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Equazioni a derivate parziali. — *One-dimensional problem for heat and mass transport in oil-wax solution.* Nota (*) di Roberto Gianni e Anna G. Petrova, presentata dal Socio M. Primicerio.

Abstract. — A mathematical model of heat and mass transport in non-isothermal partially saturated oil-wax solution was formulated by A. Fasano and M. Primicerio [1]. This paper is devoted to the study of a one-dimensional problem in the framework of that model. The existence of classical solutions in a small time interval is proved, based on the application of a fixed-point theorem to the constructed operator. The technique employed is close to the one of [3] and [4].

Key words: Free boundary problems; Parabolic equations; Oil-wax solution; Wax segregation.

Riassunto. — *Un problema mono-dimensionale per il trasporto di calore e di massa in petroli ricchi di cera.* Un modello per il trasporto di calore e di massa in idrocarburi pesanti, parzialmente saturi è stato a suo tempo proposto da A. Fasano and M. Primicerio in [1]. Questo articolo è dedicato allo studio di un problema evolutivo mono-dimensionale (costituito da un problema a frontiera libera per un sistema accoppiato di equazioni paraboliche) sviluppato nel contesto di tale modello. Viene dimostrata l'esistenza e l'unicità, in un piccolo intervallo di tempo, di una soluzione. La dimostrazione è ottenuta utilizzando il teorema delle contrazioni applicato ad un opportuno operatore.

0. Introduction

In the last years an increasing interest has aroused on the study of the behaviour of oil in pipeline, especially with regards to the rheological properties of the oil and how they affect its flow. In particular when low grade oils are involved, namely the ones with a great concentration of heavy hydrocarbons (generally called wax), a critical phenomenon is observed, *i.e.* the wax is segregated and usually it migrates towards the wall of the pipe coating it and preventing the oil from flowing freely. These kind of studies are of particular importance nowadays when oil reserves are running short and any kind of oil is of interest to the industries. A complete theoretical explanation of wax migration is missing. However it has been proposed that it is due mainly to two phenomena: the first one is wax crystals displacement (in completely saturated oil) caused by the flow and precisely the presence of a shear rate; the second one is diffusion to the wall due to a concentration gradient induced, in turn, by a thermal gradient. The last effect can be obviously observed also in a static situation and then easily checked experimental; for this reason in [1] A. Fasano and M. Primicerio proposed a model for this peculiar situation (which was also analyzed in [2]). The model, dealing with oil which is not flowing but under the effect of a thermal gradient, is a free boundary problem for a parabolic system of two coupled equations: one for the concentration of wax in oil and the other one for the temperature (which is the only

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state variable taken into account). The free boundary appears naturally as the level set for concentration being equal to a saturation concentration beyond which wax can not stay dissolved in the liquid and starts to crystallize. From the mathematical point of view what we get is a parabolic system of differential equations, coupled through the main terms, for which a free boundary problem has to be solved. A weak formulation for such problem was provided in [1], the study of which will be the subject of a following paper. In this paper we devote ourselves to the proof of the well posedness of the problem in the classical sense, in the one dimensional case. The solution thus found has a regular free boundary and is defined just in a small time interval. The technique employed relies on a linearization of the system which allows us to use the well known results on linear parabolic system by Solonnikov (see [5]) enabling us to find the solution of our problem via a contraction mapping which, in turns, ensures the uniqueness of such solution. At the end of the paper in Section 5 we will concentrate ourselves on self similar solutions, which are proved to exist under some assumptions on the coefficients. Finally we want to stress the fact that throughout the paper the notations, if not differently specified, are the ones of [1].

1. Statement of a problem

In the spirit of what was said in the introduction and using the same notations of [1] we will consider the following problem: find five functions $u_i(x,t), T_i(x,t), (i=1,2), s(t)$ such, that:

$$\frac{\partial u_1}{\partial t} = D_G \frac{\partial^2 u_1}{\partial x^2} - Q_1[T],\ \frac{\partial T_1}{\partial t} = a_1 \frac{\partial^2 T_1}{\partial x^2},\ \ 0 < x < s(t), \tag{1.1}$$

$$\frac{\partial u_2}{\partial t} = D \frac{\partial^2 u_2}{\partial x^2} - Q_2[T], \qquad \frac{\partial T_2}{\partial t} = a_2 \frac{\partial^2 T_2}{\partial x^2}, \qquad s(t) < x < 1; \tag{1.2}$$

with conditions at the interface $x = s(t)$

$$u_1 = 0, \qquad u_2 = 0, \qquad T_1 = T_2,$$

$$D_G \frac{\partial u_1}{\partial x} - D \frac{\partial u_2}{\partial x} = D c_s'(T_2) \frac{\partial T_2}{\partial x} - D c_s'(T_1) \frac{\partial T_1}{\partial x},\ \ a_1 \frac{\partial T_1}{\partial x} = a_2 \frac{\partial T_2}{\partial x}; \tag{1.3}$$

The conditions at the fixed boundaries are

$$D_G \frac{\partial u_1}{\partial x} = -(1-\chi) D \frac{\partial c_s}{\partial x}, \qquad T_1 = h_1(t), \qquad x = 0,$$

$$D \frac{\partial u_2}{\partial x} = -D \frac{\partial c_s}{\partial x}, \qquad T_2 = h_2(t), \qquad x = 1; \tag{1.4}$$

and the initial conditions are

$$s(0) = s_0,\ \ u_1(x,0) = u_{1,0}(x),\ \ T_1(x,0) = T_{1,0}(x),\ \ 0 < x < s_0,$$

$$u_2(x,0) = u_{2,0}(x),\ \ T_2(x,0) = T_{2,0}(x),\ \ s_0 < x < 1. \tag{1.5}$$

In (1.1), (1.2) $Q_i[T]$ are operators acting on the function $T(x,t)$ defined as follows:

$$Q_i[T] = c_s'(T_i)\left(\frac{\partial T_i}{\partial t} - D\frac{\partial^2 T_i}{\partial x^2}\right) - Dc_s''(T_i)\left(\frac{\partial T_i}{\partial x}\right)^2, i = 1,2\,. \tag{1.6}$$

Through (1.6) we have the coupling of the equations for u_i and T_i.

