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The mathematics of Kuramoto models which describe synchronization phenomena

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Sommario: *I cosiddetti "modelli di Kuramoto", e altri simili ad essi, rappresentano un modo paradigmatico per descrivere una serie di fenomeni di sincronizzazione, cioè stati a cui possono passare sistemi incoerenti, come capita spesso nelle transizioni di fase e in una moltitudine di casi, che vanno dalla Fisica alle Neuroscienze, dalla Biologia all'Ingegneria e persino alle Scienze Sociali. Questi fenomeni spiegano, almeno qualitativamente, una grande varietà di processi complessi. In questo articolo, passiamo in rassegna tali modelli e la matematica sottostante, mostrando alcune delle loro peculiarità.*

Abstract: *The so-called "Kuramoto models" and similar ones represent a paradigmatic way to describe a number of synchronization phenomena. These are states into which incoherent systems may go, often as it occurs in phase transition, and concern a variety of cases, ranging form Physics to Neuroscience, from Biology to Engineering and even Social Sciences. They explain, at least qualitatively, a large variety of complex processes. In this paper, we review such models and the underlying mathematics, showing some of their peculiarities.*

1. – Introduction

The so-called "Kuramoto models" (and similar models), that is, systems of ordinary differential equations (ODEs), describing coupled oscillators, seem now to be ubiquitous. They describe rather general interactions in populations of "units", which can be interpreted as oscillators undergoing *synchronization*, and hence exhibiting a collective behavior. It turns out that such synchronization may explain a number of phenomena pertaining to Biology, Medicine, Neuroscience, Chemistry, Physics, Engineering, and even Social Sciences.

We recall the picturesque story according to which the biologist Hugh M. Smith, in 1935 observed, in Thailand, swarms of a certain kind of fireflies [male Pteroptyx malaccae fireflies], one insect virtually on each leave of a certain tree, flashing in synchrony, in the dark. What Smith reported is recorded in the literature. He wrote: "Imagine a tree 35 to 40 feet high thickly covered with small ovate leaves, apparently with a firefly on every leaf, and all the fireflies flashing in perfect unison at the rate of about three times in two seconds. Between flashes the tree was in complete darkness. Imagine a tenth of a mile of river front with an unbroken line of Sonnerati trees with fireflies on every leaf flashing in synchronism, the trees at the ends of the line acting in perfect union with those between."

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However, other similar, though less spectacular phenomena of collective synchronization were observed in a number of cases: the old famous Huygens' synchronization of pendulum clocks hanging on a wall, crickets chirping in unison, pancreatic beta-cells and pacemakers heart cells, epileptic seizures in the brain, menstrual cycles in groups of women, Josephson junctions, lasers, and antennas arrays, rhythmic applauses, and many more.

While, likely, such models may not provide detailed information on specific concrete problems, e.g., in Medicine and Engineering, they are however quite powerful in shedding light on fundamental mechanisms governing a variety of collective phenomena. Indeed, developments and many applications of Kuramoto and similar models have appeared even recently. We recall complex networks of coupled oscillators, in particular the Internet and the World Wide Web, coupled biological and chemical systems, neural networks, social interacting species, computational neuroscience, dynamical organization of resting-state activity in the brain, microwave generation in magnetic nano-structures driven by spin-polarized current, synchronization of oscillations across space in ecology. But these are, again, only few examples.

From the mathematical standpoint, these models include both, "molecular dynamics"-type models, when we face finitely many (though very numerous) oscillators, that are described by systems of ordinary differential equations, as well as by certain partial differential equations (PDEs), typically describing populations of infinitely many oscillators. More precisely, the former can be formulated in terms of systems of deterministic or stochastic ODEs, the latter as integroparabolic differential equations. In this paper, we will confine ourselves only to very few aspects, according to the taste and the specific contributions of the author and his collaborators.

The widespread popularity of Kuramoto and similar models in so many fields is witnessed by the to date $1,355$ Google Scholar citations (876 in the Web of Science) of the review paper published in the Rev. Mod. Phys. in 2005 [6]. This review already covered research made over the previous 28 years, with 225 references.

