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Esteban Andruchow, Gustavo Corach

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Analisi funzionale. — *Metrics in the set of partial isometries with finite rank.* Nota (*)
di ESTEBAN ANDRUCHOW e GUSTAVO CORACH, presentata dal Socio M. Primicerio.

ABSTRACT. — Let $\mathcal{I}_{(\infty)}$ be the set of partial isometries with *finite* rank of an infinite dimensional Hilbert space $\mathcal{H}$. We show that $\mathcal{I}_{(\infty)}$ is a smooth submanifold of the Hilbert space $\mathcal{B}_2(\mathcal{H})$ of Hilbert-Schmidt operators of $\mathcal{H}$ and that each connected component is the set $\mathcal{I}_N$, which consists of all partial isometries of rank $N < \infty$. Furthermore, $\mathcal{I}_{(\infty)}$ is a homogeneous space of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$, where $\mathcal{U}(\infty)$ is the classical Banach-Lie group of unitary operators of $\mathcal{H}$, which are Hilbert-Schmidt perturbations of the identity. We introduce two Riemannian metrics in $\mathcal{I}_{(\infty)}$: one, via the ambient inner product of $\mathcal{B}_2(\mathcal{H})$, the other, by means of the group action. We show that both metrics are equivalent and complete.

KEY WORDS: Partial isometry; Projection; Riemannian metric.

RIASSUNTO. — *Metriche nell'insieme di isometrie parziali di rango finito.* Sia $\mathcal{I}_{(\infty)}$ l'insieme delle isometrie con rango *finito* di uno spazio di Hilbert $\mathcal{H}$ ad infinite dimensioni. Si prova che $\mathcal{I}_{(\infty)}$ è una sottovarietà regolare dello spazio di Hilbert $\mathcal{B}_2(\mathcal{H})$ degli operatori di Hilbert-Schmidt di $\mathcal{H}$, e che ciascuna componente connessa è l'insieme $\mathcal{I}_N$, che consiste di tutte le isometrie parziali di rango $N < \infty$. Inoltre, $\mathcal{I}_{(\infty)}$ è uno spazio omogeneo di $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$, dove $\mathcal{U}(\infty)$ è il classico gruppo di Banach-Lie degli operatori unitari di $\mathcal{H}$, che sono perturbazioni di Hilbert-Schmidt dell'identità. Si introducono due metriche Rimanniane in $\mathcal{I}_{(\infty)}$: una per mezzo del prodotto interno di $\mathcal{B}_2(\mathcal{H})$, l'altra utilizzando il gruppo d'azione. Si prova che le due metriche sono equivalenti e complete.

1. INTRODUCTION

There are several papers dealing with the geometry and the topology of the set $\mathcal{I}$ of partial isometries of a Hilbert space (see, for example [6, 9, 2, 1]). However, these papers usually endow $\mathcal{I}$ with the operator norm topology. The advantage of this approach is that it allows the study of the set $\mathcal{I}$ as a whole. A disadvantage is that the geometry provided by the operator norm is highly non Riemannian. In the present approach we deal with a smaller subset $\mathcal{I}_{(\infty)}$ of $\mathcal{I}$ which admits the structure of a Hilbertian Manifold. More precisely, this paper aims to understand the geometric structure of the set $\mathcal{I}_{(\infty)}$ of partial isometries with finite rank acting on an infinite dimensional Hilbert space $\mathcal{H}$. Note that $\mathcal{I}_{(\infty)}$ is a subset of the space $\mathcal{B}_2(\mathcal{H})$ of Hilbert-Schmidt operators, itself a Hilbert space with the trace inner product. It turns out that $\mathcal{I}_{(\infty)}$ is a C^∞ submanifold of $\mathcal{B}_2(\mathcal{H})$. This is proven by noting that two partial isometries which lie at distance less that 1 in $\mathcal{B}_2(\mathcal{H})$ have the same rank. Let us denote by $\mathcal{I}_N$ the set of partial isometries of rank $N < \infty$. Thus the local structure of $\mathcal{I}_{(\infty)}$ is that of the sets $\mathcal{I}_N$, $1 \leq N < \infty$. These sets $\mathcal{I}_N$ are the connected components of $\mathcal{I}_{(\infty)}$. Each set $\mathcal{I}_N$ carries a smooth transitive left action of the group $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$, where $\mathcal{U}(\infty)$ is the classical Banach-Lie group [4] of Hilbert-Schmidt perturbations of the identity. This action has local cross sections, a fact which implies the submanifold structure for $\mathcal{I}_N$ and $\mathcal{I}_{(\infty)}$.

(*) Pervenuta in forma definitiva all'Accademia il 22 giugno 2004.

Two Riemannian metrics can be defined in $\mathcal{I}_{(\infty)}$. First, the one induced by the ambient inner product of $\mathcal{B}_2(\mathcal{H})$, called here *ambient metric*. Second, the one pushed forward by the inner product metric of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$ on the quotient structure of each component $\mathcal{I}_N$, called *homogeneous metric*. We show that both metrics differ but are equivalent, with bounds that do not depend on the rank N. We also show that these metrics are complete in the stronger sense of the term [8]. Notice that each $\mathcal{I}_N$ is an *infinite* dimensional manifold, and therefore there are several, in general non equivalent, notions of completeness [8]. The manifold $\mathcal{I}_N$ is a complete metric space in the metric given by the (minima of) lengths of smooth curves.

The curves $\gamma(t) = e^{tX}Ve^{-tY}$, $V \in \mathcal{I}_N$, and X, Y in the Lie algebra of $\mathcal{U}(\infty)$, need not be geodesics of the homogeneous metric. This is because the homogeneous space $\mathcal{I}_N$ is not a symmetric. These curves are geodesics of the ambient metric only if X, Y satisfy a quadratic relation, which turns out to be equivalent to a system of two linear Rosenblum-type operator equations. We show in an appendix that this system in general does not have a solution, *i.e.*, the curves γ need not be geodesics of the ambient connection neither.

There are two interesting submanifolds of $\mathcal{I}_N$: the set $\mathcal{P}_N$ of projections with rank N, and, for a fixed $P \in \mathcal{P}_N$, the unitary group $\mathcal{U}(P(\mathcal{H}))$ of the N-dimensional space $P(\mathcal{H})$ (isomorphic to $\mathcal{U}(N)$, the group of unitary $N \times N$ matrices). The ambient metric for these submanifolds induces their usual Riemannian metrics. We show, via the quadratic relation cited above, that the geodesics of these manifolds are geodesics of $\mathcal{I}_N$. This fact plays a key role in the proof of the completeness of $\mathcal{I}_N$.

