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On weak Hessian determinants

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Analisi funzionale. — *On weak Hessian determinants.* Nota di LUIGI D'ONOFRIO, FLAVIA GIANNETTI e LUIGI GRECO, presentata (*) dal Socio C. Sbordone.

ABSTRACT. — We consider and study several weak formulations of the Hessian determinant, arising by formal integration by parts. Our main concern are their continuity properties. We also compare them with the Hessian measure.

KEY WORDS: Hessian determinant; Schwartz distributions; Hessian measure.

RIASSUNTO. — *Sui determinanti hessiani deboli.* Consideriamo ed esaminiamo varie formulazioni deboli del determinante hessiano, definite come distribuzioni di Schwartz mediante integrazione per parti, principalmente riguardo alle loro proprietà di continuità. Confrontiamo inoltre tali formulazioni deboli con la misura hessiana.

1. INTRODUCTION

In this paper we discuss several weak formulations of the Hessian determinant which still attract a great interest, see *e.g.* [3-5, 17, 10, 12]. We begin with the case of functions of two variables. Let $\Omega \subset \mathbb{R}^2$ be an open set and u be a function defined on Ω with second order derivatives. We denote by $\mathscr{H}u$ the Hessian determinant

$$\mathscr{H}u = \det D^2 u = \begin{vmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{vmatrix}.$$

If u is sufficiently smooth, the following identities can be easily checked:

$$\begin{aligned}\mathscr{H}u &= (u_x\, u_{yy})_x - (u_x\, u_{xy})_y = (u_y\, u_{xx})_y - (u_y\, u_{xy})_x \\ &= \frac{1}{2}(u\, u_{xx})_{yy} + \frac{1}{2}(u\, u_{yy})_{xx} - (u\, u_{xy})_{xy} \\ &= (u_x\, u_y)_{xy} - \frac{1}{2}(u_x^2)_{yy} - \frac{1}{2}(u_y^2)_{xx}.\end{aligned}$$

On the other hand, each of the above expressions can be used to define a Schwartz distribution on Ω, provided u belongs to an appropriate Sobolev space. The first one we consider is the distribution of order zero defined by the rule

$$\mathscr{H}_0 u[\varphi] = \int_\Omega \mathscr{H}u(x,y)\, \varphi(x,y)\, dx\, dy$$

for all test functions $\varphi \in \mathscr{D}(\Omega)$. Of course, the natural assumption for defining $\mathscr{H}_0 u$ is $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$, as it implies that the Hessian determinant $\mathscr{H}u$ is locally integrable on Ω.

(*) Nella seduta del 16 giugno 2005.

Other distributions will result integrating formally by parts. Precisely, we define

- $\mathscr{H}_1 u[\varphi] = \int_\Omega (u_x u_{xy} \varphi_y - u_x u_{yy} \varphi_x)\, dx\, dy$

  $= \int_\Omega (u_y u_{xy} \varphi_x - u_y u_{xx} \varphi_y)\, dx\, dy$

- $\mathscr{H}_2 u[\varphi] = \frac{1}{2} \int_\Omega (u\, u_{xx} \varphi_{yy} + u\, u_{yy} \varphi_{xx} - 2u\, u_{xy} \varphi_{xy})\, dx\, dy$

- $\mathscr{H}_2^* u[\varphi] = \frac{1}{2} \int_\Omega (2u_x u_y \varphi_{xy} - u_x^2 \varphi_{yy} - u_y^2 \varphi_{xx})\, dx\, dy$

for all $\varphi \in \mathscr{D}(\Omega)$.

In general, $\mathscr{H}_1 u$ is a distribution of order one. It is well defined for $u \in W^{2,\frac{4}{3}}_{\mathrm{loc}}$. Indeed, by Sobolev imbedding $u_x, u_y \in L^4_{\mathrm{loc}}$, thus $u_x u_{xy}$ and $u_x u_{yy}$ are locally integrable and the first integral in the definition of $\mathscr{H}_1 u$ converges. The same argument shows that the second integral also converges for $u \in W^{2,\frac{4}{3}}_{\mathrm{loc}}$. We remark that the two integrals give the same distribution. Therefore, we can also write

$$\mathscr{H}_1 u[\varphi] = -\frac{1}{2} \int_\Omega \left[\varphi_x (u_x u_{yy} - u_y u_{xy}) + \varphi_y (u_{xx} u_y - u_{xy} u_x)\right] dx\, dy\,.$$

REMARK 1.1. The definition of $\mathscr{H}_1 u$ is related to the well-known concept of weak (or distributional) Jacobian introduced by Ball [2]. Actually, $\mathscr{H}_1 u$ is precisely the weak Jacobian of the gradient map Du.

