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Carmela Currò, Domenico Fusco

Discontinuous travelling wave solutions for a class of dissipative hyperbolic models

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Fisica matematica. — *Discontinuous travelling wave solutions for a class of dissipative hyperbolic models*. Nota (*) di Carmela Currò e Domenico Fusco, presentata dal Socio T. Ruggeri.

Abstract. — Discontinuous shock structure solutions for a general system of balance laws is considered in order to investigate the problem of connecting two equilibrium states lying on different sides of a singular barrier representing a locus of irregular singular points for travelling waves. Within such a theoretical setting a governing system of monoatomic gas is considered.

Key words: Shocks; Singular phase plane; Travelling waves.

Riassunto. — *Soluzioni d'onda discontinue per una classe di sistemi iperbolici dissipativi*. Si considerano le soluzioni di tipo struttura d'urto per un sistema di equazioni di bilancio allo scopo di studiare la connessione tra due stati di equilibrio separati nello spazio delle fasi da una barriera singolare, rappresentante un luogo di punti di singolarità nello studio delle «travelling waves». Si considerano infine le equazioni che descrivono il bilancio di un gas monoatomico uni-dimensionale dedotte nell'ambito della Termodinamica Estesa.

1. Introduction and outline of the problem

In mathematical modelling irreversible phenomena within the theoretical framework of extended thermodynamics a prominent role is played by systems of balance laws of the form [1]

$$\partial_\alpha \boldsymbol{F}^\alpha(\boldsymbol{U}) = \boldsymbol{F}(\boldsymbol{U}) \tag{1}$$

where $\boldsymbol{U}, \boldsymbol{F}, \boldsymbol{F}^\alpha$ are R^N-column vectors, with $\boldsymbol{U}$ being the field depending variable, while $x^0 = t$ and x^i, $i = 1, 2, 3$, are respectively time and space coordinates. As in most applications the solutions of the system (1) are assumed to satisfy a supplementary conservation equation [1, 2]

$$\partial_\alpha \boldsymbol{h}^\alpha(\boldsymbol{U}) = \boldsymbol{\Sigma}(\boldsymbol{U}) \tag{2}$$

which in physical applications represents in fact an entropy-like law with entropy density $-\boldsymbol{h}^0(\boldsymbol{U})$ and entropy flux $-\boldsymbol{h}^i(\boldsymbol{U})$, $-\boldsymbol{\Sigma}(\boldsymbol{U})$ being a non negative entropy production.

Owing to the consistency of (1) and (2), the use of a well established approach [3-7], permits to introduce a new set of dependent variables $\boldsymbol{U}'$ (Lagrange Multipliers) such that

$$d\boldsymbol{h}^\alpha = \boldsymbol{U}' \cdot d\boldsymbol{F}^\alpha \qquad \boldsymbol{\Sigma} = \boldsymbol{U}' \cdot \boldsymbol{F} \leq \boldsymbol{0} \tag{3}$$

and four potential-like functions

$$\boldsymbol{h}'^\alpha = \boldsymbol{U}' \cdot \boldsymbol{F}^\alpha - \boldsymbol{h}^\alpha \tag{4}$$

such that

(*) Pervenuta in forma definitiva all'Accademia l'8 settembre 2004.

$$\boldsymbol{F}^a = \frac{\partial \boldsymbol{b}'^a}{\partial \boldsymbol{U}'} \tag{5}$$

whence (1) can be recast into the symmetric hyperbolic and conservative form

$$\partial_a \left(\frac{\partial \boldsymbol{b}'^a}{\partial \boldsymbol{U}'} \right) = \boldsymbol{F} \Longleftrightarrow \frac{\partial^2 \boldsymbol{b}'^a}{\partial \boldsymbol{U}' \partial \boldsymbol{U}'} \partial_a \boldsymbol{U}' = \boldsymbol{F}. \tag{6}$$

Systems belonging to the class (1) (or (6)) exhibit relevant prominent features in modeling wave propagation [6]. In particular, as in the theoretical setting of the wave hierarchies problems [8], we split $\boldsymbol{U}' \in R^N$ into two parts $\boldsymbol{U}' = (\boldsymbol{V}', \boldsymbol{W}')$, $\boldsymbol{V}' \in R^M$, $\boldsymbol{W}' \in R^{N-M}$ $(0 < M < N)$ and accordingly $\boldsymbol{F} = (\boldsymbol{f}, \boldsymbol{g})$ so that (6), in turn, adopts the form

$$\partial_a \left(\frac{\partial \boldsymbol{b}'^a}{\partial \boldsymbol{V}'} \right) = \boldsymbol{f}(\boldsymbol{V}', \boldsymbol{W}') \tag{7}$$

$$\partial_a \left(\frac{\partial \boldsymbol{b}'^a}{\partial \boldsymbol{W}'} \right) = \boldsymbol{g}(\boldsymbol{V}', \boldsymbol{W}'). \tag{8}$$