2. – The original Kuramoto model

In 1975, Y. Kuramoto [15, 16], motivated by the behavior of biological and chemical oscillators, proposed a mathematical model to describe the dynamics of their synchronization, introducing the system of ODEs

$$\text{(1)} \quad \dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, 2, \ldots, N,$$

where N is the number of the limit-cycle oscillators, the ith oscillator having its natural frequency ω_i, and $K > 0$ being a constant sizing the coupling strength (the strength of the nonlinearity). Clearly, if we set $K = 0$ in (1), we obtain a simple set of "phases", $\theta_i = \theta_i(t)$, $i = 1, 2, \ldots, N$, evolving trivially (and independently on each other) as $\theta_i(t) = \omega_i t + \theta_i(0)$. The effect of a nonzero value of the parameter K is to couple all such phases, and since this coupling is nonlinear, the ensuing phenomena may be expected to be rather complicated. Adopting a terminology borrowed from statistical mechanics, we can say that the model described by (1) is a *mean-field* model, since each oscillator is affected by the average of all the others.

More general models, often also called "Kuramoto models", can be found in the literature, for instance with K/N replaced by $K_{i,j}/N$ (hierarchical, in particular, first-neighbor coupling) under the sum.

As we said, with no coupling, i.e., setting $K = 0$, all oscillators run independently at their natural frequency, and even when the coupling is sufficiently weak, the oscillators run incoherently. Beyond some threshold, i.e., when $K > K_c$, being K_c some critical value, it is found that collective synchronization emerges spontaneously. This is a partially synchronized state, and full synchronization may occur in the limit when $K \to +\infty$.

It is convenient to associate to such ODE model the complex-valued "order parameter", defined by the relation

$$\text{(2)} \qquad r_N\, e^{i\psi_N} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j},$$

where $r_N = r_N(t)$ and $\psi_N = \psi_N(t)$ are amplitude and phase of the order parameter. By this, we can write the system in (1) in the more compact form

(3) $\dot{\theta}_i = \omega_i + K\, r_N \sin(\psi_N - \theta_i), \quad i = 1, 2, \ldots, N.$

Note that here r_N and ψ_N *depend* on the θ_i's, i.e., they are functionals of the θ_i's,

When all oscillators run independently, i.e., incoherently, then $r_N = 0$. Otherwise, whenever $r_N > 0$, its value provides an indication, a measure of the degree of coherence; $r_N \equiv 1$ means full synchronization, usually achieved only in the limit $K \to +\infty$. Transition from incoherence ($r_N \equiv 0$) to collective synchronization ($r_N > 0$) resembles a phase transition in statistical physics.

As for the case of infinitely many oscillators, it is convenient to define a distribution (a density) function, say $\rho \equiv \rho(\theta, \omega, t)$, of phases, satisfying the "continuity equation",

$$\frac{\partial \rho}{\partial t} + \frac{\partial (v\, \rho)}{\partial \theta} = 0, \tag{4}$$

where we set

$$v := \omega + K\, r \sin(\psi - \theta). \tag{5}$$

This PDE describes the evolution of the system obtained in the limit $N \to \infty$, and should be solved along with a normalization condition,

$$\int_{-\infty}^{+\infty} \rho(\theta, \omega, t)\, d\theta = 1, \tag{6}$$

and an appropriate initial condition. Actually, only normalization of the *initial* value, $\rho(\theta, \omega, 0)$, should be required, while normalization of ρ at later times follows. This hyperbolic PDE yields, however, infinitely many solutions, and, as we will see below, it represents the singular limiting-case when a small amount of noise is introduced into the system. In fact, in the latter case the PDE will become parabolic.

It should be recalled that, historically, before Kuramoto, Robert Adler, "the man who invented the TV remote control", in 1946 considered a single nonlinear oscillator with sinusoidal nonlinearity like

$$\frac{d\alpha}{dt} = \Delta\omega_0 - \frac{E_1}{E} \frac{\omega_0}{2Q} \sin \alpha, \tag{7}$$

("Adler's equation"), to describe locking phenomena [7], in a circuit containing a vacuum tube, but it was generalized in 1993 by R.A. York to include a number of similar coupled oscillators. Sometimes, in the literature, some people still refer to "Adler's type systems". It may be a nice curiosity to recall that Robert Adler, who held 180 patens for electronic devices, was the man who invented one of the first kinds of wireless TV remote control. Adler's idea was to use sound waves, and later ultrasonic waves, to transmit signals to the TV set. According to an anecdote, he had the idea while he was sitting in the living room, in front of his TV set, while his wife was in the kitchen to get him a bier. Even Alan Turing in 1952, used sinusoidal couplings in studying the chemical basis of morphogenesis [27].

