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On integral representations of q-gamma and q-beta functions

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Funzioni speciali. — *On integral representations of q-gamma and q-beta functions.* Nota (*) di Alberto De Sole e Victor G. Kac, presentata dal Socio C. De Concini.

Abstract. — We study q-integral representations of the q-gamma and the q-beta functions. As an application of these integral representations, we obtain a simple conceptual proof of a family of identities for Jacobi triple product, including Jacobi's identity, and of Ramanujan's formula for the bilateral hypergeometric series.

Key words: q-calculus; q-gamma function; q-beta function.

Riassunto. — *Sulle rappresentazioni integrali delle funzioni q-gamma e q-beta*. Studiamo la rappresentazione q-integrale delle funzioni q-gamma e q-beta. Questo studio svela una q-costante molto interessante. Come applicazione di queste rappresentazioni integrali, otteniamo una semlice dimostrazione concettuale di una famiglia di identità per il prodotto triplo di Jacobi, che include l'identità di Jacobi, e della formula di Ramanujan per le serie ipergeometriche bilaterali.

1. Introduction

There is no general rigorous definition of a «q-analogue». An intuitive definition of a q-analogue of a mathematical object $\mathcal{A}$ is a family of objects $\mathcal{A}_q$, $0 < q < 1$, such that the limit of $\mathcal{A}_q$, as q tends to 1, is $\mathcal{A}$. Under certain additional requirements the q-analogue may be unique, but sometimes it is useful to consider several q-analogues of the same object.

The *q-calculus* (*i.e.* the q-analogue of the usual calculus) begins with the definition of the q-analogue $d_q f(x)$ of the differential of a function, $df(x)$. The former is much simpler than the latter:

$$d_q f(x) \;=\; f(qx) - f(x) \;.$$

Having said this, we immediately get the q-analogue of the derivative of $f(x)$, called its *q-derivative*:

$$D_q f(x) \;:=\; \frac{d_q f(x)}{d_q x} \;=\; \frac{f(qx)-f(x)}{(q-1)x} \;. \tag{1.1}$$

Notice that the q-derivative satisfies the following q-analogue of Leibniz rule

$$D_q(f(x)g(x)) = g(x)D_q f(x) + f(qx)D_q g(x) \;. \tag{1.2}$$

Of course, the first question is what is the q-derivative of x^α, where $\alpha \in \mathbb{C}$. This is very easy:

$$D_q x^\alpha \;=\; [\alpha] x^{\alpha-1} \;,$$

(*) Pervenuta in forma definitiva all'Accademia il 27 luglio 2004.

where $[\alpha]$ is the q-analogue of α:

$$[\alpha] \;=\; \frac{1-q^\alpha}{1-q}\ .$$

Here we discover the remarkable property of a q-analogue $\mathcal{A}_q$ of an object $\mathcal{A}$ - sometimes $\mathcal{A}_q$ makes sense even when $\mathcal{A}$ does not:

$$[\infty] \;=\; \frac{1}{1-q}\ .$$

The next question is what is the q-analogue $(x+a)_q^n$ of the function $(x+a)^n$ for a non-negative integer n. The natural requirements are

$$D_q(x+a)_q^n \;=\; [n](x+a)_q^{n-1}\ ,\quad (x+a)_q^0 \;=\; 1\ . \tag{1.3}$$

It is a nice exercise to check that the (unique) answer is ($n \in \mathbb{N}$)

$$(x+a)_q^n \;=\; \prod_{j=0}^{n-1}(x+q^ja)\ . \tag{1.4}$$

The product (1.4) plays in combinatorics the most fundamental role. Again, remarkably, for $x=1$, $a=x$ this formula makes sense for $n=\infty$:

$$(1+x)_q^\infty \;=\; \prod_{j=0}^{\infty}(1+q^jx)\ . \tag{1.5}$$

Under our assumptions on q, the infinite product (1.5) is convergent. Now we may give a definition of $(1+x)_q^\alpha$ for any number α, which is consistent with (1.4):

$$(1+x)_q^\alpha \;=\; \frac{(1+x)_q^\infty}{(1+q^\alpha x)_q^\infty}\ . \tag{1.6}$$

We shall need the following generalizations of (1.3), where $n \in \mathbb{Z}$ and $a,b,t \in \mathbb{C}$,

$$\begin{cases} D_q(ax+b)_q^n \;=\; a[n](ax+b)_q^{n-1}\ , \\ D_q(a+bx)_q^n \;=\; b[n](a+bqx)_q^{n-1}\ , \\ D_q(1+bx)_q^t \;=\; b[t](1+bqx)_q^{t-1}\ . \end{cases} \tag{1.7}$$

The proof of these identities is left as a simple exercise for the reader.

There are two important q-analogues of the exponential function:

$$E_q^x = \sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{x^n}{[n]!} \;=\; (1+(1-q)x)_q^\infty\ , \tag{1.8}$$

$$e_q^x = \sum_{n=0}^{\infty}\frac{x^n}{[n]!} \;=\; \frac{1}{(1-(1-q)x)_q^\infty}\ , \tag{1.9}$$

where $[n]! = [1][2]\cdots[n]$. Note that the series in (1.8) (resp. (1.9)) converges for $|x| < \infty$ $\left(\text{resp. } |x| < [\infty] = \dfrac{1}{1-q}\right)$. The second equalities in (1.8) and (1.9) are due to Euler. The simplest proof of them is obtained by using the q-analogue of Taylor's formula (see *e.g.* [6]). The function e_q^x is uniquely defined by the requirements