To justify the definitions of $\mathscr{H}_2 u$ and $\mathscr{H}_2^* u$ we again use Sobolev imbeddings:

$$W^{2,2}_{\mathrm{loc}} \subset W^{2,\frac{4}{3}}_{\mathrm{loc}} \subset W^{2,1}_{\mathrm{loc}} \subset \widetilde{W}^{2,1}_{\mathrm{loc}} \subset \begin{cases} C^0 \\ W^{1,2}_{\mathrm{loc}} \end{cases} \tag{1.1}$$

where we denoted by $\widetilde{W}^{2,1}_{\mathrm{loc}}$ the space of $W^{1,1}_{\mathrm{loc}}$-functions whose second order distributional derivatives are Radon measures on Ω. Therefore, clearly $\mathscr{H}_2 u$ makes sense for $u \in W^{2,1}_{\mathrm{loc}}$, or even for $u \in \widetilde{W}^{2,1}_{\mathrm{loc}}$ (with obvious interpretation of u_{xx}, u_{yy} and u_{xy} Radon measures).

Surprisingly, $\mathscr{H}_2^* u$ makes sense for $u \in W^{1,2}_{\mathrm{loc}}$ and does not require second order derivatives.

Thus, we are imposing weaker and weaker conditions on u. It will be clear that, if we are in a position to define two of the above expressions, they yield the same distribution. For example, if $u \in \widetilde{W}^{2,1}_{\mathrm{loc}}$, we can define both $\mathscr{H}_2 u$ and $\mathscr{H}_2^* u$, and we have

$$\mathscr{H}_2 u[\varphi] = \mathscr{H}_2^* u[\varphi] \tag{1.2}$$

for all $\varphi \in \mathscr{D}(\Omega)$.

1.1. *Weak Hessian in* $\mathbb{R}^n$.

To deal with functions of several variables, we will use the formalism of differential forms which is quite effective. Let $\Omega \subset \mathbb{R}^n$ be an open subset and let $u \in C^\infty(\Omega)$. The Hessian determinant $\mathscr{H}u = \det D^2 u$ induces a n-form:

$$\mathscr{H}u\, dx = du_{x_1} \wedge \cdots \wedge du_{x_n} .$$

For $j = 1, \ldots, n$, set $\omega_j = du_{x_1} \wedge \cdots \wedge du \wedge \cdots \wedge du_{x_n}$, where du is at the j-th factor in the wedge product, while for $i \neq j$ the i-th factor is du_{x_i}. Then

$$\begin{aligned}\frac{\partial}{\partial x_j}\,\omega_j &= du_{x_1 x_j} \wedge \cdots \wedge du \wedge \cdots \wedge du_{x_n} \\ &\quad + \cdots + du_{x_1} \wedge \cdots \wedge du_{x_j} \wedge \cdots \wedge du_{x_n} \\ &\quad + \cdots + du_{x_1} \wedge \cdots \wedge du \wedge \cdots \wedge du_{x_n x_j} .\end{aligned} \tag{1.3}$$

The first term in the right hand side (assuming $j \neq 1$) will appear also in $\dfrac{\partial}{\partial x_1}\omega_1$, but with opposite sign because $du_{x_1 x_j} = du_{x_j x_1}$ and du are exchanged to each other. Similarly for the other terms in the right hand side of (1.3), except for

$$du_{x_1} \wedge \cdots \wedge du_{x_j} \wedge \cdots \wedge du_{x_n} = \mathscr{H}u\, dx .$$

Hence we have

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\,\omega_j = n\, du_{x_1} \wedge \cdots \wedge du_{x_n} ,$$

that is

$$du_{x_1} \wedge \cdots \wedge du_{x_n} = \frac{1}{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\, du_{x_1} \wedge \cdots \wedge du \wedge \cdots \wedge du_{x_n} . \tag{1.4}$$

