Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Edoardo Vesentini

The Gleason-Kahane-Zelazko theorem and function algebras

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **16** (2005), n.2, p. 87–108.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2005_9_16_2_87_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2005.

Rend. Mat. Acc. Lincei
s. 9, v. 16:87-108 (2005)

Analisi funzionale. — *The Gleason-Kahane-Zelazko theorem and function algebras*. Nota (*) del Socio EDOARDO VESENTINI.

ABSTRACT. — A theorem due to A. Gleason, J.-P. Kahane and W. Zelazko characterizes continuous characters within the space of all continuous linear forms of a locally multiplicatively convex, sequentially complete algebra. The present paper applies these results to investigate linear isometries of Banach algebras (with particular attention to normal uniform algebras) and of some locally multiplicatively convex algebras. The locally multiplicatively convex algebra of all holomorphic functions on a domain, will be investigated at the end of the paper.

KEY WORDS: Linear isometry; Character; Banach algebra; Locally multiplicatively convex algebra.

RIASSUNTO. — *Il teorema di Gleason-Kahane-Zelazko e le algebre di funzioni.* Un teorema dovuto a A. Gleason, J.-P. Kahane e W. Zelazko determina i caratteri continui nello spazio di tutte le forme lineari continue di un'algebra moltiplicativamente localmente convessa e sequenzialmente completa. Nel presente lavoro si applicano questi risultati allo studio delle isometrie lineari di algebre di Banach (con particolare attenzione alle algebre normali uniformi) e di algebre localmente moltiplicativamente convesse. Si studia infine l'algebra localmente moltiplicativamente convessa delle funzioni olomorfe su un dominio.

A theorem first established by A. Gleason [2] and, in a more general setting, by J.-P. Kahane and W. Zelazko [6, 17] characterizes the continuous characters (1) among the continuous linear forms of a locally multiplicatively convex, unital and sequentially complete algebra. Given two such algebras $\mathcal{A}$ and $\mathcal{B}$ and a map $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ such that

$$A(\mathcal{A}^{-1}) \subset \mathcal{B}^{-1} \tag{1}$$

(where $\mathcal{A}^{-1}$ and $\mathcal{B}^{-1}$ denote the sets of all invertible elements of $\mathcal{A}$ and $\mathcal{B}$), the Gleason-Kahane-Zelazko theorem yields a representation of A as a weighted composition operator [15].

Let $\mathcal{A}$ and $\mathcal{B}$ be the unital commutative Banach algebras of all complex-valued, continuous functions on two compact Hausdorff spaces M and N, and let A be a linear isometry of $\mathcal{A}$ into $\mathcal{B}$. It was shown in [15] that, if A maps all continuous unitary functions in $\mathcal{A}$ (*i.e.*, the continuous functions whose values have modulus one at all points of M) to continuous unitary functions in $\mathcal{B}$, then (1) holds and therefore A is a weighted composition operator. According to [14] (see also [1] for a general view), if the linear isometry A is surjective, the hypothesis concerning the behaviour of A on the unitary functions on M and on N is satisfied, and the representation of A as a weighted composition operator yields the classical Banach-Stone theorem.

In the first part of the present paper, after discussing an application of the Gleason-Kahane-Zelazko theorem to the characters of a non-unital, locally multiplicatively convex,

(*) Pervenuta in forma definitiva all'Accademia il 28 ottobre 2004.
(1) Throughout this article, a character is a homomorphism of the algebra into the complex field.

sequentially complete algebra, we will first assume $\mathcal{A}$ and $\mathcal{B}$ to be the sup-norm Banach algebras of all continuous functions vanishing at infinity on two locally compact Hausdorff spaces M and N, and further explore the connection between A being a linear isometry and (1) (where now $\mathcal{A}^{-1}$ and $\mathcal{B}^{-1}$ stand now for the sets of all quasi-regular elements of $\mathcal{A}$ and $\mathcal{B}$). In the case in which the linear isometry A is surjective, that will yield a generalization of the Banach-Stone theorem to the case of algebras of continuous functions vanishing at infinity on locally compact Hausdorff spaces. Following a line of reasoning developed by W. Holsztyński in a different context, we will describe all linear isometries of all normal uniform algebras.

The final part of this paper concerns the multiplicatively convex algebra $\mathcal{H}(D)$ of holomorphic functions on a domain of $D \subset \mathbb{C}^n$, endowed with the topology of uniform convergence on compact sets of D. Extending some preliminary results established in [15], we will investigate the continuous characters of $\mathcal{H}(D)$ in the cases in which D is either a Runge domain in $\mathbb{C}^n$ or any domain in $\mathbb{C}$. These results yield a representation as a weighted composition operator of any $A \in \mathcal{L}(\mathcal{H}(D), \mathcal{H}(D))$ mapping $\mathcal{H}(D)^{-1}$ into itself. The case in which D is the open unit disc in $\mathbb{C}$ will be explored in greater detail.

1. The Gleason-Kahane-Zelazko theorem

Let $\mathcal{A}$ be a locally multiplicatively convex and sequentially complete algebra (2) and let λ be a continuous linear form on $\mathcal{A}$.

The following theorem was established in [15].

THEOREM 1. *Let the algebra $\mathcal{A}$ be unital. If* $\ker \lambda$ *contains no invertible element of $\mathcal{A}$, there is a continuous character χ of $\mathcal{A}$ such that*

$$\langle x, \lambda \rangle = \langle 1, \lambda \rangle \langle x, \chi \rangle$$

for all $x \in \mathcal{A}$.

Assume now that $\mathcal{A}$ is not (necessarily) unital and that the continuous linear form λ is such that the affine space of equation $\langle \bullet, \lambda \rangle = 1$ contains no quasi-regular element of $\mathcal{A}$, *i.e.* [11], $\mathcal{A}$ is such that $\langle x, \lambda \rangle \neq 1$ whenever there is $y \in \mathcal{A}$ for which

$$xy - x - y = 0, xy = yx. \tag{2}$$

Let $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ be the Banach algebra obtained by adjoining an identity, denoted by 1 or $1_{\mathcal{A}}$, to $\mathcal{A}$, equipped with the norm

$$||x + \zeta||_{\tilde{\mathcal{A}}} = ||x||_{\mathcal{A}} + |\zeta| \ (x \in \mathcal{A}, \ \zeta \in \mathbb{C}).$$

If λ is a continuous linear form on $\mathcal{A}$, $\tilde{\lambda}$ is the continuous linear form defined on $\tilde{\mathcal{A}}$ by

$$\langle z + \zeta 1, \tilde{\lambda} \rangle = \langle z, \lambda \rangle + \zeta$$

for all $z \in \mathcal{A}$ and all $\zeta \in \mathbb{C}$.

(2) Throughout this article, all algebras will be assumed to be associative.

Lemma 1. *The linear form $\tilde{\lambda}$ is different from zero on all invertible elements of $\tilde{\mathcal{A}}$ if, and only if, $\langle x, \lambda \rangle \neq 1$ for every quasi-regular element $x \in \mathcal{A}$.*

Proof. The equation (2) is equivalent to

$$(1 - x)(1 - y) = 1 = (1 - y)(1 - x). \tag{3}$$

1) If $u - \alpha 1 \in \tilde{\mathcal{A}}^{-1}$ for $u \in \mathcal{A}$, $\alpha \in \mathbb{C}$, *i.e.*, if there are $v \in \mathcal{A}$ and $\beta \in \mathbb{C}$ such that

$$(\alpha 1 - u)(\beta 1 - v) = 1 = (\beta 1 - v)(\alpha 1 - u),$$

then

$$uv - \beta u - \alpha v = 0, \quad uv = vu, \quad \alpha\beta = 1$$

showing that $\alpha \neq 0$ and that $\frac{1}{\alpha} u$ is quasi-regular. Hence

$$\left\langle \frac{1}{\alpha} u, \lambda \right\rangle \neq 1,$$

i.e.

$$\langle u - \alpha 1, \tilde{\lambda} \rangle = \langle u, \lambda \rangle - \alpha \neq 0.$$

Thus $\tilde{\lambda} \neq 0$ on every element of $\tilde{\mathcal{A}}^{-1}$.

2) Let $\tilde{\lambda}$ be different from zero on every point of $\tilde{\mathcal{A}}^{-1}$, and let λ be the restriction of $\tilde{\lambda}$ to $\mathcal{A}$:

$$\langle u, \lambda \rangle = \langle u + 1, \tilde{\lambda} \rangle, \quad (u \in \mathcal{A}).$$

If $x \in \mathcal{A}$ is quasi-regular, *i.e.*, if (2) is satisfied by some $y \in \mathcal{A}$, then (3) holds. Therefore

$$\langle 1 - x, \tilde{\lambda} \rangle \neq 0,$$

and in conclusion

$$\langle x, \lambda \rangle = \langle x - 1, \tilde{\lambda} \rangle + 1 \neq 1. \qquad \square$$

If $x \in \mathcal{A}$ is quasi-regular, the element $y \in \mathcal{A}$ in (2) is called the quasi-inverse of x.

The Gleason-Kahane-Zelazko theorem (see [15, Theorem 1]) yields now the following theorem.

