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Leibniz's mathematical handling of death, catastrophes, and insurances

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Sommario: *Leibniz fu filosofo pratico che dedicò le proprie conoscenze legali e le competenze matematiche al servizio del* "commune bonum" *o benessere pubblico. L'articolo discute cinque aspetti di tale servizio: 1. Leibniz enfatizzò la necessità di creare un sistema pubblico di previdenza che era basato sul principio di solidarietà. 2. Indicò come è possibile calcolare il valore contante di una somma di denaro che deve essere pagata in futuro. 3. Leibniz riconobbe l'importanza della statistica al fine del buon governo dello stato. Tuttavia, impiegò ipotesi eccessivamente semplificatrici per il proprio modello matematico di aspettativa di vita per il calcolo dell'ammontare di rendite vitalizie. 4. Leibniz trattò tipi diversi di vitalizi e dedusse il prezzo di acquisto di una pensione grazie alla sua operazione di ribasso. Trovò la durata presunta di tre tipi distinti di associazioni. 5. Spiegò come le rendite vitalizie potessero essere impiegate per eliminare l'indebitamento eccessivo degli stati.*

Abstract: *Leibniz was a practical philosopher who devoted his legal knowledge and his mathematical competence to the service of the* commune bonum *or public welfare. The article discusses five aspects of this service: 1. Leibniz emphasized the need for the creation of a system of public insurances that was based on the principle of solidarity. 2. He taught how to calculate the cash value of a sum of money that is to be paid in the future. 3. Leibniz acknowledged the importance of statistics for the sake of good governance of a state. But he used strongly simplifying hypotheses for his mathematical model of human life in order to discuss life annuities. 4. Leibniz discussed different types of life annuities and deduced the purchase price of a pension by means of his operation of rebate. He found out the presumable life spans of three different types of associations. 5. He explained how life annuities were suitable for eliminating excessive indebtedness of states.*

Introduction

The Roman architect Vitruvius tells us the following story (1):

Aristippus philosophus Socraticus, naufragio cum ejectus ad Rhodiensium litus animadvertisset geometrica schemata descripta, exclamavisse ad comites ita dicitur: Bene speremus, hominum enim vestigial video.

When the shipwrecked Socratic philosopher Aristippus, thrown on the beach of the inhabitants of Rhodes, had noticed drawn geometrical figures he is said to have exclaimed to his companions: Let us be hopeful, since I see the traces of men.

Mathematics as a cultural force! This is exactly Leibniz's approach when he combined mathematics, law, and politics. For him mathematics proved the existence of culture and preserved it. He was the antithesis of the cloistered scholar. He occupied himself with problems of great public interest: 1. Insurance cover, 2. Justice in financial operations, 3. Demographic evolution, 4. Old-age pensions, 5. Public indebtedness.

The following article discusses these five issues.

Accettato: il 5 ottobre 2016.
(1) Vitruvius, *De architectura*, book 6, preface.

Fig. 1. – The shipwrecked Aristippus on the beach of Rhodes, CREDITS: Euclidis *quae supersunt opera omnia*, ed. David Gregory. Oxford 1703, copperplate engraving of the title page.

1. – The economy and science: mathematics as a cultural force

In about 1680 Leibniz wrote a memorandum about public insurances saying: "Hence the whole state is so to speak a ship, which is exposed to many storms and misfortunes. For that reason it is unjust that a misfortune should affect only a small number of people while the rest are not affected." (2)

Leibniz's leading idea was public welfare. In his memoranda for the Hanoverian duke John Frederick, for the Brandenburg elector Frederick III in Berlin, and for the German emperor Leopold in Vienna, he emphasized the need for the creation of a system of public insurance in the interest of a flourishing community (3) and thus in the interest of all, including the sovereign. Its purpose was to protect the individual citizen against damages, particularly those caused by fire or water, "because", he added, "one cannot demand something from people which they do not have." (4)

In another memorandum for the foundation of an academy of sciences, written on the 26th of March 1700, he emphasized: "One of the best useful things for the benefit of the country and of the people would be a reliable institution for the protection against damages caused by fire, because in the meantime one has found excellent means against that based on machines and on a mathematical foundation." (5) "Similarly, it would be necessary to establish an institution against damages caused by water...To that important end one has but to correctly use geometry. Indeed, now the art of the spirit level has been much advanced." (6) Leibniz added: "Though that is not sufficiently well known". Mathematics is a cultural force that preserves culture. This reminds of Vitruvius's story we have just heard. For that reason it its worth mentioning that in his memorandum for public insurances Leibniz just mentions the *Lex Rhodia de jactu* according to which merchandise that has been thrown away in order to lighten the ship should be restored by using the overhead costs (7).

(2) Leibniz 2000, 13: *Also ist die ganze Republick gleichsam ein schiff zu achten, welches vielen Wetter und unglück unterworffen, und daher ohnbillig, daß das unglück nur etliche wenige treffen[,] die andern aber frey ausgehen sollen.*

(3) Leibniz 2000, nos. I,1; I,2; I,3; I,4; I,5.

(4) Leibniz 2000, 13: *weil man [...] von den Leuten nicht preßen kann, was sie nicht haben.*

(5) Leibniz 2000, 25: *Zum Exempel, eines der nüzlichsten Dinge, zum Besten von Land und Leuten wäre eine gute Anstalt gegen Feuerschäden. Und weilen nunmehr vortrefliche Mittel dagegen aufgefunden, welche in Machinis und mathematischen Grund beruhen.*

(6) Leibniz 2000, 25: *Ebenmäßig wäre auch Anstalt zu machen gegen Wasserschäden [...] Zu diesem trefflichen Zweck ist nichts Anders als ein rechter Gebrauch der Geometria von Nöthen, und ist die Kunst der Wasserwaage nunmehr sehr hoch gebracht [...] obschon es insgemein nicht gnugsam bekannt.*

(7) Leibniz 2000, 13: *Wie Lege Rhodia de jactu sehr weislich geordnet worden, daß die zu erleichterung des schiffes ausgeworffene wahren aus gemeinen Kosten erstattet werden sollen.*

Leibniz admonished the sovereigns to use only means answering the purpose, and that for reasons of credibility: “Indeed, credibility is one of the most important things which has to be looked for and to be preserved. Sometimes it has to be held in higher esteem than cash in hand.” ([8]) He suggested that the surplus be deposited in the cash box of the society, that is, of the Academy of Sciences – an institution he intended to establish at that time – whose purpose was to be the promotion of public welfare. He wanted to charge it with the administration of its affairs and those of its collaborators. For Leibniz, the economy and science were dependent on each other as spheres of a community.