Noise, due to a variety of reasons, might affect the system of oscillators in equation (1), which then takes the form of a system of Langevin equations,

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \xi_i(t), \qquad i = 1, 2, \ldots, N, \tag{8}$$

where the ξ_i's are suitable stochastic processes. If these are chosen to be independent identically distributed white noises, they have mean and correlation

$$\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t) \xi_j(s) \rangle = 2D\, \delta_{ij}\, \delta(t - s), \tag{9}$$

where the parameter $D > 0$ sizes the strength of the noise.

3. – Infinitely many noisy oscillators: The Kuramoto - Sakaguchi model

The previous equations in (8) are (Ito-type) *stochastic* differential equations. We can associate to that system the *parabolic* PDE [20, 21, 22]

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial (v\, \rho)}{\partial \theta}, \tag{10}$$

satisfied by the oscillators' distribution function [the probability *density* function], $\rho(\theta, \omega, t)$; ρ is also called a "one-oscillator" distribution function. This PDE is a nonlinear Fokker-Planck equation, sometimes called Vlasov-McKean equation, and is obtained in the thermodynamic limit, $N \to \infty$.

From the mathematical standpoint, we now face a nonlinear integrodifferential parabolic PDE. An initial value,

$$\rho(\theta, \omega, 0) = \rho_0(\theta, \omega), \tag{11}$$

should be prescribed, besides 2π-periodicity (in θ), and normalization of ρ_0 (while that of ρ at later times can be proved).

The order parameter is now defined as

$$r\, e^{i\psi} := \int_0^{2\pi} \int_{-\infty}^{+\infty} e^{i\theta} \rho(\theta, t, \omega)\, g(\omega)\, d\theta\, d\omega, \tag{12}$$

where $g(\omega)$ is the natural frequencies distribution, and hence the drift term in the PDE becomes

$$v = v(\theta, t, \omega) := \omega + K\, r\, \sin(\psi - \theta). \tag{13}$$

Note that this PDE replaces the aforementioned hyperbolic PDE in (4), and may be considered more satisfactory, being more realistic. Some numerical treatment of such problems was conduced in [23, 5].

In all PDE models, it is understood that the frequency, ω, should be picked up from the support of the given distribution $g(\omega)$. For such reason some authors refer to a set of "randomly" distributed frequencies, which can be a bit misleading. Again, $r \equiv 0$ (and $\rho \equiv 1/2\pi$) means incoherence, while $r > 0$ represents some degree of synchronization. The occurrence $r_N \equiv 1$ means full synchronization, usually achieved only in the limit $K \to +\infty$.

Linear stability of the incoherent state was studied by Strogatz and Mirollo in [24]. They showed that the behavior earlier conjectured by Kuramoto that, in the infinite-N limit, the incoherent solution is stable for all $K < K_c$ and becomes unstable for $K > K_c$, was correct. In practice, this means that a partially synchronized state exists and is (linearly) stable when $K > K_c$. Moreover, the case of infinite-N limit was found to be singular with respect to the addition of a small amount of noise, since in presence of the latter, when $K < K_c$ the incoherent solution switches from (neutrally) stable, as it was, to linearly stable: in presence of noise, the solution with $r(t) \equiv 0$ is unique, while there are infinitely many solutions in the absence of noise.

4. – Bimodal frequency distributions

The interesting, richer, case of *bimodal* frequency distributions was investigated in [12, 13, 1]. In [24], the natural frequencies distribution was considered symmetric and one-humped. In [12, 13, 1] the *nonlinear* stability of incoherence was studied, along with various bifurcations. In [1], bimodal, even "asymmetric" bimodal distributions were considered.

In fig. 1, we show the stability regions for the incoherent solution in the in the parameter space $(K/D, \omega_0/D)$, for the bimodal distribution $g(\omega) = [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]/2$. Above the parabolic line but on the left of the vertical line, incoherence is linearly stable, while it is linearly unstable elsewhere in the first quadrant, see [12].

