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On the uniqueness and simplicity of the principal eigenvalue

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Equazioni a derivate parziali. — *On the uniqueness and simplicity of the principal eigenvalue*. Nota di Marcello Lucia, presentata (*) dal Socio A. Ambrosetti.

Abstract. — Given an open set Ω of $\mathbb{R}^N$ $(N > 2)$, bounded or unbounded, and a function $w \in L^{\frac{N}{2}}(\Omega)$ with $w^+ \not\equiv 0$ but allowed to change sign, we give a short proof that the positive principal eigenvalue of the problem

$$-\Delta u = \lambda w(x)u, \qquad u \in \mathcal{D}_0^{1,2}(\Omega),$$

is unique and simple. We apply this result to study unbounded Palais-Smale sequences as well as to give a priori estimates on the set of critical points of functionals of the type

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(x,u)dx, \qquad u \in \mathcal{D}_0^{1,2}(\Omega),$$

when G has a subquadratic growth at infinity.

Key words: Principal eigenvalue; Simple eigenvalue; Capacity; Palais Smale sequence.

Riassunto. — *Sull'unicità e la semplicità dell'autovalore principale*. Dato un aperto connesso Ω di $\mathbb{R}^N$ $(N > 2)$, limitato o illimitato, e una funzione $w \in L^{\frac{N}{2}}(\Omega)$ con $w^+ \not\equiv 0$ cui è consentito cambiare segno, si dimostra che l'autovalore principale positivo del problema

$$-\Delta u = \lambda w(x)u, \qquad u \in \mathcal{D}_0^{1,2}(\Omega),$$

è unico e semplice. Tale risultato viene applicato allo studio delle successioni di Palais-Smale illimitate ed utilizzato per costruire stime a priori sull'insieme dei punti critici di funzionali del tipo

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(x,u)dx, \qquad u \in \mathcal{D}_0^{1,2}(\Omega),$$

dove G ha un andamento subquadratico all'infinito.

1. Introduction

Given Ω an open and connected subset of $\mathbb{R}^N$ with $N > 2$, we consider the Beppo-Levi space $\mathcal{D}_0^{1,2}(\Omega)$ defined as the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|u\| := \left(\int_\Omega |\nabla u|^2\right)^{1/2}$. In this space, we consider the following linear problem

$$(1.1) \qquad -\triangle u = \lambda w(x)u, \qquad u \in \mathcal{D}_0^{1,2}(\Omega), \qquad u \not\equiv 0,$$

where w satisfies

$$(1.2) \qquad w \in L^{N/2}(\Omega) \qquad \text{and} \qquad w^+ \not\equiv 0.$$

We note that w is allowed to change sign on a domain Ω which may be unbounded.

(*) Nella seduta dell'11 marzo 2005.

Under assumption (1.2), we shall investigate the uniqueness and simplicity of positive principal eigenvalues in the following sense:

Definition 1.1. *We say that* $\lambda \in \mathbb{R}$ *is a «principal eigenvalue» of Problem* (1.1) *if there exists* $u \in \mathcal{D}_0^{1,2}(\Omega)$ *such that* $u \geq 0$ *a.e. in* Ω *and* (λ, u) *solves* (1.1). *We say that* $\lambda \in \mathbb{R}$ *is a «simple eigenvalue» of Problem* (1.1) *if*

$$\{u \in \mathcal{D}_0^{1,2}(\Omega) : (\lambda, u) \text{ solves } (1.1)\} \cup \{0\}$$

is of dimension 1.

For w satisfying (1.2), it is known by a result of Szulkin and Willem [15] that (1.1) has a positive principle eigenvalue Λ. If $w^- \not\equiv 0$, by applying the result of [15] to the weight function $-w$, we deduce the existence of a negative principal eigenvalue of Problem (1.1). Moreover, under some additional assumption on w, they prove that such eigenvalue is simple. The aim of this paper is to emphasize that condition (1.2) is actually enough to ensure uniqueness and simplicity of the positive principal eigenvalue.

