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## On Axiomatic Foundations Common to Classical Physics and Special Relativity

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**Matematica.** — *On Axiomatic Foundations Common to Classical Physics and Special Relativity.* Nota (*) di ALDO BRESSAN.

ABSTRACT. —
(i) The class of the axiomatic foundations mentioned in the title is called Ax Found; and its structure is treated in the introduction.
(ii) This consists of Parts A to G followed by the References.
(iii) In [17] Bressan's modal logic is treated in a consciously non-rigorous way. Instead here, as well as Ax Found, it has a rigorous treatment. Such a treatment had been appreciated by the mathematical physicist C. Truesdell in [62].
(iv) In 1953 Truesdell had a remarkable intuition, whose correctness appeared only in 1962, from Bressan's monograph [3].
(v) As a foreign member of the Lincei Academy, Truesdell supported some logical features, absent in his school, and he gave M. Pitteri a "confidential copy" involving this fact.
(vi) Since thus the present rigorous treatment of Bressan's modal logic appears strongly supported by Truesdell, it was natural to dedicate the present work to his memory.
(vii) In the introduction one says to have proved certain results (whose proof does not appear there) concerning rational mechanics or Bressan's modal logic treated rigorously.

KEY WORDS: Axiomatic Foundations; Special Relativity; Classical Physics; Continuous Media.

RIASSUNTO. — *Sui fondamenti comuni alla fisica classica e alla relatività ristretta.*
(i) La classe dei fondamenti assiomatici menzionati nel titolo è detta brevemente Ax Found; ed è trattata nell'introduzione.
(ii) Questa consiste nelle Parts A, . . . , G seguite dalle *References*.
(iii) In [17] la logica modale di Bressan è trattata in modo consciamente non rigoroso. Invece qui essa, al pari di Ax Found, ha una trattazione rigorosa. Una tale trattazione era stata apprezzata dal fisico matematico Truesdell in [62].
(iv) Nel 1953 Truesdell ebbe una notevole intuizione, la cui correttezza risultò solo nel 1962, dalla monografia [3] di Bressan.
(v) Come Socio straniero dei Lincei, Truesdell sostenne degli aspetti logici, assenti nella sua scuola. Inoltre diede a M. Pitteri una "confidential copy" involgente tale fatto.
(vi) La presente trattazione rigorosa della logica modale di Bressan appare perciò fortemente sostenuta da Truesdell. Era quindi naturale dedicare alla sua memoria il lavoro inviato.
(vii) Nell'introduzione si dice di aver ottenuto certi risultati (la cui dimostrazione non appare ivi) concernenti la Meccanica razionale o la logica modale di Bressan trattata rigorosamente.

## CONTENTS



(*) Pervenuta in forma definitiva all'Accademia il 29 settembre 2005.

## PART A: ON THE CLASS OF THE AXIOMATIC FOUNDATIONS MENTIONED IN THE TITLE

There is a unique increasing series of instants $\tau_0, \tau_1, \tau_2, \ldots$ such that, for $r = 0, 1, \ldots$, at $\tau_r$ some assertions belonging to axiomatic foundations begin to be known. These assertions constitute a work $w_r$ called contribution to Ax Found. The works $w_0, w_1, \ldots$ are briefly denoted by "Contr Ws" and for practical motives see $\text{AS}_2$ and $\text{AS}_{2,1}$ below. Of course, for $r = 0, 1, 2, \ldots$, at $\tau_r$ the contributing works $w_{r+1}, w_{r+2}, \ldots$, are not known.

The study of Ax Found – see (i) in the Abstract – is strongly based on Br&Mont$=_D$ [7] – see ($\text{AS}_{1,3}$ to $\text{AS}_{1,7}$ and mainly) $\text{AS}_{1,6}$ below – and aims at improving and amplifying [7].

**$\text{AS}_1$** The mentioned contributing works are consecutive in that they form a unique series (like if they were chapters or groups of chapters in a same book. This book is being constructed and its possible developments will generally be at most imperfectly known).

**$\text{AS}_{1,2}$** Different contributions might be published on different scientific journals.

**$\text{AS}_{1,3}$** I shall use (also but not only) 'Pitteri 03' as an abbreviation for Pitteri's paper published on the Journal of Elasticity **72**, 241-261, 2003. This fact also appears from [60].