Theorem 2. *Let λ be a continuous linear form on any locally multiplicatively convex, sequentially complete algebra $\mathcal{A}$. Then, λ is a character of $\mathcal{A}$ if, and only if,*

$$\langle x, \lambda \rangle \neq 1 \tag{4}$$

for every quasi-regular element $x \in \mathcal{A}$.

Proof. First of all,

$$\langle 1, \tilde{\lambda} \rangle = 1 - \langle 0, \lambda \rangle = 1.$$

If (4) holds for every quasi regular element $x \in \mathcal{A}$, Lemma 1 and Theorem 1 of [15] imply that $\tilde{\lambda}$ is a character of $\tilde{\mathcal{A}}$. Thus, for $u, v \in \mathcal{A}$,

$$\begin{aligned}\langle (1-u)(1-v), \tilde{\lambda}\rangle =& \langle 1-u, \tilde{\lambda}\rangle\langle 1-v, \tilde{\lambda}\rangle \\ =&(1-\langle u, \lambda\rangle)(1-\langle v, \lambda\rangle) \\ =&1-\langle u, \lambda\rangle - \langle v, \lambda\rangle + \langle u, \lambda\rangle\,\langle v, \lambda\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle (1-u)(1-v), \tilde{\lambda}\rangle =& \langle 1-u-v+uv, \tilde{\lambda}\rangle \\ =&1-\langle u, \lambda\rangle - \langle v, \lambda\rangle + \langle uv, \lambda\rangle.\end{aligned}$$

Hence,

$$\langle uv, \lambda\rangle = \langle u, \lambda\rangle\,\langle v, \lambda\rangle \quad \forall\, u, v \in \mathcal{A}.$$

Vice versa, if λ is a character of $\mathcal{A}$ and (2) holds, then

$$\langle x, \lambda\rangle\,\langle y, \lambda\rangle - \langle x, \lambda\rangle - \langle y, \lambda\rangle = 0,$$

i.e.,

$$(\langle x, \lambda\rangle - 1)\,(\langle y, \lambda\rangle - 1) = 1.$$

Hence, $\langle x, \lambda\rangle \neq 1$. □

Theorem 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two locally multiplicatively convex, sequentially complete algebras, and let $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$.*

If A maps all quasi-regular elements of $\mathcal{A}$ to quasi-regular elements of $\mathcal{B}$, for any continuous character χ of $\mathcal{B}$ there is a continuous character $\phi(\chi)$ of $\mathcal{A}$ such that

$$\langle Ax, \chi\rangle = \langle x, \phi(\chi)\rangle \qquad \forall\, x \in \mathcal{A}. \tag{5}$$

If both $\mathcal{A}$ and $\mathcal{B}$ are unital and A maps all invertible elements of $\mathcal{A}$ to invertible elements of $\mathcal{B}$, for any continuous character χ of $\mathcal{B}$ there is a continuous character $\phi(\chi)$ of $\mathcal{A}$ such that

$$\langle Ax, \chi\rangle = \langle A1_{\mathcal{A}}, \chi\rangle\langle x, \phi(\chi)\rangle \qquad \forall\, x \in \mathcal{A}.$$

Proof. If both $\mathcal{A}$ and $\mathcal{B}$ are unital, and if A maps all invertible elements of $\mathcal{A}$ to invertible elements of $\mathcal{B}$, then, by Theorem 1, for any continuous character χ of $\mathcal{B}$ the map

$$\mathcal{A} \ni x \mapsto \frac{\langle Ax, \chi\rangle}{\langle A1_{\mathcal{A}}, \chi\rangle}$$

is a continuous character of $\mathcal{A}$.

As for the first part of the theorem, if χ is any continuous character of $\mathcal{B}$, by Theorem 2

$$\langle Au, \chi\rangle \neq 1$$

for all quasi-regular elements $u \in \mathcal{A}$. Thus, $\mathcal{A} \ni x \mapsto \langle Ax, \chi\rangle$, is a continuous character, say $\phi(\chi)$, of $\mathcal{A}$. □

Let $\mathcal{A}'$ and $\mathcal{B}'$ be the topological duals of $\mathcal{A}$ and $\mathcal{B}$, endowed with the topologies defined by $\mathcal{A}$ and $\mathcal{B}$, and let $\Sigma(\mathcal{A}) \subset \mathcal{A}'$ and $\Sigma(\mathcal{B}) \subset \mathcal{B}'$ be the sets of the continuous characters of $\mathcal{A}$ and $\mathcal{B}$, endowed with the relative topology.

Lemma 2. *The map $\phi : \Sigma(\mathcal{B}) \to \Sigma(\mathcal{A})$ in Theorem 3 is continuous.*

PROOF. Let $\chi, \chi_0 \in \Sigma(\mathcal{B})$. If χ tends to χ_0, for any $x \in \mathcal{A}$ $\langle Ax, \chi\rangle$ tends to $\langle Ax, \chi_0\rangle$. Therefore $\langle x, \phi(\chi)\rangle$ tends to $\langle x, \phi(\chi_0)\rangle$. □

COROLLARY 1. *If one of the following conditions holds for* $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$:

1) *A maps all quasi-regular elements of $\mathcal{A}$ to quasi-regular elements of $\mathcal{B}$;*

2) *both $\mathcal{A}$ and $\mathcal{B}$ are unital, $A\,1_{\mathcal{A}} = 1_{\mathcal{B}}$ and A maps all invertible elements of $\mathcal{A}$ to invertible elements of $\mathcal{B}$,*

then (5) *holds.*

2. Linear homomorphisms of Banach algebras

Let $\mathcal{A}$ be a unital Banach algebra. For $x \in \mathcal{A}$, $\sigma(x)$ or $\sigma_{\mathcal{A}}(x)$ and $\rho(x)$ or $\rho_{\mathcal{A}}(x)$ will indicate respectively the spectrum and the spectral radius of x. As before, let $\mathcal{A}^{-1}$ be the set of all invertible elements of $\mathcal{A}$.

We will denote by $\kappa(x)$ or $\kappa_{\mathcal{A}}(x)$, and call *inner spectral radius* of x, the non-negative real number

$$\kappa(x) = \inf\{|\zeta| : \zeta \in \sigma(x)\}.$$

Thus, $\kappa(x) = 0$ if x is not invertible, or (by the spectral mapping theorem)

$$\kappa(x) = \frac{1}{\rho(x^{-1})}$$

if $x \in \mathcal{A}^{-1}$.

Let $\mathcal{B}$ be another unital Banach algebra, and let $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$. Then (1) holds if, and only if,

$$\kappa_{\mathcal{A}}(x) > 0 \Longrightarrow \kappa_{\mathcal{B}}(Ax) > 0. \tag{6}$$

Assume from now on, in the present section, both unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$ to be commutative (in which case the spectral radii define continuous seminorms in $\mathcal{A}$ and $\mathcal{B}$). Denoting again by $\Sigma(\mathcal{A})$ and $\Sigma(\mathcal{B})$ the sets of all (continuous) characters of $\mathcal{A}$ and $\mathcal{B}$, endowed with the Gelfand topology, Theorem 3 can be rephrased as follows:

THEOREM 4. *If* (6) *holds, there is a continuous map* $\phi : \Sigma(\mathcal{B}) \longrightarrow \Sigma(\mathcal{A})$ *such that*

$$\langle Ax, \chi\rangle = \langle A\,1_{\mathcal{A}}, \chi\rangle\, \langle x, \phi(\chi)\rangle \tag{7}$$

for all $x \in \mathcal{A}$ and all $\chi \in \Sigma(\mathcal{B})$.

Thus, for any $x \in \mathcal{A}$,

$$\begin{aligned} \kappa_{\mathcal{B}}(Ax) &= \inf\{|\langle Ax, \chi\rangle| : \chi \in \Sigma(\mathcal{B})\} \\ &= \inf\{|\langle A1_{\mathcal{A}}, \chi\rangle\, \langle x, \phi(\chi)\rangle| : \chi \in \Sigma(\mathcal{B})\} \\ &\geq \inf\{|\langle A1_{\mathcal{A}}, \chi\rangle| : \chi \in \Sigma(\mathcal{B})\} \inf\{|\langle x, \phi(\chi)\rangle| : \chi \in \Sigma(\mathcal{B})\} \\ &\geq \inf\{|\langle A1_{\mathcal{A}}, \chi\rangle| : \chi \in \Sigma(\mathcal{B})\} \inf\{|\langle x, \chi\rangle| : \chi \in \Sigma(\mathcal{A})\} \\ &= \kappa_{\mathcal{B}}(A1_{\mathcal{A}})\, \kappa_{\mathcal{A}}(x). \end{aligned} \tag{8}$$

The following proposition is a direct consequence of the previous considerations.

PROPOSITION 1. *If* (6) *is satisfied, A is a continuous homomorphism of $\mathcal{A}$ into $\mathcal{B}$ if, and only if, $A\,1_{\mathcal{A}} = 1_{\mathcal{B}}$. The homomorphism A is surjective if, and only if, ϕ is a homeomorphism.*

If A is a continuous homomorphism of $\mathcal{A}$ into $\mathcal{B}$, condition (6) is satisfied, in which case (5) holds for all $x \in \mathcal{A}$ and all $\chi \in \Sigma(\mathcal{B})$.