$$D_q\, e_q^x \;=\; e_q^x\;, \quad e_q^0 \;=\; 1\;.$$

However, due to the product expansions in (1.8) and (1.9), $(e_q^x)^{-1} = E_q^{-x}$ (not e_q^{-x}), which explains why one needs both q-analogues of the exponential function. The q-derivative of E_q^x is

$$D_q\, E_q^x \;=\; E_q^{qx}\;.$$

In the present paper we shall discuss q-analogues of Euler's gamma and beta functions $\Gamma(t)$ and $B(t,s)$. Their definitions are reminded in Section 2, equations (2.1) and (2.2) (for the unlikely reader who is not familiar with these functions). The q-gamma function was introduced by Thomae [14] and later by Jackson [4] as the infinite product

$$\Gamma_q(t) = \frac{(1-q)_q^{t-1}}{(1-q)^{t-1}}\;, \quad t > 0\;. \tag{1.10}$$

Though the literature on the q-gamma function and its applications is rather extensive (see [1-3] and references there), the authors usually avoided the use of its q-integral representation. In fact, each time when a q-integral representation was discussed, it was, as a rule, not quite right. The first correct integral representation of $\Gamma_q(t)$ that we know of is in references [11, 12]:

$$\Gamma_q(t) = \int_0^{[\infty]} x^{t-1} E_q^{-qx}\, d_q x\;, \tag{1.11}$$

where the q-integral (introduced by Thomae [14] and Jackson [5]) is defined by

$$\int_0^a f(x)\, d_q x = (1-q)\sum_{j=0}^{\infty} aq^j f(aq^j)\;. \tag{1.12}$$

Notice that the series on the right-hand side is guaranteed to be convergent as soon as the function f is such that, for some $C > 0$, $\alpha > -1$, $|f(x)| < Cx^\alpha$ in a right neighborhood of $x = 0$.

The definition (1.12) has an obvious interpretation in terms of a Riemann sum. Moreover it is easy to check that the q-integral defined in (1.12) is the q-antiderivative. Namely for any function $f(x)$ continuous at $x = 0$ we have

$$\int_0^a D_q f(x)\, d_q x \;=\; f(a) - f(0)\;, \quad D_q \int_0^x f(t)\, d_q t \;=\; f(x)\;. \tag{1.13}$$

In the following we will denote, for arbitrary numbers a, b

$$\int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x .$$

As immediate consequence q-Leibniz rule (1.2) and of equation (1.13), we get the q-analogue of the rule of integration by parts

$$\int_a^b g(x)D_q f(x)d_q x = f(x)g(x)\Big|_a^b - \int_a^b f(qx)D_q g(x)d_q x . \tag{1.14}$$

The q-beta function was more fortunate than the q-gamma function as far as the integral representations are concerned. Already in the above mentioned papers by Thomae and Jackson it was shown that the q-beta function, defined by the q-analogue of the usual formula (2.5),

$$\mathrm{B}_q(t,s) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)} , \tag{1.15}$$

has the following q-integral representation, which is a q-analogue of Euler's formula (2.2):

$$\mathrm{B}_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x , \quad t, s > 0 . \tag{1.16}$$

Jackson [5] made an attempt to give a q-analogue of a less traditional Euler's integral representation of the beta function:

$$\mathrm{B}(t,s) = \int_0^\infty \frac{x^{t-1}}{(1+x)^{t+s}} dx , \tag{1.17}$$

which is obtained from (2.2) by the substitution $x \to 1/(1+x)$. However, his definition is not quite right, since it is not quite equal to $B_q(t,s)$, as will be explained in Remark 3.5. A correct q-analogue of (1.17) is the famous Ramanujan's formula for the bilateral hypergeometric series, see [1, pp. 502-505].

In the present paper we give another q-integral representation of $\Gamma_q(t)$, based on the q-exponential function e_q^x, and a q-integral representation of $\mathrm{B}_q(t,s)$ which is a q-analogue of (1.17). Both representations are based on the following remarkable function, cf. [12]:

$$K(x,t) = \frac{x^t}{1+x}\left(1+\frac{1}{x}\right)_q^t (1+x)_q^{1-t} . \tag{1.18}$$

This function is a q-constant in x, *i.e.*

$$K(qx,t) = K(x,t) ,$$

and for t an integer it is independent of x and is equal to $q^{t(t-1)/2}$. However, for $t \in (0,1)$

this function does depend on x, since for these t one has

$$\lim_{q\to 0} K(x;t) = x^t + x^{t-1} \ .$$

Our integral representations are as follows:

$$\Gamma_q(t) = K(A,t) \int_0^{\infty/A(1-q)} x^{t-1} e_q^{-x} d_q x \ , \tag{1.19}$$

$$\mathrm{B}_q(t,s) = K(A,t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_q x \ , \tag{1.20}$$

where the improper integral, following [5] and [13], is defined by

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{n\in\mathbb{Z}} \frac{q^n}{A} f\left(\frac{q^n}{A}\right) . \tag{1.21}$$

Since $K(A,t)$ depends on A, we conclude that the integrals in both formulas do depend on A. In his formulas Jackson used the factor $q^{t(t-1)/2}$ in place of $K(A,t)$, which is correct only for an integer t.

For the proof of these integral representations, provided in Section 3, we will need the following *reciprocity relations*, which follow immediately from the definition of q-integral:

$$\int_0^A f(x) d_q x = \int_{q/A}^{\infty/A} \frac{1}{x^2} f\left(\frac{1}{x}\right) d_q x \ , \qquad \int_0^{\infty/A} f(x) d_q x = \int_0^{\infty \cdot A} \frac{1}{x^2} f\left(\frac{1}{x}\right) d_q x \ . \tag{1.22}$$

These are change of variables rules under the substitution $x \to \dfrac{1}{x}$, and similar rules are easily derived for the substitution $x \to \alpha x^\beta$. However there is no general change of variables formula for the q-integral. This is the main handicap of q-calculus. The reader may find more on q-calculus and its applications in the books [1, 3, 6].