**$\text{AS}_{1,4}$** In ref. (*i.e.* reference) [16] (on p. 261) of Pitteri 2003 it is shown how to obtain an extended version of it, which I shall denote by 'ExtendPitt' – *e.g.*, in order to obtain Extend Pitt one can also see [61].

**$\text{AS}_{1,5}$** In some entries of the References of the present §1, some abbreviations are explicitly added within parentheses: *e.g.*, such additions are performed in [3, 4] and [5] by means of '$=_D$ Met', '$=_D$ GIMC', and '$=_D$ Bressan 1974' in order to respectively introduce the abbreviations 'Met', 'GIMC' and 'Bressan 1974'.

**$\text{AS}_{1,6}$** Besides using 'Br&Mont' for [17], I can abbreviate [48] (on p. 27) in ExtendPitt, *i.e.*, (the only) Truesdell's 654 page book, by 'Truesdell 1984'.

**$\text{AS}_{1,7}$** More details on abbreviations can be found in Part F.

**$\text{AS}_{1,8}$** Let us explicitly remark that, *e.g.*, within $\text{AS}_2$ or $\text{AS}_{2,3}$, AS has to be read as *assertion* or *subassertion* respectively.

Furthermore I mean every such assertion as an assertion set or an assertion conjunction; and I regard every subassertion $\text{AS}_{r,s}$ as an assertion headed by $\text{AS}_r$ (for instance belonging to $\text{AS}_r$).

## Part B: Main reasons for changing some parts of Section 2 in (Br&Mont's$=_D$) [17]. A strange but lucky situation

**AS$_2$** On the one hand, the article Br&Mont – see AS$_{1,6}$ – treats particle systems (without constraints), is of the Mach-Painlevé type ([1]), and is based on A. Bressan's modal logic (although for practical motives this logic is sometimes consciously treated non-rigorously in Br&Mont – see ftn. 7 on p. 168 there –).

**AS$_{2,1}$** Furthermore, I believe that the aforementioned conscious toleration of mistakes of modal logic was even compulsory in Br&Mont, in order not to compel, *e.g.*, rigorous mathematical physicists interested in Br&Mont (as a work on mechanics) to know my (powerful but complex) theory of modal logic (presented in GIMC – see AS$_{1,5}$ –).

**AS$_3$** On the other hand, in the present work the logical modal calculus $MC^\nu$ presented in GIMC is instead carefully taken into account (and also with some useful results for mechanics appreciable by the well known mathematical physicist C. Truesdell – see AS$_7$ below in part C).

**AS$_4$** However, in Ax Found – see AS$_1$ – a contributing work will perform some changes improving Br&Mont not only in connection with (modal) logic; and this holds especially for Def. 2.3 and the subsequent part of Sect. 2. Furthermore these changes are in accord with the proofs written in Sects. 3 to 8 for the theorems considered in Br&Mont, while these theorems as well as their proofs strongly contrast with the original version of Def. 2.3 and the subsequent part of Sect. 2. This is the (strange) situation mentioned in (Part B)'s title.

**AS$_{4,1}$** Neither Truesdell nor Pitteri attended any technical course on modal logic. However they used this logic intuitively and correctly.

**AS$_{4,2}$** In particular, in the year 2004 I noted a remarkable intuition of Truesdell that he wrote as contributor, within a (polemic) ftn. in a paper appeared in 1953 – see AS$_6$ below in part C.

**AS$_{4,3}$** In spite of AS$_3$, by AS$_4$ to AS$_{4,2}$ (hence AS$_6$), I think that also readers not interested in modal logic may be interested in many contributions of mine to Ax Found.

## Part C: Reasons for using "Ax Found" and for writing the present work in memory of Prof. C. Truesdell

(1) In (Part B)'s title some changes needed by Br&Mont are mentioned, however together

([1]) ($\alpha$) A book or paper, *e.g.* Met$=_D$[3] is said to be *à la Mach-Painlevé* or *of the Mach-Painlevé type*, if in it notions such as mass and force (and possibly also, *e.g.*, inertial space and (inertial) instant) are not regarded as primitives, unlike what happens in most works.

($\beta$) *E.g.* Met is *à la Mach-Painlevé* at a high level, especially with respect to Painlevé 1922, *i.e.* [59].

with a strange situation (explained in $AS_4$) that strongly weakens these changes' importance.

