his very intense career an extraordinary number of outstanding results in an impressive variety of topics. Among the multifold research lines in which he acted as a trailblazer, the one focused on nonlocal minimal surfaces offered an excellent opportunity for Alessio to pioneer some of the first settlements in a brand new subject of investigation and pave the way to a broad spectrum of future research.

Nonlocal minimal surfaces are beautiful objects whose research combines motivations and methods arising in different disciplines, including mathematical analysis, differential geometry and mathematical physics, and a full understanding of their complexity requires a truly cross-disciplinary and open-minded approach. As it often occurs in the new

research lines, to understand nonlocal minimal surfaces one has to discover novel methodologies revealing the striking differences with respect to the previous knowledge, get off the beaten path, and think differently. Giving a full account of all the important progress that the field of nonlocal minimal surfaces has recently experienced is a goal which goes well beyond the purpose of this note, therefore we will simply focus here on some of the very original and important contributions given by Alessio in this field. Without any attempt of being exhaustive, other fundamental contributions provided by other authors will be only tangentially discussed: the reader who wants to dig more into the subject can consult the existing literature, including a set of lecture notes coauthored by Alessio himself [CF17].

Nonlocal minimal surfaces have been introduced in [CRS10] as the outcome of a minimization problem involving a nonlocal notion of perimeter. Roughly

Accettato: il 15 maggio 2019.

speaking, the energy functional takes into account the pointwise interactions of a set with its complement. These interactions are weighted by a kernel which is invariant under translations and rotations, and which is self-similar after scaling. One can also take into account the contributions of this energy functional with respect to a given reference domain $\Omega \subseteq \mathbb{R}^n$ (that we take with smooth boundary for the sake of simplicity). In this case the energy contributions that involve the interactions of points lying in the complement of Ω are omitted from the functional.

More precisely, fixed $s \in (0, 1)$, one defines the s -perimeter of a (measurable) set $E \subseteq \mathbb{R}^n$ with respect to the reference domain Ω as

$$(1) \quad \begin{aligned} \text{Per}_s(E, \Omega) &:= I_s(E \cap \Omega, E^c \cap \Omega) + \\ &+ I_s(E \cap \Omega, E^c \cap \Omega^c) + I_s(E \cap \Omega^c, E^c \cap \Omega), \end{aligned}$$

where we used the superscript “ c ” to denote complementary sets in \mathbb{R}^n , and $I_s(\cdot, \cdot)$ represents the set interaction given by

$$(2) \quad I_s(A, B) := s(1-s) \iint_{A \times B} \frac{dx dy}{|x-y|^{n+s}},$$

for all disjoint subsets A and B of \mathbb{R}^n ,

see Figure 1.

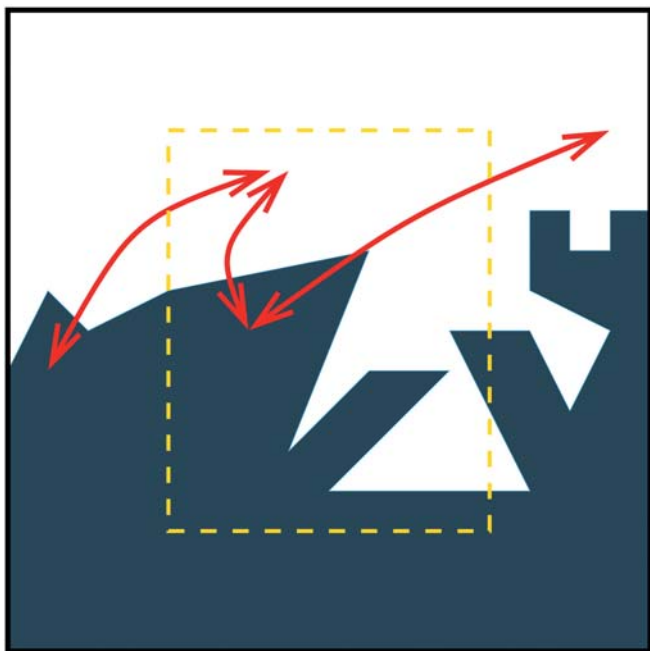


Fig. 1. – Pointwise interactions defining the s -perimeter.

