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On the geometry of moduli of curves and line bundles

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Geometria algebrica. — *On the geometry of moduli of curves and line bundles.* Nota (*) di Claudio Fontanari, presentata dal Socio C. De Concini.

Abstract. — Here we focus on the geometry of $\overline{P}_{d,g}$, the compactification of the universal Picard variety constructed by L. Caporaso. In particular, we show that the moduli space of spin curves constructed by M. Cornalba naturally injects into $\overline{P}_{d,g}$ and we give generators and relations of the rational Picard group of $\overline{P}_{d,g}$, extending previous work by A. Kouvidakis.

Key words: Universal Picard variety; Geometric invariant theory; Spin curve; Stable curve.

Riassunto. — *Sulla geometria degli spazi di moduli di curve e fibrati in rette.* Il presente lavoro è dedicato alla geometria di $\overline{P}_{d,g}$, la compattificazione della varietà di Picard universale costruita da L. Caporaso. In particolare, si dimostra che lo spazio dei moduli delle curve spin costruito da M. Cornalba si mappa iniettivamente in $\overline{P}_{d,g}$ e si esibiscono generatori e relazioni del gruppo di Picard razionale di $\overline{P}_{d,g}$, estendendo un precedente risultato di A. Kouvidakis.

1. Introduction

The universal Picard variety $P_{d,g}$ is the coarse moduli space for line bundles of degree d on smooth algebraic curves of genus g. Even though one is mainly interested in the behaviour of line bundles on smooth curves, nevertheless it is often useful to control their degenerations on singular curves. Perhaps the most celebrated example of proof by degeneration is provided by the Brill-Noether-Petri theorem (see [2, and the references therein]). Another very recent achievement of degeneration techniques is the proof given by L. Caporaso and E. Sernesi (see [7, 8]) that a general curve of genus $g \geq 3$ can be recovered from its odd theta-characteristics. In particular, in order to control degenerations of curves with prescribed theta-characteristics, a key rôle is played in [8] by the moduli space of spin curves $\overline{S}_g$ constructed by M. Cornalba in [9]. This perspective suggests the deep mathematical interest (both in itself and as a tool) of a geometrically meaningful compactification of the moduli spaces parameterizing pairs of curves and line bundles. Let $\overline{P}_{d,g}$ denote the compactification of $P_{d,g}$ constructed by L. Caporaso in [4] via geometric invariant theory. The boundary points of $\overline{P}_{d,g}$ correspond to certain line bundles on Deligne-Mumford semistable curves, while all previously known compactifications of the generalized Jacobian of an integral nodal curve used torsion free sheaves of rank one. From this point of view, a strict analogy emerges between $\overline{P}_{d,g}$ and $\overline{S}_g$: even though the techniques used in the two constructions are completely different, in both cases the resulting compactification is given in terms of line bundles on the same kind of singular curves. We will see that this analogy has a precise explanation: namely, in Section 3 we introduce a subscheme of $\overline{P}_{d,g}$ which compactifies

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the locus in $P_{d,g}$ corresponding to curves with theta characteristics and we investigate how it is related to $\overline{S}_g$. Indeed, after having established the existence of a natural morphism from $\overline{S}_g$ to $\overline{P}_{d,g}$ (see Theorem 1), we prove that it is an injection and we give an explicit combinatorial description of its image and of a locus where it is an immersion (see Theorem 2 and Theorem 3). As pointed out in [6] (see Theorem 5.4.2 and its Corollary), the above properties definitely do not hold for moduli spaces of higher spin curves, which were introduced by T.J. Jarvis in a rather different style (see [14-16]). In Section 4, instead, we obtain a complete description of the divisor class group of $\overline{P}_{d,g}$ (see Theorem 5) and of its rational Picard group (see Corollary 1). The strategy of proof is straightforward: first of all, we deduce from the basic properties of $\overline{P}_{d,g}$ a rough description of its boundary (see Proposition 4); next, we recall a theorem proved by A. Kouvidakis in [17] on the Picard group of $P_{d,g}$ (see Theorem 4). Hence the result on generation follows and in order to exclude nontrivial relations we simply lift to $\overline{P}_{d,g}$ the families of curves constructed by E. Arbarello and M. Cornalba in [1].

We work over the field $\mathbb{C}$ of complex numbers.

2. Notation and preliminaries

Let X be a Deligne-Mumford semistable curve and let E be a complete, irreducible subcurve of X. One says that E is *exceptional* if it is smooth, rational, and meets the other components in exactly two points. Moreover, one says that X is *quasistable* if any two distinct exceptional components of C are disjoint. In the sequel, $\tilde{X}$ will denote the subcurve $\overline{X \setminus \cup E_i}$ obtained from X by removing all exceptional components.

A *spin curve* of genus g (see [9, 10]) is the datum of a quasistable genus g curve X with an invertible sheaf ζ_X of degree $g-1$ on X and a homomorphism of invertible sheaves

$$\alpha_X : \zeta_X^{\otimes 2} \longrightarrow \omega_X$$

such that

(*i*) ζ_X has degree 1 on every exceptional component of X;

(*ii*) α_X is not zero at a general point of every non-exceptional component of X.

From the definition it follows that α_X vanishes identically on all exceptional components of X and induces an isomorphism

$$\tilde{\alpha}_X : \zeta_X^{\otimes 2}|_{\tilde{X}} \longrightarrow \omega_{\tilde{X}}.$$

In particular, when X is smooth, ζ_X is just a theta-characteristic on X.

By definition (see [9, § 2]), two spin curves (X, ζ_X, α_X) and $(X', \zeta_{X'}, \alpha_{X'})$ are *isomorphic* if there are isomorphisms $\sigma : X \to X'$ and $\tau : \sigma^*(\zeta'_X) \to \zeta_X$ such that τ is compatible with the natural isomorphism between $\sigma^*(\omega_{X'})$ and ω_X. However, we point out the following fact.

