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ON A CLASS OF ELLIPTIC OPERATORS WITH UNBOUNDED COEFFICIENTS IN CONVEX DOMAINS

ABSTRACT. — We study the realization A of the operator $\mathcal{A} = \frac{1}{2}\Delta - \langle DU, D\cdot\rangle$ in $L^2(\Omega, \mu)$, where Ω is a possibly unbounded convex open set in $\mathbb{R}^N$, U is a convex unbounded function such that $\lim_{x\to\partial\Omega,\, x\in\Omega} U(x) = +\infty$ and $\lim_{|x|\to+\infty,\, x\in\Omega} U(x) = +\infty$, $DU(x)$ is the element with minimal norm in the subdifferential of U at x, and $\mu(dx) = c\exp(-2U(x))\,dx$ is a probability measure, infinitesimally invariant for $\mathcal{A}$. We show that A, with domain $D(A) = \{u \in H^2(\Omega, \mu) : \langle DU, Du\rangle \in L^2(\Omega, \mu)\}$ is a dissipative self-adjoint operator in $L^2(\Omega, \mu)$. Note that the functions in the domain of A do not satisfy any particular boundary condition. Log-Sobolev and Poincaré inequalities allow then to study smoothing properties and asymptotic behavior of the semigroup generated by A.

KEY WORDS: Kolmogorov operators; Unbounded coefficients; Convex domains.

1. INTRODUCTION

In this paper we give a contribution to the theory of second order elliptic operators with unbounded coefficients, that underwent a great developement in the last few years. See *e.g.* [1, 5-8, 12, 13].

Here we consider the operator

$$\mathcal{A}u = \frac{1}{2}\Delta u - \langle DU, Du\rangle \tag{1.1}$$

in a convex open set $\Omega \subset \mathbb{R}^N$, where U is a convex function such that

$$\lim_{x\to\partial\Omega,\, x\in\Omega} U(x) = +\infty, \qquad \lim_{|x|\to+\infty,\, x\in\Omega} U(x) = +\infty. \tag{1.2}$$

Since we do not impose any growth condition on U, the usual L^p and Sobolev spaces with respect to the Lebesgue measure are not the best setting for the operator $\mathcal{A}$. It is more convenient to introduce the measure

$$\mu(dx) = \left(\int_\Omega e^{-2U(x)}\,dx\right)^{-1} e^{-2U(x)}\,dx, \tag{1.3}$$

which is infinitesimally invariant for $\mathcal{A}$, *i.e.*

$$\int_\Omega \mathcal{A}u(x)\mu(dx) = 0, \qquad u \in C_0^\infty(\mathbb{R}^N),$$

and lets $\mathcal{A}$ be formally self-adjoint in $L^2(\Omega, \mu)$, as an easy computation shows. We prove in fact that the realization A of $\mathcal{A}$ in $L^2(\Omega, \mu)$, with domain

$$D(A) = \{u \in H^2(\Omega, \mu) : \mathcal{A}u \in L^2(\Omega, \mu)\} = \{u \in H^2(\Omega, \mu) : \langle DU, Du\rangle \in L^2(\Omega, \mu)\}$$

is a self-adjoint and dissipative operator, provided $C_0^\infty(\Omega)$ is dense in $H^1(\Omega, \mu)$. We recall that $H^1(\Omega, \mu)$ is naturally defined as the set of all $u \in H^1_{loc}(\Omega)$ such that u,

$D_i u \in L^2(\Omega, \mu)$, for $i = 1, \dots, N$. While it is easy to see that $C_0^\infty(\Omega)$ is dense in $L^2(\Omega, \mu)$, well-known counterexamples show that $C_0^\infty(\Omega)$ is not dense in $H^1(\Omega, \mu)$ in general. A sufficient condition in order that $C_0^\infty(\Omega)$ be dense in $H^1(\Omega, \mu)$ is

$$DU \in L^2(\Omega, \mu). \tag{1.4}$$

Once we know that $C_0^\infty(\Omega)$ is dense in $H^1(\Omega, \mu)$, it is not hard to show that for each $u \in D(A)$ and $\psi \in H^1(\Omega, \mu)$ we have

$$\int_\Omega (\mathcal{A} u)(x)\psi(x)\mu(dx) = -\frac{1}{2}\int_\Omega \langle Du(x), D\psi(x)\rangle \mu(dx).$$