2. Differentiable structure of $\mathcal{I}_N$

Fix a positive integer $N < \infty$, and let $\mathcal{I}_N$ be the set of partial isometries of the Hilbert space $\mathcal{H}$, with rank N. Denote by $\mathcal{P}_N$ the set of selfadjoint projections of rank N. Then $\mathcal{P}_N \subset \mathcal{I}_N$. Let $\mathcal{B}_2(\mathcal{H})$ be the (Hilbert) space of Hilbert-Schmidt operators, *i.e.*

$$\mathcal{B}_2(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : \mathrm{Tr}(A^*A) < \infty\},$$

where Tr is the usual trace of $\mathcal{B}(\mathcal{H})$. Clearly $\mathcal{I}_N \subset \mathcal{B}_2(\mathcal{H})$. In this section we shall prove that $\mathcal{I}_N$ is a submanifold of $\mathcal{B}_2(\mathcal{H})$. Moreover, it will be shown that it is a homogeneous space. Denote by $\mathcal{U}(\infty)$ the group of unitaries which are Hilbert-Schmidt perturbations of the identity,

$$\mathcal{U}(\infty) = \{U = I + U' : U' \in \mathcal{B}_2(\mathcal{H}) \text{ and } U \text{ is unitary}\}.$$

Consider the following action of the group $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$ on $\mathcal{I}_N$:

$$(U, W) \cdot V = UVW^*. \tag{2.1}$$

This action is transitive and admits local cross sections. This was proven in [1] for the action of the whole unitary group.

Lemma 2.2. *The left action* (2.1) *is transitive.*

PROOF. It suffices to show that any element $V \in \mathcal{I}_N$ is of the form UPW^* for some projection $P \in \mathcal{P}_N$ and $U, W \in \mathcal{U}(\infty)$. In fact, U, W can be chosen as finite rank perturbations of the identity. The proof of this fact is left to the reader. □

The group $\mathcal{U}(\infty)$ is one of the so called *classical* Banach-Lie groups [4]. The Lie algebra is the space $\mathcal{B}_2(\mathcal{H})_{ah}$ of antihermitian operators in $\mathcal{B}_2(\mathcal{H})$. With the natural metric given by the real part of the trace inner product, $\mathcal{U}(\infty)$ is a complete Riemannian manifold, whose geodesic curves have the form

$$\mu(t) = Ue^{tX},$$

where $U \in \mathcal{U}(\infty)$ and $X \in \mathcal{B}_2(\mathcal{H})_{ah}$.

Let us prove that the action of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$ on $\mathcal{I}_N$ admits continuous local cross sections. Recall some basic facts on the geometry of the space $\mathcal{P}_N$ of selfadjoint projections of rank N. These facts are certainly well known (see, for example, [5, 3]), but we could not find a reference where the finite rank components of the space $\mathcal{P}$ are considered with the Hilbert-Schmidt metric.

REMARK 2.3. 1) The space $\mathcal{P}_N$ is a C^∞ submanifold of $\mathcal{B}_2(\mathcal{H})$. The group $\mathcal{U}(\mathcal{H})$ acts smoothly and transitively on $\mathcal{P}_N$ by means of

$$U \cdot P = UPU^*, U \in \mathcal{U}(\mathcal{H}), P \in \mathcal{P}_N.$$

This action makes $\mathcal{P}_N$ a C^∞ homogeneous space of $\mathcal{U}(\mathcal{H})$. The tangent space $(T\mathcal{P}_N)_P$ equals $\{XP - PX : X^* = -X\}$.

2) If $P, Q \in \mathcal{P}_N$ satisfy $\|P - Q\| < 1$, then there exists $Z \in (T\mathcal{P}_N)_P$, which is a smooth function of Q, such that

$$e^{iZ} P e^{-iZ} = Q.$$

Note that $(T\mathcal{P}_N)_P = \{XP - PX : X^* = -X\}$ lies inside $\mathcal{B}_2(\mathcal{H})$: in fact, it consists of operators with finite rank at most $2N$. Then the usual norm of $\mathcal{B}(\mathcal{H})$ and the Hilbert-Schmidt norm are equivalent there,

$$\|A\| \le \|A\|_2 \le \sqrt{2N}\|A\|.$$

In particular, this implies that the mapping $Q \mapsto Z$ is defined in the open ball of radius 1 around P in $\mathcal{B}_2(\mathcal{H})$, and is continuous in the Hilbert-Schmidt topology. Finally note that

$$e^{iZ} = I + iZ - \frac{1}{2}Z^2 - \frac{i}{6}Z^3 + \ldots \in \mathcal{U}(\infty).$$

PROPOSITION 2.4. *The action* (2.1) *has continuous local cross sections, with uniform radius. That is, there exists R, $R \ge \frac{1}{2}$, such that for any $V_0 \in \mathcal{I}_N$, there is a continuous map*

$$\sigma_{V_0} : \{V \in \mathcal{I}_N : \|V - V_0\|_2 < R\} \to \mathcal{U}(\infty) \times \mathcal{U}(\infty)$$

such that

$$\sigma_{V_0}(V) \cdot V_0 = V.$$

PROOF. Let us describe the procedure given in [2] for the construction of local cross sections for partial isometries in $\mathcal{B}(\mathcal{H})$, and check that it fits into our context. In [2] it is shown that if $\|V - V_0\| < 1/2$ then there exist unitaries U, W in $\mathcal{B}(\mathcal{H})$ such that $UV_0W^* = V$. These unitaries are constructed as follows. Observe first that $\|V - V_0\| < 1/2$ implies that $\|V^*V - V_0^*V_0\| < 1$ and $\|VV^* - V_0V_0^*\| < 1$. Then, by the above remark, there exist selfadjoint operators Z, Z' of finite rank, which depend continuously on V, such that

$$e^{iZ}V_0^*V_0e^{-iZ} = V^*V \text{ and } e^{iZ'}V_0V_0^*e^{-iZ'} = VV^*.$$

Let $\tilde{W} = V(e^{iZ'}V_0e^{-iZ})^* + (I - VV^*)$. Then $\tilde{W}$ is a unitary operator and a finite rank perturbation of I. Moreover, one has

$$\tilde{W}e^{iZ'}V_0e^{-iZ} = V.$$

Then, $\sigma_{V_0}(V) = (\tilde{W}e^{iZ'}, e^{iZ}) \in \mathcal{U}(\infty) \times \mathcal{U}(\infty)$ is a local cross section for the action (2.1). The map σ_{V_0} is defined on the set $\{V \in \mathcal{I}_N : \|V - V_0\| < 1/2\}$. Since $\|V - V_0\| \le \|V - V_0\|_2$, then it follows that σ_{V_0} is also defined on a ball of radius $\frac{1}{2}$ in the Hilbert-Schmidt metric. Finally, using that the action of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$ is transitive and clearly isometric for the Hilbert-Schmidt norm, the map σ can be translated to any $V_1 \in \mathcal{I}_N$, and defined on a (translated) ball with the same radius. □