2. – Negotium mathematici iuris: mathematics as a legal force

How should one calculate the current value of a sum of money that is to be paid in the future? This is a problem that concerns Law, Politics, and Mathematics. The rebate must be determined.

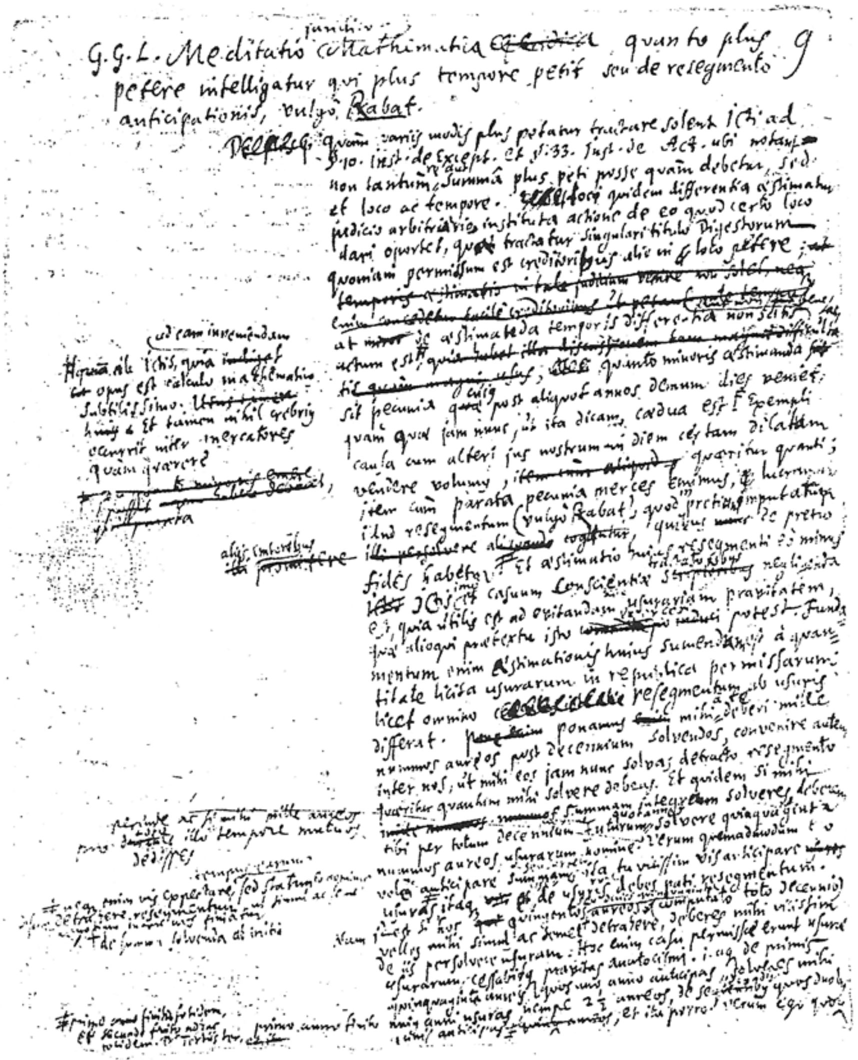


Fig. 2. – Juridico-mathematical consideration about the rebate (1st draft), CREDITS: Manuscript page kept in the Archives of the Leibniz Library Hannover, shelf mark LH II 5,1 sheet 9 obverse. By courtesy of the Leibniz Library, Hannover.

([8]) Leibniz 2000, 17f.: *Maßen Credit eines der wichtigsten dinge ist so man zu suchen und zu erhalten, und bisweilen höher als ein bahres Capital zu schäzen.*

None of these three disciplines can decide this question by itself. The just value must favour neither the debtor nor the creditor. It must conciliate the interest within the framework of commercial law and valid law of contract:

1. No composite interest.
2. The legal rate of interest is 5%.

According to civil law the following principle was valid:

Somebody who pays earlier than he is obliged to pay, has to pay less at that moment.

The legitimate rebate was called "interusurium", "interest accruing in the meantime". This notion was not defined by the Roman law. There was no explanation of how to calculate it, either. For Leibniz, there were three fundamental applications of this problematic notion:

1. Restitution of debts.
2. Sales by auctions.
3. Various kinds of insurance (old-age-insurance, etc.).

In his writings there are three ways of calculating this rebate: He found the correct solution in a number of steps and discussed it with several correspondents, including Christoph Pfautz and Johann Jacob Ferguson. This is a crucial problem for our subject because Leibniz used this rebate in order to calculate the value of a pension.

We know five preliminary drafts of the finally published article ([9]). The reproduction shows the first of them, that is, the *Juridical-mathematical consideration about the question: How much more does somebody demand according to our understanding who demands prematurely or about the deduction of anticipation, commonly called rebate* ([10]).

First solution: Carpzov

In the middle of the 17th century the famous Saxon jurist Benedict Carpzov had claimed that the rebate had to be calculated on the basis of the interest on the money that the buyer had not yet paid at the beginning of each year. When Leibniz examined this practice the result was destructive. Carpzov's scheme implied absurd consequences:

The interest on the outstanding payments could be higher than the money paid in cash. In such a case the bidder had paid less than nothing. ([11])

Leibniz was astonished that Carpzov believed to have eliminated every doubt of the reader. He added: The Saxon jurists were not sufficiently experienced in the domain of a mathematician of law (*negotium mathematici iuris*). ([12]) The first solution favoured the person who paid cash.

Second solution: Jurists (popular calculation)

Leibniz defined the interest accruing in the meantime so that it provided – together with the current value – the promised sum.

The simple "interest accruing in the meantime" concerned the current value of a single sum, while the compound "interest accruing in the meantime" concerned the current values of several sums that had to be paid at different times, as in the case of pensions:

interest accruing in the meantime	
simple	compound

Compound interest was forbidden by law. Hence Leibniz thought initially that he could not apply it in this case.

