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Characterization of the interior regularity for parabolic systems with discontinuous coefficients

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Equazioni a derivate parziali. — *Characterization of the interior regularity for parabolic systems with discontinuous coefficients.* Nota di Dian K. Palagachev e Lubomira G. Softova, presentata (*) dal Socio G. Da Prato.

Abstract. — We deal in this *Note* with linear parabolic (in sense of Petrovskij) systems of order $2b$ with discontinuous principal coefficients belonging to $VMO \cap L^\infty$. By means of a priori estimates in Sobolev-Morrey spaces we give a precise characterization of the Morrey, *BMO* and Hölder regularity of the solutions and their derivatives up to order $2b-1$.

Key words: Parabolic systems; A priori estimates; Morrey spaces; Hölder regularity; VMO.

Riassunto. — *Caratterizzazione della regolarità all'interno per sistemi parabolici con coefficienti discontinui.* In questa *Nota* si studiano dei sistemi lineari parabolici (in senso di Petrovskij) di ordine $2b$ con coefficienti principali appartenenti a $VMO \cap L^\infty$. Per mezzo di stime a priori negli spazi di Sobolev-Morrey si propone una caratterizzazione precisa della regolarità (Morrey, *BMO* e Hölder) all'interno delle soluzioni e le loro derivate fino all'ordine $2b-1$.

1. Introduction

A celebrated 1958's paper by J. Nash asserts Hölder continuity of weak solutions to *second-order, divergence* form linear parabolic equations with measurable and essentially bounded coefficients, providing this way for a parabolic counterpart to the famous elliptic result of E. De Giorgi. By employing completely different technique, N.V. Krylov and M.V. Safonov proved in the beginning of 1980's Hölder continuity of the strong solutions to *second-order, non-divergence* form equations with L^∞ coefficients. Their results opened the way for the current blossom of the theory of fully nonlinear equations with all its applications in differential geometry, stochastic control, nonlinear optimization, etc. On the other hand, it is well known that this kind of results *cannot hold,* in general, neither for systems nor for single equations of order *greater* than 2 if the principal coefficients are *merely* bounded. A notable exception is given by operators for which the eigenvalues of the principal part *do not scatter too much* – a condition originally introduced by H.O. Cordes (see [12, and the references therein]).

It turns out that appropriate regularity of the principal coefficients not only guarantees Hölder's continuity of strong solutions but even permits to develop a relevant theory of elliptic/parabolic systems in Sobolev classes built on L^p for $p > 1$. The background for these results is ensured by the possibility to estimate a priori the L^p-norms of solution's *highest* order derivatives (say $D^{2b}\boldsymbol{u}$) by means of the data. The method, associated to the names of A. Calderón and A. Zygmund, uses explicit representation formulas for $D^{2b}\boldsymbol{u}$ in terms of singular integral acting on the known right-hand side plus another one acting on the very same derivatives $D^{2b}\boldsymbol{u}$. Fortunately,

(*) Nella seduta del 14 gennaio 2005.

these derivatives appear in a singular commutator the norm of which can be made small if the coefficients have *small oscillation* over small balls. This way, *continuity* of the principal coefficients is a *sufficient* condition ensuring boundedness of the singular integral operators and therefore validity of the L^p a priori estimate. The desired regularity of $\boldsymbol{u}$ then follows by known embeddings between Sobolev and Hölder spaces for suitable values of p. We refer the reader to the seminal works [1, 2, 6] for what concerns elliptic systems, and to [7, 9, 15] in the case of parabolic operators. A relevant L^p-theory of uniformly elliptic systems with *discontinuous* principal coefficients was developed in [4]. The discontinuity is expressed in terms of appartenance to the class of functions with *vanishing mean oscillation VMO* which contains as a proper subset the space of uniformly continuous functions.

This *Note* deals with linear systems of order $2b$ which are parabolic in sense of Petrovskij. Our aim is to extend the aforementioned results to such systems with *discontinuous* principal coefficients $a_\alpha^{kj}(x,t)$. We deal with strong solutions belonging to Sobolev-Morrey's class $W^{2b,1}_{p,\lambda,\mathrm{loc}}(Q)$ where Q is a cylindrical domain in $\mathbb{R}^n \times \mathbb{R}_+$. It is proved that $a_\alpha^{kj}(x,t) \in VMO \cap L^\infty$ is a *sufficient* condition ensuring local Hölder regularity of such solutions and all their spatial derivatives up to order $2b-1$ for appropriate values of p and λ.