This crucial integration formula implies that A is symmetric and dissipative. The next step is to prove that $\lambda I - A$ is onto for $\lambda > 0$, so that A is m-dissipative. This is done by approximation, solving first, for each $\lambda > 0$ and $f \in C_0^\infty(\Omega)$,

$$\lambda u_\alpha(x) - (\mathcal{A}_\alpha u_\alpha)(x) = f(x), \qquad x \in \mathbb{R}^N, \tag{1.5}$$

where A_α is defined as $\mathcal{A}$, with U replaced by its Moreau-Yosida approximation U_α. To be more precise, first we extend f and U to the whole $\mathbb{R}^N$ setting $f(x) = 0$ and $U(x) = +\infty$ for x outside Ω; since the extension of U is lower semicontinuous and convex the Moreau-Yosida approximations U_α are well defined and differentiable with Lipschitz continuous gradient in $\mathbb{R}^N$. Then (1.5) has a unique solution $u_\alpha \in H^2(\mathbb{R}^N, \mu_\alpha)$, with $\mu_\alpha(dx) = \left(\int_{R^N} e^{-2U_\alpha(x)}dx\right)^{-1} e^{-2U_\alpha(x)}dx$, and the norm of u_α in $H^2(\mathbb{R}^N, \mu_\alpha)$ is bounded by $C(\lambda)\|f\|_{L^2(\mathbb{R}^N, \mu_\alpha)}$, where the constant $C(\lambda)$ is independent of α, due to the estimates for equations in the whole $\mathbb{R}^N$ already proved in [5]. Using the convergence properties of U_α and of DU_α to U and to DU respectively, we arrive at a solution $u \in H^2(\Omega, \mu)$ of

$$\lambda u(x) - (\mathcal{A} u)(x) = f(x), \qquad x \in \Omega, \tag{1.6}$$

that belongs to $D(A)$, satisfies $\|u\|_{H^2(\Omega, \mu)} \le C(\lambda)\|f\|_{L^2(\Omega, \mu)}$ and is the unique solution to the resolvent equation because A is dissipative. If f is just in $L^2(\Omega, \mu)$, (1.6) is solved approaching f by a sequence of functions in $C_0^\infty(\Omega)$.

A lot of nice consequences follow: A generates an analytic contraction semigroup $T(t)$ in $L^2(\Omega, \mu)$, which is a Markov semigroup and it may be extended in a standard way to a contraction semigroup in $L^p(\Omega, \mu)$ for each $p \ge 1$. The measure μ is invariant for $T(t)$, *i.e.*

$$\int_\Omega (T(t)f)(x)\mu(dx) = \int_\Omega f(x)\mu(dx), \qquad f \in L^1(\Omega, \mu),$$

and moreover $T(t)f$ converges to the mean value $\overline{f} = \int_\Omega f(x)\mu(dx)$ of f as $t \to +\infty$, for each $f \in L^2(\Omega, \mu)$.

If, in addition, $U - \omega|x|^2/2$ is still convex for some $\omega > 0$, $T(t)$ enjoys further properties. 0 comes out to be a simple isolated eigenvalue in $\sigma(A)$, the rest of the spectrum is contained in $(-\infty, -\omega]$, and $T(t)f$ converges to $\overline{f}$ at an exponential rate as $t \to +\infty$. Moreover, $T(t)$ is a bounded operator (with norm not exceeding 1) from $L^p(\Omega, \mu)$ to $L^{q(t)}(\Omega, \mu)$, with $q(t) = 1 + (p-1)e^{2\omega t}$. This hypercontractivity property is the best we can expect in weighted Lebesgue spaces with general weight, and there is no hope that $T(t)$ maps, say, $L^2(\Omega, \mu)$ into $L^\infty(\Omega)$. Similarly, Sobolev embeddings are not available in general. The best we can prove is a logarithmic Sobolev inequality,

$$\int_\Omega f^2(x) \log(f^2(x))\mu(dx) \leq \frac{1}{\omega} \int_\Omega |Df(x)|^2 \mu(dx) + \overline{f^2} \log(\overline{f^2}), \qquad f \in H^1(\Omega, \mu).$$

2. Preliminaries: operators in the whole $\mathbb{R}^N$

Let $U: \mathbb{R}^N \mapsto \mathbb{R}$ be a convex C^1 function, satisfying

$$\lim_{|x| \to +\infty} U(x) = +\infty. \tag{2.1}$$

Then there are $a \in \mathbb{R}$, $b > 0$ such that $U(x) \geq a + b|x|$, for each $x \in \mathbb{R}^N$. It follows that the probability measure $\nu(dx) = e^{-2U(x)}dx / \int_{\mathbb{R}^N} e^{-2U(x)}dx$ is well defined.

The spaces $H^1(\mathbb{R}^N, \nu)$ and $H^2(\mathbb{R}^N, \nu)$, consist of the functions $u \in H^1_{loc}(\mathbb{R}^N)$ (respectively, $u \in H^2_{loc}(\mathbb{R}^N)$) such that u and its first (resp., first and second) order derivatives are in $L^2(\mathbb{R}^N, \nu)$.