Let p be the sum of the lent money, let a be the number of years after which the sum has to be repaid, let i be the legal rate of interest, and x the current value looked for, $v = \frac{100}{i}$. In this case the following linear formula must be used: ([13])

$$x = p\,\frac{v}{v + a}$$

([9]) Leibniz 1683.

([10]) Leibniz 2000, 108f.: *Meditatio iuridico-mathematica quanto plus petere intelligatur qui plus tempore petit seu de resegmento anticipationis, vulgo Rabat.*

([11]) Leibniz 2000, nos. II.2, II.10, II.11, II.12.

([12]) Leibniz 2000, 46f.: *Nec dubito quin illi ipsi viri insignes, si viverent, accensa clarissima luce agnituri essent errorem suum praesertim in negotio mathematici juris, ubi se minus exercitatos non diffitebantur.*

([13]) Leibniz 2000, 130f.

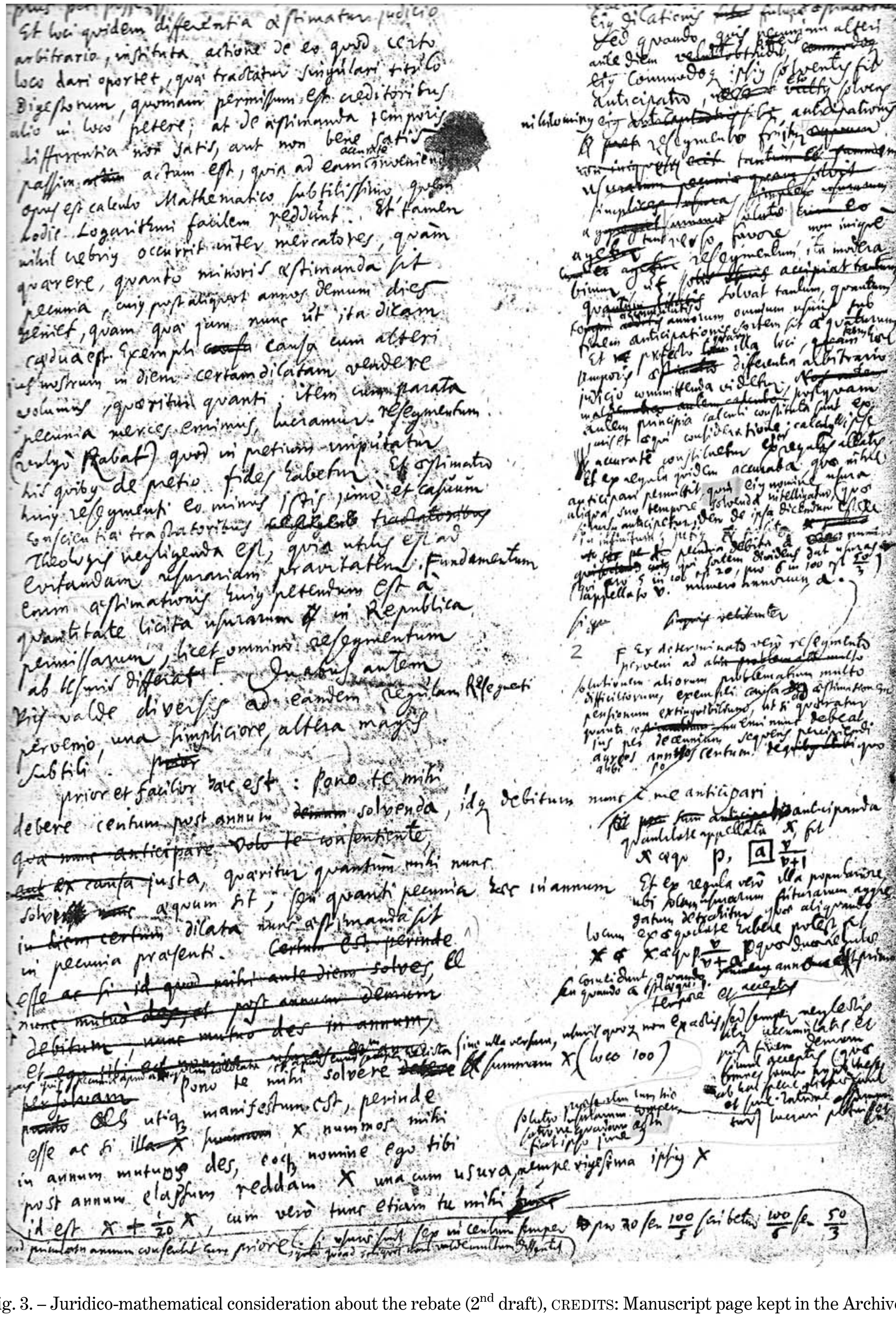


Fig. 3. – Juridico-mathematical consideration about the rebate (2nd draft), CREDITS: Manuscript page kept in the Archives of the Leibniz Library Hannover, shelf mark LH II 5,1 sheet 15 obverse. By courtesy of the Leibniz Library, Hannover.

Third solution: The exact calculations of the merchants

As Leibniz himself avowed, the second solution also implied absurd consequences when he wanted to calculate the purchase price distributed over 40 years. The second method favours the person who does not pay cash but by instalments, the debtor rather than the creditor.

The third solution provides the just value, namely, that given by the multiplicative formula ([14]):

$$x = p\left(\frac{v}{v+1}\right)^a.$$

At the end of the right column we find the third and (under it) the second solution written in Leibniz's symbolism: *aequ.* means equal to or =, the a within the square denotes the exponent of a power.

Leibniz deduced the third solution in three ways:

1. as the sum of the infinite series ([15]):

$$1\frac{p}{1} - \frac{a}{1v}\,p + \frac{a(a+1)}{1.2}\,\frac{p}{v^2} - \frac{a(a+1)(a+2)}{1.2.3}\,\frac{p}{v^3} \pm \cdots$$

considering several years a at the same time;

2. by stepwise calculating the infinite number of mutual virtual arguments, of anticipation and compensation ([16]). If $a = 1$, $p = 1$ one gets:

$$\frac{v}{v+1} = \frac{20}{21} = 1 - \frac{1}{20} + \frac{1}{400} - \frac{1}{8\,000} \pm \ldots$$

Here debtor and creditor are subject to a potentially infinite mechanism of rebates, of anticipations $-\frac{1}{20}$, $-\frac{1}{8\,000}$ etc. by the debtor and of compensations $+\frac{1}{400}$, $+\frac{1}{16\,000}$ etc. for the creditor. Leibniz told Pfautz that he was not able either to find or to demonstrate the foundation of the calculation, that is, the proportion

$$\frac{v}{v+1} = 20 : 21 = x : p,$$

without the use of infinite series. ([17])

3. By inverting the formula of compound interest: ([18])

On condition that $x\left(\frac{v+1}{v}\right)^a = p$ we get $x = p\left(\frac{v}{v+1}\right)^a$.