There was proved in [2] that given some assigned value $\boldsymbol{W}'_*(x^a)$ of $\boldsymbol{W}'$ any of the $2^N - 2$ subsystems arising from (7)-(8) is symmetric hyperbolic or in other words it inherits all the structural properties of the full original system. The latter property is relevant in studying subcharacteristic conditions and subshocks [2].

As in most applications, here our main concern is the case when system (7)-(8) consists of M conservative equations with $\boldsymbol{f} = 0$ and $N - M$ balance equations involving a source-like term $\boldsymbol{g}$.

In the following we choose as field variables the components of the main field, *i.e.* $\boldsymbol{u} \equiv \boldsymbol{U}' \equiv (\boldsymbol{v}, \boldsymbol{w})$, so that the system is equivalent to [9]

$$\frac{\partial \boldsymbol{V}(\boldsymbol{u})}{\partial t} + \frac{\partial \boldsymbol{P}^i(\boldsymbol{u})}{\partial x^i} = 0 \tag{9}$$

$$\frac{\partial \boldsymbol{W}(\boldsymbol{u})}{\partial t} + \frac{\partial \boldsymbol{R}^i(\boldsymbol{u})}{\partial x^i} = \boldsymbol{g}(\boldsymbol{u}) \tag{10}$$

where $\boldsymbol{V}, \boldsymbol{P}^i \in R^M$, $\boldsymbol{W}, \boldsymbol{R}^i \in R^{N-M}$

$$\boldsymbol{F}^0 = \begin{bmatrix} \boldsymbol{V} \\ \boldsymbol{W} \end{bmatrix}, \quad \boldsymbol{F}^i = \begin{bmatrix} \boldsymbol{P}^i \\ \boldsymbol{R}^i \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{g} \end{bmatrix}.$$

According to [1] we assume the system (9)-(10) (or equivalently (1)) to admit two equilibria

$$\boldsymbol{u}_0 = (\boldsymbol{v}_0, \boldsymbol{0}), \quad \boldsymbol{u}_1 = (\boldsymbol{v}_1, \boldsymbol{0}) \tag{11}$$

which satisfy the condition

$$\boldsymbol{D}(\boldsymbol{v}, \boldsymbol{w}) = \frac{1}{2} \left\{ \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{w}} + \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{w}} \right)^T \right\} \quad \textit{negative definite} \tag{12}$$

whereupon it is possible to show that the equilibria (11) maximize $\boldsymbol{\Sigma}$.

As far as dissipative shock wave processes are concerned, Boillat and Ruggeri [2] proved that a smooth shock structure solution connecting the two equilibria (11) does not exist when the shock velocity is greater than the maximum characteristic velocity. Under suitable assumptions on the equilibria $(\boldsymbol{v}_0, 0), (\boldsymbol{v}_1, 0)$, in this paper we show that the searched connection consists of a smooth solution relating one of the two equilibrium states with an intermediate non equilibrium state and of a shock. A general approach is outlined for the system of balance laws (9)-(10). Later a set of governing equations of non equilibrium monoatomic gas is considered.

2. Discontinuous Travelling Waves

For hyperbolic dissipative models relevant informations about shock wave behaviour are provided by shock structure solutions. Hence, in line with the analysis worked out in [10], we look for a regular solution of (9)-(10) of the form

$$(13) \qquad \boldsymbol{u} = \boldsymbol{u}(\varphi), \quad \varphi = x^i n_i - st, \quad s = \text{const}, \quad \boldsymbol{n} \equiv (n_i) = \text{const}$$

and such that

$$(14) \qquad \lim_{\varphi \to \pm\infty} \boldsymbol{u}(\varphi) = \begin{cases} \boldsymbol{u}_0 \\ \\ \boldsymbol{u}_1 \end{cases}; \quad \lim_{\varphi \to \pm\infty} \frac{d\boldsymbol{u}}{d\varphi} = 0.$$

