Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Ioannis Athanasopoulos

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **15** (2004), n.3-4, p. 345–355.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2004_9_15_3-4_345_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2004.

Rend. Mat. Acc. Lincei
s. 9, v. 15:345-355 (2004)

IOANNIS ATHANASOPOULOS

FREE BOUNDARY REGULARITY IN STEFAN TYPE PROBLEMS

ABSTRACT. — Regularity results of free boundaries for Stefan type problems are discussed. The influence that curvature may have on the behavior of the free boundary is studied and various open problems are also mentioned.

KEY WORDS: Free boundary problem; Stefan problem; Regularity; Curvature; Viscosity solution.

1. INTRODUCTION

In this *Note* I would like to present various results, which were obtained in collaboration with L.A. Caffarelli and S. Salsa, and mention some open problems about regularity of free boundaries for Stefan-type problems whose speed may or may not depend on its curvature. The problems are of local nature and classical solutions can be described as follows:

On a unit cylinder $Q_1 = B_1 \times (-1, 1)$ we have two complementary domains Ω and $Q_1 \backslash \Omega$ separated by a smooth surface $S = \partial\Omega \cap Q_1$. In Ω and $Q_1 \backslash \Omega$ we have two smooth solutions u_1 and u_2 of the heat equations

$$\Delta u_1 - D_t u_1 = 0 = \Delta u_2 - D_t u_2$$

with $u_2 \leqslant 0 \leqslant u_1$. The functions u_1 are C^1 up to S and along S both $u_i = 0$ and the interphase energy balance conditions

$$\text{(I)} \qquad \frac{(u_i)_t}{(u_i)_\nu} = G((x, t), \nu, (u_1)_\nu, (u_2)_\nu)$$

or

$$\text{(II)} \qquad \frac{(u_i)_t}{(u_i)_\nu} = G((x, t), \nu, (u_1)_\nu, (u_2)_\nu, \kappa)$$

sre satisfied for some appropriate G (see below).

The best known example of these problems is, of course, the Stefan problem *i.e.* when $G = (u_2)_\nu - (u_1)_\nu$. Problems of type II *i.e.* when G depends also on the curvature of the free boundary may be not connected to phase field models but they do occur naturally in certain biological models; see for example the work of Rubinstein [9].

To treat such problems one must start with a «weak» solution which should be constructed for all times and then would like to obtain optimal regularity for such solutions. In the case of the Stefan problem weak solutions to the inclusion

$$\Delta u \in \beta(u)_t$$

with $\beta(u) = u^+ = u^- + \operatorname{sgn} u$ are well known to exist and in 1983 were proved (see [6]) to be continuous with logarithmic modulus of continuity. Then heuristically $u_1 = u^+$ in $\Omega = \{u > 0\}$, $u_2 = -u^-$ in $\Omega^c = \{u \leqslant 0\}$ and $\partial\Omega = \{u > 0\}$ becomes the free boundary.

Once the optimal or adequate regularity of the solution is known the regularity of free boundaries may be approached in a fashion similar to that of minimal surface theory:

1. Lipschitz free boundaries are smooth.
2. «Flat» free boundaries are smooth.
3. Free boundaries are «surfaces» in some generalized sense and are smooth except on some small set.

Before discussing case 1 and 2 separately in the sections to be followed let me mention that except for the (classical) Stefan problem *i.e.* when $G = u_\nu^+ - u_\nu^-$ where free boundaries have been shown to be locally boundaries of sets of finite perimeter nothing has been done so far in case 3. Concluding this introduction let me be more precise about the notion of a weak solution. It is given in the sense of a viscosity solution which requires the definition of a classical super and sub-solution. In the sequel, if it is not necessary, we do not distinguish interphase condition (I) from (II), since (II) includes (I).

