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Understanding singularities in free boundary problems

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Sommario: *I problemi di frontiera libera sono quelli descritti da EDP che mostrano interfacce o frontiere a priori sconosciuti (liberi). L'esempio più classico è lo scioglimento del ghiaccio in acqua (problema di Stefan). In questo caso, la frontiera libera è l'interfaccia solido-liquido tra acqua e ghiaccio. Una sfida matematica centrale in questo contesto è comprendere la regolarità e le singolarità delle frontiere libere. In questo articolo introduciamo questo argomento presentando alcuni risultati classici di Luis Caffarelli, oltre ad alcuni importanti lavori recenti dovuti ad Alessio Figalli e collaboratori.*

Abstract: *Free boundary problems are those described by PDEs that exhibit a priori unknown (free) interfaces or boundaries. The most classical example is the melting of ice to water (the Stefan problem). In this case, the free boundary is the liquid-solid interface between ice and water. A central mathematical challenge in this context is to understand the regularity and singularities of free boundaries. In this paper we provide a gentle introduction to this topic by presenting some classical results of Luis Caffarelli, as well as some important recent works due to Alessio Figalli and collaborators.*

1. – Introduction

1.1 – *The Stefan problem*

The Stefan problem, dating back to the XIXth century, is the most classical and important *free boundary problem*.

First considered by Lamé and Clapeyron in 1831, aims to describe the temperature distribution in a homogeneous medium undergoing a phase change, typically a body of ice at zero degrees centigrade submerged in water. It is named after Josef Stefan, a Slovenian physicist who introduced the general class of such problems around 1890; see [52, 53, 37].

The most classical formulation of the Stefan problem is as follows: Let $\Omega \subset \mathbb{R}^3$ be some bounded domain. For concreteness, we let us think that Ω is cylindric water tank as depicted in Figure 1. We denote

$$\theta = \theta(x, t)$$

the temperature of the water at the point $x \in \Omega$ at time $t \in \mathbb{R}^+ := [0, +\infty)$. We assume that $\theta \geq 0$ in $\Omega \times \mathbb{R}^+$.

Given are the (nonnegative) initial temperature and temperature at the boundary of the tank.

The set $\{(x, t) \in \Omega \times \mathbb{R}^+ : \theta(x, t) > 0\}$, denoted for brevity $\{\theta > 0\}$, represents the water while its

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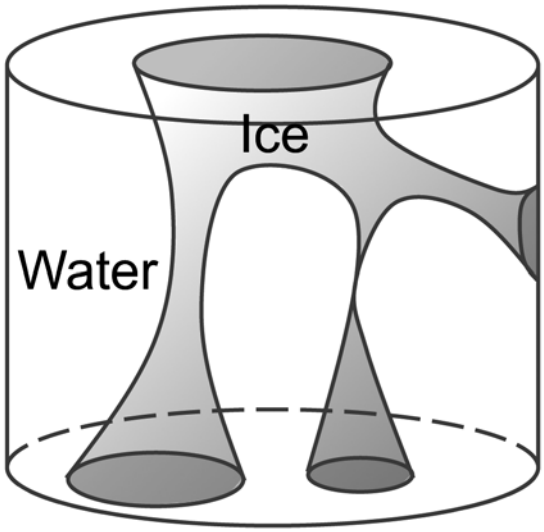

Fig. 1. – The Stefan problem.

complement, denoted $\{\theta = 0\}$, represents the ice. The temperature θ satisfies the heat equation

$$\partial_t\theta - \Delta\theta = 0 \quad \text{in the region } \{\theta > 0\},$$

while in the complement θ is just zero.

Determining where is the interphase or *free boundary* that separates the two regions (i.e., the surface $\partial\{\theta > 0\}$) is part of the problem. For it, an extra equation (or boundary condition) on the interface is needed. This is the so-called Stefan condition:

$$\partial_t\theta = |\nabla_x\theta|^2 \quad \text{on} \quad \partial\{\theta > 0\}. \tag{1.1}$$

This extra relation comes from two considerations.

First, the normal velocity of the interphase, V, is proportional to the amount of heat absorbed by it (which must be used to melt the ice). In turn, this heat which "enters" the interphase is, by Fourier law, proportional to the gradient of temperature. Thus, we have $|V| = C|\nabla\theta|$. Second, since $\theta = 0$ on the moving interphase we obtain that, on it, V and $\nabla\theta$ are parallel and $(\partial_t + V \cdot \nabla)\theta = 0$. Combining the two previous informations and choosing the physical units to make $C = 1$, we obtain the Stefan condition (1.1).