Identity (1.4) yields the definition of the distribution $\mathscr{H}_1 u$. We multiply both sides by a test function φ and integrate by parts in the right hand side:

$$\mathscr{H}_1 u[\varphi] = -\frac{1}{n}\sum_{j=1}^{n} \int_\Omega \varphi_{x_j}\, du_{x_1} \wedge \cdots \wedge du \wedge \cdots \wedge du_{x_n} . \tag{1.5}$$

Using Laplace theorem for determinants, this is easily seen to equal

$$\mathscr{H}_1 u[\varphi] = -\frac{1}{n}\sum_{j=1}^{n} \int_\Omega u_{x_j}\, du_{x_1} \wedge \cdots \wedge d\varphi \wedge \cdots \wedge du_{x_n} .$$

As above, $d\varphi$ is the j-th factor.

We proceed further from (1.4). One way is to notice that for each $j = 1, \ldots, n$ we have

$$\omega_j = (-1)^{j-1}\, d(u\, du_{x_1} \wedge \cdots \wedge du_{x_{j-1}} \wedge du_{x_{j+1}} \wedge \cdots \wedge du_{x_n}) .$$

Inserting this into (1.4), multiplying by the test function φ and integrating by parts twice will lead to the definition of $\mathscr{H}_2 u$:

$$\mathscr{H}_2 u[\varphi] = \frac{1}{n}\sum_{j=1}^{n} \int_\Omega u\, du_{x_1} \wedge \cdots \wedge du_{x_{j-1}} \wedge d\varphi_{x_j} \wedge du_{x_{j+1}} \wedge \cdots \wedge du_{x_n} . \tag{1.6}$$

To define $\mathscr{H}_2^* u$, for $i \neq j$ we write

$$\omega_j = (-1)^{i-1} d(u_{x_i}\, du_{x_1} \wedge \cdots \wedge du \wedge \cdots \wedge du_{x_{i-1}} \wedge du_{x_{i+1}} \wedge \cdots \wedge du_{x_n}) .$$

Now we sum with respect to i and insert into (1.4). In a routine manner, this gives

$$(1.7) \qquad n\,(n-1)\,\mathscr{H}_2^* u[\varphi] =$$

$$= \sum_{i \neq j} \int_\Omega u_{x_i}\, du_{x_1} \wedge \cdots \wedge du \wedge \cdots \wedge du_{x_{i-1}} \wedge d\varphi_{x_j} \wedge du_{x_{i+1}} \wedge \cdots \wedge du_{x_n} .$$

We recall that in each term du is the j-th factor and $d\varphi_{x_j}$ is the i-th factor of the wedge product.

Now we consider functions u in Sobolev spaces. If $u \in W^{2,\frac{n^2}{n+1}}_{\mathrm{loc}}(\Omega)$, then clearly the minors of order $n-1$ of $D^2 u$ are locally integrable with exponent $n^2/(n^2-1)$, while by Sobolev imbedding theorem the first order derivatives of u are locally integrable with exponent n^2. Noticing that these exponents are Hölder conjugate, we easily see that $\mathscr{H}_1 u$ makes sense:

$$\mathscr{H}_1 u \colon W^{2,\frac{n^2}{n+1}}_{\mathrm{loc}}(\Omega) \to \mathscr{D}'(\Omega) .$$

For $\mathscr{H}_2 u$ we recall that functions in $W^{2,n-1}_{\mathrm{loc}}(\Omega)$ are continuous (actually, Hölder continuous with exponent $(n-2)/(n-1)$ if $n > 2$). For $\mathscr{H}_2^* u$ we remark that if $u \in W^{2,\frac{n^2}{n+2}}_{\mathrm{loc}}(\Omega)$, then the minors of order $n-2$ of $D^2 u$ and the first order derivatives of u are locally integrable respectively with exponents $n^2/(n^2-4)$ and $n^2/2$. Therefore,

$$\mathscr{H}_2 u \colon W^{2,n-1}_{\mathrm{loc}}(\Omega) \to \mathscr{D}'(\Omega) , \qquad \mathscr{H}_2^* u \colon W^{2,\frac{n^2}{n+2}}_{\mathrm{loc}}(\Omega) \to \mathscr{D}'(\Omega) .$$

We notice that in dimension $n > 2$ the definition of $\mathscr{H}_2^* u$ requires that u has second order derivatives.