In Sections 4 and 5 we will apply equation (1.20) to find an integral representation of the q-beta function which is manifestly symmetric under the exchange of t and s, and to find a q-analogue of translation invariance of certain improper integrals. Finally, in Section 6 we will show that equation (1.19) is equivalent to a family of triple product identities, a limiting case of which is the Jacobi triple product identity:

$$(1-q)_q^\infty (1-x)_q^\infty (1-q/x)_q^\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} x^n \ , \tag{1.23}$$

and that equation (1.20) is equivalent to Ramanujan's summation formula, see [1, pp. 501-505]:

$$\sum_{n=-\infty}^{\infty} \frac{(1-a)_q^n}{(1-b)_q^n} x^n = \frac{(1-q)_q^\infty (1-b/a)_q^\infty (1-ax)_q^\infty (1-q/ax)_q^\infty}{(1-b)_q^\infty (1-q/a)_q^\infty (1-x)_q^\infty (1-b/ax)_q^\infty} \ . \tag{1.24}$$

Also, the symmetric integral representation of the q-beta function is equivalent to the following identity:

$$(1.25)\qquad \sum_{n=-\infty}^{\infty} \frac{(1-a)_q^n(1-q/a)_q^{-n}}{(1-b)_q^n(1-c)_q^{-n}} = \frac{(1-a)_q^{\infty}(1-q/a)_q^{\infty}}{(1-b)_q^{\infty}(1-c)_q^{\infty}} \frac{(1-q)_q^{\infty}(1-bc/q)_q^{\infty}}{(1-b/a)_q^{\infty}(1-ac/q)_q^{\infty}} \ .$$

Jacobi's triple product identity and Ramanujan's summation formula are among the most famous and important identities in all of Mathematics. In particular, each of these identities turned out to be the tip of the iceberg of an important representation theory: The Jacobi triple product identity (respectively a specialization of the Ramanujan's summation formula) is the «denominator identity» of the simplest affine Kac-Moody algebra, $\mathfrak{sl}(2)^{\widehat{}}$ (resp. of the simplest affine superalgebra, $\mathfrak{sl}(2,1)^{\widehat{}}$), [7, 9]. At the same time, these identities arise in quantum field theory, as an essential part of the boson-fermion correspondence and its super analogue [8].

One way or another most of the results of the paper have appeared in the literature, however our exposition seems to be more systematic.

2. Definition of q-gamma and q-beta functions

The Euler's gamma and beta functions are defined as the following definite integrals $(s, t > 0)$:

$$(2.1)\qquad \Gamma(t) = \int_0^{\infty} x^{t-1}e^{-x}dx \ ,$$

$$(2.2)\qquad \mathrm{B}(t,s) = \int_0^1 x^{t-1}(1-x)^{s-1}dx \ ,$$

$$(2.3)\qquad = \int_0^{\infty} \frac{x^{t-1}}{(1+x)^{t+s}}dx \ .$$

From the expression (2.2) it is clear that $\mathrm{B}(t,s)$ is symmetric in t and s. Recall some of the main properties of the gamma and beta functions:

$$(2.4)\qquad \Gamma(t+1) = t\Gamma(t) \ , \quad \Gamma(1) = 1 \ ,$$

$$(2.5)\qquad \mathrm{B}(t,s) = \Gamma(t)\Gamma(s)/\Gamma(t+s) \ .$$

In this paper we are interested in the q-analogue of the gamma and beta functions. They are defined in the following way.

Definition 2.1. (a) *For $t > 0$, the q-gamma function is defined to be*

$$\Gamma_q(t) = \int_0^{[\infty]} x^{t-1} E_q^{-qx} d_q x \ . \tag{2.6}$$

(b) *For $s, t > 0$, the q-beta function is*

$$\mathrm{B}_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x \ . \tag{2.7}$$

$\Gamma_q(t)$ and $\mathrm{B}_q(t,s)$ are the «correct» q-analogues of the gamma and beta functions, since they reduce to $\Gamma(t)$ and $\mathrm{B}(t,s)$ respectively in the limit $q \to 1$, and they satisfy properties analogues to (2.4) and (2.5). This is stated in the following

THEOREM 2.2. (a) *$\Gamma_q(t)$ can be equivalently expressed as*

$$\Gamma_q(t) = \frac{(1-q)_q^{t-1}}{(1-q)^{t-1}} \ . \tag{2.8}$$

In particular one has

$$\Gamma_q(t+1) = [t]\Gamma_q(t) \ , \quad \forall t > 0 \ , \quad \Gamma_q(1) = 1 \ .$$

(b) *The q-gamma and q-beta functions are related to each other by the following two equations*

$$\Gamma_q(t) = \frac{\mathrm{B}_q(t,\infty)}{(1-q)^t} \ , \tag{2.9}$$

$$\mathrm{B}_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)} \ . \tag{2.10}$$

