Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Edoardo Vesentini

On a class of inner maps

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **16** (2005), n.4, p. 215–226.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2005_9_16_4_215_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2005.

Rend. Mat. Acc. Lincei
s. 9, v. 16:215-226 (2005)

EDOARDO VESENTINI

ON A CLASS OF INNER MAPS

To Guido Zappa on the occasion of his 90th birthday

ABSTRACT. — Let f be a continuous map of the closure $\overline{\Delta}$ of the open unit disc Δ of $\mathbb{C}$ into a unital associative Banach algebra $\mathcal{A}$, whose restriction to Δ is holomorphic, and which satisfies the condition whereby $0 \notin \sigma(f(z)) \subset \overline{\Delta}$ for all $z \in \Delta$ and $\sigma(f(z)) \subset \partial\Delta$ whenever $z \in \partial\Delta$ (where $\sigma(x)$ is the spectrum of any $x \in \mathcal{A}$). One of the basic results of the present paper is that f is *spectrally constant*, that is to say, $\sigma(f(z))$ is then a compact subset of $\partial\Delta$ that does not depend on z for all $z \in \overline{\Delta}$. This fact will be applied to holomorphic self-maps of the open unit ball of some J^*-algebra and in particular of any unital C^*-algebra, investigating some cases in which not only the spectra but the maps themselves are necessarily constant.

KEY WORDS: Associative Banach algebra; Holomorphic map; Spectrum; Spectral radius.

An inner function on the open unit disc Δ of $\mathbb{C}$ defines a holomorphic map of Δ into itself such that the radial limits of $f(re^{i\theta})$ as $r \uparrow 1$ exist and have modulus one almost everywhere on the unit circle $\partial\Delta$. The inner function f is injective if and only if it is a holomorphic automorphism of Δ; hence, it has a (unique) continuous extension to $\overline{\Delta}$, which is a homeomorphism of this latter set onto itself. At the other extreme, if the inner function f is the restriction to Δ of a continuous complex-valued function on $\overline{\Delta}$ – which will be denoted by the same symbol f – and if $f(\Delta) \neq \Delta$, then, by the maximum modulus theorem, f is constant: $f(\overline{\Delta})$ is a point in $\partial\Delta$.

The situation changes radically if Δ is replaced by the open unit ball B of $\mathbb{C}^n$ (for some $n > 1$) endowed with the euclidean norm. In which case, a non-constant inner function f on B (whose existence was established by A. B. Aleksandrov in 1983; see [8] also for historical and bibliographical references) has quite an irregular behaviour on ∂B. For example, if f extends continuously to one point of ∂B, then f is constant (see, *e.g.*, [7, 8]). On the other hand, if B is the open unit polydisc in $\mathbb{C}^n$ (for some $n > 1$), non-constant holomorphic maps $B \to \Delta$ having continuous extensions of modulus one at each point of the distinguished boundary of B, do exist, for which the validity of a similar conclusion to the one stated at the beginning in the case in which $B = \Delta$ can then be investigated.

A possible explanation of these different behaviours may be found in the fact that the polydisc $\mathbb{C}^n$ is the unit ball of an algebra, whereas the unit euclidean ball is not when $n > 1$. In the present paper we shall test this explanation by replacing B by the unit ball of a Banach algebra $\mathcal{M}$ and f by a holomorphic function on B, satisfying suitable boundary conditions on ∂B, with values in a unital Banach algebra $\mathcal{A}$. These latter conditions can be weakened in the case in which $\mathcal{M}$ is any unital C^*-algebra (or, more in general, any J^*-algebra, [3], for which the set of extreme points of $\overline{B}$ is not empty).

One of the main results is expressed by the following theorem.

THEOREM. *Let B be the open unit ball of a unital C^*-algebra $\mathcal{M}$. Let f be a holomorphic map of B into a complex unital Banach algebra $\mathcal{A}$, which has a continuous extension to the set Γ of all extreme points of $\overline{B}$.*

If, for every $x \in B$, the spectrum $\sigma(f(x))$ of $f(x)$ is contained in $\overline{\Delta}$ and if

$$\sigma(f(x)) \subset \partial\Delta \qquad \forall\, x \in \Gamma \tag{1}$$

then, either $f(x)$ is not invertible in $\mathcal{A}$ at some point $x \in B$ or $\sigma(f(x))$ does not depend on x.

If f maps B into the open unit ball of $\mathcal{A}$, if (1) holds and if there is $x_0 \in B$ for which $f(x_0)$ is invertible in $\mathcal{A}$ and

$$\|f(x_0)^{-1}\| = 1, \tag{2}$$

then, either $f(x)$ is not invertible in $\mathcal{A}$ at some $x \in B$ or $f(x)$ does not depend on x.