This method does not reveal why the objection to the application of compound interest is not justified here: ([19])

"One can claim interest on interest paid before the date agreed upon, that is, prematurely." The capitalization of interest justifies this claim.

"One cannot claim interest on interest which the debtor did not pay punctually (prohibition of compound interest)."

At the end of his article published in 1683 ([20]) Leibniz promised:

De usu horum in quibusdam iuris quaestionibus apud egregios autores non satis recte definitis, aestimandisque reditibus ad vitam (ubi interusurio locus est) alio schediamate disseremus. (We will discuss the use of these things in some questions of law that are not sufficiently correctly defined in eminent authors and in the estimation of life annuities where interest accruing in the meantime plays a role in another article.)

Yet, Leibniz never published such an article. Our analysis has to be based on his unpublished manuscripts.

3. – Calculus politicus: demography

Leibniz invariably underlined the importance of statistics relating to the country and the people for the sake of good governance of the state. He called it *calculus politicus,* political calculation. ([21]) This notion corresponded to the "political arithmetic" of some of his contemporaries, such as the English

([14]) Leibniz 2000, 130f.
([15]) Leibniz 2000, 120-125, 360f., 368f.
([16]) Leibniz 2000, 266f., 278f.
([17]) Leibniz 2000, 220f.: *Hoc ego non potui invenire neque demonstrare, nisi ope serierum infinitarum, tu si communi calculo demonstraveris facies rem mihi inexpectatam.*

([18]) Leibniz 2000, 92f.
([19]) Knobloch 1999, 548; Leibniz 2000, 242f.: *Quaeritur ergo cur possim ego petere usuram de usuris quas tibi ante tempus solvo; non possim petere usuram de usuris quas mihi in tempore non solvisti.*
([20]) Leibniz 1683.
([21]) Leibniz 2000, no. III.15.

demographer William Petty (1623-1678) and John Graunt (1620-1674).

Leibniz cites their publications as well as the publications of the Dutchmen Jan de Witt (1625-1672) and Jan Hudde (1640-1704), and of the Englishman Edmond Halley (1656-1743). Thus his demographical interests represented a European interest.

In 1682 he enumerated 56 questions relevant to these interests. (22) He asked for

1. the age at which people are especially exposed to death,
2. the number of children who reach adulthood,
3. the mean duration of human life,
4. the increase and decrease of the number of humans,
5. the value of life annuities etc.

These considerations were based on experience, they were uncertain. He himself explicitly preferred general considerations, which were not based on mortality tables. He preferred hypothetical considerations in order to calculate life expectancy and the value of life annuities.

Leibniz was a pioneer of mathematical modelling of reality and was conscious of working with strongly simplifying hypotheses. He may be said to have been an extreme simplifier.

His certain hypotheses were nearly always as following: (23)

Hyp. 1 All people are equally vital.

Hyp. 2 Every age is equally fatal.

Hyp. 3 The limit of human life is 80 (70, 81) years.

Sometimes Leibniz chose 70 years. Sometimes he assumed that the 80th year might be completed and at other time he assumed that it would not be completed. The duration of real life was but a special case of a finite number of possible durations. Human life was subject to an order of mortality and to random events; it was an image of the Divine Order.

Leibniz thinks that chance is but ignorance of the chain of causes which depends on Providence. It is true that human destiny depends on Providence. Leibniz conciliates the role of Providence with equal probability of individual destinies: the risk of dying is always the same for all. (24)

Leibniz assumes a stationary population: the total number of people remains unchanged. The number of people who are born is the same as the number of those who die.

He deduces formulae for the mean duration of life of individuals or of groups of persons of an arbitrary age. (25) These assumed hypotheses are crucial for such a calculation. If one changes them, then one obtains other results. Hence these calculations include three results:

1. a simple and original formalization of mortality in terms of probability,
2. the basis of a rigorous analysis of mortality in terms of probability,
3. a philosophical approach to problems such as unity and multiplicity, certainty and probability, necessity and contingency, time and eternity, determinism and liberty. (26)

4. – Life annuities: mathematics as a political force

What is the just price of a life annuity? Studying this question Leibniz underlined the importance of demography. The duration of life can only be revealed by a prophet, by Divine Revelation. (27) As an actuary Leibniz must use the calculus of probabilities in order to attribute a

(22) Leibniz 2000, no. III.15: *Quaestiones Calculi politici circa Hominum vitam: et cognatae: Quae aetates magis mortibus obnoxiae; Quot ex infantibus ad annos confirmatos perveniant; Quae sit longitudo media vitae humanae; Incrementum aut decrementum generis humani; Quanti sit reditus ad vitam.*

(23) Leibniz 2000, 416-419, 472f. *Si octoginta anni sunt terminus vitae humanae [...], ita scilicet, ut omnes homines ponantur aeque vitales, omnesque aetates aeque fatales,* 448f.

(24) Rohrbasser & Véron 2001, 88.

(25) Leibniz 2000, 466f., 494f., 498f.

(26) Rohrbasser & Véron 2001, 88.

(27) Leibniz 2000, 414-419.

presumable duration to life annuities and thus a just purchase price. In spite of the uncertainty a certain sure and mathematical estimation of the probability is possible: ([28])

certa quaedam et mathematica probabilitatis aestimatio.