LEMMA 3. *If A is a continuous homomorphism of $\mathcal{A}$ into $\mathcal{B}$ such that $\rho_{\mathcal{B}}(Ax) = \rho_{\mathcal{A}}(x)$ for all invertible elements $x \in \mathcal{A}$, then $\kappa_{\mathcal{B}}(Ax) = \kappa_{\mathcal{A}}(x)$ whenever $x \in \mathcal{A}^{-1}$.*

PROOF. If $x \in \mathcal{A}^{-1}$,

$$\kappa_{\mathcal{B}}(Ax) = \frac{1}{\rho_{\mathcal{B}}((Ax)^{-1})} = \frac{1}{\rho_{\mathcal{B}}(Ax^{-1})} = \frac{1}{\rho_{\mathcal{A}}(x^{-1})} = \kappa_{\mathcal{A}}(x).$$

□

Let $\mathcal{A}$ be, as before, a unital, commutative Banach algebra and let $A \in \mathcal{L}(\mathcal{A})$ be such that $A(\mathcal{A}^{-1}) \subset \mathcal{A}^{-1}$. Then A is represented, for all $x \in \mathcal{A}$ and all $\chi \in \Sigma(\mathcal{A})$, by (7), where ϕ is a continuous map of $\Sigma(\mathcal{A})$ into itself.

For all $x \in \mathcal{A}$

$$\begin{aligned}
\rho(Ax) &= \sup\{|\langle Ax, \chi\rangle| : \chi \in \Sigma(\mathcal{A})\} \\
&= \sup\{|\langle A1, \chi\rangle|\,|\langle x, \phi(\chi)\rangle| : \chi \in \Sigma(\mathcal{A})\} \\
&\leq \sup\{|\langle A1, \chi\rangle| : \chi \in \Sigma(\mathcal{A})\}\sup\{|\langle x, \phi(\chi)\rangle| : \chi \in \Sigma(\mathcal{A})\} \\
&\leq \sup\{|\langle A1, \chi\rangle| : \chi \in \Sigma(\mathcal{A})\}\sup\{|\langle x, \chi\rangle| : \chi \in \Sigma(\mathcal{A})\} \\
&= \rho(A1)\,\rho(x) \leq ||A1||\,||x||
\end{aligned} \tag{9}$$

for all $x \in \mathcal{A}$.

Since, for $n \geq 2$,

$$\begin{aligned}
\langle A^n x, \chi\rangle &= \langle A A^{n-1}x, \chi\rangle = \langle A1, \chi\rangle\ \langle A^{n-1}x, \phi(\chi)\rangle \\
&= \langle A1, \chi\rangle\ \langle A1, \chi\rangle\langle A^{n-2}x, \phi^2(\chi)\rangle = \langle A1, \chi\rangle^2\ \langle A^{n-2}x, \phi^2(\chi)\rangle \\
&= \ldots = \langle A1, \chi\rangle^n\ \langle x, \phi^n(\chi)\rangle,
\end{aligned}$$

then

$$\rho(A^n x) \leq \rho(A1)^n\,\rho(x)\ \ \forall\, x \in \mathcal{A}.$$

If $\zeta \in \partial\sigma(A)$, then $\zeta^n \in \partial\sigma(A^n)$ for all positive integers n. For any such n there is a sequence $\{x_\upsilon\}$ in $\mathcal{A}$ such that $||x_\upsilon|| = 1$ for all υ, and

$$\lim_{\upsilon\to\infty}(A^n x_\upsilon - \zeta^n x_\upsilon) = 0.$$

Since ρ is a continuous seminorm in the commutative Banach algebra $\mathcal{A}$, then

$$\lim_{\upsilon\to\infty} \rho(A^n x_\upsilon - \zeta^n x_\upsilon) = 0,$$

and therefore

$$\lim_{\upsilon\to\infty} |\rho(A^n x_\upsilon) - |\zeta|^n \rho(x_\upsilon)| = 0.$$

Thus, for any $\varepsilon > 0$ there is υ_0 (depending on n and on ε) such that

$$\rho(A^n x_\upsilon) > |\zeta|^n \rho(x_\upsilon) - \varepsilon \qquad \forall\, \upsilon \geq \upsilon_0;$$

hence, by (9),

$$\rho(A1)^n > |\zeta|^n \rho(x_\upsilon) - \varepsilon \qquad \forall\, \upsilon \geq \upsilon_0. \tag{10}$$

Let the image of $\mathcal{A}$ by the Gelfand transform be closed in the space $C(\Sigma(\mathcal{A}))$ of all continuous functions on $\Sigma(\mathcal{A})$ endowed with the uniform norm (which is equivalent to the existence of a constant $c > 0$ with $||x||^2 \leq c||x^2||$ for all $x \in \mathcal{A}$). Then there is a constant $k \in (0, 1]$ such that

$$k||x|| \leq \rho(x) \leq ||x|| \qquad \forall\, x \in \mathcal{A}. \tag{11}$$

Thus, by (10),

$$|\zeta|^n \left(k - \left(\frac{\rho(A1)}{|\zeta|} \right)^n \right) < \varepsilon \;\; \forall\, \upsilon \geq \upsilon_0,$$

which implies that $|\zeta| \leq \max(1, \rho(A1))$ and proves the following

THEOREM 5. *If $\mathcal{A}$ is a unital, commutative, Banach algebra whose image by the Gelfand transform is closed in the space $C(\Sigma(\mathcal{A}))$, and if $A \in \mathcal{L}(\mathcal{A})$ is such that $A(\mathcal{A}^{-1}) \subset \mathcal{A}^{-1}$, the spectrum $\sigma(A)$ of A is contained in the closed disc in $\mathbb{C}$ with center* 0 *and radius* $\max\{1, \rho(A1)\}$.

Let Δ be the open unit disc of $\mathbb{C}$.

COROLLARY 2. *If A is a homomorphism* (3) $\mathcal{A} \to \mathcal{A}$, *then* $\sigma(A) \subset \overline{\Delta}$.
If A is an automorphism, then $\sigma(A) \subset \partial\Delta$.

REMARKS. *a)* When A is an automorphism, the proof of Theorem 5 simplifies. Indeed, if A is an automorphism, ϕ is a homeomorphism of $\Sigma(\mathcal{A})$ onto itself, and therefore (5) yields

$$\begin{aligned}\rho(Ax) &= \sup\{|\langle x, \phi(\chi)\rangle| : \chi \in \Sigma(\mathcal{A})\} \\ &= \sup\{|\langle x, \chi\rangle| : \chi \in \Sigma(\mathcal{A})\} = \rho(x)\end{aligned}$$

for all $x \in \mathcal{A}$.

(3) Since $\mathcal{A}$ is semisimple, A is continuous, see, *e.g.*, [11].

Thus, with the same notations as above, if $\zeta \in \sigma(A)$, the equation

$$\lim_{\upsilon\to\infty} |\rho(Ax_\upsilon) - |\zeta|\rho(x_\upsilon)| = 0$$

yields

$$|1 - |\zeta|\,| \lim_{\upsilon\to\infty} \rho(x_\upsilon) = 0,$$

and therefore, by (11), $|1 - |\zeta|\,| = 0$.

b) If A is a homomorphism, Corollary 2 can be established without appealing to the Gleason-Kahane-Zelazko theorem. Here is a proof.

LEMMA 4. *Let $\mathcal{A}$ be a complex Banach space, and let $A \in \mathcal{L}(\mathcal{A})$. If there is a sequence $\{x_\upsilon\}$ in $\mathcal{A}$ such that $||x_\upsilon|| = 1$ for all υ and*

$$\lim_{\upsilon\to+\infty} (Ax_\upsilon - \zeta x_\upsilon) = 0, \tag{12}$$

then $\zeta \in \sigma(A)$.

PROOF. If $\zeta \notin \sigma(A)$, then

$$x_\upsilon = (A - \zeta I)^{-1}(A - \zeta I)x_\upsilon = (A - \zeta I)^{-1}(Ax_\upsilon - \zeta x_\upsilon) \to 0$$

as $\upsilon \to +\infty$. □

Let $\mathcal{A}$ satisfy the hypotheses of Theorem 5 and let $A : \mathcal{A} \to \mathcal{A}$ be a homomorphism. If $||x|| = 1$,

$$||x^2|| \geq \rho\left(x^2\right) = \rho(x)^2 \geq k^2||x||^2 = k^2.$$

Let $\zeta \in \partial\sigma(A)$, and let $\{x_\upsilon\}$ be a sequence in $\mathcal{A}$ with $||x_\upsilon|| = 1$ for all υ, and satisfying (12).