In spite of its remarkable structural simplicity, codifying the most fundamental geometric property of a given set, the s -perimeter turns out to be one of the most difficult objects to fully understand. On the other hand, it provides a large amount of information on several models of concrete interest, including long-range phase transitions, spin models and image reconstruction.

The factor $s(1-s)$ has been included in the interaction functional in (2) for normalization purposes. In this way, one has that if E has finite classical perimeter in a neighborhood of Ω , then

$$\lim_{s \nearrow 1} \text{Per}_s(E, \Omega) = \omega_{n-1} \text{Per}(E, \bar{\Omega}),$$

where ω_n is the volume of the unit ball in \mathbb{R}^n , and “Per” denotes the classical perimeter. In this sense, up to a dimensional constant, the s -perimeter recovers the classical perimeter as $s \nearrow 1$.

The limit of the s -perimeter as $s \searrow 0$ is somehow more complicated, and it has been investigated in full details in [DFPV13]. In this work there are explicit examples of smooth sets for which such a limit does not even exist, and, in general, it is shown that the existence of the limit of the s -perimeter as $s \searrow 0$ is strictly related to the existence of the following limit:

$$(3) \quad \zeta(E) := \lim_{s \searrow 0} \frac{s}{n\omega_n} \int_{E \setminus B_1} \frac{dx}{|x|^{n+s}}.$$

Roughly speaking, the quantity $\zeta(E)$ measures the “mass” of the set E at infinity, weighted by the kernel. We observe that ζ is monotone with respect to set inclusion. Also, it is apparent that $\zeta(\emptyset) = 0$ and $\zeta(\mathbb{R}^n) = 1$. Furthermore, ζ evaluated at a half-space is exactly $1/2$ and, more generally, ζ evaluated at a cone gives the opening of the cone itself, that is if $E := \{tp, p \in \Sigma, t > 0\}$ for some $\Sigma \subseteq S^{n-1}$, then

$$\begin{aligned} \zeta(E) &= \lim_{s \searrow 0} \frac{s}{\mathcal{H}^{n-1}(S^{n-1})} \iint_{(\rho, \omega) \in (1, +\infty) \times \Sigma} \frac{\rho^{n-1}}{\rho^{n+s}} d\rho d\mathcal{H}^{n-1}(\omega) \\ &= \frac{\mathcal{H}^{n-1}(\Sigma)}{\mathcal{H}^{n-1}(S^{n-1})}. \end{aligned}$$

It turns out that we can consider $\zeta(E)$ as a “convex parameter”, and it is proved in [DFPV13] that if E has finite s_0 -perimeter in Ω for some $s_0 \in (0, 1)$ and