Our approach makes use of the Calderón-Zygmund method of expressing highest order solution's derivatives in terms of Gaussian-type potentials. These turn out to be singular integrals with *kernels of mixed homogeneity* of degree $-n-2b$ (strongly defined by the system itself) and their commutators with the multiplication by the *VMO* functions a_α^{kj} which have *small integral oscillation* over small cylinders. Employing results on boundedness of these singular integral operators in $L^{p,\lambda}$ (recently proven by the authors in [13]), we derive a priori bounds of Caccioppoli type which yield estimates for the strong solutions in $W^{2b,1}_{p,\lambda,\mathrm{loc}}(Q)$ by means of the $L^{p,\lambda}_{\mathrm{loc}}(Q)$ norm of the right-hand side plus a weaker norm of the solution itself. By virtue of embedding properties of the Sobolev-Morrey spaces into Hölder ones, these a priori bounds lead to a complete characterization of the Morrey, *BMO* and Hölder regularity of the solution and its spatial derivatives up to order $2b-1$.

2. Main results

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and define $Q = \Omega \times (0,T)$ with $T > 0$. We consider the linear system

$$(1) \qquad \mathfrak{L}\boldsymbol{u} := D_t\boldsymbol{u}(x,t) - \sum_{|\alpha|=2b} \boldsymbol{A}_\alpha(x,t) D^\alpha \boldsymbol{u}(x,t) = \boldsymbol{f}(x,t)$$

for the vector-valued function $\boldsymbol{u}: Q \to \mathbb{R}^m$ given by the transpose $\boldsymbol{u}(x,t) = \big(u_1(x,t),\dots,u_m(x,t)\big)^{\mathrm{T}}$, $\boldsymbol{f} = (f_1,\dots,f_m)^{\mathrm{T}}$, where $\boldsymbol{A}_\alpha(x,t)$ is the $m \times m$ matrix $\big\{a_\alpha^{jk}(x,t)\big\}_{j,k=1}^m$ of the measurable coefficients $a_\alpha^{jk}: Q \to \mathbb{R}$. Hereafter $b \geq 1$ is a fixed integer, $\alpha = (\alpha_1,\dots,\alpha_n)$, $D_t := \partial/\partial t$ and $D^\alpha \equiv D_x^\alpha := D_1^{\alpha_1} \dots D_n^{\alpha_n}$ with $D_i := \partial/\partial x_i$. Later on, $D^\alpha \boldsymbol{u} := \big(D^\alpha u_1, \dots, D^\alpha u_m\big)^{\mathrm{T}}$ and $D^s\boldsymbol{u}$ substitutes *any* derivative $D^\alpha \boldsymbol{u}$ with $|\alpha| = s \in \mathbb{N}$.

Suppose the system (1) is *uniformly parabolic in sense of Petrovskij* (see [7, 9, 15]). That is, for a.a. $(x,t) \in Q$ and any $\xi \in \mathbb{R}^n$ set $p_s(x,t,\xi)$, $s = 1,\dots,m$, for the eigenvalues of the $m \times m$ matrix $(-1)^b \sum_{|\alpha|=2b} \boldsymbol{A}_\alpha(x,t)\xi^\alpha$ where $\xi^\alpha := \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}$. Then parabolicity of (1) means

$$(2) \qquad \exists\, \delta > 0: \quad \operatorname{Re} p_s(x,t,\xi) \le -\delta|\xi|^{2b} \quad \text{a.a. } (x,t) \in Q,\ \forall \xi \in \mathbb{R}^n,\ \forall s = 1,\dots,m.$$

Endow $\mathbb{R}^{n+1} = \mathbb{R}^n_x \times \mathbb{R}_t$ with the parabolic metric $\rho(x,t) = \max\{|x|, |t|^{1/2b}\}$ and consider the collection of *parabolic cylinders*

$$(3) \qquad \mathcal{C}_r(x_0,t_0) := B_r(x_0) \times (t_0 - r^{2b}, t_0), \quad B_r(x_0) := \{x \in \mathbb{R}^n\colon\ |x - x_0| < r\}$$

each having Lebesgue measure $|\mathcal{C}_r|$ comparable to r^{n+2b}.