Because $\mathcal{A}$ is commutative and A is a homomorphism, then

$$Ax_\upsilon{}^2 - \zeta^2 x_\upsilon{}^2 = (Ax_\upsilon)^2 - \zeta^2 x_\upsilon{}^2 = (Ax_\upsilon - \zeta x_\upsilon)\,(Ax_\upsilon + \zeta x_\upsilon),$$

and therefore

$$||Ax_\upsilon{}^2 - \zeta^2 x_\upsilon{}^2|| \leq ||Ax_\upsilon - \zeta x_\upsilon||\,||Ax_\upsilon + \zeta x_\upsilon||.$$

Since,

$$||Ax_\upsilon + \zeta x_\upsilon|| \leq ||Ax_\upsilon|| + |\zeta|\,||x_\upsilon|| \leq ||A|| + |\zeta|,$$

then, by (12),

$$Ax_\upsilon{}^2 - \zeta^2 x_\upsilon \to 0$$

as $\upsilon \to +\infty$. Thus, setting $y_\upsilon = (1/||x_\upsilon{}^2||)x_\upsilon{}^2$,

$$||Ay_\upsilon - \zeta^2 y_\upsilon|| = \frac{1}{||x_\upsilon{}^2||}||Ax_\upsilon{}^2 - \zeta^2 x_\upsilon{}^2||$$

$$\leq \frac{1}{k^2}||Ax_\upsilon{}^2 - \zeta^2 x_\upsilon{}^2|| \to 0$$

as $\upsilon \to +\infty$.

Hence, by Lemma 4, $\zeta^2 \in \sigma(A)$. Iteration of this argument yields

LEMMA 5. *If $\zeta \in \partial\sigma(A)$, for any $n = 1, 2, \ldots$ there exists a sequence $\{u_\upsilon\}$ in $\mathcal{A}$ with $|u_\upsilon| = 1$ for all υ, and*

$$\lim_{\upsilon \to +\infty} \left(A u_\upsilon - \zeta^{2^n} u_\upsilon \right) = 0.$$

COROLLARY 3. *If $\zeta \in \partial\sigma(A)$, then $\zeta^{2^n} \in \sigma(A)$ for $n = 1, 2, \ldots$.*

In [7] (see also [13]), H. Kamowitz and S. Scheinberg have investigated the case of an automorphism A of a commutative, semisimple Banach algebra $\mathcal{A}$, and have shown that either $A^p = I$ for some integer p, in which case $\sigma(A)$ consists of a finite union of finite subgroups of $\partial\Delta$, or else $\partial\Delta \subset \sigma(A)$. As a consequence of this result, Corollary 2 yields

COROLLARY 4. *Let $\mathcal{A}$ satisfy the hypotheses of Theorem 5 and let $A : \mathcal{A} \to \mathcal{A}$ be an automorphism. If, and only if, $A^p \neq I$ for all integers $p \neq 0$, then $\sigma(A) = \partial\Delta$.*

If $A^p = I$ for some integer $p \neq 0$, $\sigma(A)$ is described in [7].

3. LINEAR ISOMETRIES OF BANACH ALGEBRAS

If the unital Banach algebra $\mathcal{A}$ is a normal uniform algebra on a compact Hausdorff space M, then [3, Theorem, p. 190] $M = \Sigma(\mathcal{A}) = \partial\mathcal{A}$, the Shilov boundary of $\mathcal{A}$. If $\lambda \in \mathcal{A}'$ is different from zero on any invertible element of $\mathcal{A}$, there is a (unique) point $t \in M$ such that

$$\langle x, \lambda \rangle = \langle 1, \lambda \rangle\, x(t) \qquad \forall\, t \in M.$$

Thus, the following theorem holds:

THEOREM 6. *If $\mathcal{A}$ and $\mathcal{B}$ are normal uniform algebras on two compact Hausdorff spaces M and N, and if $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ satisfies (6), there is a continuous map $\phi : N \to M$ such that*

$$(Ax)(s) = (A1_{\mathcal{A}})(s)\, x(\phi(s)) \;\; \forall\, x \in \mathcal{A},\, s \in N. \tag{13}$$

Let now $\mathcal{A} = C(M)$ and $\mathcal{B} = C(N)$ be the uniform algebras of all complex-valued continuous functions on two compact Hausdorff spaces M and N. As in [15], let

$$\Theta(\mathcal{A}) = \{u \in \mathcal{A} : |u(s)| = 1 \;\; \forall s \in M\},$$

$$\Theta(\mathcal{B}) = \{v \in \mathcal{B} : |v(t)| = 1 \;\; \forall t \in N\}.$$

If $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ is an isometry such that

$$\kappa_{\mathcal{B}}(Ax) = \kappa_{\mathcal{A}}(x) \;\; \forall\, x \in \mathcal{A}^{-1}, \tag{14}$$

then, for $x \in \Theta(\mathcal{A})$,

$$1 = ||x|| = ||Ax|| = \sup\{|(Ax)(s)| : s \in N\},$$

and also

$$1 = \kappa_{\mathcal{A}}(x) = \kappa_{\mathcal{B}}(Ax) = \inf\{|(Ax)(s)| : s \in N\}.$$

Hence $|(Ax)(s)| = 1$ for all $s \in N$, *i.e.*, $Ax \in \Theta(\mathcal{B})$, proving thereby the first part of the following lemma.

LEMMA 6. *If* $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ *is an isometry, then*

$$A(\Theta(\mathcal{A})) \subset \Theta(\mathcal{B}) \tag{15}$$

if, and only if, (14) *holds.*

PROOF. It was shown in [15] that (15) implies (1), *i.e.*,

$$\kappa_{\mathcal{A}}(x) > 0 \Longrightarrow \kappa_{\mathcal{B}}(Ax) > 0.$$

Since, by (15), $\kappa(A\,1_{\mathcal{A}}) = 1$, (8) yields

$$\kappa_{\mathcal{B}}(Ax) \geq \kappa_{\mathcal{A}}(x) \quad \forall\, x \in C(M). \tag{16}$$

On the other hand, if $x \in \mathcal{A}^{-1}$, then $x^{-1}(t) = 1/x(t)$ for all $t \in M$, and therefore

$$||x^{-1}|| = ||A(x^{-1})|| = \sup\left\{\frac{1}{|x(\phi(s))|} : s \in N\right\}$$

$$\leq \sup\left\{\frac{1}{|x(t)|} : t \in M\right\} = ||x^{-1}||.$$

Hence

$$\sup\left\{\frac{1}{|x(\phi(s))|} : s \in N\right\} = \sup\left\{\frac{1}{|x(t)|} : t \in M\right\},$$

that is,

$$\inf\{|x(\phi(s))| : s \in N\} = \inf\{|x(t)| : t \in M\},$$

showing that (16) implies (14). □

Theorem 4 yields then the Banach-Stone theorem, [14, 1].

THEOREM 7. i) *If the isometry* $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ *satisfies* (15)*, there is a continuous surjective map* $\phi : N \to M$ *such that*

$$(Ax)(s) = (A1_{\mathcal{A}})(s)\, x(\phi(s)) \tag{17}$$

for all $x \in \mathcal{A}$ *and all* $s \in N$.

ii) *If an isometry* A *is expressed by* (17)*, and if* $A1_{\mathcal{A}} \in \Theta(\mathcal{B})$*, then* (15) *holds.*

PROOF. *i*) The existence of a continuous map $\phi : N \to M$ satisfying (17) for all $x \in \mathcal{A}$ and all $s \in N$ is a consequence of Theorem 4. Since ϕ is continuous, $\phi(N)$ is compact in M. Thus, A being an isometry implies that $\phi(N) = M$.

The proof of *ii*) is trivial. □

REMARK. Since ϕ is surjective, (17) implies that (14) holds for all $x \in \mathcal{A}$.

As was noted, *e.g.*, in [14], if the linear isometry A is surjective, (17) holds. If that is the case, ϕ is a homeomorphism of N onto M which defines a topological isomorphism of $\mathcal{A}$ onto $\mathcal{B}$, as was remarked by M. Nagasawa in [10].

If the linear isometry $A : C(M) \to C(N)$ is not surjective, $\sigma(A) = \overline{\Delta}$ and Δ is contained in the residual spectrum of A (see, *e.g.*, [16, Corollary 1.12.10 and Lemma 1.12.11]). If the isometry A is surjective, then $\sigma(A) \subset \partial\Delta$. According to Corollary 4, if the isometry A is an isomorphism, either $A^p = I$ for some integer $p \neq 0$, and $\sigma(A)$ is a finite set in $\partial\Delta$, or $A^p \neq I$ for all integers $p \neq 0$, and $\sigma(A) = \partial\Delta$.

PROPOSITION 2. *Let M be a compact metric space. A surjective map $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ such that $A(\mathcal{A}^{-1}) \subset \mathcal{B}^{-1}$ is an isometry if, and only if,*

$$\sigma(A1_{\mathcal{A}}) = (A1_{\mathcal{A}})(N) \subset \partial\Delta. \tag{18}$$

PROOF. If this latter condition holds, then

$$|(Ax)(s)| = |x(\phi(s))| \ \forall\, s \in N,$$

and therefore

$$||Ax|| = \max\{|x(\phi(s))| : s \in N\} = \max\{|x(t)| : t \in M\} = ||x||$$

because ϕ is surjective.

Vice versa, if A is an isometry, then $\sigma(A1_{\mathcal{A}}) \subset \overline{\Delta}$. If $\sigma(A1) \not\subset \partial\Delta$, there is $s_0 \in N$ such that $|(A1_{\mathcal{A}})(s_0)| < 1$.