We recall some results proved in [5] on the realization A of $\mathcal{A}$ in $L^2(\mathbb{R}^N, \nu)$. It is defined by

$$\begin{cases} D(A) = \{u \in H^2(\mathbb{R}^N, \nu) : \mathcal{A}u \in L^2(\mathbb{R}^N, \nu)\} \\ \qquad\qquad = \{u \in H^2(\mathbb{R}^N, \nu) : \langle DU, Du \rangle \in L^2(\mathbb{R}^N, \nu)\}, \\ (Au)(x) = \mathcal{A}u(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{2.2}$$

Theorem 2.1. *Let $U : \mathbb{R}^N \mapsto \mathbb{R}$ be a convex function satisfying assumption* (2.1). *Then the resolvent set of A contains $(0, +\infty)$ and*

$$\begin{cases} (i) & \|R(\lambda, A)f\|_{L^2(\mathbb{R}^N, \nu)} \leq \dfrac{1}{\lambda} \|f\|_{L^2(\mathbb{R}^N, \nu)}, \\ (ii) & \||DR(\lambda, A)f|\|_{L^2(\mathbb{R}^N, \nu)} \leq \dfrac{2}{\sqrt{\lambda}} \|f\|_{L^2(\mathbb{R}^N, \nu)}, \\ (iii) & \||D^2R(\lambda, A)f|\|_{L^2(\mathbb{R}^N, \nu)} \leq 4\|f\|_{L^2(\mathbb{R}^N, \nu)}. \end{cases} \tag{2.3}$$

Theorem 2.2. *Let $U : \mathbb{R}^N \mapsto \mathbb{R}$ satisfy* (2.1), *and be such that $x \mapsto U(x) - \omega|x|^2/2$ is convex, for some $\omega > 0$. Then, setting $\overline{u} = \int_{\mathbb{R}^N} u(x)\nu(dx)$, we have*

$$\int_{\mathbb{R}^N} |u(x) - \overline{u}|^2 \nu(dx) \leq \frac{1}{2\omega} \int_{\mathbb{R}^N} |Du(x)|^2 \nu(dx),$$

$$\int_{\mathbb{R}^N} u^2(x) \log(u^2(x))\nu(dx) \leq \frac{1}{\omega} \int_{\mathbb{R}^N} |Du(x)|^2 \nu(dx) + \overline{u^2} \log(\overline{u^2}),$$

for each $u \in H^1(\mathbb{R}^N, \nu)$ (we adopt the convention $0 \log 0 = 0$).

3. The operator A

Let $U : \Omega \mapsto \mathbb{R}$ be a convex function satisfying assumption (1.2), and let us extend it to the whole $\mathbb{R}^N$ setting

$$U(x) = +\infty, \qquad x \notin \Omega. \tag{3.1}$$

The extension, that we shall still call U, is lower semicontinuous and convex. For each $x \in \mathbb{R}^N$, the subdifferential $\partial U(x)$ of U at x is the set $\{y \in \mathbb{R}^N : U(\xi) \geq U(x) + \langle y, \xi - x\rangle, \forall \xi \in \mathbb{R}^N\}$. At each $x \in \Omega$, since U is real valued and continuous, $\partial U(x)$ is not empty and it has a unique element with minimal norm, that we denote by $DU(x)$. Of course if U is differentiable at x, $DU(x)$ is the usual gradient. At each $x \notin \Omega$, $\partial U(x)$ is empty and $DU(x)$ is not defined.

Lemma 3.1. *There are $a \in \mathbb{R}$, $b > 0$ such that $U(x) \geq a + b|x|$ for each $x \in \Omega$.*

Proof. The statement is obvious if Ω is bounded. If Ω is unbounded, we may assume without loss of generality that $0 \in \Omega$. Assume by contradiction that there is a sequence x_n with $|x_n| \to +\infty$ such that $\lim_{n\to\infty} U(x_n)/|x_n| = 0$. Let R be so large that $\min\{U(x) - U(0) : x \in \Omega,\ |x| = R\} > 0$. Since U is convex, for n large enough we have

$$U\left(\frac{R}{|x_n|}x_n\right) \leq \frac{R}{|x_n|}U(x_n) + \left(1 - \frac{R}{|x_n|}\right)U(0)$$

so that

$$\limsup_{n\to\infty} U\left(\frac{R}{|x_n|}x_n\right) - U(0) \leq \lim_{n\to\infty} \frac{R}{|x_n|}U(x_n) - \frac{R}{|x_n|}U(0) = 0,$$

a contradiction. □

We set as usual $e^{-\infty} = 0$. The function

$$x \mapsto e^{-2U(x)}, \qquad x \in \mathbb{R}^N,$$

is continuous, it is positive in Ω, and it vanishes outside Ω. Lemma 3.1 implies that it is in $L^1(\Omega)$. Therefore, the probability measure (1.3) is well defined, and it has Ω as support.

Lemma 3.2. *$C_0^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N, \mu)$, in $H^1(\mathbb{R}^N, \mu)$ and in $H^2(\mathbb{R}^N, \mu)$. Moreover,*

(*i*) *$C_0^\infty(\Omega)$ is dense in $L^2(\Omega, \mu)$;*

(*ii*) *If* (1.4) *holds, then $C_0^\infty(\Omega)$ is dense in $H^1(\Omega, \mu)$.*

Proof. The proof of the first statement is identical to the proof of [5, Lemma 2.1], and we omit it.