Assuming in particular $\mathcal{A} = \mathcal{M} = \mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, *if $\|f(x)\| \le 1$ for all $x \in B$, if $f(x_0)$ is invertible and (2) holds for some $x_0 \in B$, then either $f(x)$ is not invertible for some $x \in B$ or there is a linear isometric automorphism y of $\mathcal{H}$ such that $f(x) = y_{|B}$ for all $x \in B$.*

The basic ideas in the proofs, which are already present in the case $B = \Delta$ that will be discussed in Section 2, rely heavily on maximum theorems for spectra and on properties of holomorphic families of linear automorphisms.

The results of Section 2 are instrumental in investigating the general case in which the role of Δ is played by the open unit ball of a J^*-algebra, leading in particular to the theorem stated above for unital C^*-algebras and to a similar result holding in the case of E. Cartan's spin factors.

1. INNER SPECTRAL RADIUS

Let $\mathcal{A}$ be an associative unital Banach algebra (1). For $u \in \mathcal{A}$, $\sigma(u)$ or $\sigma_{\mathcal{A}}(u)$ and $\rho(u)$ or $\rho_{\mathcal{A}}(u)$ will indicate respectively the spectrum and the spectral radius of u. Let $\mathcal{A}^{-1}$ be the set of all invertible elements of $\mathcal{A}$.

We will denote by $\kappa(u)$ or $\kappa_{\mathcal{A}}(u)$, and call *inner spectral radius* of u, the non-negative real number

$$\kappa(u) = \inf\{|\zeta| : \zeta \in \sigma(u)\}.$$

Thus, $\kappa(u) = 0$ if u is not invertible, or (by the spectral mapping theorem)

$$\kappa(u) = \frac{1}{\rho(u^{-1})}$$

if $u \in \mathcal{A}^{-1}$.

(1) Throughout this article, all Banach algebras will be tacitly assumed to be associative.

Assume now that $\mathcal{A}$ is not unital, and recall that $u \in \mathcal{A}$ is a quasi-regular element of $\mathcal{A}$ if there is $v \in \mathcal{A}$ (which is unique and is called sometimes the quasi-inverse of u) for which

$$uv - u - v = 0, uv = vu. \tag{3}$$

Let $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ be the Banach algebra obtained by adjoining an identity (denoted by 1 or $1_{\tilde{\mathcal{A}}}$) to $\mathcal{A}$, equipped with the norm

$$\|u + \zeta 1_{\tilde{\mathcal{A}}}\|_{\tilde{\mathcal{A}}} = \|u\|_{\mathcal{A}} + |\zeta| \; (u \in \mathcal{A},\ \zeta \in \mathbb{C}).$$

As is well known, if $\mathcal{A}$ is non-unital, for any $u \in \mathcal{A}$, $0 \in \sigma_{\mathcal{A}}(u) = \sigma_{\tilde{\mathcal{A}}}(u)$, and therefore $\kappa(u) = 0$ for all $u \in \mathcal{A}$.

On the other hand, by (3) $u \in \mathcal{A}$ is quasi-regular if, and only if, $u - 1$ is invertible in $\tilde{\mathcal{A}}$ (in which case

$$(u - 1_{\tilde{\mathcal{A}}})^{-1} = v - 1_{\tilde{\mathcal{A}}}),$$

so that u is quasi-regular if, and only if, $1 \notin \sigma_{\mathcal{A}}(u)$. Thus, in the case of a non-unital Banach algebra $\mathcal{A}$, some of the roles of the inner spectral radius and of the spectral radius are played by two numerical indicators $\beta(u)$ and $\gamma(u)$, where:

$\beta(u)$ is the distance, in $\mathbb{C}$, of 1 from $\sigma_{\mathcal{A}}(u)$, *i.e.*,

$$\beta(u) = \inf\{|\zeta - 1| : \zeta \in \sigma_{\mathcal{A}}(u)\},$$

and $\gamma(u)$ is the supremum of the distances, in $\mathbb{C}$, from 1 to the points of $\sigma_{\mathcal{A}}(u)$:

$$\gamma(u) = \sup\{|\zeta - 1| : \zeta \in \sigma_{\mathcal{A}}(u)\}.$$

Hence,

$$\beta(u) \leq \gamma(u),$$

and, by the spectral mapping theorem,

$$\begin{aligned}\gamma(u) &= \sup\{|\zeta - 1| : \zeta \in \sigma_{\mathcal{A}}(u)\} = \sup\{|\zeta| : \zeta \in \sigma_{\tilde{\mathcal{A}}}(u) - 1\} \\ &\leq \rho(u) + 1.\end{aligned}$$

2. One complex variable

Let Δ be the open unit disc in $\mathbb{C}$, let $\mathcal{A}$ be a unital Banach algebra and let g be a holomorphic map of Δ into $\mathcal{A}$.