One can also see that, by the "maximum principle", the ice $\{\theta = 0\}$ shrinks with time. In other words, if at some point of the tank there is liquid water for some given time then that point will remain liquid at all future times.

It can be shown that, after the transformation

$$u(x,t) := \int_0^t \theta(x,\tau)d\tau,$$

(see [2, 18]) the new function

$$u : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$$

satisfies

$$\begin{aligned} \partial_t u - \Delta u &= -\chi_{\{u>0\}} \\ u &\geq 0 \\ \partial_t u &\geq 0, \end{aligned} \tag{1.2}$$

where χ_A denotes the characteristic function of the set A.

Since we can easily recover θ from u by computing its time derivative, we see that (1.2) is an equivalent formulation of the Stefan problem. The new formulation is useful because it enjoys better mathematical properties (it has the structure of "variational inequality") than the original formulation with θ. For instance, while from the original formulation with θ it is unclear how to show existence and uniqueness of solution (there was no rigurous proof for more than a century!), it is much easier to do it with the equivalent formulation (1.2).

The stationary version of (1.2) is the well-known *obstacle problem*:

$$\begin{aligned} \Delta u &= \chi_{\{u>0\}} \\ u &\geq 0. \end{aligned} \tag{1.3}$$

It is among the most famous problems in elliptic PDE, as it arises in a variety of situations.

1.2 – *Motivations and applications*

Both the Stefan problem and the obstacle problem appear in many different models in physics, industry, biology, or finance. We next briefly comment on some of them, and refer to the books [19, 34, 43, 27, 42] for more details and further applications of obstacle-type problems.

- **Phase transitions**. In the classical Stefan problem, as explained in the previous subsection, the solution u of (1.2) is the integral of the temperature of a solid undergoing a phase transition, such as ice melting to water.

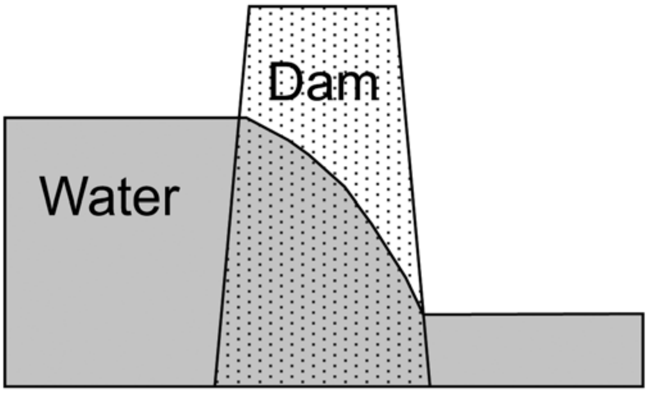

Fig. 2. – The Dam problem.

- **Fluid filtration**. The so-called Dam problem aims to describe the filtration of water inside a porous dam. One considers a porous dam separating two reservoirs of water at two different heights; see Figure 2. Then, the interior of the dam has a wet part, where water flows, and a dry part. In this context, an integral of the pressure solves the obstacle problem (1.3), and the free boundary corresponds to the interface between the wet and dry parts of the dam.
- **Hele-Shaw flow**. This model, dating back to 1898, describes a fluid flow between two flat parallel plates separated by a very thin gap. Various problems in fluid mechanics can be approximated to Hele-Shaw flows, and that is why understanding these flows is important.

  A Hele-Shaw cell (see Figure 3) is an experimental device in which a viscous fluid is sandwiched in a narrow gap between two parallel plates. In certain regions, the gap is filled with fluid while in others the gap is filled with air. When liquid is injected inside the device though some sinks (e.g. though a small hole on the top plate) the region filled with liquid grows. In this context, an integral of the pressure solves, for each fixed time t, the obstacle problem (1.3). Similarly as in the Dam problem, the free boundary corresponds to the interface between the fluid and the air regions.

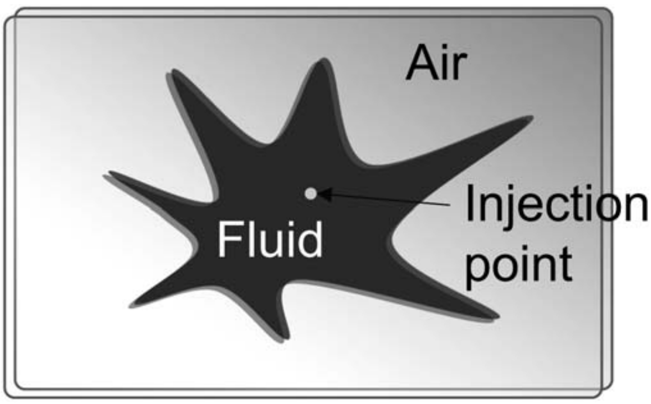

Fig. 3. – A Helle-Shaw cell.