Let $\theta_n : \mathbb{R} \mapsto \mathbb{R}$ be a sequence of smooth functions such that $0 \leq \theta_n(y) \leq 1$ for each y, $\theta_n \equiv 1$ for $y \leqslant n$, $\theta_n \equiv 0$ for $y \geqslant 2n$, and such that

$$|\theta_n'(y)| \leqslant \frac{C}{n}, \qquad y \in \mathbb{R}.$$

For each $u \in L^2(\Omega, \mu)$ set

$$u_n(x) = u(x)\theta_n(U(x)), \quad x \in \Omega, \qquad u_n(x) = 0, \quad x \notin \Omega. \tag{3.2}$$

Then u_n has compact support, and $u_n \to u$ in $L^2(\mathbb{R}^N, \mu)$. Indeed,

$$\int_{\mathbb{R}^N} |u_n - u|^2 \mu(dx) \leqslant \int_{\{x \in \Omega:\ U(x) \geqslant n\}} |u|^2 \mu(dx)$$

which goes to 0 as $n \to \infty$. In its turn, u_n may be approximated in $L^2(\Omega)$ by a sequence of $C_0^\infty(\Omega)$ functions obtained by convolution with smooth mollifiers. Since u_n has compact support, such a sequence approximates u_n also in $L^2(\Omega, \mu)$, and statement (*i*) follows.

Statement (*ii*) is proved in three steps. First, we note that any $u \in H^1(\Omega, \mu)$ may be approached by functions in $H^1(\Omega, \mu) \cap L^\infty(\Omega)$. Then we approach any function in $H^1(\Omega, \mu) \cap L^\infty(\Omega)$ by functions in $H^1(\Omega, \mu)$ with compact support. Third, any function in $H^1(\Omega, \mu)$ with compact support is approximated by a sequence of $C_0^\infty(\Omega)$ functions obtained as above by convolution with smooth mollifiers.

For any $u \in H^1(\Omega, \mu)$ we set

$$u_\varepsilon(x) = \frac{u(x)}{1 + \varepsilon u(x)^2}.$$

Then

$$\int_\Omega |u - u_\varepsilon|^2 \mu(dx) = \int_\Omega u^2 \left(1 - \frac{1}{1 + \varepsilon u^2}\right)^2 \mu(dx)$$

goes to 0 as $\varepsilon \to 0$, and

$$Du_\varepsilon = \frac{Du}{1 + \varepsilon u^2} - \frac{2\varepsilon u^2 Du}{(1 + \varepsilon u^2)^2}$$

so that $|Du - Du_\varepsilon|$ goes to 0 in $L^2(\Omega, \mu)$ as well. So, u is approximated by a sequence of bounded H^1 functions.

Now, let $u \in H^1(\Omega, \mu) \cap L^\infty(\Omega)$, and define u_n by (3.2).

Since U is convex, it is locally Lipschitz continuous, so that it is differentiable almost everywhere with locally L^∞ gradient. It follows that u_n is differentiable a.e. and for almost each x in Ω we have

$$Du_n(x) = Du(x)\theta_n(U(x)) + u(x)\theta_n'(U(x))DU(x).$$

Here $Du\theta_n(U)$ goes to Du in $L^2(\Omega, \mu)$, and $u\theta_n'(U)DU$ goes to 0 in $L^2(\Omega, \mu)$ as $n \to \infty$ because $u \in L^\infty$, $DU \in L^2(\Omega, \mu)$ and $|\theta_n'| \leqslant C/n$. Statement (*ii*) follows. □

We remark that in general $C_0^\infty(\Omega)$ is not dense in $H^1(\mathbb{R}^N, \mu)$. See next Example 4.1.

We introduce now the main tool in our study, *i.e.* the *Moreau-Yosida approxima-*

tions of U,

$$U_\alpha(x) = \inf\left\{U(y) + \frac{1}{2\alpha}|x-y|^2 : y \in \mathbb{R}^N\right\}, \qquad x \in \mathbb{R}^N, \quad \alpha > 0,$$

that are real valued on the whole $\mathbb{R}^N$ and enjoy good regularity properties: they are convex, differentiable, and for each $x \in \mathbb{R}^N$ we have (see *e.g.* [2, Prop. 2.6, Prop. 2.11])

$$U_\alpha(x) \leq U(x), \ |DU_\alpha(x)| \leq |DU(x)|, \qquad \lim_{\alpha\to 0} U_\alpha(x) = U(x), \quad x \in \mathbb{R}^N,$$

$$\lim_{\alpha\to 0} DU_\alpha(x) = DU(x), \quad x \in \Omega; \qquad \lim_{\alpha\to 0}|DU_\alpha(x)| = +\infty, \quad x \notin \Omega.$$

Moreover DU_α is Lipschitz continuous for each α, with Lipschitz constant $1/\alpha$.