Theorem 1. *If g satisfies the following conditions* (*):

i) $\rho(g(z)) \leq 1 \; \forall \, z \in \Delta$;

(*) *Note added in proofs.* In a forthcoming article (*Inner maps and Banach algebras*), condition *iii)* has been replaced by the weaker hypotheses:

there exist $k > 0$ and $r_0 \in (0, 1)$ such that

$$1 > |z| > r_0 \Longrightarrow \kappa(g(z)) \geq k;$$

there exist a measurable set $H \subset [0, 2\pi]$, with Lebesgue measure 2π, such that

$$\lim_{r \uparrow 1} \kappa(g(re^{i\theta})) = 1$$

for all $\theta \in H$.

ii) $g(z)$ *is invertible,* i.e. $\kappa(g(z)) > 0 \ \forall \ z \in \Delta$;

iii) *for every* $\varepsilon \in (0, 1)$ *there is some* $\delta \in (0, 1)$ *such that*

(4) $$1 > |z| > 1 - \delta \Longrightarrow \kappa(g(z)) > 1 - \varepsilon,$$

then $\rho(g(z)) = \kappa(g(z)) = 1$ *at all* $z \in \Delta$ *and* $\sigma(g(z))$ *is a compact subset of* $\partial\Delta$ *that does not depend on* z.

Proof. The function $h : \Delta \ni z \mapsto g(z)^{-1} \in \mathcal{A}$ is holomorphic;

$$\rho(h(z)) = \rho\left(g(z)^{-1}\right) = \frac{1}{\kappa(g(z))} > 0 \ \forall \ z \in \Delta.$$

If $\rho(h(z)) < 1$ at some point $z \in \Delta$, there is some $\tau \in \Delta$, $\tau \neq 0$, such that $\tau \in \sigma(h(z))$. Since

$$\frac{1}{\tau} \in \sigma(g(z)),$$

then $\rho(g(z)) > 1$, contradicting *i)* and showing thereby that

(5) $$\rho(h(z)) \geq 1 \ \forall z \in \Delta.$$

For any $\varepsilon \in (0, 1)$ there is $\delta \in (0, 1)$ satisfying (4).

The function $\mu : \overline{\Delta} \to \mathbb{R}_+$ defined by

$$\mu(z) = \begin{cases} \rho(h(z)) & \text{if } z \in \Delta \\ 1 & \text{if } z \in \partial\Delta, \end{cases}$$

is upper-semicontinuous on $\overline{\Delta}$, and therefore it reaches a maximum at some point of $\overline{\Delta}$. By (5) and the maximum theorem for the spectral radius, [11, 12], $\rho(h(z)) = 1$ for all $z \in \Delta$ and the peripheral spectrum of $h(z)$ (*i.e.* the set $\partial\Delta \cap \sigma(h(z))$) is a compact subset of $\partial\Delta$ which does not depend on z.

By (4), the entire spectrum of $h(z)$ is a compact subset of $\partial\Delta$ which does not depend on z. The same conclusion holds for $\sigma(g(z))$. □

Corollary 1. *Let* g *be a continuous map of* $\overline{\Delta}$ *into* $\mathcal{A}$ *such that* $g_{|\Delta}$ *is holomorphic. If* g *satisfies conditions* i) *and* ii), *and is such that*

iv) $\kappa(g(z)) = 1 \ \forall \ z \in \partial\Delta$,

then $\kappa(g(z)) = 1$ *at all* $z \in \Delta$ *and* $\sigma(g(z))$ *is a compact subset of* $\partial\Delta$ *that does not depend on* z.

Remark. If $\mathcal{A} = \mathbb{C}$, in Corollary 1 – where g is a (scalar valued) inner function, and therefore $\sigma(g(z)) = g(z)$, $\rho(g(z)) = \kappa(g(z)) = |g(z)|$ – then *i)* and *ii)* are expressed by:

v) $0 < |g(z)| \leq 1 \ \forall \ z \in \Delta$,

and *iii)* reads:

vi) for every $\varepsilon \in (0, 1)$ there is some $\delta \in (0, 1)$ such that

$$1 > |z| > 1 - \delta \Longrightarrow |g(z)| > 1 - \varepsilon.$$

The conclusion of Theorem 1, whereby g is now equal to a constant of modulus one on Δ, can be reached through a direct application of the maximum modulus principle for scalar-valued holomorphic functions.

Following [4] we will come to the same conclusion showing, by a different argument, that if an inner function g does not vanish on Δ and satisfies condition *vi*), then it is constant.

The first hypothesis implies that g is a singular function, *i.e.*, up to the product by a complex number of modulus one, g is represented by the integral

$$g(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right),$$

where μ is a singular positive measure on $\partial\Delta$. The holomorphic function h on Δ expressed by

$$h(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta),$$

is such that

$$|g(z)| = \quad e^{-\Re h(z)} \ \forall\ z \in \Delta.$$

If $\{z_\nu\}$ is a sequence in Δ, converging non tangentially to any pre-assigned point of $\partial\Delta$, then, by condition *vi*), $\Re h(z_\nu) \to 0$, *i.e.* the non-tangential limits of $\Re h$ vanish at all points of $\partial\Delta$. That implies that the derivative $d\mu/d\theta$ vanishes identically on the unit circle, proving thereby that g is constant.

