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SALVATORE RIONERO

L^2-STABILITY OF THE SOLUTIONS TO A NONLINEAR BINARY REACTION-DIFFUSION SYSTEM OF P.D.ES.

To Guido Zappa on the occasion of his 90th birthday

ABSTRACT. — The L^2-stability (instability) of a binary nonlinear reaction diffusion system of P.D.Es. - either under Dirichlet or Neumann boundary data - is considered. Conditions allowing the reduction to a stability (instability) problem for a linear binary system of O.D.Es. are furnished. A peculiar Liapunov functional V linked (together with the time derivative along the solutions) by direct simple relations to the eigenvalues, is used.

KEY WORDS: Nonlinear Stability; Liapunov Direct Method; Reaction - Diffusion Systems.

1. INTRODUCTION

Let $\Omega \subset \Re^3$ be a bounded smooth domain. The nonlinear stability analysis of an equilibrium state in Ω of two «substances» diffusing in Ω can be traced back to the nonlinear stability analysis of the zero solution of a dimensionless binary system of P.D.Es. like

$$(1) \qquad \begin{cases} u_t = a_1 u + a_2 v + \gamma_1 \Delta u + f(u, v, \nabla u, \nabla v) \\ v_t = a_3 u + a_4 v + \gamma_2 \Delta v + g(u, v, \nabla u, \nabla v) \end{cases}$$

with f and g nonlinear and

$$(2) \qquad \begin{cases} a_i = \text{const.}\ (i = 1,2,3,4) \\ \gamma_i = \text{const.} > 0 (i = 1,2) \\ (u = v = 0) \Rightarrow f = g = 0 \\ u : (\boldsymbol{x}, t) \in \Omega \times \Re^+ \to u(\boldsymbol{x}, t) \in \Re \\ v : (\boldsymbol{x}, t) \in \Omega \times \Re^+ \to v(\boldsymbol{x}, t) \in \Re \end{cases}$$

under Dirichlet boundary conditions

$$(3) \qquad u = v = 0 \quad \text{on } \partial\Omega \times \Re^+$$

or Neumann boundary conditions ($\mathbf{n}$ being the unit outward normal to $\partial\Omega$)

$$\frac{du}{d\mathbf{n}} = \frac{dv}{d\mathbf{n}} = 0 \text{ on } \partial\Omega \times \Re^+ \tag{4}$$

with the additional conditions

$$\int_\Omega u d\Omega = \int_\Omega v d\Omega = 0\,, \forall t \in \Re^+, \tag{5}$$

in the case (4). The stability problems (1)-(5) are encountered in many models of real world phenomena like fluid motion in porous media, heat conduction, spatial ecology (see [1-8] and references quoted therein).

Denoting by

$\langle\cdot,\cdot\rangle$ the scalar product in $L^2(\Omega)$;

$\|\cdot\|$ the $L^2(\Omega)$-norm;

$H_0^1(\Omega)$ the Sobolev space such that

$$\varphi \in H_0^1(\Omega) \to \left\{\varphi^2 + (\nabla\varphi)^2 \in L(\Omega), \varphi = 0 \text{ on } \partial\Omega\right\};$$

$H_*^1(\Omega)$ the Sobolev space such that

$$\varphi \in H_*^1(\Omega) \to \left\{\varphi^2 + (\nabla\varphi)^2 \in L(\Omega), \frac{d\varphi}{d\mathbf{n}} = 0 \text{ on } \partial\Omega, \int_\Omega \varphi d\Omega = 0\right\};$$

the L^2-stability of $(u_* = v_* = 0)$ with respect to the perturbation (u, v) belonging, $\forall t \in \Re^+$, to $\left[H_0^1(\Omega)\right]^2$ in the case (3) and to $\left[H_*^1(\Omega)\right]^2$ in the case (4)-(5), has been studied in [7, 8] under the assumptions

$$\begin{cases} \|f\| + \|g\| = o\left[(\|u\|^2 + \|v\|^2)^{1/2}\right] \\ b_1 = a_1 - \bar{\alpha}\gamma_1 < 0 \\ b_4 = a_4 - \bar{\alpha}\gamma_2 < 0 \end{cases} \tag{6}$$