The function $c_s(T)$ is a given, increasing, sufficiently smooth function; D, D_G, a_1, a_2 are positive constants, and $0 \leq \chi \leq 1$. Note that, in this context, the latent heat of the segregated phase is assumed to be negligible (as it is the case in most physical situations).

We now state the main result of the paper, *i.e.* a local in time classical existence theorem for problem (1.1)-(1.5)

THEOREM 1.1. *Assume that the following hypotheses are satisfied for some positive τ and $a \in (0,1)$:*

a) $h_i \in H^{2+a}([0,\tau]), (i=1,2); T_{1,0}(x) \in H^{3+a}([0,s_0]), T_{2,0}(x) \in H^{3+a}([s_0,1]);$

$$u_{1,0}(x) \in H^{3+a}([0,s_0]),$$

$$u_{2,0}(x) \in H^{3+a}([s_0,1]); 0 < s_0 < 1; u_{i,0}'(s_0) < 0, (i=1,2);$$

$$h_1(t) < h_2(t), (t \in [0,\tau]); u_{1,0}(x) \geq 0, (0 \leq x \leq s_0); u_{2,0}(x) \leq 0, (s_0 \leq x \leq 1). \tag{1.7}$$

b) *The function $c_s(T)$ possesses continuous derivatives up to 4-th order satisfying inequalities $|c_s^{(k)}(T)| \leq M_1, 0 \leq k \leq 4$ for T such that*

$$\min\{\inf_{t\in[0,\tau]} h_1(t), \inf_{x\in[0,s_0]} T_{1,0}(x)\} \leq T \leq \max\{\sup_{t\in[0,\tau]} h_2(t), \sup_{x\in[s_0,1]} T_{2,0}(x)\}; \tag{1.8}$$

c) *All the needed compatibility conditions necessary to have solutions having continuous second order derivatives in space and first order in time are satisfied.*

d) *The inequality corresponding to the complementing condition* (see § 611, Chapter 7 of [5]) *applied to the boundary value problem for the parabolic system obtained linearizing our problem is satisfied,* i.e.:

$$\det\begin{pmatrix} \sqrt{D_G} & -\sqrt{D} & (\sqrt{D_G a_1} + D)/(\sqrt{a_1} + \sqrt{D_G}) & -\sqrt{D} \\ 0 & 0 & \sqrt{a_1} & -\sqrt{a_2} \\ 1 & W & 0 & 0 \\ 0 & (D - D_G W)/D & 1 & 1 \end{pmatrix} \neq 0 \tag{1.9}$$

with

$$W = \frac{du_{1,0}}{dx}(s_0)/\frac{du_{2,0}}{dx}(s_0).$$

Then there exists a time interval $[0,t^*], t^* < \tau$, such that the one-dimensional problem (1.1)-(1.6) *has a unique solution $(u_1, u_2, T_1, T_2, s(t))$ such that*:

$$u_1(x,t) \in H^{3+a,\frac{3+a}{2}}([0,s(t)] \times [0,t^*]), \quad u_2(x,t) \in H^{3+a,\frac{3+a}{2}}([s(t),1] \times [0,t^*]),$$

$$T_1(x,t) \in H^{3+\alpha,\frac{3+\alpha}{2}}([0,s(t)] \times [0,t^*]), \quad T_2(x,t) \in H^{3+\alpha,\frac{3+\alpha}{2}}([s(t),1] \times [0,t^*]);$$

$$s(t) \in H^{\frac{3+\alpha}{2}}([0,t^*]).$$

Remark 1.1. In the simplest physical meaningful case $a_1 = a_2 = \kappa > D > D_G$ condition (1.9) takes form

$$\frac{-\sqrt{\kappa}}{D} \cdot \left(2\sqrt{D_G} \cdot DW + D\sqrt{D} + D \cdot \frac{(\sqrt{D_G \kappa} + D)}{\sqrt{\kappa} + \sqrt{D_G}} + D_G W \cdot \frac{(\sqrt{\kappa} - \sqrt{D})(\sqrt{D} - \sqrt{D_G})}{\sqrt{\kappa} + \sqrt{D_G}} \right) \neq 0$$

and is obviously fulfilled.

2. Formulation of the modified problem

In order to prove Theorem 1.1 we will transform problem (1.1)-(1.6) in an equivalent one in which an explicit differential equation for $s(t)$ is present. This will be done by differentiating formally equation (1.1), (1.2) with respect to x and making use of the following relations easily obtained from (1.1), (1.2) together with (1.6):

$$Q_i[T] = c_s'(T_i)(a_i - D) \frac{\partial^2 T_i}{\partial x^2} - D c_s''(T_i) \left(\frac{\partial T_i}{\partial x} \right)^2, \quad i = 1,2.$$

Letting:

$$v_i = \frac{\partial u_i}{\partial x}, (i = 1,2), \qquad v_3 = \frac{\partial T_1}{\partial x}, \qquad v_4 = \frac{\partial T_2}{\partial x}.$$