In fact, when the natural frequency distribution is bimodal, new features appear. It turns out that when the coupling is sufficiently small, namely $K < 2D$, the incoherent solution is linearly stable for all values of ω_0, while it becomes linearly unstable for sufficiently large couplings, $K > 4D$. For intermediate values of the coupling, i.e., for $2D < K < 4D$, the incoherent solution may become linearly unstable in two ways:

(i) for $\omega_0 > D$, a pair of complex conjugate eigenvalues cross the imaginary axis onto the right half-

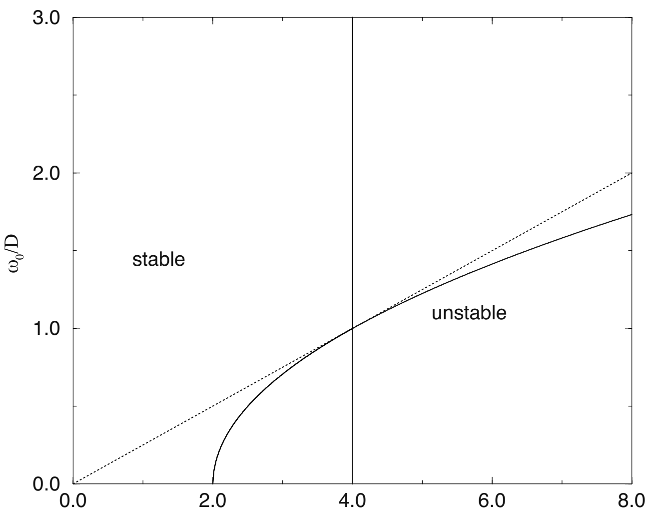

Fig. 1. – Stability regions for the incoherent solution in the parameter space $(K/D, \omega_0/D)$, for the bimodal distributions $g(\omega) = [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]/2$. Above the parabolic line but on the left of the vertical line, incoherence is linearly stable, while it is linearly unstable elsewhere.

plane, as K becomes larger than the critical value $K_c := 4D$. When $K = K_c$, a branch of solutions with a time-periodic order parameter bifurcates from the incoherent solution.

(ii) for $\omega_0 < D$, one eigenvalue becomes positive as K becomes larger than some K_c', being $K_c'/D := 2(1+\omega_0^2/D^2)$. A stationary state branches off the incoherent solution at such value, K_c'.

5. – Analyzing the Kuramoto-Sakaguchi PDE

A basic rigorous analysis on the Kuramoto-Sakaguchi model, represented by the integroparabolic PDE,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (v\,\rho)}{\partial \theta} = D\frac{\partial^2 \rho}{\partial \theta^2}, \tag{14}$$

where

$$v := \omega + K\,r\,\sin(\psi - \theta) \tag{15}$$

(see (10), (13)), requiring 2π-periodicity in θ, and prescribing an initial value as well as its normalization, i.e.,

$$\int_{-\infty}^{+\infty} \rho(\theta, \omega, t=0)\,d\theta = 1, \tag{16}$$

was made in 1999, 2000, 2004, and in 2014, see [17, 18, 19]. In [17], existence and uniqueness of solutions to such a problem were established, while in [18] time-independent estimates were derived and a comparison theorem for certain nonlinear integroparabolic equation of the Fokker-Planck type was also proved. In [19], existence, uniqueness, and regularity for the Kuramoto-Sakaguchi PDE were established when the oscillators' natural frequency distribution is assumed to be unbounded.

It should be observed that a number of pathologies affect such a model, making it highly nonstandard:

- the PDE is nonlinear and integrodifferential;
- BCs appear in the form of a periodicity condition;
- the support of the frequency distribution $g(\omega)$ in (12) could be, in general, unbounded;
- the variable ω, which enters the PDE, not in its derivatives but as an integration variable, runs over the full real line (hence, it is an unbounded coefficient).

Global in time existence and uniqueness of classical solutions were established in [17], and regularity and time-independent estimates were obtained in [18]. In [19], the limitation to frequency distributions with a compact support was removed.

