Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Luisa Moschini, Alberto Tesei

Harnack inequality and heat kernel estimates for the Schrödinger operator with Hardy potential

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **16** (2005), n.3, p. 171–180.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2005_9_16_3_171_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2005.

Rend. Mat. Acc. Lincei
s. 9, v. 16:171-180 (2005)

Equazioni a derivate parziali. — *Harnack inequality and heat kernel estimates for the Schrödinger operator with Hardy potential.* Nota di Luisa Moschini e Alberto Tesei, presentata (*) dal Socio A. Tesei.

Abstract. — In this preliminary *Note* we outline some results of the forthcoming paper [11], concerning positive solutions of the equation $\partial_t u = \Delta u + \frac{c}{|x|^2}\, u \left(0 < c < \frac{(n-2)^2}{4};\, n \geq 3\right)$. A parabolic Harnack inequality is proved, which in particular implies a sharp two-sided estimate for the associated heat kernel. Our approach relies on the unitary equivalence of the Schrödinger operator $Hu = -\Delta u - \frac{c}{|x|^2}\, u$ with the opposite of the weighted Laplacian $\Delta_\lambda v = \frac{1}{|x|^\lambda} div(|x|^\lambda \nabla v)$ when $\lambda = 2 - n + 2\sqrt{c_0 - c}$.

Key words: Harnack inequality; Moser iteration technique; Weighted Laplace-Beltrami operator; Sharp heat kernel estimate; Hardy potential.

Riassunto. — *Disuguaglianza di Harnack e stime sul nucleo del calore per l'operatore di Schrödinger con potenziale di Hardy.* In questa *Nota* preliminare si presentano alcuni risultati del successivo lavoro [11], riguardanti soluzioni positive dell'equazione $\partial_t u = \Delta u + \frac{c}{|x|^2}\, u \left(0 < c < \frac{(n-2)^2}{4};\, n \geq 3\right)$. Si dimostra una disuguaglianza di Harnack parabolica, che in particolare implica una stima bilatera sul nucleo del calore associato. Il nostro approccio si basa sull'equivalenza unitaria dell'operatore di Schrödinger $Hu = -\Delta u - \frac{c}{|x|^2}\, u$ con l'opposto dell'operatore di Laplace pesato $\Delta_\lambda v = \frac{1}{|x|^\lambda} div(|x|^\lambda \nabla v)$ quando $\lambda = 2 - n + 2\sqrt{c_0 - c}$.

1. Introduction

We deal with positive solutions of the parabolic equation

$$\partial_t u = \Delta u + \frac{c}{|x|^2}\, u \tag{1.1}$$

in $\mathbb{R}^n \times (0, T]$ $(n \geq 3)$; here $c \in (0, c_0)$, $c_0 := \frac{(n-2)^2}{4}$ denoting the best constant in the Hardy inequality. The main purpose of this *Note* is to present a parabolic Harnack inequality for equation (1.1). As is well known, a number of interesting consequences can be drawn from such inequality; in particular, we point out a sharp two-sided estimate for the associated heat kernel (see Theorem 4.3 below).

Our approach relies on the change of unknown $u \to v := \frac{u}{\varphi}$, where $\varphi(x) = |x|^{\frac{\lambda_+}{2}}$ and

$$\lambda_+ := 2 - n + 2\sqrt{c_0 - c} \in (2 - n, 0)\ . \tag{1.2}$$

(*) Nella seduta del 16 giugno 2005.

This recasts equation (1.1) into the form

$$\partial_t v = \frac{1}{|x|^{\lambda_+}} div\left(|x|^{\lambda_+} \nabla v\right) \tag{1.3}$$

–namely, into the heat equation for the *weighted Laplacian* Δ_{λ_+} on the *weighted manifold* $(\mathbb{R}^n, |x|^{\lambda_+} dx)$. This suggests to derive results concerning equation (1.1) from those concerning heat semigroups on weighted manifolds (see [7] for an up-to-date account of the subject).