We have that such functions satisfy the following system of differential equations:

$$\frac{\partial v_1}{\partial t} = D_G \frac{\partial^2 v_1}{\partial x^2} - c_s'(T_1)(a_1 - D) \cdot \frac{\partial^2 v_3}{\partial x^2} +$$

$$+ c_s''(T_1)(3D - a_1) v_3 \cdot \frac{\partial v_3}{\partial x} + D c_s'''(T_1) v_3^3, \qquad 0 < x < s(t), \tag{2.1}$$

$$\frac{\partial v_2}{\partial t} = D \frac{\partial^2 v_2}{\partial x^2} - c_s'(T_2)(a_2 - D) \frac{\partial^2 v_4}{\partial x^2} +$$

$$+ c_s''(T_2)(3D - a_2) v_4 \cdot \frac{\partial v_4}{\partial x} + D c_s'''(T_2) v_4^3, \qquad s(t) < x < 1, \tag{2.2}$$

$$\frac{\partial v_3}{\partial t} = a_1 \cdot \frac{\partial^2 v_3}{\partial x^2}, \qquad 0 < x < s(t), \tag{2.3}$$

$$\frac{\partial v_4}{\partial t} = a_2 \frac{\partial^2 v_4}{\partial x^2}, \qquad s(t) < x < 1. \tag{2.4}$$

To obtain the boundary conditions at $x = s(t)$ for the new functions we rewrite 4-th and 5-th equality of (1.3) in terms of v_i and differentiate the first three equality of (1.3) with respect to t, obtaining:

$$D_G v_1 - D v_2 = D c_s'(T)(v_4 - v_3), \tag{2.5}$$

$$a_1 v_3 = a_2 v_4, \tag{2.6}$$

$$D_G \frac{\partial v_1}{\partial x} - \left(c_s'(T_1)(a_1 - D)\frac{\partial v_3}{\partial x} - D c_s''(T_1) v_3^2 \right) + v_1 s'(t) = 0, \tag{2.7}$$

$$D \frac{\partial v_2}{\partial x} - \left(c_s'(T_2)(a_2 - D)\frac{\partial v_4}{\partial x} - D c_s''(T_2) v_4^2 \right) + v_2 s'(t) = 0, \tag{2.8}$$

$$a_1 \frac{\partial v_3}{\partial x} + v_3 s'(t) = a_2 \frac{\partial v_4}{\partial x} + v_4 s'(t). \tag{2.9}$$

Now we express $s'(t)$ from equation (2.8):

$$s'(t) = (v_2)^{-1}\left(-D\frac{\partial v_2}{\partial x} + c_s'(T_2)(a_2 - D)\frac{\partial v_4}{\partial x} - D c_s''(T_2) v_4^2 \right), \tag{2.8'}$$

and, with the use of (2.5) and (2.6), rewrite conditions (2.7), (2.9) in the form

$$D_G \frac{\partial v_1}{\partial x} - \frac{v_1}{v_2} D \frac{\partial v_2}{\partial x} - c_s'(T_1)(a_1 - D) \cdot \frac{\partial v_3}{\partial x} + \frac{v_1}{v_2} c_s'(T_2)(a_2 - D)\frac{\partial v_4}{\partial x} = -D c_s''(T_1) v_3^2 - \frac{D c_s''(T_2) v_4^2 v_1}{v_2}, \tag{2.7'}$$

$$-\frac{D - D_G v_1/v_2}{c_s'(T_1)} \cdot \frac{\partial v_2}{\partial x} + a_1 \cdot \frac{\partial v_3}{\partial x} - \frac{D^2 + D_G(a_2 - D)v_1/v_2}{D} \cdot \frac{\partial v_4}{\partial x} = \frac{(D v_2 - D_G v_1) c_s''(T_2) v_4^2}{v_2 c_s'(T_2)}. \tag{2.9'}$$

Boundary conditions at $x = 0$ and $x = 1$ are obtained in similar way. Namely: at $x = 0$ we get:

$$D_G v_1 - D(1 - \chi) c_s'(h_1) v_3 = 0, \tag{2.10}$$

$$a_1 \cdot \frac{\partial v_3}{\partial x} = h_1'(t); \tag{2.11}$$

and at $x = 1$:

$$v_2 + c_s'(h_2) v_4 = 0, \tag{2.12}$$

$$a_2 \frac{\partial v_4}{\partial x} = h_2'(t). \tag{2.13}$$

The initial conditions are clearly:

$$v_1(x,0) = u_{1,0}'(x), \qquad v_3(x,0) = T_{1,0}'(x), \qquad (0 < x < s_0);$$

$$v_2(x,0) = u_{2,0}'(x), \qquad v_4(x,0) = T_{2,0}'(x), \qquad (s_0 < x < 1). \tag{2.14}$$

For the sake of brevity, in notations $c_s'(T_i), c_s''(T_i)$ and $c_s'''(T_i)$ the functions T_i stand for

$$T_1(x,t) = \int_0^x v_3(x,t)dt + h_1(t), \qquad T_2(x,t) = -\int_x^1 v_4(x,t)dt + h_2(t),$$

so that, in fact, the new problem thus obtained is a integro-differential problem.

System (2.1)-(2.14) will be named modified problem (MP).

3. Formulation of the auxiliary problem

We now introduce the auxiliary problem which will be used to prove the existence theorem. To this purpose first we perform a change of coordinates passing to the space variable $\xi = x/s(t)$ in the domain $0 < x < s(t)$ and to $\xi = (1-x)/(1-s(t))$ in the domain $s(t) < x < 1$. Then we set $w_i(\xi,t) = v_i(x,t)$ $(i = 1,2)$, replace $s(t)$ with given function $r(t)$ everywhere except of (2.8'), replace w_3, w_4 in the nonlinear terms of the parabolic equations with new, given functions $\theta_1(\xi,t), \theta_2(\xi,t)$ and replace w_1, w_2 in the nonlinear terms of boundary conditions at $\xi = 1$ with given V_1, V_2 . Thus we get the system of equations:

$$\frac{\partial w_1}{\partial t} = \frac{D_G}{r^2(t)} \cdot \frac{\partial^2 w_1}{\partial \xi^2} - \frac{c_s'(T_1)(a_1 - D)}{rh^2(t)} \cdot \frac{\partial^2 w_3}{\partial \xi^2} + \frac{\xi}{r(t)} \cdot r'(t) \cdot \frac{\partial w_1}{\partial \xi} + + \frac{c_s''(T_1)}{r(t)}(3D - a_1)\theta_1 \cdot \frac{\partial w_3}{\partial \xi} + f_1(\xi,t), \tag{3.1}$$

$$\frac{\partial w_2}{\partial t} = \frac{D}{(1 - r(t)^2)} \frac{\partial^2 w_2}{\partial \xi^2} - \frac{c_s'(T_2)(a_2 - D)}{(1 - r(t))^2} \cdot \frac{\partial^2 w_4}{\partial \xi^2} - \frac{\xi}{1 - r(t)} \cdot r'(t) \frac{\partial w_4}{\partial \xi} - - \frac{c_s''(T_2)(3D - a_2)}{1 - r(t)} \cdot \theta_2 \frac{\partial w_4}{\partial \xi} + f_2(\xi,t), \tag{3.2}$$

$$\frac{\partial w_3}{\partial t} = a_1 \cdot \frac{\partial^2 w_3}{\partial \xi^2} + \frac{\xi}{r(t)} \cdot r'(t) \frac{\partial w_3}{\partial \xi}, \tag{3.3}$$

$$\frac{\partial w_4}{\partial t} = \frac{a_2}{(1 - r(t)^2)} \frac{\partial^2 w_4}{\partial \xi^2} - \frac{\xi}{1 - r(t)} \cdot r'(t) \frac{\partial w_4}{\partial \xi}, \tag{3.4}$$

$$s'(t) = V_2^{-1}\left(\frac{D}{1 - r(t)} \cdot \frac{\partial w_2}{\partial \xi} - \frac{c_s'(T_2)(a_2 - D)}{1 - r(t)} \cdot \frac{\partial w_4}{\partial \xi} - Dc_s''(T_2)\theta_2^2\right). \tag{3.5}$$

Boundary conditions at $\xi = 1$:

$$D_G w_1 - D w_2 - D c_s'(T_1)(w_4 - w_3) = 0 \quad at \quad \xi = 1, \tag{3.6}$$

$$a_1 w_3 - a_2 w_4 = 0 \quad at \quad \xi = 1. \tag{3.7}$$

$$\frac{D_G}{r(t)}\cdot\frac{\partial w_1}{\partial\xi}+\frac{V_1}{V_2}\cdot\frac{D}{1-r(t)}\cdot\frac{\partial w_2}{\partial\xi}-\frac{c_s'(T_1)(a_1-D)}{r(t)}\cdot\frac{\partial w_3}{\partial\xi}-$$
$$-\frac{V_1}{V_2}\cdot\frac{1}{1-r(t)}\cdot c_s'(T_2)(a_2-D)\frac{\partial w_4}{\partial\xi}=\phi_3, \tag{3.8}$$

$$\frac{D-D_G V_1/V_2}{c_s'(T_1)(1-r(t))}\cdot\frac{\partial w_2}{\partial\xi}+\frac{a_1}{r(t)}\cdot\frac{\partial w_3}{\partial\xi}+\frac{D^2+D_G(a_2-D)V_1/V_2}{D(1-r(t))}\cdot\frac{\partial w_4}{\partial\xi}=\phi_4, \tag{3.9}$$

Boundary conditions at $\xi=0$:

$$D_G w_1 - D(1-\chi)c_s'(h_1)v_3 = 0, \tag{3.10}$$

$$\frac{a_1}{s(t)}\frac{\partial w_3}{\partial\xi}=\psi_1=-D_G c_s''(h_1)\theta_1^2+h_1'(t), \tag{3.11}$$

$$w_2+c_s'(h_2)w_4=0, \tag{3.12}$$

$$\frac{a_2}{1-h(t)}\frac{\partial w_4}{\partial\xi}=\psi_2=h_2'(t). \tag{3.13}$$

Initial conditions:

$$w_1(\xi,0)=u_{1,0}'(\xi\cdot s_0),\qquad w_2(\xi,0)=u_{2,0}'(1-\xi\cdot(1-s_0)),$$
$$w_3(\xi,0)=T_{1,0}'(\xi\cdot s_0),\qquad w_4(\xi,0)=T_{2,0}'(1-\xi\cdot(1-s_0)). \tag{3.14}$$

In problem (3.1)-(3.14)

$$f_1=Dc_s'''(T_1)\theta_1^3,\ \ f_2=Dc_s'''(T_2)\theta_2^3,$$

and

$$\phi_3=-Dc_s''(T_1(1,t))\theta_1^2(1,t)+\frac{Dc_s''(T_2(1,t))\theta_2^2(1,t)V_1}{V_2},$$

$$\phi_4=\frac{(DV_2-D_GV_1)c_s''(T_2(1,t))\theta_2^2(1,t)}{V_2c_s'(T_2(1,t))}.$$

For the sake of brevity in this new system we have maintained the same notation T_i as arguments for the functions c_s and its derivatives. However in this framework

$$T_1(\xi,t)=\int_0^\xi r(t)\cdot\theta_1(\xi,t)dt+h_1(t)\quad\text{and}\quad T_2(\xi,t)=\int_0^\xi(1-r(t))\cdot\theta_2(\xi,t)dt+h_2(t).$$