(2) Hence Br&Mont substantially is, I think, the more advanced among our (published) articles on particle systems.

(3) I think that ElimPrF – see [16] in the References following Part F – is one among my best results of the Mach-Painlevé type (see ftn. 1 placed on $AS_2$); it was found in Sept 04 and it is yet unpublished.

(4) The difference between the title of the present work (Ax Found) and Br&Mont's is not substantial; but only in Ax Found both terms such as "kinematics" or "dynamics" fail to appear (essentially). Thus, briefly, (3) is asserted in that [16], abbreviated by ElimPrF and concerning PrF (proprietà fisica, *i.e.*, physical property), allows us to somehow effectively perform the elimination mentioned in [16] itself.

**AS**$_5$ Let me add that the notion of physical necessity, which (briefly) is a possible sense for the logical symbol $N(=_D \Box)$ of the language $MC^v$ (presented in GIMC) ([2]) and which strongly affects many works of A. Bressan and, *e.g.* of his pupils, is important (*e.g.* in Met) within both kinematics and dynamics. Obviously it is also essential in ElimPrF.

**AS**$_6$ In 1953, Truesdell communicated an article, precisely reference [24] in ExtendPitt, and in a source footnote he expressed a complete disagreement (shared by G. Hamel) with this article; but Truesdell explained that "publication of this article may arouse the interest of *students of mechanics and logic alike*, thus perhaps leading to a proper solution of this outstanding but neglected problem". I regarded the above phrase italicized by me, as a remarkable intuition because the first paper giving a particular solution to the above problem is my (long) 1962-article Met; and it totally complies with that phrase.

**AS**$_7$ Besides $AS_6$, let me note that in his 1986-letter (in Italian) to the Accademia dei Lincei, Truesdell supported a research program (already started by me); and he strongly appreciated the logical features of some articles of mine (or of my pupils). Hence the present work, which often (rigorously) deals with above logical features, appears strongly supported by him – see $AS_{7,2}$ –. Therefore I regard it correct to write it in memory of prof. C. Truesdell.

**AS**$_{7,1}$ **(i)** Truesdell gave to Pitteri a 'confidential copy' of his aforementioned 1986-letter (which is written in Italian) ([3]). Furthermore its main part written by me below is substantially included in its version printed in item (D) on pp. 21-22 of ExtendPitt.

([2]) Physical necessity is a rigorous and strongly specified notion related to A.W. Burk's notion of causal implication – see [18].

([3]) Much information related with this letter can be obtained in the following way. Consider (1°) the notion of physical necessity – see the preceding footnote – (2°) the extended version of Pitteri 2003 – see [61] –. In this see in particular ftn. 3 on p. 19, the second paragraph of Sect. 5 (on p. 19), furthermore (on pp. 19 to 21) the items (A), (B) and (D).

Remark that this strongly justifies what I have written here. Furthermore **(ii)** the aforementioned version printed in (D) belongs to the article Pitteri 2003 – see $AS_{1,3}$ – printed in ExtendPitt. **(iii)** Lastly I note that its main part written below gives us various useful information (possibly not directly related with Truesdell's letter) even if it fails to be completed by ExtendPitt, so that it is not fully understandable.

**$AS_{7,2}$** **(i)** Let me specify that the Memoria mentioned in assertion (a) below is the Memoria Lincea mentioned in [12] on p. 24 of ExtendPitt. In his 1986-letter Truesdell strongly supported this memoria as a starting point for the extension of Met to continuous media ([4]).

**(ii)** Truesdell's appreciation of Bressan appears on p. 259 of the work Pitteri 2003 – see $AS_{1,3}$ – from line -14 to line -6, as well as from (D) on p. 21 of ExtendPitt (mainly from (D)'s part (e)). Furthermore

**(iii)** for his appreciation of Bressan in general, one can see Truesdell 1984 – see $AS_{1,7}$ – and, *e.g.*, item 14 in the Contents on its p. 503. There 'Fear of Real Work in Formal Logic' is mentioned; and the mentioned work is performed by Bressan. Incidentally, to see N. Belnap's observation in 10 within page xxiv of GIMC's preface may be useful too.

**(iv)** Lastly Truesdell's appreciation of Bressan also appears from (some parts of) the pp. 533 to 534 in Truesdell 1984. (He knew, *e.g.* ftn. 7 on p. 19 of ExtendPitt).