Definition 1.1. Let v be a continuous function in $Q_1 := B_1 \times (-1, 1)$, where $B_1 := B_1(0) = \{x \in \mathbb{R}^n\colon |x| < 1\}$ Then v is called a classical subsolution (supersolution) to a free boundary problem if

i) $\Delta v - v_t \geqslant 0 \ (\leqslant 0)$ in $\Omega^+ := Q_1 \cap \{v > 0\}$

ii) $\Delta v - v_t \geqslant 0 \ (\leqslant 0)$ in $\Omega^- := Q_1 \cap \{v < 0\}$

iii) $v \in C^1(\overline{\Omega}^+ \cap C^1(\overline{\Omega}^-)$

iv) for any $(x,t) \in \partial\Omega^+ \cap Q_1$, $\nabla_x v^+(x,t)) \neq 0$, and $V_\nu \geqslant -G((x,t), \nu, v_\nu^+, v_\nu^-, \kappa)(\leqslant)$ where V_ν is the speed of the surface $F_t := \partial\Omega^+ \cap \{t\}$ in the direction $\nu := \dfrac{\nabla_x v^+}{|\nabla_x v^+|}$, κ the curvature of F_t (taken positive if F_t is convex in the direction of ν) and G is increasing in v_ν^+, decreasing in v_ν^- and κ, continuous in all of the arguments with $G \to +\infty$ if $v_\nu^+ - v_\nu^- - \kappa \to +\infty$.

We say that v is a classical solution to a free boundary problem if it is both a subsolution and a supersolution. The set $\partial\Omega^+ \cap Q_1$ is called the free boundary.

Remark. In the above definition condition (*iv*) can be replaced by

$$\frac{v_t^+}{v_\nu^+} \leqslant G((x,\, t),\, \nu,\, v_\nu^+,\, v_\nu^-,\, \kappa)(\geqslant).$$

Definition 1.2. Let u be a continuous function in Q_1. Then u is called a viscosity subsolution (supersolution) to a free boundary problem if, for any subcylinder Q of Q_1 and every classical supersolution (subsolution) v in Q with $u \leqslant v$ $(u \geqslant v)$ on $\partial_p Q$ (∂_p-parabolic boundary) implies that $u \leqslant v$ $(u \geqslant v)$ in Q. The function u is called a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

Notice that the above mentioned weak solution to the Stefan problem is also a solution in the viscosity sense.

Notation: In the next sections we use $x^+ = \max(x, 0)$, $x^- = -\min(x, 0)$ but when $+$ or $-$ appear as indices have no independent meaning.

2. Lipschitz free boundaries

Why one would like to start with Lipschitz? One of the reasons is that the techniques developed under this assumption may be refined to be applied to the other cases. Of course, it has also its own interest since under special geometry of the domain and/or initial data lead to solutions which provide directly Lipschitz free boundaries (*i.e.* monotone solution along some space directions).

By Lipschitz free boundaries we mean Lipschitz in space and time. This is not the natural homogeneity balance for the study of parabolic equations but it is so for the study of the phase transition relations of the form

$$F(u_\nu^+, u_\nu^-, V_\nu) = 0$$

along $\partial\Omega^+$.

Under this hypothesis one can obtain the optimal regularity of the solution *i.e.* Lipschitz continuity. It is the optimal since the jump conditions (I) or (II) hold across the free boundary. The main ingredients for the proof of this result are local properties given below that nonnegative solutions to the heat equation possess in Lipschitz domains for their proof we refer to [2]. The first one is that the solution is monotone in a space-time cone determined by the Lipschitz constant of the free boundary *i.e.*

Cone of monotonicity. *Any nonnegative solution u to the heat equation in $Q_1 \cap \Omega$ with $u = 0$ on $Q_1 \cap \partial\Omega$ is increasing along every direction $\mu \in \Gamma(e_n, \overline{\theta}) :=$ $:= \{\mu \in \mathbb{R}^{n+1} \colon |\mu| = 1,\ \mu \cdot e_n < \cos\theta\}$ where $\overline{\theta} = \frac{1}{2}\cot^{-1}(L)$ in $Q_\delta \cap \Omega$ for $\delta =$ $= \delta\left(n, L, \frac{m}{M}, \|\nabla u\|_{L^2}\right)$ small enough where n is the dimension, L the Lipschitz constant of $\partial\Omega := \{x_n = f(x', t) : x' \in \mathbb{R}^{n-1}\}$, $0 < m := u(x_0, t_0)$ for (x_0, t_0) a fixed point away from $\partial\Omega$, and M the supremum of u.*

This cone of monotonicity allows one to control the time derivative by the space gradient and this establishes the «almost harmonicity» of u at each time level, more precisely:

ALMOST HARMONICITY. *Let u be as above, then there exists $\varepsilon > 0$ and $\delta > 0$ depending only on n and L such that*

$$w_+ := u + u^{1+\varepsilon} \qquad w_- := u - u^{1+\varepsilon}$$

are subharmonic and superharmonic, respectively, in $Q_\delta \cap \Omega \cap \{t = 0\}$.