- **Optimal stopping, finance**. In probability and finance, both the Stefan problem (1.2) and the obstacle problem (1.3) appear when considering optimal stopping problems for stochastic processes.

  A typical example is the Black-Scholes model for pricing of American options. An American option is a contract that entitles its owner to buy some financial asset (typically a share of some company) at some specified price (the"strike price") at any time before some specified date (the "maturity date"'). This option has some value, since in case that the always fluctuating market price of the asset goes higher than the strike price then the option can be "exercised" to buy the asset at the lower price.

  The Black-Sholes model aims to calculate the rational price $u = u(x, t)$ of an option at any time t prior to the maturity date and depending on the price $x \in \mathbb{R}^+$ of the financial asset. Since the option can be exercised at any time before maturity, determining the "exercise region", i.e. the pairs (x, t) for which it is better to exercise the option, is a part of the problem. Interestingly, this problem leads to a Stefan problem of the type (1.2), and the free boundary corresponds to the boundary of the exercise region.
- **Interacting particle systems**. Large systems of interacting particles arise in physical, biological, or material sciences.

  In some some models the particles attract each other when they are far, but experience a repulsive force when they are close [14]. In other related models in statistical mechanics, the particles (e.g. electrons) repel with a Coulomb force and one wants to understand their behaviour in presence of some external field that confines them [48].

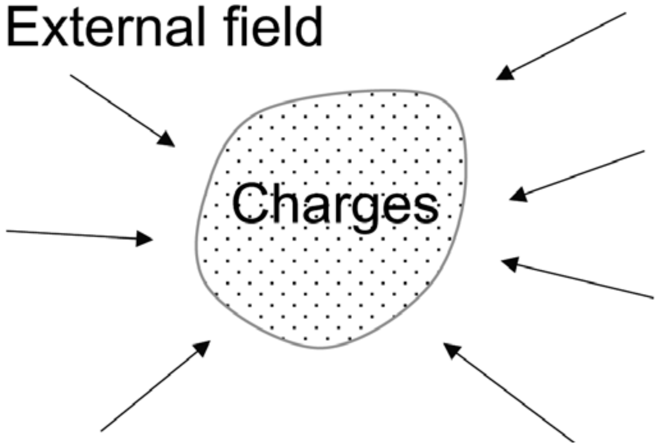

Fig. 4. – The equilibrium configuration for a Coulomb system.

In the previous models, a natural and interesting question is to determine the "equilibrium configurations". For instance, in Coulomb systems the charges accumulate in some region with a well definite boundary; see Figure 4. Interestingly, this problems are equivalent to obstacle problems of the type (1.3) — for instance the electric potential $u = u(x)$ generated by the charges solves a problem like (1.3) and free boundary corresponds to the boundary of the region in which the particles concentrate.

- **Elasticity**. Let us consider the equilibrium position $v(x)$ of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle $\varphi(x)$.

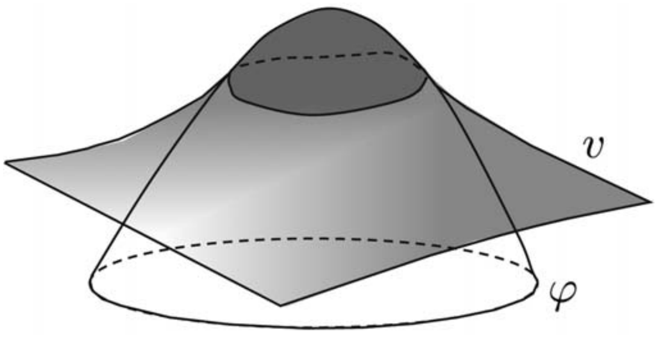

Fig. 5. – An elastic membrane above an obstacle φ.

In the region where the membrane is above the obstacle φ, the solution v solves a PDE (i.e., $\Delta v = 0$ in $\{v > \varphi\}$), while in the other region the membrane coincides with the obstacle (i.e., $v = \varphi$). By considering the function $u := v - \varphi \geq 0$, we are led to an obstacle problem of the type (1.3).

2. – Regularity of free boundaries

From the mathematical point of view, a central question in the Stefan problem (1.2) and the obstacle problem (1.3) is to understand the *regularity of free boundaries* [12, 42].