Let us define now the realization A of $\mathcal{A}$ in $L^2(\Omega, \mu)$ by

$$\begin{cases} D(A) = \{u \in H^2(\Omega, \mu) : \langle DU, Du\rangle \in L^2(\Omega, \mu)\}, \\ (Au)(x) = \mathcal{A}u(x), \quad x \in \Omega. \end{cases} \tag{3.3}$$

We shall show that A is a self-adjoint dissipative operator, provided $C_0^\infty(\Omega)$ is dense in $H^1(\mathbb{R}^N, \mu)$. The fact that A is symmetric is a consequence of the next lemma.

Lemma 3.3. *If $C_0^\infty(\Omega)$ is dense in $H^1(\mathbb{R}^N, \mu)$, then for each $u \in D(A)$, $\psi \in H^1(\mathbb{R}^N, \mu)$ we have*

$$\int_\Omega (\mathcal{A}u)(x)\psi(x)\nu(dx) = -\frac{1}{2}\int_\Omega \langle Du(x), D\psi(x)\rangle \mu(dx). \tag{3.4}$$

Proof. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N, \mu)$ it is sufficient to show that (3.4) hold for each $\psi \in C_0^\infty(\mathbb{R}^N)$.

If $\psi \in C_0^\infty(\Omega)$, then the function $\psi \exp(-2U)$ is continuously differentiable and it has compact support in Ω. Integrating by parts $(\Delta u)(x)\psi(x)\exp(-2U(x))$ we get

$$\frac{1}{2}\int_\Omega (\Delta u)(x)\psi(x)e^{-2U(x)}dx = -\frac{1}{2}\int_\Omega \langle Du(x), D(\psi(x)e^{-2U(x)})\rangle dx =$$

$$= -\frac{1}{2}\int_\Omega \langle Du(x), D\psi(x)\rangle e^{-2U(x)}dx + \frac{1}{2}\int_\Omega \langle Du(x), 2DU(x)\rangle \psi(x)e^{-2U(x)}dx$$

so that (3.4) holds. □

Taking $\psi = u$ in (3.4) shows that A is symmetric.

Once we have the integration formula (3.4) and the powerful tool of the Moreau-Yosida approximations at our disposal, the proof of the dissipativity of A is similar to the proof of Theorem 2.4 of [5]. However we write down all the details for the reader's convenience.

THEOREM 3.4. *Let $U : \Omega \mapsto \mathbb{R}$ be a convex function satisfying assumption* (1.2), *and be such that $C_0^\infty(\Omega)$ is dense in $H^1(\Omega, \mu)$. Then the resolvent set of A contains* $(0, +\infty)$ *and*

$$
(3.5) \qquad \begin{cases} (i) & \|R(\lambda, A) f\|_{L^2(\Omega, \mu)} \le \dfrac{1}{\lambda} \|f\|_{L^2(\Omega, \mu)}, \\ (ii) & \| |DR(\lambda, A) f| \|_{L^2(\Omega, \mu)} \le \dfrac{2}{\sqrt{\lambda}} \|f\|_{L^2(\Omega, \mu)}, \\ (iii) & \| |D^2 R(\lambda, A) f| \|_{L^2(\Omega, \mu)} \le 4 \|f\|_{L^2(\Omega, \mu)}. \end{cases}
$$

Moreover the resolvent $R(\lambda, A)$ is positivity preserving, and $R(\lambda, A)\mathbb{1} = \mathbb{1}/\lambda$.

PROOF. For $\lambda > 0$ and $f \in L^2(\Omega, \mu)$ consider the resolvent equation

$$
(3.6) \qquad \lambda u - Au = f.
$$

It has at most a solution, because if $u \in D(A)$ satisfies $\lambda u = Au$ then by (3.4) we have

$$
\int_\Omega \lambda (u(x))^2 \mu(dx) = \int_\Omega (Au)(x) u(x) \mu(dx) = -\frac{1}{2} \int_\Omega |Du(x)|^2 \mu(dx) \le 0,
$$

so that $u = 0$.

To find a solution to (3.6), we approximate U by the Moreau-Yosida approximations U_α defined above, we consider the measures $\nu_\alpha(dx) = e^{-2U_\alpha(x)} dx / \int_{\mathbb{R}^N} e^{-2U_\alpha(x)} dx$ in $\mathbb{R}^N$ and the operators $\mathcal{A}_\alpha$ defined by $\mathcal{A}_\alpha u = \Delta u/2 - \langle DU_\alpha, Du \rangle$.

Since the functions U_α are convex and satisfy (2.1), the results of Theorem 2.1 hold for the operators A_α: $D(A_\alpha) = H^2(\mathbb{R}^N, \nu_\alpha) \mapsto L^2(\mathbb{R}^N, \nu_\alpha)$. In particular, for each $f \in C_0^\infty(\mathbb{R}^N)$ with support contained in Ω, the equation

$$
(3.7) \qquad \lambda u_\alpha - A_\alpha u_\alpha = f,
$$

has a unique solution $u_\alpha \in D(A_\alpha)$. Moreover, each u_α is bounded with bounded and Hölder continuous second order derivatives, thanks to the Schauder estimates and the maximum principle that hold for operators with Lipschitz continuous coefficients, see [10].