For $V_0 \in \mathcal{I}_N$, denote by π_{V_0} the surjective map

$$\pi_{V_0} : \mathcal{U}(\infty) \times \mathcal{U}(\infty) \to \mathcal{I}_N, \;\; \pi_{V_0}(U, W) = UV_0W^*.$$

The proposition above states that π_{V_0} has continuous local cross sections. Clearly this map is C^∞ as a map from $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$ to $\mathcal{B}_2(\mathcal{H})$. The differential at I can be explicitely computed:

$$\delta_{V_0} := d(\pi_{V_0})_I : \mathcal{B}_2(\mathcal{H})_{ah} \times \mathcal{B}_2(\mathcal{H})_{ah} \to \mathcal{B}_2(\mathcal{H}), \quad \delta_{V_0}(X, Y) = XV_0 - V_0Y.$$

The isotropy group G_{V_0} at V_0 is

$$G_{V_0} = \{(G, H) \in \mathcal{U}(\infty) \times \mathcal{U}(\infty) : GV_0 = V_0H\}.$$

Proposition 2.5. *The space $\mathcal{I}_N$ is a C^∞ submanifold of $\mathcal{B}_2(\mathcal{H})$, and the map π_{V_0} is a C^∞ submersion. In particular, $\mathcal{I}_N$ is a C^∞ homogeneous space of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$.*

Proof. We shall use a fine result in [10], which states sufficient conditions on a left action from a Banach-Lie group on a Banach space, in order that the orbits of the action become submanifolds of the ambient Banach space, and smooth homogeneous spaces of the Banach-Lie group. In our context, Raeburn's conditions amount to the following:
1. $\pi_{V_0} : \mathcal{U}(\infty) \times \mathcal{U}(\infty) \to \mathcal{I}_N$ is an open map,
2. $\delta_{V_0} : \mathcal{B}_2(\mathcal{H})_{ah} \times \mathcal{B}_2(\mathcal{H})_{ah} \to \mathcal{B}_2(\mathcal{H})$ has closed and complemented range, and
3. δ_{V_0} has closed and complemented kernel.

If that is the case, then $\mathcal{I}_N \subset \mathcal{B}_2(\mathcal{H})$ is a C^∞ submanifold, and the map π_{V_0} is a submersion.

The first condition is fulfilled: in fact, π_{V_0} is open because it has continuous local cross sections by the proposition above.

Note that $\ker \delta_{V_0}$ is a real subspace of the real Hilbert space $\mathcal{B}_2(\mathcal{H})_{ah} \times \mathcal{B}_2(\mathcal{H})_{ah}$, and that $R(\delta_{V_0})$ is a real subspace of the real Hilbert space structure of $\mathcal{B}_2(\mathcal{H})$. In both cases, the inner product is given by the real part of the trace Tr. Therefore, to prove the second and third conditions, it suffices to show that the range and the kernel of δ_{V_0} are closed. The kernel of δ_{V_0} is closed, because δ_{V_0} is continuous. Let us examine the range of δ_{V_0}. Consider the real linear map $\mathcal{K}_{V_0}$,

$$(2.6)\quad \begin{cases} \mathcal{K}_{V_0} : \mathcal{B}_2(\mathcal{H}) \to \mathcal{B}_2(\mathcal{H}) \times \mathcal{B}_2(\mathcal{H}),\ \mathcal{K}_{V_0}(A) = (\kappa_1, \kappa_2), \\ \kappa_1 = \frac{1}{4} V_0 V_0^* A V_0^* - \frac{1}{4} V_0 A^* V_0 V_0^* + (I - V_0 V_0^*) A V_0^* - V_0 A^* (I - V_0 V_0^*), \\ \kappa_2 = -\frac{1}{4} V_0^* A V_0^* V_0 + \frac{1}{4} V_0^* V_0 A^* V_0 - V_0^* A (I - V_0^* V_0) + (I - V_0^* V_0) A^* V_0\,. \end{cases}$$

Straightforward computations show that

$$\delta_{V_0} \circ \mathcal{K}_{V_0} \circ \delta_{V_0} = \delta_{V_0}.$$

This implies that $\delta_{V_0} \circ \mathcal{K}_{V_0}$ is an idempotent operator on $\mathcal{B}_2(\mathcal{H})$, whose range equals the range of δ_{V_0}, which is therefore closed. □

We shall return to this linear operator $\mathcal{K}_{V_0}$ in the next section.

Let us denote by $\mathcal{I}_{(\infty)}$ the set of all partial isometries of finite rank:

$$\mathcal{I}_{(\infty)} = \cup_{N \ge 1} \mathcal{I}_N.$$

The set $\mathcal{I}_{(\infty)}$ is a discrete union of connected submanifolds of $\mathcal{B}_2(\mathcal{H})$. Moreover, it is known (see [9]), that two partial isometries V_0, V_1 such that $\|V_0 - V_1\| < 1$ are conjugate by the action of $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$. Therefore, if $\|V_0 - V_1\|_2 < 1$, then V_0 and V_1 belong to the same component of $\mathcal{I}_{(\infty)}$. In other words, $d(\mathcal{I}_N, \mathcal{I}_M) \ge 1$ if $N \ne M$.