For example, in the Stefan problem: is there a regularization mechanism that smoothes out the free boundary, independently of the initial data? (Notice that a priori the free boundary could be a very irregular set, even a fractal set!) Such type of questions are usually very hard, and even in the simplest cases almost nothing was known before the 1970s. The development of the regularity theory for free boundaries started in the late seventies, and since then it has been a very active area of research.

2.1 – *Some examples of Schaeffer*

The first thing one might try is to construct some explicit solutions to the obstacle problem, and see how their free boundaries behave. What one finds is that, in most simple cases, free boundaries seem to be very smooth.

It was Schaeffer who first realized that with a bit more effort one can actually construct several free boundaries with some singularities. Namely, he constructed different solutions to the obstacle problem in R^2 in which the free boundary has a cusp.

These examples are actually constructed by using complex analysis, and the cusps are represented by the curves

$$x_2 = \pm x_1^{2k+\frac{1}{2}}, \quad 0 \leq x_1 \leq 1.$$

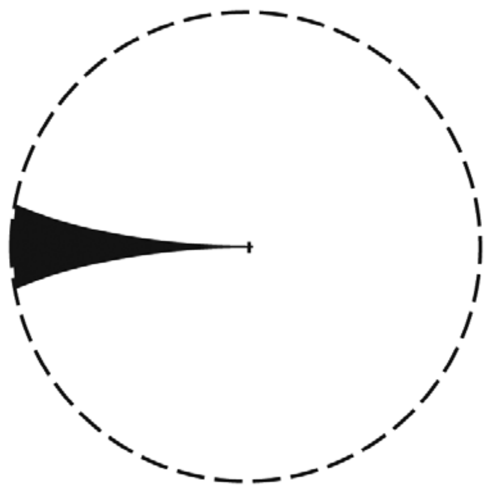

Fig. 6. – A one-sided cusp.

The set $\{u > 0\}$ is actually the image of $\{|z| \leq 1,\ \mathrm{Im}\, z > 0\}$ under the conformal mapping $f(z) = z^2 + i\, z^{4k+1}$, and u satisfies near the origin

$$u(z) \approx \frac{x_2^2}{2} + c_k \,\mathrm{Im}\big(z^{2k+\frac{3}{2}}\big) + ...,$$

where $z = x_1 + ix_2$.

Another type of singularities (a two-sided cusp) was also constructed by Schaeffer.

In this case, these two-sided cusps are represented by the curves

$$x_2 = \pm\, |x_1|^{2k}, \quad -1 \leq x_1 \leq 1.$$

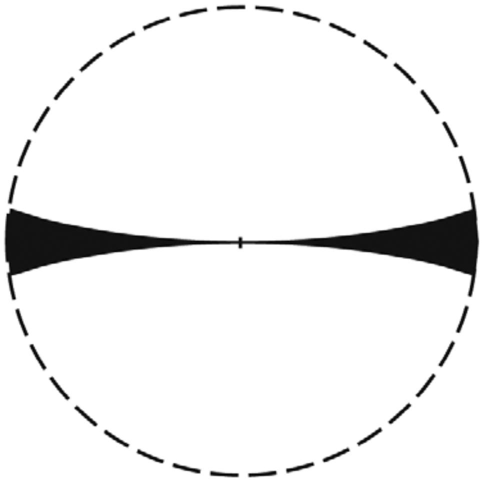

Fig. 7. – A two-sided cusp.

By considering slightly more general obstacle problems of the type (1.3) in $\mathbb{R}^2$, Schaeffer noticed that it is even possible to construct examples in which the free boundary has infinitely many cusps.

Fig. 8. – The free boundary could even create an infinite number of cusps.

2.2 – *The breakthrough of Caffarelli*

Despite all these examples, before the late 1970s there was no general regularity result for free boundaries. A natural guess, given these examples, could be that free boundaries are smooth outside a certain set of "singular points". However, no result of this type was known, and this seemed to be an extremely challenging open problem.

This changed in 1977 with the groundbreaking paper of Luis Caffarelli [6]. Such work developed for the first time the regularity theory for free boundaries in both the obstacle problem (1.3) and the Stefan problem (1.2).