An important consequence of this is that the monotonicity formula of [1] will hold in our situation.

MONOTONICITY FORMULA. *Let w_1, w_2 be continuous subharmonic functions in the unit ball $B_1 \subset \mathbb{R}^n$ such that $w_1(0) = w_2(0) = 0$ and $w_1 \cdot w_2 = 0$ in B_1. Then*

$$g(r) := \frac{1}{r^4} \int_{B_r} \frac{|\nabla w_1|^2}{|x|^{n-2}} dx \int_{B_r} \frac{|\nabla w_2|^2}{|x|^{n-2}} dx$$

is an increasing function of r. Moreover, if $w := w_1 + w_2$ for $r < \frac{1}{2}$ we have $g(r) \leqslant c(n) \|w\|^4_{L^\infty}$.

By setting $w_1 := u^+ + (u^+)^{1+\varepsilon}$ and $w_2 := u^- + (u^-)^{1-\varepsilon}$ one can easily see that the monotonicity formula applies.

Finally, the careful analysis of the behavior of the solution near particular points of the free boundary is required.

ASYMPTOTIC DEVELOPMENT. *Let u be a viscosity solution and monotone for every $\sigma \in \Gamma(e_n, \overline{\theta})$. If there is an $(n+1)$-dimensional ball $B \subset \Omega^+$* (resp. $B \subset \Omega^-$) *such that $B \cap \partial\Omega^+ = \{(0, 0)\}$ then*

$$u(x, t) \geqslant (\beta t + \alpha \langle x, \nu \rangle)^+ + o(d(x, t))$$

$$(\text{resp.} \quad u(x, t) \leqslant (\beta t + \alpha \langle x, \nu \rangle)^+ + o(d(x, t))$$

for some $\beta \in \mathbb{R}$, $\alpha \in (0, \infty]$ (resp. $\alpha \in [0, \infty)$) *where ν denotes the inward* (resp. *outward*) *radial direction of $B \cap \{t = 0\}$ at $(0, 0)$, $d(x, t)$ the distance between (x, t) and $(0, 0)$, and $< \cdot, \cdot >$ the inner product. Furthermore, equality holds on the hyperplane $t = 0$.*

Now, we have everything we need to prove the Lipschitz continuity of our solution. In general, a solution to a free boundary problem of parabolic type is not expected to be Lipschitz continuous (see for example [6]). The idea of the proof of both cases *i.e.* (I) and (II) is essentially the same. The difference lies only in the construction of a certain classical subsolution. We sketch their proof below (for complete proofs see [2] and [5]).

THEOREM 2.1. *Let u be a viscosity solution to our free boundary problem* (I) *or* (II) *in Q_1. If $\partial\Omega^+$ is Lipschitz in some space direction then u is Lipschitz in $Q_{1/2}$.*

SKETCH OF PROOF. *Type (I).* Since we have a cone of monotonicity for the solution it is enough to prove that the space gradient is bounded. We suppose that at point $(x_0, 0)$ of distance h from the free boundary at $t=0$ level, $u(x_0, 0) = Mh$. We want to bound M independently. For this purpose we construct locally about the origin a classical subsolution v (see [2]).

At $t=0$ level in $B_\delta(0)$ for $\delta > 0$ small we have $\{v>0\} \cap B_\delta(0) \subset \subset B_h(x_0) \subset \{u>0\}$ with $\partial\{v>0\}$ touching $\partial\{u>0\}$ at the origin and $v_\nu^+(0,0) = M$. On the complement of $\{v>0\} \cap B_\delta(0)$ in $B_\delta(0)$ v is negative and $v_\nu^-(0,0) = \alpha$ where α is the coefficient of the first order term of the asymptotic development of our solution u. Since by the monotonicity formula $M^2\alpha^2 < c$, α will be small if M is large.

Finally, the speed of its free boundary at the origin is given by $\frac{\beta}{M}$ where β is chosen so that $\frac{\beta}{M} \sim G((0,0), \nu, M, \alpha)$. Now, we have a classical subsolution v whose free boundary touches at the origin from the right the free boundary of u. If M is large then α is small and the speed of the free boundary of v at $(0,0)$ is large. Since the speed of the free boundary of the solution is bounded, the free boundaries will cross each other. A contradiction, since v is a subsolution for small times.