Definition 2.1. Set $Q_r := \mathcal{C}_r(x_0,t_0) \cap Q$ and suppose there is a positive constant A such that $|Q_r| \ge Ar^{n+2b}$ whenever $(x_0,t_0) \in Q$ (for example, this is clearly verified if Ω has the interior cone property). Let $p \in (1,\infty)$ and $\lambda \in (0, n+2b)$. The function $u \in L^p(Q)$ belongs to the parabolic Morrey space $L^{p,\lambda}(Q)$ if

$$\|u\|_{p,\lambda;Q} = \left(\sup_{r>0}\ \sup_{(x_0,t_0)\in Q} \frac{1}{r^\lambda} \int_{Q_r} |u(x,t)|^p\, dxdt \right)^{1/p} < \infty.$$

The Sobolev-Morrey space $W^{2b,1}_{p,\lambda}(Q)$ consists of all functions $u\colon Q \to \mathbb{R}$ belonging to the Sobolev space $W^{2b,1}_p(Q)$ with derivatives $D_t u$, $D^\alpha_x u$, $|\alpha| \le 2b$, lying in $L^{p,\lambda}(Q)$. The norm in $W^{2b,1}_{p,\lambda}(Q)$ is given by $\|u\|_{W^{2b,1}_{p,\lambda}(Q)} := \|D_t u\|_{p,\lambda;Q} + \sum_{s=0}^{2b} \sum_{|\alpha|=s} \|D^\alpha_x u\|_{p,\lambda;Q}$.

For the sake of brevity, we denote the cross-product of m copies of $L^{p,\lambda}(Q)$ by the same symbol. Thus, $\boldsymbol{u} = (u_1,\dots,u_m)^{\mathrm{T}} \in L^p(Q)$ means $u_k \in L^p(Q)$ for all $k = 1,\dots,m$, and $\|\boldsymbol{u}\|_{p,\lambda;Q} := \sum_{k=1}^m \|u_k\|_{p,\lambda;Q}$. Further on, $u \in W^{2b,1}_{p,\lambda,\mathrm{loc}}(Q)$ if $u \in W^{2b,1}_{p,\lambda}(\Omega' \times (0,T))$ for any $\Omega' \Subset \Omega$.

For an integrable function $f\colon\ Q \to \mathbb{R}$ define

$$\eta_f(R) := \sup_{r\le R}\ \sup_{(x_0,t_0)\in Q} \frac{1}{|Q_r|} \int_{Q_r} |f(y,\tau) - f_{Q_r}|\, dyd\tau \quad \forall\, R > 0,$$

where f_{Q_r} is the average $|Q_r|^{-1} \int_{Q_r} f(y,\tau)\, dyd\tau$. Then:

- $f \in BMO(Q)$ (cf. [10]) if $\|f\|_{*;Q} := \sup_R \eta_f(R) < \infty$;
- $f \in VMO(Q)$(see [14]) if $f \in BMO(Q)$ and $\lim_{R\downarrow 0} \eta_f(R) = 0$. The quantity $\eta_f(R)$ is referred to as *VMO*-modulus of f.

Our main result provides an a priori estimate in the Sobolev-Morrey scale for each strong solution of the system (1) with zero initial trace on $\{t = 0\}$.

Theorem 2.2. *Suppose* (2), $a^{jk}_\alpha \in VMO(Q) \cap L^\infty(Q)$, $p \in (1,\infty)$, $\lambda \in (0, n+2b)$ *and let* $\boldsymbol{u} \in W^{2b,1}_{p,\lambda,\mathrm{loc}}(Q)$ *be a strong solution to* (1) *such that* $\boldsymbol{u}(x,0) = \boldsymbol{0}$.