Lemma 1. *Let (X, ζ_X, α_X) and $(X', \zeta_{X'}, \alpha_{X'})$ be two spin curves and assume that there are isomorphisms $\sigma : X \to X'$ and $\tau : \sigma^*(\zeta_{X'}) \to \zeta_X$. Then (X, ζ_X, α_X) and $(X', \zeta_{X'}, \alpha_{X'})$ are isomorphic as spin curves.*

PROOF. Let $\beta_X : \zeta_X^{\otimes 2} \to \omega_X$ be defined by the following commutative diagram:

$$\begin{array}{ccc} \zeta_X^{\otimes 2} & \xrightarrow{\beta_X} & \omega_X \\ \uparrow \tau^{\otimes 2} & & \uparrow \cong \\ (\sigma^* \zeta_{X'})^{\otimes 2} = \sigma^*(\zeta_{X'}^{\otimes 2}) & \xrightarrow{\sigma^* \alpha_{X'}} & \sigma^* \omega_{X'}. \end{array}$$

Then (X, ζ_X, β_X) is a spin curve, which is isomorphic to $(X', \zeta_{X'}, \alpha_{X'})$ by definition and to (X, ζ_X, α_X) by [9, Lemma (2.1)], so the claim follows. □

A *family of spin curves* is a flat family of quasistable curves $f : \mathcal{X} \to S$ with an invertible sheaf ζ_f on $\mathcal{X}$ and a homomorphism

$$\alpha_f : \zeta_f^{\otimes 2} \longrightarrow \omega_f$$

such that the restriction of these data to any fiber of f gives rise to a spin curve.

Two families of spin curves $f : \mathcal{X} \to S$ and $f' : \mathcal{X}' \to S$ are *isomorphic* if there are isomorphisms $\sigma : \mathcal{X} \longrightarrow \mathcal{X}'$ and $\tau : \sigma^*(\zeta_{f'}) \longrightarrow \zeta_f$ such that $f = f' \circ \sigma$ and τ is compatible with the natural isomorphism between $\sigma^*(\omega_{f'})$ and ω_f (see [10 p. 212]).

Let $\overline{\mathcal{S}}_g$ be the contravariant functor from schemes to sets, which to every scheme S associates the set $\overline{\mathcal{S}}_g(S)$ of isomorphism classes of families of spin curves of genus g.

Let $\overline{S}_g$ be the set of isomorphism classes of spin curves of genus g and S_g be the subset consisting of classes of smooth curves. One can define a natural structure of analytic variety on $\overline{S}_g$ (see [9, §5]) and from [9, Proposition (4.6)], it follows that $\overline{S}_g$ is a coarse moduli variety for $\overline{\mathcal{S}}_g$.

Let now $g \geq 3$. For every integer d, there is a *universal Picard variety*

$$\psi_d : P_{d,g} \longrightarrow \mathcal{M}_g^0$$

whose fiber $J^d(X)$ over a point X of $\mathcal{M}_g^0$ parametrizes line bundles on X of degree d modulo isomorphism.

Assume $d \geq 20(g-1)$, but notice that this is not a real restriction because of the natural isomorphism $P_{d,g} \cong P_{d+n(2g-2),g}$. Then $P_{d,g}$ has a natural compactification

$$\phi_d : \overline{P}_{d,g} \longrightarrow \overline{\mathcal{M}}_g$$

such that $\phi_d^{-1}(\mathcal{M}_g^0) = P_{d,g}$. Namely, let $\mathrm{Hilb}_{d-g}^{dx-g+1}$ be the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^{d-g}$ having Hilbert polynomial $dx - g + 1$, fix $G = SL(d-g+1)$ and set $H_d := \{h \in \mathrm{Hilb}_{d-g}^{dx-g+1} : h$ is G-semistable and the corresponding curve is connected$\}$. Then $\overline{P}_{d,g}$ was constructed in [4] as a GIT quotient

$$\pi_d : H_d \longrightarrow H_d/G = \overline{P}_{d,g}.$$

Moreover, one can define (see [4, § 8.1]) the contravariant functor $\overline{\mathcal{P}}_{d,g}$ from schemes to sets, which to every scheme S associates the set $\overline{\mathcal{P}}_{d,g}(S)$ of equivalence classes of polarized families of quasistable curves of genus g

$$f : (\mathcal{X}, \mathcal{L}) \longrightarrow S$$

such that $\mathcal{L}$ is a relatively very ample line bundle of degree d whose multidegree satisfies the following Basic Inequality on each fiber.

Definition 1. Let $X = \bigcup_{i=1}^{n} X_i$ be a projective, nodal, connected curve of arithmetic genus g, where the X_i's are the irreducible components of X. We say that the multidegree $(d_1, \ldots, d_n)$ satisfies the Basic Inequality if for every complete subcurve Y of X of arithmetic genus g_Y we have

$$m_Y \leq d_Y \leq m_Y + k_Y$$

where

$$d_Y = \sum_{X_i \subseteq Y} d_i$$

$$k_Y = |Y \cap \overline{X \setminus Y}|$$

$$m_Y = \frac{d}{g-1}\left(g_Y - 1 + \frac{k_Y}{2}\right) - \frac{k_Y}{2}$$

(see [4, pp. 611 and 614]).

Two families over S, $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}', \mathcal{L}')$ are *equivalent* if there exists an S-isomorphism

$$\sigma : \mathcal{X} \longrightarrow \mathcal{X}'$$

and a line bundle M on S such that

$$\sigma^* \mathcal{L}' \cong \mathcal{L} \otimes f^* M.$$

By [4, Proposition 8.1], there is a morphism of functors:

$$\overline{\mathcal{P}}_{d,g} \longrightarrow \mathbf{Hom}(\ .\ ,\ \overline{P}_{d,g}) \tag{1}$$

and $\overline{P}_{d,g}$ coarsely represents $\overline{\mathcal{P}}_{d,g}$ if and only if

$$(d - g + 1, 2g - 2) = 1. \tag{2}$$

The relationship between $\overline{S}_g$ and $\overline{P}_{d,g}$ can be expressed as follows.