PROOF. We reproduce here the proof of Kac and Cheung [6, pp. 76-79], because similar arguments will be used to prove the results in the next section. If we put $s = \infty$ in the definition of the q-beta function, use (1.8) and the change of variable $x = (1-q)y$, we get

$$\begin{aligned} \mathrm{B}_q(t,\infty) &= \int_0^1 x^{t-1}(1-qx)_q^{\infty} d_q x = \int_0^1 x^{t-1} E_q^{-\frac{qx}{1-q}} d_q x \\ &= (1-q)^t \int_0^{1/(1-q)} y^{t-1} E_q^{-qy} d_q y = (1-q)^t \Gamma_q(t) \ , \end{aligned}$$

which proves (2.9). It follows from q-integration by parts (1.14) and equation (1.7) that $\mathrm{B}_q(t,s)$ satisfies the following recurrence relations ($t, s > 0$):

$$\text{(2.11)}\qquad \begin{aligned} B_q(t+1,s) &= -\frac{1}{[s]}\int_0^1 x^t D_q(1-x)_q^s d_qx \\ &= \frac{1}{[s]}\int_0^1 (D_qx^t)(1-qx)_q^s d_qx \\ &= \frac{[t]}{[s]}B_q(t,s+1)\ , \\ B_q(t,s+1) &= \int_0^1 x^{t-1}(1-qx)^s d_qx \\ &= \int_0^1 x^{t-1}(1-q^sx)(1-qx)_q^{s-1} d_qx \\ &= B_q(t,s)-q^sB_q(t+1,s)\ . \end{aligned}$$

The above equations (2.11) imply

$$B_q(t,s+1)=\frac{[s]}{[s+t]}B_q(t,s)\ .$$

Since clearly $B_q(t,1)=\frac{1}{[t]}$, we get, for $t>0$ and any positive integer n,

$$\text{(2.12)}\qquad \begin{aligned} B_q(t,n) &= \frac{[n-1]\dots[1]}{[t+n-1]\dots[t]} = (1-q)\frac{(1-q)_q^{n-1}}{(1-q^t)_q^n} \\ &= (1-q)\frac{(1-q)_q^{n-1}(1-q)_q^{t-1}}{(1-q)_q^{t+n-1}}\ . \end{aligned}$$

Taking the limit for $n\to\infty$ in this expression we get

$$B_q(t,\infty)=(1-q)(1-q)_q^{t-1}\ .$$

This together with (2.9) proves (2.8).

Now we prove (2.10). By comparing (2.12) and (2.8) we have that (2.10) is true for any positive integer value of s. To conclude that (2.10) holds for non integer values of s we will use the following simple argument. If we substitute $a=q^s$ and $b=q^t$ in (2.10) we can write the left-hand side as

$$(1-q)\sum_{n\geq 0} b^n\,\frac{(1-q^n)_q^\infty}{(1-aq^{n-1})_q^\infty}\ ,$$

and the right-hand side as

$$(1-q)\frac{(1-q)_q^\infty(1-ab)_q^\infty}{(1-a)_q^\infty(1-b)_q^\infty}\ .$$

Both these expressions can be viewed as formal power series in q with coefficients rational functions in a and b. Since we already know that they coincide, for any given b, for infinitely many values of a (of the form $a = q^n$, with positive integer n), it follows that they must be equal for every value of a and b. This concludes the proof of the theorem. □

3. An equivalent definition of q-gamma and q-beta functions

In the previous section the definition of $\Gamma_q(t)$ was obtained from the integral expression (2.1) of the Euler's gamma function, simply by replacing the integral with a Jackson integral and the exponential function e^{-x} with its q-analogue E_q^{-qx}. It is natural to ask what happens if we use the other q-exponential function. In other words, we want to study the following function ($A > 0$):

$$\gamma_q^{(A)}(t) = \int_0^{\infty/A(1-q)} x^{t-1} e_q^{-x} d_q x \; . \tag{3.1}$$

Similarly, the function $\mathrm{B}_q(t,s)$ was obtained by taking the q-analogue of the integral expression (2.2) of the Euler's beta function. We now want to study the q-analogue of the integral expression (2.3). We thus define

$$\beta_q^{(A)}(t,s) = \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_q x \; . \tag{3.2}$$

In this section we will show how the functions $\gamma_q^{(A)}(t)$ and $\beta_q^{(A)}(t,s)$ are related to the q-gamma and q-beta function respectively. We will adapt in this situation the same arguments used for the proof of Theorem 2.2. Recall the main steps used there

1. put $s = \infty$ in the definition of $\mathrm{B}_q(t,s)$ to derive equation (2.9),
2. use q-integration by parts to derive recurrence relations for $\mathrm{B}_q(t,s)$,
3. notice that $\mathrm{B}_q(t,s)$ is a formal power series in q with coefficients rational functions in $a = q^s$ and $b = q^t$.

Step 1. By taking the limit $s \to \infty$ in the definition of $\beta_q^{(A)}(t,s)$, using the infinite product expansion of e_q^x and making the change of variables $x = (1-q)y$, we get

$$\begin{aligned}\beta_q^{(A)}(t,\infty) = \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{\infty}} d_q x \; &= \int_0^{\infty/A} x^{t-1} e_q^{-\frac{x}{1-q}} d_q x \\ =&(1-q)^t \int_0^{\infty/A(1-q)} y^{t-1} e_q^{-y} d_q y \; = \; (1-q)^t \gamma_q^{(A)}(t) \; .\end{aligned}$$

We therefore proved

$$\gamma_q^{(A)}(t) = \frac{1}{(1-q)^t}\beta_q^{(A)}(t,\infty)\ . \tag{3.3}$$

Step 2. We now want to find recursive relations for $\gamma_q^{(A)}(t)$ and $\beta_q^{(A)}(t,s)$. By integration by parts (1.14) we get

$$\gamma_q^{(A)}(t+1) = q^{-t}[t]\gamma_q^{(A)}(t)\ .$$

Here we used the fact that $x^t e_q^{-x}$ tends to zero as $x \to 0$ and $x \to +\infty$ (the second fact follows from the identity $e_q^{-x} = 1/E_q^x$). Since obviously $\gamma_q^{(A)}(1) = 1$, we conclude that for every positive integer n (and any value of $A > 0$),

$$q^{n(n-1)/2}\gamma_q^{(A)}(n) = [n-1]! = \Gamma_q(n)\ . \tag{3.4}$$

Let us now consider the function $\beta_q^{(A)}(t,s)$. In order to perform integration by parts we will need the following