$\bar{\alpha}$ being the positive constant appearing in the Poincaré - Wirtinger inequality (¹)

$$\|\nabla\varphi\|^2 \geq \bar{\alpha}\|\varphi\|^2 \tag{7}$$

holding both in the spaces $H_0^1(\Omega)$, $H_*^1(\Omega)$. As it is well known, $\bar{\alpha} = \bar{\alpha}(\Omega) > 0$ is the lowest eigenvalues λ of

$$\Delta\phi + \lambda\phi = 0$$

(¹) When Ω is a «cell of periodicity» in three dimensions like

$$\Omega : \boldsymbol{x} = (x, y, z) \in \Omega \Rightarrow 0 \leq x \leq a, 0 \leq y \leq b, |z| \leq \frac{1}{2}$$

with u and v periodic in x and y directions of period a and b respectively, then (3) (4) are required only on $|z| = \frac{1}{2}$ ([4, p. 237] and [5, pp. 387-388]).

respectively in $H_0^1(\Omega)$ and $H_*^1(\Omega)$ (*i.e.* the principal eigenvalue of $-\Delta$). In the present paper we reconsider the problem requiring $(6)_1$ and only $b_1 + b_4 < 0$. Our aim is to show that the stability (instability) of the critical point ($u_* = v_* = 0$) of (1) is implied by the stability (instability) of the critical point $\xi_* = \eta_* = 0$ of the linear binary system of O.D.Es.

$$\begin{cases} \dfrac{d\xi}{dt} = b_1\xi + a_2\eta \\ \dfrac{d\eta}{dt} = a_3\xi + b_4\eta\,, \end{cases} \tag{8}$$

without requiring $a_2 = a_3$, *i.e.* the symmetry of the linear operator acting in (1) [see *iv*) of Section 5].

The plan of the paper is as follows. In Section 2 we introduce a suitable rescaling transformation for u and v and a basic Liapunov functional V such that the sign of $\dfrac{dV}{dt}$ along the solutions of (1) is linked directly to the eigenvalues of (8). Section 3 is dedicated to the stability, while the instability is considered in Section 4. The paper ends with some final remarks (Section 5).

2. Preliminaries

Denoting by α and β two rescaling constants to be chosen suitably later, and setting

$$u = \alpha\bar{u},\ v = \beta\bar{v},\ f^* = \gamma_1(\Delta u + \bar{\alpha}u),\ g^* = \gamma_2(\Delta v + \bar{\alpha}v)\,, \tag{9}$$

in view of (1), we obtain

$$\begin{cases} \bar{u}_t = b_1\bar{u} + b_2\bar{v} + \bar{f}^* + \bar{f} \\ \bar{v}_t = b_3\bar{u} + b_4\bar{v} + \bar{g}^* + \bar{g} \end{cases} \tag{10}$$

with

$$\begin{cases} \bar{f}^* = \dfrac{1}{\alpha}f^*\Big|_{(u=\alpha\bar{u})}, & \bar{g}^* = \dfrac{1}{\beta}g^*\Big|_{(v=\beta\bar{v})}, & b_2 = \dfrac{\beta}{\alpha}a_2 \\ \bar{f}^* = \dfrac{1}{\alpha}f^*\Big|_{(u=\alpha\bar{u})}, & \bar{g}^* = \dfrac{1}{\beta}g^*\Big|_{(v=\beta\bar{v})}, & b_3 = \dfrac{\alpha}{\beta}a_3\,. \end{cases} \tag{11}$$

In the sequel we will use essentially the following peculiar Liapunov functional

$$V(\bar{u},\bar{v}) = \frac{1}{2}\left[A(\|\bar{u}\|^2 + \|\bar{v}\|^2) + \|b_1\bar{v} - b_3\bar{u}\|^2 + \|b_2\bar{v} - b_4\bar{u}\|^2\right], \tag{12}$$

with

$$A = b_1b_4 - b_2b_3 = b_1b_4 - a_2a_3, I = b_1 + b_4. \tag{13}$$

By virtue of

(14) $$\frac{dV}{dt} = (A+b_3^2+b_4^2)\langle\bar{u},\bar{u}_t\rangle + (A+b_1^2+b_2^2)\langle\bar{v},\bar{v}_t\rangle - (b_1b_3+b_2b_4)(\langle\bar{u},\bar{u}_t\rangle + \langle\bar{v},\bar{u}_t\rangle).$$