Corollary 2.7. *The set $\mathcal{I}_{(\infty)}$ of partial isometries of finite rank is a C^∞ submanifold of $\mathcal{B}_2(\mathcal{H})$, and a discrete union of homogeneous spaces of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$.*

3. The ambient Riemannian metric of $\mathcal{I}_{(\infty)}$

By the argument closing the preceding section, the local structure of $\mathcal{I}_{(\infty)}$ is that of $\mathcal{I}_N$. So we shall focus this study in each component. Fix $N \ge 1$ and $V_0 \in \mathcal{I}_N$. Since the map π_{V_0} is a submersion, the tangent space of $\mathcal{I}_N$ (or $\mathcal{I}_{(\infty)}$ for that matter) is

$$(T\mathcal{I}_N)_{V_0} = R(\delta_{V_0}) = \{XV_0 - V_0 Y : X, Y \in \mathcal{B}_2(\mathcal{H})_{ah}\}.$$

Recall the map $\mathcal{K}_{V_0}$ (2.6). Note that $\mathcal{K}_{V_0}$ takes values in $\mathcal{B}_2(\mathcal{H})_{ah} \times \mathcal{B}_2(\mathcal{H})_{ah}$. It was noted that $P_{V_0} = \delta_{V_0} \circ \mathcal{K}_{V_0}$ is an idempotent real linear operator on $\mathcal{B}_2(\mathcal{H})$ which is the identity when restricted to the tangent space $(T\mathcal{I}_N)_{V_0}$. Explicitly

$$(3.1) \quad P_{V_0}(A) = \frac{1}{2} V_0 V_0^* A V_0^* V_0 - \frac{1}{2} V_0 A^* V_0 + \\ +(I - V_0 V_0^*) A V_0^* V_0 + V_0 V_0^* A (I - V_0^* V_0).$$

Clearly P_{V_0} is the identity when restricted to $(T\mathcal{I}_N)_{V_0}$, and because the extension of $\mathcal{K}_{V_0}$ takes antihermitian values, it follows that the range of P_{V_0} is contained in $(T\mathcal{I}_N)_{V_0}$. In other words, P_{V_0} is a real linear idempotent operator of $\mathcal{B}_2(\mathcal{H})$ with range equal to the tangent space $(T\mathcal{I}_N)_{V_0}$. $(T\mathcal{I}_N)_{V_0}$ is a real subspace of the real Hilbert space $\mathcal{B}_2(\mathcal{H})$ with inner product $< A, B >_{\mathbb{R}} = Re\,\mathrm{Tr}(B^* A)$.

LEMMA 3.2. *The linear map P_{V_0} of* (3.1) *is the orthogonal projection onto $(T\mathcal{I}_N)_{V_0}$ for the inner product $< \ , \ >_{\mathbb{R}}$.*

PROOF. The proof is straightforward, it consists in showing that P_{V_0} is symmetric for the inner product $< \ , \ >_{\mathbb{R}}$. □

Let us define the Riemannian metric of $\mathcal{I}_N$ induced by the ambient metric $< \ , \ >_{\mathbb{R}}$. For $V_0 \in \mathcal{I}_N$ and $X, Y \in (T\mathcal{I}_N)_{V_0}$, define

$$(3.3) \qquad g^a_{V_0}(X, Y) = < X, Y >_{\mathbb{R}} = Re\mathrm{Tr}(Y^* X).$$

The Riemannian connection induced by this metric is therefore defined as follows: given tangent vector fields $\mathcal{X}, \mathcal{Y}$ of $\mathcal{I}_N$, then

$$(3.4) \qquad \nabla^a_{\mathcal{X}} \mathcal{Y}_V = P_V\big(\mathcal{X}(\mathcal{Y})_V\big), \quad V \in \mathcal{I}_N.$$

In particular, a curve $\gamma \in \mathcal{I}_N$ is a geodesic for this metric if

$$(3.5) \qquad 0 = P_\gamma(\ddot{\gamma}) = \frac{1}{2}\gamma\gamma^*\ddot{\gamma}\gamma^*\gamma - \frac{1}{2}\gamma\ddot{\gamma}^*\gamma + (I - \gamma\gamma^*)\ddot{\gamma}\gamma^*\gamma + \gamma\gamma^*\ddot{\gamma}(I - \gamma^*\gamma).$$

LEMMA 3.6. *Fix a projection $P \in \mathcal{I}_N$, let $X, Y \in \mathcal{B}_2(\mathcal{H})_{ah}$. The curve $\gamma(t) = e^{tX} P e^{-tY}$, $t \in \mathbb{R}$, is a geodesic of the connection* (3.4) *if and only if*

$$(3.7) \qquad X^2 P - 2XPY + PY^2$$

PROOF. Clearly $\dot{\gamma} = e^{tX}(XP - PY)e^{-tY}$ and $\ddot{\gamma} = e^{tX}(X^2P - 2XPY + PY^2)e^{-tY}$. Also $\gamma^*\gamma = e^{tY} P e^{-tY}$ and $\gamma\gamma^* = e^{tX} P e^{-tX}$. Using these expressions one obtains that the equation (3.5) is equivalent to

$$(I - P)(X^2P - 2XPY + PY^2)P + P(X^2P - 2XPY + PY^2)(I - P) = 0.$$

Apparently, this in turn is equivalent to the condition that $X^2P - 2XPY + PY^2$ commutes with P. □

The homogeneous Riemannian manifold $\mathcal{P}_N$ (of projections of rank N) is a submanifold of $\mathcal{I}_N$. Another interesting submanifold of $\mathcal{I}_N$ is the set of partial isometries with *initial* and *final* spaces equal to the range of P, or equivalently, unitary operators of $P(\mathcal{H})$. Let us denote it by $\mathcal{U}(P(\mathcal{H}))$. This set clearly identifies with the group $\mathcal{U}(N)$ of $N \times N$ unitaries. Consider these submanifolds with the ambient metric of $\mathcal{I}_N$ (or the *real* $\mathcal{B}_2(\mathcal{H})$) and the Riemannian connections induced by these metrics.