Thanks to such auxiliary problem an operator $\boldsymbol{S}$ acting on

$$(r(t),V_1(t),V_2(t),\theta_1(\xi,t),\theta_2(\xi,t))\in\boldsymbol{H}(\tau)$$

with:

$$\boldsymbol{H}(\tau)=H^{1+\frac{a}{2}}([0,\tau])\times H^{\frac{1+a}{2}}([0,\tau])\times H^{\frac{1+a}{2}}([0,t^*])\times$$
$$\times H^{1+a,\frac{1+a}{2}}([0,1]\times[0,\tau])\times H^{1+a,\frac{1+a}{2}}([0,1]\times[0,\tau])$$

can be defined as follows:

$$\boldsymbol{S}(r(t), V_1(t), V_2(t), \theta_1(\xi,t), \theta_2(\xi,t)) = \left(\widehat{s}(t), \widehat{V}_1(t), \widehat{V}_2(t), \widehat{\theta}_1(\xi,t), \widehat{\theta}_2(\xi,t)\right),$$

where:

$$\widehat{s}(t) = s_0 +$$

$$+\int_0^t \left(\frac{D}{V_2(1-r)}\frac{\partial w_2}{\partial \xi}(1,t) - \frac{c_s'(T_2(1,t))(a_2-D)}{V_2(t)(1-r(t))}\frac{\partial w_4}{\partial \xi}(1,t) - \frac{Dc_s''(T_2(1,t))\theta_2^2(1,t)}{V_2(t)}\right)dt,$$

$$\widehat{V}_i(t) = w_i(1,t), (i=1,2), \quad \widehat{\theta}_1(\xi,t) = w_3(\xi,t), \quad \widehat{\theta}_2(\xi,t) = w_4(\xi,t),$$

and $w_i(\xi,t)$ is the solution of the auxiliary problem.

Obviously the previous definition makes sense if the auxiliary problem admits a unique classical solution. This will be proved in the following section.

We will prove the existence of a fixed point for the operator $\boldsymbol{S}$ in the closed convex subset $\boldsymbol{M}_{t^*} \in \boldsymbol{H}(t^*)$ made of the elements of $\boldsymbol{H}(t^*)$ having norm bounded by a suitable constant M and satisfying the conditions:

$$r(t) \in H^{1+a/2}([0,t^*]), V_i(t) \in H^{\frac{1+a}{2}}([0,t^*]), \theta_i(\xi,t) \in H^{1+a,\frac{1+a}{2}}([0,1]\times[0,t^*]), (i=1,2),$$

$$r(0) = s_0, \quad V_1(0) = u_{1,0}'(s_0), \quad V_2(0) = u_{2,0}'(s_0),$$

$$\theta_1(\xi,0) = T_{1,0}'(\xi \cdot s_0), \quad \theta_2(\xi,0) = T_{2,0}'(1-\xi\cdot(1-s_0)); \tag{3.15}$$

$$a_1\theta_1(1,t) = a_2\theta_2(1,t), \quad t\in[0,t^*]; \tag{3.16}$$

$$|r(t)-s_0| \le \delta, \quad \delta < \max\{s_0, 1-s_0\}, \quad t \in [0,t^*]; \tag{3.17}$$

$$|V_2(t)| \ge |u_{2,0}'(s_0)|/2\ , \quad t\in[0,t^*]; \tag{3.18}$$

satisfying the compatibility conditions for the auxiliary problem and also the following condition:

$$\det\begin{pmatrix} \sqrt{D_G} & -\sqrt{D} & (\sqrt{D_G a_1}+D)/(\sqrt{a_1}+\sqrt{D_G}) & -\sqrt{D} \\ 0 & 0 & \sqrt{a_1} & -\sqrt{a_2} \\ 1 & V_1(t)/V_2(t) & 0 & 0 \\ 0 & 1-D_G V_1(t)/DV_2(t) & 1 & 1 \end{pmatrix} \neq 0 \tag{3.19}$$

for $t\in[0,t^*]$.

Note that the fixed point which we will obtain is also a solution of our original free boundary problem since it solves a system of equations which can easily be obtained from (2.1)-(2.6), (2.7′), (2.8′), (2.9′), (2.10)-(2.14) passing to the new space variables $\xi = x/s(t)$ in the domain $0<x<s(t)$ and $\xi = (1-x)/(1-s(t))$ in the domain $s(t)<x<1$.

4. Solvability of the auxiliary problem

Lemma 4.1. *If problem* (1.1)-(1.5) *possesses a classical solution* u_1, u_2, s, T_1, T_2, *then* T_i $(i = 1, 2)$ *satisfy inequality* (1.8).

Proof of Lemma 4.1. It is a standard application of maximum principle for scalar parabolic equations (applied to the equations for T_i) as can be found in Theorem 2.9 p. 23 of [5]. In this regard we use the boundary conditions at the free boundary to rule out the possibility that a maximum or a minimum appears on $x = s(t)$.

Lemma 4.2. *Let conditions* (1.7)-(1.9) *be fulfilled. Then for any choice of*

$$(r(t), V_1(t), V_2(t), \theta_1(\xi,t), \theta_2(\xi,t)) \in \boldsymbol{M}_{t^*}$$

the auxiliary problem (3.1)-(3.4), (3.6)-(3.14) *has a unique solution* $w_i(\xi,t) \in H^{2+\alpha,1+\alpha/2}$, *and this solution satisfies the inequality*:

$$\sum_{i=1}^{4} |w_i|^{(2+\alpha)}_{[0,1]\times[0,t^*]} \le C(M,t^*)\cdot\left(\sum_{i=1}^{2} |f_i|^{(\alpha)}_{[0,1]\times[0,t^*]} + \right. \tag{4.1}$$

$$\left. + \sum_{i=1}^{2}\left(|\phi_{i+2}|^{(1+\alpha)/2}_{[0,t^*]} + |\psi_i|^{(1+\alpha)/2}_{[0,t^*]}\right) + |u'_{1,0}|^{(2+\alpha)}_{[0,s_0]} + |u'_{2,0}|^{(2+\alpha)}_{[s_0,1]} + |T'_{1,0}|^{(2+\alpha)}_{[0,s_0]} + |T'_{2,0}|^{(2+\alpha)}_{[s_0,1]}\right),$$