The representation of an inner function as the product of a singular function and of a Blaschke product yields then

LEMMA 1. *An inner function g is the restriction to Δ of a continuous function on $\overline{\Delta}$ if, and only if, it satisfies condition* vi) *and vanishes on a finite set of points of Δ.*

If the unital Banach algebra $\mathcal{A}$ is commutative, for any character χ of $\mathcal{A}$ the function $\chi \circ g$ is a scalar-valued holomorphic function which satisfies all the hypotheses stated above for g, and therefore is a constant of modulus one on Δ. That yields a different proof of Theorem 1, in the case of commutative unital Banach algebras.

EXAMPLE. If $\mathcal{A}$ is the uniform algebra $C(T)$ of all continuous functions on a compact Hausdorff space T, for every $x \in C(T)$ $\sigma(x)$ is the image $x(T)$ of T by x. Thus, by Theorem 1, if the holomorphic map $g : \Delta \to C(T)$ is such that

$$0 \notin g(z)(T) \subset \overline{\Delta} \,\forall\, z \in \Delta,$$

and if, for every $\varepsilon \in (0,1)$ there is $\delta \in (0,1)$ for which

$$1 - \delta < |z| < 1 \Longrightarrow 1 - \varepsilon < |g(z)(t)| \le 1 \,\forall\, t \in T,$$

then there is a function $y \in C(T)$ with $y(T) \subset \partial\Delta$, such that $g(z)(T) = y(T)$ for all $z \in \Delta$. We shall come back to this example in Proposition 6.

REMARK. When $\mathcal{A} = \mathbb{C}$, the conclusion of Theorem 1 involves the values of the function g and not only some gauges of those values, as in the general case. This gap can be overcome by appealing to the theory of holomorphic set-valued functions developed by K. Oka ([6], see also [1]), *i.e.* to functions k defined on Δ, such that, for any $z \in \Delta$, the set $k(z) \subset \mathbb{C}$ is compact,

$$\{(z,\zeta) : z \in \Delta, \tau \notin k(z)\} \subset \Delta \times \mathbb{C}$$

is a domain of holomorphy and the compact set-valued function k is upper semi-continuos.

According to a theorem of Z. Slodkowski ([10], Theorem IV, 365, 378-386), if

$$l = \sup\{\max\{|\tau| : \tau \in k(z)\} : z \in \Delta\} < \infty, \tag{6}$$

there is a separable complex Hilbert space $\mathcal{H}$ and a holomorphic map $F : \Delta \to \mathcal{L}(\mathcal{H})$ such that $k(z) = \sigma(F(z))$ for all $z \in \Delta$.

Hence, Theorem 1 yields

PROPOSITION 1. *Let k be an Oka-analytic set-valued function defined on Δ and satisfying* (6) *with $l \leq 1$. If, for every $\varepsilon \in (0,1)$ there is some $\delta \in (0,1)$ such that, whenever $1-\delta < |z| < 1$, $k(z)$ is contained in the annulus $\{\zeta \in \mathbb{C} : 1-\varepsilon < \zeta \leq 1\}$, then $k(z)$ is a compact subset of $\partial\Delta$ which does not depend on z, for all $z \in \Delta$.*

Suppose now that $\mathcal{A}$ is a closed unital subalgebra of the algebra $\mathcal{L}(\mathcal{E})$, of all bounded linear operators on a complex Banach space $\mathcal{E}$.

Let $z_0 \in \Delta$ and let $g(z_0)$ be a linear isometry of $\mathcal{E}$. Since $\sigma(g(z_0)) = \overline{\Delta}$ or $\sigma(g(z_0)) \subset \partial\Delta$ if $g(z_0)$ is respectively non-surjective or surjective, then the peripheral spectrum of $g(z_0)$ covers the entire unit circle if $g(z_0)$ is not surjective and coincides with $\sigma(g(z_0))$ if $g(z_0)$ is surjective. The maximum principles for the spectral radius and for the peripheral spectrum, [11, 12], yield

LEMMA 2. *If the holomorphic map $g : \Delta \to \mathcal{A}$ is such that $\rho(g(z)) \leq 1$ for all $z \in \Delta$ and if $g(z_0)$ is a linear isometry for some $z_0 \in \Delta$, then $\rho(g(z)) = 1$ for all $z \in \Delta$, and the peripheral spectrum of $g(z)$ is a compact subset of $\partial\Delta$ which does not depend on z.*

We will now investigate under which conditions the function g itself is constant.

First of all, if g satisfies *i*), *ii*) and *iii*), the conclusions of Theorem 1 hold also for the map $z \mapsto g(z)^{-1}$ (and the constant compact subsets $\sigma(g(z)^{-1})$ is the image of $\sigma(g(z))$ by the map $\zeta \mapsto \overline{\zeta}$).

If *i*) is replaced by the stronger condition

i') $\|g(z)\| \leq 1 \; \forall \, z \in \Delta$,

Theorem 1 implies that, if g satisfies *i'*), *ii*) and *iii*), then

$$\|g(z)\| = 1 \quad \forall \, z \in \Delta.$$

As for $g(z)^{-1}$, one can only say that $\|g(z)^{-1}\| \geq 1$ at all $z \in \Delta$.