Corollary 3.8. *The geodesics of $\mathcal{P}_N$ are geodesics of $\mathcal{I}_N$. The geodesics of $\mathcal{U}(P(\mathcal{H}))$ are geodesics of $\mathcal{I}_N$.*

Proof. Geodesics of $\mathcal{P}_N$ are of the form [3]

$$e^{tX}Pe^{-tX},$$

with $X \in \mathcal{B}_2(\mathcal{H})_{ah}$ such that $X = PX(I-P) + (I-P)XP$. In other words, when written as a 2×2 matrix in terms of the projection P, X is codiagonal. Then, by the lemma above in the case $X = Y$, one needs to show that (here $X = Y$) $X^2P - 2XPX + PX^2$ commutes with P. Since X^2 is a diagonal matrix in terms of P, it commutes with P. The element XPX is a product of two codiagonal matrices with a diagonal one, therefore it also commutes with P. Geodesics of (the natural Riemannian connection) of the unitary group $\mathcal{U}(P(\mathcal{H}))$ of $P(\mathcal{H})$ have the form

$$Pe^{tX}P = e^{tX}P = Pe^{tX}$$

with X an antihermitian operator in $P(\mathcal{H})$. It fits in the description of the lemma above, putting $Y = 0$, because X commutes with P. □

Remark 3.9. The lemma does not give a complete characterization of the geodesics of $\mathcal{I}_N$. The curves $\gamma = e^{tX}Pe^{-tY}$ can be translated using the action of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$, in order to obtain curves that start at any chosen point of $\mathcal{I}_N$ (note that the action is isometric). However, not every possible tangent vector A of $(T\mathcal{I}_N)_P$ is of the form $\dot{\gamma}(0) = XP - PY$, with X, Y satisfying the condition 3.7 of the lemma. Therefore the curves γ above do not characterize all possible geodesics of $\mathcal{I}_N$. We work out this fact in the second appendix.

4. Completeness of $\mathcal{I}_N$ in the ambient Riemannian metric

Let $\imath$ be the map

$$\imath : \mathcal{I}_N \to \mathcal{P}_N, \quad \imath(V) = V^*V.$$

Clearly $\imath$ is smooth. The differential of $\imath$ at $V \in \mathcal{I}_N$ is

$$d\imath_V : (T\mathcal{I}_N)_V \to (T\mathcal{P}_N)_{V^*V}, \quad d\imath_V(A) = A^*V + V^*A.$$

LEMMA 4.1. *Let $V \in \mathcal{I}_N$ and $A \in (T\mathcal{I}_N)_V$. Then*

$$\|d\imath_V(A)\|_2 \leq \sqrt{2}\|A\|_2.$$

PROOF.

$$\|d\imath_V(A)\|_2^2 = \mathrm{Tr}(V^*AA^*V + V^*AV^*A + A^*VA^*V + A^*VV^*A) \tag{4.2}$$

If γ is a curve in $\mathcal{I}_N$, then $\gamma\gamma^*\gamma = \gamma$. Differentiating we get $\dot{\gamma}\gamma^*\gamma + \gamma\dot{\gamma}^*\gamma + \gamma\gamma^*\dot{\gamma} = \dot{\gamma}$. If γ is a curve with $\gamma(0) = V$ and $\dot{\gamma}(0) = A$, we get $AV^*V + VA^*V + VV^*A = A$. Using this relation in (4.2) above, one obtains

$$\|d\imath_V(A)\|_2^2 = 2\mathrm{Tr}(A^*A - V^*VA^*A) = 2\mathrm{Tr}(A(I - V^*V)A^*) \leq 2\mathrm{Tr}(AA^*),$$

because $I - V^*V \leq I$. □

Then, if γ is a curve in $\mathcal{I}_N$, the length of the curve $\gamma^*\gamma$ (measured in $\mathcal{P}_N$) is bounded by $\sqrt{2}$ times the length of γ (measured in $\mathcal{I}_N$). If $(\mathcal{M}, g)$ is a Riemannian manifold and $A, B \in \mathcal{M}$, let us denote by $d_{\mathcal{M}}(A, B)$ the geodesic distance, defined as the infimum of the lengths of the curves in $\mathcal{M}$ joining A and B. The above remark clearly implies that if $V_0, V_1 \in \mathcal{I}_N$, then

$$d_{\mathcal{I}_N}(V_0, V_1) \leq \sqrt{2}\ d_{\mathcal{P}_N}(\imath(V_0), \imath(V_1)). \tag{4.3}$$

Analogously, we can define the map

$$\varphi : \mathcal{I}_N \to \mathcal{P}_N,\ \ \varphi(V) = VV^*.$$

Clearly this map has the same properties as $\imath$:

$$d_{\mathcal{I}_N}(V_0, V_1) \leq \sqrt{2}\ d_{\mathcal{P}_N}(\varphi(V_0), \varphi(V_1)). \tag{4.4}$$

THEOREM 4.5. *$\mathcal{I}_N$ is a complete metric space in the geodesic distance $d_{\mathcal{I}_N}$.*

PROOF. Let $\{V_n\}$ be a Cauchy sequence in $\mathcal{I}_N$ for the metric $d_{\mathcal{I}_N}$. By the above remarks, it follows that $\{\imath(V_n)\}$ and $\{\varphi(V_n)\}$ are Cauchy sequences of $\mathcal{P}_N$ for the metric $d_{\mathcal{P}_N}$. It is known that $\mathcal{P}_N$ is complete for the geodesic distance. Then there exist $P, Q \in \mathcal{P}_N$ such that

$$\imath(V_n) = V_n^*V_n \to P, \quad \varphi(V_n) = V_nV_n^* \to Q.$$

The action of $\mathcal{U}(\infty)$ on $\mathcal{P}_N$ admits continuous local cross sections, which are defined on balls of radius 1 around each point of $\mathcal{P}_N$ (2.3). It follows that there exist unitaries $U_n, W_n \in \mathcal{U}(\infty)$ such that $V_nV_n^* = U_nPU_n^*$ and $V_n^*V_n = W_nQW_n^*$, with $U_n \to I$ and $W_n \to I$.

Since P, Q are conjugate by the action of $\mathcal{U}(\infty)$, there exists $U_0 \in \mathcal{U}(\infty)$ such that $Q = U_0PU_0^*$. Let $\tilde{V}_n = U_0^*U_n^*V_nW_n$. Then straightforward computations show that $\tilde{V}_n\tilde{V}_n{}^* = P$ and $\tilde{V}_n{}^*\tilde{V}_n = P$. That is, $\tilde{V}_n$ is a unitary operator of $P(\mathcal{H})$.