LEMMA 3.1. *For arbitrary numbers α, β we have*

$$D_q \frac{x^\alpha}{(1+x)_q^\beta} \ = \ [\alpha]\frac{x^{\alpha-1}}{(1+x)_q^{\beta+1}} - ([\beta]-[\alpha])\frac{x^\alpha}{(1+x)_q^{\beta+1}}\ .$$

PROOF. The lemma follows immediately from the definition (1.1) of q-derivative, and the definition (1.6) of $(1+x)_q^\alpha$. □

It then follows from integration by parts (1.14) and Lemma 3.1 that, for $t, s > 0$

$$\begin{aligned} \beta_q^{(A)}(t+1,s) &= -\frac{1}{[t+s]}q^{-t}\int_0^{\infty/A} (qx)^t D_q \frac{1}{(1+x)_q^{t+s}} d_q x \\ &= q^{-t}\frac{1}{[t+s]}\int_0^{\infty/A} \frac{1}{(1+x)_q^{t+s}} D_q x^t d_q x \\ &= q^{-t}\frac{[t]}{[t+s]}\beta_q^{(A)}(t,s)\ . \end{aligned} \tag{3.5}$$

For $t = 1$ we have

$$\beta_q^{(A)}(1,s) = \int_0^{\infty/A} \frac{1}{(1+x)_q^{s+1}} d_q x = \frac{1}{[s]}\ . \tag{3.6}$$

Formulas (3.5) and (3.6) imply ($s > 0,\ n \in \mathbb{Z}_+$)

$$q^{n(n-1)/2}\beta_q^{(A)}(n,s) = (1-q)\frac{(1-q)_q^{s-1}(1-q)_q^{n-1}}{(1-q)_q^{s+n-1}} = \mathrm{B}_q(n,s)\ . \tag{3.7}$$

Similarly we have by integration by parts (1.14) and Lemma 3.1

$$\beta_q^{(A)}(t,s+1) = \frac{1}{[t+s]}q^s \int_0^{\infty/A} \frac{1}{(qx)^s} D_q \frac{x^{t+s}}{(1+x)_q^{t+s}} d_q x \tag{3.8}$$

$$= -q^s \frac{1}{[t+s]} \int_0^{\infty/A} \frac{x^{t+s}}{(1+x)_q^{t+s}} D_q \frac{1}{x^s} d_q x$$

$$= \frac{[s]}{[t+s]} \beta_q^{(A)}(t,s) \ .$$

We now need to compute $\beta_q^{(A)}(t,1)$. By definition of $\beta_q^{(A)}(t,s)$ and Lemma 3.1 we have

$$\beta_q^{(A)}(t,1) = \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+1}} d_q x = \frac{1}{[t]} \int_0^{\infty/A} D_q \frac{x^t}{(1+x)_q^t} d_q x \ . \tag{3.9}$$

When using the fundamental theorem of q-calculus to compute the right-hand side, we have to be careful, since the limit for $x \to +\infty$ of the function $F(x) = \frac{x^t}{(1+x)_q^t}$ does not exist. On the other hand, by definition of q-derivative and Jackson integral, we have

$$\int_0^{\infty/A} D_q F(x) d_q x = \lim_{N\to\infty} F\left(\frac{1}{Aq^N}\right) - \lim_{N\to\infty} F\left(\frac{q^N}{A}\right) ,$$

where the limits on the right-hand side are taken over the sequence of integer numbers N. We then have from (3.9)

$$\beta_q^{(A)}(t,1) = \frac{1}{[t]} \left(\lim_{N\to\infty} (Aq^N)^t (1 + \frac{1}{Aq^N})_q^t \right)^{-1} \ . \tag{3.10}$$

Let us denote by $K(A;t)$ the limit in parenthesis in the right-hand side of (3.10). We can compute the limit for $N \to \infty$ in the expression of $K(A,t)$, simply using the definition (1.6) of $(1+x)_q^\alpha$:

$$K(A;t) = A^t \lim_{N\to\infty} q^{Nt} \frac{(1+q^{-N}/A)\cdots(1+q^{-1}/A)}{(1+q^{t-N}/A)\cdots(1+q^{t-1}/A)} \frac{(1+1/A)_q^\infty}{(1+q^t/A)_q^\infty}$$

$$= A^t \left(1+\frac{1}{A}\right)_q^t \lim_{N\to\infty} \frac{(1+Aq)\cdots(1+Aq^N)}{(1+Aq^{1-t})\cdots(1+Aq^{N-t})}$$

$$= \frac{1}{1+A} A^t \left(1+\frac{1}{A}\right)_q^t (1+A)_q^{1-t} \ .$$

From (3.8) and (3.10) we conclude that for any $t > 0$ and positive integer n

$$K(A;t)\beta_q^{(A)}(t,n) = (1-q) \frac{(1-q)_q^{n-1}(1-q)_q^{t-1}}{(1-q)_q^{n+t-1}} = \mathrm{B}_q(t,n) \ . \tag{3.11}$$

In the following lemma we enumerate some interesting properties of the function

$$K(x;t) = \frac{1}{1+x} x^t \left(1+\frac{1}{x}\right)_q^t (1+x)_q^{1-t} \ .$$

LEMMA 3.2. (a) *In the limits $q \to 1$ and $q \to 0$ we have*

$$\lim_{q\to 1} K(x;t) \ = \ 1 \ , \qquad \forall\, x,t \in \mathbb{R}$$

$$\lim_{q\to 0} K(x;t) \ = \ x^t + x^{t-1} \ , \qquad \forall\, t \in (0,1), \ x \in \mathbb{R} \ .$$

In particular $K(x,t)$ is not constant in x.