Assume now that there is $z_0 \in \varDelta$ at which

$$\|g(z_0)^{-1}\| = 1. \tag{7}$$

Since, for any vector $\xi \in \mathcal{E}\backslash\{0\}$,

$$\|\xi\| = \|g(z_0)^{-1} g(z_0)\,\xi\| \leq \|g(z_0)\,\xi\| \leq \|\xi\|,$$

then $\|g(z_0)\xi\| = \|\xi\|$ for all $\xi \in \mathcal{E}$, *i.e.* the holomorphic family of linear contractions $z \mapsto g(z)$ of $\mathcal{E}$ contains the automorphism $g(z_0)$. Hence, [2, Proposition V.1.10], $g(z)$ is independent of z, and the following theorem holds.

THEOREM 2. *Let $g : \varDelta \to \mathcal{A} \subset \mathcal{L}(\mathcal{E})$ be a holomorphic function mapping $\varDelta$ into the open unit ball B of $\mathcal{A}$. If $g(z)$ is invertible in $\mathcal{A}$ for all $z \in \varDelta$, if, for every $\varepsilon \in (0,1)$ there is some $\delta \in (0,1)$ such that, whenever $1-\delta < |z| < 1$, $\sigma(g(z))$ is contained in the annulus $\{\zeta \in \mathbb{C} : 1-\varepsilon < |\zeta| \leq 1\}$ and if moreover, (7) holds at some point $z_0 \in \varDelta$, then $g(z)$ is (the restriction to B of) a linear isometric automorphism of $\mathcal{E}$ which does not depend on z.*

REMARK. A similar statement to Theorem 1 in the case in which the Banach algebra $\mathcal{A}$ is not unital can be established substituting invertible elements with quasi-regular elements and replacing the hypotheses *i*), *ii*), *iii*) by the following two conditions: *vii*) $\sigma_{\mathcal{A}}(g(z)) \subset \overline{\varDelta(1,1)}\backslash\{1\}$ $\forall z \in \varDelta$ (where $\varDelta(1,1)$ is the open disc in $\mathbb{C}$ with center 1 and radius 1); *viii*) for every $\varepsilon \in (0,1)$ the is some $\delta \in (0,1)$ such that

$$1 > |z| > 1-\delta \Longrightarrow \sigma_{\mathcal{A}}(g(z)) \subset \{\zeta : \zeta \in \overline{\varDelta(1,1)}\backslash\{1\}, |\zeta - 1| < \varepsilon\}. \tag{8}$$

Theorem 1 yields then

PROPOSITION 2. *If g satisfies both conditions* vii) *and* viii), *then $\sigma_{\mathcal{A}}(g(z))$ is a compact subset of $\partial\Delta(1,1)$ which does not depend on $z \in \overline{\varDelta}$.*

3. BANACH ALGEBRAS

Here and in the following σ, ρ and κ will stand for the spectrum, the spectral radius and the inner spectral radius in $\mathcal{A}$.

Let D be a bounded, convex, circular domain in a complex Banach algebra $\mathcal{B}$ and let f be a holomorphic map of D into $\mathcal{A}$ such that

$$\rho(f(x)) \leq 1 \;\forall\, x \in D \tag{9}$$

and that, given any $\varepsilon \in (0,1)$ there exists in $\mathcal{B}$ an open set $U_\varepsilon \supset \partial D$ satisfying the conditions:

$$x \in U_\varepsilon \cap D \Longrightarrow \kappa(f(x)) > 1-\varepsilon. \tag{10}$$

The intersection of D with the complex affine line in $\mathcal{B}$ defined by any two distinct points x_1 and x_2 in D is a bounded convex domain $D(x_1,x_2)$ which is biholomorphically equivalent to $\varDelta$. By Theorem 1 applied to the holomorphic function $g = f_{|D(x_1,x_2)}$, either

$0 \in \sigma(f(x))$ for some $x \in D(x_1, x_2)$ or there is a compact set $K \subset \partial\Delta$ such that $\sigma(f(x)) = K$ for all $x \in D(x_1, x_2)$. Letting x_2 vary in D, we obtain the following proposition.

PROPOSITION 3. *If* (9) *and* (10) *are satisfied, either* $f(x)$ *is not invertible in* $\mathcal{A}$ *for some* $x \in D$ *or there is a compact set* $K \subset \partial\Delta$ *such that* $\sigma(f(x)) = K$ *for all* $x \in D$.

Assume now, as in Section 2, $\mathcal{A}$ to be a closed unital subalgebra of the Banach algebra $\mathcal{L}(\mathcal{E})$, where $\mathcal{E}$ is, as before, a complex Banach space. Replacing D by the open unit ball B of $\mathcal{B}$, a similar argument to the one leading to Proposition 3, based now on Theorem 2, yields

PROPOSITION 4. *If* (9) *and* (10) *hold for all* $x \in B$, *if* f *maps* B *into the closed unit ball of* $\mathcal{A}$ *and if there is* $x_0 \in B$ *for which* $f(x_0)$ *is invertible in* $\mathcal{A}$, *and*

$$\|f(x_0)^{-1}\| = 1, \tag{11}$$

then either $f(x)$ *is not invertible at some* $x \in B$ *or* $f(x)$ *is a linear isometric automorphism of* $\mathcal{E}$ *which does not depend on* x.