Let $t_0 = \phi(s_0)$ and let $x \in C(M)$ be such that

$$|x(t)| < |x(t_0)| \ \ \forall\, t \in M \backslash \{t_0\}.$$

Since

$$|(Ax)(s_0)| = |(A1_{\mathcal{A}})(s_0)|\,|x(\phi(s_0))| < |x(\phi(s_0))| = ||x||,$$

and (if $(A1_{\mathcal{A}})(s) \neq 0$)

$$\begin{aligned} |(Ax)(s)| =& |(A1_{\mathcal{A}})(s)|\,|x(\phi(s))| < |(A1_{\mathcal{A}})(s)|\,|x(t_0)| \\ \leq& |x(t_0)| = ||x||, \end{aligned}$$

then $||Ax|| < ||x||$. □

We will now extend the Banach-Stone theorem to the algebras of all continuous functions vanishing at infinity on locally compact Hausdorff spaces.

Let M be a locally compact Hausdorff space, and let $\mathcal{A} = C_o(M)$ be the function algebra of all continuous functions vanishing at infinity on M. We will denote by $\Xi(\mathcal{A})$ the set all quasi-regular elements in $C_o(M)$ whose quasi-inverses are their conjugates:

$$\Xi(\mathcal{A}) = \{x \in C_o(M) : |x(t)|^2 - 2\Re x(t) = 0 \ \ \forall\, t \in M\},$$

i.e.,

$$\Xi(\mathcal{A}) = \{x \in C_o(M) : |x(t) - 1| = 1 \ \ \forall\, t \in M\}.$$

For $x \in \mathcal{A}$, let $\tilde{x} \in \tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ be defined by

$$\tilde{x}(t) = x(t) - 1.$$

Then

$$x \in \Xi(\mathcal{A}) \Longleftrightarrow \tilde{x} \in \Theta(\tilde{\mathcal{A}}). \tag{19}$$

Let N be a locally compact Hausdorff space, and let $\mathcal{B} = C_o(N)$ be the function algebra of all continuous functions vanishing at infinity on N. For $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$, let $\tilde{A} \in \mathcal{L}(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ be defined on $x + \zeta \in \mathcal{A} \oplus \mathbb{C}$ by

$$\tilde{A}(x + \zeta) = Ax + \zeta.$$

If, and only if, A is an isometry, $\tilde{A}$ is an isometry. Hence, (19) and Theorem 7 yield

Theorem 8. *If A is a linear isometry of $\mathcal{A} = C_o(M)$ into $\mathcal{B} = C_o(N)$ such that*

$$A(\Xi(\mathcal{A})) \subset \Xi(\mathcal{B}), \tag{20}$$

then A is a continuous homomorphism of the algebra $\mathcal{A}$ into the algebra $\mathcal{B}$ which is expressed on any $x \in \mathcal{A}$ by

$$Ax = x \circ \phi,$$

where ϕ is a continuous map of N onto M.

By (19) and Lemma 1 in [14], if A is surjective, (20) holds, implying the following

Corollary 5. *If the linear isometry A in Theorem 8 is surjective, then ϕ is a homeomorphism of N onto M, and A is a continuous isomorphism.*

In the case in which M and N are compact, the action of *any* linear isometry of $C(M)$ into $C(N)$ was described by W. Holsztyński in [4]. Holsztyński's results will now be extended to linear isometries of normal uniform algebras into unital commutative Banach algebras.

Let $\mathcal{A}$ be a unital, commutative Banach algebra. As before let $\Sigma(\mathcal{A})$ be the space of all characters of $\mathcal{A}$ endowed with the Gelfand topology, and let $\partial\mathcal{A} \subset \Sigma(\mathcal{A})$ be its Shilov boundary. For $\chi \in \partial\mathcal{A}$, let

$$\Omega(\mathcal{A}, \chi) = \{x \in \mathcal{A} : |\langle x, \chi\rangle| = ||x|| = 1\}.$$

If $\mathcal{B}$ is a unital, commutative Banach algebra, and $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ is an isometry, for $\chi \in \partial\mathcal{A}$ let

$$\Upsilon(\mathcal{B}, \chi) = \{\lambda \in \partial\mathcal{B} : |\langle Ax, \lambda\rangle| = ||x|| = 1 \ \forall \ x \in \Omega(\mathcal{A}, \chi)\}.$$

Lemma 7. *For any $\chi_0 \in \partial\mathcal{A}$, $\Upsilon(\mathcal{B}, \chi_0) \neq \emptyset$.*

Proof. [4] For $n \geq 1$ and $x_1, \ldots, x_n \in \Omega(\mathcal{A}, \chi_0)$, let

$$x = \sum_{j=1}^{n} \overline{\langle x_j, \chi_0\rangle} x_j.$$

For $\chi \in \partial\mathcal{A}$,

$$\langle x, \chi\rangle = \sum_{j=1}^{n} \overline{\langle x_j, \chi_0\rangle}\, \langle x_j, \chi\rangle.$$

Since

$$|\langle x_j, \chi_0\rangle| = 1 \qquad \text{for } j = 1, \dots, n,$$

then

$$|\langle x, \chi\rangle| \le \sum_{j=1}^{n} |\langle x_j, \chi\rangle| \le \sum_{j=1}^{n} ||x_j|| = n,$$

$$|\langle x, \chi_0\rangle| = \sum_{j=1}^{n} |\langle x_j, \chi_0\rangle|^2 = n,$$

and therefore $||x|| = n$.

There is $\lambda_0 \in \partial\mathcal{B}$ for which

$$|\langle Ax, \lambda_0\rangle| = ||Ax|| = n.$$

Hence,

$$n = \left|\sum_{j=1}^{n} \overline{\langle x_j, \chi_0\rangle}\, \langle Ax_j, \lambda_0\rangle\right| \le \sum_{j=1}^{n} |\langle Ax_j, \lambda_0\rangle|$$

$$\le \sum_{j=1}^{n} ||Ax_j|| = \sum_{j=1}^{n} ||x_j|| = n,$$

and therefore

$$\left|\langle Ax_j, \lambda_0\rangle\right| = 1 \qquad \text{for } j = 1, \dots, n.$$

Thus, for any $n \ge 1$ and any choice of $x_1, \dots, x_n \in \Omega(\mathcal{A}, \chi_0)$, the set

$$\{\lambda \in \partial\mathcal{B} : \left|\langle Ax_j, \lambda\rangle\right| = 1 \quad \text{for } j = 1, \dots n\}$$

is not empty. Since $\partial\mathcal{B}$ is compact, then

$$\{\lambda \in \partial\mathcal{B} : |\langle Ax, \lambda\rangle| = 1 \ \ \forall\, x \in \Omega(\mathcal{A}, \chi_0)\} \neq \emptyset. \qquad \square$$

We assume now $\mathcal{A}$ to be a normal function algebra on a compact, Hausdorff space M. Then, [3, Theorem, p. 190], $M = \Sigma(\mathcal{A}) = \partial\mathcal{A}$.

Lemma 8. *If $x \in \mathcal{A}\backslash\{0\}$ and if $\langle x, \chi\rangle = 0$ for every χ in an open neighbourhood U of $\chi_0 \in \partial\mathcal{A}$ in $\partial\mathcal{A}$, then $\langle Ax, \lambda\rangle = 0$ for every $\lambda \in \Upsilon(\mathcal{B}, \chi_0)$.*

Proof. Since $x \neq 0$, there is $\chi \in \partial\mathcal{A}$ for which $\langle x, \chi\rangle \neq 0$. As a consequence, $\overline{U} \neq \partial\mathcal{A}$, and there is $\tau \in V := \partial\mathcal{A}\backslash\overline{U}$ such that $\langle x, \tau\rangle = ||x||$.

Normalize x such that $||x|| = 1$, and let $z \in \mathcal{A}\backslash\{0\}$ be such that $\operatorname{Supp} z \subset U$ and

$$\langle z, \chi_0\rangle = ||z|| \ge ||x|| = 1.$$

Then $(1/||z||)\,z \in \Omega(\mathcal{A}, \chi_0)$, and therefore

$$|\langle Az, \lambda\rangle| = ||z|| \;\forall\, \lambda \in \Upsilon(\mathcal{B}, \chi_0).$$

Let $\zeta \in \Delta$ and let

$$w = \zeta x + z \quad (\zeta \in \Delta).$$

If $\chi \in \overline{U}$,

$$\langle w, \chi\rangle = \zeta\langle x, \chi\rangle + \langle z, \chi\rangle = \langle z, \chi\rangle,$$

and, if $\tau \in V$,

$$\langle w, \tau\rangle = \zeta\langle x, \tau\rangle + \langle z, \tau\rangle = \zeta\langle x, \tau\rangle.$$

Hence,

$$||w|| = \max\{|\zeta|\,||x||, ||z||\} = ||z||$$

and

$$\langle w, \chi_0\rangle = \zeta\langle x, \chi_0\rangle + \langle z, \chi_0\rangle = ||z|| = ||w||.$$

As a consequence,

$$|\langle Aw, \lambda\rangle| = ||w|| = ||z|| = |\langle Az, \lambda\rangle|,$$

i.e.,

$$|\zeta\langle Ax, \lambda\rangle + \langle Az, \lambda\rangle| = |\langle Az, \lambda\rangle|$$

for all $\zeta \in \Delta$. Thus,

$$\langle Ax, \lambda\rangle = 0 \quad \forall\, \lambda \in \Upsilon(\mathcal{B}, \chi_0).$$ □

LEMMA 9. *If $\langle x, \chi_0\rangle = 0$ for some $\chi_0 \in \partial\mathcal{A}$, then $\langle Ax, \lambda_0\rangle = 0$ for all $\lambda_0 \in \Upsilon(\mathcal{B}, \chi_0)$.*

PROOF. Let $\varepsilon > 0$, and let U be the open neighbourhood of χ_0 in $\partial\mathcal{A}$ defined by

$$U = \{\chi \in \partial\mathcal{A} : |\langle x, \chi\rangle| < \varepsilon\}.$$

Let $u \in \mathcal{A}$. Denoting by $\hat{u}$ the Gelfand transform of u, let u be such that: $||u|| = 1$, $\operatorname{Supp}\hat{u} \subset U$ and such that $\hat{u} = 1$ in a neighbourhood V of χ_0 in $\partial\mathcal{A}$.