The study of linear problem with Dirichlet boundary conditions and weight allowed to change sign as well as to have singularities was to our knowledge initiated by Manes-Micheletti [13]. In their work, the simplicity of the first positive eigenvalue was proved for weight $w \in L^p(\Omega)$ with $p > \dfrac{N}{2}$, $w^+ \not\equiv 0$. By assuming moreover that $w \in L^\infty(\Omega)$ and the domain to be sufficiently regular, a proof of the uniqueness of the positive principal eigenvalue can be found in [8, Proposition 1.15]. Similar results have been obtained for elliptic operators having nondivergence form by Hess-Kato [11] and Berestycki, Nirenberg, Varadhan [3], by assuming mainly the weight w to be bounded. When $\Omega = \mathbb{R}^N$, sufficient conditions on w ensuring this existence of principal eigenvalues have been given by Brown, Cosner and Fleckinger [5], Allegretto [1], Tertikas [16]. In [6], beside the question of existence, Brown-Stavrakakis prove uniqueness and simplicity of the positive principal eigenvalue for weight assumed to be sufficiently regular.

The paper is organized as follows.

In Section 2, we prove that the positive principal eigenvalue of Problem (1.1) is unique and simple when $w \in L^{N/2}(\Omega)$ and $w^+ \not\equiv 0$. Though we shall use classical technics, no proof to our knowledge has been given to handle such weights. Let us also emphasize that the same procedure allows to handle more general weights. But to keep the discussion short we are just considering the case of weights of class $L^{N/2}(\Omega)$. In Section 3, we apply this uniqueness property of the principal eigenvalue to analyze the behavior of unbounded Palais-Smale sequences of a functional having a subquadratic growth at infinity. The same approach gives a priori estimates for the set of critical points of such functional.

2. Proof of the uniqueness

For Problem (1.1), the question of existence has been studied by Szulkin and Willem. By applying their result (see Theorem 2.2, [15]) we get the following:

PROPOSITION 2.1. *Let Ω be an open subset of $\mathbb{R}^N$, $w \in L^{\frac{N}{2}}(\Omega)$ be such that $w^+ \not\equiv 0$ and consider*

$$\Lambda := \inf_{\varphi \in \mathcal{D}_0^{1,2}(\Omega)} \left\{ \int_\Omega |\nabla \varphi|^2 : \int_\Omega w\varphi^2 = 1 \right\}. \tag{2.3}$$

Then, there exists $\Phi \in \mathcal{D}_0^{1,2}(\Omega)$, $\Phi \geq 0$ solving the minimizing Problem (2.3). *Hence, Λ is a positive principal eigenvalue of Problem* (1.1).

Let us note that if $w \in L^p_{loc}(\Omega)$ with $p > \frac{N}{2}$, then the strong maximum principle applies. Therefore, for such weights w, any (λ, u) solving (1.1) with $u \geq 0$ has the property $u > 0$ (see [14]). When $w \in L^{N/2}(\Omega)$, we cannot exclude the existence of points where u vanishes. However, we can use a result due to Ancona [2] (see also the paper of Brezis-Ponce [4]), that in our setting can be stated as follows:

THEOREM 2.2. *Assume $w \in L^{N/2}(\Omega)$. Let (λ, u) be a solution of* (1.1) *with $u \geq 0$ and consider its «precise representative» $\tilde{u}$. Then, $\{x : \tilde{u}(x) = 0\}$ is of H^1-capacity zero and in particular of Lebesgue measure zero.*

It is known that any element u of a Sobolev space has a «precise representative», whose main property is to be quasicontinuous (see [9]). Therefore, in the sequel, it will be implicitely assumed that we work with such special representative.

Another difficulty to overcome when Problem (1.1) is considered with a function $w \in L^{N/2}(\Omega)$ is the possible lack of regularity of the solutions. For example, consider the function $\phi(x) = \log(|x|)$ on the ball $\Omega := B\left(0, \frac{1}{2}\right)$. A straightforward calculation shows that ϕ solves:

$$-\Delta\phi = w(x)\phi,$$

with $w(x) := \dfrac{N-2}{|x|^2 \log|x|}$. One check easily that $w \in L^{N/2}(\Omega)$, but $\phi \notin L^\infty(\Omega)$.

Nevertheless, by refining slightly some argument found in [7], combined with Theorem 2.2, we are able to prove the uniqueness of the positive principle eigenvalue.