In particular, the implication (*ii*) $\Rightarrow$ (*i*) of Theorem 2.5 below suggests to prove both the *doubling property* and the *Poincaré inequality* in $(\mathbb{R}^n, |x|^{\lambda_+} dx)$, in order to demonstrate the parabolic Harnack inequality for equation (1.3), thus for equation (1.1). To do so, a technical obstruction is given by the lack of regularity of the weight $|x|^{\lambda_+}$; in fact, a standard assumption underlying Theorem 2.5 is that the weighted manifold $(\mathcal{M}, \mu)$ be endowed with a measure $d\mu = \varphi^2 dv$, where dv is the Riemannian measure and φ is a *smooth* positive function on $\mathcal{M}$ (see Section 2).

However, checking the above implication in the present case offers no difficulties, since the Moser iteration technique applies to measurable coefficients (in this connection, see [13]). Hence we prove that the doubling property and the Poincaré inequality are satisfied; then the result follows.

2. Mathematical framework

(*a*) Let $\Omega \subseteq \mathbb{R}^n$ be a domain containing the origin ($n \geq 3$). The Schrödinger operator $H = -\Delta - \frac{c}{|x|^2}$ (with Dirichlet homogeneous boundary conditions if $\partial\Omega$ is nonempty; $c \in (0, c_0)$) is defined in $L^2(\Omega)$ as the generator of the symmetric form

$$\mathcal{H}[u_1, u_2] := \int_\Omega \left(\nabla u_1 \nabla u_2 - \frac{c}{|x|^2} u_1\, u_2\right) dx$$

with domain $D(\mathcal{H}) := H_0^1(\Omega)$. In view of the Hardy inequality, the form $\mathcal{H}$ is nonnegative and $C_0^\infty(\Omega \setminus \{0\})$ is a core for it. The operator H is nonnegative and self-adjoint, so that $-H$ is the generator of a contraction holomorphic semigroup $\{e^{-Ht}\}_{t \geq 0}$ on $L^2(\Omega)$.

Let us recall the following well-known construction (*e.g.*, see [1]). Let $\varphi \in$ $\in C^\infty(\Omega \setminus \{0\})$, $\varphi > 0$; consider the weighted space $L^2_\varphi(\Omega) \equiv L^2(\Omega, \varphi^2 dx)$ and the unitary map

$$\Phi : L^2_\varphi(\Omega) \to L^2(\Omega), \qquad \Phi v := \varphi v \qquad (v \in L^2_\varphi(\Omega)) .$$

Define the nonnegative, self-adjoint operator

$$H_\varphi := \Phi^* H \Phi . \tag{2.1}$$

in $L^2_\varphi(\Omega)$. Clearly,

$$e^{-H_\varphi t} = \Phi^* e^{-Ht} \Phi \quad \text{for any } t \geq 0 , \tag{2.2}$$

$\{e^{-H_\varphi t}\}_{t \geq 0}$ denoting the semigroup on $L^2_\varphi(\Omega)$ generated by $-H_\varphi$; moreover, H_φ is the generator of the symmetric form

$$\text{(2.3)} \qquad \begin{cases} D(\mathcal{H}_\varphi) := \left\{ v \in L^2_\varphi(\Omega) \,\middle|\, \Phi v \in H^1_0(\Omega) \right\} \\ \mathcal{H}_\varphi[v_1, v_2] := \mathcal{H}[\Phi v_1, \Phi v_2] . \end{cases}$$

Observe that $C_0^\infty(\Omega \setminus \{0\})$ is also a core of $\mathcal{H}_\varphi$. It is easily checked that on this core there holds:

$$\text{(2.4)} \quad \mathcal{H}_\varphi[v_1, v_2] := \int_\Omega \left\{ \nabla(\varphi v_1)(\nabla \varphi v_2) - \frac{c}{|x|^2}\, v_1\, v_2\, \varphi^2 \right\} dx =$$

$$= \int_\Omega \left\{ \nabla v_1 \nabla v_2 - \left(\frac{\Delta\varphi}{\varphi} + \frac{c}{|x|^2} \right) v_1 v_2 \right\} \varphi^2 dx .$$