1.2. *The Hessian measure.*

Another concept of weak Hessian is classically introduced for defining generalized solutions of the Monge-Ampère equation, see [9] and references therein; see also [18] for further developments. Now, we assume that $\Omega \subset \mathbb{R}^n$ is a convex set and u is a real-valued convex function defined on Ω. We denote by ∂u the subdifferential (also called normal mapping) of u and by $\partial u(E)$ the image by this multifunction of the subset $E \subseteq \Omega$:

$$\partial u(E) = \bigcup_{x \in E} \partial u(x) .$$

The Hessian measure of u, which we denote by Mu, is a Borel measure on Ω defined by

$$(1.8) \qquad Mu(E) = \mathscr{L}^n(\partial u(E))$$

for each Borel subset E. $\mathscr{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$. It is well known that for $u \in C^2(\Omega)$ the Hessian measure coincides with the Hessian determinant $\mathscr{H} u$ in the sense that the following equality holds

$$(1.9) \qquad \int_{\Omega} \varphi \, dMu = \int_{\Omega} \varphi \, \mathscr{H} u \, dx \,,$$

for all continuous φ with compact support.

We recall that a convex function u is locally Lipschitz and its gradient Du has locally bounded variation, see [6]. Hence $u \in \widetilde{W}^{2,1}_{\mathrm{loc}}$ and in dimension 2 we can consider $\mathscr{H}_2 u$. In the following theorem we compare these notions.

THEOREM 1.2. *If u is in the domain of two of the above weak formulations of the Hessian, these formulation coincide.*

As an illustration, in dimension $n = 2$ if u is convex, then $Mu = \mathscr{H}_2 u = \mathscr{H}^*_2 u$. Theorem 1.2 will be an easy consequence of the continuity properties we discuss in the next section.

2. CONTINUITY PROPERTIES

One of our concern is to elucidate on the continuity properties of those weak formulations of the Hessian. First, we note trivially that $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}^*_2$ are continuous in their corresponding domains with respect to the strong convergences. More precisely:

- If $u_h \to u$ in $W^{2,\frac{n^2}{n+1}}_{\mathrm{loc}}$, then $\mathscr{H}_1 u_h \to \mathscr{H}_1 u$ in the sense of distributions. This means that $\mathscr{H}_1 u_h[\varphi] \to \mathscr{H}_1 u[\varphi]$, for all $\varphi \in \mathscr{D}(\Omega)$;
- If $u_h \to u$ in $W^{2,n-1}_{\mathrm{loc}}$, then $\mathscr{H}_2 u_h \to \mathscr{H}_2 u$ in the sense of distributions.
- For $n = 2$, if $u_h \to u$ in $W^{1,2}_{\mathrm{loc}}$, then $\mathscr{H}^*_2 u_h \to \mathscr{H}^*_2 u$ in the sense of distributions.
- For $n > 2$, if $u_h \to u$ in $W^{2,\frac{n^2}{n+2}}_{\mathrm{loc}}$, then $\mathscr{H}^*_2 u_h \to \mathscr{H}^*_2 u$ in the sense of distributions.

It is also easy to prove convergence in $\mathscr{D}'$ of the weak Hessians of u_h coupling a suitable weak convergence of the minors of the Hessian matrix $D^2 u_h$ with a strong convergence of the function u_h or its first order derivatives Du_h. As an example, we have

- If $Du_h \to Du$ strongly in $L^{\frac{n^2}{2}}_{\mathrm{loc}}$ and (for $n > 2$) the minors of order $n - 2$ of $D^2 u_h$ converge to corresponding minors of $D^2 u$ weakly in $L^{\frac{n^2}{n^2-4}}_{\mathrm{loc}}$, then $\mathscr{H}^*_2 u_h \to \mathscr{H}^*_2 u$ in the sense of distributions.