PROPOSITION 2.3. *Assume $w \in L^{N/2}(\Omega)$ and $w^+ \not\equiv 0$. Let (Λ, Φ) be a solution of the minimizing Problem* (2.3) *and consider a solution (λ, φ) of* (1.1) *with $\lambda \geq 0$. Then, $\lambda \geq \Lambda$. If furthermore $\varphi \geq 0$, then $\lambda = \Lambda$.*

PROOF. From the definition of Λ, we deduce easily that $\lambda \geq \Lambda$. Assume now that (λ, φ) solves (1.1) with

$$\lambda \geq \Lambda \quad \text{and} \quad \varphi \geq 0.$$

Hence, we have

$$-\Delta\Phi = \Lambda w \Phi, \quad \Phi \geq 0, \tag{2.4}$$

$$-\Delta\varphi = \lambda w \varphi, \quad \varphi \geq 0. \tag{2.5}$$

For each $k \geq 0$, let us consider the function

$$\Phi_k(x) := \begin{cases} k & \text{if } \Phi(x) \geq k, \\ \Phi(x) & \text{if } \Phi(x) \in [0, k). \end{cases}$$

Clearly $\Phi_k \in L^\infty(\Omega)$ and it is well-known that $\Phi_k \in \mathcal{D}_0^{1,2}(\Omega)$ (see [10]). Given $\varepsilon > 0$, we consider also the function $\dfrac{\Phi_k^2}{\varphi + \varepsilon}$ which is in $\mathcal{D}_0^{1,2}(\Omega)$. Since Φ_k and $\dfrac{\Phi_k^2}{\varphi + \varepsilon}$, are legitimate test function in (2.4), respectively (2.5), we get

$$\int_\Omega \nabla\Phi\nabla\Phi_k - \int_\Omega \nabla\varphi\nabla\left(\frac{\Phi_k^2}{\varphi + \varepsilon}\right) = \int_\Omega \Lambda w\Phi\Phi_k - \int_\Omega \lambda w\varphi\frac{\Phi_k^2}{\varphi + \varepsilon},$$

which is equivalent to

$$\int_\Omega \left\{|\nabla\Phi_k|^2 - \nabla\varphi\nabla\left(\frac{\Phi_k^2}{\varphi + \varepsilon}\right)\right\} = \int_\Omega \left\{\Lambda w\Phi\Phi_k - \lambda w\varphi\frac{\Phi_k^2}{\varphi + \varepsilon}\right\}. \tag{2.6}$$

But, a direct calculation shows that the following «Picone's identity» holds:

$$|\nabla\Phi_k|^2 - \nabla\varphi\nabla\left(\frac{\Phi_k^2}{\varphi + \varepsilon}\right) = \left|\nabla\Phi_k - (\frac{\Phi_k}{\varphi + \varepsilon})\nabla\varphi\right|^2. \tag{2.7}$$

By plugging (2.7) in (2.6), we get

$$0 \leq \int_\Omega \left|\nabla\Phi_k - (\frac{\Phi_k}{\varphi + \varepsilon})\nabla\varphi\right|^2 = \int_\Omega \left\{\Lambda w\Phi\Phi_k - \lambda w\varphi\frac{\Phi_k^2}{\varphi + \varepsilon}\right\}. \tag{2.8}$$

Since by Theorem 2.2 the set $\{\varphi = 0\}$ is of measure zero, (2.8) is equivalent to

$$0 \leq \int_{\{\varphi>0\}} \left|\nabla\Phi_k - \left(\frac{\Phi_k}{\varphi + \varepsilon}\right)\nabla\varphi\right|^2 = \int_{\{\varphi>0\}} \left\{\Lambda w\Phi\Phi_k - \lambda w\varphi\frac{\Phi_k^2}{\varphi + \varepsilon}\right\}. \tag{2.9}$$

Now, letting $\varepsilon \to 0$ and $k \to \infty$ in (2.9) and applying Lebesgue dominated Theorem to the right handside, we get

$$0 \leq (\Lambda - \lambda)\int_\Omega w\Phi^2. \tag{2.10}$$

Since, $\lambda \geq \Lambda$ and $\int_\Omega w(x)\Phi^2 = 1$, (2.10) implies $\lambda = \Lambda$. □

We conclude this section by proving that the principle eigenvalue (2.3) has to be simple.