Estimates (2.3) imply that

$$
(3.8) \qquad \begin{cases} \|u_\alpha\|_{L^2(\mathbb{R}^N, \nu_\alpha)} \le \dfrac{1}{\lambda} \|f\|_{L^2(\mathbb{R}^N, \nu_\alpha)}, \\ \| |Du_\alpha| \|_{L^2(\mathbb{R}^N, \nu_\alpha)} \le \dfrac{2}{\sqrt{\lambda}} \|f\|_{L^2(\mathbb{R}^N, \nu_\alpha)}, \\ \| |D^2 u_\alpha| \|_{L^2(\mathbb{R}^N, \nu_\alpha)} \le 4 \|f\|_{L^2(\mathbb{R}^N, \nu_\alpha)}, \end{cases}
$$

so that

$$
\|u_\alpha\|_{H^2(\mathbb{R}^N, \nu_\alpha)} \le C \|f\|_{L^2(\mathbb{R}^N, \nu_\alpha)}
$$

with $C = C(\lambda)$ independent of α. Since $U_\alpha(x)$ goes to $U(x)$ monotonically as $\alpha \to 0$, then $\exp(-2U_\alpha(x))$ goes to $\exp(-2U(x))$ monotonically, and $\left(\int_{\mathbb{R}^N} e^{-2U_\alpha(x)} dx \right)^{-1}$

goes to $\left(\int_{\mathbb{R}^N} e^{-2U(x)} dx \right)^{-1}$, $\|f\|_{L^2(\mathbb{R}^N, \nu_\alpha)}$ goes to $\|f\|_{L^2(\mathbb{R}^N, \mu)}$ as $\alpha \to 0$. It follows that the norm $\|u_\alpha\|_{H^2(\mathbb{R}^N, \nu_\alpha)}$ is bounded by a constant independent of α, and consequently also the norm $\|u_\alpha\|_{H^2(\mathbb{R}^N, \mu)}$ is bounded by a constant independent of α. Therefore there is a sequence u_{α_n} that converges weakly in $H^2(\mathbb{R}^N, \mu)$ to a function $u \in H^2(\mathbb{R}^N, \mu)$, and converges to u in $H^1(K)$ for each compact subset $K \subset \Omega$. This implies easily that u solves (3.6). Indeed, let $\phi \in C_0^\infty(\Omega)$. For each $n \in \mathbb{N}$ we have

$$\int_{\mathbb{R}^N} \left(\lambda u_{\alpha_n} - \frac{1}{2} \Delta u_{\alpha_n} + \langle DU_{\alpha_n}, Du_{\alpha_n} \rangle - f \right) \phi e^{-2U} dx = 0.$$

Letting $n \to \infty$, we get immediately that $\int_{\mathbb{R}^N} \left(\lambda u_{\alpha_n} - \frac{1}{2} \Delta u_{\alpha_n} \right) \phi e^{-2U(x)} dx$ goes to $\int_{\mathbb{R}^N} \left(\lambda u - \frac{1}{2} \Delta u \right) \phi e^{-2U(x)} dx$. Moreover $\int_{\mathbb{R}^N} \langle DU_{\alpha_n}, Du_{\alpha_n} \rangle \phi e^{-2U(x)} dx$ goes to $\int_{\mathbb{R}^N} \langle DU, Du \rangle \phi e^{-2U(x)} dx$ because DU_{α_n} goes to DU in $L^2(\operatorname{supp} \phi)$. Therefore letting $n \to \infty$ we get

$$\int_{\mathbb{R}^N} (\lambda u - \mathcal{A} u - f) \phi e^{-2U} dx = 0$$

for each $\phi \in C_0^\infty(\mathbb{R}^N)$, and hence $\lambda u - \mathcal{A} u = f$ almost everywhere in Ω. So, $u_{|\Omega} \in D(A)$ is the solution of the resolvent equation, and letting $\alpha \to 0$ in (3.8) we get

$$\text{(3.9)} \qquad \begin{cases} \|u\|_{L^2(\Omega, \mu)} \leq \dfrac{1}{\lambda} \|f\|_{L^2(\Omega, \mu)}, \quad \| |Du| \|_{L^2(\Omega, \mu)} \leq \dfrac{2}{\sqrt{\lambda}} \|f\|_{L^2(\Omega, \nu)}, \\ \| |D^2 u| \|_{L^2(\Omega, \mu)} \leq 4 \|f\|_{L^2(\Omega, \mu)}. \end{cases}$$

Let now $f \in L^2(\Omega, \mu)$ and let f_n be a sequence of $C_0^\infty(\Omega)$ functions going to f in $L^2(\Omega, \mu)$ as $n \to \infty$. Thanks to estimates (3.9), the solutions u_n of

$$\lambda u_n - A u_n = f_n$$

are a Cauchy sequence in $H^2(\Omega, \mu)$, and converge to a solution $u \in H^2(\Omega, \mu)$ of (3.6). Due again to estimates (3.9), u satisfies (3.5).