6. – Models with inertia: The "adaptive frequency model"

In 1991, B. Ermentrout introduced a model with inertia, that is with some mass, formulated as a system of second-order ODEs for phases [14]. This was an "adaptive frequency" model, which implies that the natural frequencies of all oscillators are allowed to vary with time. H.A. Tanaka, A.J. Lichtenberg, and S. Oishi, in 1997, also suggested something similar, in the framework of mean-field coupling, with sinusoidal nonlinearities [25, 26]. Starting from such contributions, in 1998 the system of second-order Langevin equations,

$$m\,\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + K\,r_N\,\sin(\psi_N - \theta_i) + \xi_i(t), \qquad i = 1, 2, \ldots, N, \tag{17}$$

where $m > 0$ is like a mass (an inertial term), was proposed in [2, 4], and the ξ_i's take into account some noise. This equation is equivalent to the system

$$\begin{cases} \dot{\theta}_i = \omega_i \\ \dot{\omega}_i = \dfrac{1}{m}[-\omega_i + \Omega_i + K\,r_N \sin(\psi_N - \theta_i)] + \dfrac{1}{m}\xi_i(t), \end{cases} \tag{18}$$

for $i = 1, 2, \ldots, N$.

In the thermodynamic limit $N \to \infty$, the "adaptive model" takes on the form of the nonlinear integrodifferential PDE (of the Fokker-Planck type) [2]

$$\frac{\partial \rho}{\partial t} = \frac{D}{m^2}\frac{\partial^2 \rho}{\partial \omega^2} - \frac{1}{m}\frac{\partial}{\partial \omega}[(-\omega + \Omega + Kr\sin(\psi - \theta))\rho] - \omega\frac{\partial \rho}{\partial \theta}, \tag{19}$$

and the order parameter is now defined by

$$r\,e^{i\psi} = \int_{-\infty}^{+\infty} d\omega \int_{0}^{2\pi} d\theta \int_{-\infty}^{+\infty} d\Omega\, g(\Omega) e^{i\theta} \rho(\theta, \omega, \Omega, t), \tag{20}$$

where $g(\Omega)$ is the natural frequencies distribution.

Models with inertial effects seem to better explain certain aftereffects in alterations of circadian cycles in mammalians as well as the behavior of some "Josephson junctions" arrays. Josephson junctions are quantum mechanical devices, made by sandwiching a thin layer of a nonsuperconducting material between two layers of superconducting materials. There are several applications of Josephson junctions to electronic circuits, and it seems that very fast computers could be built using them. A noteworthy application occurs in some special devices called SQUIDs (Superconducting Quantum Interference Devices), which are capable to measure extremely weak magnetic fields, such as those in the human brain.

7. – An uncertainty principle

A meaningful synchronization phenomenon should actually concern both *phase* and *frequency* synchronization, hence the adaptive frequency model should receive a special attention.

An "uncertainty principle" was found for the original Kuramoto model (with no inertia) [3]. This term is reminiscent of the celebrated principle of Quantum Mechanics, according to which it is impossible to determine with arbitrary accuracy at the same time both, position and velocity, e.g., of an electron. Indeed, we found that, within the Kuramoto model, it is impossible to do the same with phase and frequency. We first defined the spreads

$$\begin{aligned} (\Delta\theta)^2 &:= \langle (\theta - \psi)^2 \rangle_\rho - (\langle \theta - \psi \rangle_\rho)^2, \\ (\Delta v)^2 &:= \langle v^2 \rangle_\sigma - (\langle v \rangle_\sigma)^2, \end{aligned} \tag{21}$$

where $\langle \cdot \rangle_\rho$ denotes taking an average with respect to the phase distribution, $\rho = \rho(\theta, \omega, t)$, and $\sigma = \sigma(v, \omega, t)$ denotes the corresponding frequency distribution. Here the spread of the drift, v, was used since the "instantaneous frequency", $\dot{\theta}$, in some sense can be considered not observable, having the same nature of the white noise, while the drift term has the same nature of the Brownian motion (see (8). Then, it was found that, for large K's,

$$(\Delta\theta)^2 \sim \frac{D}{K}, \qquad (\Delta v)^2 \sim D\,K, \tag{22}$$

and hence

$$\Delta\theta\, \Delta v \sim D. \tag{23}$$

This shows the aforementioned "uncertainty principle": the more accurate is the phase, namely the smaller is $\Delta\theta$, the larger will be the inaccuracy of the drift, v.

For the model with inertia, where the oscillators have some mass, m, defining

$$(\Delta\omega)^2 := \langle \omega^2 \rangle_\rho - (\langle \omega \rangle_\rho)^2, \tag{24}$$

we found instead

$$\Delta\theta\, \Delta\omega \sim \frac{D}{(mK)^{1/2}}, \tag{25}$$

for large K. Hence, a large inertia and a large coupling strength, tend to reduce the uncertainty, while the noise tends to make it more significant. We actually found the two separate relations

$$(\Delta\theta)^2 \sim \frac{D}{K}, \qquad (\Delta\omega)^2 \sim \frac{D}{m}, \tag{26}$$

for large K, well confirmed by numerical simulations, see the figs. 2-5 below.