Next, we examine continuity properties with respect to weak convergences. It is enough to consider the weakest formulation; *i.e.* $\mathscr{H}^*_2$. We first consider the case $n = 2$. In [10] it is shown by an example that $\mathscr{H}^*_2$ is not continuous with respect to weak convergence in $W^{1,2}_{\mathrm{loc}}$. Here we present another simple example of this type.

EXAMPLE 2.1. The functions

$$u_h = u_h(x, y) = \frac{1}{h} \sin h(x^2 + y^2)$$

clearly converge uniformly to zero and Du_h weakly $*$ in L^∞. On the other hand, $\mathscr{H}_2^* u_h$ are not converging to zero in the sense of distributions. To see this we consider a test function φ supported in the unit disk $D \subset \mathbb{R}^2$, and compute

$$\mathscr{H}_2^* u_h[\varphi] = \int_D (4xy\varphi_{xy} - 2x^2\varphi_{yy} - 2y^2\varphi_{xx}) \cos^2 h(x^2+y^2)\, dx\, dy\,.$$

Integrating in polar co-ordinates and then by parts, we find that

$$\lim_h \mathscr{H}_2^* u_h[\varphi] = \int_D (2xy\varphi_{xy} - x^2\varphi_{yy} - y^2\varphi_{xx})\, dx\, dy = 2\int_D \varphi\, dx\, dy\,.$$

The above example and the following arguments indicate clearly that in order to ensure convergence of $\mathscr{H}_2^* u_h$ to $\mathscr{H}_2^* u$ one really needs strong convergence in $W^{1,2}_{\text{loc}}$.

THEOREM 2.2. *Let $u_h \in W^{2,1}_{\text{loc}}$, $h = 1, 2, \ldots$, and $u \in W^{2,1}_{\text{loc}}$ be given. Then the distribution $\mathscr{H}_2 u_h = \mathscr{H}_2^* u_h$ converge to $\mathscr{H}_2 u = \mathscr{H}_2^* u$ under each of the assumptions below*:

(a) *$u_h \rightharpoonup u$ weakly in $W^{2,1}_{\text{loc}}$*;

(b) *the sequence $\{u_h\}$ is bounded in $W^{2,1}_{\text{loc}}$ and converges to u uniformly on compact subsets.*

The proof follows easily when we realize that both assumptions (*a*) and (*b*) yield $u_h \to u$ strongly in $W^{1,2}_{\text{loc}}$. Actually, we have

LEMMA 2.3. (a) $\Rightarrow$ (b) $\Rightarrow$ *$u_h \to u$ strongly in $W^{1,2}_{\text{loc}}$.*

We remark that both imbeddings of $W^{2,1}_{\text{loc}} \subset C^0$ and $W^{2,1}_{\text{loc}} \subset W^{1,2}_{\text{loc}}$ are not compact, in the sense that bounded sequences in $W^{2,1}_{\text{loc}}$ may not have subsequences converging locally uniformly, or strongly in $W^{1,2}_{\text{loc}}$.

EXAMPLE 2.4. Given a function in $u \in W^{2,1}(\mathbb{R}^2)$ with $u \not\equiv 0$ and $\lim_{z\to\infty} u(z) = 0$, we define $u_h(z) = u(hz)$, $\forall h \in \mathbb{N}$. Then $u_h(z) \to 0$, $\forall z \in \mathbb{R}^2 - \{0\}$, $Du_h \to 0$ in L^1, $\|D^2 u_h\|_1 = \|D^2 u\|_1$ for all $h \in \mathbb{N}$ and $D^2 u_h \overset{*}{\rightharpoonup} C\delta$ weakly in the sense of measures, with C a constant matrix. Moreover $\{u_h\}$ has no subsequence converging locally uniformly and, as $\|Du_h\|_2 = \|Du\|_2 > 0$ for all h, no subsequence strongly converging in $W^{1,2}_{\text{loc}}$.

On the other hand, the assumption of weak convergence in $W^{2,1}_{\text{loc}}$ is essentially stronger than mere boundedness. By Ascoli-Arzelà theorem, under assumption (*a*) we can prove uniform convergence $u_h \to u$ on compact subsets, hence condition (*b*) holds.

PROOF OF LEMMA 2.3. There is no loss of generality in assuming $u = 0$, otherwise we consider the functions $u_h - u$.