taking into account that along the solutions of (10) one immediately obtains

(15) $$\begin{cases} \langle\bar{u},\bar{u}_t\rangle = b_1\langle\bar{u},\bar{u}_t\rangle + b_2\langle\bar{u},\bar{v}\rangle + \langle\bar{u},\bar{f}^* + \bar{f}\rangle \\ \langle\bar{v},\bar{v}_t\rangle = b_3\langle\bar{u},\bar{v}\rangle + b_4\langle\bar{v},\bar{v}\rangle + \langle\bar{v},\bar{g}^* + \bar{g}\rangle \\ \langle\bar{v},\bar{u}_t\rangle = b_1\langle\bar{u},\bar{v}\rangle + b_2\langle\bar{v},\bar{v}\rangle + \langle\bar{v},\bar{f}^* + \bar{f}\rangle \\ \langle\bar{u},\bar{v}_t\rangle = b_3\langle\bar{u},\bar{u}_t\rangle + b_4\langle\bar{u},\bar{v}\rangle + \langle\bar{u},\bar{g}^* + \bar{g}\rangle, \end{cases}$$

by straightforward calculations it turns out that along the solution of (10)

(16) $$\frac{dV}{dt} = AI(\|\bar{u}\|^2 + \|\bar{v}\|^2) + \Psi^* + \Psi$$

with

(17) $$\begin{cases} \Psi^* = \langle a_1\bar{u} - a_3\bar{v},\bar{f}^*\rangle + \langle a_2\bar{v} - a_3\bar{u},\bar{g}^*\rangle \\ \Psi = \langle a_1\bar{u} - a_3\bar{v},\bar{f}\rangle + \langle a_2\bar{v} - a_3\bar{u},\bar{g}\rangle \\ a_1 = A + b_3^2 + b_4^2,\ a_2 = A + b_1^2 + b_2^2 \\ a_3 = b_1b_3 + b_2b_4. \end{cases}$$

REMARK 1. We observe that

i) the eigenvalues of (8) are given by

(18) $$\lambda = \frac{I\sqrt{I^2 - 4A}}{2},$$

hence

(19) $$\begin{cases} I = \lambda_1 + \lambda_2 \\ A = \lambda_1\lambda_2 \\ AI = (\lambda_1 + \lambda_2)\lambda_1\lambda_2. \end{cases}$$

Therefore

(20) $$I < 0$$

imply the asymptotic exponential stability of the null solution of (8), while either

(21) $$I > 0$$

or

$$A < 0\,, \tag{22}$$

imply the instability. In fact let (22) hold. Then the eigenvalues of (8) are real positive numbers. Analogously when (21) hold with $A \geq 0$, at least one of the eigenvalues of (8) is a real positive number (case $I \geq 4A$), or has positive real part (case $I < 4A$).

ii) The rescaling $\{u = \alpha\overline{u}, v = \beta\overline{v}\}$ does not influence A and I.

3. $L^2(\Omega)$-STABILITY

LEMMA 1. *Let*

$$\begin{cases} \gamma_1 = \gamma_2 \\ A > 0\,. \end{cases} \tag{23}$$

Then

$$\Psi^* \leq 0\,. \tag{24}$$

PROOF. In view of (17) and $(22)_2$ it follows that

$$a_i > 0 \quad i = 1, 2. \tag{25}$$

$$\Psi^*(\Omega) = \gamma_1 a_1\left[-\|\nabla\overline{u}\|^2 + \overline{a}\|\overline{u}\|^2\right] + \gamma_2 a_2\left[-\|\nabla\overline{v}\|^2 + \overline{a}\|\overline{v}\|^2\right] + (\gamma_1+\gamma_2)a_3[\langle\nabla\overline{v}, \nabla\overline{u}\rangle - \overline{a}\langle\overline{u}, \overline{v}\rangle]. \tag{26}$$

For $\gamma_1 = \gamma_2 = \gamma$, it follows that

$$\Psi^*(\Omega) = \begin{cases} -A\gamma\left[\|\nabla\overline{u}\|^2 + \|\nabla\overline{v}\|^2 - \overline{a}\left(\|\overline{u}\|^2 + \|\overline{v}\|^2\right)\right] \\ -\gamma\left[\|\nabla(b_1\overline{u} + b_3\overline{v})\|^2 - \overline{a}\|b_1\overline{u} + b_3\overline{v}\|^2\right] + \\ -\gamma\left[\|\nabla(b_2\overline{u} + b_4\overline{v})\|^2 - \overline{a}\|b_2\overline{u} + b_4\overline{v}\|^2\right]. \end{cases} \tag{27}$$

Let $\gamma_1 \neq \gamma_2$ and assume, for the sake of concreteness, $\gamma_1 < \gamma_2$. Then the following Lemmas hold.