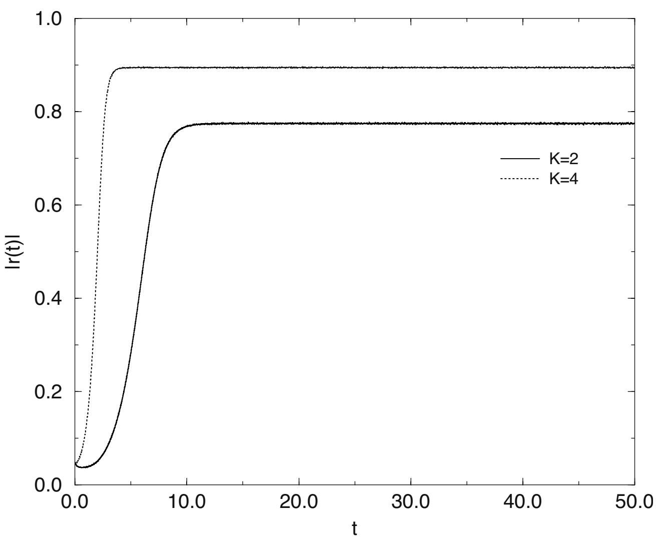

Fig. 2. – Time evolution of the amplitudes of the order parameters, with $N = 20{,}000$ oscillators, no noise ($D = 0$), Lorenzian frequency distribution, $g(\omega) = (\gamma/\pi)/(\omega^2 + \gamma^2)$ with $\gamma = 0.4$, coupling $K = 2$ (continuous line), and $K = 4$ (dashed line).

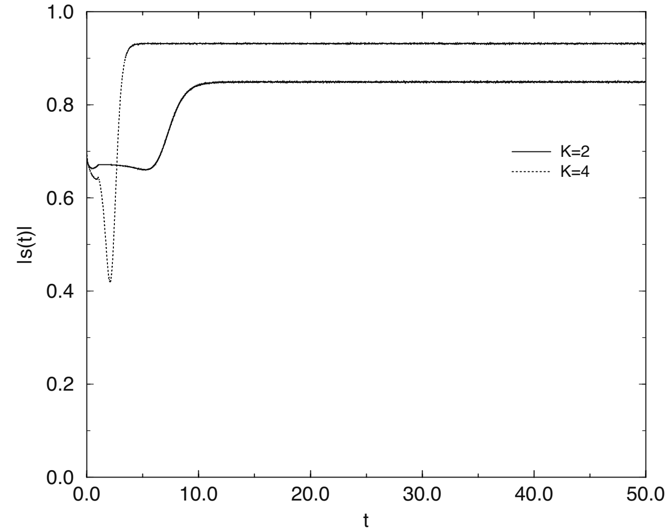

Fig. 3. – See previous caption: here there is $|s(t)|$ vs. t.

In summary, clearly (for given inertia and coupling), the *noise is responsible* for such uncertainty. Cf. the role of the Planck constant in Quantum Mechanics, and also the noise as a possible explanation put forth in Nelson's stochastic mechanics. Hence, using stochastic instead of deterministic ODEs may explain how uncertainty affects an otherwise deterministic world. Generally speaking, uncertainty is intrinsic to Quantum Mechanics, and, according to E. Nelson, noise may provide a possible interpretation for it.

It is convenient to define a complex *order parameter* for the frequencies besides that for phases,

$$(27)\quad r_N e^{i\psi_N} := \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}, \qquad s_N e^{i\phi_N} := \frac{1}{N}\sum_{j=1}^{N} e^{i\omega_j}$$

(*phase and frequency order parameters, when* $N < \infty$),

$$(28)\qquad re^{i\psi} := \int_{-\infty}^{+\infty} d\omega \int_{0}^{2\pi} d\theta\, g(\omega)\, e^{i\theta} \rho(\theta, \omega, t)$$

(*phase order parameter, when* $N \to \infty$),

$$(29)\qquad se^{i\phi} := \int_{-\infty}^{+\infty} d\omega \int_{\omega - Kr}^{\omega + Kr} dv\, g(\omega)\, e^{iv} \sigma(v, \omega, t)$$

(*frequency order parameter, when* $N \to \infty$).