while $s(t)$ *satisfies, using equation* (3.5), *the following inequality*:

$$|s|^{(1+(\alpha+1)/2)}_{[0,t^*]} \le C(M,t^*). \tag{4.2}$$

These two inequalities in fact prove the compactness of the operator. This fact allows us to write down easily an inequality of the following form:

$$\|\big(\widehat{s}(t), \widehat{V}_1(t), \widehat{V}_2(t), \widehat{\theta}_1(\xi,t), \widehat{\theta}_2(\xi,t)\big)\|_{\boldsymbol{M}_{t^*}} \le \tag{4.3}$$

$$\le C\|(r(t), V_1(t), V_2(t), \theta_1(\xi,t), \theta_2(\xi,t))\|_{\boldsymbol{M}_{t^*}} (t^*)^{\varepsilon} + C_1$$

for some positive ε and constants C, C_1, depending only on the boundary data and known coefficients.

Proof of Lemma 4.2. Immediately follows from Theorem 10.1 [5] if we note that Lopatinski condition is obviously fulfilled at $\xi = 0$ (see § 611, Chapter 7 of [5] and [6]). Moreover, condition (3.19) is nothing but Lopatinski condition at $\xi = 1$.

Lemma 4.3. *Operator* $\boldsymbol{S}$ *is continuous compact, contractive operator transforming* $\boldsymbol{M}_{t^*}$ *into itself.*

Proof of Lemma 4.3. The compactness of $\boldsymbol{S}$ follows from inequalities (4.1), (4.2).

Operator $\boldsymbol{S}$ is easily proved to be Lipschitz continuous from the metric space $\boldsymbol{M}_{t^*}$ endowed with its norm to a more regular one compactly embedded in $\boldsymbol{M}_{t^*}$ (see (4.1) and (4.2)). In fact, working as we did to get (4.3), we can prove:

$$(4.4)\quad \|\widehat{s^1}-\widehat{s^2},\ \widehat{v_1^1}-\widehat{v_1^2},\ \widehat{v_2^1}-\widehat{v_2^2},\ \widehat{\theta_1^1}-\widehat{\theta_1^2},\ \widehat{\theta_2^1}-\widehat{\theta_2^2}\|_{\boldsymbol{M}_{t^*}}\le$$

$$\le C\|r^1-r^2,\ \ v_1^1-v_1^2,\ \ v_2^1-v_2^2,\ \ \theta_1^1-\theta_1^2,\ \ \theta_2^1-\theta_2^2\|_{\boldsymbol{M}_{t^*}}\cdot(t^*)^{\varepsilon}$$

Conditions (3.15)-(3.19) are obviously fulfilled for images $\widehat{\theta_i}, \widehat{V_2}$ and $\widehat{s}$ of θ_i, V_2 and r respectively, at least for small t^*; and estimate (4.3) easily implies that $\boldsymbol{S}(\boldsymbol{M}_{t^*})\subseteq\boldsymbol{M}_{t^*}$ at least for a suitable large M and small t^*. Note in this regard that the equations satisfied by the difference of two solutions corresponding to two different entries are linear and with zero initial data. Obviously (4.4) implies the contractiveness of the operator.

Proof of Theorem 1.1. It suffices to note that the fixed point of $\boldsymbol{S}$ is the unique solution of the modified problem (M.P.) (2.1)-(2.6), (2.7′), (2.8′), (2.9′), (2.10)-(2.14) after it has undergone the change of coordinates defined at beginning of Section 3, namely $\xi = x/s(t)$ for $0<x<s(t)$ and $\xi=(1-x)/(1-s(t))$ for $s(t)<x<1$.

Finally the result follows observing that (M.P.) and Problem (1.1)-(1.6) are equivalent, once we set

$$T_1(x,t)=\int_0^x v_3(x,t)dx+T_1(t),\qquad T_2(x,t)=-\int_x^1 v_4(x,t)dx+T_2(t),$$

$$u_1(x,t)=-\int_x^{s(t)} v_1(x,t)dx,\qquad u_2(x,t)=\int_{s(t)}^x v_2(x,t)dx.$$

The only fact we have to ensure is that u_i and T_i $(i=1,2)$ belong to $H^{3+\alpha,\frac{3+\alpha}{2}}$. Actually, if we differentiate the first of conditions (1.3) with respect to t, we obtain

$$\frac{\partial u_1}{\partial t}(x,t)=-s'(t)\cdot v_1(s(t),t)-\int_x^{s(t)}\frac{\partial v_1}{\partial t}(x,t)dx.$$

The first term of right-hand part obviously belongs to $H^{1+\alpha,\frac{1+\alpha}{2}}$. Taking into account (2.1), the second integral term can be written as

$$D_G\frac{\partial v_1}{\partial x}(x,t)-D_G\frac{\partial v_1}{\partial x}(s(t),t)+c_s'(T_1(s(t)))(a_1-D)\frac{\partial v_3}{\partial x}(s(t))-$$

$$-Dc_s''(T_1(s(t)))v_3^2(s(t))-c_s'(T_1(x))(a_1-D)\frac{\partial v_3}{\partial x}(x)+Dc_s''(T_1(x))v_3^2(x),$$

where every term belongs to $H^{1+\alpha,\frac{1+\alpha}{2}}$. Analogously, $\partial u_2/\partial t$, $\partial T_1/\partial t$, $\partial T_2/\partial t$ belong to $H^{1+\alpha,\frac{1+\alpha}{2}}$. Thus the existence and uniqueness theorem is proved.

5. Self-similar solutions

Governing equations.