Theorem 1. *For every integer* $t \geq 10$ *there is a natural morphism:*

$$f_t : \overline{S}_g \longrightarrow \overline{P}_{(2t+1)(g-1),g}.$$

Proof. First of all, notice that in this case (2) does not hold, so the points of $\overline{P}_{d,g}$ are *not* in one-to-one correspondence with equivalence classes of very ample line bundles of degree d on quasistable curves, satisfying the Basic Inequality (see [4, p. 654]). However, we claim that the result can be deduced from the existence of a morphism of functors:

$$F_t : \overline{\mathcal{S}}_g \longrightarrow \overline{\mathcal{P}}_{(2t+1)(g-1),g}. \tag{3}$$

Indeed, since $\overline{S}_g$ coarsely represents $\overline{\mathcal{S}}_g$, any morphism of functors $\overline{\mathcal{S}}_g \to \mathbf{Hom}(\ .\ ,\ S)$

induces a morphism of schemes $\overline{S}_g \to S$, so the claim follows from (1). Now, a morphism of functors as (3) is the datum for any scheme S of a set-theoretical map

$$F_t(S) : \overline{\mathcal{S}}_g(S) \longrightarrow \overline{\mathcal{P}}_{(2t+1)(g-1),g}(S),$$

satisfying obvious compatibility conditions. Let us define

$$F_t(S)([f : \mathcal{X} \to S, \zeta_f, \alpha_f]) := [f : (\mathcal{X}, \zeta_f \otimes \omega_f^{\otimes t}) \to S].$$

In order to prove that $F_t(S)$ is well-defined, the only non-trivial matter is to check that the multidegree of $\zeta_f \otimes \omega_f^{\otimes t}$ satisfies the Basic Inequality on each fiber, so the result follows from the next Lemma. □

Lemma 2. *If Y is a complete subcurve of X and d_Y is the degree of $\zeta_X \otimes \omega_X^{\otimes t}|_Y$, then $m_Y \leq d_Y \leq m_Y + k_Y$ in the notation of the Basic Inequality. Moreover, if we set*

$$\tilde{k}_Y := |\tilde{Y} \cap \overline{\tilde{X} \setminus \tilde{Y}}|,$$

we have $d_Y = m_Y$ if and only if $\tilde{k}_Y = 0$ and all exceptional components in Y do not intersect $\overline{X \setminus Y}$, and $d_Y = m_Y + k_Y$ if and only if $\tilde{k}_Y = 0$ and all exceptional components in $\overline{X \setminus Y}$ do not intersect Y.

Proof. Let $Y_1, \ldots, Y_\upsilon$ be the irreducible components of Y, of arithmetic genus $g_1, \ldots, g_\upsilon$ respectively. We may assume that the first $\tilde{\upsilon}$ ones are non-exceptional and the last $(\upsilon - \tilde{\upsilon})$ ones are exceptional, so that $\tilde{Y} = Y_1 \cup \ldots \cup Y_{\tilde{\upsilon}}$. Next, let $\{p_1, \ldots p_\delta\}$ be the points of intersection between two distinct irreducible components of Y. Again, we may assume that the first $\tilde{\delta}$ ones involve two non-exceptional components and the last $(\delta - \tilde{\delta})$ ones are between a non-exceptional and an exceptional component. We have

$$g_Y = \sum_{i=1}^{\upsilon} g_i + \delta - \upsilon + 1 = \sum_{i=1}^{\tilde{\upsilon}} g_i + \delta - \upsilon + 1$$

and since $\zeta_X^{\otimes 2}|_{\tilde{Y}} \cong \omega_{\tilde{X}}|_{\tilde{Y}}$ we may compute

$$\deg \zeta_X|_{\tilde{Y}} = \frac{1}{2} \deg \omega_{\tilde{X}}|_{\tilde{Y}} = \frac{1}{2}\left(\sum_{i=1}^{\tilde{\upsilon}} (2g_i - 2) + 2\tilde{\delta} + \tilde{k}_Y\right).$$

Hence we deduce

$$\begin{aligned} d_Y &= \deg\big(\zeta_X \otimes \omega_X^{\otimes t}\big)|_Y = \deg \zeta_X|_Y + t \deg \omega_X|_Y = \\ &= \deg \zeta_X|_{\tilde{Y}} + \deg \zeta_X|_{\overline{Y \setminus \tilde{Y}}} + t \deg \omega_X|_Y = \\ &= \frac{1}{2}\left(\sum_{i=1}^{\tilde{\upsilon}} (2g_i - 2) + 2\tilde{\delta} + \tilde{k}_Y\right) + (\upsilon - \tilde{\upsilon}) + t(2g_Y - 2 + k_Y) = \\ &= g_Y - 1 - (\delta - \tilde{\delta}) + 2(\upsilon - \tilde{\upsilon}) + \frac{\tilde{k}_Y}{2} + 2t\left(g_Y - 1 + \frac{k_Y}{2}\right). \end{aligned}$$

On the other hand,

$$m_Y = (2t+1)\left(g_Y - 1 + \frac{k_Y}{2}\right) - \frac{k_Y}{2} = 2t\left(g_Y - 1 + \frac{k_Y}{2}\right) + g_Y - 1,$$

so

$$d_Y = m_Y - (\delta - \tilde{\delta}) + 2(v - \tilde{v}) + \frac{\tilde{k}_Y}{2}$$

and the Basic Inequality is satisfied if and only if

$$0 \leq 2(v - \tilde{v}) - (\delta - \tilde{\delta}) + \frac{\tilde{k}_Y}{2} \leq k_Y.$$

Now, since every exceptional component meets the other components in exactly two points, there are obvious inequalities

$$\delta - \tilde{\delta} \leq 2(v - \tilde{v})$$

and

$$(\delta - \tilde{\delta}) + (k_Y - \tilde{k}_Y) \geq 2(v - \tilde{v}),$$

hence the claim follows. □

REMARK 1. If t_1 and t_2 are integers ≥ 10, then Lemma 8.1 of [4] yields an isomorphism $\tau : \overline{P}_{(2t_1+1)(g-1),g} \longrightarrow \overline{P}_{(2t_2+1)(g-1),g}$. We point out that by the definitions of τ (see [4, proof of Lemma 8.1]) and of f_t (see proof of Theorem 1) there is a commutative diagram:

$$\begin{array}{ccc} \overline{S}_g & \xrightarrow{f_{t_1}} & \overline{P}_{(2t_1+1)(g-1),g} \\ \| & & \downarrow \tau \\ \overline{S}_g & \xrightarrow{f_{t_2}} & \overline{P}_{(2t_2+1)(g-1),g}. \end{array}$$