Type (II). The idea of the proof here is the same. The difference lies in the extra care of the free boundary of v. The free boundary surface must be constructed in such a way that it has bounded mean curvature independently of M. This is done using the distance function (see [5]). □

Now knowing that u is Lipschitz we can make sense in a weak way the free boundary relation at good points of the free boundary.

FREE BOUNDARY RELATION. *Let u be a viscosity solution to a free boundary problem in $D_1 = B_1(0) \times (-1, 1)$ with Lipschitz free boundary and $(0,0) \in \partial\{u>0\}$. Suppose that near the origin and for $t \leq 0$, the following asymptotic inequality holds*:

$$(1)\quad \begin{cases} u(x,t) \geqslant (\alpha_+(x,\nu) + \beta_+ t)^+ - (\alpha_-(x,\nu) + \beta t)^- + o\left(\sqrt{|x|^2+t^2}\right) \\ \qquad\qquad \text{with } \alpha_+ > 0,\ \alpha_- \geqslant 0 \\ (\text{resp. } u(x,t) \leqslant (\alpha_+(x,\nu) + \beta_+ t)^+ - (\alpha_-(x,\nu) + \beta_- t)^- + o\left(\sqrt{|x|^2+t^2}\right) \\ \qquad\qquad \text{with } \alpha_+ \geqslant 0, \alpha_- > 0) \end{cases}$$

with equality holding when $t = 0$. Then

$$(2)\quad \frac{\beta_+}{\alpha_+} \geqslant G((0,0), \nu, \alpha_+, \alpha_-), \qquad \left(\text{resp. } \frac{\beta_-}{\alpha_-} \leqslant G(0,0), \nu, \alpha_+, \alpha_-)\right),$$

when ν denotes an inward (resp. *outward*) *spatial direction with respect to $\{u>0\}$ at $(0,0)$.*

2.1. *Type (I).*

Now, we have all we need to start the free boundary regularity. In problems with free boundary condition (I) though we are confronted with a «hyperbolic» phenomenon *i.e.* free boundaries may not regularize instantaneously. Such phenomena were known to occur in degenerate parabolic equation but not in linear ones. For simplicity we give a counterexample for the one-phase Stefan problem, for a two-phase counterexample see §10 of [3].

COUNTEREXAMPLE. Let $n = 2$ and $f(t)$ be a real value function such that $f(t) > 2$ and $f'(t) \leq 0$ for every $t \geq 0$. Define in polar coordinates,

$$v(r,\theta) := \left(r^{f(t)} \cos\left(f(t)\theta\right)\right)^{+}$$

One can easily check that this function is a supersolution to the one-phase Stefan problem for $r \leq r_o$, with r_o small and any $t \geq 0$. Now, let u be the solution to the one-phase Stefan problem in $B_{r_o} \times (0, T]$ with initial and boundary values equal to those of v when testricted to the parabolic boundary of $B_{r_o} \times (0, T]$. It is clear that the free boundary of u which goes through the origin must stay to the right of the free boundary of v. Since for the one-phase problem we have $u_t \geq 0$, the free boundary of u can move only to the left. Therefore the «corner» stays a «corner» or not depends on $f(t)$.

Once the corner becomes «flat» enough then it smooths out (see Section 3 of this *Note*). At which angles this really occurs is an open question! Consequently we have to assume some kind of «non-degeneracy» in order to avoid the situation just described. This is condition (*ii*) or (*ii*)′ in the following theorem.