Set $\varphi(x) = |x|^{\frac{\lambda_+}{2}}$ with λ_+ defined in (1.2). Since $|x|^{\frac{\lambda_+}{2}}$ is a weak solution of the equation

$$\Delta\varphi + \frac{c}{|x|^2}\, \varphi = 0 \quad \text{ in } \Omega ,$$

the form (2.3) reads in this case:

$$\text{(2.5)} \qquad \begin{cases} D(\mathcal{H}_{\lambda_+}) := \{ v \in L^2_{\lambda_+}(\Omega) \,|\, |x|^{\frac{\lambda_+}{2}} v \in H^1_0(\Omega) \} \\ \mathcal{H}_{\lambda_+}[v_1, v_2] := \displaystyle\int_\Omega \nabla v_1 \nabla v_2\, |x|^{\lambda_+}\, dx \end{cases}$$

(here the subscript λ_+ instead of φ has been used). Since $C_0^\infty(\Omega \setminus \{0\})$ is a core of the above form and the norm induced on this core is equivalent to the norm

$$\text{(2.6)} \qquad v \to \|v\|_{H^1_{\lambda_+}} := \left\{ \int_\Omega \left(|\nabla v|^2 + v^2 \right) |x|^{\lambda_+}\, dx \right\}^{\frac{1}{2}},$$

the generator H_{λ_+} of the form (2.5) is the opposite of the weighted Laplacian Δ_{λ_+} on $(\Omega, |x|^{\lambda_+} dx)$, which is defined as follows.

Let $\lambda \in (2 - n, 0)$. Consider the symmetric form in $L^2_\lambda(\Omega) \equiv L^2(\Omega, |x|^\lambda dx)$

$$\mathcal{H}_\lambda[v_1, v_2] := \int_\Omega \nabla v_1 \nabla v_2\, |x|^\lambda\, dx$$

with domain $D(\mathcal{H}_\lambda) := H^1_{0,\lambda}(\Omega)$; the latter is defined as the closure of $C_0^\infty(\Omega \setminus \{0\})$ in the norm (2.6) (which coincides with the closure of $C_0^\infty(\Omega)$ in the same norm, since $\lambda > 2 - n$). Then the weighted Laplacian Δ_λ (with Dirichlet homogeneous boundary conditions if $\partial\Omega$ is nonempty) is defined in $L^2_\lambda(\Omega)$ as the opposite of the generator of the form $\mathcal{H}_\lambda$, namely:

$$\text{(2.7)} \qquad \begin{cases} D(\Delta_\lambda) := \left\{ v \in H^1_{0,\lambda}(\Omega) \,\middle|\, \dfrac{1}{|x|^\lambda}\, div\left(|x|^\lambda\, \nabla v \right) \in L^2_\lambda(\Omega) \right\} \\ \Delta_\lambda v := \dfrac{1}{|x|^\lambda}\, div\left(|x|^\lambda\, \nabla v \right) \qquad \text{for any } v \in D(\Delta_\lambda) . \end{cases}$$

We shall denote by $\{e^{\Delta_\lambda t}\}_{t\geq 0}$ the contraction holomorphic semigroup generated by Δ_λ in $L^2_\lambda(\Omega)$.

(β) Let us recall some relevant results concerning smoothly weighted manifolds (*e.g.*, see [7]).

A weighted manifold $(\mathcal{M},\mu) \equiv (\mathcal{M},g,\mu)$ is a Riemannian manifold $\mathcal{M}$ with metric g, endowed with a measure $d\mu = \varphi^2 dv$, where dv is the Riemannian measure and φ is a smooth positive function on $\mathcal{M}$ (observe that the same definition makes sense if φ is measurable and positive). The weighted manifold $(\mathcal{M},\mu)$ is complete if the metric space $(\mathcal{M},d)$, where d denotes the geodesic distance induced by the metric g, is complete.

The weighted Laplace-Beltrami operator on $(\mathcal{M},\mu)$

$$\Delta_\mu v := \frac{1}{\varphi^2} div(\varphi^2 \nabla v) = \frac{1}{\varphi^2\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\varphi^2 \sqrt{|g|}\, g^{i,j} \frac{\partial v}{\partial x_j}\right),$$