We claim that $\tilde{V}_n$ is a Cauchy sequence in $\mathcal{I}_N$. To prove this, it suffices to show that if V_n is a Cauchy sequence in $\mathcal{I}_N$ and G_n is a convergent (to G) sequence of $\mathcal{U}(\infty)$, then both G_nV_n and V_nG_n are Cauchy sequences in $\mathcal{I}_N$. Let us prove the first of these assertions, the second is analogous. Observe first that

$$d_{\mathcal{I}_N}(V_nG_n, V_mG_m) \leq d_{\mathcal{I}_N}(V_nG_n, V_nG) + d_{\mathcal{I}_N}(V_nG, V_mG) + d_{\mathcal{I}_N}(V_mG, V_mG_m).$$

The terms in the middle $d_{\mathcal{I}_N}(V_nG, V_mG) = d_{\mathcal{I}_N}(V_n, V_m)$ tend to zero. The first and third term are dealt analogously, let us proceed with the first. Since the action of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$ on $\mathcal{I}_N$ is isometric, we can multiply on the right by G^*, or equivalently, suppose that $G = I$. We may also suppose n big enough so that G_n lies in a normal neighbourhood of I in $\mathcal{U}(\infty)$. That is, there exists $X_n \in \mathcal{B}_2(\mathcal{H})_{ah}$ such that $G_n = e^{X_n}$ and $\mu_n(t) = e^{tX_n}$ is a minimizing geodesic of $\mathcal{U}(\infty)$ joining I and G_n. Then $\gamma_n = V_n\mu_n$ is a curve joining V_n and V_nG_n in $\mathcal{I}_N$ and

$$d_{\mathcal{I}_N}(V_nG_n, V_n) \leq length(\gamma_n) = \int_0^1 g_{\gamma_n}(\dot{\gamma}_n)^{1/2}\, dt = \|V_nX_n\|_2.$$

Note that $\|V_nX_n\|_2 = \mathrm{Tr}(X_n^*V_n^*V_nX_n)^{1/2}$, which together with $V_n^*V_n \leq I$, imply that

$$\|V_nX_n\|_2 \leq \mathrm{Tr}(X_n^*X_n)^{1/2} = \|X_n\|_2 = d_{\mathcal{U}(\infty)}(G_n, I) \to 0.$$

In fact, we proved that $d_{\mathcal{I}_N}(V_nG_n, V_nG) \leq d_{\mathcal{U}(\infty)}(G_n, G)$. Therefore our claim is verified, and $\tilde{V}_n$ is a Cauchy sequence in $\mathcal{I}_N$, which lies in the submanifold $\mathcal{U}(P(\mathcal{H}))$. Since the geodesics of $\mathcal{U}(P(\mathcal{H}))$ are geodesics of the ambient $\mathcal{I}_N$, it follows that $\tilde{V}_n$ is a Cauchy sequence in $\mathcal{U}(P(\mathcal{H}))$. This manifold is isometrically diffeomorphic to $\mathcal{U}(N)$, which is complete. Therefore $\tilde{V}_n$ is convergent in $\mathcal{U}(P(\mathcal{H}))$, and there exists $\tilde{V} \in \mathcal{U}(P(\mathcal{H}))$ such that $\tilde{V}_n \to \tilde{V} \in \mathcal{U}(P(\mathcal{H}))$. Then

$$V_n = U_0U_n\tilde{V}_nW_n^* \to U_0\tilde{V} \in \mathcal{I}_N. \qquad \square$$

5. A METRIC INDUCED BY THE ACTION

The manifold $\mathcal{I}_N$ is a homogeneous space, namely, for any fixed $V_0 \in \mathcal{I}_N$,

$$\mathcal{I}_N \simeq \mathcal{U}(\infty) \times \mathcal{U}(\infty)/G_{V_0},$$

where G_{V_0} is the subgroup of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$ given by

$$G_{V_0} = \{(H, K) \in \mathcal{U}(\infty) \times \mathcal{U}(\infty) : HV_0 = V_0K\}.$$

We introduce a new metric in $\mathcal{I}_N$ via the natural metric in the Lie algebra $\mathcal{B}_2(\mathcal{H})_{ah} \times \mathcal{B}_2(\mathcal{H})_{ah}$ of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$, as follows. Let $\mathcal{G}_{V_0}$ be the Lie algebra of G_{V_0}. Note that $\mathcal{G}_{V_0} = \ker \delta_{V_0}$. It follows that

$$\delta_{V_0}|_{\ker \delta_{V_0}^{\perp}} : \ker \delta_{V_0}^{\perp} \to (T\mathcal{I}_N)_{V_0}$$

is an isomorphism. Here $\ker \delta_{V_0}^{\perp}$ is the orthogonal complement with respect to the inner product of $\mathcal{B}_2(\mathcal{H})_{ah} \times \mathcal{B}_2(\mathcal{H})_{ah}$ given by the real part of the trace:

$< (A,B),(A',B') >= Re\ \mathrm{Tr}(A'^*A + B'^*B)$. We induce a metric in $(T\mathcal{I}_N)_{V_0}$ by requiring that $\delta_{V_0}|_{\ker \delta_{V_0}^{\perp}}$ be an *isometric* isomorphism, for all $V_0 \in \mathcal{I}_N$. Let us describe this metric explicitely.

We denote

$$\mathcal{O}_{V_0} = \ker \delta_{V_0}^{\perp}. \tag{5.1}$$

Recall the map $\mathcal{K}_{V_0}$ of (2.6). It is a relative inverse for δ_{V_0}. We claim that it is the relative inverse with range equal to $\mathcal{O}_{V_0}$. We do this by showing that both distributions $V \mapsto \delta_V$ and $V \mapsto \mathcal{K}_V$ are equivariant with respect to the action of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$.

LEMMA 5.2. *Let $V \in \mathcal{I}_N$ and $U, W \in \mathcal{U}(\infty)$. Then*

$$\delta_{UVW^*}(X,Y) = (U,W) \cdot \big(\delta_V(Ad(U^*,W^*)(X,Y))\big),\ X,Y \in \mathcal{B}_2(\mathcal{H})_{ah},$$

and

$$\mathcal{K}_{UVW^*}(A) = Ad(U,W)\big(\delta_V((U,W)\cdot A)\big),\ A \in (T\mathcal{I}_N)_V.$$

PROOF. The proof is a straightforward computation. □

PROPOSITION 5.3. *The map $\mathcal{K}_V$ of* (2.6) *is the relative inverse of δ_V with range equal to $\mathcal{O}_V$.*

PROOF. By the above lemma, and the fact that the actions involved are isometric, it suffices to prove the proposition for the case $V = P$. Note that the isotropy group G_P consists of pairs of unitaries $(H,K) \in \mathcal{U}(\infty) \times \mathcal{U}(\infty)$ such that $HP = PK$. This implies that $PH^* = K^*P$, and then $KP = PH$. Then $PHP = HP = PH$ and analogously for K. Then G_P can be characterized as follows

$$G_P = \{(H,K) \in \mathcal{U}(\infty) \times \mathcal{U}(\infty) : H,K \text{ commute with } P \text{ and } PHP = PKP\}. \tag{5.4}$$