He calculated the purchase price of a pension by means of his operation of rebate. It was a matter of the current values of payments made at different times for a common date of purchase. Let a be the number of years, x the purchase price, $v = \frac{100}{i}$, i the rate of interest, and p the annual pension. Leibniz had to calculate the sum of a terms of a finite geometric series:

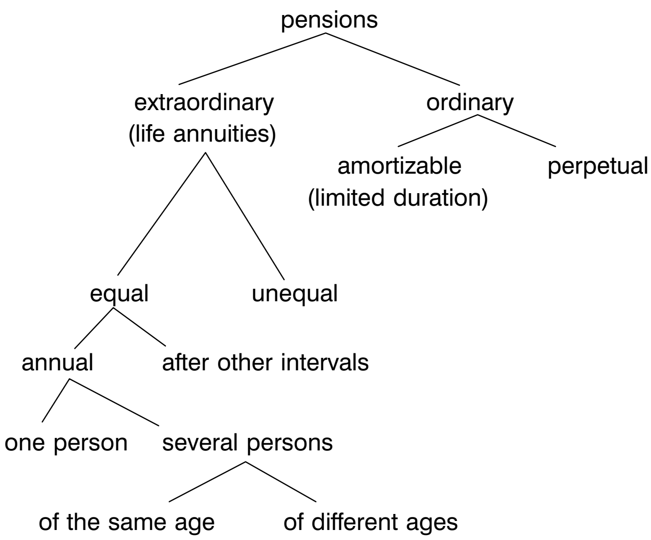
$$x = p\left(\frac{v}{v+1}\right)^1 + p\left(\frac{v}{v+1}\right)^2 + \ldots + p\left(\frac{v}{v+1}\right)^a.$$

Thus Leibniz deduced the formula:

$$x = \left(1 - \left(\frac{v}{v+1}\right)^a\right)vp$$

four times ([29]) without publishing anything regarding his ample consideration of life-annuities.

The end of such calculations is the transformation of these pensions into ordinary pensions, that is, Leibniz used the following division:



([28]) Leibniz 2000, 446f.; see also Leibniz 2000, 416f.
([29]) Leibniz 2000, nos. II.10, II.11, II.12, III.17.

The diagram illustrates the Leibnizian method: First, he presupposes that as many quantities as possible are constant and equal:

1. The pensions are always equal.
2. The payments are made after one year.
3. The money is given to one person.
4. If it is a matter of several persons, these persons are of the same age.

He generalizes these conditions stepwise:

1. The pensions are unequal.
2. The time intervals between the payments are shorter than one year.
3. The money is given to associations with members who might be of different ages.

He carried out long and complicated calculations pertaining to pensions being unequal at different times. He called life annuities of associations of men of different ages the apogee of this study (*huius inquisitionis fastigium*) ([30]) but did not publish anything relating to these results.

In order to calculate the just value of the purchase price of an extraordinary pension, that is, of a life-annuity, Leibniz had to transform it into an ordinary pension, or a pension limited in time. In other words, he had to calculate the expectation of life of an individual or of associations of persons.

The expectation of life defines the presumed life span of a single person or of an association of persons. The presumed lifetime defines the duration of payments of the life-annuity.

But how could Leibniz determine this expectation of life? We know that he did not rely on mortality tables. He consciously wanted to avoid accidental circumstances of reality in order to make possible an exact calculation based on certain hypotheses.

I would like to discuss the problem of associations. We will see that Leibniz's approach was

([30]) Leibniz 2000, 468f.

fundamentally based on combinatorics. He enumerates the possible cases. Their completeness is guaranteed by a table ordered according to the possible cases.

In order to present the relevant details we must introduce two Leibnizian definitions: ([31])

Def. 1 The life span of an association is the upper limit of the individual life spans of its members. An association survives up to the death of its last member.

Def. 2 The presumed life span of an association of n arbitrary persons is the arithmetical mean of the life spans on n-tuples.

Leibniz determined the life expectations of a group of the same age as well as those of persons of different ages. His hypothesis 2 (every year is equally fatal) implies exactly that one person dies at every age:

One of n persons lives	0 years
Another	1 year
Still another	2 years
The n-th person	$n-1$ years

Finally in order to facilitate his task, Leibniz only considers groups of persons consisting of no more than 81 persons. That is, according to hypothesis 3, $n-1=80$ or $n=81$.

Even if n should be larger, all persons must have died after 80 years. Let us consider associations of several persons, for example of 2 or 3. As noted, Leibniz's approach is based on the enumeration of cases. The presupposed conditions are decisive. Once one has calculated the presumed life span of such an association one has to insert it into the formula for the price of a life annuity. There the calculated value has to replace a.

([31]) Leibniz 2000, nos. III.9 (p. 420-424): *Sed maioris operae est definire vivacitatem praesumptivam alicuius collegii quod in uno conservatur seu durationem pensionis in plurium vitam constitutae, nec nisi omnibus extinctis finiendae [...] Quae summa aestimationum aeque possibilium secundum omnes combinationes collectarum dividenda est per numerum ipsarum aestimationum sive casuum aeque possibilium [...] quae est longaevitas media seu praesumptiva*, III.11.

First case ([32])

1. Let a (75) be the same age of a group of n (6) persons.
2. All n persons have different life spans (0, 1, ..., $n-1$ years of life).
3. Let $x = 80$ years be the maximal life span.
4. Leibniz needs four steps in order to deduce the presumed life span of such an association:
   - 4.1 He looks for all possible associations (combinations) of k persons (of k-tuples).
   - 4.2 He determines the life spans of the associations (combinations) (pairs, triples, . . . , n-tuples).
   - 4.3 He calculates the total number of years of the life spans.
   - 4.4 He calculates the presumed life span of k persons.

Let us solve these four problems.

4.1 The possible k-tuples

To facilitate our task, we assume that $k = 3$ and that the total number of persons is 6. Let the persons be A, B, C, D, E, F.

Hence we have the following triples: ([33])

ABC	ABD	ABE	ABF
	ACD	ACE	ADF
		ADE	ADF
			AEF
	BCD	BCE	BCF
		BDE	BDF
			BEF
		CDE	CDF
			CEF
			DEF

There are $\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}$, i.e., 20 triples.
They are

a) "without repetition": repetitions are excluded (impossible, because the associations consist of different persons). Every person has another presumable life span according to our hypothesis 2.

([32]) Leibniz 2000, no. III.14.
([33]) Leibniz 2000, 508f.

b) "unarranged": the case ABC is the same as the cases ACB, BCA, etc., because it is always a matter of the same association, and consequently always of the same life span of that association. We are only interested in this life span.

4.2 Life spans of associations

We suppose, that all 6 persons are dead after 6 years:

A dies in the course of the first year.
B dies in the course of the second year.
C dies in the course of the third year.
D dies in the course of the fourth year.
E dies in the course of the fifth year.
F dies in the course of the sixth year.

