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Harnack inequality and heat kernel estimates for the Schrödinger operator with Hardy potential


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Abstract. — In this preliminary Note we outline some results of the forthcoming paper [11], concerning positive solutions of the equation \( \partial_t u = \Delta u + \frac{c}{|x|^2} u \left( 0 < c < \frac{(n-2)^2}{4} ; n \geq 3 \right) \). A parabolic Harnack inequality is proved, which in particular implies a sharp two-sided estimate for the associated heat kernel. Our approach relies on the unitary equivalence of the Schrödinger operator \( Hu = -\Delta u - \frac{c}{|x|^2} u \) with the opposite of the weighted Laplacian \( A_2 v = \frac{1}{|x|^2} \text{div}(|x|^2 \nabla v) \) when \( \lambda = 2 - n + 2\sqrt{c_0 - c} \).

Key words: Harnack inequality; Moser iteration technique; Weighted Laplace-Beltrami operator; Sharp heat kernel estimate; Hardy potential.

Riassunto. — Disuguaglianza di Harnack e stime sul nucleo del calore per l’operatore di Schrödinger con potenziale di Hardy. In questa Nota preliminare si presentano alcuni risultati del successivo lavoro [11], riguardanti soluzioni positive dell’equazione \( \partial_t u = \Delta u + \frac{c}{|x|^2} u \left( 0 < c < \frac{(n-2)^2}{4} ; n \geq 3 \right) \). Si dimostra una disuguaglianza di Harnack parabolica, che in particolare implica una stima bilatera sul nucleo del calore associato. Il nostro approccio si basa sull’equivalenza unitaria dell’operatore di Schrödinger \( Hu = -\Delta u - \frac{c}{|x|^2} u \) con l’opposto dell’operatore di Laplace pesato \( A_2 v = \frac{1}{|x|^2} \text{div}(|x|^2 \nabla v) \) quando \( \lambda = 2 - n + 2\sqrt{c_0 - c} \).

1. Introduction

We deal with positive solutions of the parabolic equation

\[
\partial_t u = \Delta u + \frac{c}{|x|^2} u
\]

in \( \mathbb{R}^n \times (0, T) \) \( (n \geq 3) \); here \( c \in (0, c_0) \), \( c_0 := \frac{(n-2)^2}{4} \) denoting the best constant in the Hardy inequality. The main purpose of this Note is to present a parabolic Harnack inequality for equation (1.1). As is well known, a number of interesting consequences can be drawn from such inequality; in particular, we point out a sharp two-sided estimate for the associated heat kernel (see Theorem 4.3 below).

Our approach relies on the change of unknown \( u \rightarrow v := \frac{u}{\varphi} \), where \( \varphi(x) = |x|^{\frac{n}{2}} \) and

\[
\lambda_+ := 2 - n + 2\sqrt{c_0 - c} \in (2 - n, 0)
\]

(*) Nella seduta del 16 giugno 2005.
This recasts equation (1.1) into the form

\begin{equation}
\partial_t \nu = \frac{1}{|x|^{2+}} \text{div} \left( \frac{1}{|x|^{2+}} \nabla \nu \right)
\end{equation}

– namely, into the heat equation for the *weighted Laplacian* $\Delta_{k+}$ on the *weighted manifold* $(\mathbb{R}^n, |x|^{2+} \, dx)$. This suggests to derive results concerning equation (1.1) from those concerning heat semigroups on weighted manifolds (see [7] for an up-to-date account of the subject).

In particular, the implication $(ii) \Rightarrow (i)$ of Theorem 2.5 below suggests to prove both the doubling property and the Poincaré inequality in $(\mathbb{R}^n, |x|^{2+} \, dx)$, in order to demonstrate the parabolic Harnack inequality for equation (1.3), thus for equation (1.1). To do so, a technical obstruction is given by the lack of regularity of the weight $|x|^{2+}$; in fact, a standard assumption underlying Theorem 2.5 is that the weighted manifold $(\mathcal{M}, \mu)$ be endowed with a measure $d\mu = \varphi^2 \, dv$, where $dv$ is the Riemannian measure and $\varphi$ is a *smooth* positive function on $\mathcal{M}$ (see Section 2).