If $v = ux$,

$$\begin{aligned} ||v|| = ||ux|| &\le \sup\{|\langle ux, \chi\rangle| : \chi \in \partial\mathcal{A}\} = \sup\{|\langle ux, \chi\rangle| : \chi \in U\} \\ &\le \sup\{|\langle x, \chi\rangle| : \chi \in \partial\mathcal{A}\} < \varepsilon, \end{aligned}$$

and therefore

$$|\langle Av, \lambda_0\rangle| \le ||Av|| = ||v|| < \varepsilon.$$

Let $w = x - v = (1 - u)x$. Since, for any $\chi \in V$,

$$\langle w, \chi\rangle = \langle 1 - u, \chi\rangle\,\langle x, \chi\rangle = 0,$$

by Lemma 8, $\langle Aw, \lambda_0\rangle = 0$, *i.e.*,

$$\langle Ax, \lambda_0\rangle = \langle Av, \lambda_0\rangle.$$

Thus, $|\langle Ax, \lambda_0\rangle| < \varepsilon$ for all $\varepsilon > 0$, and, in conclusion, $\langle Ax, \lambda_0\rangle = 0$. □

For $x_1, x_2 \in \mathcal{A}$ and $\chi \in \partial\mathcal{A}$, set

$$x = \langle x_2, \chi\rangle x_1 - \langle x_1, \chi\rangle x_2.$$

Then, $\langle x, \chi\rangle = 0$, and therefore, for any $\lambda \in \Upsilon(\mathcal{B}, \chi)$, $\langle Ax, \lambda\rangle = 0$, *i.e.*,

$$\langle x_2, \chi\rangle \, \langle Ax_1, \lambda\rangle = \langle x_1, \chi\rangle \, \langle Ax_2, \lambda\rangle.$$

If $x_1 \in \Omega(\mathcal{A}, \chi)$, then

$$|\langle Ax_1, \lambda\rangle| = |\langle x_1, \chi\rangle| = 1,$$

and therefore

$$|\langle x_2, \chi\rangle| = |\langle x_2, \chi\rangle| \, |\langle Ax_1, \lambda\rangle| = |\langle Ax_2, \lambda\rangle|.$$

That proves

PROPOSITION 3. *If $\chi \in \partial\mathcal{A}$ and $\lambda \in \Upsilon(\mathcal{B}, \chi)$, then*

$$|\langle Ax, \lambda\rangle| = |\langle x, \chi\rangle|$$

for all $x \in \mathcal{A}$.

Let

$$\Upsilon(\mathcal{B}) = \bigcup \{\Upsilon(\mathcal{B}, \chi) : \chi \in \partial\mathcal{A}\}.$$

COROLLARY 6. *If $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ is an isometry, then*

$$\langle Ax, \lambda\rangle \neq 0 \tag{21}$$

for all $x \in \mathcal{A}^{-1}$ and all $\lambda \in \Upsilon(\mathcal{B})$,

PROOF. For $\lambda \in \Upsilon(\mathcal{B})$, let $\chi \in \partial\mathcal{A}$ be such that $\lambda \in \Upsilon(\mathcal{B}, \chi)$. Because $(1/||x||)x \in \Omega(\mathcal{A}, \chi)$, then (21) holds. □

By Theorem 1, for every $\lambda \in \Upsilon(\mathcal{B})$, the linear form

$$\phi(\lambda) : \mathcal{A} \ni x \mapsto \frac{\langle Ax, \lambda\rangle}{\langle A1, \lambda\rangle}$$

is a character of $\mathcal{A}$.

THEOREM 9. *Let $\mathcal{A}$ be a normal uniform algebra on a compact Hausdorff space, and let $\mathcal{B}$ be a unital, commutative Banach algebra. If A is a linear isometry of $\mathcal{A}$ into $\mathcal{B}$, there exist a closed set $\Upsilon(\mathcal{B}) \subset \partial\mathcal{B}$ and a continuous surjective map $\phi : \Upsilon(\mathcal{B}) \to \partial\mathcal{A}$ such that*

$$\langle Ax, \lambda\rangle = \langle A1, \lambda\rangle \, \langle x, \phi(\lambda)\rangle$$

for all $x \in \mathcal{A}$ and all $\lambda \in \Upsilon(\mathcal{B})$.

PROOF. In view of Corollary 6 and of Theorem 3 what is left to prove is the continuity of ϕ and the fact that $\Upsilon(\mathcal{B})$ is closed.

As for the continuity of ϕ, let $\lambda, \lambda_0 \in \Upsilon(\mathcal{B})$. If $\lambda \to \lambda_0$ for the weak-star topology, for every $x \in \mathcal{A}$, $\langle A1, \lambda \rangle$ tends to $\langle A1, \lambda_0 \rangle$ and $\langle Ax, \lambda \rangle$ tends to $\langle Ax, \lambda_0 \rangle$. Thus, $\langle x, \phi(\lambda) \rangle$ tends to $\langle x, \phi(\lambda_0) \rangle$.

If $\lambda_0 \in \overline{\Upsilon(\mathcal{B})}$, for every $x \in \mathcal{A}$ there is a sequence $\{\lambda_n,\ n = 1, 2, \ldots\}$, with $\lambda_n \in \Upsilon(\mathcal{B}, \chi_n)$, $\chi_n \in \partial\mathcal{A}$ such that

$$|\langle Ax, \lambda_0 \rangle - \langle Ax, \lambda_n \rangle| < \frac{1}{n},$$

i.e.,

$$|\langle Ax, \lambda_0 \rangle - \langle A1, \lambda_n \rangle \langle x, \phi(\lambda_n) \rangle| < \frac{1}{n},$$

and

$$|\langle A1, \lambda_0 \rangle - \langle A1, \lambda_n \rangle| |\langle x, \phi(\lambda_n) \rangle| < \frac{1}{n}$$

for $n = 1, 2, \ldots$.

Because $\partial\mathcal{A}$ is compact, there exists a subsequence of $\{\phi(\lambda_n)\}$ – which we will assume to be the entire sequence $\{\phi(\lambda_n)\}$ – converging to some character χ_0 of $\mathcal{A}$. For every $\varepsilon > 0$ there is a neighbourhood U of χ_0 in $\partial\mathcal{A}$ such that, if $\chi \in U$, then

$$|\langle A1, \lambda_0 \rangle| \, |\langle x, \chi_0 \rangle - \langle x, \chi \rangle| < \varepsilon.$$

If $n > 1/\varepsilon$, then

$$\begin{aligned} |\langle Ax, \lambda_0 \rangle - \langle A1, \lambda_0 \rangle \langle x, \chi_0 \rangle| \leq & |\langle Ax, \lambda_0 \rangle - \langle A1, \lambda_n \rangle \langle x, \phi(\lambda_n) \rangle| + \\ & |\langle A1, \lambda_n \rangle - \langle A1, \lambda_0 \rangle| \, |\langle x, \phi(\lambda_n) \rangle| + \\ & |\langle A1, \lambda_0 \rangle| \, |\langle x, \phi(\lambda_n) \rangle - \langle x, \chi_0 \rangle| \\ & < \frac{1}{n} + \frac{1}{n} + \varepsilon < 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then

$$\langle Ax, \lambda_0 \rangle = \langle A1, \lambda_0 \rangle \langle x, \chi_0 \rangle \tag{22}$$

for all $x \in \mathcal{A}$.

Because $1 \in \Omega(\mathcal{A}, \chi_n)$ and $\lambda_n \in \Upsilon(\mathcal{B}, \chi_n)$, then $\langle A1, \lambda_n \rangle \in \partial\Delta$ for all $n \geq 1$.