Let us prove $(a) \Rightarrow (b)$. For simplicity, we assume $\Omega =]0,1[^2$. For $\varphi \in C_0^\infty(\Omega)$, we define $v_h = \varphi u_h$, $h = 1,2,\dots$, and we have

$$Dv_h = u_h D\varphi + \varphi Du_h\,,$$

$$D^2 v_h = u_h D^2\varphi + D\varphi \otimes Du_h + Du_h \otimes D\varphi + \varphi D^2 u_h\,,$$

hence $v_h \rightharpoonup 0$ weakly in $W^{2,1}(\Omega)$.

The point is that weak convergence of $\{u_h\}$ gives equi-integrability of $\{Du_h\}$. It is well known that $W^{2,1}(\Omega)$ is continuously imbedded into $C(\overline{\Omega})$, see [1, p. 100]. If v is a continuous function of class $W^{2,1}(\Omega)$ vanishing on $\partial\Omega$, then for all (x_1,y_1) and (x_2,y_2) in $\overline{\Omega}$, we have

$$|v(x_1,y_1) - v(x_2,y_2)| \leqslant \left|\int_0^{x_1} ds \int_{y_2}^{y_1} v_{xy}(s,t)\,dt\right| + \left|\int_{x_2}^{x_1} ds \int_0^{y_2} v_{xy}(s,t)\,dt\right|.$$

In particular,

$$\|v\|_\infty \leqslant \|v_{xy}\|_1\,.$$

These inequalities yield equiboundedness and equicontinuity of any bounded family of functions in $W_0^{2,1}(\Omega)$ with equi-integrable second order derivatives. We are in a position to apply Ascoli-Arzelà theorem and get $v_h \to 0$ uniformly in $\overline{\Omega}$. This implies uniform convergence $u_h \to 0$ on compact subsets of Ω. If $K \subset \Omega$ is compact, it suffices to choose $\varphi \equiv 1$ on K.

Now we assume that condition (b) holds and prove strong convergence in $W^{1,2}_{\mathrm{loc}}(\Omega)$. If $v \in W_0^{2,1}(\Omega)$, integrating by parts we find

$$\int_\Omega |Dv|^2 = -\int_\Omega v\,\Delta v \leqslant \|v\|_\infty\,\|D^2 v\|_1\,.$$

Therefore, defining $v_h = \varphi u_h$ as above, we see that $v_h \to 0$ strongly in $W^{1,2}(\Omega)$. To deduce from this strong convergence of u_h on an arbitrary compact $K \subset \Omega$, as before we only need to take $\varphi \equiv 1$ on K.

Notice that (b) implies strong convergence of $\{u_h\}$ in $W^{1,2}_{\mathrm{loc}}$ can be seen also as a consequence of inequality (1.16) of [13]. □

Theorem 2.2 can be slightly generalized.

THEOREM 2.5. *Let $\{u_h\}$ be a sequence bounded in $\widetilde{W}^{2,1}_{\mathrm{loc}}$ and converging to u uniformly on compact subsets. Then $u \in \widetilde{W}^{2,1}_{\mathrm{loc}}$ and $\mathscr{H}_2 u_h = \mathscr{H}^*_2 u_h$ converge to $\mathscr{H}_2 u = \mathscr{H}^*_2 u$ in the sense of distributions.*

Now we assume $n > 2$. We have the following continuity property.

THEOREM 2.6. *If $u_h \rightharpoonup u$ weakly in $W^{2,p}_{\mathrm{loc}}$ for some $p > \frac{n^2}{n+2}$, then $\mathscr{H}^*_2 u_h \to \mathscr{H}^*_2 u$ in the sense of distributions.*

PROOF. As $p > \frac{n^2}{n+2}$ the imbedding $W^{2,p}_{\mathrm{loc}} \subset W^{1,\frac{n^2}{2}}_{\mathrm{loc}}$ is compact. Then $Du_h \to Du$ strongly in $L^{\frac{n^2}{2}}_{\mathrm{loc}}$. On the other hand, the minors of order $n-2$ of D^2u_h converge to corresponding minors of D^2u in the sense of distributions, see [11, Theorem 8.2.1, p. 173], and are bounded in $L^{\frac{p}{n-2}}_{\mathrm{loc}}$, where we notice that $\frac{p}{n-2} > \frac{n^2}{n^2-4} > 1$, hence they are weakly converging in $L^{\frac{p}{n-2}}_{\mathrm{loc}}$ and therefore we can pass to the limit in the expression defining $\mathscr{H}^*_2 u[\varphi]$. □