4. J^*-ALGEBRAS

In the case in which the role of $\mathcal{B}$ is played by a class of J^*-algebras, some of the foregoing results can be improved by weakening the hypotheses on the boundary behaviour of f.

Let $\mathcal{M}$ be a J^*-algebra, [3], let B be its open unit ball and let Γ be the set of the extreme points of $\overline{B}$, which will be always assumed to be non-empty. Let $\mathcal{A}$ be a closed unital subalgebra of the algebra $\mathcal{L}(\mathcal{E})$, where $\mathcal{E}$ is a complex Banach space, and let $f : B \to \mathcal{A}$ be a holomorphic map; as before, σ, ρ and κ will stand for the spectrum, the spectral radius and the inner spectral radius in $\mathcal{A}$.

THEOREM 3. *If*

$$\rho(f(x)) \leq 1 \; \forall \, x \in B \tag{12}$$

(equivalently, if $\sigma(f(x)) \in \overline{\Delta} \; \forall \, x \in B$*) and if,*

ix) *for every* $\varepsilon \in (0, 1)$, *there is an open set* $U_\varepsilon \supset \Gamma$ *such that*

$$x \in U_\varepsilon \cap B \Longrightarrow \kappa(f(x)) > 1 - \varepsilon, \tag{13}$$

then either $f(x) \notin \mathcal{A}^{-1}$ (i.e. $0 \in \sigma(f(x))$) *for some* $x \in B$ *or, for all* $x \in B$, $\sigma(f(x))$ *is a compact subset of* $\partial\Delta$ *which does not depend on* x.

PROOF. Suppose that $0 \notin \sigma(f(x))$ for all $x \in B$.

a) For any $w \in \Gamma$ let $g : \Delta \to \mathcal{A}$ be defined by $g : z \mapsto f(zw)$. By Theorem 1,

$\sigma(g(z)) = \sigma(f(zw))$ is a compact subset $K \subset \partial\Delta$ which does not depend on $z \in \Delta$. Since $\sigma(g(0)) = \sigma(f(0))$, K is the same for all $w \in \Gamma$.

b) For any $x \in B$, let M be a Moebius transformation of B mapping 0 to x. Denoting by the same symbol M the continuous extension of M to $B \cup \Gamma$, and letting $v \in M^{-1}(w)$, for any $w \in \Gamma$, the map

$$\Delta \ni z \mapsto zv$$

is the unique complex geodesic for the Carathéodory metric of B whose support S is such that $0 \in S$ and $v \in \overline{S}$ ([13], see also [5]).

The image by M of the disc $\{zv : z \in \Delta\}$ is the only complex geodesic in B the closure of whose support contains x and w. More exactly, if Λ is the support of a complex geodesic $\lambda : \Delta \to B$ such that $x \in \Lambda$ and $w \in \overline{\Lambda}$, then $\Lambda = M(S)$ and there is a Moebius transformation φ of Δ onto Δ such that

$$M(\varphi(z)v) = \lambda(z) \;\; \forall\, z \in \Delta.$$

Setting

$$g : \Delta \ni z \mapsto f \circ M(zv),$$

a) yields the conclusion. □

As a consequence of Lemma 2 the following proposition holds.

PROPOSITION 5. *If* (12) *is satisfied and if* $f(x_0)$ *is a linear isometry for some* $x_0 \in B$, *then either* $f(x) \notin \mathcal{A}^{-1}$ *for some* $x \in B$ *or* $\rho(f(x)) = 1$ *for all* $x \in B$, *and the peripheral spectrum of* $f(x)$ *is a compact subset of* $\partial\Delta$ *which does not depend on* x.

The peripheral spectrum of $f(x)$ covers the entire unit circle if $f(x_0)$ is not surjective and coincides with $\sigma(f(x)) = \sigma(f(x_0))$ if $f(x_0)$ is surjective. A similar argument replacing Theorem 1 by Theorem 2 yields

THEOREM 4. *Let* f *be a holomorphic map of the open unit ball* B *of* $\mathcal{M}$ *into the open unit ball of* $\mathcal{A} \subset \mathcal{L}(\mathcal{E})$. *If, condition* ix) *is satisfied, if there is* $x_0 \in B$ *for which* $f(x_0)$ *is invertible in* $\mathcal{A}$ *and*

$$\|f(x_0)^{-1}\| = 1, \tag{14}$$

then either $f(x)$ *is not invertible at some* $x \in B$ *or* $f(x)$ *is (the restriction to* B *of) a linear isometric automorphism of* $\mathcal{E}$ *which does not depend on* x.