However, checking the above implication in the present case offers no difficulties, since the Moser iteration technique applies to measurable coefficients (in this connection, see [13]). Hence we prove that the doubling property and the Poincaré inequality are satisfied; then the result follows.

2. Mathematical framework

(a) Let $\Omega \subseteq \mathbb{R}^n$ be a domain containing the origin $(n \geq 3)$. The Schrödinger operator $H = -\Delta - \frac{c}{|x|^2}$ (with Dirichlet homogeneous boundary conditions if $\partial \Omega$ is nonempty; $c \in (0, c_0)$) is defined in $L^2(\Omega)$ as the generator of the symmetric form

$$\mathcal{H}[u_1, u_2] := \int_{\Omega} \left( \nabla u_1 \nabla u_2 - \frac{c}{|x|^2} u_1 u_2 \right) \, dx$$

with domain $D(\mathcal{H}) := H^1_0(\Omega)$. In view of the Hardy inequality, the form $\mathcal{H}$ is nonnegative and $C_0^\infty(\Omega \setminus \{0\})$ is a core for it. The operator $H$ is nonnegative and self-adjoint, so that $-H$ is the generator of a contraction holomorphic semigroup $\{e^{-Ht}\}_{t \geq 0}$ on $L^2(\Omega)$.

Let us recall the following well-known construction (e.g., see [1]). Let $\varphi \in C^\infty(\Omega \setminus \{0\})$, $\varphi > 0$; consider the weighted space $L^2_\varphi(\Omega) \equiv L^2(\Omega, \varphi^2 \, dx)$ and the unitary map

$$\Phi : L^2_\varphi(\Omega) \rightarrow L^2(\Omega), \quad \Phi \nu := \varphi \nu \quad (\nu \in L^2_\varphi(\Omega)).$$

Define the nonnegative, self-adjoint operator

$$H_\varphi := \Phi^* H \Phi.$$

in $L^2_\varphi(\Omega).$ Clearly,

\begin{equation}
\Phi e^{-H_{\varphi t}} = e^{-H_{\varphi t}} \Phi \quad \text{for any} \ t \geq 0,
\end{equation}

$\{e^{-H_{\varphi t}}\}_{t \geq 0}$ denoting the semigroup on $L^2_\varphi(\Omega)$ generated by $-H_\varphi$; moreover, $H_\varphi$ is the
generator of the symmetric form

\[
\begin{align*}
\mathcal{H}_\phi[v_1, v_2] & := \int_\Omega \left\{ -\frac{c}{|x|^2} \, v_1 \, v_2 \, \phi^2 \right\} \, dx = \\
& = \int_\Omega \left\{ \nabla v_1 \nabla v_2 - \left( \frac{\Delta \phi}{\phi} + \frac{c}{|x|^2} \right) \, v_1 v_2 \right\} \phi^2 \, dx.
\end{align*}
\]

Set \( \varphi(x) = |x|^{\frac{n}{2}} \) with \( \lambda_+ \) defined in (1.2). Since \( |x|^{\frac{n}{2}} \) is a weak solution of the equation

\[
\Delta \varphi + \frac{c}{|x|^2} \varphi = 0 \quad \text{in} \quad \Omega,
\]

the form (2.3) reads in this case:

\[
\begin{align*}
\mathcal{H}_{\lambda_+}[v_1, v_2] & := \int_\Omega \left\{ \nabla v_1 \nabla v_2 \, |x|^{\lambda_+} \right\} \, dx \\
\mathcal{H}_{\lambda_-}[v_1, v_2] & := \int_\Omega \left\{ \nabla v_1 \nabla v_2 \, |x|^{\lambda_-} \right\} \, dx
\end{align*}
\]

(here the subscript \( \lambda_+ \) instead of \( \phi \) has been used). Since \( C_0^\infty(\Omega \setminus \{0\}) \) is a core of the above form and the norm induced on this core is equivalent to the norm

\[
\|v\|_{\mathcal{H}_{\lambda_+}} := \left\{ \int_\Omega \left[ |\nabla v|^2 + v^2 \right] |x|^{\lambda_+} \, dx \right\}^{\frac{1}{2}},
\]

the generator \( H_{\lambda_+} \) of the form (2.5) is the opposite of the weighted Laplacian \( A_{\lambda_+} \) on \( \Omega, |x|^{\lambda_+} \, dx \), which is defined as follows.