□

3. SPIN CURVES IN $\overline{P}_{d,g}$

For every integer $t \geq 10$ we define

$$K_{(2t+1)(g-1)} := \{h \in \mathrm{Hilb}^{(2t+1)(g-1)x-g+1}_{(2t+1)(g-1)-g} : \text{ there is a spin curve } (X, \zeta_X, \alpha_X) \text{ and an embedding } h_t : X \to \mathbb{P}^{(2t+1)(g-1)-g} \text{ induced by } \zeta_X \otimes \omega_X^{\otimes t} \text{ such that } h = h_t(X)\}.$$

By applying [4, Proposition 6.1], from the first part of Lemma 2 we deduce

$$K_{(2t+1)(g-1)} \subset H_{(2t+1)(g-1)}$$

(the definition of H_d was recalled above in Section 2). Moreover, we claim that $K_{(2t+1)(g-1)}$ is a constructible set but not a scheme. Indeed, let

$$J_t := \pi^{-1}_{(2t+1)(g-1)}(\pi_{(2t+1)(g-1)}(K_{(2t+1)(g-1)})),$$

let A_t be the stable locus in J_t and let B_t the locus in J_t corresponding to strictly semistable points with closed orbits. Since A_t is open in J_t and B_t is closed in J_t, it turns out that $K_{(2t+1)(g-1)} = A_t \cup B_t$ is constructible. On the other hand, if $K_{(2t+1)(g-1)}$ were locally closed in J_t, it should be $K_{(2t+1)(g-1)} = C_t \cap U_t$ with C_t closed in J_t and U_t open in J_t. This would imply $J_t = \overline{A_t} \subseteq C_t \subseteq J_t$ and $K_{(2t+1)(g-1)} = U_t$, contradiction. We also point out that the second part of Lemma 2 provides a great amount of information on Hilbert points corresponding to spin curves.

Proposition 1. *Let (X, ζ_X, α_X) be a spin curve. Then $h_t(X)$ is GIT-stable if and only if $\tilde{X}$ is connected.*

Proof. We are going to apply the stability criterion of [4, Lemma 6.1], which says that $h_t(X)$ is GIT-stable if and only if the only subcurves Y of X such that $d_Y = m_Y + k_Y$ are union of exceptional components.

If $\tilde{X}$ is connected, then for every subcurve Y of X which is not union of exceptional components we have $\tilde{k}_Y > 0$, so from Lemma 2 it follows that $d_Y < m_Y + k_Y$ and $h_t(X)$ turns out to be GIT-stable.

If instead $\tilde{X}$ is not connected, pick any connected component Z of $\tilde{X}$ and take Y to be the union of Z with all exceptional components of X intersecting Z. It follows that $\tilde{k}_Y = 0$ and all exceptional components in $\overline{X \setminus Y}$ do not intersect Y, so Lemma 2 yields $d_Y = m_Y + k_Y$ and $h_t(X)$ is not GIT-stable. □

Proposition 2. *If (X, ζ_X, α_X) is a spin curve, then the orbit of $h_t(X)$ is closed in the semistable locus.*

Proof. Just recall the first part of [4, Lemma 6.1], which says that the orbit of $h_t(X)$ is closed in the semistable locus if and only if $\tilde{k}_Y = 0$ for every subcurve Y of X such that $d_Y = m_Y$, so the result is a direct consequence of Lemma 2. □

The sublocus of $\overline{P}_{(2t+1)(g-1),g}$ obtained by projection from $K_{(2t+1)(g-1)}$ is indeed the GIT analogue of $\overline{S}_g$ we are looking for. Namely, if we set

$$\Sigma_t := \pi_{(2t+1)(g-1)}(K_{(2t+1)(g-1)})$$

then the following holds.

Theorem 2. *The morphism f_t induces a bijection*

$$f_t : \overline{S}_g \hookrightarrow \Sigma_t$$

which is an immersion at all points corresponding to spin curves (X, ζ_X, α_X) such that $\tilde{X}$ is connected.

Proof. It is easy to check that $f_t(\overline{S}_g) = \Sigma_t$. Indeed, if $[(X, \zeta_X, \alpha_X)] \in \overline{S}_g$, then any choice of a base for $H^0(X, \zeta_X \otimes \omega_X^{\otimes t})$ induces an embedding $h_t : X \to \mathbb{P}^{(2t+1)(g-1)-g}$ and

$f_t([(X', \zeta_{X'}, \alpha_{X'})]) = \pi_{(2t+1)(g-1)}(h_t(X)) \in \Sigma_t$; conversely, if $\pi_{(2t+1)(g-1)}(h) \in \Sigma_t$, then there is a spin curve (X, ζ_X, α_X) and an embedding $h_t : X \to \mathbb{P}^{(2t+1)(g-1)-g}$ such that $h = h_t(X)$ and $f_t([(X', \zeta_{X'}, \alpha_{X'})]) = \pi_{(2t+1)(g-1)}(h)$.

Next we claim that f_t is injective. Indeed, let (X, ζ_X, α_X) and $(X', \zeta_{X'}, \alpha_{X'})$ be two spin curves and assume that $f_t([(X, \zeta_X, \alpha_X)]) = f_t([(X', \zeta_{X'}, \alpha_{X'})])$. Choose bases for $H^0(X, \zeta_X \otimes \omega_X^{\otimes t})$ and $H^0(X', \zeta_{X'} \otimes \omega_{X'}^{\otimes t})$ and embed X and X' in $\mathbb{P}^{(2t+1)(g-1)-g}$. If $h_t(X)$ and $h_t(X')$ are the corresponding Hilbert points, then $\pi_{(2t+1)(g-1)}(h(X)) = \pi_{(2t+1)(g-1)}(h(X'))$ and the Fundamental Theorem of GIT implies that $\overline{O_G(h_t(X))}$ and $\overline{O_G(h_t(X'))}$ intersect in the semistable locus. It follows from Proposition 2 that $O_G(h_t(X)) \cap O_G(h_t(X')) \neq \emptyset$, so $O_G(h_t(X)) = O_G(h_t(X'))$ and there are isomorphisms $\sigma : X \to X'$ and $\tau : \sigma^*(\zeta_{X'}) \to \zeta_X$. Now the claim follows from Lemma 1.