Therefore the elements of $\mathcal{G}_P = \ker \delta_P$ are pairs of 2×2 *diagonal* matrices (in terms of P) which have the same $1,1$ entry. Apparently, the orthogonal complement of this space is the set of pairs of matrices of the form

$$\left(\begin{pmatrix} A & B \\ -B^* & 0 \end{pmatrix}, \begin{pmatrix} -A & C \\ -C^* & 0 \end{pmatrix} \right),$$

where A is an antihermitian operator in $P(\mathcal{H})$. In the case at hand ($V = P$), the map $\mathcal{K}_p : (T\mathcal{I}_N)_P \to \mathcal{B}_2(\mathcal{H}) \times \mathcal{B}_2(\mathcal{H})$ is given by

$$\mathcal{K}_P(A) = \big(\frac{1}{2}PAP + (I-P)AP - PA^*(I-P), -\frac{1}{2}PAP - PA(I-P) + (I-P)A^*P\big).$$

It is clear that the range of this map equals $\mathcal{O}_P$. □

Let us define a second Riemannian metric in $\mathcal{I}_N$, the one induced by the isomorphisms $\mathcal{K}_V$, $V \in \mathcal{I}_N$. If $A, B \in (T\mathcal{I}_N)_V$, then

$$(5.5)\quad g^b_V(A,B) =< \mathcal{K}_V(A), \mathcal{K}_V(B) >_{\mathcal{B}_2(\mathcal{H})_{ab}\times\mathcal{B}_2(\mathcal{H})_{ab}}=$$

$$= Re\ \mathrm{Tr}\big(-\frac{1}{2}V^*AV^*B + 2B^*(I-VV^*)A + 2A(I-V^*V)B^*\big) =$$

$$= Re\ \mathrm{Tr}\big(-\frac{1}{2}V^*AV^*B + 4AB^* - 2B^*VV^*A - 2AV^*VB\big).$$

By Lemma 5.2 it is clear that $\mathcal{U}(\infty)\times\mathcal{U}(\infty)$ also acts isometrically for this metric g^b.

Let us show that $\mathcal{I}_N$ is complete with the homogeneous metric as well. In order to do this, we shall see that both metrics g^a and g^b are equivalent.

Proposition 5.6. *Let $V\in\mathcal{I}_N$ and $X\in(T\mathcal{I}_N)_V$. Then*

$$\frac{1}{2}g^a_V(X,X) \le g^b_V(X,X) \le 2g^a_V(X,X).$$

Proof. Let $\mathbb{V}$ be the (complex) subspace of $\mathcal{B}_2(\mathcal{H})$ given by

$$\mathbb{V} = \{X\in\mathcal{B}_2(\mathcal{H}) : (I-P)X(I-P)=0\},$$

and

$$\Pi : \mathbb{V}\to\mathbb{V},\ \Pi(X) = \frac{1}{2}PXP + 2X(I-P) + 2(I-P)X.$$

Clearly $\Pi(\mathbb{V})\subset\mathbb{V}$. Note that Π is an isomorphism with inverse

$$\Pi^{-1}(X) = 2PXP + \frac{1}{2}X(I-P) + \frac{1}{2}(I-P)X.$$

Also it is apparent that $\|\Pi\|\le 2$ and $\|\Pi^{-1}\|\le 2$. Consider first the case $V=P$. Let $X\in(T\mathcal{I}_N)_P$. Then X is antihermitian. Compute

$$g^b_P(X,X) = Re\mathrm{Tr}\left(-\frac{1}{2}PXPX + 2X^*(I-P)X + 2X(I-P)X^*\right) =$$

$$= Re\mathrm{Tr}\left(\frac{1}{2}PXPX^* + 2(I-P)XX^* + 2X(I-P)X^*\right) =$$

$$= \mathrm{Tr}\left(\left[\frac{1}{2}PXP + 2(I-P)X + 2X(I-P)\right]X^*\right).$$

Since $(I-P)X(I-P)=0$, then $(I-P)X=(I-P)XP$ and $X(I-P)=PX(I-P)$. Therefore

$$g^b_P(X,X) =< \Pi(X), X >_{\mathbb{V}}.$$

On the other hand, $< X,X >_{\mathbb{V}} = g^a_P(X,X)$. The bounds $\|\Pi\|\le 2$ and $\|\Pi^{-1}\|\le 2$ imply

$$\frac{1}{2} < X,X >_{\mathbb{V}} \le < \Pi(X),X >_{\mathbb{V}} \le 2 < X,X >_{\mathbb{V}},$$

or equivalently,

$$\frac{1}{2}g^a_P(X,X) \le g^b_P(X,X) \le 2g^a_P(X,X).$$

At other points $V \in \mathcal{I}_N$, the inequality is proven by means of the transitive action of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$, which is isometric for both metrics. □

COROLLARY 5.7. *The manifold $\mathcal{I}_N$ is complete in the Riemannian metric g^b.*

6. APPENDIX: $\mathcal{I}_N$ IS SIMPLY CONNECTED

We may extend the action of $\mathcal{U}(\infty) \times \mathcal{U}(\infty)$ to the whole unitary groups $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$. By Kuiper's theorem [7], this group is contractible. In particular, the transitivity of the action implies that the map

$$\pi_P : \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) \to \mathcal{I}_N, \quad \pi_P(U, W) = UPW^*$$

is surjective. Therefore $\mathcal{I}_N$ is connected. It was also shown that this map has continuous local cross sections. This implies that it is a locally trivial fibre bundle. The fibre of this bundle is the subgroup $\bar{G}_P$, consisting of *all* pairs of unitaries (G, H) such that $GP = PH$. This group can be characterized analogously as in (5.4), and consists of pairs of unitaries (G, H) which commute with P and verify $PGP = PHP$. In matrix form (in terms of P):

$$G = \begin{pmatrix} U_0 & 0 \\ 0 & G_\infty \end{pmatrix}, \quad H = \begin{pmatrix} U_0 & 0 \\ 0 & H_\infty \end{pmatrix},$$

where U_0 is a unitary operator in $P(\mathcal{H})$ (of dimension N) and G_∞, H_∞ are unitary operators in $P(\mathcal{H})^\perp$. Both $\mathcal{U}(N)$ and $\mathcal{U}(P(\mathcal{H})^\perp)$ are connected, and therefore $\bar{G}_P$ is connected. In fact,

$$\bar{G}_P \simeq \mathcal{U}(N) \times \mathcal{U}(P(\mathcal{H})^\perp) \times \mathcal{U}(P(\mathcal{H})^\perp).$$

Examining the homotopy exact sequence of the bundle π_P, using that $P(\mathcal{H})^\perp$ is infinite dimensional, it follows that

$$\pi_{n+1}(\mathcal{I}_N) \simeq \pi_n(\bar{G}_P) \simeq \pi_n(\mathcal{U}(N)).$$

In particular, for $n = 0$, $\pi_1(\mathcal{I}_N) = 0$.