Upon inserting (13) into (9)-(10) we have

$$(15) \qquad \frac{d}{d\varphi}(-s\boldsymbol{V}(\boldsymbol{u}) + \boldsymbol{P}_n(\boldsymbol{u})) = 0, \quad \boldsymbol{P}_n = \boldsymbol{P}^i n_i$$

$$(16) \qquad \frac{d}{d\varphi}(-s\boldsymbol{W}(\boldsymbol{u}) + \boldsymbol{R}_n(\boldsymbol{u})) = \boldsymbol{g}(\boldsymbol{u}), \quad \boldsymbol{R}_n = \boldsymbol{R}^i n_i.$$

Next by taking into account the boundary conditions (14), from (15) we get

$$(17) \qquad -s(\boldsymbol{V}(\boldsymbol{v}_1, 0) - \boldsymbol{V}(\boldsymbol{v}_0, 0)) + \boldsymbol{P}_n(\boldsymbol{v}_1, 0) - \boldsymbol{P}_n(\boldsymbol{v}_0, 0) = 0$$

i.e. the Rankine-Hugoniot equations for the sub-system arising from (9)-(10) by assigning the value $\boldsymbol{w} = 0$, namely

$$(18) \qquad \frac{\partial \boldsymbol{V}(\boldsymbol{v},0)}{\partial t} + \frac{\partial \boldsymbol{P}_i(\boldsymbol{v},0)}{\partial x^i} = 0.$$

Owing to the relations (17) the equilibria (11) of the full system (9)-(10) turn out to be also shock states for the sub-system (18) such that

$$(19) \qquad \boldsymbol{v}_1 = \boldsymbol{v}_1(\boldsymbol{v}_0, s), \quad \lim_{s \to 0} \boldsymbol{v}_1 = \boldsymbol{v}_0$$

namely $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ are the states ahead and behind the shock respectively. If $\lambda^{\max}(\boldsymbol{u})$ is the greatest of the characteristic velocities of the system (9)-(10), Boillat and Ruggeri proved that a C^1 shock wave structure propagating with velocity $s : s > \lambda^{\max}(\boldsymbol{u}_0)$ cannot exist; in other words smooth solutions of the form (13)-(14) may exist only if $s \leq \lambda^{\max}(\boldsymbol{u}_0)$. Therefore the value $\lambda^{\max}(\boldsymbol{u}_0)$ can be used to define a range of validity of a hyperbolic model of interest. For hyperbolic models based upon extended-thermodynamics Weiss

[11, 1] through numerical approaches pointed out that a singular surface (subshock) arises in front of the shock if $s > \lambda^{\max}(\boldsymbol{u}_0)$. Moreover there was also pointed out that there are systems (35-moments case) for which a second subshock appears as Mach number increases with respect to $\lambda^{\max}(\boldsymbol{u}_1)$ [11, 1].

The main aim of this work is to outline a systematical approach in order to investigate how the equilibria $(\boldsymbol{v}_0, 0)$, $(\boldsymbol{v}_1, 0)$ can be connected in the state space for a general system of balance laws (9)-(10). Later we prove that if $\lambda^{\max}(\boldsymbol{u}_0) < s < \lambda^{\max}(\boldsymbol{u}_1)$ the searched connection consists of a smooth solution relating one of the two equilibrium states with an intermediate non equilibrium state and of a shock.

Under the assumption

$$\det(\nabla_{\boldsymbol{v}}\boldsymbol{P}_n - s\nabla_{\boldsymbol{v}}\boldsymbol{V}) \neq 0 \tag{20}$$

the equation (15) yields $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{w})$ which inserted into (16) gives rise to

$$K(\boldsymbol{v}(\boldsymbol{w}), \boldsymbol{w})\frac{d\boldsymbol{w}}{d\varphi} = \boldsymbol{g}(\boldsymbol{v}(\boldsymbol{w}), \boldsymbol{w}) \tag{21}$$

where

$$K(\boldsymbol{v}(\boldsymbol{w}), \boldsymbol{w}) = (\nabla_{\boldsymbol{w}}\boldsymbol{R}_n - s\nabla_{\boldsymbol{w}}\boldsymbol{W}) - \\ -(\nabla_{\boldsymbol{v}}\boldsymbol{R}_n - s\nabla_{\boldsymbol{v}}\boldsymbol{W})(\nabla_{\boldsymbol{v}}\boldsymbol{P}_n - s\nabla_{\boldsymbol{v}}\boldsymbol{V})^{-1}(\nabla_{\boldsymbol{w}}\boldsymbol{P}_n - s\nabla_{\boldsymbol{w}}\boldsymbol{V}). \tag{22}$$