Finally, we are going to prove that f_t^{-1} is a morphism at all points in Σ_t corresponding to GIT-stable points in $K_{(2t+1)(g-1)}$. Recall that the Hilbert scheme $\mathrm{Hilb}_r^{p(x)}$ carries a universal polarized family $\mathcal{U} \xrightarrow{u} \mathrm{Hilb}_r^{p(x)}$ (see for instance [4] on p. 601). Since $\overline{S}_g$ coarsely represents $\overline{\mathcal{S}}_g$, there is a morphism of functors $\Xi : \overline{\mathcal{S}}_g \longrightarrow \boldsymbol{Hom}(\ .\ , \overline{S}_g)$. We look at the image under $\Xi(A_t) : \overline{\mathcal{S}}_g(A_t) \longrightarrow \boldsymbol{Hom}(A_t, \overline{S}_g)$ of $F_t(A_t)^{-1}([\mathcal{U}_{|A_t}])$, where F_t is the morphism of functors (3), $A_t \subset K_{(2t+1)(g-1)}$ is the stable locus defined above, and $\mathcal{U}$ is the universal family on $\mathrm{Hilb}_{(2t+1)(g-1)-g}^{(2t+1)(g-1)x-g+1}$. This construction yields a G-invariant morphism $A_t \longrightarrow \overline{S}_g$ which by universality of the GIT quotient induces the morphism

$$f_t^{-1} : \pi_{(2t+1)(g-1)}(A_t) \longrightarrow \overline{S}_g.$$

In order to conclude, just apply Proposition 1. □

REMARK 2. Maybe it could be useful to restate the above result in different words. Indeed, as pointed out in the Conclusion of [14], Cornalba's moduli space $\overline{S}_g$ can be identified with the special case of Jarvis' compactification parametrizing triples $(X, \mathcal{E}, b)$, where X is a stable curve, $\mathcal{E}$ is a rank one torsion free sheaf on X, and $b : \mathcal{E}^{\otimes 2} \to \omega_X$ is a suitable $\mathcal{O}_X$-module homomorphism (see [14, Definition 2.1.2]). On the other hand, as shown by Pandharipande (see [18, Theorem 8.2.1 and Theorem 10.3.1]), $\overline{P}_{d,g}$ parametrizes equivalence classes modulo automorphisms of torsion-free sheaves of rank one and degree d on stable curves of genus g. In this set-up, our Theorem 2 simply says that the natural forgetful map $(X, \mathcal{E}, b) \mapsto (X, \mathcal{E})$ is injective. On the other hand, we do not know whether the morphism f_t is an immersion everywhere. It is clear that the final part of our proof does not work with A_t replaced with $K_{(2t+1)(g-1)}$ since, as we have already remarked, $K_{(2t+1)(g-1)}$ is not a scheme. As a matter of fact, we are not able to find either an alternate proof or a counterexample.

Next we are going to derive an explicit combinatorial description of Σ_t. We omit the proof of the following easy Lemma, referring to the proof of Lemma 2 for a similar computation.

Lemma 3. *Let (X, ζ_X, α_X) be a spin curve. Fix a decomposition*

$$X = \bigcup_{i=1}^{n} \tilde{X}_i \cup \bigcup_{j=1}^{m} E_j$$

where the $\tilde{X}_i$'s are the irreducible components of $\tilde{X}$ and the E_j's are the exceptional components of X. Set $k_i := |\tilde{X}_i \cap X \setminus \tilde{X}_i|$ and $\tilde{k}_i := |\tilde{X}_i \cap \tilde{X} \setminus \tilde{X}_i|$. Then the multidegree of $\zeta_X \otimes \omega_X^{\otimes t}$ on X is $\underline{d} = (d_1, \ldots, d_n, d_{n+1}, \ldots, d_{n+m})$ with

$$\begin{aligned} d_i &= (2t+1)(p_a(\tilde{X}_i) - 1) + tk_i + \frac{1}{2}\tilde{k}_i & \quad 1 \leq i \leq n \\ d_{n+j} &= 1 & \quad 1 \leq j \leq m. \end{aligned}$$

□

As in [4, § 5.1], we set

$$M_C^{\underline{d}} := \{h \in H_d, h = \mathrm{hilb}(C, L) : \underline{\deg} \, L = \underline{d}\}$$

and

$$V_C^{\underline{d}} := \overline{M_C^{\underline{d}}} \cap H_d.$$

By [4, Corollary 5.1] (but see also on p. 627), if $[C] \in \overline{\mathcal{M}}_g$ then the $V_C^{\underline{d}}$'s are exactly the irreducible components of the fiber over $[C]$ of the natural morphism

$$\psi_d : H_d \longrightarrow \overline{\mathcal{M}}_g.$$

Proposition 3. *Let $\underline{d} = (d_1, \ldots, d_n)$ be a multidegree and let $C = \bigcup_{i=1}^n C_i$ be a stable curve, where the C_i's are the irreducible components of C. Set $g_i := p_a(C_i)$ and $k_{ij} := |C_i \cap C_j|$ if $i \neq j$, $k_{ij} := 0$ if $i = j$.*