Hence we get the following life spans: $0, 1, 2, 3, 4, 5$ or the following triples of life spans:

012	013	014	015
	023	024	025
		034	035
			045
	123	124	125
		134	135
			145
		234	235
			245
			345

According to Leibniz's first definition, the life spans of our associations are:

2	3	4	5
	3	4	5
	4	5	
		5	
	3	4	5
	4	5	
		5	
	4	5	
		5	
		5	

4.3 The total number of life spans is $1.2 + 3.3 + 6.4 + 10.5 = 85$.

The left factors are the triangular numbers, or $\binom{n}{2}$

$$\binom{2}{2}2 + \binom{3}{2}3 + \binom{4}{2}4 + \binom{5}{2}5 = 85$$

4.4 The presumed life span of three persons chosen at random in a population of six persons is

$$\frac{85}{20} = 4{,}25 \text{ years} =$$

$$= \frac{3}{4} \cdot 2 + 2 = \frac{17}{4} = \frac{n}{n+1}(x-n) + (n-1).$$

Or

$$\frac{\binom{2}{2}2 + \binom{3}{2}3 + \binom{4}{2}4 + \binom{5}{2}5}{\binom{6}{3}} = \frac{85}{20}.$$

In order to deduce the general formula we replace 3 by n and 6 by 80:

Presumably, n persons will live

$$\frac{n}{n+1}(80-n) + (n-1) = \frac{80n-1}{n+1} \text{ years.}$$

Leibniz gives this very formula elsewhere. ([34]) The underlying hypotheses are fundamental for such a calculation. If they are changed, we get other results.

Second case ([35])

1. Let $a = 76$ be the same age of a group of 4 persons, $l = 80$ the limit of life. In this case all persons will die in the course of 4 years. No one can exceed the limit of life.

2. This time Leibniz admits equal life times.

3. Let $t = 79$ be the maximal life span.

4.1 The possible k-tuples

Leibniz considers pairs as in the manuscript shown above or triples. Let us take $n = 3$ persons who must die in the course of the first, second, third, fourth year but who might have the same life spans, that is, 0, 1, 2, 3 years. Hence we must look for the different triples

a) with repetition
b) unarranged.

([34]) Leibniz 2000, 498f.
([35]) Leibniz 2000, no. III.9.

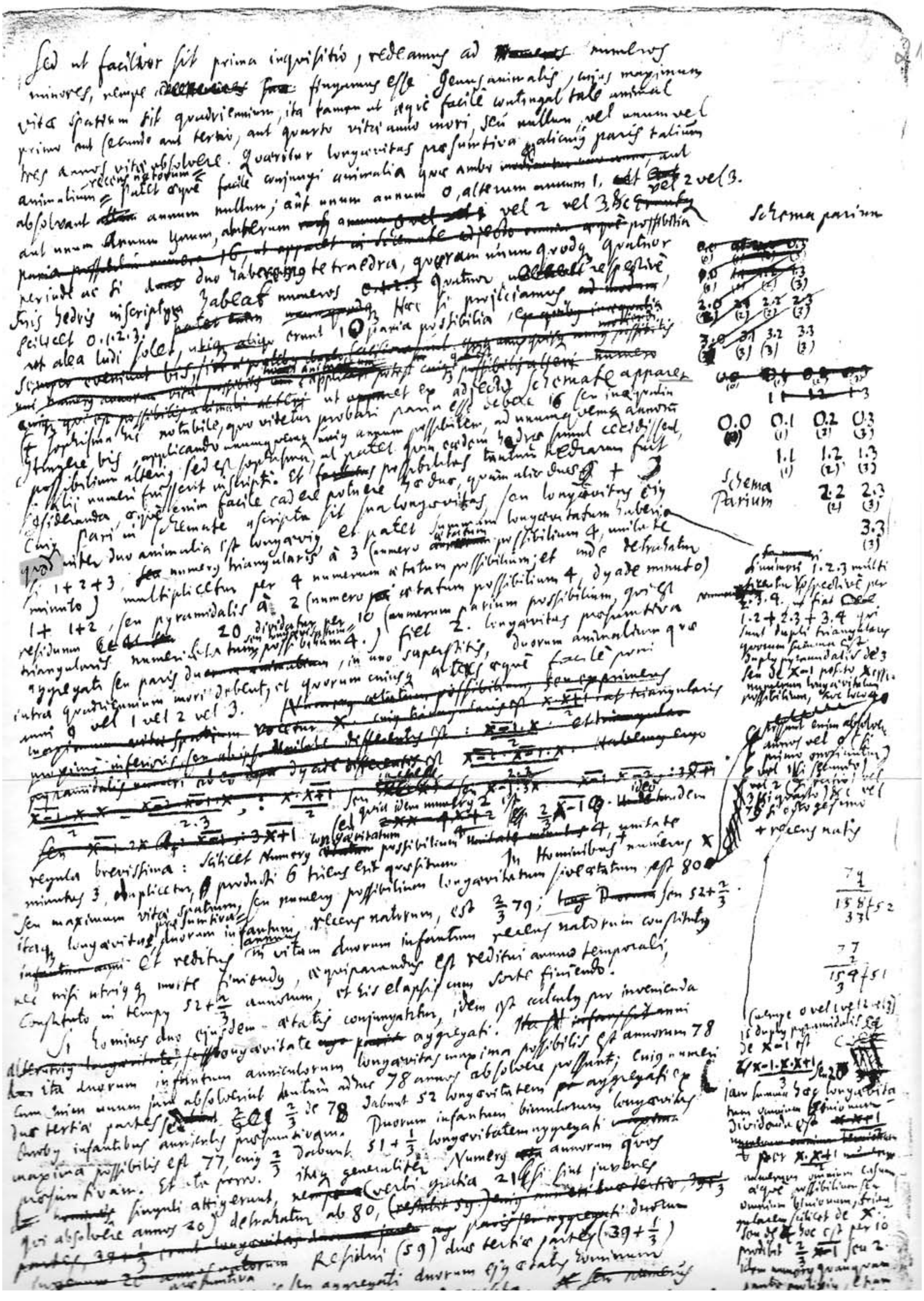


Fig. 4. – Pairs of persons when equal life spans are admitted, CREDITS: Manuscript page kept in the Archives of the Leibniz Library Hannover, shelf mark LH II 5,2 sheet 21 obverse. By courtesy of the Leibniz Library, Hannover.