Proposition 2.4. *Let $w \in L^{N/2}(\Omega)$ be such that $w^+ \not\equiv 0$. Consider Λ defined by* (2.3) *and the eigenspace $V(\Lambda) := \{\Phi \in \mathcal{D}_0^{1,2}(\Omega) : (\Lambda, \Phi)$* solves (1.1)$\} \cup \{0\}$. *Then,*

1. *For any $\Phi \in V(\Lambda) \setminus \{0\}$, we have*

$$\Phi > 0 \ a.e. \quad or \quad \Phi < 0 \ a.e. \tag{2.11}$$

2. dim $V(\Lambda) = 1$.

Proof. (1) Let $\Phi \in V(\Lambda)$ be such that $\int_\Omega w\Phi^2 = 1$.We note (as in Theorem 2.5 of [15]) that

$$\Phi^+ = \Phi + |\Phi| \in V(\Lambda). \tag{2.12}$$

Indeed, since

$$\int_\Omega |\nabla\Phi|^2 = \int_\Omega |\nabla|\Phi||^2 = \Lambda \quad \text{and} \quad \int_\Omega w|\Phi|^2 = \int_\Omega w\Phi^2 = 1,$$

Φ and $|\Phi|$ solve the minimization Problem (2.3). In particular, $|\Phi| \in V(\Lambda)$ and (2.12) follows immediately. Now, Theorem 2.2 implies that either $\Phi^+ \equiv 0$, or $\Phi^+ > 0$ a.e., which proves (2.11).

(2) We know that dim $V(\Lambda) \geq 1$ (by Theorem 2.1). The proof that the dimension is exactly 1, can be done using the idea given in Lemma 7 of [13].

Let $\Phi_1, \Phi_2 \in V(\Lambda)$. By part (1), we can assume without loss of generality that $\Phi_1, \Phi_2 > 0$ a.e. Let us consider the set

$$T := \{t \in \mathbb{R} : \Phi_1 + t\Phi_2 > 0 \text{ a.e.}\}.$$

We shall show that $\Phi_1 + t_0\Phi_2 \equiv 0$ when $t_0 = \inf T$.

Claim 1. $T \neq \emptyset$, $\inf T > -\infty$.
Since $0 \in T$, we see that $T \neq \emptyset$. To prove that this set is bounded from below, let us consider for each $\delta, M > 0$, the set

$$A_{\delta,M} := \{\Phi_1 < M,\ \Phi_2 > \delta\}.$$

Since $\Phi_2 > 0$ a.e., there exists $\widetilde{\delta} > 0$ such that $|\{\Phi_2 > \widetilde{\delta}\}| > 0$. Moreover, since $\cup_{M>0} A_{\tilde{\delta},M} = \{\Phi_2 > \widetilde{\delta}\}$, we deduce the existence of $\widetilde{M}$ such that $|A_{\tilde{\delta},\tilde{M}}| > 0$. Now, for any $x \in A_{\tilde{\delta},\tilde{M}}$ and $t < -\dfrac{\widetilde{M}}{\widetilde{\delta}}$, we get

$$(\Phi_1 + t\Phi_2)(x) < \widetilde{M} + t\widetilde{\delta} < 0.$$

Hence, when $t < -\dfrac{\widetilde{M}}{\widetilde{\delta}}$, the function $\Phi_1 + t\Phi_2$ is negative on a set of positive measure, which proves $\inf T > -\infty$.