Then there exists a spin curve (X, ζ_X, α_X) with an embedding $h_t : X \to \mathbb{P}^{(2t+1)(g-1)-g}$ induced by $\zeta_X \otimes \omega_X^{\otimes t}$ such that $h_t(X) \in V_C^{\underline{d}}$ if and only if for every $1 \leq i,j \leq n$ there are integers s_{ij} and σ_{ij} with

$$0 \leq s_{ij} \leq k_{ij} \qquad s_{ij} = s_{ji} \qquad \sum_{j=1}^{n} (k_{ij} - s_{ij}) \equiv 0 \text{ mod. } 2$$

$$0 \leq \sigma_{ij} \leq s_{ij} \qquad \sigma_{ij} + \sigma_{ji} = s_{ij}$$

such that

$$d_i = (2t+1)(g_i - 1) + t\sum_{j=1}^{n} k_{ij} + \frac{1}{2}\sum_{j=1}^{n} (k_{ij} - s_{ij}) + \sum_{j=1}^{n} \sigma_{ij}.$$

Proof. To get a quasistable curve X starting from $C = \bigcup_{i=1}^n C_i$, for every $1 \leq i,j \leq n$ choose r_i nodes of C_i and s_{ij} contact points between C_i and C_j and blow them up, by adding a smooth rational component connecting the branches. In the notation of Lemma 3, notice that $p_a(\tilde{X}_i) = g_i - r_i$, $k_i = \sum_{j=1}^n k_{ij} + 2r_i$ and $\tilde{k}_i = \sum_{j=1}^n (k_{ij} - s_{ij})$.

As pointed out in [9, § 3], in order for a spin curve having X as underlying curve to exist, a necessary and sufficient condition is that $\tilde{k}_i \equiv 0$ mod. 2 for every $1 \leq i \leq n$. Moreover, by [4, Proposition 5.1], $h_t(X) \in V_C^{\underline{d}}$ if and only if there is a partition $X = \bigcup_{i=1}^n X_i$ such that X_i is a complete connected subcurve of X whose stable model is C_i and $d_i = \deg_{X_i}(\zeta_X \otimes \omega_X^{\otimes t})$. So the claim follows from Lemma 3. □

Let $V_C := \bigcup_{\underline{d}} V_C^{\underline{d}}$ and let $\overline{P}_{d,C} := \phi_d^{-1}(C)$. By [4, proof of Corollary 5.1], we have

$$\overline{P}_{d,C} = V_C/G$$

and for every irreducible component I of $\overline{P}_{d,C}$ there is a unique multidegree $\underline{d}$ such that $V_C^{\underline{d}}$ dominates I via the quotient map

$$V_C \longrightarrow \overline{P}_{d,C}.$$

Theorem 3. *Let $C = \bigcup_{i=1}^n C_i$ be a stable curve, where the C_i's are the irreducible components of C. Set $g_i := p_a(C_i)$ and $k_{ij} := |C_i \cap C_j|$ if $i \neq j$, $k_{ij} := 0$ if $i = j$. Let I be an irreducible component of $\overline{P}_{d,C}$ and let $\underline{d} = (d_1, \dots, d_n)$ be the multidegree such that I is dominated by $V_C^{\underline{d}}$.*

Then there exists a spin curve (X, ζ_X, α_X) such that $f_t([(X, \zeta_X, \alpha_X)]) \in I$ if and only if for every $1 \leq i, j \leq n$ there are integers s_{ij} and σ_{ij} with

$$0 \leq s_{ij} \leq k_{ij} \qquad s_{ij} = s_{ji} \qquad \sum_{j=1}^n (k_{ij} - s_{ij}) \equiv 0 \text{ mod. } 2$$

$$0 \leq \sigma_{ij} \leq s_{ij} \qquad \sigma_{ij} + \sigma_{ji} = s_{ij}$$

such that

$$d_i = (2t+1)(g_i - 1) + t\sum_{j=1}^n k_{ij} + \frac{1}{2}\sum_{j=1}^n (k_{ij} - s_{ij}) + \sum_{j=1}^n \sigma_{ij}.$$

Proof. By the Fundamental Theorem of GIT,

$$f_t([(X, \zeta_X, \alpha_X)]) = \pi_{(2t+1)(g-1)}(h_t(X)) \in I$$

if and only if there is $h \in V_C^{\underline{d}}$ such that $\overline{O_G(h_t(X))}$ and $\overline{O_G(h)}$ intersect in the semistable locus.

Since $O_G(h_t(X))$ is closed in the semistable locus by Proposition 2, we have

$$O_G(h_t(X)) \cap \overline{O_G(h)} \neq \emptyset$$

and since $\overline{O_G(h)}$ is a union of orbits we may rephrase the above condition as

$$h_t(X) \in \overline{O_G(h)}.$$

On the other hand, we have $V_C^{\underline{d}} = \bigcup_{h \in V_C^{\underline{d}}} O_G(h)$ since $V_C^{\underline{d}}$ is G-invariant and $V_C^{\underline{d}} = \bigcup_{h \in V_C^{\underline{d}}} \overline{O_G(h)}$ since $V_C^{\underline{d}}$ is closed.

Summing up, we see that $f_t([(X, \zeta_X, \alpha_X)]) \in I$ if and only if $h_t(X) \in V_C^{\underline{d}}$. Now the claim follows from Proposition 3. □

EXAMPLE 1. Let C be a *split curve* of genus g, *i.e.* the union of two nonsingular rational curves meeting transversally at $g+1$ points. Such curves are particularly interesting, for a number of reasons (see [5, 8]). According to Theorem 3, Σ_t meets an irreducible component I of $\overline{P}_{d,C}$ if and only if I corresponds to a bidegree (d_1, d_2) with

$$d_1 = \left(t + \frac{1}{2}\right)(g+1) - (2t+1) - \frac{1}{2}s + \sigma$$
$$d_2 = \left(t + \frac{1}{2}\right)(g+1) - (2t+1) + \frac{1}{2}s - \sigma$$

where s and σ are nonnegative integers satisfying

$$\sigma \leq s \leq g+1 \qquad s \equiv g+1 \text{ mod. } 2.$$

Moreover, from the proof of Proposition 3 it follows that s is the number of exceptional components of a quasi-stable curve X underlying a spin curve (X, ζ_X, α_X) such that $f_t([(X, \zeta_X, \alpha_X)]) \in \Sigma_t \cap I$.