In figs. 2 and 3, the time evolution of amplitudes of both order parameters, $r(t)$ and $s(t)$, is plotted for the classical (no inertia) Kuramoto model, with $N = 20{,}000$ oscillators, a Lorentzian frequency distribution $g(\omega) = (\gamma/\pi)/(\omega^2 + \gamma^2)$ with $\gamma = 0.4$, $K = 2$ (continuous line), and $K = 4$ (dash line), and no noise ($D = 0$). These figures show that, in absense of noise, increasing the coupling parameter, K, larger values of $|r(t)|$ *as well as* of $|s(t)|$ are observed, hence a higher degree of phase *and* frequency synchronization occurs. Things are vey different when noise is present: Figs. 4 and 5 show that larger K's provide larger values of $|r(t)|$ (a smaller spread, according to the first relation in (22)), and smaller values of $|s(t)|$.

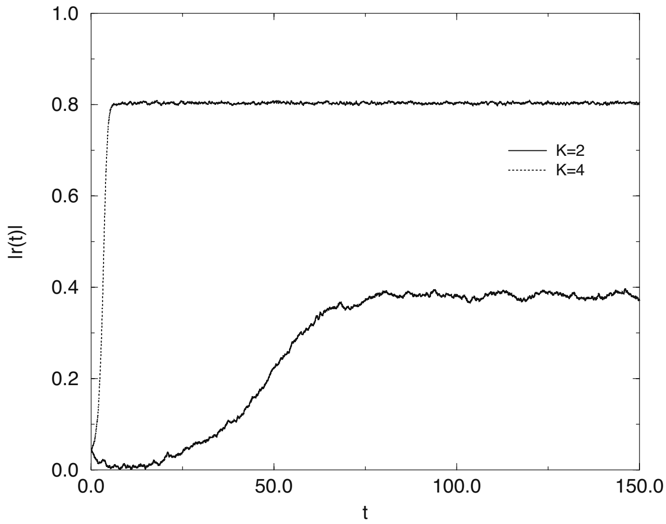

Fig. 4. – As above, with $N = 20{,}000$ oscillators, noise $D = 0.5$, Lorenzian frequency distribution, $g(\omega) = (\gamma/\pi)/(\omega^2 + \gamma^2)$ with $\gamma = 0.4$, coupling $K = 2$ (continuous line), and $K = 4$ (dashed line).

8. – Mathematical analysis of the adaptive frequency model

The mathematical analysis of the so-called "adaptive frequency model", i.e.,

$$(30)\quad \frac{\partial \rho}{\partial t} = \frac{D}{m^2}\frac{\partial^2 \rho}{\partial \omega^2} - \frac{1}{m}\frac{\partial}{\partial \omega}[(-\omega + \Omega + Kr\sin(\psi - \theta))\rho] - \omega\frac{\partial \rho}{\partial \theta},$$

is much more challenging than that of the classical Kuramoto model. It required extensive preliminary work, see [8, 9, 10, 11]. This equation represents an evolution equation for the oscillator density,

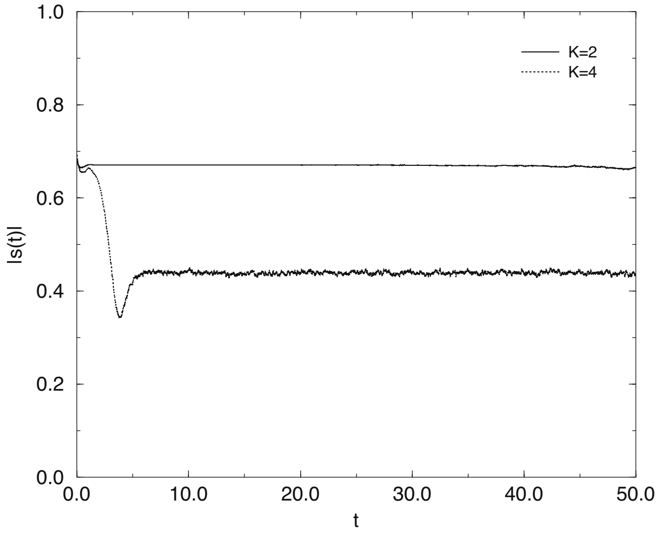

Fig. 5. – See previous caption: here there is $|s(t)|$ vs. t.