Then, for any $Q' = \Omega' \times (0,T)$ *and* $Q'' = \Omega'' \times (0,T)$, $\Omega' \Subset \Omega'' \Subset \Omega$, *there is a constant*

$C = C(n, m, b, \delta, p, \lambda, \|a_\alpha^{jk}\|_{\infty;Q}, \eta_{a_\alpha^{jk}}, \mathrm{dist}\,(\Omega', \partial\Omega''))$ *such that*

$$\|\boldsymbol{u}\|_{W^{2b,1}_{p,\lambda}(Q')} \le C(\|\mathfrak{L}\boldsymbol{u}\|_{p,\lambda;Q''} + \|\boldsymbol{u}\|_{p,\lambda;Q''}). \tag{4}$$

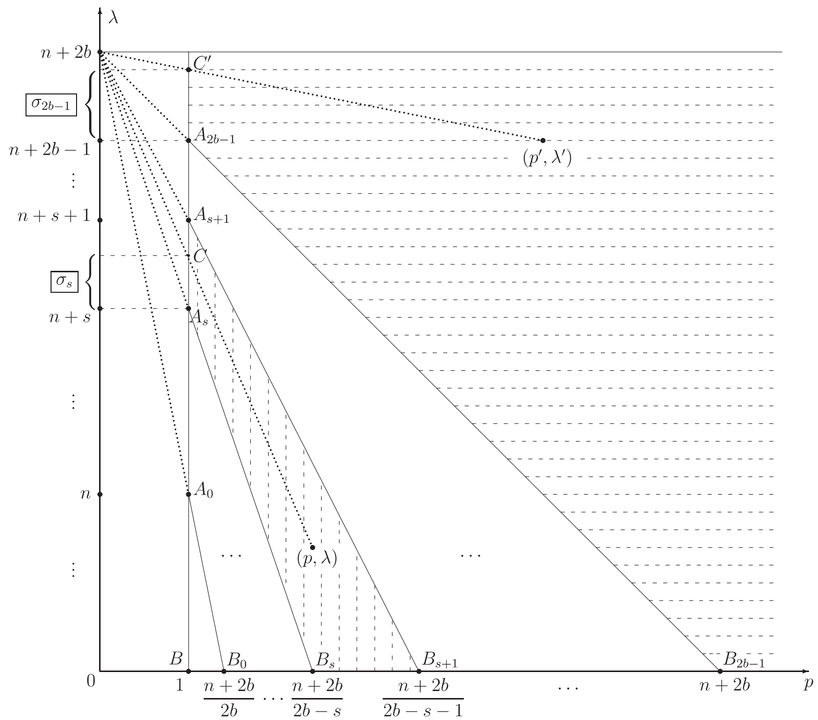
As consequence of (4) we obtain a precise characterization of the Morrey, *BMO* and Hölder regularity of the derivatives $D^s\boldsymbol{u}$ with $s \in \{0, 1, \dots, 2b-1\}$. Namely,

Corollary 2.3. *Under the hypotheses of Theorem 2.2 set* $\mathcal{K}_{p,\lambda;Q''}(\boldsymbol{u}) := \|\boldsymbol{u}\|_{p,\lambda;Q''} + \|\mathfrak{L}\boldsymbol{u}\|_{p,\lambda;Q''}$ *and fix an* $s \in \{0, 1, \dots, 2b-1\}$. *There is a constant C such that:*

a) if $p \in \left(1, \dfrac{n+2b-\lambda}{2b-s}\right)$ *then* $D^s\boldsymbol{u} \in L^{p,(2b-s)p+\lambda}(Q')$ *and* $\|D^s\boldsymbol{u}\|_{p,(2b-s)p+\lambda;Q'} \le C\mathcal{K}_{p,\lambda;Q''}(\boldsymbol{u})$;

b) if $p = \dfrac{n+2b-\lambda}{2b-s}$ *then* $D^s\boldsymbol{u} \in BMO(Q')$ *and* $\|D^s\boldsymbol{u}\|_{*;Q'} \le C\mathcal{K}_{p,\lambda;Q''}(\boldsymbol{u})$;

c) if $p \in \left(\dfrac{n+2b-\lambda}{2b-s}, \dfrac{n+2b-\lambda}{2b-s-1}\right)$ (1) *then* $D^s\boldsymbol{u} \in C^{\sigma_s,\sigma_s/2b}(Q')$ *with* $\sigma_s = 2b - s - \dfrac{n+2b-\lambda}{p}$ *and* $\displaystyle\sup_{(x,t),\,(x',t')\in Q';(x,t)\neq(x',t')} \frac{|D^s\boldsymbol{u}(x,t) - D^s\boldsymbol{u}(x',t')|}{(|x-x'| + |t-t'|^{1/2b})^{\sigma_s}} \le C\mathcal{K}_{p,\lambda;Q''}(\boldsymbol{u})$.