Bearing in mind (20), it is simple matter to ascertain that

$$\det(K(\boldsymbol{v}(\boldsymbol{w}), \boldsymbol{w})) = \det(\nabla_{\boldsymbol{u}}\boldsymbol{F}_n - s\nabla_{\boldsymbol{u}}\boldsymbol{F}), \quad \boldsymbol{F}_n = \boldsymbol{F}^i n_i \tag{23}$$

provided that s is not a characteristic velocity of the system (1), so that (21) can be recast into the normal form

$$\frac{d\boldsymbol{w}}{d\varphi} = (K(\boldsymbol{v}(\boldsymbol{w}), \boldsymbol{w}))^{-1}\boldsymbol{g}(\boldsymbol{v}(\boldsymbol{w}), \boldsymbol{w}). \tag{24}$$

Let s be a fixed value different from a characteristic speed of (1), the equation

$$\Delta(\boldsymbol{w}) = \det(K(\boldsymbol{v}(\boldsymbol{w}), \boldsymbol{w})) = 0 \tag{25}$$

defines in the state space a locus of irregular singular points [12, 13] where the solutions of (21) exhibit singularities. Such a locus is usually named *singular barrier* [14-16]. A C^1 shock-wave structure connecting in the state space the points $(\boldsymbol{v}_0, 0), (\boldsymbol{v}_1, 0)$ may exist only if the equilibria lie in the interior of the region bounded by the singular barrier; such a situation is related to the condition $s \leq \lambda^{\max}(\boldsymbol{u}_0)$. If the barrier lies between the two equilibria, as well known travelling wave solutions have no meaning and the resulting phase trajectories can cross the singular barrier only at non equilibrium points where $\boldsymbol{g}(\boldsymbol{v}(\boldsymbol{w}), \boldsymbol{w}) = 0$, $\nabla_u \boldsymbol{g}(\boldsymbol{v}(\boldsymbol{w}), \boldsymbol{w}) \neq 0$. The latter points are usually indicated as «holes in the wall» [17]. Furthermore, if $s > \lambda^{\max}(\boldsymbol{u}_1)$, the equilibria in point belong to the exterior of the region bounded by the singular barrier. That is the case where for the 35-moments model the occurence of a second subshock was evidentiated [11, 1].

In order to the equilibria lie on different sides of the singular barrier we require the shock speed s to fulfill the relation

$$(\Delta(\boldsymbol{w}))_{\boldsymbol{u}_0}(\Delta(\boldsymbol{w}))_{\boldsymbol{u}_1} < 0. \tag{26}$$

Let $\lambda_1(\boldsymbol{u}) < \lambda_2(\boldsymbol{u}) < \ldots < \lambda_N(\boldsymbol{u})$ be the real eigenvalues of the hyperbolic system (9)-(10). It is straightforward to ascertain that if s satisfies the Lax conditions [18]

$$\lambda_1(\boldsymbol{u}_1) <\ldots < \lambda_{p-1}(\boldsymbol{u}_1) < s < \lambda_p(\boldsymbol{u}_1) < \ldots < \lambda_N(\boldsymbol{u}_1) \tag{27}$$

$$\lambda_1(\boldsymbol{u}_0) <\ldots. < \lambda_p(\boldsymbol{u}_0) < s < \lambda_{p+1}(\boldsymbol{u}_0) < \ldots. < \lambda_N(\boldsymbol{u}_0) \tag{28}$$

$$p = 1, 2, \ldots, N$$

then the condition (26) holds.

In passing we notice that for the fastest wave $p = N$ we get $s > \lambda^{\max}(\boldsymbol{u}_0)$ whence in agreement with the analysis worked out in [10] there results that the equilibria cannot be connected in the state space by a C^1 regular solution. In view of investigating the possible connection of the two equilibria (11) we consider first the singular barrier as a shock surface for the full governing system (9)-(10) so that the connection in point can be completed by using appropriate Rankine-Hugoniot conditions for discontinuous plane wave solutions (13). The scope of our later analysis is to determine which of the two equilibria $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ can be connected through the singular barrier by means of R-H conditions to another state representing in the state space the end point or, alternatively, the initial point of a smooth travelling wave trajectory. If $\boldsymbol{u}_R$ and $\boldsymbol{u}_L$ represent, respectively, states ahead (unperturbed) and behind (perturbed) the singular barrier the following R-H conditions must be fulfilled

$$-s[\boldsymbol{V}] + [\boldsymbol{P}_n] = 0 \tag{29}$$

$$-s[\boldsymbol{W}] + [\boldsymbol{R}_n] = 0 \tag{30}$$

where $[\cdot] = (\cdot)_L - (\cdot)_R$ indicates the jump and $(\cdot)_R, (\cdot)_L$ are the limit values of a quantity across the shock front evaluated respectively in the unperturbed and perturbed state, furthermore s has to satisfy the Lax conditions