7. APPENDIX II: AN EXAMPLE

In this section we show an example. In order to construct this example we need a lemma which translates the condition (3.7) (for a curve $e^{tX}Pe^{-tY}$ to be a geodesic of g^a) into a linear system of operator equations. The example will show that there are directions (*i.e.* vectors in $(T\mathcal{I}_N)_P$) which are not velocity vectors of geodesics of the type $e^{tX}Pe^{-tY}$. In other words, there are geodesics starting at P which are not of this type. Any $V \in (T\mathcal{I}_N)_P$ is of the form $V = \delta_P(A, B)$, with $A, B \in \mathcal{B}_2(\mathcal{H})_{ah}$,

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta^* & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -\alpha & \gamma \\ -\gamma^* & 0 \end{pmatrix}.$$

LEMMA 7.1. *Let $V = AP - PB$ with A, B as above. Then there exist $X_V, Y_V \in \mathcal{B}_2(\mathcal{H})_{ah}$ such that $X_V P - PY_V = V$ and $X_V^2 P - 2X_V P Y_V + PY_V^2$ commutes with P if and only if the system*

$$\begin{cases} \gamma Z - X\gamma & = 3a\gamma \\ \beta Y - X\beta & = -3a\beta \end{cases} \tag{7.2}$$

has a solution, where the operators $X : P(\mathcal{H}) \to P(\mathcal{H})$ and $Y, Z : P(\mathcal{H})^\perp \to P(\mathcal{H})^\perp$ are antihermitian. If X, Y, Z provide a solution, then putting

$$X_V = \begin{pmatrix} a + X & \beta \\ -\beta^* & Y \end{pmatrix}, \ Y_V = \begin{pmatrix} -a - X & \gamma \\ -\gamma^* & Z \end{pmatrix}$$

gives the geodesic pair which satisfies the quadratic relation (3.7)*, with $\delta_P(X_V, Y_V) = V$.*

PROOF. Note that the pairs

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \begin{pmatrix} -X & 0 \\ 0 & Z \end{pmatrix}$$

with X, Y, Z as above, parametrize $\ker \delta_P$. It follows that

$$A' = \begin{pmatrix} a + X & \beta \\ -\beta^* & Y \end{pmatrix}, \ B' = \begin{pmatrix} -a - X & \gamma \\ -\gamma^* & Z \end{pmatrix}$$

parametrize all pairs (A', B') such that $\delta(A', B') = V$. One arrives at the system (7.2) by routine matrix calculations, using that the solutions X, Y, Z must be antihermitian. □

Notice in the first equation of (7.2) that the solution Z must leave both $\ker \gamma$ and $(\ker \gamma)^\perp$ invariant. Indeed, if $\xi \in \ker \gamma$, then

$$0 = 3a\gamma\xi = \gamma Z\xi - X\gamma\xi = \gamma Z\xi.$$

Since Z is a priori antihermitian, it leaves invariant also the orthogonal complement. Analogously, from the second equation, it follows that any solution Y leaves invariant $\ker \beta$ and $(\ker \beta)^\perp$.

Both γ, β have closed (finite dimensional) ranks. Therefore, they both have bounded Moore-Penrose pseudo-inverses $\gamma^\dagger, \beta^\dagger$,

$$\gamma^\dagger\gamma = P_{(\ker \gamma)^\perp}, \ \gamma\gamma^\dagger = P_{R(\gamma)}, \ \beta^\dagger\beta = P_{(\ker \beta)^\perp}, \ \beta\beta^\dagger = P_{R(\beta)}.$$

Multiplying the first equation of 7.2 by $\gamma^\dagger$ on the left we obtain

$$P_{(\ker \gamma)^\perp} Z = \gamma^\dagger (X + 3a)\gamma.$$

Since Z is antihermitian and leaves $(\ker \gamma)^\perp$ invariant, it follows that Z and $P_{(\ker \gamma)^\perp}$ commute. Then $\gamma^\dagger(X + 3a)\gamma$ is antihermitian. Reasoning analogously with the second equation of (7.2), one obtains that $\beta^\dagger(X - 3a)\beta$ is antihermitian.

These two facts provide the clue to find an example of a direction V which is not the velocity vector of any geodesic of the form $e^{tX}Pe^{-tY}$.

EXAMPLE 7.3. Put $N = 2$, $\mathcal{H} = \ell^2(\mathbb{N})$ and let $\{\varepsilon_n : n \geq 1\}$ be the canonical basis of $\ell^2(\mathbb{N})$. Put P the projection onto the subspace spanned by the first two vectors of the basis. Let

$$\gamma : P(\mathcal{H})^\perp \to P(\mathcal{H}), \gamma(0,0,x_3,x_4,x_5,x_6,\dots) = (x_3, 2x_4, 0, \dots).$$

Clearly $\gamma^\dagger$ is given by $\gamma^\dagger(x_1, x_2, 0, \dots) = (0, 0, x_1, \frac{1}{2}x_2, 0, \dots)$.

By the remarks above, if X is part of a solution of the system (7.2), then both $X + 3a$ and $\gamma^\dagger(X + 3a)\gamma$ are antihermitian. A straightforward calculation shows that for this γ just defined, an operator C (in fact, a 2×2 matrix) is antihermitian with $\gamma^\dagger C \gamma$ also antihermitian, only if C is diagonal. It follows that $X + 3a$ must be diagonal. Putting $\beta = \gamma$ and reasoning analogously with the second equation, one obtains that also $X - 3a$ is diagonal. This implies that the data a must be diagonal, a fact which need not happen.

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E. Andruchow:
Instituto de Ciencias
Universidad Nacional de General Sarmiento
J.M. Gutiérrez, 1150 - 1613 LOS POLVORINES (Argentina)
eandruch@ungs.edu.ar

G. Corach:
Instituto Argentino de Matemática
CONICET
Saavedra 15, 3er. piso - 1083 BUENOS AIRES (Argentina)
gcorach@fi.uba.ar