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Global boundary regularity for the $\overline{partial}$-equation on q-pseudo-convex domains

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Analisi matematica. — *Global boundary regularity for the $\bar\partial$-equation on q-pseudoconvex domains.* Nota (*) di Heungju Ahn, presentata dal Socio C. De Concini.

Abstract. — For a bounded domain D of $\mathbb{C}^n$, we introduce a notion of «q-pseudoconvexity» of new type and prove that for a given $\bar\partial$-closed (p, r)-form f that is smooth up to the boundary on D, and for $r \geq q$, there exists a $(p, r-1)$-form u smooth up to the boundary on D which is a solution of the equation $\bar\partial u = f$.

Key words: $\bar\partial$-equation; q-pseudoconvexity; Cauchy-Riemann system.

Riassunto. — *Regolarità globale per il sistema $\bar\partial$ sopra domini q-pseudoconvessi di $\mathbb{C}^n$.* Si introduce una nuova nozione di «q-pseudoconvessità» per un dominio D di $\mathbb{C}^n$. Per un tale D, e per ogni forma $\bar\partial$-chiusa f di tipo (p, r) con $r \geq q$, che è C^∞ fino al bordo di D, si prova che esiste una forma u anch'essa C^∞ in $\overline{D}$ che risolve l'equazione $\bar\partial u = f$.

1. Introduction and main result

For a domain D of $\mathbb{C}^n$ we introduce the property of boundary global regularity. This means that for any $\bar\partial$-closed form f which is smooth up to the boundary of D, we can find a solution u of $\bar\partial u = f$ which is also smooth up to the boundary. Kohn [3] completely solved this problem on pseudoconvex domains by introducing the $\bar\partial$-Neumann operator with weight in order to gain higher differentiability: The equation $\bar\partial u = f$ for $\bar\partial f = 0$ is solvable for forms f of any degree $r \geq 1$. His method also applies to the so-called strictly q-pseudoconvex domains: in this case the $\bar\partial$-equation is solvable in any degree $r \geq q$. When the domain is neither pseudoconvex nor strictly q-pseudoconvex, very little is known. We introduce a quite general condition of weak q-pseudoconvexity that can be naturally applied to the global boundary regularity problem.

Let D be a bounded domain in $\mathbb{C}^n$ with smooth boundary bD and let ρ be its defining function. For a given boundary point $z_0 \in bD$, we consider a boundary complex frame which means an orthonormal basis $\omega^1, \dots, \omega^n = \partial\rho$ of (1,0)-forms with C^∞ coefficients on a small neighborhood U of z_0. We denote by $(\rho_{jk}(z))_{j,k=1}^{n-1}$ the matrix of the Levi form $\partial\bar\partial\rho(z)$ in the complex tangential direction at z with respect to the basis $\omega^1, \dots, \omega^n$. Let $\lambda_1(z) \leq \cdots \leq \lambda_{n-1}(z)$ be the eigenvalues of $(\rho_{jk}(z))_{j,k=1}^{n-1}$. We assume that for a suitable choice of the boundary complex frame $\omega^1, \dots, \omega^n$, there exists an integer q with $1 \leq q \leq n-1$ such that for all r with $q \leq r \leq n-1$ and for some q_0 with $0 \leq q_0 < q$ we have

$$(1) \qquad \sum_{j=1}^{r} \lambda_j(z) \geq \sum_{j=1}^{q_0} \rho_{jj}(z), \quad z \in bD \cap U$$

(here we conventionally set $\sum_{j=1}^{q_0} \rho_{jj}(z) = 0$ if $q_0 = 0$). When 1 holds we say that D is q-

(*) Pervenuta in forma definitiva all'Accademia il 20 settembre 2004.

pseudoconvex at z_0. We say that D is *q-pseudoconvex* when it is q-pseudoconvex at every boundary point. Now we can state our main theorem.

THEOREM 1.1. *Let D be a bounded q-pseudoconvex domain in $\mathbb{C}^n$. Then for every $\bar\partial$-closed form $f \in C^\infty_{(p,r)}(\overline{D})$, with $q \le r \le n$, we can find $u \in C^\infty_{(p,r-1)}(\overline{D})$ satisfying $\bar\partial u = f$.*

Ho [2] introduced another kind of q-convexity: There is an integer q with $1 \le q \le n-1$ such that for all boundary point z we have $\sum_{j=1}^q \lambda_j(z) \ge 0$. Under this condition he proved global boundary regularity for any degree $r \ge q$. Note here that the q-convexity by Ho is a special case of the q-pseudoconvexity in our sense. Zampieri also [6] introduced q-pseudoconvexity as follows: Under a suitable choice of the frame $\omega^1, \dots, \omega^n$, we have

$$(2) \qquad (\rho_{jk}(z))_{j,k\le q-1} \le 0, \quad (\rho_{jk}(z))_{j,k=q}^{n-1} \ge 0, \quad \rho_{jk}(z) = 0 \text{ if } j \le q-1,\ q \le k.$$

This clearly implies q-pseudoconvexity. Under this condition he proved local boundary regularity for any degree $r \ge q$. A large class of q-pseudoconvex domains is given by the domains whose Levi form has a constant number of $q-1$ negative eigenvalues (cf. [5]). For these domains 2 is trivially satisfied.