$$\lambda_1(\boldsymbol{u}_L) <\ldots < \lambda_{p-1}(\boldsymbol{u}_L) < s < \lambda_p(\boldsymbol{u}_L) < \ldots < \lambda_N(\boldsymbol{u}_L) \tag{31}$$

$$\lambda_1(\boldsymbol{u}_R) <\ldots. < \lambda_p(\boldsymbol{u}_R) < s < \lambda_{p+1}(\boldsymbol{u}_R) < \ldots. < \lambda_N(\boldsymbol{u}_R) \tag{32}$$

$$p = 1, 2, \ldots, N$$

whereupon

$$\lambda_p(\boldsymbol{u}_R) < s < \lambda_p(\boldsymbol{u}_L). \tag{33}$$

In line with the analysis developed hitherto, we focus our attention on two different cases concerning the states $\boldsymbol{u}_R$ and $\boldsymbol{u}_L$:

1. First we assume $\boldsymbol{u}_L = \boldsymbol{u}_1$ so that the Rankine Hugoniot conditions provide $\boldsymbol{u}_R = \boldsymbol{u}_R(\boldsymbol{u}_1, s)$ whereupon the searched connection between $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ consists of a smooth solution which starts from the unstable state $\boldsymbol{u}_0$ and terminates in $\boldsymbol{u}_R$ and of a shock which jumps from $\boldsymbol{u}_R$ to the steady state $\boldsymbol{u}_1$.

2. Let $\boldsymbol{u}_R = \boldsymbol{u}_0$ whence the relations (29)-(30) yield $\boldsymbol{u}_L = \boldsymbol{u}_L(\boldsymbol{u}_0, s)$ so that the resulting connection consists of a shock which jumps from $\boldsymbol{u}_0$ to $\boldsymbol{u}_L$ and a smooth solution starting from $\boldsymbol{u}_L$ and connecting to the stable state $\boldsymbol{u}_1$.

We remark that if $s > \lambda^{\max}(\boldsymbol{u}_0)$ case 1 cannot hold because, according to [10], C^1 solutions starting from $\boldsymbol{u}_0$ do not exist.

3. Monoatomic gas

According to Extended Thermodynamics, we consider the system of balance laws describing a non equilibrium monoatomic gas in one space dimension [1] with the right-hand side calculated for Maxwell molecules

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial}{\partial x^*}(\rho^* v^*) = 0 \tag{34}$$

$$\frac{\partial}{\partial t^*}(\rho^* v^*) + \frac{\partial}{\partial x^*}\left(\rho^* v^{*2} + \frac{3}{5}\rho^* T^* + \frac{3}{5}\sigma^*\right) = 0 \tag{35}$$

$$\frac{\partial}{\partial t^*}\left(\rho^* v^{*2} + \frac{9}{5}\rho^* T^*\right) + \frac{\partial}{\partial x^*}\left((v^{*2} + 3T^*)\rho^* v^* + \frac{6}{5}q^* + \frac{6}{5}\sigma^* v^*\right) = 0 \tag{36}$$

$$\frac{\partial}{\partial t^*}\left(\frac{10}{9}\rho^* v^2 + \sigma^*\right) + \frac{\partial}{\partial x^*}\left(\left(\frac{10}{9}v^{*2} + \frac{4}{3}T^*\right)\rho^* v^* + \frac{7}{3}\sigma^* v^* + \frac{8}{15}q^*\right) = -\rho^*\sigma^* \tag{37}$$

$$\begin{aligned}\frac{\partial}{\partial t^*}\left(\frac{5}{3}\rho^* v^{*3} + 5\rho^* T^* v^* + 2\sigma^* v^* + 2q^*\right) + \frac{\partial}{\partial x^*}\left(\left(\frac{5}{3}\rho^* v^{*2} + 8\rho^* T^* + 5\sigma^*\right)v^{*2} + \right.\\ \left. + \frac{32}{5}q^* v^* + \frac{3}{5}T^*(5\rho^* T^* + 7\sigma^*)\right) = -2\rho^*\left(\sigma^* v^* + \frac{2}{3}q^*\right)\end{aligned} \tag{38}$$

where ρ^*, v^*, T^*, q^* are, respectively, the dimensionless mass density, velocity, absolute temperature, heat flux, while σ^* is the dimensionless component of the pressure deviator related to the physical variables $\rho, v, T, q, \sigma = -t_{\langle 11 \rangle}$ through the relations:

$$\begin{cases} \rho^* = \dfrac{\rho}{\rho_0}, \quad v^* = \dfrac{v}{a_0}, \quad T^* = \dfrac{T}{T_0}, \quad \sigma^* = \dfrac{5\sigma}{3\rho_0 a_0^2}, \quad q^* = \dfrac{5q}{3\rho_0 a_0^3}, \\ x^* = \dfrac{\rho_0 \alpha}{a_0}x, \quad t^* = \rho_0 \alpha t, \quad a_0^2 = \dfrac{5k}{3m}T_0. \end{cases} \tag{39}$$

As is well known for monoatomic gas the pressure p and the internal energy e are related by the equation $p = \frac{2}{3}\rho e = \frac{k}{m}\rho T$, α is a constant for Maxwellian molecules.

A strict wave analysis was worked out in [19, 1] in order to validate the governing model (34)-(38). As far as acceleration waves and hyperbolicity regions are concerned, the characteristic polynomial is given by

$$\widehat{\lambda}\left\{\widehat{\lambda}^4 - \left(\frac{62}{15}\widehat{\sigma} + \frac{78}{25}\right)\widehat{\lambda}^2 - \frac{96}{25}\widehat{q}\widehat{\lambda} + \frac{27}{25} + \frac{18}{5}\widehat{\sigma} + \frac{21}{5}\widehat{\sigma}^2\right\} = 0 \tag{40}$$

where

$$\widehat{\lambda} = \frac{\lambda^* - v^*}{\sqrt{T^*}}, \quad \widehat{\sigma} = \frac{3}{5}\frac{\sigma^*}{\rho^* T^*}, \quad \widehat{q} = \frac{3}{5}\frac{q^*}{\rho^* \sqrt{T^{*3}}}, \quad \lambda^* = \frac{\lambda}{a_0} \tag{41}$$

being λ the characteristic wave speed.

Among others there was shown [1], that by assuming a state close to equilibrium the equation (40) provides real characteristic speeds.

Let $(\rho_0, v_0, T_0, 0, 0)$ denote a generic equilibrium state. We consider a plane wave propagating with dimensionless velocity $s^* = \dfrac{s}{a_0}$ and introduce the relative velocity u^* and the Mach number M_0

$$u^* = s^* - v^*, \quad M_0 = s^* - v_0^*, \quad z = x^* - s^* t^*, \quad s^* > 0. \tag{42}$$

In line with the theoretical framework and the method of approach outlined in Section 2, by identifying

$$\boldsymbol{v} = \begin{bmatrix} \rho^* \\ \sigma^* \\ q^* \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} u^* \\ T^* \end{bmatrix} \tag{43}$$

we get

$$\rho^* = \frac{M_0}{u^*} \tag{44}$$

$$\sigma^* = 1 + \frac{5}{3} M_0 (M_0 - u^*) - M_0 \frac{T^*}{u^*} \tag{45}$$

$$q^* = -\frac{M_0}{2}\left(\frac{5}{3}(M_0 - u^*)^2 - 3T^* + 5\right) + u^* \tag{46}$$

and the system (21) specializes to the set of equations

$$\left(-\frac{14}{3} M_0 u^* + 3M_0^2 + \frac{9}{5}\right)\frac{du^*}{dz} - \frac{9}{5} M_0 \frac{dT^*}{dz} = \frac{M_0}{u^*}\left(1 + \frac{5}{3} M_0 (M_0 - u^*) - M_0 \frac{T^*}{u^*}\right) \tag{47}$$

$$\begin{aligned}
&\left(-2M_0 u^{*2} - \left(\frac{7}{3} M_0^2 + \frac{7}{5}\right) u^* - \frac{34}{5} M_0 T^* + \frac{8}{3} M_0 \left(M_0^2 + 3\right) + \frac{3}{5} M_0 \frac{T^{*2}}{u^{*2}}\right)\frac{du^*}{dz} + \\
&+\left(-\frac{34}{5} M_0 u^* + \frac{7}{2}\left(M_0^2 + \frac{3}{5}\right) - \frac{6}{5} M_0 \frac{T^*}{u^*}\right)\frac{dT^*}{dz} = \\
&= \frac{M_0}{9u^*}\left(-10 M_0 u^{*2} + \left(3 + 5M_0^2\right) u^* + M_0\left(15 + 5M_0^2 - 18T^*\right)\right).
\end{aligned}$$