LEMMA 2. *Let*

$$\begin{cases} \gamma_1 < \gamma_2 \\ A > 0\,. \end{cases} \tag{28}$$

If exists a constant μ such that choosing

$$\frac{\alpha}{\beta} = \mu \tag{29}$$

it turns out that

$$\frac{|a_3|}{\sqrt{a_1a_2}} \leq 2\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_1+\gamma_2}, \tag{30}$$

then (24) *holds.*

Proof. (30) implies either

$$(\gamma_1+\gamma_2)a_3 = \pm 2\sqrt{\gamma_1\gamma_2 a_1 a_2} \tag{31}$$

or

$$(\gamma_1+\gamma_2)a_3 = \pm 2\sqrt{\gamma_1\bar{\gamma} a_1 a_2} \tag{32}$$

with

$$\gamma_1 \leq \bar{\gamma} = \frac{(\gamma_1+\gamma_2)^2 a_3^2}{4\gamma_1 a_1 a_2} < \gamma_2. \tag{33}$$

Then in view of (31) one obtains

$$\Psi^* = -\left[\|\nabla(\sqrt{a_1\gamma_1}\,\bar{u} \mp \sqrt{a_2\gamma_2\bar{v}})\|^2 - \bar{a}\|\sqrt{a_1\gamma_1}\bar{u} \mp \bar{a}\sqrt{a_2\gamma_2\bar{v}}\|^2\right] < 0. \tag{34}$$

Analogously – in the case (32) – setting

$$\varepsilon = \gamma_2 - \bar{\gamma}$$

it follows that

$$\Psi^* = -\varepsilon a_2\left[\|\nabla\bar{v}\|^2 - \bar{a}\|\bar{v}\|^2\right] - \left[\|\nabla\left(\sqrt{a_1\gamma_1}\bar{u} \mp \sqrt{a_2\bar{\gamma}\bar{v}}\right)\|^2 \mp \bar{a}\|\sqrt{a_1\gamma_1}\bar{u} \mp \bar{a}\sqrt{a_2\bar{\gamma}\bar{v}}\|^2\right] < 0. \tag{35}$$

Lemma 3. *Let* (28) *and*

$$b_1a_2a_3b_4 < 0 \tag{36}$$

hold. Then choosing

$$\mu = \frac{\alpha}{\beta} = \left|\frac{a_2b_4}{b_1a_3}\right|^{\frac{1}{2}} \tag{37}$$

(24) *holds.*

Proof. In fact (37) implies $a_3 = 0$ and (24) is immediately implied by (26).

Lemma 4. *Let* (28) *and either*

$$\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}|b_4| < 2\sqrt{A+b_4^2} \tag{38}$$

or

$$\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}|b_1| < 2\sqrt{A+b_1^2} \tag{39}$$

hold. Then (24) *holds.*

PROOF. (30) – in view of (29) – can be written

$$|b_1 a_3 \mu^2 + a_2 b_4| \leq \frac{2\sqrt{\gamma_1\gamma_2}}{\gamma_1+\gamma_2}\sqrt{(A+\mu^2 a_3^2 + b_4^2)\left[\mu^2(A+b_1^2)+a_2^2\right]}. \tag{40}$$

Therefore (38) implies that (40) is verified strictly as inequality for $\mu = 0$, hence exists a μ_1 such that $\mu < \mu_1$ implies that (40) is verified. Analogously (39) implies that (40) is verified strictly as inequality in the limit $\mu \to \infty$. Therefore exists a μ_2 such that for $\mu > \mu_2$ Lemma 2 holds.