The main results from [6] (see also [33]) may be summarized as follows:

- The free boundary splits into *regular* points and *singular* points.
- The set of *regular points* is an open subset of the free boundary and it is C^∞.
- *Singular points* x_0 can be characterized as those at which the set "contact set" $\{u = 0\}$ has density zero (as in a cusp), i.e.

$$\lim_{r\to 0} \frac{|\{u = 0\} \cap B_r(x_0)|}{|B_r(x_0)|} = 0. \tag{2.1}$$

In other words:

> *The free boundary is smooth, outside of a certain set of cusp-like singularities.*

This is one of the main results for which Caffarelli received the Wolf Prize in 2012 and the Shaw Prize in 2018.

2.3 – *Blow-ups*

To prove such regularity result, one considers *blow-ups*. This is a key idea that is common in many problems in PDEs and Geometric Analysis.

Given a free boundary point x_0, one first considers the rescaled functions

$$u_r(x) := \frac{u(x_0 + rx)}{r^2},$$

for $r \in (0, 1)$. Notice that, when taking $r > 0$ smaller and smaller, we are zooming in the solution u around the point x_0. Caffarelli showed that for any free boundary point x_0 and $r > 0$ one has

$$cr^2 \leq \sup_{B_r(x_0)} u \leq Cr^2$$

where c and C are positive constants. Thus, the rescaling factor r^{-2} is just taken so that the functions u_r satisfy $c \leq \max_{B_1} u_r \leq C$.

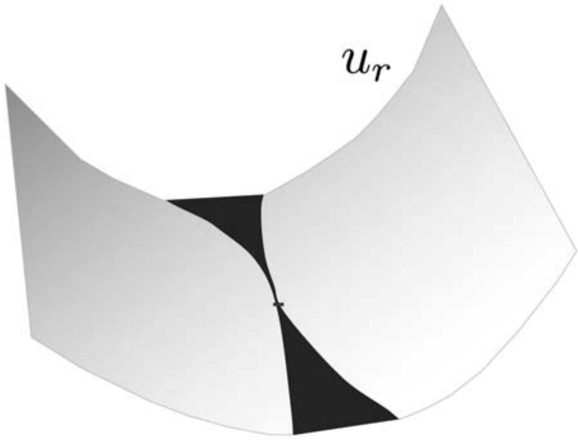

Fig. 9. – The rescaled function u_r at a singular point.

Then, the idea is to take the limit $r \downarrow 0$, and prove that

$$u_r(x) := \frac{u(x_0 + rx)}{r^2} \xrightarrow{r \downarrow 0} u_0(x)$$

for some function u_0 which is a *global* solution of the obstacle problem in the entire space. The function u_0 is called a blow-up of u at x_0.

Roughly speaking, the function u_0 should give us information about how the solution u looks like at the point x_0.

The main difficulty is actually to *classify blow-ups*, i.e., to show that either

(a) u_0 is 1D solution of the form

$$u_0(x) = (x \cdot e)_+^2,$$

where $e \in S^{n-1}$ is some unit vector;

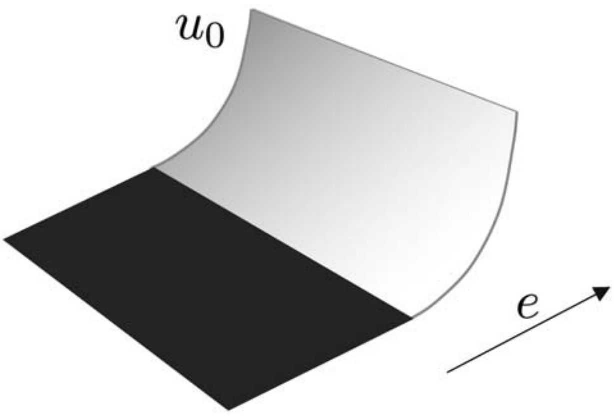

Fig. 10. – A 1D solution u_0.

(b) u_0 is a quadratic polynomial of the form

$$u_0(x) = \frac{1}{2} x^T A x,$$

where $A \geq 0$ is any positive semidefinite matrix satisfying $\mathrm{tr}(A) = 1$.

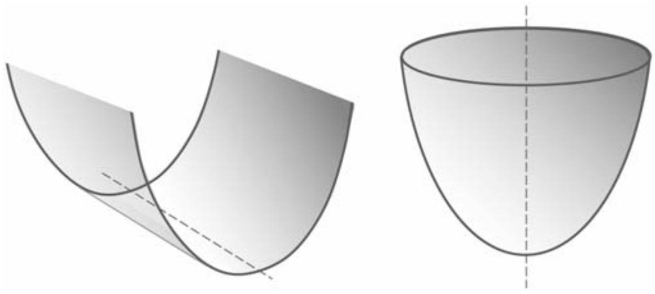

Fig. 11. – A quadratic polynomial u_0.