Once M_0 is fixed, a direct inspection of (47), shows that the equilibria are given by

$$S_0 \equiv (M_0, 1), \quad S_1 \equiv \left(\frac{M_0^2 + 3}{4M_0}, \frac{\left(M_0^2 + 3\right)\left(5M_0^2 - 1\right)}{16 M_0^2}\right). \tag{48}$$

The singular barrier is defined by the condition

$$\Delta(u^*,T^*)=\frac{422}{15}M_0^2u^{*2}-\frac{614}{15}M_0\left(M_0^2+\frac{3}{5}\right)u^*-\frac{166}{25}M_0^2T^*-\tag{49}$$

$$-\frac{18}{5}M_0\left(M_0^2+\frac{3}{5}\right)\frac{T^*}{u^*}+\frac{27}{25}M_0^2\frac{T^{*2}}{u^{*2}}+\frac{153}{10}M_0^4+27M_0^2+\frac{189}{50}=0.$$

The equation (49) evaluated at the equilibria (48) reduces respectively to

$$\Delta_0=\frac{5}{2}M_0^4-\frac{39}{5}M_0^2+\frac{27}{10}\tag{50}$$

$$\Delta_1=\frac{31}{16}M_0^4-\frac{303}{40}M_0^2+\frac{243}{80}.\tag{51}$$

The vanishing of Δ_0 and Δ_1 gives, respectively, the following values for the Mach number

$$M_0=\pm 0.62972.... \qquad M_0=\pm 1.65029....$$

$$M_0=\pm 0.67351.... \qquad M_0=\pm 1.85905....$$

hence it is straightforward to ascertain that in the range

$$1.65029...<M_0<1.85905...\tag{52}$$

the condition (26) is fulfilled. As pointed out in [1], we notice that condition (52) corresponds to $s>\lambda^{\max}(\boldsymbol{u}_0)$. According to Weiss [1, pp. 277-308] in the range (52) the equilibria (48) are stable nodes.

The Rankine-Hugoniot conditions (29)-(30) associated to the full governing system are given by

$$[\rho^*u^*]=0\tag{53}$$

$$\left[\rho^*u^{*2}+\frac{3}{5}\rho^*T^*+\frac{3}{5}\sigma^*\right]=0\tag{54}$$

$$\left[2q^*-5\rho^*T^*u^*-2\sigma^*u^*-\frac{5}{3}\rho^*u^{*3}\right]=0\tag{55}$$

$$\left[\frac{8}{15}q^*+\frac{4}{3}\rho^*T^*u^*+\frac{7}{3}\sigma^*u^*+\frac{10}{9}\rho^*u^{*3}\right]=0\tag{56}$$

$$\left[\frac{5}{3}\rho^*u^{*4}+3\rho^*T^{*2}+\frac{21}{5}T^*\sigma^*+\frac{32}{5}u^*q^*+u^{*2}(8\rho^*T^*+5\sigma^*)\right]=0.\tag{57}$$

In view of connecting the two equilibria lying in opposite sides of the singular barrier and along the lines of investigation indicated in Section 1, bearing in mind the range (52) the Rankine-Hugoniot conditions (53)-(57) for $\boldsymbol{u}_0=\boldsymbol{u}_R$, give rise to

$$\rho_L^* = \frac{M_0}{u_L^*} \tag{58}$$

$$T_L^* = -\frac{35}{27}u_L^{*2} + \frac{5}{3M_0}\left(M_0^2 + \frac{3}{5}\right)u_L^* - \frac{10}{27}M_0^2 \tag{59}$$

$$\sigma_L^* = \frac{10}{27}\frac{M_0}{u_L^*}\left(M_0^2 - u_L^{*2}\right) \tag{60}$$

$$q_L^* = -\frac{25}{18}M_0\left(M_0^2 + \frac{9}{5}\right) + \frac{25}{6}u_L^*\left(M_0^2 + \frac{3}{5}\right) - \frac{25}{9}M_0 u_L^{*2} \tag{61}$$

with u_L^* and M_0 satisfying the equation:

$$3470M_0^2u_L^{*3} - 3550M_0^3u_L^{*2} - 4212M_0u_L^{*2} + \\ +\left(729 + 2106M_0^2 + 715M_0^4\right)u_L^* + 40M_0^5 = 0. \tag{62}$$

Since M_0 obeys the restriction (52) as proved in [19] the Lax conditions

$$s > \lambda_4(\boldsymbol{u}_0) > \lambda_3(\boldsymbol{u}_0) > \lambda_2(\boldsymbol{u}_0) > \lambda_1(\boldsymbol{u}_0)$$

$$\lambda_4(\boldsymbol{u}_L) > s > \lambda_3(\boldsymbol{u}_L) > \lambda_2(\boldsymbol{u}_L) > \lambda_1(\boldsymbol{u}_L)$$

are fulfilled. Thus within the theoretical framework outlined in section 1 it is possible to connect the two equilibria by a shock which jumps from $\boldsymbol{u}_0$ to $\boldsymbol{u}_L$ through the singular barrier and by a smooth orbit which starts from $\boldsymbol{u}_L$ and terminates in the stable state $\boldsymbol{u}_1$.