(where $(x_1,..,x_n)$ are local coordinates, $(g_{i,j}) = (g_{j,i})$, $|g| \equiv det\ (g_{i,j})$ and $(g^{i,j}) \equiv (g_{i,j})^{-1}$) with initial domain $C_0^\infty(\mathcal{M})$ is symmetric and nonpositive, in view of the Green formulas with respect to the measure μ. Moreover, it has a self-adjoint extension (unique if the manifold $\mathcal{M}$ is complete) in the weighted space $L^2(\mathcal{M},\mu)$; hence the *heat semigroup* $\{e^{\Delta_\mu t}\}_{t\geq 0}$ is defined in $L^2(\mathcal{M},\mu)$. It turns out to be a semigroup of integral operators – namely, there exists a smooth positive *heat kernel* $p_\mu(x,y,t)$ such that

$$\left(e^{\Delta_\mu t} v_0\right)(x) = \int_{\mathcal{M}} p_\mu(x,y,t) v_0(y)\, d\mu(y)$$

for any $v_0 \in L^2(\mathcal{M},\mu)$, $x \in \mathcal{M}$ and $t > 0$.

Let $B(x_0,r)$ denote the geodesic ball of $(\mathcal{M},g)$ centered at x_0 with radius r; set $V_{\mathcal{M}}(x_0,r) := \mu(B(x_0,r))$. Let us recall the following definitions.

DEFINITION 2.1. $(\mathcal{M},\mu)$ satisfies the doubling property if there exists $C_D > 0$ such that

$$V_{\mathcal{M}}(x_0,2r) \leq C_D V_{\mathcal{M}}(x_0,r)$$

for any $x_0 \in \mathcal{M}$, $r > 0$.

DEFINITION 2.2. $(\mathcal{M},\mu)$ satisfies the Poincaré inequality with parameter $\delta \in (0,1]$ and constant $C_P > 0$ if

$$\inf_{\xi\in\mathbb{R}} \int_{B(x_0,\delta r)} |f-\xi|^2 d\mu \leq C_P\, r^2 \int_{B(x_0,r)} |\nabla f|^2 d\mu$$

for any $x_0 \in \mathcal{M}$, $r > 0$ and $f \in C^1(\overline{B(x_0,r)})$.

DEFINITION 2.3. $(\mathcal{M},\mu)$ satisfies the parabolic Harnack inequality if there exists $C_H > 0$ such that, for any $x_0 \in \mathcal{M}, r > 0$, $s \in \mathbb{R}$, any positive solution v to the heat equation

$$\partial_t v = \Delta_\mu v$$

in $Q \equiv Q(x_0, r) := B(x_0, r) \times (0, r^2)$ satisfies

$$\operatorname*{ess\,sup}_{(x,t)\in Q_-} v(x,t) \le C_H \operatorname*{ess\,inf}_{(x,t)\in Q_+} v(x,t),$$

where

$$Q_- \equiv Q_-(x_0, r) := B\Big(x_0, \frac{r}{2}\Big) \times \left(\frac{r^2}{4}, \frac{r^2}{2}\right),$$

$$Q_+ \equiv Q_+(x_0, r) := B\Big(x_0, \frac{r}{2}\Big) \times \left(\frac{3}{4}r^2, r^2\right).$$

Definition 2.4. The heat kernel on $(\mathcal{M}, \mu)$ satisfies the Li-Yau estimate if there exist $C_1, C_2 > 0$ such that

$$\frac{C_1\ e^{-C_2\frac{d^2(x,y)}{t}}}{V_{\mathcal{M}}(x,\sqrt{t})^{\frac{1}{2}} V_{\mathcal{M}}(y,\sqrt{t})^{\frac{1}{2}}} \le p_\mu(t,x,y) \le \frac{C_2\ e^{-C_1\frac{d^2(x,y)}{t}}}{V_{\mathcal{M}}(x,\sqrt{t})^{\frac{1}{2}} V_{\mathcal{M}}(y,\sqrt{t})^{\frac{1}{2}}}$$

for any $x, y \in \mathcal{M}$ and $t > 0$.

The following result can be proved (see [8, Theorem 2.7]).