THEOREM 2.2. *Let u be a viscosity solution of a free boundary problem in Q_2, whose free boundary, F, is given by the graph of a Lipschitz function $x_n = f(x', t)$ with Lipschitz constant L. Assume that $M = \sup_{Q_2} u\left(e_n, -\frac{3}{2}\right) = 1$, $(0, 0) \in F$, and that*

(*i*) *$G = G(\nu, a, b) : \partial B_1 \times \mathbf{R}^2 \to \mathbf{R}$ is a Lipschitz function in all of its arguments, with Lipschitz constant L_G, and for some positive number c^*,*

$$D_a G \geq c^* \quad \text{and} \quad D_b G \leq -c^*,$$

(*ii*) (*non-degeneracy condition*) *there exists $m > 0$ such that, if $(x_0, t_0) \in F$ is a regular point from the right or from the left, then, for any small r,*

$$\fint_{B_r(x_0)} |u| \geq mr$$

or

(*ii*)′ *there exists $L_0 = L_0(n)$ such that the Lipschitz constant in space of f $L_1 \leq L_0$.*

Then, the following conclusion hold:

(1) *In Q_1 the free boundary is a C^1 graph in space and time. Moreover, for any η, $0 < \eta$, there exists a positive constant $C_1 = C_1(n, L, M, L_G, c^\star, m, a, a_2, \eta)$ such that, for every $(x', x_n, t), (y', y_n, s) \in F$,*

$$|\nabla_{x'} f(x', t) - \nabla_{x'} f(y', t)| \leqslant C_1(-\log|x' - y'|)^{-\frac{3}{2+\eta}}, \tag{1}$$

$$|D_t f(x', t) - D_t f(x', s)| \leqslant C_1(-\log|t - s|)^{-\frac{1}{2+\eta}} \tag{2}$$

(2) *$u \in C^1(\overline{\Omega}^+) \cup C^1(\overline{\Omega}^-)$ and on $F \cap Q_1$.*

$$u_\nu^+ \geqslant C_2 > 0$$

with $C_2 = C_2(n, L, M, L_G, c^*, m, a_1, a_2, \eta)$.
Therefore u is a classical solution.

Sketch of the proof. The result under assumption $(ii)'$ is really a corollary to Theorem 3.1 of the next section. Thus we restrict ourselves to the assumption (ii). As we have seen the Lipschitz assumption of the free boundary determines a cone of monotonicity. To prove that the free boundary is C^1 it is enough to prove the existence of a tangent hyperplane uniformly at each point of the free boundary (or of any level surface) which is equivalent to having a cone of monotonicity at that point of aperture π. Therefore the proof consists in showing that on a sequence of dyadically shrinking cylinders around a free boundary point u becomes monotone on a sequence of cones of increasing aperture which in the limit is π.

The free boundary relation scales hyperbolically and the heat equation scales parabolically. To balance both homogeneities the choice of the shrinking cylinders is neither hyperbolic nor parabolic. This is a delicate matter and is reflected in the estimates of each step.

Briefly, each step is achieved as follows: Away from the free boundary in a region of parabolic size via the parabolic interior Hournack inequality we can have a uniform increase of the cone. One has to refine then the estimates so that this increase remains valid in a hyperbolic region. Next this information has to be propagated to the free boundary. This is done via a family of perturbations to the free boundary.

Finally, to have uC^1 up to the free boundary, we observe that at each time level *i.e.* $t_0 \in (-1, 1)$, $\Omega^\pm \cap \{t = t_0\}$ is a Liapunov-Dini domain. Since u_t is bounded the results of K.-O. Widman [11] apply and therefore $\nabla_x u^\pm$ are continuous up to the free boundary at each time level. Finally using the boundary condition the result follows. □

2.2. *Type (II).*

The presence of curvature in the free boundary condition (II) does not allow any kind of waiting time phenomena to occur such as the one encountered in the previous section. Actually, since we have Lipschitz continuity of the free boundary, the curvature plays a more predominant role with respect to the caloric part of the solution. As a corollary to the Theorem 2.1 we have:

COROLLARY. *The mean curvature of the free boundary is bounded for every t in the viscosity sense.*

The proof of this Corollary is done by contradiction in a similar fashion as that of the Lipschitz continuity of u *i.e.* Theorem 2.1. As a matter of fact, if the curvature of the free boundary is large enough in the viscosity sense we construct a classical subsolution whose free boundary touches at a point from one side the free boundary of the solution. At later times though, if the curvature is large, the free boundaries cross each other, a contradiction.

Since the mean curvature is bounded in the viscosity sense for each t, we apply the result in [7] to conclude that the free boundary is a $C^{1,\alpha}$ surface in $\mathbb{R}^n$ for each t. Hence, since by assumption it is also Lipschitz in time, the solution u is $C^{1,\alpha}$ in space and C^{β} in time up to the free boundary from both sides in a neighborhood of the free boundary. Furthermore, this implies that the space gradient of u is Holder continuous in time, too.