(1) This inclusion rewrites as $p \in (n + 2b - \lambda, \infty)$ when $s = 2b - 1$.

The picture illustrates geometrically the results of Corollary 2.3 with the couple (p,λ) lying in the semistrip $\{(p,\lambda)\colon\ p>1,\ 0<\lambda<n+2b\}$ and $s\in\{0,1,\dots,2b-1\}$. The points B_s on the p-axis are $B_s=\left(\dfrac{n+2b}{2b-s},0\right)$, $B=(1,0)$, and $A_s=(1,n+s)$ is the intersection of the line passing through $(0,n+2b)$ and B_s with the vertical line $\{p=1\}$.

We have $D^s\boldsymbol{u}\in L^{p,(2b-s)p+\lambda}(Q')$ if (p,λ) belongs to the open right triangle BB_sA_s (case a)). In particular, $(p,\lambda)\in\triangle BB_0A_0$ yields $\boldsymbol{u}\in L^{p,2bp+\lambda}(Q')$.

Let $s\in\{0,\dots,2b-2\}$ and take (p,λ) in the interior of the quadrilateral $R_s:=B_sB_{s+1}A_{s+1}A_s$. Then the spatial derivatives $D^s\boldsymbol{u}$ are Hölder continuous with exponent σ_s (case c)), while $D^{s+1}\boldsymbol{u}\in L^{p,(2b-s-1)p+\lambda}(Q')$. Moreover, σ_s is the length $|CA_s|$ of the segment CA_s where $C=C(p,\lambda)$ is the intersection of the vertical line $\{p=1\}$ with the line connecting the points (p,λ) and $(0,n+2b)$. If $(p,\lambda)\in A_sB_s$ (the open line segment) we have $D^s\boldsymbol{u}\in BMO(Q')$ (case b)), whereas $(p,\lambda)\in A_{s+1}B_{s+1}$ implies $D^{s+1}\boldsymbol{u}\in BMO(Q')$.

Similarly, let R_{2b-1} be the shadowed *unbounded region* on the picture. Then $(p',\lambda')\in R_{2b-1}$ gives that $D^{2b-1}\boldsymbol{u}$ are Hölder continuous with exponent $\sigma_{2b-1}= =|C'A_{2b-1}|$, $C'=C'(p',\lambda')$, while $(p',\lambda')\in A_{2b-1}B_{2b-1}$ yields $D^{2b-1}\boldsymbol{u}\in BMO(Q')$.

3. Sketch of the proofs

Proof of Theorem 2.2. Set $\boldsymbol{\Gamma}$ for the fundamental matrix of the operator $\mathfrak{L}$ given by (see [7, 15])

$$\boldsymbol{\Gamma}(x,t;y,\tau)=\begin{cases}\dfrac{1}{(2\pi)^n\tau^{n/2b}}\displaystyle\int_{\mathbb{R}^n} e^{i(y\tau^{-1/2b}\cdot\xi)+(-1)^b\sum_{|\alpha|=2b}\boldsymbol{A}_\alpha(x,t)\xi^\alpha}d\xi & \text{for } \tau>0,\\ 0 & \text{for } \tau\le 0.\end{cases}$$

Let $\boldsymbol{v}\in C^\infty(\mathbb{R}^{n+1})$ be a vector-valued function which is compactly supported in x and $\boldsymbol{v}(x,0)=\mathbf{0}$. Employing standard arguments (freezing a^{ij}_α's at a point $(x_0,t_0)\in\operatorname{supp}\boldsymbol{v}$, expressing $\boldsymbol{v}$ as a Gaussian potential, taking the D^{2b}-derivatives of $\boldsymbol{v}$ and then unfreezing the coefficients, cf. [1, 2, 4, 6] in case of elliptic systems and [7, 9, 12, 15] in the parabolic case), it is not hard to get the representation