Theorem 2.5. *For any complete weighted manifold* $(\mathcal{M}, \mu)$ *the following properties are equivalent*:

(i) $(\mathcal{M}, \mu)$ *satisfies the parabolic Harnack inequality*;

(ii) $(\mathcal{M}, \mu)$ *satisfies the doubling property and the Poincaré inequality for some* $\delta \in (0, 1]$;

(iii) *the heat kernel on* $(\mathcal{M}, \mu)$ *satisfies the Li-Yau estimate.*

The main part of Theorem 2.5 is implication $(ii) \Rightarrow (i)$, which was proved independently in [6, 12] (the inverse implication $(i) \Rightarrow (ii)$ was proved in [12]). The equivalence $(i) \Leftrightarrow (iii)$ goes back to [5].

3. Parabolic Harnack inequality

The results of this section make use of the methods of the latter subsection in the present non-smooth case. Let $B(x_0, r) := \{x \in \mathbb{R}^n \,|\, |x - x_0| < r\}$, $Q \equiv Q(x_0, r) := B(x_0, r) \times (0, r^2)$ $(x_0 \in \mathbb{R}^n, r > 0)$. Set

$$V(x_0, r) := \int_{B(x_0,r)} |y|^\lambda dy \qquad (x_0 \in \mathbb{R}^n, r > 0).$$

The complete weighted manifold $(\mathbb{R}^n, |x|^\lambda dx)$ $(\lambda \in (-n, 0))$ satisfies both the doubling property and the Poincaré inequality; this is the content of the following

Theorem 3.1. *Let* $\lambda \in (-n, 0)$. *Then*:

(i) *there exists* $C_D > 0$ *such that*

$$V(x_0, 2r) \le C_D V(x_0, r) \tag{3.1}$$

for any $x_0 \in \mathbb{R}^n$, $r > 0$;

(ii) *there exist* $\delta \in (0,1]$, $C_P > 0$ *such that*

$$\int_{B(x_0,\delta r)} |f(y) - \hat{f}|^2 \, |y|^\lambda dy \le C_P \, r^2 \int_{B(x_0,r)} |\nabla f|^2(y) \, |y|^\lambda dy \tag{3.2}$$

for any $x_0 \in \mathbb{R}^n$, $r > 0$ *and* $f \in C^1(\overline{B(x_0,r)})$; *here*

$$\hat{f} := \frac{1}{V(x_0,\delta r)} \int_{B(x_0,\delta r)} f(y)|y|^\lambda dy \, .$$

Let us make the following definitions.

Definition 3.2. By a *solution* to equation (1.3) in Q we mean any $v \in C^1((0,r^2); L^2_\lambda(B(x_0,r))) \cap C((0,r^2); H^1_\lambda(B(x_0,r)))$ such that $\partial_t v$, $\nabla v \in L^2(Q, |x|^\lambda dx\,dt)$ and there holds

$$\iint_Q \left\{ \partial_t v \, \chi + \nabla v \nabla \chi \right\} |x|^\lambda dx\,dt \;=\; 0$$

for any $\chi \in C([0,r^2]; C_0^\infty(B(x_0,r)))$.

Definition 3.3. By a *solution* to equation (1.1) in Q we mean any $u \in C^1((0,r^2); L^2(B(x_0,r))) \cap C((0,r^2); H^1(B(x_0,r)))$ such that $\partial_t u$, ∇u, $\dfrac{u}{|x|^2} \in L^2(Q, dx\,dt)$ and there holds

$$\iint_Q \left\{ \partial_t u \, \chi + \nabla u \nabla \chi - \frac{c}{|x|^2} \, u\chi \right\} dx\,dt \;=\; 0$$

for any $\chi \in C([0,r^2]; C_0^\infty(B(x_0,r)))$.

Remark 3.4. Observe that Definition 3.3 excludes stationary solutions of (1.1) in $\mathcal{D}'(\mathbb{R}^n)$, which behave like $|x|^{\frac{\lambda_-}{2}}$ with

$$\lambda_- := 2 - n - 2\sqrt{c_0 - c}$$

as $|x| \to 0$ (see [2]).