THEOREM 1. *Let* $(6)_1$ *and* (24) *hold. Then*

$$\begin{cases} I < 0 \\ A > 0 \end{cases} \tag{41}$$

imply the (local) L^2*-asymptotic exponential stability of the null solution of* (1).

PROOF. In view of (16), it follows that

$$\frac{dV}{dt} \leq -AI\left(\|\bar{u}\|^2 + \|\bar{v}\|^2\right) + \Psi. \tag{42}$$

By virtue of $(41)_2$, V is positive definite, further from (12) it easily follows that V is a measure equivalent to the $L^2(\Omega)$-norm. In fact (12) implies

$$k_1\left(\|\bar{u}\|^2 + \|\bar{v}\|^2\right) < V < k_2\left(\|\bar{u}\|^2 + \|\bar{v}\|^2\right) \tag{43}$$

with

$$\begin{cases} k_1 = \frac{1}{2}A \\ k_2 = \frac{A}{2} + \sum_1^4 b_i^2 . \end{cases} \tag{44}$$

On the other hand – by virtue of (6) – it follows that exist two positive constant k and δ such that

$$\|\bar{f}\| + \|\bar{g}\| \leq \delta\left(\|\bar{u}\|^2 + \|\bar{v}\|^2\right)^{k+\frac{1}{2}} \tag{45}$$

hence

$$\begin{cases} \langle a_1\bar{u} - a_3\bar{v}, \bar{f}\rangle \leq \delta(a_1 + |a_3|)\left(\|\bar{u}\|^2 + \|\bar{v}\|^2\right)^{1+k} \\ \langle a_2\bar{v} - a_3\bar{u}, \bar{g}\rangle \leq \delta(a_2 + |a_3|)\left(\|\bar{u}\|^2 + \|\bar{v}\|^2\right)^{1+k} \\ \Psi \leq \delta_1\left(\|\bar{u}\|^2 + \|\bar{v}\|^2\right)^{1+k} \end{cases} \tag{46}$$

with

$$\delta_1 = \delta \max(a_1 + |a_3|, a_2 + |a_3|). \tag{47}$$

Therefore (42)-(47) imply

$$\frac{dV}{dt} \le -dV + d_1 V^{1+k} \tag{48}$$

with

$$d = \frac{A|I|}{k_2}, d_1 = \frac{\delta_1}{k_1^{1+k}} V^{1+k}. \tag{49}$$

It follows that

$$V_0^k < \frac{d}{d_1} \tag{50}$$

implies

$$\frac{dV}{dt} < -\eta V \tag{51}$$

with

$$\eta = d\left(1 - \frac{d_1}{d} V_0^k\right) \tag{52}$$

and hence

$$V \le V_0 e^{-\eta t}. \tag{53}$$

4. Instability

We consider now the linear instability of the null solution of (1). Precisely, let $\{\bar{a}_n, \varphi_n\}$, $(n = 1, 2, ..; \bar{a} = a_1)$ be the sequence of the eigenvalues (with the associated eigenfunctions in $H_0^1(\Omega)$ and $H_*^1(\Omega)$ according to (3) and (4)-(5), respectively) of (1). We study the instability of the null solution of

$$\begin{cases} u_{,t} = a_1 u + a_2 v + \gamma_1 \Delta u \\ v_{,t} = a_3 u + a_4 v + \gamma_2 \Delta v \end{cases} \tag{54}$$

with respect to the perturbations

$$\begin{cases} u = \sum_{n=1}^{\infty} u_n , & v = \sum_{n=1}^{\infty} v_n \\ u_n = X_n(t)\varphi_n , & v_n = Y_n(t)\varphi_n \\ X_n \in C^1(\Re^+) , & Y_n \in C^1(\Re^+) . \end{cases} \tag{55}$$

Then, by virtue of the linearity and

$$\Delta\varphi_n = -\bar{a}_n\,\varphi_n \tag{56}$$

(8) gives

$$\begin{cases} \dfrac{dX_n}{dt} = b_{1n}X_n + a_2Y_n \\ \dfrac{dY_n}{dt} = a_3X_n + b_{4n}Y_n \end{cases} \tag{57}$$

with

$$\begin{cases} b_{1n} = a_1 - \gamma_1\bar{a}_n \\ b_{4n} = a_4 - \gamma_2\bar{a}_n \end{cases}. \tag{58}$$