REMARK 2.7. In [5] it is shown that the condition $p > \frac{n^2}{n+2}$ in Theorem 2.6 is sharp, in the sense that the conclusion cannot be drawn in general (in dimension $n > 2$) assuming

$$u_h \rightharpoonup u \quad \text{weakly in} \quad W^{2,\frac{n^2}{n+2}}_{\mathrm{loc}}. \tag{2.1}$$

With this condition, we still have weak convergence in $L^{\frac{n^2}{n^2-4}}_{\mathrm{loc}}$ of the minors of order $n-2$ of D^2u_h, but we need to reinforce (2.1) to guarantee the strong convergence in $L^{\frac{n^2}{2}}_{\mathrm{loc}}$ of first derivatives.

For $1 < \frac{n}{2} < r < n$, we have the Sobolev imbeddings

$$W^{2,r}_{\mathrm{loc}} \subset W^{1,r^*}_{\mathrm{loc}} \subset C^{0,2-\frac{n}{r}}_{\mathrm{loc}},$$

neither of which is compact. Clearly, if $u_h \to u$ strongly in W^{1,r^*}_{loc}, then it converges also strongly in the Hölder space. On the other hand, by the interpolation inequality (5) of [16], for a bounded sequence $\{u_h\}$ in $W^{2,r}_{\mathrm{loc}}$ the convergence in the Hölder space implies strong convergence of first derivatives, that is, the two convergences are equivalent. For $r = \frac{n^2}{n+2}$, we have the following generalization of Theorem 2.6, in the spirit of Theorem 2.5:

Let condition (2.1) *hold together with $u_h \to u$ strongly in $C^{0,1-\frac{2}{n}}_{\mathrm{loc}}$. Then $\mathscr{H}^*_2 u_h$ converge to $\mathscr{H}^*_2 u$ in the sense of distributions.*

We conclude this section by recalling some continuity properties of the Hessian measure, see [9].

THEOREM 2.8. *Let u_h, $h = 1, 2, \ldots$, and u be convex functions and let the sequence u_h converge to u uniformly on compact subsets. Then Mu_h converges to Mu weakly $*$ in the sense of measure, that is,*

$$\int_\Omega \varphi \, dMu_h \to \int_\Omega \varphi \, dMu$$

for all continuous functions φ with compact support.

It is clear that in the two-dimensional context Theorem 2.8 follows from Theorem 2.5.

3. The L^1-part of the Hessian

One concern of the study of the weak formulations of Jacobian and Hessian determinant is to relate them with the point-wise definitions. In [14] it is proved that the point-wise Jacobian coincides with the distributional one, provided the latter is a locally integrable function. More generally, assuming that the distributional Jacobian is a Radon measure, then the point-wise Jacobian is the density of its absolutely continuous part with respect to the Lebesgue measure. In a different direction, in [8] the identity between distributional and point-wise Jacobian is established without assuming a priori that the former is a distribution of order 0, but under suitable integrability conditions for the differential matrix.

We mention also that in [15] examples are given showing that the support of the singular part of the distributional Jacobian in $\mathbb{R}^n$ can be a set of any prescribed Hausdorff dimension less than n.

The result of [14] immediately gives that the point-wise Hessian is the density of the absolutely continuous part of $\mathcal{H}_1 u$ for $u \in W^{2,\frac{n^2}{n+1}}_{\mathrm{loc}}$, assuming that $\mathcal{H}_1 u$ is a measure. This is generalized in [10] showing that the point-wise Hessian is the regular part of the distribution $\mathcal{H}_2 u$. A further extension is given in [12] for $\mathcal{H}_2^* u$. Here we remark that an analogous result holds for the Hessian measure Mu. Given a Radon measure μ on Ω, we denote by μ_a the density of its absolutely continuous part with respect to Lebesgue measure. We recall that

$$\mu_a(x) = \lim_h \mu * \rho_h(x)\,, \quad \text{for a.e. } x \in \Omega\,,$$

where $\{\rho_h\}$ is a sequence of mollifiers. If u is a convex function, its second order derivative $D^2 u$ is a (matrix-valued) measure.