Theorem 3.1 entails the Harnack inequality in $(\mathbb{R}^n, |x|^\lambda dx)$. This is the content of the following

Theorem 3.5. *Let* $\lambda \in (-n, 0)$. *Then there exists* $C_H > 0$ *such that, for any* $x_0 \in \mathbb{R}^n$, $r > 0$, *any positive solution* v *to equation* (1.3) *in* $Q(x_0,r)$ *satisfies*

$$\operatorname*{ess\,sup}_{(x,t)\in Q_-} v(x,t) \;\le\; C_H \operatorname*{ess\,inf}_{(x,t)\in Q_+} v(x,t),$$

where

$$Q_- \equiv Q_-(x_0,r) := B\Big(x_0, \frac{r}{2}\Big) \times \left(\frac{r^2}{4}, \frac{r^2}{2}\right) ,$$

$$Q_+ \equiv Q_+(x_0, r) := B\Big(x_0, \frac{r}{2}\Big) \times \Big(\frac{3}{4}r^2, r^2\Big) .$$

As a consequence of Theorem 3.5, we have:

THEOREM 3.6. *Let $c \in (0, c_0)$. Then there exists $C_H > 0$ such that, for any ball $B(x_0, r)$, any positive solution u to equation* (1.1) *in $Q(x_0, r)$ satisfies the inequality*

$$\operatorname*{ess\,sup}_{(x,t)\in Q_-}\Big(|x|^{\frac{|\lambda_+|}{2}} u(x,t)\Big) \le C_H \operatorname*{ess\,inf}_{(x,t)\in Q_+}\Big(|x|^{\frac{|\lambda_+|}{2}} u(x,t)\Big) , \tag{3.4}$$

where

$$Q_- \equiv Q_-(x_0, r) := B\Big(x_0, \frac{r}{2}\Big) \times \Big(\frac{r^2}{4}, \frac{r^2}{2}\Big) ,$$

$$Q_+ \equiv Q_+(x_0, r) := B\Big(x_0, \frac{r}{2}\Big) \times \Big(\frac{3}{4}r^2, r^2\Big) .$$

Let us briefly discuss the proof of the above results. Theorem 3.5 follows from Theorem 3.1 using the Moser iteration technique as in [12]. In turn, Theorem 3.6 follows from Theorem 3.5 by the transformation $u(x,t) \to v(x,t) = |x|^{\frac{|\lambda_+|}{2}} u(x,t)$ discussed at length in Section 2. Concerning Theorem 3.1, claim (i) follows from the following estimates:

$$D_1 r^{\lambda+n} \le V(x,r) \le D_2 r^{\lambda+n} \qquad \text{if } |x| < 2r , \tag{3.5}$$

$$D_1 r^n(|x|+r)^{\lambda} \le V(x,r) \le D_2 r^n(|x|+r)^{\lambda} \qquad \text{if } |x| \ge 2r , \tag{3.6}$$

which are proved to hold, for $\lambda \in (-n, 0)$ and suitable $D_1, D_2 > 0$, for any $x \in \mathbb{R}^n$ and $r > 0$. As for claim (ii), we first prove it both for *anchored balls* $B(0, r)$ and for *remote balls* $B(x, r)$, $|x| \ge 2r$, for any $r > 0$; then the general case follows as in Proposition 4.2 in [8].

It is worth pointing out a slightly different way to reach the same conclusions. In view of estimates (3.5)-(3.6), it is easily seen that the weight $|x|^{\lambda}$, $\lambda \in (-n, 0)$ belongs to the Muckenhoupt class A_2 (the same is true for $\lambda \in (0, n)$); thus the parabolic Harnack inequality for equation (1.3) follows from the results in [3]. Although more direct, this approach applies to a more restricted class of weights.

REMARK 3.7. In view of the perturbative arguments in [8] (see also [13]), inequality (3.3) is seen to hold for positive solutions in Q to the more general equation

$$\partial_t v = \frac{1}{|x|^{\lambda}} \sum_{i,j=1}^{n} \partial_{x_i}\Big(p_{ij}(x)|x|^{\lambda}\partial_{x_j} v\Big) ,$$

where: (i) $p \equiv (p_{ij}) = (p_{ji}) : \mathbb{R}^n \to \mathbb{R}^{2n}$ is measurable; (ii) there exist $0 < \alpha \le \beta < \infty$ such that $\alpha|\xi|^2 \le \sum_{i,j=1}^{n} p_{ij}(x)\xi_i\xi_j \le \beta|\xi|^2$ for any $x \in \mathbb{R}^n$, $\xi \equiv (\xi_1, .., \xi_n) \in \mathbb{R}^n$.