(b) *Viewed as a function of t, $K(x;t)$ satisfies the following recurrence relation:*

$$K(x;t+1) = q^t K(x;t) \ .$$

Since obviously $K(x;0) = K(x;1) = 1$, we have in particular that for any positive integer n

$$K(x;n) = q^{n(n-1)/2} \ .$$

(c) *As function of x, $K(x;t)$ is a «q-constant», namely*

$$D_q K(x,t) = 0 \ , \quad \forall\, t,x \in \mathbb{R} \ .$$

In other words $K(q^n x;t) = K(x;t)$ for every integer n.

PROOF. The limit for $q \to 1$ of $K(x;t)$ is obviously 1. In the limit $q \to 0$ we have, for any $\alpha > 0$,

$$(1+x)_q^\alpha = (1+x)\frac{(1+qx)_q^\infty}{(1+q^\alpha x)_q^\infty} \ \longrightarrow \ (1+x) \ .$$

We therefore have, for $t \in (0,1)$: $\lim_{q\to 0} K(x;t) = x^t\left(1+\frac{1}{x}\right)$. For part (*b*) we need to use the following obvious identity

$$(1+y)_q^{\alpha+1} \ = \ (1+q^\alpha y)(1+y)_q^\alpha \ .$$

It follows from the definition of $K(x;t)$ that

$$\begin{aligned} K(x;t+1) \ &= \ \frac{1}{1+x} x^{t+1}\left(1+\frac{1}{x}\right)_q^{t+1} (1+x)_q^{-t} \\ &= \ x\left(1+\frac{q^t}{x}\right)\frac{1}{(1+q^{-t}x)} K(x;t) \ = \ q^t K(x;t) \ . \end{aligned}$$

For part (*c*) it suffices to prove that $K(qx;t) = K(x;t)$. By definition

$$K(qx;t) = \frac{1}{1+qx}(qx)^t\left(1+\frac{1}{qx}\right)_q^t (1+qx)_q^{1-t} \ .$$

We can replace in the right-hand side

$$\left(1+\tfrac{1}{qx}\right)_q^t = \frac{1+1/(qx)}{1+q^t/(qx)}\left(1+\frac{1}{x}\right)_q^t ,$$

$$(1+qx)_q^{1-t} = \frac{1+q^{1-t}x}{1+x}(1+x)_q^{1-t} .$$

The claim follows from the following trivial identity

$$\frac{q^t\left(1+\frac{1}{qx}\right)(1+q^{1-t}x)}{(1+qx)\left(1+\frac{1}{q^{1-t}x}\right)} = 1 .$$

This concludes the proof of the lemma. □

Remark 3.3. The function $K(x;t)$ is an interesting example of a function which is not constant in x and with q-derivative identically zero.

Step 3. It follows from (3.7), (3.11) and Lemma 3.2 that the functions $K(A;t)\beta_q^{(A)}(t,s)$ and $\mathrm{B}_q(t,s)$ coincide for any $A>0$ as soon as either t or s is a positive integer. We want to prove that they actually coincide for every $t,s>0$.

Theorem 3.4. *For every $A,t,s>0$ one has:*

$$K(A;t)\gamma_q^{(A)}(t) = \Gamma_q(t) , \tag{3.12}$$

$$K(A;t)\beta_q^{(A)}(t,s) = \mathrm{B}_q(t,s) . \tag{3.13}$$

Remark 3.5. This result corrects and generalizes a similar statement of Jackson [5]. There (3.13) is proved in the special case in which $s+t$ is a positive integer, But, due to a computational mistake, the factor $K(A;t)$ is missing.

Proof of Theorem 3.4. Equation (3.12) is an immediate corollary of (2.9), (3.3) and (3.11). As in the proof of Theorem 2.2, in order to prove (3.13) it suffices to show that $K(A;t)\beta_q^{(A)}(t,s)$ can be written as formal power series in q with coefficients rational functions in $a=q^s$ and $b=q^t$. After performing a change of variable $y=Ax$, we get

$$K(A;t)\beta_q^{(A)}(t,s) = \frac{1}{1+A}\left(1+\frac{1}{A}\right)_q^t (1+A)_q^{1-t}\int_0^{\infty/1}\frac{y^{t-1}}{\left(1+\frac{y}{A}\right)_q^{t+s}}d_qy . \tag{3.14}$$

Fix $A > 0$. After letting $b = q^t$, we can rewrite the factor in front of the integral as

$$\frac{1}{1+A}\frac{\left(1+\frac{1}{A}\right)_q^\infty}{\left(1+\frac{b}{A}\right)_q^\infty}\frac{(1+A)_q^\infty}{\left(1+\frac{qA}{b}\right)_q^\infty} ,$$

and this is manifestly a formal power series in q with coefficients rational functions in b. We then only need to study the integral term in (3.14), which we decompose as

$$\int_0^1 \frac{y^{t-1}}{\left(1+\frac{y}{A}\right)_q^{t+s}} d_q y + \int_1^{\infty/1} \frac{y^{t-1}}{\left(1+\frac{y}{A}\right)_q^{t+s}} d_q y. \tag{3.15}$$