Claim 2. Setting $t_0 := \inf T$, we have $\Phi_1 + t_0\Phi_2 \equiv 0$.
From part (1) of this proposition, we have the following alternatives:

$$(a)\ \Phi_1 + t_0\Phi_2 > 0, \quad (b)\ \Phi_1 + t_0\Phi_2 < 0, \quad (c)\ \Phi_1 + t_0\Phi_2 \equiv 0.$$

Assume (a) holds. By setting

$$E_{\delta,M} := \{\Phi_1 + t_0\Phi_2 > \delta,\ \Phi_2 < M\},$$

and arguing as in claim (1), we prove $|E_{\tilde{\delta},\tilde{M}}| > 0$ for some $\widetilde{\delta}, \widetilde{M} > 0$. Thus, for any $x \in E_{\tilde{\delta},\tilde{M}}$ and $\varepsilon \in \left(0, \dfrac{\widetilde{\delta}}{\widetilde{M}}\right)$, we get

$$(\Phi_1 + (t_0 - \varepsilon)\Phi_2)(x) > \widetilde{\delta} - \varepsilon\widetilde{M} > 0.$$

From part (1), we deduce that $\Phi_1 + (t_0 - \varepsilon)\Phi_2 > 0$ a.e., in contradiction with the definition of t_0. A similar contradiction is reached if we assume (*b*). Therefore, the alternative (*c*) holds, which concludes the proof of the proposition. $\square$

3. Application to a problem asymptotically linear at infinity

Given Ω an open set of $\mathbb{R}^N$ (bounded or unbounded), we consider the functional:

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(x,u)dx, \quad u \in \mathcal{D}_0^{1,2}(\Omega), \tag{3.13}$$

where $G(x,s) = \int_0^s g(x,\xi)d\xi$ and g is assumed to satisfy the following assumptions.

(*H1*) $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies $g(x,s) = 0$ a.e. $x \in \Omega$, $\forall s \leq 0$;

(*H2*) There exists $\gamma \in L^{N/2}(\Omega)$ such that $|g(x,s)| \leq \gamma(x)s$ a.e. $x \in \Omega$, $\forall s \geq 0$;

(*H3*) The function g is «asymptotically linear» at infinity in the sense:

$$g_\infty(x) := \lim_{s\to\infty} \frac{g(x,s)}{s} \text{ exists a.e. } x \in \Omega.$$

The existence of non-trivial critical points for I under such assumptions was done for example in [17, 12]. An important step in those papers is to study the compactness of Palais-Smale sequences. Actually, simple examples like

$$G(x,s) = \begin{cases} \lambda s^2 & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases}$$

show that under the assumptions (*H1*) to (*H3*), the functional I can have unbounded Palais-Smale sequences. In [17] such analysis was done for positive and bounded g on a bounded domain, and it was noticed in [12] that changing-sign nonlinearity could also be considered. In this section, we show how those arguments can be extended to nonlinearity, not necessarily positive, of class $L^{N/2}(\Omega)$ where Ω can be bounded or unbounded. We start with the following Lemma:

Lemma 3.1. *Let $p > 1$ and consider a sequence $(\alpha_n, \beta_n) \in \mathcal{D}_0^{1,2}(\Omega) \times L^p(\Omega)$ such that*

$$\alpha_n \rightharpoonup \alpha \text{ in } \mathcal{D}_0^{1,2}(\Omega) \quad \text{and} \quad \beta_n \rightharpoonup \beta \text{ in } L^p(\Omega).$$

Assume $\frac{1}{q} := \frac{1}{p} + \frac{1}{2^*} < 1$, then $\alpha_n\beta_n \rightharpoonup \alpha\beta$ in $L^q(\Omega)$.

Proof. We check easily that $\alpha_n\beta_n$ is bounded in $L^q(\Omega)$. Therefore, $\alpha_n\beta_n \rightharpoonup v$ in $L^q(\Omega)$. To prove the lemma, it is enough to show that $v = \alpha\beta$ on each open set $\tilde{\Omega} \subset\subset \Omega$.

Let $\varphi \in C_0^\infty(\widetilde{\Omega})$. On the one hand, we have

$$\int_{\tilde{\Omega}} \alpha_n \beta_n \varphi \to \int_{\tilde{\Omega}} v\varphi \tag{3.14}$$

On the other hand,

$$\int_{\tilde{\Omega}} |\alpha_n\beta_n - \alpha\beta||\varphi| \le \int_{\tilde{\Omega}} |\alpha_n - \alpha||\beta_n\varphi| + \int_{\tilde{\Omega}} |\beta_n - \beta||\alpha\phi| \tag{3.15}$$