4. DIVISORS ON $\overline{P}_{d,g}$

In order to understand the boundary of $\overline{P}_{d,g}$, we recall the morphism $\phi_d : \overline{P}_{d,g} \longrightarrow \overline{\mathcal{M}}_g$ and the decomposition of the boundary of $\overline{\mathcal{M}}_g$ into its irreducible components:

$$\partial\overline{\mathcal{M}}_g = \Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_{\lfloor g/2 \rfloor}.$$

Next we define

$$D_i := \phi_d^{-1}(\Delta_i)$$

for $i = 0, \ldots \lfloor g/2 \rfloor$, and we notice that, since ϕ_d is surjective, each D_i turns out to be a divisor on $\overline{P}_{d,g}$. Moreover, if $X \in \overline{\mathcal{M}}_g$, we set as usual $\overline{P}_{d,X} := \phi_d^{-1}(X)$.

LEMMA 4. *For every i, if X is a general element in Δ_i then $\overline{P}_{d,X}$ is irreducible.*

PROOF. If $i \geq 1$, a general element X of Δ_i is the union of two smooth curves X_1 and X_2 meeting at one node, so X is of compact type and $\overline{P}_{d,X}$ is irreducible (see [4, footnote on p. 594]). If instead $i = 0$, a general element $X \in \Delta_0$ is an irreducible curve and also $\overline{P}_{d,X}$ turns out to be irreducible (see [4, 7.1]). □

As a consequence, we obtain a complete description of the boundary of $\overline{P}_{d,g}$.

PROPOSITION 4. *For every i, D_i is irreducible.*

PROOF. By [4, Corollary 5.1 (2)], for every $X \in \overline{\mathcal{M}}_g$ all irreducible components of $\phi_d^{-1}(X)$ have dimension g. Let I be an irreducible component of D_i; by applying the theorem on the dimensions of the fibers (see for instance [13, II, Ex. 3.22 (b), p. 95]) to the map

$$\phi_{d|I} : I \longrightarrow \phi_d(I) \subseteq \Delta_i$$

we obtain

$$\dim I - \dim \phi_d(I) \leq \dim (\phi_d^{-1}(X) \cap I) \leq g.$$

Hence

$$\begin{aligned} 3g - 4 = \dim \Delta_i \geq \dim \phi_d(I) &\geq 4g - 4 - \dim (\phi_d^{-1}(X) \cap I) \\ &\geq 3g - 4 \end{aligned}$$

and all the above inequalities turn out to be equalities. In particular, we have

$$\dim (\phi_d^{-1}(X) \cap I) = g$$

and $\phi_d^{-1}(X) \cap I$ is a closed subscheme of $\phi_d^{-1}(X)$ of maximal dimension. By Lemma 4 there is a dense open subset $U_i \subset \Delta_i$ such that $\phi_d^{-1}(X)$ is irreducible for every $X \in U_i$. It follows that $\phi_d^{-1}(X) \subset I$ for every $X \in U_i$ and $\phi_d^{-1}(U_i)$ is a dense open subset of I. Hence $I = \overline{\phi_d^{-1}(U_i)}$ is uniquely determined. □

We recall that the class of any line bundle $\mathcal{L}$ on $P_{d,g}$ restricted to a fiber $J^d(X)$ is a multiple $m\theta$ of the class θ of the Θ divisor (see [17, p. 840]); it seems therefore natural to define the *class* of $\mathcal{L}$ to be the integer m.

The Picard group of $P_{d,g}$ is completely described by the following result, due to A. Kouvidakis (see [17, Theorem 4, p. 849]).

THEOREM 4. *If $g \geq 3$ then the Picard group of the universal Picard variety $\psi_d : P_{d,g} \to \mathcal{M}_g^0$ is freely generated over $\mathbb{Z}$ by the line bundles $\mathcal{L}_{d,g}$ and $\psi_d^*(\lambda)$, where $\mathcal{L}_{d,g}$ is any line bundle on $P_{d,g}$ with class*

$$k_{d,g} = \frac{2g - 2}{\gcd (2g - 2, g + d - 1)}$$

and λ is the Hodge bundle on $\mathcal{M}_g^0$.

The analogous result for $\overline{P}_{d,g}$ is the following:

THEOREM 5. *Assume $g \geq 3$ and $d \geq 20(g - 1)$. Then the divisor class group of the universal Picard variety $\phi_d : \overline{P}_{d,g} \to \overline{\mathcal{M}}_g$ is freely generated over $\mathbb{Z}$ by the classes $\mathcal{L}_{d,g}$, $\phi_d^*(\lambda)$ and $\mathcal{D}_i$ ($i = 0, \ldots \lfloor g/2 \rfloor$), where $\mathcal{D}_i$ denotes the linear equivalence class of D_i and $\mathcal{L}_{d,g}$ is the class of the closure in $\overline{P}_{d,g}$ of any Weil divisor with class $\mathcal{L}_{d,g}$ on $P_{d,g}$.*

PROOF. Since $P_{d,g}$ is smooth and irreducible, there is a natural identification $\mathrm{Pic}(P_{d,g}) = A_{4g-4}(P_{d,g})$, where A_n denotes the Chow group of n-dimensional cycles modulo rational equivalence. Hence using the exact sequence

$$A_{4g-4}(\overline{P}_{d,g} \setminus P_{d,g}) \to A_{4g-4}(\overline{P}_{d,g}) \to A_{4g-4}(P_{d,g}) \to 0$$

we may deduce from Theorem 4 that $A_{4g-4}(\overline{P}_{d,g})$ is generated by $\mathcal{L}_{d,g}$, $\phi_d^*(\lambda)$ and $A_{4g-4}(\overline{P}_{d,g} \setminus P_{d,g})$. Now, we have

$$\overline{P}_{d,g} \setminus P_{d,g} = D_0 \cup \ldots \cup D_{\lfloor g/2 \rfloor} \cup \phi_d^{-1}(\mathcal{M}_g \setminus \mathcal{M}_g^0). \tag{4}$$