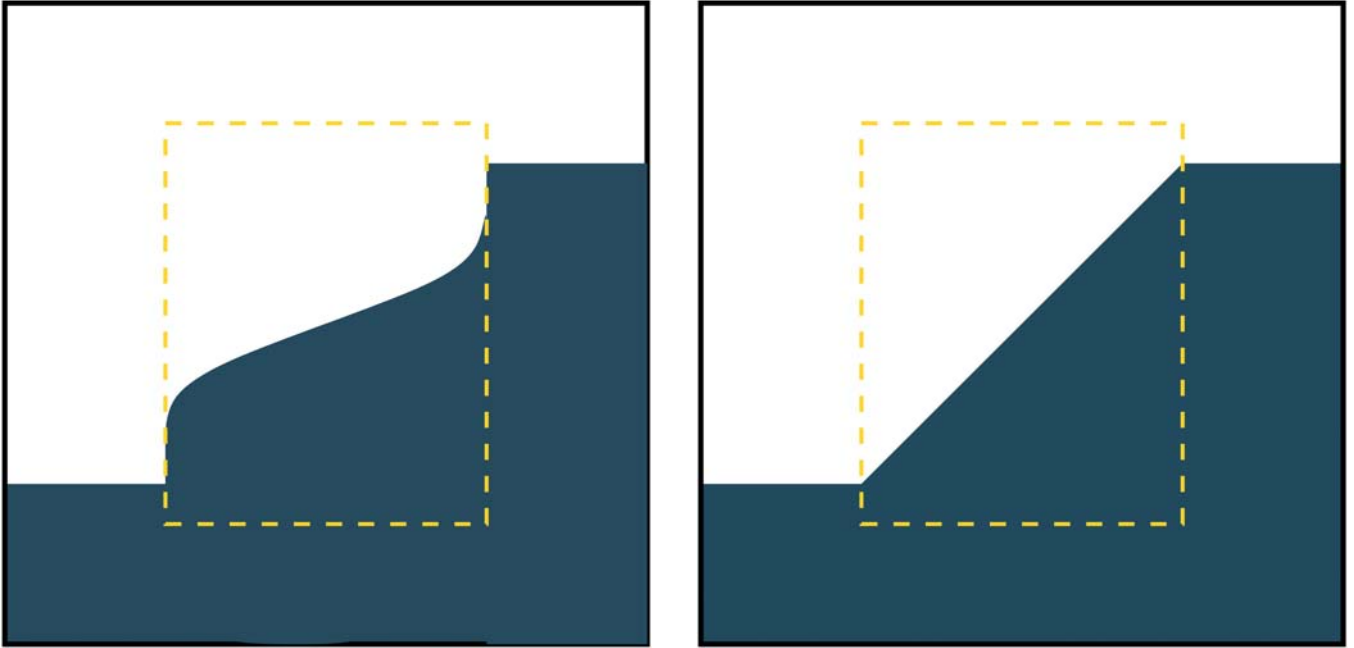


Fig. 2. – Classical (on the left) and nonlocal (on the right) minimal surfaces inside the dashed box, given the outside data.

the limit in (3) exists, then the limit as $s \searrow 0$ of the s -perimeter of E in Ω also exists, and

$$(4) \lim_{s \searrow 0} \text{Per}_s(E, \Omega) = (1 - \zeta(E)) |E \cap \Omega| + \zeta(E) |E^c \cap \Omega|,$$

where we denoted by $|\cdot|$ the volume of a set.

In the special case in which $|E \cap \Omega| = |E^c \cap \Omega|$, it is also shown in [DFPV13] that (4) is true independently on the existence of the limit in (3), since in this case

$$\lim_{s \searrow 0} \text{Per}_s(E, \Omega) = |E \cap \Omega| = |E^c \cap \Omega|.$$

On the other hand, if $|E \cap \Omega| \neq |E^c \cap \Omega|$ the existence of the limit in (4) is shown to be equivalent to that in (3).

Besides its importance in the foundation of a new field of research, these results contributed in the development of several new lines of investigation: in particular, the quantity introduced in (3) has been later efficiently utilized in order to detect a rather surprising phenomenon occurring for the minimizers of the s -perimeter, namely their strong tendency to “stick” at the boundary of the domain, in sharp contrast with the classical case, see e.g. Figure 2 – but this is somehow a different story.

Let us now go back to the variational problem related to the s -perimeter. Local minimizers of the s -perimeter are called s -minimal sets, and their boundaries are called s -minimal surfaces. An s -minimal set which is the subgraph of a function (in a given direction) is called an s -minimal graph. Also, an s -minimal set which is a cone (i.e. a point p belongs to it if and only if tp belongs to it for all $t > 0$) is called an s -minimal cone. The empty set, the full space \mathbb{R}^n , and the half-spaces $\{\omega \cdot x > 0\}$ with $\omega \in S^{n-1}$ are examples of s -minimal cones. The other cones necessarily exhibit a singularity at the origin, and therefore are called “singular”.