In order to proceed further we need additional structure assumptions on G *i.e.* we suppose that

$$G\big((x,t), \nu, v_\nu^+(x,t), v_\nu^-(x,t), \kappa\big) = -H\big((x,t), \nu, M\big) + g(x,t)$$

where g records the dependence on v_ν^+ and v_ν^-, and M is an $n \times n$ matrix representation of the second fundamental form for the free boundary for each time t. H is assumed to be uniformly elliptic *i.e.* for $\mu > 0$, $e \perp \nu$, $|e| = 1$

$$\lambda\mu \leq H\big(M + \mu(e \otimes e), \nu, (x,t)\big) - H\big(M, \nu, (x,t)\big) \leq \Lambda\mu$$

where λ, Λ positive constant and H is Lipschitz continuous in ν.

Now, we can write our free boundary condition as

$$f_t - H(D^2 f, Df, x, t) = g(x,t)$$

where f is the function representing the free boundary, *i.e.* $x_n = f(x', t)$ for $x' \in \mathbb{R}^{n-1}$, and $g \in C^{\gamma}$. Hence, by [10], f is $C^{1,\alpha}$ in space and time and by classical results we conclude that u is $C^{1,\beta}$ in space and time.

Therefore we have shown the following.

THEOREM 2.3. *Under the above assumptions on G the free boundary is a $C^{1,\alpha}$ surface and the solution is $C^{1,\beta}$ up to the free boundary from both sides. Thus we have a classical solution.*

3. «FLAT» FREE BOUNDARIES

The nondegeneracy condition (*ii*) of Theorem 2.2 actually prevents simultaneous vanishing of the heat flow from both sides of the free boundary. On the other hand, condition (*ii*)′ of the same theorem says that obtuse enough angles do not persist *i.e.* we are in a «nondegenerate» situation enough for regularization. Also, according to Theorem 2.3 when the curvature is present in the free boundary condition we are again in a «nondegenerate» situation, enough for regularization.

It turns out that we can refine our techniques so that we do not require for a free boundary to be a Lipschitz graph or a graph at all. It is enough to have a suitable flatness condition which we express it as an ε-monotonicity:

Given an $\varepsilon > 0$, *a function* u *is called* ε-*monotone in the direction* τ *if* $u(p + \lambda\tau) \geqslant u(p)\ \forall\ \lambda \geqslant \varepsilon$.

Actually, if a free boundary «surface» is flat in the context of minimal surface theory, then one can easily see that its solution is ε-monotone in a cone of directions.

Under this flatness condition we have smoothness of the free boundary for type (I) (Theorem 3.1), but for type (II) this is still an open problem! It is interesting to notice that, with the presence of curvature, the proof of «Lipschitz free boundaries are smooth» is in some sense «easier» than the one when the curvature is absent. But when one considers «flat» free boundaries the situation seems to be reversed.

THEOREM 3.1. *Let* u *be a viscosity solution of a free boundary problem in* Q_2 *that is* ε-*monotone along all directions* $\tau \in \Gamma_x(\theta_0^x, e_n) \cup \Gamma_t(\theta_0^t, \nu)$ *with* $\theta_0^t - \alpha(e_n, \nu) \equiv \theta^* > 0$. *Assume that* $M_0 = \sup_{Q_2} u$, $u\left(e_n, -\frac{3}{2}\right) = 1$, $(0, 0) \in F$, *and* $G = G(\nu, a, b): \partial B_1 \times \mathbb{R}^2 \to \mathbb{R}$ *is a Lipschitz function in all its arguments with Lipschitz constant* L_G *and, for some positive number* c^*,

$$D_a G \geqslant c^*, \qquad D_b G \leqslant -c^*.$$

Then if ε *and* $\delta_0 := \frac{\pi}{2} - \theta_0^x$ *are small enough, depending on* n *and* θ^*, *the following conclusion hold*:

1. *In* Q_1 *the free boundary is a* C^1 *graph in space and time. Moreover, there exists a positive constant* $C_1 = C_1(n, M_0, L_G, c^*, a_1, a_2, \theta^*)$ *such that, for every* (x', x_n, t), $(y', y_n, s) \in F$,
   (*a*) $|\nabla_{x'} f(x', t) - \nabla_{x'} f(y', t)| \leqslant C_1(-\log|x' - y'|)^{-4/3}$
   (*b*) $D_t f(x', t) - D_t f(x', s)| \leqslant C_1(\log|t - s|)^{-1/3}$.
2. $u \in C^1(\overline{\Omega}^+) \cup C^1(\overline{\Omega}^-)$ *and, on* $F \cap Q_1$,
$$u_\nu^+ \geqslant C_2 > 0$$
   with $C_2 = C_2(n, M_0, L_G, c^*, a_1, a_2, \theta^*)$.