4. Two-sided heat kernel estimates

The following result shows that $\{e^{-Ht}\}_{t\geq 0}$, $\{e^{\Delta_\lambda t}\}_{t\geq 0}$ are semigroups of integral operators.

THEOREM 4.1. (i) *Let* $\lambda \in (2-n, 0)$. *Then there exists a positive function* $K_\lambda = K_\lambda(x,y,t) = K_\lambda(y,x,t)$ $(x,y\in\Omega; t>0)$ *such that*

$$\left(e^{\Delta_\lambda t} v_0\right)(x) = \int_\Omega K_\lambda(x,y,t) v_0(y)\, |y|^\lambda\, dy \qquad (x\in\Omega)$$

for any $v_0 \in L^2_\lambda(\Omega)$.

(ii) *Let* $c\in(0,c_0)$. *Then there exists a positive function* $K = K(x,y,t) = = K(y,x,t)$ $(x,y\in\Omega; t>0)$ *such that*

$$\left(e^{-Ht}\ u_0\right)(x) = \int_\Omega K(x,y,t) u_0(y) dy \qquad (x\in\Omega)$$

for any $u_0\in L^2(\Omega)$. *Moreover,*

$$K(x,y,t) = |xy|^{\frac{\lambda_+}{2}} K_{\lambda_+}(x,y,t) \qquad (x,y\in\Omega, t>0)\,. \tag{4.1}$$

The proof of the above theorem relies on the ultracontractivity of the semigroup generated by $H_\psi = \Psi^* H \Psi$ in the weighted space $L^2_\psi(\Omega) \equiv L^2(\Omega, \psi^2 dx)$; here Ψ is the unitary map

$$\Psi : L^2_\psi(\Omega) \to L^2(\Omega), \qquad u = \Psi w := \psi w\,,$$

ψ being a suitable truncation of $|x|^{\frac{\lambda_+}{2}}$ (see [4, Theorem 2.4.6] and [9] for details). Then the claims follow by the unitary equivalence of the semigroups under consideration.

The functions K_λ, K are referred to as the heat kernel of the semigroup $\{e^{\Delta_\lambda t}\}_{t\geq 0}$, respectively $\{e^{-Ht}\}_{t\geq 0}$. As in the proof of the equivalence $(i) \Leftrightarrow (iii)$ of Theorem 2.5 (*e.g.*, see [14]), from Theorem 3.5 and estimates (3.5)-(3.6) we obtain the following two-sided heat kernel estimate.

THEOREM 4.2. *Let* $\lambda\in(2-n,0)$; *let* K_λ *be the heat kernel of the semigroup* $\{e^{\Delta_\lambda t}\}_{t\geq 0}$ *in* $L^2_\lambda(\mathbb{R}^n)$. *Then there exist* $C_1, C_2 > 0$ *such that*

$$C_1\, t^{-\frac{n}{2}}\, e^{-C_2\frac{|x-y|^2}{t}} k_\lambda(x,t)\, k_\lambda(y,t) \;\leq\; K_\lambda(x,y,t) \leq C_2\; t^{-\frac{n}{2}}\; e^{-C_1\frac{|x-y|^2}{t}} k_\lambda(x,t)\, k_\lambda(y,t)\,, \tag{4.2}$$

where $k_\lambda(x,t) := t^{\frac{n}{4}} V(x,\sqrt{t})^{-\frac{1}{2}}$ $(x,y\in\mathbb{R}^n, t>0)$. *Moreover,*

$$k_\lambda(x,t) \sim \begin{cases} (|x|+\sqrt{t})^{\frac{|\lambda|}{2}} & \textit{if} \quad |x|\geq 2\sqrt{t}\,, \\ t^{\frac{|\lambda|}{4}} & \textit{if} \quad |x| < 2\sqrt{t}\,. \end{cases}$$