In this setting, the regularity of the s -minimal surfaces turns out to be one of the most interesting and challenging topic related to nonlocal problems, which still presents many open fundamental questions.

As usual, Alessio attacked the problem vigorously, obtaining pioneering results on the topic. In previous works, in low dimension and for special ranges of the nonlocal exponent, a regularity in class $C^{1,\alpha}$ for all $\alpha \in (0, s)$ was obtained. Then, in [BFV14] a new and general bootstrap result for fully nonlinear nonlocal equations was provided, whose special application to the geometric case of s -minimal surfaces established

that if an s -minimal surface is locally $C^{1,\alpha}$ for all $\alpha \in (0, s)$ (or even just locally Lipschitz, as remarked in [FV17]) then it must be locally C^∞ .

This, combined with previous results, give that s -minimal surfaces are C^∞ in $\Omega \subset \mathbb{R}^n$ at least in two cases:

- if $n \leq 2$,
- if $n \leq 7$ and $s \in (s_0, 1)$, for some $s_0 \in (0, 1)$.

It has still to be determined whether or not s -minimal surfaces are smooth in dimension $n \leq 7$ for s not close to 1 (say, s smaller than the above mention s_0 , which, in principle, could be very close to 1), and in dimension $n \geq 8$. In particular, it is still not known an example of singular s -minimal cone, not even in very high dimension. It is also an open problem to establish whether s -minimal surfaces are analytic.

The regularity of s -minimal surface is also related to the flatness of s -minimal graphs. Namely, we say that the s -Bernstein property holds true in \mathbb{R}^n if all the s -minimal graphs in \mathbb{R}^n are necessarily affine. In [FV17], a general result is given which states that if there are no singular s -minimal cones in dimension $n - 1$, then the s -Bernstein property holds true in \mathbb{R}^n . From this one obtains that the s -Bernstein property holds true in \mathbb{R}^n at least in two cases:

- if $n \leq 3$,
- if $n \leq 8$ and $s \in (s_0, 1)$, for some $s_0 \in (0, 1)$.

Once again, the general picture remains rather mysterious, namely it is not known whether or not the s -Bernstein property holds true in \mathbb{R}^n when $n \leq 8$ and $s \leq s_0$, and when $n \geq 9$, hence we hope that the work of Alessio will also stimulate new results in these directions.

We observe that the s -perimeter defined in (1) can be also considered in the case $\Omega = \mathbb{R}^n$, in which case one simply has that

$$(5) \quad \text{Per}_s(E, \mathbb{R}^n) = I_s(E, E^c).$$

A well-investigated question in this setting is the isoperimetric problem, consisting in detecting the minimizers of the functional in (5) for a prescribed volume. As in the classical case, these minimizers turn out to be balls, namely, given the scale-invariance of the s -perimeter

$$(6) \quad \frac{\text{Per}_s(E, \mathbb{R}^n)}{|E|^{\frac{n-s}{n}}} \geq \frac{\text{Per}_s(B_1, \mathbb{R}^n)}{|B_1|^{\frac{n-s}{n}}},$$

being B_1 the unit ball of \mathbb{R}^n .

In [FFM⁺15], a number of important questions related to the s -isoperimetric inequality in (6) are addressed. First of all, a “stable” version of (6) is obtained, stating that if a set “almost attains” the minimal possible value in the fractional isoperimetric ratio, then it must be necessarily “almost a ball”. More precisely, one considers the so-called Fraenkel asymmetry of a set E , which measures the L^1 -distance of E from the set of balls of volume $|E|$ and is defined by

$$A(E) := \inf_{x \in \mathbb{R}^n} \frac{|E \Delta B_{r_E}(x)|}{|E|},$$

where $r_E > 0$ is such that $|B_{r_E}| = |E|$. In this setting, it is shown in [FFM+15] that for any $s_0 \in (0, 1)$ and any $s \in [s_0, 1)$ there exists a positive constant $C(n, s_0)$ such that

$$(7) \quad \frac{\text{Per}_s(E, \mathbb{R}^n)}{|E|^{\frac{n-s}{n}}} \geq \frac{\text{Per}_s(B_1, \mathbb{R}^n)}{|B_1|^{\frac{n-s}{n}}} \left(1 + \frac{A^2(E)}{C(n, s_0)} \right).$$