The so called Truesdell's volume is the volume on which Pitteri 2003 is published, *i.e.* the volume containing the following

> Symposium on recent advances and new directions in Mechanics, Continuum Thermodynamics, and Kinetic Theory in Memory of Clifford A. Truesdell III, within the 14th U.S.A. National Congress of Applied Mechanics, Blacksburg, June 23-28, 2002.

**$AS_{7,3}$** Below is our main part (a) to (h) of Truesdell's 1986-letter: the main points (a) to (f) (on which ftns. (a) to (c) are placed for explanations) and Truesdell's additions, mainly (g) to (h).

> (a) The subject of this Memoria belongs to Hilbert's sixth problem, 'Mathematische Behandlung der Axiome der Physik ...', which presently is only partially solved.
> Truesdell mentions the remarkable contributions given to its solution by Noll, Williams, Gurtin, Appleby, Šilhavý, Ziemer, Matolecsi, among others; then he adds:
>
> (b) All of them have used only the procedure of mathematical analysis. Their axioms are proposed as *necessary conditions* to construct mathematical structures fit for consistent and clear applications to physical systems and present-day technology ...

([4]) On p. 19 of ExtendPitt see the 1st paragraph of the text and ftn. 3 (where '[12]' just refers to the above Memoria).

(c) In 1962 prof. Bressan, considering only mass-point systems, in a deep work of well known difficulty, reached an impeccable solution in the sense of mathematical logic and he was the first to do so.

(d) In his more recent studies he treats classical mechanics in a universe that contains mathematical objects called 'observers', and

(d') he adds a suitable *existence axiom* that was lacking (5).

(e) In addition, he constructs a formal concept of *physical possibility* which abstractly represents a physical experiment (6).

(f) Presently prof. Bressan is completing his logical axiomatization of mechanics with some of his pupils (Montanaro, Pitteri, etc.), and in a way capable of including the mechanics of continuous media in the sense of Noll and his followers.

(g) After a digression Truesdell recommends the publication of a Bressan's paper, the one that appeared in 1987 as [10], because 'it carries out a remarkable step' in the program mentioned in (f), the main circumstances relevant for it having been sketched in (a) to (e).

(h) Lastly Truesdell adds that Bressan is the unique person (as far as he knows) capable of mastering mathematical logic, rational mechanics, electromagnetism and special relativity; in his opinion Italy and the Accademia dei Lincei can be proud of him.

* * *

Here, and as well as in his book Truesdell 1984, Truesdell mentions fields outside physics, like logic and philosophy of science. His remarks about Bressan's high standing in these fields get a very authoritative support by earlier assertions of the philosopher of science N. Belnap in his Foreword to GIMC, [...] (7).

(5) The lacking axiom is Ammissione 10.2 in Met, p. 106 (also recalled in GIMC, footnote 3 on pp. 110-111). This is framed as a possibility condition, but (d'), where *existence* axiom substantially stands for *possibility axiom*, is correct because of some peculiar technical features of Met; and this shows that Truesdell had grasped the main technical semantic features of that paper.

(6) This construction, performed in Met and refined later in GIMC and Bressan 1985, agrees with the views of Hamel 1908 and Hamel 1927.

(7) In his foreword, pp. xiii to xxv of GIMC, Belnap preliminary says that GIMC 'is the most important contribution to date concerning the introduction of quantifiers into modal logic. It surpasses any article or book in the generality of its conceptions, the degree of their development and the profundity of their analysis'(p. xiii, line 3). Belnap especially points out Bressan's 'new analysis of predication' (p. xiv, line 10).

In the subsequent sections 1 to 8 Belnap details and motivates his preliminary assertions. For instance he says (p. xiv, sect. 1, line 3): '... that [Bressan's modal language] $ML^{\nu}$ is – uniquely among modal logics – a complete type theory with no upper limit on its types, is extremely important'. Indeed, as is broadly mentioned on p. xvii, line -9, by the lack of this limit GIMC solves positively the problem considered in Carnap 1954 – see [19] –, pp. 195 to 196. Among Bressan's 'distinctive semantic features' (pp. xviii to xxiii) Belnap mentions attributes, lambda abstraction, and definite descriptions. Furthermore, in New Directions (p. xxiv, lines 1 to 6), Belnap especially points out Bressan's notion of *absolute attribute*, and says: 'The articulation and deployment of this notion is extremely important to Bressan's enterprise ...'. This is

**AS**$_{7,4}$ The work ElimPrF – see [16] –, mentioned above $AS_5$, is strongly related with Met, in Italian; and incidentally Br&Mont (written in English) appears, *e.g.*, in the contents at the outset of the present §1. Then it is useful, first, to give the information on Met in $AS_4$ below, and second, to write (in $AS_8$ below) an Italian-English translation followed by certain comments on some features that I believe important – see $AS_{8,1}$ –.