Since

$$\langle A1, \lambda_0 \rangle = \lim_{n \to +\infty} \langle A1, \lambda_n \rangle,$$

then $|\langle A1, \lambda_0 \rangle| = 1$, and (22) yields

$$|\langle Ax, \lambda_0 \rangle| = |\langle x, \chi_0 \rangle|$$

for all $x \in \mathcal{A}$. Hence, if $x \in \Omega(\mathcal{A}, \chi_0)$, then

$$|\langle Ax, \lambda_0 \rangle| = ||x|| \, ,$$

showing that

$$\lambda_0 \in \Upsilon(\mathcal{B}, \chi_0) \subset \Upsilon(\mathcal{B}).$$

□

REMARK. The condition $\Upsilon(\mathcal{B}) = \partial\mathcal{B}$ is equivalent to requiring that, for every $\lambda \in \partial\mathcal{B}$ there exists $\chi \in \partial\mathcal{A}$ such that, if $||x|| = |\langle x, \chi\rangle| = 1$, then $|\langle Ax, \lambda\rangle| = 1$.

4. HOLOMORPHIC FUNCTIONS

Let D be a domain in $\mathbb{C}^n$ ($n \geq 1$), and let $\mathcal{H}(D)$ be the locally multiplicatively convex, sequentially complete, unital algebra of all holomorphic functions on D, endowed with the topology of uniform convergence on all compact subsets of D (with respect to which it is a Fréchet space). The topological dual $\mathcal{H}(D)'$ of $\mathcal{H}(D)$ is the space of all linear analytic functionals on D.

A similar argument to the proof of Lemma 6 in [15] yields

LEMMA 10. *If D is a Runge domain in $\mathbb{C}^n$, any continuous character χ of $\mathcal{H}(D)$ is a point evaluation.*

PROOF. As a consequence of the Hahn-Banach theorem [8, Proposition 1.1, pp. 9-10], there is a compactly supported measure μ on D such that

$$\langle x, \chi\rangle = \int x\, d\mu \ \ \forall\, x \in \mathcal{H}(D).$$

Let $s^1, \dots, s^n$ be the cartesian coordinates of $s \in D \subset \mathbb{C}^n$, let $\imath_1, \dots, \imath_n \in \mathcal{H}(D)$ be the coordinate functions: $\imath_j(s) = s^j$, $j = 1, \dots, n$. Let

$$s_\chi = \langle \imath_1, \chi\rangle, \dots, \langle \imath_n, \chi\rangle \in \mathbb{C}^n.$$

Since D is a Runge domain, every $x \in \mathcal{H}(D)$ can be approximated by polynomials uniformly on every compact subset of D, [5]. Hence, there is a sequence $\{P_\upsilon\}$ of polynomials P_υ on $\mathbb{C}^n$ converging to x uniformly on the support of μ.

Since $\langle P_\upsilon, \chi\rangle = P_\upsilon(s_\chi)$, then

$$\begin{aligned} \langle x, \chi\rangle &= \lim \int P_\upsilon\, d\mu = \lim \langle P_\upsilon, \chi\rangle \\ &= \lim P_\upsilon(s_\chi) = x(s_\chi). \end{aligned} \tag{23}$$

□

Let now $n = 1$ and let R be a rational function on $\mathbb{C}$: $R = \dfrac{P}{Q}$, with P, Q polynomials and $Q \neq 0$ at all points of D. Since $P = QR$, with $R \in \mathcal{H}(D)$, then

$$\begin{aligned} P(s_\chi) &= \langle P, \chi\rangle = \langle Q, \chi\rangle\, \langle R, \chi\rangle \\ &= Q(s_\chi)\, \langle R, \chi\rangle, \end{aligned}$$

and therefore

$$\langle R, \chi\rangle = \frac{P(s_\chi)}{Q(s_\chi)} = R(s_\chi).$$

If $E \subset \mathbb{C}$ is a set which has one point in each connected component of $\overline{\mathbb{C}}\backslash D$, there is a sequence $\{R_\nu\}$ of rational functions, with poles only in E, converging to x uniformly on every compact subset of D. Since (23) holds with P_ν replaced by R_ν, the following proposition has been established.

PROPOSITION 4. *If $n > 1$ and D is a Runge domain in $\mathbb{C}^n$, or if $n = 1$ and D is a domain in $\mathbb{C}$, every analytic character of $\mathcal{H}(D)$ is a point evaluation.*

Theorem 1 yields then

THEOREM 10. *Let $n > 1$ and D be a Runge domain in $\mathbb{C}^n$, or let $n = 1$ and D be any domain in $\mathbb{C}$. If λ is a linear analytic functional on D such that* ker λ *contains no invertible element of $\mathcal{H}(D)$, there is a (unique) point $s \in D$ such that*

$$\langle x, \lambda\rangle = \langle 1, \lambda\rangle\, x(s)$$

for all $x \in \mathcal{H}(D)$.

Let D_1 be a domain in $\mathbb{C}^{n_1}$. In the following, both D and D_1 will be tacitly assumed to satisfy the conditions stated for D in Theorem 10.

Theorem 3 yields

THEOREM 11. *Any $A \in \mathcal{L}(\mathcal{H}(D), \mathcal{H}(D_1))$ such that*

$$A((\mathcal{H}(D))^{-1}) \subset (\mathcal{H}(D_1))^{-1} \tag{24}$$

is the weighted composition operator expressed by

$$(A\,x)(t) = (A\,1)(t)\,x(\phi(t)) \tag{25}$$

for all $x \in \mathcal{H}(D)$ and all $t \in D_1$, where the map $\phi : D_1 \to D$ is holomorphic.

COROLLARY 7. *The operator $A \in \mathcal{L}(\mathcal{H}(D), \mathcal{H}(D_1))$ is a homomorphism of the algebra $\mathcal{H}(D)$ into the algebra $\mathcal{H}(D_1)$ if, and only if, $A\,1 = 1$ and* (24) *holds. If these conditions are satisfied, there is a holomorphic map $\phi : D_1 \to D$ such that*

$$A\,x = x \circ \phi \tag{26}$$

for all $x \in \mathcal{H}(D)$.

Let $\{x_\nu,\ \nu = 1, 2, \ldots\}$ be a sequence in $\mathcal{H}(D)^{-1}$ uniformly convergent on all compact sets in D to $x \in \mathcal{H}(D)$. According to a theorem of Hurwitz, either $x \in \mathcal{H}(D)^{-1}$ or $x = 0$. That proves

LEMMA 11. *If A is injective, a sufficient condition for* (24) *to hold is the existence of a dense subset of $\mathcal{H}(D)^{-1}$ whose image by A is contained in $(\mathcal{H}(D_1))^{-1}$.*

If $D_1 = \mathbb{C}^{n_1}$ and D is bounded, by the Liouville theorem there is a point $s_0 \in D$ such that $\phi(D_1) = \{s_0\}$. Thus, (25) yields

$$x(s_0)Ay = y(s_0)Ax \ \ \forall\, x, y \in \mathcal{H}(D),$$

and the following proposition holds.

PROPOSITION 5. *If $D_1 = \mathbb{C}^{n_1}$, D is bounded and* (24) *holds, then the range of A consists of the scalar multiples of an entire function.*

Consider now the Banach algebras $H^\infty(D)$ and $H^\infty(D_1)$ of all bounded holomorphic functions on D and on D_1 endowed with the sup-norm. The following theorem is a direct consequence of (25).

THEOREM 12. *If* (24) *holds, then $A(H^\infty(D)) \subset H^\infty(D_1)$ if, and only if, $A\,1 \in H^\infty(D_1)$. If these conditions are satisfied, $A|H^\infty(D)$ is a continuous linear map of $H^\infty(D)$ into $H^\infty(D_1)$ whose norm does not exceed $||A\,1||_{H^\infty(D_1)}$.*

COROLLARY 8. *If A is a continuous homomorphism of $\mathcal{H}(D)$ into $\mathcal{H}(D_1)$, and if $A(H^\infty(D)) \subset H^\infty(D_1)$, the restriction of A to $H^\infty(D)$ is a continuous Banach algebra homomorphism of $H^\infty(D)$ into $H^\infty(D_1)$.*

The same argument as in the proof of Theorem 6 of [15] yields the following

THEOREM 13. *If $A \in \mathcal{L}(\mathcal{H}(D), \mathcal{H}(D_1))$ is bijective and satisfies* (24), *then $n = m$, ϕ is a biholomorphic homeomorphism of D_1 onto D, A is expressed by* (25), *and*

$$(A^{-1}y)(s) = \frac{1}{(A\,1)(\phi^{-1}(s))}\, y(\phi^{-1}(s))$$

for all $y \in \mathcal{H}(D_1)$ and all $s \in D$.

COROLLARY 9. *If $A \in \mathcal{L}(\mathcal{H}(D), \mathcal{H}(D_1))$ is an isomorphism of the algebra $\mathcal{H}(D)$ onto the algebra $\mathcal{H}(D_1)$, then $n = m$, ϕ is a biholomorphic homeomorphism of D_1 onto D such that A and A^{-1} are expressed by* (26) *and by*

$$(A^{-1}y)(s) = y(\phi^{-1}(s))$$

for all $y \in \mathcal{H}(D_1)$ and all $s \in D$.

COROLLARY 10. *If D is a Runge domain in $\mathbb{C}^n$ or a domain in $\mathbb{C}$, the group of all linear automorphisms of the algebra $\mathcal{H}(D)$ is isomorphic to the group of all holomorphic automorphisms of D.*

For any point r in D or in D_1, δ_r will be, as before, the point evaluation at r.
The following statement characterizes the case in which (24) holds.