Notice that, after the blow-up, the contact set $\{u_0 = 0\}$ becomes either a half-space –in the case (a)– or it is a linear proper sub-space of $\mathbb{R}^n$ and hence it has zero measure –in the case (b). The first case is what we expect at regular points, while the second is what we expect at singular points. This intuition is not only correct but actually is translated into a rigorous mathematical definition: we say that a free boundary point x_0 is *regular* if the blow-up u_0 at x_0 is of the type (a) and we say that it is *singular* if the blow-up u_0 at x_0 is of the type (b).

To complete the prove of Caffarelli's theorem, one has to "transfer information" from u_0 to u, and show that if x_0 was a regular point, then the free boundary is C^1 in a neighborhood of x_0. This is delicate and is another of the main achievements in [6]. On the other hand, the fact that $\{u = 0\}$ satisfies 2.1 at singular points is a more immediate consequence of the definition of singular point and follows easily using the fact that the convergence of u_r to u_0 is uniform and the that $u_0 > 0$ outside of a proper linear sub-space of $\mathbb{R}^n$. We refer to [7, 8] or [42] for more details.

3. – Understanding singularities

After the results of Caffarelli [6], a natural question is to understand better the set of singular points.

The fine understanding of singularities is actually a central research topic in a number of areas related to nonlinear PDE and Geometric Analysis. In the present context, it is natural to ask:

- How large may the singular set be?
- In case it is a large set, does it enjoy some nice regularity properties?

3.1 – *The 2D case*

The first results in this direction were obtained by L. Caffarelli and N. Riviere in two dimensions [11]. They proved that, at every singular point x_0, any solution u to the obstacle problem in $\mathbb{R}^2$ satisfies

$$u(x) = p_2(x) + o\big(|x - x_0|^2\big), \tag{3.1}$$

where p_2 is a quadratic polynomial satisfying $p_2 \geq 0$ and $\Delta p_2 = 1$.

Moreover, this implies that the singular set is contained in a C^1 curve.

It took a long time before these results could be improved. The next important result in this direction was obtained by Sakai in 1991 [44, 45], who proved that the isolated cusps of Schaeffer (described above) are essentially the only ones that may appear for the obstacle problem (1.3) in $\mathbb{R}^2$. This nice and sharp result gives a complete picture for the obstacle problem in the plane, and its proof uses crucially tools from complex analysis.

3.2 – *Higher dimensions*

In dimensions $n \geq 3$, where complex analysis is of no use, the first results on the singular set were established by Caffarelli in 1998 [8] (see also [40]). He proved that, if u is any solution to the obstacle problem (1.3) in $\mathbb{R}^n$, then (3.1) holds at every singular point x_0.

Moreover, this implies that the singular set is contained in a C^1 manifold of dimension $(n-1)$.

For almost two decades, this was the best known result for the singular set in dimensions $n \geq 3$. Still, it was an open question to understand whether (3.1) can be improved or not.

In two dimensions, because of the results of Sakai, at every singular free boundary point we have that

$$u(x) = p_2(x) + O\big(|x - x_0|^3\big). \tag{3.2}$$

It is however important to notice that Sakai's methods, based on complex anaysis, cannot work in higher dimensions (nor for the Stefan problem). Thus, improving (3.1) in dimensions $n \geq 3$ would require completely different ideas.

The first new result in this direction for dimensions $n \geq 3$ was established by Colombo, Spolaor, and Velichkov in 2017 [15], by improving and refining the methods of Weiss [54].

They proved that at every singular point the expansion (3.1) holds with an additional logarithmic modulus of continuity on the term $o\big(|x - x_0|^2\big)$.

Independently and with different methods, Figalli and the second author proved in [23] the following result for the obstacle problem in $\mathbb{R}^n$:

THEOREM 1 ([23]). – *Outside a small set of Hausdorff dimension $n-3$,* [3.2] *holds.*

In other words, in higher dimensions, [3.2] holds at *most* singular points x_0.

It is important to remark that the fine description of u [3.2] actually gives a fine description of the cusp at x_0. Namely, it follows from the results of [23] that, at most singular points x_0, (possibly after a rotation) near x_0 we have

$$\{u = 0\} \subseteq \{|x_n| \leq C|x'|^2\},$$

where $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$.

Furthermore, it was shown in [23] that the result is sharp, in the sense that there exist (isolated) singular points in $\mathbb{R}^3$ at which (3.2) fails. In fact, at these points, the logarithmic modulus of continuity of [15] cannot be improved.