$$\begin{aligned}(5)\qquad D^\alpha\boldsymbol{v}(x,t)&=\underbrace{p.v.\int_{\mathbb{R}^{n+1}}D^\alpha\boldsymbol{\Gamma}(x,t;x-y,t-\tau)\mathfrak{L}\boldsymbol{v}(y,\tau)dyd\tau}_{=:\,\mathfrak{K}_\alpha(\mathfrak{L}\boldsymbol{v})}\\ &+\underbrace{\sum_{|\alpha'|=2b}p.v.\int_{\mathbb{R}^{n+1}}D^\alpha\boldsymbol{\Gamma}(x,t;x-y,t-\tau)\big(\boldsymbol{A}_{\alpha'}(y,\tau)-\boldsymbol{A}_{\alpha'}(x,t)\big)D^{\alpha'}_y\boldsymbol{v}(y,\tau)dyd\tau}_{=:\,\mathfrak{C}_\alpha[\boldsymbol{A}_{\alpha'},D^{\alpha'}\boldsymbol{v}]}\\ &+\int_{\mathbb{S}^n}D^{\beta^s}\boldsymbol{\Gamma}(x,t;y,\tau)\upsilon_s d\sigma_{(y,\tau)}\mathfrak{L}\boldsymbol{v}(x,t)\qquad\forall\alpha\colon|\alpha|=2b\end{aligned}$$

where the derivatives $D^\alpha \boldsymbol{\Gamma}(\cdot,\cdot;\cdot,\cdot)$ are taken with respect to the third variable, $\beta^s := (\alpha_1, \dots, \alpha_{s-1}, \alpha_s - 1, \alpha_{s+1}, \dots, \alpha_n)$ and υ_s stands for the s-th entry of the outward normal to the unit Euclidean sphere $\mathbb{S}^n := \{(x,t) \in \mathbb{R}^{n+1}\colon\ |x|^2 + t^2 = 1\}$. Further on, $D^\alpha \boldsymbol{\Gamma}(x,t;\mu y, \mu^{2b}\tau) = \mu^{-(n+2b)} D^\alpha \boldsymbol{\Gamma}(x,t;y,\tau) \quad \forall \mu > 0$ and $\int_{\mathbb{S}^n} D^\alpha \boldsymbol{\Gamma}(x,t;y,\tau) d\sigma_{(y,\tau)} = \mathbf{0}$ whence each entry of the $m \times m$ matrix $D^\alpha \boldsymbol{\Gamma}(x,t;y,\tau)$ is a *variable kernel of Calderón-Zygmund type with mixed homogeneity* (see [8, 13]). The singular integral operators $\mathfrak{K}_\alpha$ and $\mathfrak{C}_\alpha[\boldsymbol{A}_{\alpha'}, \cdot]$ are bounded on $L^p(Q)$ because of $\boldsymbol{A}_\alpha \in L^\infty(Q)$ (cf. [8]) and therefore the representation formula (5) still holds true (a.e. in Q) for *compactly supported in x* functions $\boldsymbol{v} \in W^{2b,1}_p$ such that $\boldsymbol{v}(x,0) = \mathbf{0}$. Moreover, $\mathfrak{K}_\alpha$ and $\mathfrak{C}_\alpha[\boldsymbol{A}_{\alpha'}, \cdot]$ act continuously from $L^{p,\lambda}(Q)$ into itself and the norm of the commutator $\mathfrak{C}_\alpha[\boldsymbol{A}_{\alpha'}, \cdot]$ is comparable to $\|\boldsymbol{A}_{\alpha'}\|_{*;Q}$ (see [13, Theorem 2.1, Corollary 2.7]).

Fix an arbitrary point $(x_0,t_0) \in Q$ and consider the parabolic cylinder $\mathcal{C}_r := \mathcal{C}_r(x_0,t_0)$ (cf. (3)). Let $\boldsymbol{v} \in W^{2b,1}_{p,\lambda}(\mathcal{C}_r)$, $\operatorname{supp} \boldsymbol{v} \subset \mathcal{C}_r$ and $\boldsymbol{v}(x, t_0 - r^{2b}) = \mathbf{0}$. Since $\boldsymbol{A}_\alpha \in VMO$, the norm $\|\mathfrak{C}_\alpha[\boldsymbol{A}_{\alpha'}, D^{\alpha'}\boldsymbol{v}]\|_{p,\lambda;\mathcal{C}_r}$ can be made arbitrary small if $\mathcal{C}_r$ shrinks to (x_0,t_0). Therefore, for each $\varepsilon > 0$ there is an r_0 depending on ε and on the VMO-moduli $\eta_{\boldsymbol{A}_\alpha}$ of the coefficients, such that $\|D^{2b}\boldsymbol{v}\|_{p,\lambda;\mathcal{C}_r} \le C(\|\mathfrak{L}\boldsymbol{v}\|_{p,\lambda;\mathcal{C}_r} + \varepsilon\|D^{2b}\boldsymbol{v}\|_{p,\lambda;\mathcal{C}_r})$ whenever $r < r_0$ (cf. [13, Corollary 2.8]). Hence, choosing ε small enough we get