**AS**$_{7,5}$ My long article Met belongs to classical physics and deals with particle systems (mainly) without constraints; and it is written in an *unusual extensional* part of the Italian language, but it can deal with (the modal notion of) causal possibility ([8]). Furthermore, it is *à la Mach-Painlevé* – see ftn. 1 placed on $AS_2$ (in part B) –. By using these features, in Met (certain) rigorous foundations of classical particle mechanics *à la Mach-Painlevé* have been stated.

**AS**$_{7,6}$ Truesdell strongly supported a research program (already started by A. Bressan) to extend Met to continuous systems. As was in part already noted, this can be easily seen from (i) above Pitteri's confidential copy in $AS_{7,1}$, or (ii) from Section 5 of Pitteri 2003 – see $AS_{1,3}$ –, p. 18, or better (iii) from its extended version: see ftn. 3 on p. 19 of ExtendPitt (and the part of Sect. 5 on pp. 18 to 19).

**AS**$_8$ Here is the aforementioned (Italian-English) translation related to some symbols used in Met.

| | | |
|---|---|---|
| 1) | PE (punto evento, or cronotopo) | EP (event point, or space time) |
| 2) | PM (punto materiale, or particle) | MP (mass point, or particle) |
| 3) | CMP (caso meccanicamente possibile) | MPC (mechanically possible case) |
| 3') | CMP-caso | MPC-case |
| 4) | Ist (istante) | Inst (instant) |
| 5) | PEO($M, \gamma$) (punto evento occupato dal punto materiale $M$ nel CMP-caso $\gamma$, pensato come funzione PEO di $M$ e $\gamma$) | EPO($M, \gamma$) (event point occupied by the mass point $M$ in the MPC-case $\gamma$, regarded as a function EPO of $M$ and $\gamma$) |
| 6) | Preced (relazione di *precedenza temporale* fra punti eventi) | Preced (relation of *time precedence* between event points) |
| 7) | PrF (*proprietà fisica* non cinematica e *concernente un punto materiale* ad un istante ed in un CMP-caso) | PrF (non-kinematic *physical property* for a particle at an instant and in an MPC-case) |

**AS**$_{8,1}$ Let us now emphasize that the above translations 5) to 7) are chosen suitably,

confirmed, for instance, by Bacon 1980. Lastly, let us note that the citesmanship assertion in Truesdell 1984 – see $AS_{1,7}$ –, p. 532, line -12, is supported by Section 16, p. xxiv, line -13 in GIMC, where Belnap says that GIMC 'does not itself contain an axiomatization of physics (Bressan has written on this elsewhere in a somewhat different form ...) (he refers to Met, where modalities are phrased in a rather unusual but extensional language).

([8]) Causal possibility is considered in ftn. 17 on p. 22 of ExtendPitt. (See also item (E) on the same page).

looking forward to a simplified version of Met. Actually, Met contains much more complex definitions analogous to above 1) to 7) ([9]).

## PART D: SOME RESULTS READY FOR INCLUSION IN AX FOUND . A VERY SERIOUS DIFFICULTY OVERCOME BY A. ZANARDO

**AS**$_9$ I have (intuitively) characterized (certain) primitive notions common to classical physics and special relativity. Thus a relevant lack in Br&Mont has a (yet unpublished) remedy.

**AS**$_{10}$ In Sect. 3 on p. 169 of Br&Mont, the assertion involving (3.3) is a non-trivial theorem, whose proof is not even hinted at. Now, among other things, a (yet unpublished) rigorous version of it is written in Italian (*i.e.* ElimPrF).

**AS**$_{11}$ From the title of Part B, (explained in AS$_4$), it is obvious that the proof of some theorems asserted in Sects. 3 to 8 of Br&Mont – see AS$_4$ – need a (rigorous) completion taking into account the changes of Definition 2.3 considered in Ax Found. Now in some sense this completion is ready.