If in addition $f(x) \geq 0$ a.e. in Ω, we may take $f_n(x) \geq 0$ in Ω, see the proof of Lemma 3.2. Each u_α, solution to (3.7) with f replaced by f_n, has nonnegative values thanks to the maximum principle for elliptic operators with Lipschitz continuous coefficients proved in [10]. Our limiting procedure gives $R(\lambda, A) f_n(x) \geq 0$ for each x, and $R(\lambda, A) f(x) \geq 0$ for each x. So, $R(\lambda, A)$ is a positivity preserving operator. □

4. Examples and consequences

Example 4.1. Let Ω be the unit open ball in $\mathbb{R}^N$, and let $U(x) = -\frac{\alpha}{2} \log(1 - |x|)$ for $x \in \Omega$, with $\alpha > 0$. Then

$$\exp(-2U(x)) = (1 - |x|)^\alpha, \quad DU(x) = \frac{\alpha x}{2|x|(1 - |x|)}, \quad 0 < |x| < 1,$$

and it is known that $C_0^\infty(\Omega)$ is dense in $H^1(\Omega, \mu)$ iff $\alpha \geq 1$. See *e.g.* [14, Theorem 3.6.1]. In this case the result of Theorem 3.4 holds, and A is a self-adjoint dissipative operator in $L^2(\Omega, \mu)$.

Note that assumption (1.4) is satisfied only for $\alpha > 1$. This shows that assumption (1.4) is not equivalent to the fact that $C_0^\infty(\Omega)$ is dense in $H^1(\mathbb{R}^N, \mu)$, however it is not very far. □

Under the assumptions of Theorem 3.4, A is the infinitesimal generator of an analytic contraction semigroup $T(t)$ in $L^2(\Omega, \mu)$.

Since the resolvent $R(\lambda, A)$ is positivity preserving for $\lambda > 0$, also $T(t)$ is positivity preserving. Since $R(\lambda, A)\,\mathbb{1} = \mathbb{1}/\lambda$, then $T(t)\,\mathbb{1} = \mathbb{1}$ for each $t > 0$. Therefore, $T(t)$ is a Markov semigroup and it may be extended in a standard way to a contraction semigroup (that we shall still call $T(t)$) in $L^p(\Omega, \mu)$, $1 \leq p \leq \infty$. $T(t)$ is strongly continuous in $L^p(\Omega, \mu)$ for $1 \leq p < \infty$, and it is analytic for $1 < p < \infty$. See *e.g.* [4, Chapter 1]. The infinitesimal generator of $T(t)$ in $L^p(\Omega, \mu)$ is denoted by A_p. The characterization of the domain of A_p in $L^p(\Omega, \mu)$ for $p \neq 2$ is an interesting open problem.

An important optimal regularity result for evolution equations follows, see [9].

COROLLARY 4.2. *Let $1 < p < \infty$, $T > 0$. For each $f \in L^p((0, T); L^p(\Omega, \mu))$* (i.e. $(t, x) \mapsto f(t)(x) \in L^p((0, T) \times \Omega; dt \times \mu)$) *the problem*

$$\begin{cases} u'(t) = A_p u(t) + f(t), & 0 < t < T, \\ u(0) = 0, \end{cases}$$

has a unique solution $u \in L^p((0, T); D(A_p)) \cap W^{1,p}((0, T); L^p(\Omega, \mu))$.

From Lemma 3.3 we get, taking $\psi \equiv 1$,

$$\int_\Omega Au\,\mu(dx) = 0, \qquad u \in D(A),$$

and hence,

$$\int_\Omega T(t) f \mu(dx) = \int_\Omega f\mu(dx), \qquad t > 0,$$

for each $f \in L^2(\Omega, \mu)$. Since $L^2(\Omega, \mu)$ is dense in $L^1(\Omega, \mu)$, the above equality holds for each $f \in L^1(\Omega, \mu)$. In other words, μ is an invariant measure for the semigroup $T(t)$.

From Lemma 3.3 we get also

$$u \in D(A), \quad Au = 0 \Rightarrow Du = 0,$$

and hence the kernel of A consists of the constant functions. Let us prove now that

$$\lim_{t \to +\infty} T(t) f = \int_\Omega f(y)\mu(dy) \quad \text{in } L^2(\Omega, \mu), \tag{4.1}$$

for all $f \in L^2(\Omega, \mu)$.