Let us consider problem (1.1)-(1.6) in the semi-infinite space interval and suppose that:

$$u_i(x,t) = u_i(\xi), \quad T_i(x,t) = T_i(\xi), \quad \xi = \frac{x}{\sqrt{t}}, \quad s(t) = \beta\sqrt{t}.$$

Denoting

$$Q_i(\xi) = c_s'(T_i)\big(-\frac{\xi}{2}T_i'(\xi) - DT_i''(\xi)\big) - Dc_s''(T_i)(T_i'(\xi))^2,$$

we obtain the self-similar version of the one-dimensional problem (1.1)-(1.6):

$$-\frac{\xi}{2}u_1' = D_G u_1'' - Q_1, \qquad 0 < \xi < \beta, \tag{5.1}$$

$$-\frac{\xi}{2}T_1' = a_1 T_1'', \qquad 0 < \xi < \beta, \tag{5.2}$$

$$-\frac{\xi}{2}u_2' = Du_2'' - Q_2, \qquad \beta < \xi < \infty, \tag{5.3}$$

$$-\frac{\xi}{2}T_2' = a_2 T_2'', \qquad \beta < \xi < \infty, \tag{5.4}$$

$$T_1(\beta) = T_2(\beta), \tag{5.5}$$

$$a_1 T_1'(\beta) = a_2 T_2'(\beta), \tag{5.6}$$

$$u_1(\beta) = u_2(\beta) = 0, \tag{5.7}$$

$$D_G u_1'(\beta) + Dc_s'(T_1(\beta))T_1'(\beta) = Du_2'(\beta) + Dc_s'(T_2(\beta))T_2'(\beta), \tag{5.8}$$

$$T_1(0) = T_0, \tag{5.9}$$

$$T_2(\infty) = T_\infty, \tag{5.10}$$

$$D_G u_1'(0) = -(1-\chi)Dc_s'(T_0)T_1'(0), \qquad (\chi = \text{const}, 0 \le \chi \le 1), \tag{5.11}$$

$$u_2(\infty) + c_s(T_\infty) = c_\infty, \tag{5.12}$$

where a_i, D, D_G, are given positive constants.

To give the problem physical meaning we should suppose that

$$0 < T_0 < T_\infty, \quad 0 < c_\infty < 1. \tag{5.13}$$

Also we assume that the function $c_s(T)$ is a continuously differentiable, nonnegative function with positive derivative, satisfying inequality

$$c_s(T_0) < c_\infty < c_s(T_\infty). \tag{5.14}$$

We will denote by (SS) problem (5.1)-(5.12), with data satisfying conditions (5.13), (5.14).

Together with problem (SS) we consider problem (SS*), which differs from (SS) in the boundary condition (5.11) which now takes the form:

$$D_G u_1'(0) = -(1 - \chi(u_1(0)))Dc_s'(T_0)T_1'(0), \tag{5.11*}$$

The coefficient χ represents the fraction of incoming mass to the wall $x = 0$ which leaves the system as solid deposit. It seems to be quite natural to consider this fraction as positive, nondecreasing continuous function of concentration of solid wax $u_1(0)$ on the wall $x = 0$, equal to zero at $u_1(0) \leq 0$ and not exceeding 1.

We define self similar solutions of problem (1.1)-(1.5) the solution $\beta, T_i(\xi), u_i(\xi)$ of (SS) or (SS*), such that $\beta > 0$ and

$$u_1(\xi) \in [0, 1 - c_s(T_1(\xi))] \text{ for } \xi \in [0, \beta]; \quad u_1(0) > 0;$$

$$-c_s(T_2(\xi)) \leq u_2(\xi) \leq 0 \text{ for } \xi \in [\beta, \infty). \tag{5.15}$$

Note that conditions (5.15) are imposed by the physical meaning of the problem.

In fact, as we will see later, problem (SS*) appears to be more physical, since its solution u_1 is automatically between zero and $1 - c_s(T_1)$; while in the case of problem (SS) this natural bound can be only obtained imposing some extra assumption on data which are not natural.

The simplest case.

To point out the peculiarities of the problem, caused by condition (5.11) and by the properties of function $c_s(T)$, we consider the case of equal diffusivities: $D_G = D = a_1 = = a_2 = a$. In this case the temperature in both phases is given by the formula:

$$T(\xi) = \frac{T_\infty - T_0}{\sqrt{\pi \cdot a}} \int_0^\xi \exp\left(-\xi^2/4a\right) d\xi + T_0,$$

and the total concentration by:

$$u + c_s(T) = c_\infty - \chi c_s'(T_0)(T_\infty - T(\xi)). \tag{5.16}$$

Here we stipulate that $T_i = T, u_i = u$, both if $\xi \in [0, \beta)$ and if $\xi \in (\beta, \infty)$. The unknown constant β is determined as the root of eq. u=0, that is:

$$\chi c_s'(T_0) = \frac{c_\infty - c_s(T(\beta))}{T_\infty - T(\beta)}, \tag{5.17}$$

where $T(\beta) = \frac{T_\infty - T_0}{\sqrt{\pi \cdot a}} \int_0^\beta \exp\left(-\xi^2/4a\right) d\xi + T_0$.

To guarantee the existence of a solution β fulfilling the second requirement of (5.15), we must impose the restriction:

$$\chi c_s'(T_0) < \frac{c_\infty - c_s(T_0)}{T_\infty - T_0} \tag{5.18}$$

on the data. However, the physical meaning of the problem does not impose such restriction which, for this reason, looks unnatural. Nevertheless it can be dropped if we study problem (SS*). Namely, it follows from eq. (5.16) that

$$u(0) + \chi(u(0))c_s'(T_0)(T_\infty - T_0) = c_\infty - c_s(T_0).$$

This equation has a unique root $0 < u(0) < 1 - c_s(T_0)$ satisfying

$$\chi(u(0)) = \frac{c_\infty - c_s(T_0) - u(0)}{c_s'(T_0)(T_\infty - T_0)},$$

consequently inequality (5.18) is automatically fulfilled. The rest of requirements (5.15) are satisfied provided eq. (5.17) has only one root. This means that the curve $c_s(T)$, when considered for $T \in [T_0, T_\infty]$, must intersect the straight line
$c_\infty - \chi c_s'(T_0)(T_\infty - T)$ in an unique point

$$T = T(\beta) = \frac{T_\infty - T_0}{\sqrt{\pi \cdot a}} \int_0^\beta \exp(-\xi^2/4a)d\xi + T_0.$$