Theorem 3.1. *For every convex function u the density of the absolutely continuous part of the Hessian measure Mu is $\det(D^2 u)_a$, that is*

$$(Mu)_a = \det(D^2 u)_a\,. \tag{3.1}$$

In particular, $\det(D^2 u)_a$ is locally integrable.

Denoting Mu by $\det D^2 u$, we formally rewrite (3.1) as

$$(\det D^2 u)_a = \det(D^2 u)_a\,.$$

Therefore, we can state the result of Theorem 3.1 by saying that the two operations of computing the Hessian determinant and of passing to the density of the absolutely continuous part commute.

To prove Theorem 3.1, we begin with a lemma.

Lemma 3.2. *Let $f_h \in L^1_{\mathrm{loc}}(\Omega)$, $h = 1, 2, \dots,$ be functions verifying $f_h \geqslant 0$, $\forall h$, $f_h(x) \to f(x)$ a.e. and $f_h \overset{*}{\rightharpoonup} \mu$ weakly in the sense of measures. Then we have $f(x) \leqslant \mu_a(x)$ a.e. in Ω.*

PROOF. For any continuous function $\varphi \geqslant 0$ with compact support, by Fatou lemma we find

$$\int_\Omega \varphi f\, dx \leqslant \lim_h \int_\Omega \varphi f_h\, dx = \int_\Omega \varphi\, d\mu .$$

This inequality with arbitrary φ implies that $f \in L^1_{\mathrm{loc}}(\Omega)$. Moreover, considering a sequence $\{\rho_h\}$ of mollifiers, for a.e. $x \in \Omega$ we have

$$f(x) = \lim_h f * \rho_h(x) \leqslant \lim_h \mu * \rho_h(x) = \mu_a(x) .$$

□

More generally, if μ_h, $h = 1, 2, \ldots$, are nonnegative Radon measures and $\mu_h \overset{*}{\rightharpoonup} \mu$, then

$$\liminf_h (\mu_h)_a \leqslant \mu_a .$$

PROOF OF THEOREM 3.1. By convolution, we can find a sequence of smooth convex functions u_h such that $u_h \to u$ uniformly on compact subsets of Ω, $D^2 u_h \to (D^2 u)_a$ a.e. in Ω and hence also $\det D^2 u_h \to \det (D^2 u)_a$ a.e. in Ω. Notice that for all h we have $Mu_h = \det D^2 u_h$. Recalling the continuity property of Theorem 2.8, by Lemma 3.2 we get

$$\det (D^2 u)_a \leqslant (Mu)_a , \quad \text{a.e. in } \Omega .$$

To prove the opposite inequality, we use the change of variable formula, see [7, Theorem 1, p. 72]. Accordingly, there exists a Borel set $R_u \subseteq \Omega$ such that u is twice differentiable in R_u, $\mathscr{L}^n(\Omega \setminus R_u) = 0$, and

$$\int_A \det (D^2 u)_a\, dx = \int_{\mathbb{R}^n} N_u(y, A)\, dy , \tag{3.2}$$

for every measurable subset A of Ω. Here $N_u(y, A)$ denotes the Banach's indicatrix

$$N_u(y, A) = \#\{\, x \in A \cap R_u \ :\ Du(x) = y \,\} .$$

From (3.2) we deduce

$$\int_A \det (D^2 u)_a\, dx \geqslant \mathscr{L}^n\big(Du(A \cap R_u)\big) = Mu(A \cap R_u) ,$$

which clearly implies

$$\det (D^2 u)_a \geqslant (Mu)_a .$$

□

REMARK 3.3. Taking into account the continuity property of Hessian measures stated in Theorem 2.8, it is interesting to notice that weak $*$ convergence of measures does not imply point-wise convergence of the densities of absolutely continuous parts. As an example, we recall that the Lebesgue integral of a continuous function is the limit of integral sums, which can be considered as integrals with respect to purely atomic measures.

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