2. THE WEIGHTED $\bar\partial$-NEWMANN OPERATORS

If $f, g \in L^2_{(p,q)}(D)$ and $\varphi \in C(\overline{D})$ the weighted L^2-inner product and norms are defined by

$$(f,g)_\varphi = \int_D \langle f, g\rangle e^{-\varphi}\, dV \quad \text{and} \quad \|f\|^2_\varphi = (f,f)_\varphi,$$

where dV is the volume element and $\langle f, g\rangle$ is the inner product on (p,q)-forms induced by the hermitian metric. We denote the formal adjoint of $\bar\partial$ by ϑ_φ so that $(f, \bar\partial g)_\varphi = (\vartheta_\varphi f, g)_\varphi$ for every $g \in C^\infty_{(p,q-1)}(D)$ with compact support in D. We write $f = \sum'_{|I|=p,|J|=q} f_{I,J}\, \omega^I \wedge \bar\omega^J$ where $\sum'$ denotes sum over ordered multi-indices I and J. It follows from the definitions of $\bar\partial$ and ϑ_φ that

$$(3) \qquad \bar\partial f = \sum_{I,J}{}' \sum_{k=1}^n \overline{L}_k f_{I,J}\, \bar\omega^k \wedge \omega^I \wedge \bar\omega^J + \cdots = Af + \cdots$$

$$(4) \qquad \vartheta_\varphi f = (-1)^p \sum_{\substack{|I|=p,\\ |K|=q-1}}{}' \sum_{j=1}^n \delta_j^\varphi f_{I,jK}\, \omega^I \wedge \bar\omega^K + \cdots = Bf + \cdots,$$

Here $\delta_j^\varphi u = e^\varphi L_j(e^{-\varphi} u)$, and the dots indicate terms in which no $f_{I,J}$ and $f_{I,jK}$ are differentiated and which do not involve φ. With these notations we can prove the following Kohn-Hörmander type inequality (we refer to [1] in case D is pseudoconvex in usual sense).

PROPOSITION 2.1. *If* $f = \sum'_{I,J} f_{I,J}\,\omega^I \wedge \bar{\omega}^J \in C^\infty_{(p,q)}(\overline{D}) \cap$ *with* $\operatorname{supp} f \in \overline{D} \cap U$ *and* $\varphi \in C^2(\overline{D})$, *then we have for every integer s with* $0 \le s \le n-1$

$$2(\|\bar{\partial} f\|^2_\varphi + \|\vartheta_\varphi f\|^2_\varphi) + C\|f\|_\varphi \ge \frac{1}{2} \sum_{I,J}{}' \left[\sum_{j \ge s+1} \|\overline{L}_j f_{I,J}\|^2_\varphi + \sum_{j \le s} \|\delta^\varphi_j f_{I,J}\|^2_\varphi \right] +$$

$$+ \sum_{I,K}{}' \sum_{j,k} (\varphi_{jk} f_{I,\,jK}, f_{I,\,kK})_\varphi - \sum_{I,J}{}' \sum_{j \le s} (\varphi_{jj} f_{I,J}, f_{I,J})_\varphi +$$

$$+ \sum_{I,K}{}' \sum_{j,k} \int_{bD} \rho_{jk} f_{I,jK} \overline{f_{I,kK}}\, e^{-\varphi}\, dS - \sum_{I,J}{}' \sum_{j \le s} \int_{bD} \rho_{jj} |f_{I,J}| e^{-\varphi}\, dS.$$

From now on we fix $\varphi_t(z) = t|z|^2,\ t \ge 0$ and use the notation $\|\cdot\|_{(t)} = \|\cdot\|_{\varphi_t}$, $(\ ,\ )_{(t)} = (\ ,\ )_{\varphi_t}$ and $\vartheta_t = \vartheta_{\varphi_t}$ and etc. The above Kohn-Hörmander type inequality implies the L^2-existence for the $\bar{\partial}$. Let $\Box_t = \bar{\partial}\bar{\partial}^*_t + \bar{\partial}^*_t\bar{\partial}$ and take $f \in \operatorname{Dom}(\Box_t)$ of degree $r \ge q$, then we have for every $t > C$

$$(t - C)\|f\|^2_{(t)} \le 4(\|\bar{\partial} f\|^2_{(t)} + \|\bar{\partial}^*_t f\|^2_{(t)}) = 4(\Box_t f, f)_{(t)} \le 4\|\Box_t f\|_{(t)}\|f\|_{(t)}$$

or

$$(t - C)\|f\|_{(t)} \le 4\|\Box_t f\|_{(t)}, \quad f \in \operatorname{Dom}(\Box_t).$$

This means that Range $(\Box_t)$ is closed and $\Box_t$ is injective. Therefore we can establish the existence theorem of the inverse of $\Box_t$ the so called weighted $\bar{\partial}$-Neumann operator N_t.