Theorem 14. *If, and only if, for every $t \in D_1$ there exist $c \in \mathbb{C}\setminus\{0\}$ and $s \in D$ such that*

$$A'\delta_t = c\,\delta_s, \tag{27}$$

then (24) *holds.*

Proof. If (24) is satisfied, then

$$\begin{aligned}\langle x, A'\delta_t\rangle =&\langle A\,x, \delta_t\rangle = (A\,x)(t)\\ =&(A\,1)(t)\,x(\phi(t)) = \langle x, (A\,1)(t)\delta_{\phi(t)}\rangle\end{aligned}$$

for all $x \in \mathcal{H}(D)$, which implies (27).

Vice versa, if, given $t \in D_1$, (27) holds for some $s \in D$ and some $c \in \mathbb{C}\setminus\{0\}$, then, for any $x \in (\mathcal{H}(D))^{-1}$,

$$\begin{aligned}(A\,x)(t) =&\langle x, A'\delta_t\rangle = c\,\langle x, \delta_s\rangle\\ =&c\,x(s) \neq 0,\end{aligned}$$

showing that $A\,x \in (\mathcal{H}(D_1))^{-1}$. □

Let now $n = m = 1$, and, as before, let $A \in \mathcal{L}(\mathcal{H}(D), \mathcal{H}(D_1))$ satisfy (24).

Since ϕ is either constant or an open map, (25) yields

Lemma 12. *The holomorphic map $\phi : D_1 \to D$ is constant if, and only if, Ax is a scalar multiple of $A1$ for all $x \in \mathcal{H}(D)$.*

The linear map A is injective if, and only if, ϕ is not constant, if, and only if, $\dim_{\mathbb{C}} A(\mathcal{H}(D)) > 1$.

If $D_1 = \mathbb{C}$, the map ϕ of Theorem 11 is an entire function. By the Picard theorem, either $D \supset \mathbb{C}\setminus\{s_0\}$ for some $s_0 \in \mathbb{C}$, or $\phi(\mathbb{C}) = \{c\}$ with $c \in \mathbb{C}$.

Let $D = D_1 = \mathbb{C}$. It is easily seen, [15], that, if A is surjective, there are $a \in \mathbb{C}\setminus\{0\}$ and $b \in \mathbb{C}$ such that $\phi(t) = at + b$ for all $t \in \mathbb{C}$.

We will consider now the case in which $n = m = 1, D = D_1 = \Delta, A \in \mathcal{L}(\mathcal{H}(\Delta))$ is such that

$$A((\mathcal{H}(\Delta))^{-1}) \subset (\mathcal{H}(\Delta))^{-1}. \tag{28}$$

The linear map A is represented by (25), where $\phi : \Delta \to \Delta$ is holomorphic, and therefore is either constant or an open map.

Suppose that ϕ is not constant. There is measurable set $K \subset \partial\Delta$, with Lebesgue measure $m(K) = 2\pi$, such that, for every $k \in K$, the non-tangential limit

$$\lim_{t\to k} \phi(t) := \phi^*(k)$$

exists, and $\phi^*(k) \in \overline{\Delta}$.

If $\phi(\Delta) \neq D$, there exists a measurable set $L \subset \partial\Delta$ such that $l = m(L) > 0$ and $\phi^*(L) \subset D$. By the Lusin theorem, for every $\varepsilon > 0$ there exists a measurable set $L_\varepsilon \subset L$ with $m(L_\varepsilon) < \varepsilon$ and a continuous function $\omega : \partial\Delta \to \mathbb{C}$ satisfying the following conditions:

$$\sup\{|\omega(t)| : t \in K\} = \sup\{|\phi^*(t)| : t \in K\};$$

letting

$$L_\varepsilon := \{t \in K : \omega(t) \neq \phi^*(t)\},$$

then

$$m(L_\varepsilon) < \varepsilon.$$

Since $L_\varepsilon \subset L$, then $m(L_\varepsilon) < l$ and

$$m(L \backslash L_\varepsilon) > l - \varepsilon.$$

For every $\delta > 0$ there exist an open set V and a closed set F in $\partial\Delta$ such that $F \subset L\backslash L_\varepsilon \subset V$ and $m(V\backslash F) < \delta$. If F is a finite set, $m(V\backslash F) = m(V) > l - \varepsilon$. Hence, choosing $\delta > 0$ sufficiently small, F contains infinite points. Furthermore, being

$$m(L\backslash L_\varepsilon) - m(F) < m((L\backslash L_\varepsilon)\backslash F) \leq m(V\backslash F) < \delta,$$

if δ is sufficiently small, then $m(F) > 0$.

Because the holomorphic function ϕ is not constant, $\phi^*(F) = \omega(F)$ is a compact subset of Δ containing infinite points. Let Z be an infinite, countable subset of F whose image by ϕ^* is an infinite subset of the compact set $\omega(F) \subset \Delta$. By the Fatou theorem, [3], there is a function y, holomorphic on Δ, continuous on $\overline{\Delta}$, whose zero-set is Z. On the other hand, the holomorphic function $y \circ \phi$ vanishes on the set $\phi^*(Z) \subset \Delta$, and therefore vanishes identically in Δ, contradicting the fact that – ϕ being non-constant – A is injective. That proves

Theorem 15. *Let $A \in \mathcal{L}(\mathcal{H}(\Delta))$ satisfy* (28)*, and therefore be represented by* (25)*. The map $\phi : \Delta \to \Delta$ is either constant, or an inner function.*

Denote now $H^\infty(\Delta)$ by H^∞ and let the restriction of $A \in \mathcal{L}(\mathcal{H}(\Delta))$ to H^∞ be a surjective isometry of the Banach space H^∞ onto itself. As was shown by K. Hoffman [3, p. 147], there exist a constant $c \in \partial\Delta$ and a Moebius transformation ϕ of Δ such that

$$(Au)(t) = c\, u(\phi(t))$$

for all $t \in \Delta$ and all $u \in H^\infty$. Note that $c = A1$.

Let now $x \in \mathcal{H}(\Delta)$. For any $a \in (0, 1)$, the function $x_a : \Delta \ni s \mapsto x(as)$ is contained in H^∞, and therefore

$$(A\, x_a)(t) = (A\, 1)(t)\, x(a\phi(t)) \;\; \forall\, x \in \mathcal{H}(\Delta), t \in \Delta_1. \tag{29}$$

As $a \uparrow 1$, $x_a \to x$, uniformly on all compact sets in Δ, and therefore $Ax_a \to Ax$. Thus, (29) yields (25), and the following proposition holds.

Proposition 6. *If $A|H^\infty$ is a Banach space-isometry mapping H^∞ onto itself, then* (24) *holds.*

References

[1] R.J. Fleming - J.E. Jamison, *Isometries on Banach spaces: function spaces*. Chapman and Hall/CRC Press, Boca Raton, Florida, 2002, 208 pp.
[2] A. Gleason, *A characterization of maximal ideals*. J. Analyse Math., 19, 1967, 171-172.
[3] K. Hoffman, *Banach spaces of analytic functions*. Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1962, 217 pp.
[4] W. Holsztyński, *Continuous mappings induced by isometries of spaces of continuous functions*. Studia Math., 26, 1966, 133-136.
[5] L. Hörmander, *An introduction to complex analysis in several variables*. Van Nostrand, Princeton, N.J., 1966, 208 pp.
[6] J.-P. Kahane - W. Zelazko, *A characterization of maximal ideals in commutative Banach algebras*. Studia Math., 29, 1968, 339-343.
[7] H. Kamowitz - S. Scheinberg, *The spectrum of automorphisms of Banach algebras*. J. Functional Analysis, 4, 1949, 268-276.
[8] A. Martineau, *Sur les fonctionnelles analytiques et la transformation de Fourier-Borel*. J. Analyse Math., 9, 1963, 1-164.
[9] E. Michael, *Locally multiplicatively-convex topological algebras*. Mem. Amer. Math. Soc., 11, 1952, 82 pp.
[10] M. Nagasawa, *Isomorphisms between commutative Banach algebras with an application to rings of analytic functions*. Kodai Math. Sem. Rep., 11, 1959, 182-188.
[11] C.E. Rickart, *General theory of Banach algebras*. Van Nostrand, Princeton, N.J., 1960.
[12] W. Rudin, *Functional analysis*. McGraw Hill, New York 1973.
[13] S. Scheinberg, *Automorphisms of commutative Banach algebras*. In: R.C. Gunning (ed.), *Problems in analysis. A symposium in honor of Salomon Bochner*. Princeton University Press, Princeton, N.J., 1970, 319-323.
[14] E. Vesentini, *On the Banach-Stone Theorem*. Advances in Math., 112, 1995, 135-146.
[15] E. Vesentini, *Weighted composition operators and the Gleason-Kahane-Zelazko theorem*. Advances in Math., 191, 2005, 423-445.
[16] E. Vesentini, *Introduction to continuous semigroups*. Scuola Normale Superiore, Pisa 2002.
[17] W. Zelazko, *A characterization of multiplicative linear functionals in complex Banach algebras*. Studia Math., 29, 1968, 339-343.

Pervenuta il 29 luglio 2004,
in forma definitiva il 28 ottobre 2004.

Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi, 24 - 10129 Torino