$$\|D^{2b}\boldsymbol{v}\|_{p,\lambda;\mathcal{C}_r} \le C\|\mathfrak{L}\boldsymbol{v}\|_{p,\lambda;\mathcal{C}_r}. \tag{6}$$

Let $r \in (0,r_0)$, $\theta \in (0,1)$, $\theta' = \theta(3-\theta)/2 > \theta$ and define the cut-off function $\varphi(x,t) := \varphi_1(x)\varphi_2(t)$, $0 \le \varphi \le 1$, with $\varphi_1 \in C_0^\infty(B_r(x_0))$ and $\varphi_2 \in C^\infty(\mathbb{R})$ such that

$$\varphi_1(x) = \begin{cases} 1 & x \in B_{\theta r}(x_0) \\ 0 & x \notin B_{\theta' r}(x_0) \end{cases} \qquad \varphi_2(t) = \begin{cases} 1 & t \in (t_0 - (\theta r)^{2b}, t_0] \\ 0 & t < t_0 - (\theta' r)^{2b}. \end{cases}$$

Since $\theta' - \theta = \theta(1-\theta)/2$, we have $|D_x^s \varphi| \le C(s)[\theta(1-\theta)r]^{-s}$ for any $1 \le s \le 2b$ and $|D_t\varphi| \le C[\theta(1-\theta)r]^{-2b}$. Thus, applying (6) to $\boldsymbol{v} := \varphi\boldsymbol{u}$, we obtain

$$\Theta_{2b} \le C\left(r^{2b}|\mathfrak{L}\boldsymbol{u}|_{p,\lambda;\mathcal{C}_r} + \sum_{s=1}^{2b-1} \Theta_s + \Theta_0 \right),$$

where Θ_s stands for the Morrey seminorm $\Theta_s := \sup_{0<\theta<1} [\theta(1-\theta)r]^s \|D^s\boldsymbol{u}\|_{p,\lambda;\mathcal{C}_{\theta r}}$, $s \in \{0, \dots, 2b\}$. Now interpolate the intermediate seminorms above (cf. [15, eq. (5.6)] and [13, Proposition 3.2]) and fix $\theta = 1/2$, in order to get the next *Caccioppoli-type* estimate

$$\|D^{2b}\boldsymbol{u}\|_{p,\lambda;\mathcal{C}_{r/2}} \le C(\|\mathfrak{L}\boldsymbol{u}\|_{p,\lambda;\mathcal{C}_r} + Cr^{-2b}\|\boldsymbol{u}\|_{p,\lambda;\mathcal{C}_r}), \tag{7}$$

which holds for $\|D_t\boldsymbol{u}\|_{p,\lambda;\mathcal{C}_{r/2}}$ as well by virtue of the parabolic structure of (1).

Therefore (7) implies (4) by means of a finite covering of Q by cylinders $\mathcal{C}_{r/2}$, $r < \operatorname{dist}(\Omega', \partial\Omega'')$.

REMARK 3.1. Employing the representation formula (5) and suitable homotopy arguments (cf. [13, Section 3]) it is possible to prove also that the operator $\mathfrak{L}$ *improves* *integrability.* Namely, being in the framework of Theorem 2.2 let $q \in [p, \infty)$ and suppose $\boldsymbol{u} \in W^{2b,1}_{q,\mathrm{loc}}(Q)$ be a strong solution of (1) such that $\boldsymbol{u}(x, 0) = \mathbf{0}$. Then $\mathfrak{L}\boldsymbol{u} \in L^{p,\lambda}_{\mathrm{loc}}(Q)$ implies $\boldsymbol{u} \in W^{2b,1}_{p,\lambda,\mathrm{loc}}(Q)$.