After letting $a = q^s$ and $b = q^t$ the first term in (3.15) can be written as

$$(1-q)\sum_{n\geq 0} b^n \frac{\left(1+ab\frac{q^n}{A}\right)}{\left(1+\frac{q^n}{A}\right)} ,$$

and this is a formal power series in q with coefficients rational functions in a and b. We are left to consider the second term in (3.15). By relation (1.22) we can rewrite it as

$$\int_0^q \frac{x^{s-1}}{x^{t+s}\left(1+\frac{1}{Ax}\right)_q^{t+s}} d_q x . \tag{3.16}$$

Recalling the definition of $K(x;t)$, we have the identity

$$\frac{1}{x^{t+s}\left(1+\frac{1}{Ax}\right)_q^{t+s}} = \frac{1}{1+Ax}(1+Ax)_q^{1-t-s}\frac{A^{t+s}}{K(Ax;t+s)} .$$

The main observation is that, even though $K(Ax; t+s)$ is not constant in x, by Lemma 3.2 $K(Aq^n; t+s) = K(A; t+s)$, $\forall\, n \in \mathbb{Z}$, therefore inside the Jackson integral it can be treated as a constant. Using this fact, we can rewrite (3.16) as

$$\frac{A^{t+s}}{K(A;t+s)}\int_0^q \frac{1}{1+Ax}x^{s-1}(1+Ax)_q^{1-t-s} d_q x . \tag{3.17}$$

After letting $a = q^s$ and $b = q^t$ we can finally rewrite the first factor in (3.17) as

$$\frac{(1+A)\left(1+\frac{ab}{A}\right)_q^\infty\left(1+\frac{qA}{ab}\right)_q^\infty}{\left(1+\frac{1}{A}\right)_q^\infty(1+A)_q^\infty} , \tag{3.18}$$

and the integral term in (3.17) as

$$(1-q)\sum_{n\geq 0} a^{n+1}\frac{(1+Aq^{n+2})_q^\infty}{\left(1+\frac{Aq^{n+2}}{ab}\right)_q^\infty} . \tag{3.19}$$

Clearly both expression (3.18) and (3.19) are formal power series in q with coefficients rational functions in a and b. This concludes the proof of the theorem. □

REMARK 3.6. The proof of Theorem 3.4 is more complicated than one may have expected. This is due to the fact that we do not have a formula for the change of variables $x \to x+1$. Notice that, instead, we took advantage of the fact that one can take out of the sign of q-integral an arbitrary q-constant, not just a constant as for ordinary integrals.

4. APPLICATION 1: AN INTEGRAL EXPRESSION OF THE q-BETA FUNCTION MANIFESTLY SYMMETRIC IN t AND s

Theorem 2.2 implies that $B_q(t,s)$ is a symmetric function in t and s. This is not obvious from its integral expression (2.7). We now want to use Theorem 3.4 to find an integral expression for $B_q(t,s)$ which is manifestly symmetric under the exchange of t and s. By Theorem 3.4 we have that, for any $A>0$

$$B_q(t,s) = K(A;t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_q x \ . \tag{4.1}$$

By definition of $K(x,t)$ we get, after simple algebraic manipulations

$$\frac{1}{x^t}(1+x)_q^{t+s} = K\left(\frac{1}{x};t\right)\left(1+\frac{q}{q^t x}\right)_q^t (1+q^t x)_q^s \ . \tag{4.2}$$

Since by Lemma 3.2 we have

$$K\left(\frac{1}{x};t\right) = K(A;t) \ , \quad \forall\, x = \frac{q^n}{A} \ , \ n \in \mathbb{Z} \ ,$$

when we substitute (4.2) back into (4.1) we get, after a change of variable $y = q^t x$,

$$B_q(t,s) = \int_0^{\infty/\alpha} \frac{1}{y\left(1+\frac{q}{y}\right)_q^t (1+y)_q^s} d_q y \ , \quad \forall\, \alpha > 0 \ . \tag{4.3}$$

To conclude, we just notice that this integral expression of $B_q(t,s)$ is manifestly symmetric in t and s, since performing the change of variable $x = \frac{q}{y}$ (namely applying the reciprocity relation (1.22)) gives the same integral with α replaced by $1/\alpha$ and s replaced by t.

5. APPLICATION 2: TRANSLATION INVARIANCE OF A CERTAIN TYPE OF IMPROPER INTEGRALS

One of the main failures of the Jackson integral is that there is no analogue of the translation invariance identity

$$\int_0^a f(x)dx = \int_c^{a+c} f(x-c)dx \ ,$$

obviously true for «classical» integrals. By using Theorem 3.4 we are able to write a q-analogue of translation invariance for improper integrals of a special class of function, namely of the form $x^\alpha/(1+x)_q^\beta$. More precisely we want to prove the following

COROLLARY 5.1. *For* $\alpha > 0$ *and* $\beta > \alpha + 1$ *we have*

$$\int_0^{\infty/A} \frac{x^\alpha}{(1+x)_q^\beta} d_q x = \frac{q}{q^\beta K(A,\alpha)} \int_1^{\infty/1} \frac{x^\alpha \left(1 - \frac{1}{x}\right)_q^\alpha}{x^\beta} d_q x \ . \tag{5.1}$$

REMARK 5.2. In the «classical» limit $q = 1$, the right-hand side is obtained from the left-hand side by translating $x \to x - 1$.