As illustrative example, for a given numerical value of M_0 the behaviour of the resulting trajectory in the phase plane as well as travelling wave profiles are reported in figs. 1, 2.

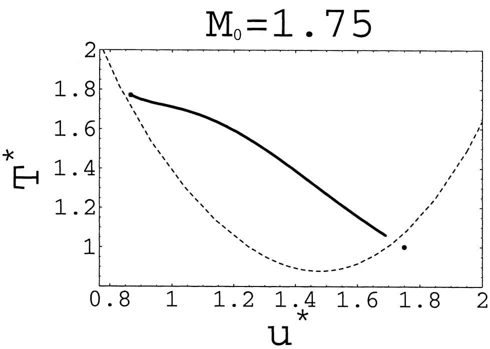

Fig. 1. – Phase plane plot of a discontinuous travelling wave solution $M_0 = 1.75$, $u_L^* = 1.69$, $T_L^* = 1.06$. The solid line is the phase trajectory starting from $\left(u_L^*, T_L^*\right)$ and connecting to the stable state S_1. The dotted line is the singular barrier which lies between the two equilibria S_0, S_1 (marked by $\cdot$). A jump connects S_1 to $\left(u_L^*, T_L^*\right)$ through the singular barrier.

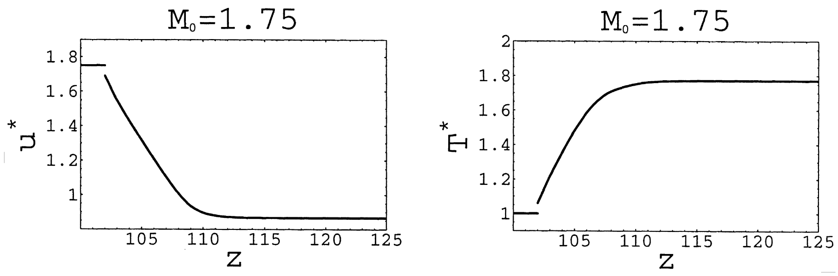

Fig. 2. – Solution profiles of a discontinuous travelling wave. The smooth solutions start, respectively, at $u_L^* = 1.69$, $T_L^* = 1.06$ and terminate in the stable states u_1^*, T_1^*. The discontinuities in u^* and T^* are pointed out.

4. Conclusions and general remarks

Since a C^1 shock wave structure propagating with velocity $s > \lambda^{\max}(\boldsymbol{u}_0)$ cannot exist [2], a strict approach was developed in order to investigate discontinuous travelling wave solutions and to establish the connection of the two equilibria (48) assumed to lie on different sides of the singular barrier. The latter one in the state space represents a locus of points where the solutions of (21) exhibit singularities. The singular barrier was considered as a shock surface for the full governing system (9)-(10) so that appropriate Rankine-Hugoniot conditions for discontinuous plane wave solutions (13) were used. Thus the searched connection through the singular barrier was achieved by means of a smooth solution relating one of the equilibria with an intermediate non-equilibrium state and by a shock. Furthermore the method of approach in point was used to investigating a dissipative model of a monoatomic gas based upon the theory of Extended Thermodynamics. The connection of the resulting equilibria was discussed in terms of the Mach number and some illustrative plots of trajectory's behaviour in the phase plane and of travelling wave profiles were given.

Finally, about the equilibria (48) a remark is in order. Owing to the leading modeling assumptions of the Extended Thermodynamics, if $s > \lambda^{\max}(\boldsymbol{u}_0)$ and condition (12) is required to be fulfilled the case 1 of Section 2 cannot hold [2]. However, for systems of governing equations (9)-(10) arising from different theoretical frameworks where the resulting equilibria are not subjected to (12), the case 1 is to be taken into account [20].

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Dipartimento di Matematica
Università degli Studi di Messina
Salita Sperone, 31 - 98166 MESSINA
curro@mat520.unime.it
fusco@mat520.unime.it