Let \( \lambda \in (2 - n, 0) \). Consider the symmetric form in \( L^2(\Omega) \equiv L^2(\Omega, |x|^{\lambda} \, dx) \)

\[
\mathcal{H}_\lambda[v_1, v_2] := \int_\Omega \left\{ \nabla v_1 \nabla v_2 \, |x|^{\lambda} \right\} \, dx
\]

with domain \( \mathcal{D}(\mathcal{H}_\lambda) := H_{0,\lambda}^1(\Omega) \); the latter is defined as the closure of \( C_0^\infty(\Omega \setminus \{0\}) \) in the norm (2.6) (which coincides with the closure of \( C_0^\infty(\Omega) \) in the same norm, since \( \lambda > 2 - n \)). Then the weighted Laplacian \( A_\lambda \) (with Dirichlet homogeneous boundary conditions if \( \partial \Omega \) is nonempty) is defined in \( L^2_\lambda(\Omega) \) as the opposite of the generator of the form \( \mathcal{H}_\lambda \), namely:

\[
\begin{align*}
\mathcal{D}(A_\lambda) & := \left\{ v \in H^1_{0,\lambda}(\Omega) \left| \frac{1}{|x|^\lambda} \, \text{div} \left( |x|^{\lambda} \nabla v \right) \in L^2_\lambda(\Omega) \right. \right\} \\
A_\lambda v & := \frac{1}{|x|^\lambda} \, \text{div} \left( |x|^{\lambda} \nabla v \right) \quad \text{for any} \quad v \in \mathcal{D}(A_\lambda).
\end{align*}
\]
We shall denote by $\{e^{A_t}\}_{t \geq 0}$ the contraction holomorphic semigroup generated by $A_t$ in $L^2(\Omega)$.

(β) Let us recall some relevant results concerning smoothly weighted manifolds (e.g., see [7]).

A weighted manifold $(\mathcal{M}, \mu) \equiv (\mathcal{M}, g, \mu)$ is a Riemannian manifold $\mathcal{M}$ with metric $g$, endowed with a measure $d\mu = \varphi^2 dv$, where $dv$ is the Riemannian measure and $\varphi$ is a smooth positive function on $\mathcal{M}$ (observe that the same definition makes sense if $\varphi$ is measurable and positive). The weighted manifold $(\mathcal{M}, \mu)$ is complete if the metric space $(\mathcal{M}, d)$, where $d$ denotes the geodesic distance induced by the metric $g$, is complete.

The weighted Laplace-Beltrami operator on $(\mathcal{M}, \mu)$

$$A_\mu v := \frac{1}{\varphi^2} \text{div}(\varphi^2 \nabla v) = \frac{1}{\varphi^2 \sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( \varphi^2 \sqrt{|g|} g^{i,j} \frac{\partial v}{\partial x_j} \right),$$

(where $(x_1, ..., x_n)$ are local coordinates, $(g_{i,j}) = (g_{i,j})$, $|g| \equiv \det (g_{i,j})$ and $(g^{i,j}) \equiv (g_{i,j})^{-1}$) with initial domain $C_0^\infty(\mathcal{M})$ is symmetric and nonpositive, in view of the Green formulas with respect to the measure $\mu$. Moreover, it has a self-adjoint extension (unique if the manifold $\mathcal{M}$ is complete) in the weighted space $L^2(\mathcal{M}, \mu)$; hence the heat semigroup $\{e^{A_t}\}_{t \geq 0}$ is defined in $L^2(\mathcal{M}, \mu)$. It turns out to be a semigroup of integral operators – namely, there exists a smooth positive heat kernel $p_\mu(x, y, t)$ such that

$$(e^{A_t} \nu_0)(x) = \int_{\mathcal{M}} p_\mu(x, y, t) \nu_0(y) \, d\mu(y)$$

for any $\nu_0 \in L^2(\mathcal{M}, \mu)$, $x \in \mathcal{M}$ and $t > 0$.