This gives a very complete picture of singularities for the obstacle problem in all dimensions.

3.3 – *The Stefan problem*

As explained above, a major question in PDE problems where singularities appear is to establish estimates for the *size* of the singular set.

Some famous results in this direction include those on minimal surfaces [50, 1], on the Navier-Stokes equations [10], and on mean curvature flow [56].

In the Stefan problem (1.2), the same techniques used in the study of the (elliptic) obstacle problem yield that for problem (1.2) the singular set Σ_t is $(n-1)$-dimensional for every t; see [55, 5, 38]. This result is optimal in space, in the sense that one can construct examples in which the singular set contains some $(n-1)$-dimensional hypersurface (for instance a sphere) for some fixed time $t = t_0$. In this respect it would seem that the set of singular points can be as large as the regular points, which is always $(n-1)$-dimensional. However, one could still hope that, jointly in space and time, the size of singular set should be in some sense much smaller than the set of regular points!

For instance, there are no examples in which singular points appear for every time t, but it is (or was) not clear a priori if this could happen or not. In case this cannot happen, the next question is to establish estimates on the size of the set of *singular times*. Namely, if we let Σ_t be the set of singular points at time t, then we may denote

$$\mathcal{S} = \{t : \Sigma_t \neq \varnothing\}$$

the set of all singular times (i.e., the set of all times at which singularities appear).

The following result was announced at the ICM lecture of Alessio Figalli [22], and proved in the paper [24]:

THEOREM 2 ([24]). – *Let $u(x,t)$ be any solution to the Stefan problem* (1.2) *in $\mathbb{R}^3$. Assume that*

$$u_t > 0 \quad \text{in} \quad \{u > 0\}.$$

Then,

$$\dim_{\mathcal{H}}(\mathcal{S}) \leq \frac{1}{2}.$$

Here, $\dim_{\mathcal{H}}(\mathcal{S})$ denotes the Hausdorff dimension of the set $\mathcal{S}$.

In particular, the previous result implies that, for the Stefan problem in $\mathbb{R}^3$:

The free boundary is C^∞ for almost every time t.

This is the first result on the size of the set of singular times for the Stefan problem.

The result is stated for simplicity in the physical dimension $n = 3$, but [24] actually establishes new results for the Stefan problem in $\mathbb{R}^n$ for any dimension $n \geq 2$.

As said above, prior to this result it was not even known if solutions to the Stefan problem (1.2) in $\mathbb{R}^3$ could have singularities for *all* times t.

It is not known whether the dimension bound $\frac{1}{2}$ is optimal or not. What is clear from the proofs is that such exponent is *critical* in several ways. This is somewhat similar (even though the results and proofs are very different) to what happens in the Navier-Stokes equations; see the classical result of Caffarelli, Kohn, and Nirenberg [10].

4. – Generic regularity

A second major question in the understanding of singularities is the development of methods to prove *generic regularity* results. (1) This is one of the big challenges in contemporary PDE theory.

Indeed, in PDE problems in which singularities may appear, it is important to understand whether these singularities appear "often", or if instead "most" solutions have no singularities.

In the context of the obstacle problem (1.3), the key question is to understand the generic regularity of free boundaries. As we saw earlier, explicit examples show that singular points in the obstacle problem can form a very large set, of dimension $n-1$ (as large as the regular set). Still, singular points are expected to be rare ([46]):

CONJECTURE 4.1 — (Schaeffer, 1974):
Generically, the weak solution of the obstacle problem is also a strong solution, in the sense that the free boundary is a C^∞ manifold.

(1) Here, by generic regularity we mean that there is an open and dense set of boundary datums for which the corresponding solution has no singularities.

In other words, the conjecture states that, generically, the free boundary has *no* singular points.

The conjecture was only known to hold in the plane $\mathbb{R}^2$ [40], and until very recently nothing was known in $\mathbb{R}^3$ or in higher dimensions.

Notice that in the obstacle problem the question of generic regularity is particularly relevant, since the singular set can be as large as the regular set — while in other problems the singular set has lower Hausdorff dimension [31]. Also, from the point of view of applications, it is particularly relevant to understand the problem in the physical space $\mathbb{R}^3$.

A key result established by Figalli and the authors in [24] is the following:

THEOREM 3 ([24]). – *Schaeffer's conjecture holds in* $\mathbb{R}^3$.

We remark that very few results are known in this direction in elliptic PDE, and most of them deal only with the cases in which the singular set is known to be very small (e.g. the obstacle problem in $\mathbb{R}^2$ [40], or area-minimizing hypersurfaces in $\mathbb{R}^8$ [51]).