Proof of Corollary 2.3. It relies on the next result which is a parabolic version of the classical Poincaré inequality.

Lemma 3.2. *Let $\boldsymbol{u} \in W^{2b,1}_p(\mathcal{C}_r)$ where $\mathcal{C}_r$ is any parabolic cylinder* (3). *Then for each $s \in \{0, 1, \dots, 2b-1\}$ there is a constant $C = C(p, m, n, s)$ such that*

$$\int_{\mathcal{C}_r} |D^s\boldsymbol{u}(x,t) - (D^s\boldsymbol{u})_{\mathcal{C}_r}|^p\, dxdt \le C(r^{(2b-s)p}(\|D^{2b}\boldsymbol{u}\|^p_{L^p(\mathcal{C}_r)} + \|D_t\boldsymbol{u}\|^p_{L^p(\mathcal{C}_r)}) + r^p\|D^{s+1}\boldsymbol{u}\|^p_{L^p(\mathcal{C}_r)}).$$

To begin with, let $s = 2b-1$. Direct calculations based on Lemma 3.2 lead to

$$\frac{1}{r^{p+\lambda}}\int_{Q_r} |D^{2b-1}\boldsymbol{u}(x,t) - (D^{2b-1}\boldsymbol{u})_{Q_r}|^p\, dxdt \le C(\|D^{2b}\boldsymbol{u}\|^p_{p,\lambda;\tilde{Q}'} + \|D_t\boldsymbol{u}\|^p_{p,\lambda;\tilde{Q}'}),$$

where $Q_r = \mathcal{C}_r \cap Q'$ with $2r < \mathbf{dist}\,(\Omega', \partial\Omega'')$, $\tilde{Q}' = \tilde{\Omega}' \times (0, T)$ and $\Omega' \Subset \tilde{\Omega}' \Subset \Omega''$. Taking the supremum with respect to r above we get the *Campanato seminorm* of $D^{2b-1}\boldsymbol{u}$ on the left-hand side which, in view of (4), turns out to be bounded by the Morrey norms of $\mathfrak{L}\boldsymbol{u}$ and $\boldsymbol{u}$ in Q''. Now, employing the embedding properties of Campanato classes on spaces of homogeneous type recently proved in [11] (cf. also [3, 5] in the case $b = 1$) we obtain as follows. If $p + \lambda < n + 2b$ then $D^{2b-1}\boldsymbol{u} \in L^{p,p+\lambda}(Q')$ and $\|D^{2b-1}\boldsymbol{u}\|_{p,p+\lambda;Q'}$ is controlled in terms of $\|\mathfrak{L}\boldsymbol{u}\|_{p,\lambda;Q''}$ and $\|\boldsymbol{u}\|_{p,\lambda;Q''}$. If $p + \lambda > n + 2b$ then $D^{2b-1}\boldsymbol{u} \in C^{\sigma_{2b-1},\sigma_{2b-1}/2b}(Q')$ with $\sigma_{2b-1} = 1 - \dfrac{n+2b-\lambda}{p}$ (cf. [11, Corollary 1]). Finally, if $p + \lambda = n + 2b$ we first apply the Hölder inequality to $\|D^{2b-1}\boldsymbol{u} - (D^{2b-1}\boldsymbol{u})_{\mathcal{C}_r}\|_{L^1(Q_r)}$ and then Lemma 3.2 in order to get $D^{2b-1}\boldsymbol{u} \in BMO(Q')$.

The proof of Corollary 2.3 completes in the same manner, running induction for decreasing values of s.

Remark 3.3. The above results are easily extendable, modulo unessential technicalities, to parabolic systems with lower order terms

$$D_t\boldsymbol{u} - \sum_{|\alpha|=2b} \boldsymbol{A}_\alpha(x,t)D^\alpha\boldsymbol{u} + \sum_{|\beta|\le 2b-1} \boldsymbol{B}_\beta(x,t)D^\beta\boldsymbol{u} = \boldsymbol{f}(x,t)$$

with $\boldsymbol{B}_\beta(x,t) = \{b^{kj}_\beta(x,t)\}^m_{k,j=1}$ and b^{kj}_β belonging to suitable Morrey classes.

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