**AS**$_{12}$ On p. 47 of GIMC, it is said that "... one might object that AS12.21 [accepted in the logical calculus $MC^v$] is confusing [for certain reasons concerning descriptions and hinted on p. 47 of GIMC, at line 2" and that it would be better to replace it with AS12.21* [which is incompatible with AS12.21]. However, AS12.21 was preferred because, *e.g.*, "It is more useful for ... showing" the validity of Theorem 63.1 (a main result in GIMC). Unfortunately, later, Bressan noted a serious non-acceptable consequence of AS12.21 – see, *e.g.*, (a) in the part (c) of Zanardo 2004 and the part of its p. 10 below line 10 –. Furthermore, the (practically) only way seen by Bressan to overcome this difficulty was to ask A. Zanardo (who professionally was a logician) whether some previous results of him could be extended in a certain way. In Zanardo 2004 a very satisfactory answer could be given, by which the last difficulty completely disappeared as well as those related (above) to descriptions and AS12.21.

([9]) ($\alpha$) In the last 10 or 20 years I elaborated a relevant simplification of Met (also with broader hypotheses and briefer notations) but practically as efficient as its original version in connection with (possible evolutions of) the real world. Furthermore,

($\beta$) In 2004 Pitteri delivered a conference on Met at the Dip. of Structural Engineering of the University of Pisa (Official laboratory for experiments on building materials); and he was successful. In fact in a letter of 19/VII/2004 Piero Villaggio wrote to me to be much impressed by the ideas presented in Met and their coherence; furthermore he suggested me to publish an English version of Met (1962) on some well known scientific journal, regretting that Met was little known.

($\gamma$) In my reply I agreed that Met was little known to rigorous mathematical physicists (and for reasonable motives also valid for Br&Mont – see AS$_2$ and AS$_{2,1}$ –). However I stressed that Met was appreciated by different scientific communities. (Obviously Villaggio ignored, *e.g.*, Sect. 5 of ExtendPitt.

($\delta$) Obviously, in my reply I considered ($\alpha$). However I had not yet known my work ElimPrF, mentioned above AS$_5$ in §1 and strongly related with Met.

## PART E: ON MONTANARO'S COLLABORATION TO BR&MONT, SUBSTANTIALLY WITHIN HIS DEGREE THESIS; AND ON HIS SUBSEQUENT PAPERS, IN PART VERY IMPORTANT OR EVEN SURPRISING

Since Ax Found is strongly based on Br&Mont to which A. Montanaro collaborated, (especially the end of) the title of Part E renders it natural to reserve a whole part of our Introduction for him.

**AS**$_{13}$ When the subject of a degree thesis is very complex, like the one of Br&Mont, it is natural to receive many suggestions.

**AS**$_{13,1}$ However, let us now specify that among Montanaro's papers without my collaboration, *e.g.*, [29] to [30], [33] and [55] are (at least) important, [51] to [53] are very important, [34] is surprising, and [48] is most important (I note that my long monograph Met – see AS$_{7,5}$ – is cited in its references), and Montanaro's result obtained in [33] is even surprising.

**AS**$_{13,2}$ More specifically, I want to emphasize that, while Br&Mont is a work of mechanics (or mathematical physics) *à la Mach-Painlevé* – see footnote 1 on AS$_2$ – Montanaro's "surprising results" were not even known in their versions belonging to the usual mechanics (not *à la Mach-Painlevé*).

## PART F: SOME TECHNICAL PRELIMINARIES. HOW ASSERTIONS CAN BE REFERRED TO AND HOW CONTENTS ARE USED IN THE INTRODUCTION OR IN CONTRIBUTING WORKS. ON SYMBOLS USED, *e.g.*, IN SPEAKING IN ENGLISH ABOUT WORKS WRITTEN IN ITALIAN. ON ABBREVIATIONS

**AS**$_{14}$ Here, in Ax Found, we use the notations of GIMC – see AS$_{1,5}$ and/or [4] after Part F, and also ftn. 1($a$) placed on AS$_2$ – added

(a) with set notations very customary in (extensional) mathematics but possibly including modal logic, and with

(b) division dots (following Zanardo 1981 or better Zanardo 2004).