Now, we note that by assumption $p' < 2^*$. Hence,

$$\|\alpha_n - \alpha\|_{L^p(\tilde{\Omega})} \to 0 \quad \text{and} \quad \alpha\phi \in L^{2^*}(\widetilde{\Omega}) \subset L^{p'}(\widetilde{\Omega}). \tag{3.16}$$

From (3.15) and (3.16) we deduce that

$$\int_{\tilde{\Omega}} \alpha_n \beta_n \varphi \to \int_{\tilde{\Omega}} \alpha\beta\varphi. \tag{3.17}$$

From (3.14) and (3.17), we get

$$\int_{\tilde{\Omega}} v\varphi = \int_{\tilde{\Omega}} \alpha\beta\varphi, \quad \forall \varphi \in C_0^\infty(\tilde{\Omega}),$$

which shows that $v = \alpha\beta$ on $\tilde{\Omega}$. □

In the sequel, we will denote by $\Lambda(g_\infty)$ the positive principal eigenvalue of the problem: $-\Delta\phi = \lambda g_\infty \phi$.

Proposition 3.2. *Let* (H1) *to* (H3) *be satisfied. Assume the existence of a sequence* $\{u_n\}$ *such that*

$$\|DI_{(u_n)}\| \xrightarrow{n} 0 \text{ in } (\mathcal{D}_0^{1,2}(\Omega))' \quad \text{and} \quad \|u_n\| \to \infty. \tag{3.18}$$

Then, $\dfrac{u_n}{\|u_n\|} \rightharpoonup w$ *in* $\mathcal{D}_0^{1,2}(\Omega)$*; moreover,* $w > 0$ *a.e. and satisfies*

$$-\Delta w = g_\infty(x) w \text{ in } \Omega.$$

In particular $\Lambda(g_\infty) = 1$.

Proof. With the help of Lemma 3.1, we can adapt the arguments of [17]. Let us set $w_n := \dfrac{u_n}{\|u_n\|}$. Since $\|w_n\| = 1$, we have (up to a subsequence)

$$w_n \rightharpoonup w \text{ in } \mathcal{D}_0^{1,2}(\Omega), \quad w_n \rightharpoonup w \text{ in } L^{2^*}(\Omega), \quad w_n \to w \text{ a.e.} \tag{3.19}$$

Claim 1. $w \not\equiv 0$

From (3.18), we have

$$\frac{\sup_{\|\varphi\|=1}\left|\int_\Omega \langle \nabla u_n, \nabla\varphi\rangle - \int_\Omega g(\cdot, u_n)\varphi\right|}{\|u_n\|} \to 0.$$

In particular, we get

$$\frac{\left|\int_\Omega \langle \nabla u_n, \nabla w_n\rangle - \int_\Omega g(\cdot, u_n) w_n\right|}{\|u_n\|} \to 0,$$

from which we deduce

$$\left|1 - \int_\Omega \frac{g(\cdot, u_n)}{\|u_n\|} w_n\right| \to 0. \tag{3.20}$$

On the other hand, from (*H2*), we have

$$\left|\int_\Omega \frac{g(\cdot, u_n)}{\|u_n\|} w_n\right| \leq \int_\Omega \gamma \frac{|u_n|}{\|u_n\|} w_n = \int_\Omega \gamma w_n^2. \tag{3.21}$$

Now, assume by contradiction $w_n \rightharpoonup 0$ in $\mathcal{D}_0^{1,2}(\Omega)$. Then, $w_n \rightharpoonup 0$ in $L^{2^*}(\Omega)$ and Lemma 3.1 implies that $w_n^2 \rightharpoonup 0$ in $L^{2^*/2}(\Omega) = (L^{N/2}(\Omega))'$. Hence, since $\gamma \in L^{N/2}(\Omega)$, the right handside of (3.21) should tend to zero. So,

$$\int_\Omega \frac{g(\cdot, u_n)}{\|u_n\|} w_n \to 0. \tag{3.22}$$

Since (3.22) contradicts (3.20), we must have $w_n \rightharpoonup w \not\equiv 0$ in $\mathcal{D}_0^{1,2}(\Omega)$.