Therefore u *is a classical solution.*

In particular, the theorem holds for 0-monotone situations *i.e.* fully monotone. Thus Theorem 2.1 under assumption (*ii*)′ is a corollary to this theorem.

SKETCH OF THE PROOF (see [4] for complete proof). Having an ε-monotonicity in a cone of direction one tries to get in a smaller region a larger cone. Away from the free boundary, this is done actually for every $\varepsilon > 0$. To transfer this information to the free boundary we need an estimate for the normal derivative of u at «regular» points of the free boundary to counterbalance the lack of nondegenerecy. Since ε-monotonicity implies full monotonicity $\sqrt{\varepsilon}$ away from the free boundary, one can improve ε-monotonicity everywhere by giving up a small portion of the enlarged cone. For this decrease in ε one needs to construct a different family of subsolutions. With these tools at hand, we can perform a double-iteration that consists at each step of a cone enlargement and of an ε-monotonicity improvement in a sequence of shrinking domains that are neither hyperbolic nor parabolic and this gives rise to the logarithmic modulus of continuity. □

To illustrate when can this ε-monotonicity occur we give two examples for the Stefan problem *i.e.* when $G = u_\nu^+ - u_\nu^-$. Note that harmonic functions are stationary solutions of the Stefan problem.

EXAMPLE 1. *Suppose u is a solution of the two-phase Stefan problem in $\overline{B_1} \times [0, +\infty)$ converging for $t \to +\infty$ to a harmonic function $u_\infty = u_\infty(x)$, $x \in B_1$, uniformly in any compact subset of B_2. Suppose that at $x_0 \in F(u_\infty)$, $|\nabla u_\infty(x_0)| \neq 0$. Then there exists $T^* > 0$ and a neighborhood V of x_0 such that in $V \times [T^*, +\infty)$, $F(u)$ is a C^1 graph and u is a classical solution.*

PROOF. In a neighborhood V of x_0, u_∞ is monotone along the directions in a cone $\Gamma(\theta^\infty, \nu)$, with $\delta_\infty < \delta_0$, δ_0 small as in Theorem 3.1; therefore, choosing V such that $|\nabla u_\infty(x)| \geq c > 0$ in V, if T^* is large enough, $u(x, t)$, $t \geq T^*$, is ε-monotone along the directions of $\Gamma(\theta^\infty, \nu)$ and the directions of a space-time cone $\Gamma_t(\theta^t, \nu)$ (with the same ν and with θ^t also large). The conclusions follow now from the main theorem. □

EXAMPLE 2. *Let $u(x, 0)$ be a compact perturbation of a traveling wave initial data*; i.e., *there exists*

$$u_0(x, 0) = (A+1)(e^{-x_n} - 1)^+ - A(e^{-x_n} - 1)^-$$

and a $\varphi_0(x)$ with compact support such that

$$u_0(x, 0) - \varphi_0(x) \leq u(x, 0) \leq u_0(x, 0) + \varphi_0(x).$$

Then, after a finite time T^*, $T^*(u_0, \varphi_0)$, the free boundary of the solution, $u(x, t)$ to the Stefan problem, with initial data $u(x, 0)$ is a smooth graph

$$x_n = g(x', t).$$

Its proof relies to the fact that the solution will stay L^∞ close to

$$u_0(x, t) = (A+1)(e^{t - x_n} - 1)^+ - A(e^{t - x_n} - 1)^-$$

i.e. a traveling wave for the Stefan problem with the stated initial value $u_0(x, 0)$ (see [4]).

Acknowledgements

The author was partially supported by RTN project HPRN-CT-2002-00274.

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Department of Applied Mathematics
University of Crete
71409 Heraklion Crete (Grecia)
athan@tem.uoc.gr