Let $B(x_0, r)$ denote the geodesic ball of $(\mathcal{M}, g)$ centered at $x_0$ with radius $r$; set $V_\mathcal{M}(x_0, r) := \mu(B(x_0, r))$. Let us recall the following definitions.

**Definition 2.1.** $(\mathcal{M}, \mu)$ satisfies the doubling property if there exists $C_D > 0$ such that

$$V_\mathcal{M}(x_0, 2r) \leq C_D V_\mathcal{M}(x_0, r)$$

for any $x_0 \in \mathcal{M}$, $r > 0$.

**Definition 2.2.** $(\mathcal{M}, \mu)$ satisfies the Poincaré inequality with parameter $\delta \in (0, 1]$ and constant $C_p > 0$ if

$$\inf_{\xi \in \mathbb{R}} \int_{B(x_0, \delta r)} |f - \xi|^2 \, d\mu \leq C_p \ r^2 \ \int_{B(x_0, r)} |\nabla f|^2 \, d\mu$$

for any $x_0 \in \mathcal{M}$, $r > 0$ and $f \in C^1(B(x_0, r))$.

**Definition 2.3.** $(\mathcal{M}, \mu)$ satisfies the parabolic Harnack inequality if there exists $C_H > 0$ such that, for any $x_0 \in \mathcal{M}$, $r > 0$, $s \in \mathbb{R}$, any positive solution $\nu$ to the heat equation

$$\partial_t \nu = A_\mu \nu$$

satisfies the inequality

$$\frac{\nu(x_0 + s, t + r)}{\nu(x_0, t)} \leq C_H \frac{\nu(x_0 + s, t)}{\nu(x_0, t)}$$
in \( Q \equiv Q(x_0, r) := B(x_0, r) \times (0, r^2) \) satisfies
\[
\text{ess sup}_{(x,t) \in Q_-} \nu(x,t) \leq C_H \text{ess inf}_{(x,t) \in Q_+} \nu(x,t),
\]
where
\[
Q_- \equiv Q_- (x_0, r) := B \left( x_0, \frac{r}{2} \right) \times \left( \frac{r^2}{4}, \frac{r^2}{2} \right),
\]
\[
Q_+ \equiv Q_+ (x_0, r) := B \left( x_0, \frac{r}{2} \right) \times \left( \frac{3}{4} r^2, r^2 \right).
\]

**Definition 2.4.** The heat kernel on \((\mathcal{M}, \mu)\) satisfies the Li-Yau estimate if there exist \( C_1, C_2 > 0 \) such that
\[
\frac{C_1 e^{-C_2 \frac{d^2(x_0)}{r}}}{V_M(x, \sqrt{t}) V_M(y, \sqrt{t})} \leq p_\mu(t, x, y) \leq \frac{C_2 e^{-C_1 \frac{d^2(x_0)}{r}}}{V_M(x, \sqrt{t}) V_M(y, \sqrt{t})}
\]
for any \( x, y \in \mathcal{M} \) and \( t > 0 \).

The following result can be proved (see [8, Theorem 2.7]).

**Theorem 2.5.** For any complete weighted manifold \((\mathcal{M}, \mu)\) the following properties are equivalent:
(i) \((\mathcal{M}, \mu)\) satisfies the parabolic Harnack inequality;
(ii) \((\mathcal{M}, \mu)\) satisfies the doubling property and the Poincaré inequality for some \( \delta \in (0, 1] \);
(iii) the heat kernel on \((\mathcal{M}, \mu)\) satisfies the Li-Yau estimate.

The main part of Theorem 2.5 is implication \((ii) \Rightarrow (i)\), which was proved independently in [6, 12] (the inverse implication \((i) \Rightarrow (ii)\) was proved in [12]). The equivalence \((i) \Leftrightarrow (iii)\) goes back to [5].