Let x be the number of possible life expectancies. Then we have to look for the number of combinations of the n-th class with repetition:

The $x=4$ life spans of $n=3$ persons result in

$$\binom{x+n-1}{x-1} = \frac{(x+n-1)!}{(x-1)!n!}$$

combinations ([36]) or

$$\binom{4+3-1}{4-1} = 20 :$$

000	001	002	003
	011	012	013
		022	023
			033
	111	112	113
		122	123
			133
		222	223
			233
			333

We cannot form triples of persons because we cannot repeat the same person, but only the life spans.

4.2 Life spans of associations

0	1	2	3
	1	2	3
	2	3	
		3	
	1	2	3
	2	3	
		3	
	2	3	
		3	
		3	

4.3 The total number of life spans

$$3.1 + 3.4 + 3.10 = 3(1+4+10) = 3\binom{6}{4} =$$

$$(4-1)\frac{(x+2)(x+1)x(x-1)}{4!} = 3\binom{x+n-1}{x}$$

([36]) Leibniz 2000, 424-427, 484-487.

4.4 The presumed life span of three persons who must die in the course of four years

$$\frac{3\binom{x+2}{4}}{\binom{x+2}{3}} = (x-1)\frac{3}{4} = 3\cdot\frac{3}{4} = 2\cdot\frac{1}{4} years = (79-76)\frac{3}{4}$$

Leibniz generalizes this result by replacing 76 by a, 3 by n and finds that:

The presumed life span of n persons of a years will be ([37]) $\frac{n}{n+1}(79-a)$.

Third case ([38])

1. An association consists of two persons P_1, P_2 of different ages (74 and 75 years, respectively). One person, P_1, must die after $r=5$ years at the latest, the other, P_2, must die after $x=4$ years. P_1, P_2 belong to different associations of 6 or 5 persons whose members can be exactly characterized by the fact that they must die after 5 or 4 years.
2. Equal life spans can occur.
3. Let $t=79$ again be the limit of life.

While up to now we knew the life spans of the selected persons, we no longer have this knowledge.

This time, Leibniz does not consider arbitrary subsets or combinations in the modern sense of the word but pairs: one element belongs to the first set, the other to the second set. We do not know whether P_1 will die in the course of the first year and P_2 in the course of the second year or vice versa: the ignorance changes the calculus of probabilities.

4.1 The possible *k*-tuples

Let us denote the members of the first group by A, B, C, D, E, F: they die before the end of the first, second, third, fourth, fifth, sixth year, that is, they live 0, 1, 2, 3, 4, 5 years.

([37]) Leibniz 2000, 486f.
([38]) Leibniz 2000, 468-473.

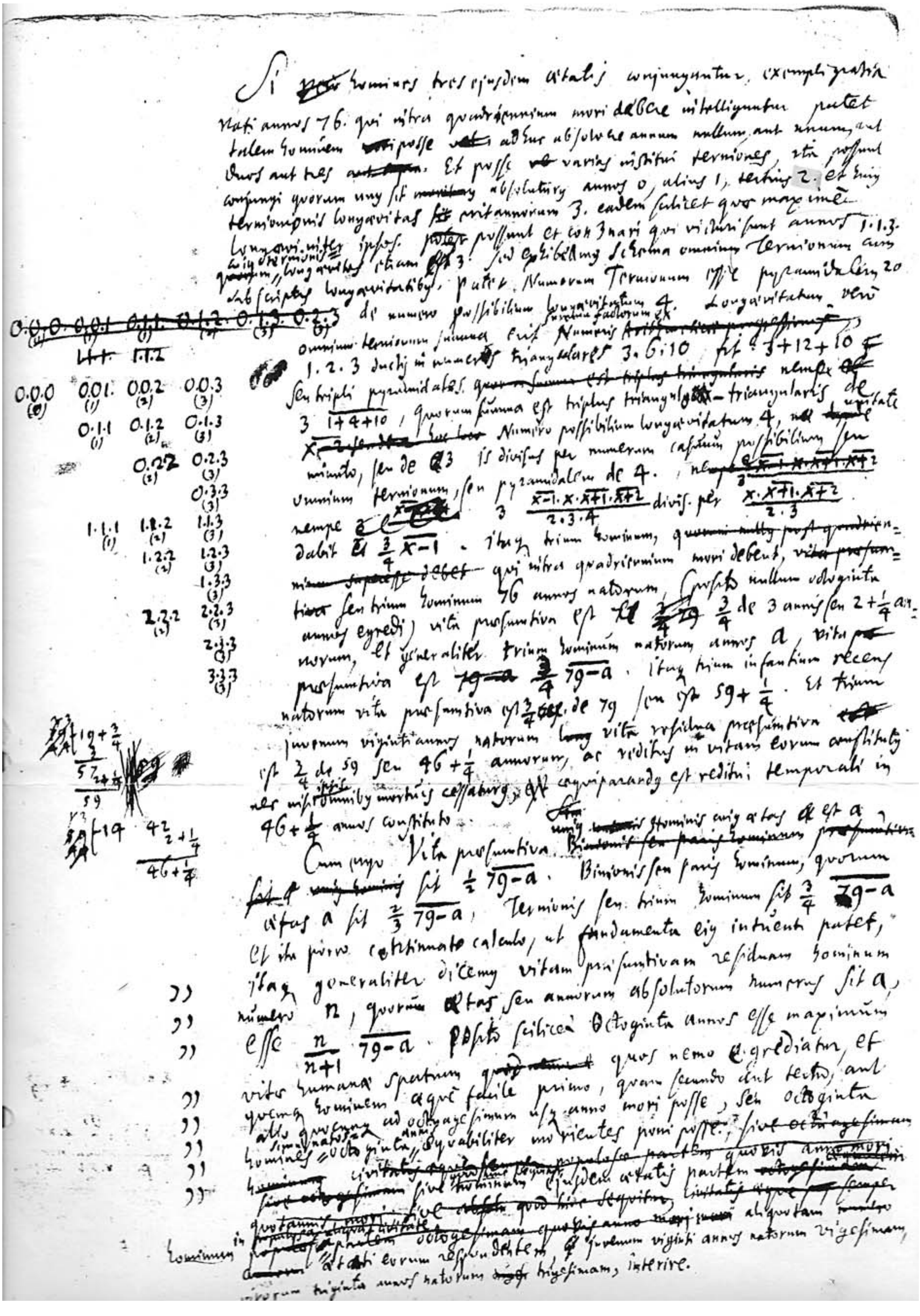


Fig. 5. – Triples of persons when equal life spans are admitted, CREDITS: Manuscript page kept in the Archives of the Leibniz Library Hannover, shelf mark LH II 5,2 sheet 21 reverse. By courtesy of the Leibniz Library, Hannover.