(c) Following above Zanardo's papers here we (often) weaken the admissibility conditions used in GIMC on every universe for the extensional $\upsilon$-sorted language $EL^{\upsilon}$ (*i.e.* the extensional part of the modal language $ML^{\upsilon}$ constructed in $N3$ of GIMC) ([10]).

([10]) ($a$) The extensional semantics is based on $\upsilon$ sets $D_1$ to $D_{\upsilon}$ to be called *individual domains* – see in GIMC from p. 18, line 15 to formula (9) p. 21 –. On p. 18, line -10 $D_i$ is assumed to have at least two objects, one of which, $a_i^{\upsilon}$, represents the non-existing object ($i = 1, \dots, \upsilon$). Furthermore, $D_{\upsilon}$ is identified with the class $\Gamma$ of *elementary possible cases* for $ML^{\upsilon-1}$ – see p. 18, line -2 and $N5$ p. 16, line 4 to line 8.

(d) The semantics for $ML^{\upsilon}$ is based on the extensional semantics for $EL^{\upsilon+1}$, on the extensional correspondent $t \vdash t^{\eta}$, $(t \in \bar{\tau})$, and on formula (9) on p. 21 ([11]). In connection with the *new* weakest admissibility conditions for $EL^{\upsilon+1}$, this semantics turns out to be extensional (while in GIMC, line -1, p. 18, it ought to be essentially modal); I now regard as advantageous the possibility of treating modal and extensional semantics uniformly following Zanardo 1981 and Zanardo 2004 (*i.e.*, [63] and [66]) ([12]).

**AS**$_{15}$ In the present introduction we have used a unique series **AS**$_1$, **AS**$_2$,..., to mark some relevant parts of it. *E.g.*, AS$_r$ has exactly one entry written in bold character; in the part marked by it not yet known explications are introduced. For obvious motives AS$_r$ has some entries both after and before it. The latter show readers where they can find a help to understand yet unclear writings. The same holds for the possible parts AS$_{r,s}$, $(s = 1, 2, \dots)$ of every element **AS**$_r$ of above series.

**AS**$_{15,1}$ In the initial pages of this introduction its contents is included. The analogue can be done in any contributing work. (Each of these has a series of, *e.g.*, **AS**$_r$ starting with $r = 1$).

**AS**$_{15,2}$ Any contributing work is divided in sections marked by (§r) (briefly sections (§r)), and (§r) is divided in numbers (1), (2),..., and each number (s) in *items* (i), (ii),... *E.g.* the sub-item ($ii_3$) (if it exists) can be used instead of (ii) for greater precision.

**AS**$_{15,3}$ *E.g.* in (§r) "see (s)(ii)" or in (§r)(s) "see (ii)" means: see (§r)(s)(ii).

**AS**$_{15,4}$ For more clarity, one can be superabundant *both* in placing (above) marks for sections, numbers, and items (in bold character) *and* in referring to them (in normal character).

**AS**$_{16}$ *E.g.*, by 'Bressan 19RS' I abbreviate the unique Bressan's paper appeared in the year 19RS (provided such paper exists). If in that year a number $\upsilon > 0$ of Bressan's papers appeared, then (being $\upsilon < +\infty$) I label them with the indexes $a_1$ to $a_\upsilon$ and I abbreviate them by 'Bressan$_{a_i}$ 19RS' $(i = 1, \dots, \upsilon)$.

([11]) ($\alpha$) The class $QI$ of $QIs$ (quasi intensions) of type $t \in \bar{\tau}^{\upsilon}$ – see N2 at p. 40 of GIMC – is determined by ($a_1$) $QI_t\upsilon = Q^{\upsilon+1}_{t^{\eta}}$ – see formula (9) on p. 21 – where $t \vdash t^{\eta}$, $(t \in \bar{\tau})$ is defined by the recursive definition

$$\mathcal{RD} \qquad O^{\eta} = (\eta + 1), \qquad r^{\eta} = (\eta + 1, r) \qquad (r = 1, \dots, \upsilon);$$
$$(t_1, \dots, t_n)^{\eta} = (t_1^{\eta}, \dots, t_n^{\eta}, \upsilon + 1), \quad (t_1, \dots, t_n : t_0)^{\eta} = (t_1^{\eta}, \dots, t_n^{\eta} : t_0^{\eta}).$$

($\beta$) Here we mention two corrections to be done in GIMC:
($1^{st}$) in (b) on p. 19: $\xi_1 \in O^{\upsilon}_{t_n} \to \xi_1 \in O^{\upsilon}_{t_1}, \dots, \xi_n \in O^{\upsilon}_{t_n}$.
($2^{nd}$) definition (8) on p. 21 must (obviously) be changed into $\mathcal{RD}$.