PROOF. From the definition of $\mathrm{B}_q(t,s)$ we have

$$\begin{aligned} \mathrm{B}_q(t,s) &= \int_0^1 x^{s-1}(1-qx)_q^{t-1} d_q x \\ &= \int_q^{\infty/1} \frac{1}{x^{s+1}} \left(1 - \frac{q}{x}\right)_q^{t-1} d_q x \\ &= \frac{1}{q^s} \int_1^{\infty/1} \frac{x^{t-1}\left(1 - \frac{1}{x}\right)_q^{t-1}}{x^{t+s}} d_q x \ . \end{aligned} \tag{5.2}$$

The first identity was obtained by applying (1.22) and the second by a change of variable $y = x/q$. From Theorem 3.4 we also have

$$\mathrm{B}_q(t,s) = K(A;t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_q x \ . \tag{5.3}$$

Equation (5.1) is obtained by comparing (5.2) and (5.3), after letting $\alpha = t-1$, $\beta = t+s$ and using the fact that $K(A;\alpha+1) = q^\alpha K(A;\alpha)$. □

REMARK 5.3. As we have remarked above, the q-integral is not translation invariant. However, as pointed out in [10] (see also [13]), translation invariance can sometimes be restored by working with q-commuting variables. T.H. Koornwinder also pointed out to us that formula (3.13) is essentially equivalent to exercise 6.17(i) from [3].

6. Application 3: identities

If we rewrite equations (3.12), (3.13) and (4.3) using the definition of improper integrals, we get some interesting identities involving q-bilateral series.

After using the infinite product expansion (1.9) of the q-exponential function e_q^x, the expression (2.8) of the q-gamma function, the definition (1.21) of the improper Jackson integral and simple algebraic manipulations, we can rewrite equation (3.12) as

$$(6.1)\quad (1-q)_q^{\infty}(1+q^t/A)_q^{\infty}(1+qA/q^t)_q^{\infty} = (1+qA)_q^{\infty}(1-q^t)_q^{\infty}\sum_{n=-\infty}^{\infty} q^{tn}(1+1/A)_q^n .$$

If we let $x=-q^t/A$ in equation (6.1) we get

$$(6.2)\quad (1-q)_q^{\infty}(1-x)_q^{\infty}(1-q/x)_q^{\infty} = (1+qA)_q^{\infty}(1+Ax)_q^{\infty}\sum_{n=-\infty}^{\infty} (-x)^n A^n (1+1/A)_q^n .$$

This is a 1-parameter family of identities for the well known Jacobi triple product $(1-q)_q^{\infty}(1-x)_q^{\infty}(1-q/x)_q^{\infty}$, parametrized by A. Notice that

$$\lim_{A\to 0} A^n(1+1/A)_q^n = q^{n(n-1)/2} .$$

This implies that, in the limit $A\to 0$, equation (6.2) reduces to the famous Jacobi triple product identity (1.23).

Let us consider now equation (3.13). After using the definition (1.21) of improper q-integral, the expression (1.15) for the q-beta function and simple algebraic manipulations, we can rewrite it as

$$(6.3)\qquad \sum_{n=-\infty}^{\infty} \frac{(1+1/A)_q^n}{(1+q^{t+s}/A)_q^n} q^{tn} = \frac{(1-q)_q^{\infty}(1-q^{t+s})_q^{\infty}(1+q^t/A)_q^{\infty}(1+qA/q^t)_q^{\infty}}{(1+q^{t+s}/A)_q^{\infty}(1+qA)_q^{\infty}(1-q^t)_q^{\infty}(1-q^s)_q^{\infty}} .$$

Notice that, after letting $a=-1/A,\ b=-q^{t+s}/A,\ x=q^t$, equation (6.3) is equivalent to the famous Ramanujan's identity (1.24). In other words, the proof of Theorem 3.4 in Section 3 can be viewed as a new conceptual proof of Ramanujan's identity.

Finally we can rewrite equation (4.3) as

$$(6.4)\qquad \sum_{n=-\infty}^{\infty} \frac{(1+1/\alpha)_q^n(1+q\alpha)_q^{-n}}{(1+q^s/\alpha)_q^n(1+q^{t+1}\alpha)_q^{-n}} = \frac{(1+1/\alpha)_q^{\infty}(1+q\alpha)_q^{\infty}}{(1+q^s/\alpha)_q^{\infty}(1+q^{t+1}\alpha)_q^{\infty}}\frac{(1-q)_q^{\infty}(1-q^{t+s})_q^{\infty}}{(1-q^s)_q^{\infty}(1-q^t)_q^{\infty}} .$$

After letting $a=-1/\alpha,\ b=-q^s/\alpha,\ c=-q^{t+1}\alpha$, equation (6.4) reduces to (1.25).

Remark 6.1. It is also interesting to see what are the identities corresponding to the original integral expression of the q-gamma and q-beta functions. For this, expand the integral expression (2.6) of the q-gamma function using the definition of q-integral, use the infinite product representation (1.8) of the q-exponential function and compare the result with (2.8). The resulting identity is:

$$\frac{1}{(1-q^t)_q^\infty} = \sum_{n\geq 0} \frac{q^{nt}}{(1-q)_q^n} ,$$

which is, after replacing $x = q^t/(1-q)$, the same as Euler's identity (1.9). Similarly, expand the integral expression (2.7) of the q-beta function and use (2.10) to get

$$\sum_{n\geq 0} q^{nt} \frac{(1-q^s)_q^n}{(1-q)_q^n} = \frac{(1-q^{s+t})_q^\infty}{(1-q^t)_q^\infty} .$$

If we then make the change of variables $a = q^s,\ x = q^t$, we get the well known Heine's product formula for a q-hypergeometric series:

$$\frac{(1-ax)_q^\infty}{(1-x)_q^\infty} = \sum_{n\geq 0} \frac{(1-a)_q^n}{(1-q)_q^n} x^n .$$

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