Setting

$$\begin{cases} A_n = b_{1n}b_{4n} - a_2a_3 \\ I_n = b_{1n} + b_{4n} \end{cases} \tag{59}$$

it follows that (for $\gamma_2 \geq \gamma_1$, $\gamma_2 = \gamma_1 + \xi$)

$$\begin{cases} A_n = A_1 + [\gamma_1^2(a_n - \bar{a}) + \xi(\bar{a}_n\gamma_1 - a_1) - \gamma_1 I_1](\bar{a}_n - \bar{a}) \\ I_n = I_1 - (\gamma_1 + \gamma_2)(\bar{a}_n - \bar{a})\,. \end{cases} \tag{60}$$

THEOREM 2. *The linear instability of the null solution of* (1) *is implied by each n such that either*

$$I_n > 0 \tag{61}$$

or

$$A_n < 0\,. \tag{62}$$

PROOF. See *i*) of Remark 1.

REMARK 2.

i) Generally the coefficients a_i depend on some dimensionless parameters characteristic of the phenomenon at hands. Assuming that the parameters are only two and denoted by R and C, (61)-(62) can be written:

$$I(n,R,C) = b_{1n} + b_{4n} > 0 \tag{63}$$

$$A(n,R,C) = b_{1n}b_{4n} - a_2a_3 < 0 \tag{64}$$

respectively. Let (63)-(64) imply respectively

$$R \leq F(n,C) \tag{65}$$

$$R \leq G(n,C) \tag{66}$$

and set

$$(67)\qquad \begin{cases} R_c^{(1)} = \inf\limits_{N^+} F(n, C) \\ R_c^{(2)} = \inf\limits_{N^+} G(n, C) \end{cases}.$$

Then the critical value $R^{(c)}$ of R guaranteeing that $R > R_C$ implies instability is given by

$$(68)\qquad R^{(c)} = \inf\,(R_c^{(1)}, R_c^{(2)})\,.$$

ii) By virtue of $(60)_2$, I_n is a decreasing function of $\bar{a}_n - \bar{a}$. Hence exists an $\bar{n}$ such that

$$0 \le I_{\bar{n}}\,, \quad I_{\bar{n}+1} < 0,$$

which implies

$$R_c^{(1)} = \inf_{n \le \bar{n}} F(n, C)\,.$$

Analogously, in view of $(60)_1$, it follows that exists a n^* such that

$$A_{n^*} < 0, \qquad A_{n^*+1} \ge 0$$

which imply

$$R_c^{(2)} = \inf_{n \le n^*} G(n, C)\,.$$

iii) In the case $\gamma_1 \neq \gamma_2$ the destabilizing effect of diffusion can appear. We refer to [7-8] for the details.

5. Final remarks

i) The L^2-asymptotic stability implies the analogous stability with respect to the essential sup, in the weak sense of the asymptotic (Lebesgue) measure stability. In fact denoting by $\widehat{\Omega}(\varepsilon, |\varphi(\boldsymbol{x}, t)|)$ the largest subdomain of Ω on each point of which, at time t, $|\varphi|$ is bigger than $\varepsilon > 0$ and by $\widetilde{\mu}(\varepsilon, |\varphi(\boldsymbol{x}, t)|)$ the Lebesgue measure of $\widehat{\Omega}$, for $p \ge 1$, the following inequality holds [9]

$$(69)\qquad \widetilde{\mu}\left(\|\varphi(\boldsymbol{x}, t)\|_p^{\frac{p}{p+1}}, |\varphi(\boldsymbol{x}, t)| \right) \le \|\varphi(\boldsymbol{x}, t)\|_p^{\frac{p}{p+1}}, \forall t \ge 0.$$

In particular for $p = 2$, it follows that

$$(70)\qquad \widetilde{\mu}\left(\|\varphi(\boldsymbol{x}, t)\|_2^{\frac{2}{3}}, |\varphi(\boldsymbol{x}, t)| \right) \le \|\varphi(\boldsymbol{x}, t)\|_2^{\frac{2}{3}}$$

and hence

$$(71)\qquad \forall\,\varepsilon > 0\,, \quad \lim_{t\to\infty} \|\varphi(\boldsymbol{x}, t)\| = 0 \quad \Rightarrow \quad \lim_{t\to\infty} \widetilde{\mu}(\varepsilon, |\varphi(\boldsymbol{x}, t)|) = 0\,.$$

ii) If $\Psi \le 0$, then Theorem 1 guarantees global L^2-asymptotic exponential stability.