The counterpart of the above result for equation (1.1) is the following

THEOREM 4.3. *Let $c \in (0, c_0)$; let K be the heat kernel of the semigroup $\{e^{-Ht}\}_{t\geq 0}$ in $L^2(\mathbb{R}^n)$. Then there exist $C_1, C_2 > 0$ such that*

$$C_1\, t^{-\frac{n}{2}}\, e^{-C_2\frac{|x-y|^2}{t}} k(x,t)\, k(y,t) \;\leq\; K(x,y,t) \leq C_2\, t^{-\frac{n}{2}}\, e^{-C_1\frac{|x-y|^2}{t}} k(x,t)\, k(y,t)\,, \tag{4.3}$$

where $k(x,t) := |x|^{\frac{\lambda_+}{2}} t^{\frac{n}{4}} V(x, \sqrt{t})^{-\frac{1}{2}}$ $(x, y \in \mathbb{R}^n, t > 0)$. Moreover,

$$k(x,t) \sim \begin{cases} \left(1 + \dfrac{\sqrt{t}}{|x|}\right)^{\frac{|\lambda_+|}{2}} & \text{if} \quad |x| \geq 2\sqrt{t}\,, \\[2ex] \left(\dfrac{\sqrt{t}}{|x|}\right)^{\frac{|\lambda_+|}{2}} & \text{if} \quad |x| < 2\sqrt{t}\,. \end{cases}$$

Acknowledgements

The work has been partially supported by RTN Contract HPRN-CT-2002-00274.

After the above results were obtained, we learned about the paper [10], where estimates like (4.3) are proved by different methods.

References

[1] G. Barbatis - S. Filippas - A. Tertikas, *Critical heat kernel estimates for Schrödinger operators via Hardy-Sobolev inequalities.* J. Funct. Anal., 208, 2004, 1-30.

[2] H. Brezis - L. Dupaigne - A. Tesei, *On a semilinear elliptic equation with inverse-square potential.* Selecta Math., 11, 2005, 1-7.

[3] F.M. Chiarenza - R.P. Serapioni, *A remark on a Harnack inequality for degenerate parabolic equations.* Rend. Sem. Mat. Univ. Padova, 73, 1985, 179-190.

[4] E.B. Davies, *Heat Kernels and Spectral Theory.* Cambridge Tracts in Mathematics, 92, Cambridge University Press, 1989.

[5] E.D. Fabes - D.W. Stroock, *A new proof of Moser's parabolic Harnack inequality via the old ideas of Nash.* Arch. Rat. Mech. Anal., 96, 1986, 327-338.

[6] A. Grigoryan, *The heat equation on non-compact Riemannian manifolds.* Mat. Sb., 182, 1991, 55-87 (in Russian); Engl. transl.: Math. USSR Sb., 72, 1992, 47-77.

[7] A. Grigoryan, *Heat kernels on weighted manifolds and applications.* Cont. Math., to appear; http://www.ma.ic.ac.uk/~grigor/wma.pdf

[8] A. Grigoryan - L. Saloff-Coste, *Stability results for Harnack inequalities.* Ann. Inst. Fourier (Grenoble), 55, 2005, to appear; http://www.ma.ic.ac.uk/~grigor/vc1eps.pdf

[9] P.D. Milman - Y.A. Semenov, *Heat kernel bounds and desingularizing weights.* J. Funct. Anal., 202, 2003, 1-24.

[10] P.D. Milman - Y.A. Semenov, *Global heat kernel bounds via desingularizing weights.* J. Funct. Anal., 212, 2004, 373-398.

[11] L. Moschini - A. Tesei, *Parabolic Harnack Inequality for the Heat Equation with Inverse-Square Potential.* Forum Math., to appear.

[12] L. SALOFF-COSTE, *A note on Poincaré, Sobolev and Harnack inequalities.* Int. Math. Res. Notes, 2, 1992, 27-38.
[13] L. SALOFF-COSTE, *Parabolic Harnack inequality for divergence form second order differential operators.* Potential Anal., 4, 1995, 429-467.
[14] L. SALOFF-COSTE, *Aspects of Sobolev-Type Inequalities.* London Math. Soc. Lecture Notes, 289, Cambridge University Press, 2002.

Pervenuta il 25 aprile 2005,
in forma definitiva il 19 maggio 2005.

Dipartimento di Matematica «G. Castelnuovo»
Università degli Studi di Roma «La Sapienza»
Piazzale A. Moro, 5 - 00185 ROMA
moschini@mat.uniroma1.it
tesei@mat.uniroma1.it