If $\mathcal{A}$ is not unital, Proposition 2 and a similar argument to the proof of Theorem 3 leads to the following Theorem.

THEOREM 5. *If the holomorphic map* $f : B \to \mathcal{A}$ *is such that:*

$$\sigma_{\mathcal{A}}(f(x)) \subset \overline{\Delta(1,1)} \;\forall x \in B,$$

for every $\varepsilon \in (0,1)$*, there is an open set* $U_\varepsilon \supset \Gamma$ *such that*

$$x \in U_\varepsilon \cap B \Longrightarrow \sigma(f(x)) \subset \{\zeta : \zeta \in \overline{\Delta(1,1)}, |\zeta - 1| < \varepsilon\},$$

then either $f(x)$ *is not quasi-regular for some* $x \in B$ *or, for all* $x \in B$, $\sigma(f(x))$ *is a compact subset of the circle* $\partial\Delta(1,1)$ *which does not depend on* x.

Relevant examples of J^*-algebras to which Theorems 3 and 4 apply are unital C^*-algebras. Given such an algebra $\mathcal{M}$, which will be identified with one of its *-isomorphic images as a uniformly closed, unital, self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$ on some complex Hilbert space $\mathcal{H}$ (see, *e.g.*, [9]), we will now assume $\mathcal{M} = \mathcal{A}$ in Theorems 3 and 4. Theorem 4 implies then

COROLLARY 2. *If* $f : B \to B$ *is a holomorphic map satisfying condition* ix) *and if there is some* $x_0 \in B$ *for which* $f(x_0)$ *is invertible in the unital* C^**-algebra* $\mathcal{M}$ *and* (14) *is satisfied, then either* $f(x)$ *is not invertible at some* $x \in B$ *or* $f(x)$ *is (the restriction to* B *of) an isometric automorphism of* $\mathcal{M}$ *which does not depend on* x.

If the unital C^* algebra $\mathcal{M}$ is commutative, by the Gelfand Theorem $\mathcal{M}$ is isometrically *-isomorphic to the function algebra $C(T)$ of all complex-valued continuous functions on a compact Hausdorff space T, endowed with the uniform norm $\|x\| =$ $= \max\{|x(t)| : t \in T\}$.

Since in this case $\|x\| = \rho(x)$ for all $x \in C(T)$, the following proposition holds.

PROPOSITION 6. *Let* B *be the open unit ball of* $C(T)$*, and let* $f : \overline{B} \to C(T)$ *be a continuous map whose restriction to* B *is holomorphic. If*

$$|x(t)| = 1 \ \forall\, t \in T \Longrightarrow |f(x)(t)| = 1 \ \forall\, t \in T,$$

then either $f(x)(t) = 0$ *for some* $x \in B$ *and some* $t \in T$ *or there is* $y \in C(T)$ *with* $|y(t)| = 1 \ \forall\, t \in T$*, such that* $f(x) = y$ *for all* $x \in B$.

5. SPIN FACTORS

As examples of J^*-algebras *stricto sensu* we will now consider spin factors. A spin factor – or Cartan factor of type four (2) – is a closed, self-adjoint linear subspace $\mathcal{M}$ of $\mathcal{L}(\mathcal{K})$ (where $\mathcal{K}$ is a complex Hilbert space) such that, if $u \in \mathcal{M}$, u^2 is a scalar multiple of the identity operator I in $\mathcal{L}(\mathcal{K})$:

$$u^2 = aI \quad \text{for some } a \in \mathbb{C}. \tag{15}$$

The space $\mathcal{M}$ is endowed with two norms, with respect to which it is complete:

the operator norm $|\!|\!|\ |\!|\!|$ in $\mathcal{L}(\mathcal{K})$ and the Hilbert-space norm $\|\ \|$ associated to the inner product in $\mathcal{M}$ defined on $u, v \in \mathcal{M}$ by

$$2(u|v)I = uv^* + v^*u,$$

(2) See [3, 14] for definitions and basic results.

where u^* is the adjoint of v (³).

The two norms are equivalent and are related by the formula

$$|||u|||^2 = \|u\|^2 + \sqrt{\|u\|^4 + |(u|u^*)|^2}.$$

Therefore the open unit ball $B = \{u \in \mathcal{M} : |||u||| < 1\}$ is defined also by

$$B = \left\{ u \in \mathcal{M} : \|u\|^2 < \frac{1 + |(u|u^*)|^2}{2} < 1 \right\}.$$

It turns out that the set Γ of all extreme points of $\overline{B}$, [14], is given by

$$\Gamma = \{e^{i\theta}u : \theta \in \mathbb{R}, u \in \mathcal{M}, u = u^*, u^2 = I\}.$$