iii) The stability-instability theorems 1-2 continue to hold for the more general system

$$\begin{cases} u_t = a_1 u + a_2 v + \boldsymbol{e} \cdot \nabla u + \gamma_1 \Delta u + f \\ v_t = a_3 u + a_4 v + \boldsymbol{b} \cdot \nabla v + \gamma_2 \Delta v + g \end{cases} \tag{72}$$

with $\boldsymbol{e}$ and $\boldsymbol{b}$ divergence free vectors, at least when either $a_3 = 0$ or $\boldsymbol{e} = \boldsymbol{b}$ in the case (3). In fact the contribution of $\boldsymbol{e} \cdot \nabla u, \boldsymbol{e} \cdot \nabla v$ to $\dfrac{dV}{dt}$ is

$$\begin{aligned} &\langle a_1 u - a_3 v, \boldsymbol{e} \cdot \nabla u \rangle + \langle a_2 v - a_3 u, \boldsymbol{b} \cdot \nabla v \rangle = \\ &= \frac{1}{2} \left[\langle a_1, \boldsymbol{e} \cdot \nabla u^2 \rangle + \langle u, \boldsymbol{b} \cdot \nabla v \rangle \right] = \\ &= - a_3 \langle v, (\boldsymbol{e} - \boldsymbol{b}) \cdot \nabla u \rangle. \end{aligned}$$

In the case (4), the additional conditions $\boldsymbol{e} \cdot \boldsymbol{n} = \boldsymbol{b} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ are needed.

iv) By virtue of theorems 1-2 it turns out that either when Lemma 1 or Lemma 3 hold, the coincidence between the condition of linear and nonlinear stability is reached without restriction on γ_1, γ_2. This coincidence – without restriction on γ_1, γ_2 – can be obtained also in the case

$$a_2 a_3 > 0 \tag{73}$$

by choosing as Liapunov functional $E = \dfrac{1}{2} \left[\|\bar{u}\|^2 + \|\bar{v}\|^2 \right]$ with a suitable choice of $\dfrac{\alpha}{\beta}$. In fact (1), in view of $(9)_1$, $(9)_2$ can be written

$$\bar{\boldsymbol{u}}_t = \mathcal{L}\bar{\boldsymbol{u}} + \mathcal{N}\bar{\boldsymbol{u}} \tag{74}$$

with

$$\mathcal{L} = \begin{pmatrix} a_1 + \gamma_1 \Delta & \dfrac{\beta}{\alpha} a_2 \\ \dfrac{\alpha}{\beta} a_3 & a_4 + \gamma_2 \Delta \end{pmatrix} \tag{75}$$

$$\mathcal{N} = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}, \qquad \bar{\boldsymbol{u}} = \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}. \tag{76}$$

In the case (73) the linear operator $\mathcal{L}$ can be symmetrized by choosing

$$\frac{\alpha}{\beta} = \left(\frac{a_2}{a_3} \right)^{1/2}. \tag{77}$$

This choice allows to obtain the coincidence between linear and nonlinear stability in the $L^2(\Omega)$-norm (we refer to [4, pp. 80-82] for the proof). Further from

$$\frac{1}{2} \frac{d}{dt} \|\boldsymbol{u}\|^2 = < \mathcal{L}\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}} > + < \mathcal{N}\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}} >, \tag{78}$$

it follows that if

$$< \mathcal{N}\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}} > \leq 0 \tag{79}$$

then one obtains the *global stability*. This happens for instance in the case

$$f = \boldsymbol{e} \cdot \nabla \bar{u}, \qquad g = \boldsymbol{e} \cdot \nabla \bar{v} \tag{80}$$

with $\boldsymbol{e}$ divergence free vector depending on $(\bar{u}, \bar{v})$, under the additional condition $\boldsymbol{e} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ when (4) hold. In fact it follows that

$$< \mathcal{N}\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}} > = \frac{1}{2} < \boldsymbol{e}, \nabla(\bar{u}^2 + \bar{v}^2) > = 0. \tag{81}$$

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