$\rho = \rho(\theta, \omega, t, \Omega)$, and may be compared with a Fokker-Planck equation, where space is the *angular* variable θ, and, correspondingly, the velocity ω is an *angular* velocity.

This PDE can be considered as an *ultraparabolic* integrodifferential PDE. In fact, it can be viewed as parabolic in both variables, θ and ω (space-like variables), but diffusion in θ is then missing. It can also be considered, instead, as having "two times", t and θ. Treated as a degenerate parabolic PDE, a *regularized* parabolic equation with bounded coefficients was first considered, where a small spatial diffusion was incorporated and the unbounded coefficients suitably bounded.

Estimates, uniform in the regularization parameters, were obtained in [8, 9], which allowed to pass to the limit, thus identifying a *classical* solution. Existence and uniqueness of the latter were finally established in [10].

The wanted solution to such a PDE is required to be 2π-periodic (in θ), and satisfy a given normalized initial value. Again, it should be pointed out that such a problem, is quite pathological, since:

- the PDE is nonlinear, integrodifferential, and now three integrals appear (instead of two) in the definition of the order parameter;
- BCs appear in the form of a periodicity condition in θ;
- the support of the frequency distribution (the $g(\Omega)$) could be, in general, unbounded;
- there is no derivative with respect to the variable Ω, but this enters the PDE as an integration variable, and runs over the full real line.

Rather cumbersome estimates were obtained for the solution in suitable *anisotropic* Sobolev spaces (the solution belongs to L^p spaces with values of p different as function of the each variable). The PDE problem can be rewritten in the more suitable form,

$$(31)\quad \frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial \omega^2} + \frac{\partial}{\partial \omega}\left[\left(\omega - \Omega - \mathcal{K}_\rho(\theta, t)\right)\rho\right] - \omega \frac{\partial \rho}{\partial \theta},$$

where we set

$$(32)\quad \mathcal{K}_\rho(\theta, t) :=$$

$$K \int_{-G}^{G} \int_{-\infty}^{+\infty} \int_{0}^{2\pi} g(\Omega') \,\sin(\theta' - \theta)\, \rho(\theta', \omega', t, \Omega')\, d\theta' d\omega' d\Omega',$$

with the BC and IC

$$(33)\qquad \rho|_{\theta=0} = \rho|_{\theta=2\pi}, \quad \rho|_{t=0} = \rho_0(\theta, \omega, \Omega).$$

As for the other assumptions, the initial profile, $\rho_0(\theta, \omega, \Omega)$, was assumed to belong to a suitable Hölder space, to be 2π periodic as a function of θ, nonnegative, normalized according to

$$(34)\qquad \int_0^{2\pi} \int_{-\infty}^{+\infty} \rho_0(\theta, \omega, \Omega)\, d\omega d\theta = 1$$

for every Ω in the support of the natural frequency distribution, and exponentially decaying (along with some of its partial derivatives), as $\omega \to \pm\infty$, being of order of $e^{-M\omega^2}$, for some $M > 0$.

In [11], it was established that global in time strong solutions are of order of $\mathcal{O}\left(e^{-M\omega^2}\right)$ and that this decay estimate is optimal.

9. – Concluding summary

Kuramoto-type models have been shown to be ubiquitous in a number of fields, too many to mention, since they describe the very general phenomenon of *synchronization* occurring in large populations of coupled "oscillators". The latter can be, in practice, individuals, particles, neurons, circuits, or other things. While large populations of nonlinearly coupled oscillators can be better described in the limit of infinitely many oscillators, by partial differential equations, yet affected by several peculiarities, analyzing the more natural case of finitely many but very numerous oscillators, is rather challenging. Numerical simulations here may play an essential role.

A model, somewhat different but similar in spirit, due to F. Cucker and S. Smale, should mentioned. In 2007, they studied the emergent behavior in flocks of birds flying together, and determined when the convergence of all of them to a common value of velocity could be achieved. Clearly, one may think of systems of any kind of interacting "agents", the case of birds of fish being just an example. The core of the question is that of "reaching of consensus without a central direction".

As recalled in the 2007 Cucker-Smale paper, examples of this situation is the emergence of a common belief in a price system, when activity takes place in a given market, another is that of the emergence of common languages in primitive societies.

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