This is surely the case if the function $c_s(T)$ does not change the sign of convexity in $[T_0, T_\infty]$ and inequalities (5.14), (5.18) are fulfilled. Thus we obtained

PROPOSITION 5.1. *In the case, when $D_G = D = a_1 = a_2 = a$ and $c_s(T)$ does not change the sign of convexity in $[T_0, T_\infty]$, problem* (SS*) *has an unique solution.*

If, in addition, inequality (5.18) *is fulfilled for given constant χ, then also problem* (SS) *has an unique solution.*

The general case.

Let us begin with finding the temperature solving (5.2), (5.4), (5.5), (5.6), (5.9), (5.10) for a given $\beta > 0$.

LEMMA 5.1. *For every $\beta > 0$ there exist twice differentiable monotonously increasing functions $T_i(\xi), (i = 1, 2)$, continuously depending on β, being the solution for the following problem*:

$$a_1 T_1'' + \frac{\xi}{2} T_1' = 0, \quad 0 < \xi < \beta$$

$$a_2 T_2'' + \frac{\xi}{2} T_2' = 0, \quad \beta < \xi < \infty,$$

$$T_1(\beta) = T_2(\beta), \qquad a_1 T_1'(\beta) = a_2 T_2'(\beta),$$

$$T_1(0) = T_0,$$

$$T_2(\infty) = T_\infty, \tag{5.19}$$

i.e. *problem* (5.2), (5.4), (5.5), (5.6), (5.9), (5.10).

PROOF. It is easy to obtain from (5.19) that

$$T_1(\xi) = \frac{T_0 \int_\xi^\beta \exp(-\xi^2/4a_1)d\xi + B \int_0^\xi \exp(-\xi^2/4a_1)d\xi}{\int_0^\beta \exp(-\xi^2/4a_1)d\xi} \tag{5.20}$$

and

$$T_2(\xi) = \frac{T_\infty - B}{\int_\beta^\infty \exp(-\xi^2/4a_2)d\xi} \cdot \int_\beta^\xi \exp(-\xi^2/4a_2)d\xi + B\,, \tag{5.21}$$

where $B = T_1(\beta) = T_2(\beta)$ is the unique root of equation

$$\frac{a_2 \cdot (T_\infty - B)}{(B - T_0)} = a_1 \cdot \frac{\exp(-\beta^2/4a_1) \int_\beta^\infty \exp(-\xi^2/4a_2)d\xi \cdot \exp(\beta^2/4a_2)}{\int_0^\beta \exp(-\xi^2/4a_1)d\xi}\,. \tag{5.22}$$

Obviously, $T_0 < B < T_\infty$.

PROPOSITION 5.2. *Let* $D \geq D_G$ *and*

$$\frac{c_\infty - c_s(T_0)}{T_\infty - T_0} > \chi c_s'(T_0)\frac{\sqrt{D}\sqrt{a_2}}{a_1}\,.$$

Then problem (5.1)-(5.12) *admits at least one solution with* $0 < \beta < \beta^*$, *where* β^* *is the unique root of equation* $c_s(T(\beta)) = c_\infty$.

PROOF. By integrating (5.1) and (5.3) we obtain

$$u_1'(\xi) = A_1 \exp\left(\frac{-\xi^2}{4D_G}\right) + \frac{1}{D_G} \cdot \exp\left(\frac{-\xi^2}{4D_G}\right) \int_0^\xi Q_1(\eta) \exp\left(\frac{\eta^2}{4D_G}\right) d\eta,$$

$$u_2'(\xi) = A_2 \exp\left(\frac{-\xi^2}{4D}\right) - c_s'(T_2(\xi))T_2'(\xi).$$

It follows from condition (5.11) that

$$D_G A_1 = -(1-\chi)D c_s'(T_0)T_1'(0)$$

and from (5.7), (5.10), (5.12) that

$$DA_2 = D\frac{c_\infty - c_s(T_2(\beta))}{\int_\beta^\infty \exp(-\xi^2/4D)d\xi}\,.$$

So condition (5.8) takes the form

$$(5.23)\quad \chi c_s'(T_0)T_1'(0) + \int_0^{\beta} c_s'(T(\xi))T_1'(\xi)\frac{\xi}{2}\frac{D-D_G}{DD_G}\exp(\xi^2/4D_G)d\xi =$$

$$= \frac{c_\infty - c_s(T_2(\beta))}{\int_\beta^\infty \exp(-\xi^2/4D)d\xi}\exp(\beta^2(D-D_G)/4DD_G),$$

where (see (5.20)-(5.22))

$$T_1'(\xi) = \frac{a_2(T_\infty - T_0)\exp(-\xi^2/4a_1)}{a_2\int_0^\beta \exp(-\xi^2/4a_1)d\xi + a_1\exp(-\beta^2/4a_1 + \beta^2/4a_2)\int_\beta^\infty \exp(-\xi^2/4a_2)d\xi}.$$

The existence of a solution β of equation (5.23) such, that $0 < \beta < \beta^*$ can be obtained with the use of simple analysis of the formula, taking into account conditions of Proposition 5.2 and making use of the elementary Weistrass theorem.

REMARK 5.2. The fulfillment of requirements (5.15) in this more general case is still an open question. The release of condition (5.15) lead to appearance of «supersaturated» or «undersaturated» zones whose physical meaning is unclear. Similarly it remains an open question whether the solution is unique at least in the case when the coefficients of diffusion differ.

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