THEOREM 2.2. *Let D be a bounded q-pseudoconvex domain in* $\mathbb{C}^n$. *Then for any* $q \le r \le n$ *and* $t > C$, *there exists a bounded operator* $N_t : L^2_{(p,r)}(D) \to L^2_{(p,r)}(D)$ *with the following properties:*

(i) Range $(N_t) \subset \operatorname{Dom}(\Box_t)$, $\quad N_t\Box_t = \Box_t N_t = I \quad$ *on* Dom $(\Box_t)$,

(ii) $f = \bar{\partial}\bar{\partial}^*_t N_t f \oplus \bar{\partial}^*_t \bar{\partial} N_t f, \quad f \in L^2_{(p,r)(D)}$,

(iii) $\bar{\partial} N_t = N_t \bar{\partial}$, $q \le r \le n-1$ *and* $\bar{\partial}^*_t N_t = N_t \bar{\partial}^*_t$, $q \le r \le n$, $r \ge 2$

(iv) *For all* $f \in L^2_{(p,r)(D)}$ *we have the estimates*

$$(t - C)\|N_t f\|_{(t)} \le 4\|f\|_{(t)},$$

$$\sqrt{t - C}\|\bar{\partial} N_t f\|_{(t)} + \sqrt{t - C}\|\bar{\partial}^*_t N_t f\|_{(t)} \le 4\|f\|_{(t)},$$

(v) *If* $\bar{\partial} f = 0$, *then* $u_t = \bar{\partial}^*_t N_t f$ *solves the equation* $\bar{\partial} u_t = f$.

3. A PRIORI ESTIMATES AND PROOF OF THE MAIN THEOREM

For nonnegative integer s we define Sobolev space $H^s_{(p,r)}(D) = \{f \in L^2_{(p,r)}(D) : \|f\|_s < +\infty\}$, where the Sobolev norm of order s is defined by

$$\|f\|_s^2 = \sum_{|\alpha|\le s} \int_D |D^\alpha f|^2 e^{-\varphi_t}\, dV.$$

THEOREM 3.1. *If* $f \in C^\infty_{(p,r)}(\overline{D})$ *with* $r \geq q$ *and* $N_t f \in C^\infty_{(p,r)}(\overline{D})$*, then for any nonnegative integer s there exist constants* C_s *and* T_s *so that for every* $t > T_s$ *we have*

$$\|N_t f\|_s \leq C_s \|f\|_s. \tag{5}$$

PROOF. The proof is the same as in [3].

With this a priori estimates and the elliptic regularization method which was used in [3] we can also prove the following actual estimates.

THEOREM 3.2. *For every integer* $s \geq 0$ *and real* $t > T_s > 0$ *the weighted* $\bar\partial$*-Neumann operator* N_t *is bounded from* $H^s_{(p,r)}(D)$ *into itself for* $q \leq r \leq n$*.*

By Theorem 2.2 (v), Theorem 3.2 and the density of $C^\infty_{(p,r)}(\overline{D})$ in $H^s_{(p,r)}(D)$ the following is immediate.

COROLLARY 3.3. *If* $f \in H^s_{(p,r)}(D)$*,* $s = 0, 1, 2, \dots$ *satisfies* $\bar\partial f = 0$*, where* $q \leq r \leq n$*, then there exists* $u \in H^s_{(p,r-1)}(D)$ *so that* $\bar\partial u = f$ *on D with the estimate* $\|u\|_s \leq C_s \|f\|_s$*.*

END OF PROOF OF THEOREM 1.1. We note that for any $r \geq q$ and for any $k = 0, 1, \dots$ there are solutions $u_k \in H^k_{(p,r-1)}(D)$ of $\bar\partial u_k = f$ such that

$$\|u_k - u_{k+1}\|_k \leq 2^{-k}.$$

This is a consequence of Corollary 3.3 through a sophisticated inductive argument due to [4, p. 230].

Setting $u_\infty = u_N + \sum_{k=N}^\infty (u_{k+1} - u_k)$, we have $u_\infty \in H^N_{(p,r-1)}(D)$ for every $N \in \mathbb{N}$. By the Sobolev embedding theorem, $u_\infty \in C^\infty_{(p,r-1)}(\overline{D})$.

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