### 3. **Parabolic Harnack Inequality**

The results of this section make use of the methods of the latter subsection in the present non-smooth case. Let \( B(x_0, r) := \{ x \in \mathbb{R}^n \mid |x - x_0| < r \}, \ Q \equiv Q(x_0, r) := B(x_0, r) \times (0, r^2) \ (x_0 \in \mathbb{R}^n, r > 0) \). Set
\[
V(x_0, r) := \int_{B(x_0, r)} |y|^2 dy \quad (x_0 \in \mathbb{R}^n, r > 0).
\]
The complete weighted manifold \((\mathbb{R}^n, |x|^2 dx) \ (\lambda \in (-n, 0))\) satisfies both the doubling property and the Poincaré inequality; this is the content of the following

**Theorem 3.1.** Let \( \lambda \in (-n, 0) \). Then:
(i) there exists \( C_D > 0 \) such that
\[
V(x_0, 2r) \leq C_D V(x_0, r)
\]
for any \( x_0 \in \mathbb{R}^n, r > 0; \)
(ii) there exist $\delta \in (0, 1]$, $C_p > 0$ such that
\begin{equation}
\int_{B(x_0, \delta r)} |f(y) - \hat{f}|^2 |y|^4 \, dy \leq C_p r^2 \int_{B(x_0, r)} |\nabla f|^2(y) |y|^4 \, dy
\end{equation}
for any $x_0 \in \mathbb{R}^n$, $r > 0$ and $f \in C^1(\bar{B}(x_0, r))$; here
\[
\hat{f} := \frac{1}{V(x_0, \delta r)} \int_{B(x_0, \delta r)} f(y) |y|^4 \, dy.
\]
Let us make the following definitions.

**Definition 3.2.** By a solution to equation (1.3) in $Q$ we mean any $v \in C^1((0, r^2); L^2(B(x_0, r))) \cap C((0, r^2); H^1(B(x_0, r)))$ such that $\partial_t v$, $\nabla v$ $\in L^2(Q, |x|^2 \, dx \, dt)$ and there holds
\[
\iint_Q \left\{ \partial_t v \chi + \nabla v \nabla \chi \right\} |x|^2 \, dx \, dt = 0
\]
for any $\chi \in C([0, r^2]; C^\infty_0(B(x_0, r)))$.

**Definition 3.3.** By a solution to equation (1.1) in $Q$ we mean any $u \in C^1((0, r^2); L^2(B(x_0, r))) \cap C((0, r^2); H^1(B(x_0, r)))$ such that $\partial_t u$, $\nabla u$, $\frac{u}{|x|^2} \in L^2(Q, dx \, dt)$ and there holds
\[
\iint_Q \left\{ \partial_t u \chi + \nabla u \nabla \chi - \frac{c}{|x|^2} u \chi \right\} \, dx \, dt = 0
\]
for any $\chi \in C([0, r^2]; C^\infty_0(B(x_0, r)))$.

**Remark 3.4.** Observe that Definition 3.3 excludes stationary solutions of (1.1) in $\mathcal{D}'(\mathbb{R}^n)$, which behave like $|x|^{\frac{2}{\lambda_-}}$ with
\[
\lambda_- := 2n - 2\sqrt{c_0 - c}
\]
as $|x| \to 0$ (see [2]).

Theorem 3.1 entails the Harnack inequality in $(\mathbb{R}^n, |x|^2 \, dx)$. This is the content of the following

**Theorem 3.5.** Let $\lambda \in (-n, 0)$. Then there exists $C_H > 0$ such that, for any $x_0 \in \mathbb{R}^n$, $r > 0$, any positive solution $v$ to equation (1.3) in $Q(x_0, r)$ satisfies
\[
\text{ess sup}_{(x,t) \in Q_-} v(x,t) \leq C_H \text{ess inf}_{(x,t) \in Q_+} v(x,t),
\]
where
\[
Q_- \equiv Q_-(x_0, r) := B(x_0, r) \times \left( \frac{r^2}{4}, \frac{r^2}{2} \right),
\]
\[
Q_+ \equiv Q_+(x_0, r) := B \left( x_0, \frac{r}{2} \right) \times \left( \frac{3}{4} r^2, r^2 \right).
\]

As a consequence of Theorem 3.5, we have:

**Theorem 3.6.** Let \( c \in (0, c_0) \). Then there exists \( C_{H} > 0 \) such that, for any ball \( B(x_0, r) \), any positive solution \( u \) to equation (1.1) in \( Q(x_0, r) \) satisfies the inequality

\[
\text{ess sup}_{(x, t) \in Q_-} \left( |x|^{\frac{\lambda}{2}} u(x, t) \right) \leq C_{H} \text{ ess inf}_{(x, t) \in Q_+} \left( |x|^{\frac{\lambda}{2}} u(x, t) \right),
\]

where

\[
Q_- \equiv Q_-(x_0, r) := B \left( x_0, \frac{r}{2} \right) \times \left( \frac{r^2}{4}, r^2 \right),
\]

\[
Q_+ \equiv Q_+(x_0, r) := B \left( x_0, \frac{r}{2} \right) \times \left( \frac{3}{4} r^2, r^2 \right).
\]

Let us briefly discuss the proof of the above results. Theorem 3.5 follows from Theorem 3.1 using the Moser iteration technique as in [12]. In turn, Theorem 3.6 follows from Theorem 3.5 by the transformation \( u(x, t) \to v(x, t) = |x|^{\frac{\lambda}{2}} u(x, t) \) discussed at length in Section 2. Concerning Theorem 3.1, claim (i) follows from the following estimates:

\[
D_1 r^{\lambda+n} \leq V(x, r) \leq D_2 r^{\lambda+n} \quad \text{if } |x| < 2r,
\]

\[
D_1 r^{\lambda} (|x| + r)^{\lambda} \leq V(x, r) \leq D_2 r^{\lambda} (|x| + r)^{\lambda} \quad \text{if } |x| \geq 2r,
\]

which are proved to hold, for \( \lambda \in (-n, 0) \) and suitable \( D_1, D_2 > 0 \), for any \( x \in \mathbb{R}^n \) and \( r > 0 \). As for claim (ii), we first prove it both for anchored balls \( B(0, r) \) and for remote balls \( B(x, r), |x| \geq 2r \), for any \( r > 0 \); then the general case follows as in Proposition 4.2 in [8].

It is worth pointing out a slightly different way to reach the same conclusions. In view of estimates (3.5)-(3.6), it is easily seen that the weight \( |x|^\lambda, \lambda \in (-n, 0) \) belongs to the Muckenhoupt class \( A_2 \) (the same is true for \( \lambda \in (0, n) \)); thus the parabolic Harnack inequality for equation (1.3) follows from the results in [3]. Although more direct, this approach applies to a more restricted class of weights.

**Remark 3.7.** In view of the perturbative arguments in [8] (see also [13]), inequality (3.3) is seen to hold for positive solutions in \( Q \) to the more general equation

\[
\partial_t u = \frac{1}{|x|^2} \sum_{i,j=1}^n \partial_{x_i} \left( p_{ij}(x) |x|^\lambda \partial_{x_j} u \right),
\]

where: (i) \( p \equiv (p_{ij}) = (p_{ij}) : \mathbb{R}^n \to \mathbb{R}^{2n} \) is measurable; (ii) there exist \( 0 < a \leq \beta < \infty \) such that \( a|\xi|^2 \leq \sum_{i,j=1}^n p_{ij}(x) \xi_i \xi_j \leq \beta|\xi|^2 \) for any \( x \in \mathbb{R}^n, \xi \equiv (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \).
4. TWO-SIDED HEAT KERNEL ESTIMATES

The following result shows that \( \{e^{-\lambda t}\}_{t \geq 0}, \{e^{\lambda t}\}_{t \geq 0} \) are semigroups of integral operators.

**Theorem 4.1.** (i) Let \( \lambda \in (2 - n, 0) \). Then there exists a positive function 
\( K_\lambda = K_\lambda(x, y, t) = K_\lambda(y, x, t) \) \((x, y \in \Omega, t > 0)\) such that
\[
(e^{\lambda t} v_0)(x) = \int_{\Omega} K_\lambda(x, y, t) v_0(y)|y|^{\frac{\lambda}{2}} \, dy \quad (x \in \Omega)
\]
for any \( v_0 \in L^2_\lambda(\Omega) \).