Of course, (6) is now a particular case of (7). Also, the result in [FFM+15] carries on to the case $s \nearrow 1$. As usual, the case $s \searrow 0$ is more tricky, and it is conjectured in [FFM+15] that

$$C(n, s_0) \sim \frac{1}{s_0} \quad \text{as } s_0 \searrow 0.$$

The second variation of the fractional perimeter has been also computed in [FFM+15]: this formula can be considered as a nonlocal counterpart of the classical Jacobi equation, in which the classical Laplace-Beltrami operator is replaced by an integral operator along the boundary of the domain, namely (up to normalization constants) an operator (acting on a given function f) of the form

$$\int_{\partial E} \frac{f(x) - f(y)}{|x - y|^{n+s}} d\mathcal{H}^{n-1}(y),$$

and the norm of the second fundamental form is replaced by a weighted L^2 -norm of the normal ν , such as

$$\int_{\partial E} \frac{|\nu(x) - \nu(y)|^2}{|x - y|^{n+s}} d\mathcal{H}^{n-1}(y).$$

Once again, the nonlocal problems reveal an intrinsic geometric structure which can be related to

the classical objects in the limit as $s \nearrow 1$.

Moreover, in [FFM+15] several variational problems in which the “aggregating” effect of the fractional perimeter is compensated by a “disaggregating” term are also considered, with special attention to the case given by Riesz potential E . In this framework, minimizers with small volume are necessarily balls, and the case of volumes which are not necessarily small demands further investigation.

Interestingly, minimizers of the s -perimeter in \mathbb{R}^n for a fixed volume satisfy an Euler-Lagrange equation which can be considered as a prescribed s -mean curvature equation. Namely, for any $x \in \partial E$, one can consider the s -mean curvature of E at x , defined by

$$\mathcal{H}_E^s(x) := s(1-s) \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+s}} dy,$$

where, as customary, $\chi_A : \mathbb{R}^n \rightarrow \{0, 1\}$ denotes the characteristic function of the set A . By symmetry, one can easily see that the s -mean curvature of a ball of radius R is constant along its boundary, and, by scaling, it is equal to a constant depending on n and s divided by R^s .

As $s \nearrow 1$, the s -mean curvature approaches the classical mean curvature.

One can show that if E is a minimizer for the s -perimeter in \mathbb{R}^n for a fixed volume, then its s -mean curvature is constant along ∂E . A natural question in this setting is to determine the shape of the sets which possess constant s -mean curvature along their boundaries. In the classical setting, this was a classical result due to the famous Russian mathematician (and mountaineer) Aleksandr Danilovich Aleksandrov, stating that a smooth, connected and closed hypersurface with constant mean curvature is necessarily a sphere (hence, soap bubbles are round).

In [CFMN18] the nonlocal counterpart of the Aleksandrov’s result is obtained, proving that if a bounded open set with smooth boundary has constant s -mean curvature, then it is necessarily a sphere. It is interesting to remark that the nonlocal version of such a result is somehow stronger than the classical case, since the set is not assumed to be

connected. This shows one of the special features of the nonlocal environment, in which remote interactions give significant contributions to the problem, and, in this case, they rule out the possibility of multiple connected components (for instance, two disconnected balls do not have constant s -mean curvature). In other words, the nonlocal setting, in this case, turns out to be much more rigid than the classical one, since, even without any connectedness assumption, a set with constant s -mean curvature is a single sphere, whereas of course any disjoint union of balls with equal radii has constant mean curvature in the classical sense.