Claim 2. $w > 0$ a.e. on Ω.
Knowing that

$$\frac{DI_{(u_n)}(\varphi)}{\|u_n\|\|\varphi\|} \to 0, \quad \forall\, \varphi \in \mathcal{D}_0^{1,2}(\Omega),$$

we deduce

$$\frac{\int_\Omega \langle \nabla u_n, \nabla \varphi\rangle dx - \int_\Omega g(x, u_n)\varphi dx}{\|u_n\|} \to 0.$$

Since $g(x, s) = 0$ for $s \leq 0$,

$$\frac{\int_\Omega \langle \nabla u_n, \nabla \varphi\rangle dx - \int_\Omega g(x, u_n^+)\varphi dx}{\|u_n\|} \to 0. \tag{3.23}$$

By setting

$$\tilde{g}_n(x) := \begin{cases} \dfrac{g(x, u_n^+(x))}{u_n^+(x)} & \text{if } u_n^+(x) \neq 0, \\ 0 & \text{if } u_n^+(x) = 0, \end{cases}$$

we can rewrite (3.23) as follows

$$\int_\Omega \langle \nabla w_n, \nabla \varphi\rangle dx - \int_\Omega \tilde{g}_n(x) w_n^+ \varphi dx \to 0, \quad \forall \varphi \in \mathcal{D}_0^{1,2}(\Omega). \tag{3.24}$$

Now, from (*H2*) and (3.19), we respectively get

$$\tilde{g}_n \rightharpoonup \tilde{\gamma} \text{ in } L^{N/2}(\Omega) \quad \text{and} \quad w_n^+ \rightharpoonup w^+ \text{ in } \mathcal{D}_0^{1,2}(\Omega). \tag{3.25}$$

Therefore, by Lemma 3.1, we deduce

$$\tilde{g}_n w_n^+ \rightharpoonup \tilde{\gamma} w^+ \text{ in } L^p(\Omega) \quad \text{with } \frac{1}{p} = \frac{2}{N} + \frac{1}{2^*} = \frac{1}{(2^*)'}.$$

Hence,

$$\int_\Omega \tilde{g}_n w_n^+ \varphi \to \int_\Omega \tilde{\gamma} w^+ \varphi \qquad \forall \varphi \in L^{2^*}(\Omega). \tag{3.26}$$

From (3.24) and (3.26), we have

$$\int_\Omega \langle \nabla w, \nabla \varphi \rangle dx - \int_\Omega \tilde{\gamma}(x) w^+ \varphi dx = 0 \qquad \forall \varphi \in \mathcal{D}_0^{1,2}(\Omega). \tag{3.27}$$

Choosing $\varphi = w^-$ in (3.27) one gets $\int_\Omega |\nabla w^-|^2 dx = 0$. Thus, $w \geq 0$ and solves

$$-\Delta w = \tilde{\gamma}(x) w \qquad \text{in } \Omega. \tag{3.28}$$

Since $w \not\equiv 0$ (by claim 1), the version of the strong maximum given in Theorem 2.2 implies $w > 0$ a.e.

Claim 3. $\tilde{\gamma} = g_\infty$ and $-\Delta w = g_\infty(x) w$.
Since $w > 0$ a.e. (by claim 2), we get $u_n \to +\infty$ a.e. in Ω. Using (*H3*) and (3.25) we then have

$$\frac{g(x, u_n)}{u_n} \to g_\infty(x) \text{ a.e.} \quad \text{and} \quad \frac{g(x, u_n)}{u_n} \rightharpoonup \tilde{\gamma}(x) \text{ in } L^{N/2}(\Omega).$$

This yields $\tilde{\gamma} = g_\infty$ a.e., and using (3.28), we obtain

$$-\Delta w = g_\infty(x) w, \quad w > 0 \text{ a.e.}$$

The positive principal eigenvalue beeing unique (Theorem 2.3), we get $\Lambda(g_\infty) = 1$. □

From this proposition, we derive immediately the following corollary:

COROLLARY 3.3. *Assume* (H1) *to* (H3) *and* $\Lambda(g_\infty) \neq 1$. *Then*,
1. *Every Palais-Smale sequence of the functional* (3.13) *is bounded*;
2. *The set of critical points of* (3.13) *is bounded in* $\mathcal{D}_0^{1,2}(\Omega)$.

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