(ii) Let \( c \in (0, c_0) \). Then there exists a positive function \( K = K(x, y, t) = K(y, x, t) \) \((x, y \in \Omega, t > 0)\) such that
\[
(e^{-\lambda t} u_0)(x) = \int_{\Omega} K(x, y, t) u_0(y)dy \quad (x \in \Omega)
\]
for any \( u_0 \in L^2(\Omega) \). Moreover,
\[
(4.1) \quad K(x, y, t) = |xy|^{\frac{\lambda}{2}} K_\lambda(x, y, t) \quad (x, y \in \Omega, t > 0).
\]

The proof of the above theorem relies on the ultracontractivity of the semigroup generated by \( H_\psi = \Psi^* H \Psi \) in the weighted space \( L^2_\psi(\Omega) \equiv L^2(\Omega, \psi^2 dx) \); here \( \Psi \) is the unitary map
\[
\Psi : L^2_\psi(\Omega) \rightarrow L^2(\Omega), \quad u = \Psi w := \psi w,
\]
\( \psi \) being a suitable truncation of \(|x|^{\frac{\lambda}{2}}\) (see [4, Theorem 2.4.6] and [9] for details). Then the claims follow by the unitary equivalence of the semigroups under consideration.

The functions \( K_\lambda, K \) are referred to as the heat kernel of the semigroup \( \{e^{\lambda t}\}_{t \geq 0} \), respectively \( \{e^{-\lambda t}\}_{t \geq 0} \). As in the proof of the equivalence \((i) \Leftrightarrow (iii)\) of Theorem 2.5 (e.g., see [14]), from Theorem 3.5 and estimates (3.5)-(3.6) we obtain the following two-sided heat kernel estimate.

**Theorem 4.2.** Let \( \lambda \in (2 - n, 0) \); let \( K_\lambda \) be the heat kernel of the semigroup \( \{e^{\lambda t}\}_{t \geq 0} \) in \( L^2_\lambda(\mathbb{R}^n) \). Then there exist \( C_1, C_2 > 0 \) such that
\[
(4.2) \quad C_1 t^{-\frac{n}{2}} e^{-\frac{C_2 |x-y|^2}{t}} k_\lambda(x, t) k_\lambda(y, t) \leq K_\lambda(x, y, t) \leq C_2 t^{-\frac{n}{2}} e^{-\frac{C_2 |x-y|^2}{t}} k_\lambda(x, t) k_\lambda(y, t),
\]
where \( k_\lambda(x, t) := \bar{t} V(x, \sqrt{t})^{-\frac{1}{2}} \) \((x, y \in \mathbb{R}^n, t > 0)\). Moreover,
\[
k_\lambda(x, t) \sim \begin{cases} 
(\frac{|x| + \sqrt{t}|}{t})^{\frac{n}{4}} & \text{if } |x| \geq 2\sqrt{t} , \\
\bar{t}^{|x|} & \text{if } |x| < 2\sqrt{t}.
\end{cases}
\]
The counterpart of the above result for equation (1.1) is the following

**Theorem 4.3.** Let $c \in (0, c_0)$; let $K$ be the heat kernel of the semigroup $\{e^{-Ht}\}_{t \geq 0}$ in $L^2(\mathbb{R}^n)$. Then there exist $C_1, C_2 > 0$ such that

$$
(4.3) \quad C_1 t^{-\frac{n}{2}} e^{-C_2\frac{|x-y|^2}{t}} k(x, t) k(y, t) \leq K(x, y, t) \leq C_2 t^{-\frac{n}{2}} e^{-C_1\frac{|x-y|^2}{t}} k(x, t) k(y, t),
$$

where $k(x, t) := |x|^{\frac{n}{2}} t^{\frac{n-1}{2}} V(x, \sqrt{t})^{-1} (x, y \in \mathbb{R}^n, t > 0)$. Moreover,

$$
k(x, t) \sim \begin{cases} 
\left(1 + \frac{\sqrt{t}}{|x|}\right)^{n+1} & \text{if } |x| \geq 2\sqrt{t}, \\
\left(\frac{\sqrt{t}}{|x|}\right)^{n+1} & \text{if } |x| < 2\sqrt{t}.
\end{cases}
$$

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After the above results were obtained, we learned about the paper [10], where estimates like (4.3) are proved by different methods.

**References**


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