Moreover, by applying the theorem on the dimensions of the fibers to the map

$$\phi_{d|\phi_d^{-1}(\mathcal{M}_g \setminus \mathcal{M}_g^0)} : \phi_d^{-1}(\mathcal{M}_g \setminus \mathcal{M}_g^0) \longrightarrow \mathcal{M}_g \setminus \mathcal{M}_g^0$$

we obtain that $\operatorname{codim}(\phi_d^{-1}(\mathcal{M}_g \setminus \mathcal{M}_g^0), \overline{P}_{d,g}) \geq \operatorname{codim}(\mathcal{M}_g \setminus \mathcal{M}_g^0, \overline{\mathcal{M}}_g) = g-2$. If $g \geq 4$, then $\phi_d^{-1}(\mathcal{M}_g \setminus \mathcal{M}_g^0)$ cannot contain any divisorial component, so from (1) and Proposition 4 it follows that $A_{4g-4}(\overline{P}_{d,g} \setminus P_{d,g})$ is generated by the $\mathcal{D}_i$'s. If, instead, $g = 3$, we recall that the hyperelliptic locus H is the unique divisor in $\mathcal{M}_3$ contained in $\mathcal{M}_3 \setminus \mathcal{M}_3^0$ (see [12, Ex. 2.27, 3]). Since $[H] = 18\lambda$ in $\operatorname{Pic}(\mathcal{M}_3 \otimes \mathbb{Q})$ (see [12, p. 164]), we have $[\phi_d^{-1}(H)] = 18\phi_d^*(\lambda)$ and the result on generation is completely proved. As for relations, let

$$a\mathcal{L}_{d,g} + b\phi_d^*(\lambda) + \sum c_i \mathcal{D}_i = 0 \tag{5}$$

be a relation in $A_{4g-4}(\overline{P}_{d,g})$. If $J^d(X)$ is the fiber over a curve X in $\mathcal{M}_g^0$, then restricting (5) to $J^d(X)$ yields $ak_{d,g}\theta = 0$ and we get $a = 0$. So we may rephrase (5) as $\phi_d^*(b\lambda + \sum c_i\delta_i) = 0$. Recall now that a natural way to check that the Hodge class λ and the boundary classes δ_i on $\overline{\mathcal{M}}_g$ are independent is to construct families of stable curves $h : X \to S$, with S a smooth complete curve, such that the vectors $(\deg_h(\lambda), \deg_h(\delta_0), \ldots)$ are linearly independent. Such a construction is carried out in detail in [1] and it is applied in [9] to show the injectivity of $\chi^* : \operatorname{Pic}(\overline{\mathcal{M}}_g) \to \operatorname{Pic}(\overline{S}_g)$, where $\chi : \overline{S}_g \to \overline{\mathcal{M}}_g$ is the natural projection. We are going to mimic the same idea in our case. Namely, in order to prove that $\phi_d^*(\lambda)$ and the $\phi_d^*(\delta_i)$'s are independent in $A_{4g-4}(\overline{P}_{d,g})$, we will lift to $\overline{P}_{d,g}$ the families $h : X \to S$ constructed in [1]. The key observation is that each of them is equipped with many sections passing through the smooth locus of the general curve C of the family. Indeed, for every irreducible component C_i of C we easily find a section σ_i of h which cuts on C a smooth point $P_i \in C_i$. Next, we decompose the integer d as a sum of d_i's in such a way that the multidegree determined by the d_i's satisfies the Basic Inequality. Finally, we endow C with the line bundle $\mathcal{L} := \otimes_i \mathcal{O}_{C_i}(d_iP_i)$. Since $\overline{P}_{d,g}$ is proper over $\overline{\mathcal{M}}_g$, this construction uniquely determines a lifting of $h : X \to S$, so the proof is over. □

Corollary 1. *Assume $g \geq 3$ and $d \geq 20(g-1)$. Then $\operatorname{Pic}(\overline{P}_{d,g})$ is freely generated over $\mathbb{Q}$ by $\mathcal{L}_{d,g}$, $\phi_d^*(\lambda)$ and $\mathcal{D}_i$ ($i = 0, \ldots \lfloor g/2 \rfloor$).*

Proof. By [4, Lemma 2.2 (1)], $\overline{P}_{d,g}$ is the quotient of a nonsingular scheme, so in particular it is normal and there is an injection:

$$\operatorname{Pic}(\overline{P}_{d,g}) \hookrightarrow A_{4g-4}(\overline{P}_{d,g}).$$

On the other hand, since the pull-back of a $\mathbb{Q}$-Cartier divisor is $\mathbb{Q}$-Cartier, both $\phi_d^*(\lambda)$ and the $\mathcal{D}_i$'s are $\mathbb{Q}$-Cartier; moreover, since $\overline{P}_{d,g} \to \overline{\mathcal{M}}_g$ is projective, $\overline{P}_{d,g}$ carries a relatively ample line bundle, which by Theorem 1 has to be a linear combination of $\mathcal{L}_{d,g}$, $\phi_d^*(\lambda)$ and the $\mathcal{D}_i$'s such that the coefficient of $\mathcal{L}_{d,g}$ is non-zero. It follows that also $\mathcal{L}_{d,g}$ is $\mathbb{Q}$-Cartier and $A_{4g-4}(\overline{P}_{d,g}) \otimes \mathbb{Q} \subseteq \mathrm{Pic}(\overline{P}_{d,g}) \otimes \mathbb{Q}$, so the proof is over. $\square$

REMARK 3. For the sake of completeness, we point out that the rational Picard group of $\overline{S}_g$ for $g \geq 9$ has been explicitly determined in [3, Corollary 1], as a consequence of previous work by Cornalba [9, Proposition (7.2)] and Harer [11, Corollary 1.3]. Indeed, for any family $f : \mathcal{X} \to B$ of spin curves, $M_f := \det Rf_* \zeta_f$ is a line bundle on B. Let M denote the corresponding line bundle on $\overline{S}_g$ associated to the universal family on $\overline{S}_g$. It turns out that $\mathrm{Pic}(\overline{S}_g)$ is freely generated over $\mathbb{Q}$ by the set of boundary classes together with the class μ of M.

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