Quantitative formulations of the Aleksandrov’s result are also provided in [CFMN18]. In particular, it is shown that bounded sets with almost-constant s -mean curvature are necessarily close to a single ball, and moreover the Lipschitz constant of the s -mean curvature controls the C^2 -distance from a single sphere.

Once again, these results reveal some special features of the nonlocal universe. Indeed, while in the classical case a connected boundary with almost-constant mean curvature may be close to a compound of nearby spheres of equal radii, the nonlocal case turns out to be more rigid and the quantitative results for almost-constant s -mean curvature sets are obtained without the need of imposing any extra geometric constraint. This also points out an interesting feature of the nonlocal case, which prevents bubbling phenomena.

Needless to say, the contributions of Alessio in this field have been pivotal also to trigger new research related to sets of constant (and, in particular, zero) s -mean curvature, and to a number of evolution problems of geometric type (e.g., the ones in which a sets evolves with normal velocity given by its s -mean curvature).

As customary, Alessio was an avant-garde investigator of nonlocal geometric and variational problems. His results, and his research style, will certainly leave an indelible footprint in the international scenario, and the future research will certainly count on his extraordinary talent to solve new questions, open new lines of research and expand knowledge way beyond the present frontiers.

References

- [BFV14] BEGOÑA BARRIOS, ALESSIO FIGALLI, and ENRICO VALDINOCI, *Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **13** (2014), no. 3, 609-639. MR3331523
- [CRS10] L. CAFFARELLI, J.-M. ROQUEJOFFRE, and O. SAVIN, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. **63** (2010), no. 9, 1111-1144, DOI 10.1002/cpa.20331. MR2675483
- [CFMN18] GIULIO CIRAULO, ALESSIO FIGALLI, FRANCESCO MAGGI, and MATTEO NOVAGA, *Rigidity and sharp stability estimates for hypersurfaces with constant and almost-constant nonlocal mean curvature*, J. Reine Angew. Math. **741** (2018), 275-294, DOI 10.1515/crelle-2015-0088. MR3836150
- [CF17] MATTEO COZZI and ALESSIO FIGALLI, *Regularity theory for local and nonlocal minimal surfaces: an overview*, Nonlocal and nonlinear dislocations and interactions: new methods and directions, Lecture Notes in Math., vol. 2186, Springer, Cham, 2017, pp. 117-158. MR3588123
- [DFPV13] SERENA DIPIERRO, ALESSIO FIGALLI, GIAMPIERO PALATUCCI, and ENRICO VALDINOCI, *Asymptotics of the s -perimeter as $s \searrow 0$* , Discrete Contin. Dyn. Syst. **33** (2013), no. 7, 2777-2790, DOI 10.3934/dcds.2013.33.2777. MR3007726
- [FFM⁺15] A. FIGALLI, N. FUSCO, F. MAGGI, V. MILLOT, and M. Morini, *Isoperimetry and stability properties of balls with respect to nonlocal energies*, Comm. Math. Phys. **336** (2015), no. 1, 441-507, DOI 10.1007/s00220-014-2244-1. MR3322379
- [FV17] ALESSIO FIGALLI and ENRICO VALDINOCI, *Regularity and Bernstein-type results for nonlocal minimal surfaces*, J. Reine Angew. Math. **729** (2017), 263-273, DOI 10.1515/crelle-2015-0006. MR3680376



Serena Dipierro

Serena Dipierro received her PhD from SISSA (Trieste) in 2012, and she is currently Associate Professor at the Department of Mathematics and Statistics of the University of Western Australia. Her research interests include partial differential equations, free boundary problems, nonlinear analysis, nonlocal equations, and calculus of variations.



Enrico Valdinoci

Enrico Valdinoci received his PhD from the University of Texas at Austin in 2002, and he is currently Professor of Mathematics at the Department of Mathematics and Statistics of the University of Western Australia. His research interests include partial differential equations, nonlocal equations, calculus of variations, and dynamical systems.