([12]) See p. 47 of Zanardo 1981, from line 7 to line 10. They are improved, within Zanardo 2004, by Definition 1.1, Remark (e) and Definition 1.2 in part (A). These definitions allow us to embody extensional logic into the logical calculus $MC^{\upsilon}$ after having weakened, following Zanardo, the requirement $\mathbf{card}\Gamma \geq 2$ used in GIMC into $\mathbf{card}\Gamma \geq 1$.

**AS**$_{16,1}$ If the year 19RS is not earlier than 2000, then in the cases $\upsilon = 1$ and $\upsilon > 1$ I regard Bressan RS = Bressan 19RS and Bressan$_{a_i}$ RS = Bressan$_{a_i}$ 19RS $(i = 1, \ldots, \upsilon)$ respectively.

**AS**$_{16,2}$ The above rules, when $\upsilon = 1$, include [5] and more generally [1, 2] and **AS**$_{1,7}$ too; and when $\upsilon > 1$ they include [9] and [10].

**AS**$_{16,3}$ Note that [9] and [10] are the only cases with labels whose abbreviations are explicitly mentioned in our references.

**AS**$_{17}$ The series of works called Ax Found includes the present Introduction, which is classified by §1 – see below the preceding Contents.

**AS**$_{17,1}$ Usually a *paragraph* of a given page is meant as a certain set of lines.

**AS**$_{17,2}$ *I state* that in every contributing work (to Ax Found), the *bold* entry of any paragraph of the form $\S_{t_1}$ – see AS$_{15}$ – *must be printed at the line where this paragraph starts*. Furthermore, ($a$) following a University of Torino (Turin), the not bold AS$_r$ [AS$_{r,s}$] can (everywhere) be used to mention the bold **AS**$_r$ [**AS**$_{r,s}$] – cf. AS$_{15}$. Hence, in order to mention a particular occurrence of AS$_r$ [AS$_{r,s}$], one has to specify it by using: “AS$_r$” [“AS$_{r,s}$”].

**AS**$_{18}$ The works interesting Ax Found, *e.g.*, contributing works, form a book being constructed – see AS$_{1,1}$ –. Furthermore

**AS**$_{18,1}$ generally they mainly have certain features hinted at in AS$_1$ to AS$_{1,2}$, and more clearly shown in AS$_2$ to AS$_3$, AS$_{4,3}$, AS$_{7,3}$ (especially in (g)), in AS$_{7,5}$, and lastly in (the remark) AS$_{7,6}$ dealing with my long article Met.

**AS**$_{18,2}$ ($\alpha$) In AS$_1$ one speaks, *e.g.*, of a *book being constructed,* with developments imperfectly known.

($\beta$) In the present paper the notations §1, §2, etc., are meant, I think, in a manner *wider* than the usual one, and hence *different* from this.

($\gamma$) By ($\alpha$) the analogues of the properties considered for the sequence **AS**$_1$,. . . , **AS**$_{1,8}$ in §1 hold for every similar sequence **AS**$_1$,. . ., **AS**$_{1,l}$,. . . , written in any paper later than §1, (*e.g.*, called §2.)

($\delta$) Assume now that §2 has been printed. Then

($\delta_1$) an arbitrary mark **AS**$_\lambda$ or **AS**$_{\lambda,\mu}$, written in §1 determines the pages of §2 where it has been printed. Furthermore

($\delta_2$) the following *advantages* appear: (1°) this mark gives information more precise than the one given by the pages where the mark is printed; (2°) it *may* be used, *e.g.*, in the paper §2 (even if in §2 this mark is not used) – cf. the point (3) above **AS**$_5$ in §1 –.

($\delta_3$) Now fix an arbitrary $s \in Z_{>0}$ and consider the analogue for paper §$s$, of what has been said about §1 (the paper §$s$ being supposed to exist).

($\delta_4$) Thus, briefly, a sequence of papers $\S_{(2)}, \S_{(3)}, \ldots$ similar to §2 can be considered.

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