It follows from (15) that $\sigma(u)$ is contained in the set $\{-a^{1/2}, a^{1/2}\}$, and therefore $\rho(u) = |a|^{1/2}$. Furthermore, the spectrum of u coincides with the point-spectrum $p\sigma(u)$, [14]; moreover, if $a \neq 0$, if $a^{1/2} \in p\sigma(u)$ and if Π_u is the spectral projector associated to u and $\{a^{1/2}\}$, then

$$u = a^{1/2}(2\,\Pi_u - I).$$

If $\sigma(u) = \{a^{1/2}\}$, then $\Pi_u = I$ and $u = a^{1/2}I$, whilst, if $a = 0$, *i.e.* $\sigma(u) = \{0\}$, then $u = 0$. Theorems 3 and 4 imply

THEOREM 6. *Let f be a holomorphic map of the open unit ball B of $\mathcal{M}$ into $\mathcal{L}(\mathcal{K})$ with $\rho(f(x)) \leq 1$ for all $x \in B$. If, for every $\varepsilon \in (0,1)$, there is an open set $U_\varepsilon \supset \Gamma$ such that*

$$x \in U_\varepsilon \cap B \Longrightarrow \rho(f(x)) > 1 - \varepsilon,$$

then, either $f(x) = 0$ for some $x \in B$ or there exists $v \in \mathcal{M}$ with

$$b \in \sigma(v) \subset \{b, -b\}$$

for some $b \in \partial\Delta$, such that, for all $x \in B$,

$$f(x) = b(2\,\Pi_v - I) = b\begin{pmatrix} I_1 & 0 \\ 0 & -I_2 \end{pmatrix},$$

for all $x \in B$, where Π_v is the spectral projector associated to b and to v.

PROOF. By Theorem 3 there is $b \in \partial\Delta$ such that

(16) $$f(x) = b\big(2\,\Pi_{f(x)} - I\big) = b\begin{pmatrix} I_1 & 0 \\ 0 & -I_2 \end{pmatrix},$$

where: $\Pi_{f(x)}$ is the spectral projector associated to b and $f(x)$; I_1 and I_2 are the identity operators on $\mathcal{K}_1 = \operatorname{Ran}\,\Pi_{f(x)}$ and $\mathcal{K}_2 = \operatorname{Ker}\,\Pi_{f(x)}$.

(³) Since

$$(u + v^*)^2 = u^2 + (v^*)^2 + uv^* + v^*u,$$

$uv^* + v^*u$ is a scalar multiple of I.

Since

$$\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2,$$

setting, for $\xi \in \mathcal{K}$, $\xi_1 = \Pi_{f(x)}\xi$, $\xi_2 = \xi - \Pi_{f(x)}\xi$, then, by (16),

$$\|f(x)\xi\|^2_{\mathcal{K}} = \|\xi_1\|^2_{\mathcal{K}_1} + \|\xi_2\|^2_{\mathcal{K}_2} = \|\xi\|^2_{\mathcal{K}}$$

for all $\xi \in \mathcal{K}$. Theorem 4 yields the conclusion. □

References

[1] B. Aupetit, *Propriétés spectrales des algèbres de Banach*. Lecture Notes in Mathematics, 735, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[2] T. Franzoni - E. Vesentini, *Holomorphic maps and invariant distances*. North Holland, Amsterdam-New York-Oxford, 1980.

[3] L.A. Harris, *Bounded symmetric homogeneous domains in infinite dimensional spaces*. In: T.L. Hayden - T.J. Suffridge (eds.), *Proceedings on infinite dimensional holomorphy, University of Kentucky 1973*. Lecture Notes in Mathematics, 364, Springer, Berlin 1974, 13-40.

[4] K. Hoffman, *Banach spaces of analytic functions*. Prentice-Hall, Englewood Cliffs, N.J., 1962.

[5] M. Jarnicki - P. Pflug, *Invariant distances and metrics in complex analysis*. Walter de Gruyter, Berlin-New York 1993.

[6] K. Oka, *Note sur les familles de fonctions analytiques multiformes etc.* J. Sci. Hiroshima Univ. Ser. A, 4, 1934, 93-98.

[7] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$. Springer-Verlag, New York-Heidelberg-Berlin 1980.

[8] W. Rudin, *New constructions of functions holomorphic in the unit ball of* $\mathbb{C}^n$. Conference Board Math. Sci., 63, 1985.

[9] S. Sakai, *C*-algebras and W*-algebras*. Springer-Verlag, New York-Heidelberg-Berlin 1971.

[10] Z. Slodkowski, *Analytic set-valued functions and spectra*. Math. Ann., 256, 1981, 363-386.

[11] E. Vesentini, *Maximum theorems for spectra*. Essays on topology and related topics, Memoires dédiés à Georges de Rham, Springer-Verlag, Berlin-Heidelberg-New York 1970, 111-117.

[12] E. Vesentini, *Maximum theorems for vector valued holomorphic functions*. University of Maryland Technical Report, 69-132, 1969; Rend. Sem. Mat. Fis. Milano, 40, 1970, 1-34.

[13] E. Vesentini, *Complex geodesics*. Compositio Math., 44, 1981, 375-394.

[14] E. Vesentini, *Holomorphic isometries of spin-factors*. Rend. Sem. Mat. Univ. Politec. Torino, 50, 4, 1992, 427-455.

Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi, 24 - 10129 Torino