Let us denote the members of the second group by L, M, N, O, P: they live 0, 1, 2, 3, 4 years. Hence, Leibniz gets 6.5=30 pairs: ([39])

AL	BL	CL	DL	EL	FL
AM	BM	CM	DM	EM	FM
AN	BN	CN	DN	EN	FN
AO	BO	CO	DO	EO	DO
AP	BP	CP	DP	EP	FP

4.2 Life spans of associations

0	1	2	3	4	5
1	1	2	3	4	5
2	2	2	3	4	5
3	3	3	3	4	5
4	4	4	4	4	5

4.3 The total number of life spans

0.1 + 1.3 + 2.5 + 3.7 + 4.9 +25 = 95

4.4 The presumed life span of two persons chosen in two groups of persons

In order to obtain the presumed remaining life span of each of these two persons, that is, $\frac{95}{30}$, we must divide this number by the number of pairs (30).

5. – Public indebtedness

For Leibniz, life annuities, or other amortizable pensions, seemed to be the appropriate means for eliminating excessive indebtedness of states or for providing the necessary money for cities, states, and sovereigns, and that in such a way that the creditor did not suffer any injustice. ([40])

Mathematics teaches us how to find the just purchase price of a pension which must be conceded to the creditor. The aim of the action is justice. It concerns not only the percentage but also the question of which kind of indebtedness can be settled in this way. Leibniz explicitly explains what he thinks about politics: public welfare is more important than individual welfare. While we cannot compel an individual against his will to accept a pension that is an instalment, so that the debtor can settle his debt, a state which got into financial straits must have this right: ([41])

Salutis enim publicae maxima semper ratio habenda est.

For public welfare must always play the most important role.

In fact, in case of need and for reasons of equity we might concede a higher percentage than that dictated by mathematics. One has to reckon with, so to speak, a payment of damages. Leibniz severely criticized Johann Joachim Becher, a chemist and economist. Becher had advised the emperor to borrow one million from Dutch merchants and pay a fixed percentage of 20% for 40 years. ([42])

For mathematical reasons, about 6% would have been reasonable. For political rather than legal reasons, one could have conceded 10% or 14% in order to grant compensation for a risk that is hardly calculable for a private creditor.

Leibniz discusses the example of a city whose revenues are 24000. It loses 5000 because of interest and spends 20000 for public responsibilities. ([43]) In order to settle this difficulty, Leibniz suggested financial support for a period of 10 years to be paid by the citizens and a temporary restriction of public expenses. In this case, the creditor could get from 13000 to 15000 a year. After 10 years the debts would be redeemed.

What would Leibniz have said about the situation of Berlin in 2001, which was 1,3 million times worse? The expenses amounted to 40 billion, the revenues to 34,2 billion. Hence there was a yearly deficit of 5,8 billion. The debt amounted to 69,12 billion, which implied yearly interest of about 4 billion.

([39]) Leibniz 2000, 470f.
([40]) Leibniz 1995, 36.
([41]) Leibniz 2000, 384f.
([42]) Leibniz 2000, 380f.
([43]) Leibniz 2000, 386f.

Epilogue

In 1997 Walter Hauser published his PhD dissertation *On the origins of the calculus of probabilities.* ([44]) He amply discussed the pioneer works by Jan de Witt, John Graunt, William Petty on political arithmetic, on the order of mortality, on demography, on life annuities, on insurance problems which Leibniz knew, cited, and used. He did not say anything about the relevant Leibnizian works. Apart from Parmentier's booklet, which was used by Mora Charles, ([45]) most of these works had not been published at that time.

Since then the situation has changed completely. The bilingual volume containing Leibniz's 50 most important papers dealing with this subject appeared in 2000. The present article is largely based on that volume. ([46]) In 2001 Jean-Marc Rohrbasser and Jacques Véron from the Institut National d'Études Démographiques in Paris published their booklet *Leibniz et les raisonnements sur la vie humaine.* Marc Barbut added a preface. ([47]) It demonstrates the quick reception of, and the great interest in, these Leibnizian studies.

Apart from the article published in 1683 these studies could not become influential because they were not published. As we have heard Leibniz did not publish a single article on pensions, annuities or life expectancies. One might ask whether he would have influenced the mathematical handling of such questions. This seems to be doubtful. Though he emphasized the importance of descriptive statistics he was reluctant to engage in mathematical statistics based on empirical material. His objections against Jakob Bernoulli's law of large numbers essentially contributed to Bernoulli's decision to postpone the publication of his *Ars conjectandi*.

John Graunt's book *Natural and Political Observations mentioned in a following Index, and made upon the Bills of Mortality* referring to the inhabitants of London, published in 1662 and again still four times up to 1676, became utmost influential. It was the model of bills of mortality in France, in the Netherlands, and in Germany. Life tables became a basic tool in medical statistics, demography, and actuarial science ([48]).

Leibniz knew Graunt's monograph as we have mentioned above but preferred to elaborate his mathematical model of human life without relying on such empirical data. Yet, the most important authors after Leibniz's death who dealt with demography, annuities, and insurances were Abraham de Moivre and his rival Thomas Simpson. De Moivre laid the foundation of modern life insurances. Simpson stressed that his results were based on real observations. Both succeeded in developing a rather complete theory of single life annuities supplemented with adequate tables ([49]). They paved the way for other or later authors like Daniel Bernoulli, Johann Heinrich Lambert, and Pierre Simon Marquis de Laplace.

Hence one might conclude that these authors would have rejected Leibniz's theoretical approach though his use of combinatorial and probabilistic argumentations would have been interesting and useful for them.

([44]) Hauser 1997.
([45]) Leibniz 1995; Mora Charles 2002.
([46]) Leibniz 2000.
([47]) Rohrbasser & Véron 2001.
([48]) Hald 1990, 104.
([49]) Hald 1990, 527.

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