Indeed, since the function $t \to \varphi(t) = \int_\Omega (T(t)f)^2 \mu(dx)$ is nonincreasing and bounded, there exists the limit $\lim_{t\to+\infty} \varphi(t) = \lim_{t\to+\infty} \langle T(2t)f, f\rangle_{L^2(\Omega,\mu)}$. By a standard argument it follows that there exists a symmetric nonnegative operator $Q \in \mathcal{L}(L^2(\Omega,\mu))$ such that

$$\lim_{t\to+\infty} T(t)f = Qf, \qquad f \in L^2(H,\mu).$$

On the other hand, using the Mean Ergodic Theorem in Hilbert space (see *e.g.* [11, p. 24]) we get easily

$$\lim_{t\to+\infty} T(t)f = P\left(\int_0^1 T(s)f ds\right),$$

where P is the orthogonal projection on the kernel of A. Since the kernel of A consists of the constant functions, (4.1) follows.

From now on we make a strict convexity assumption on U:

$$\exists\, \omega > 0 \text{ such that } x \mapsto U(x) - \omega|x|^2/2 \text{ is convex.} \tag{4.2}$$

This will allow us to prove further properties for $T(t)$, through Poincaré and Log-Sobolev inequalities.

If (Λ, m) is any measure space and $u \in L^1(\Lambda, m)$ we set

$$\overline{u}_m = \int_\Lambda u(x)\, m(dx). \tag{4.3}$$

PROPOSITION 4.3. *Let the assumptions of Theorem* 3.4 *and* (4.2) *hold. Then*

$$\int_\Omega |u(x) - \overline{u}_\mu|^2 \mu(dx) \leq \frac{1}{2\omega}\int_\Omega |Du(x)|^2 d\mu(dx), \qquad u \in H^1(\Omega,\mu), \tag{4.4}$$

and

$$\int_\Omega u^2(x)\log(u^2(x))\mu(dx) \leq \frac{1}{\omega}\int_\Omega |Du(x)|^2 \mu(dx) + \overline{u^2}_\mu \log(\overline{u^2}_\mu), \qquad u \in H^1(\Omega,\mu). \tag{4.5}$$

PROOF. Let $u \in C_0^\infty(\mathbb{R}^N)$ have support in Ω. Let U_α be the Moreau-Yosida approximations of U, and set as usual $\nu_\alpha(dx) = \left(\int_{\mathbb{R}^N} e^{-2U_\alpha(x)}dx\right)^{-1} e^{-2U_\alpha(x)}dx$. Since $x \mapsto U_\alpha(x) - \omega(1-\alpha)|x|^2$ is convex in the whole $\mathbb{R}^N$, by Theorem 2.2 we have, for $\alpha \in (0,1)$,

$$\int_{\mathbb{R}^N} |u(x) - \overline{u}_\alpha|^2 \nu_\alpha(dx) \leq \frac{1}{2\omega(1-\alpha)}\int_{\mathbb{R}^N} |Du(x)|^2 \nu_\alpha(dx), \tag{4.6}$$

(where $\overline{u}_\alpha$ stands for $\overline{u}_{\nu_\alpha}$) and

$$(4.7)\ \int_{\mathbb{R}^N} u^2(x)\log(u^2(x))\,\nu_\alpha(dx) \leq \frac{1}{\omega(1-\alpha)}\int_{\mathbb{R}^N} |Du(x)|^2\,\nu_\alpha(dx) + \overline{u^2}_\alpha \log(\overline{u^2}_\alpha).$$

Since

$$\lim_{\alpha\to 0} U_\alpha(x) = \begin{cases} U(x) & \text{if } x\in\Omega \\ +\infty & \text{if } x\notin\Omega, \end{cases}$$

then $\overline{u}_\alpha$ goes to $\overline{u}_\mu = \int_\Omega u(x)\mu(dx)$, $\overline{u^2}_\alpha$ goes to $\overline{u^2}_\mu$ as α goes to 0, and letting α go to 0 in (4.6), (4.7) we obtain that u satisfies (4.4) and (4.5). Since $C_0^\infty(\Omega)$ is dense in $H^1(\Omega,\mu)$, the statement follows. □

Proposition 4.3 yields other properties of $T(t)$, listed in the next corollary. The proof is identical to the proof of [5, Corollary 4.3], and we omit it.

COROLLARY 4.4. *Let the assumptions of Theorem* 3.4 *and* (4.2) *hold. Then* 0 *is a simple isolated eigenvalue of A. The rest of the spectrum,* $\sigma(A)\setminus\{0\}$ *is contained in* $(-\infty, -\omega]$, *and*

$$(4.8)\qquad \|T(t)u-\overline{u}_\mu\|_{L^2(\Omega,\mu)} \leq e^{-\omega t}\|u-\overline{u}_\mu\|_{L^2(\Omega,\mu)},\quad u\in L^2(\Omega,\mu),\quad t>0.$$

Moreover we have

$$(4.9)\qquad \|T(t)\varphi\|_{L^{q(t)}(\Omega,\mu)} \leq \|\varphi\|_{L^p(\Omega,\mu)},\quad p\geq 2,\quad \varphi\in L^p(\Omega,\mu),$$

where

$$(4.10)\qquad q(t) = 1+(p-1)e^{2\omega t},\quad t>0.$$

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