Due to the general character of the proofs in [24], one can even apply these results to the *Hele-Shaw flow* (explained above).

COROLLARY 4 ([24]). *Let* $u(x,t)$ *be any solution to the Hele-Shaw flow in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$.

Then, the free boundary is C^∞ *for almost every time* t.

Such result is completely new even in the 2D case (the most relevant one for this model). Moreover, as in the Stefan problem, [24] establishes a new bound on the size of the singular set: for the Hele-Shaw flow in $\mathbb{R}^2$, the set of singular times has Hausdorff dimension at most $\frac{1}{4}$.

5. – The fractional obstacle problem

5.1 – *Optimal stopping and Finance*

As explained in subsection 1.2, a nice motivation of the study of obstacle problems comes from Probability and Finance. In particular they arise in pricing of American options. Let us explain it in more detail next.

We recall that an American option entitles its owner with the possibility to buy a given asset at a fixed price at any time before the maturity date. In the Black-Scholes model, the (logarithmic) price of the underlying asset X_t is modelled as a Wiener process (or Brownian motion) with a drift parameter that recreates long-term growth and a variance parameter that matches the volatility of the asset. Under this assumption, as said in subsection 1.2, the rational price of the option solves a parabolic obstacle problem like (1.2). However, in finance price fluctuations are often better modelled by more general stochastic processes X_t: Lévy processes (see for instance [16]). These are stochastic processes with jumps, and were introduced in option pricing models by the Nobel Prize winner R. Merton [39] in the 1970s.

Under this more general assumption, the rational price of an American option still solves an obstacle problem similar to (1.2) but where the Lapacian is replaced by some other operator (actually the so-called infinitesimal generator of the process X_t, which is an elliptic operator of integro-differential type). The most important and canonical example of a Lévy process (other than the Brownian motion) corresponds to the case in which the law of X_t is rotationally invariant and satisfies a scaling property. In such case, what we get is the obstacle problem for the *fractional Laplacian*

$$(-\Delta)^s u(x) := c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}}\, dy,$$

with $s \in (0,1)$.

The obstacle problem for the fractional Laplacian is often called the fractional obstacle problem. Because of its connection to Probability and Finance, in the last decade there have been considerable efforts to understand the fractional obstacle problem (both the stationary and the evolutionary problem).

5.2 – *Known results*

The main questions in this context are:

- What is the optimal regularity of solutions?
- Can one prove the regularity of free boundaries?

The first results in this direction were obtained by Silvestre in [49] in the stationary setting, who established the almost-optimal regularity of solutions, $u \in C^{1,s-\varepsilon}$ for all $\varepsilon > 0$. The optimal regularity of solutions was established later by Caffarelli, Salsa, and Silvestre [13]:

Solutions u are $C^{1,s}$.

Furthermore, they also established the following:

The free boundary is smooth, possibly outside a certain set of degenerate points.

More precisely, they proved that if u solves the obstacle problem for the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$, then $u \in C^{1,s}$, and for every free boundary point x_0 one has a dichotomy: either the solution u is degenerate at x_0, or the free boundary is smooth around such point.

After the results of [13], many more results have been obtained for the (stationary) fractional obstacle problem: the higher regularity of free boundaries [35, 17, 36, 32], the study of singular points [28, 3, 25, 30, 20, 26], or the case of operators with drift [41, 29, 21].

Still, there was one question that was much less understood: what happens in the evolutionary setting?

5.3 – *The parabolic case*

Despite many known results for the fractional obstacle problem in the stationary setting, much less was known in the parabolic case, in which the solution evolves with time.

The first result in this direction was established by L. Caffarelli and A. Figalli in [9], where they established the optimal regularity of solutions:

THEOREM 5 ([9]). – *Let $u(x,t)$ be any solution to the parabolic fractional obstacle problem.*

Then, u is $C^{1,s}$ in x.

The question of free boundary regularity was still open for some years.

The key difficulty here was that in the stationary setting the proofs rely very strongly on certain monotonicity formulas which do not seem to exist in the parabolic context. Thus, a quite different argument must be developed.

This was accomplished in [4], where B. Barrios, A. Figalli, and the first author established the following:

THEOREM 6 ([4]). – *Let $s > \dfrac{1}{2}$, and $u(x,t)$ be any solution to the parabolic fractional obstacle problem.*

Then, the free boundary is smooth, possibly outside a certain set of degenerate points.

This extended for the first time the results of [13] to the parabolic setting, with a completely different proof.

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