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ABSTRACT. — We study the realization $A$ of the operator $A u = \frac{1}{2} \Delta u - \langle DU, Du \rangle$ in $L^2(\Omega, \mu)$, where $\Omega$ is a possibly unbounded convex open set in $\mathbb{R}^N$, $U$ is a convex unbounded function such that
\[
\lim_{x \to \partial \Omega, x \in \Omega} U(x) = +\infty \quad \text{and} \quad \lim_{|x| \to +\infty, x \in \Omega} U(x) = +\infty,
\]
$DU(x)$ is the element with minimal norm in the subdifferential of $U$ at $x$, and $\mu(dx) = e^{\exp(-2U(x))}dx$ is a probability measure, infinitesimally invariant for $\mathcal{A}$. We show that $A$, with domain $D(A) = \{ u \in H^2(\Omega, \mu) : \langle DU, Du \rangle \in L^2(\Omega, \mu) \}$ is a dissipative self-adjoint operator in $L^2(\Omega, \mu)$. Note that the functions in the domain of $A$ do not satisfy any particular boundary condition. Log-Sobolev and Poincaré inequalities allow then to study smoothing properties and asymptotic behavior of the semigroup generated by $A$.

KEY WORDS: Kolmogorov operators; Unbounded coefficients; Convex domains.

1. INTRODUCTION

In this paper we give a contribution to the theory of second order elliptic operators with unbounded coefficients, that underwent a great development in the last few years. See e.g. [1, 5-8, 12, 13].

Here we consider the operator
\[
\mathcal{A} u = \frac{1}{2} \Delta u - \langle DU, Du \rangle
\]
in a convex open set $\Omega \subset \mathbb{R}^N$, where $U$ is a convex unbounded function such that
\[
\lim_{x \to \partial \Omega, x \in \Omega} U(x) = +\infty \quad \text{and} \quad \lim_{|x| \to +\infty, x \in \Omega} U(x) = +\infty.
\]
Since we do not impose any growth condition on $U$, the usual $L^p$ and Sobolev spaces with respect to the Lebesgue measure are not the best setting for the operator $\mathcal{A}$. It is more convenient to introduce the measure
\[
\mu(dx) = \left( \frac{1}{\int_{\Omega} e^{-2U(x)} \, dx} \right)^{-1} e^{-2U(x)} \, dx,
\]
which is infinitesimally invariant for $\mathcal{A}$, i.e.
\[
\int_{\Omega} \mathcal{A} u(x) \mu(dx) = 0, \quad u \in C_0^\infty (\mathbb{R}^N),
\]
and lets $\mathcal{A}$ be formally self-adjoint in $L^2(\Omega, \mu)$, as an easy computation shows. We prove in fact that the realization $A$ of $\mathcal{A}$ in $L^2(\Omega, \mu)$, with domain
\[
D(A) = \{ u \in H^2(\Omega, \mu) : \mathcal{A} u \in L^2(\Omega, \mu) \} = \{ u \in H^2(\Omega, \mu) : \langle DU, Du \rangle \in L^2(\Omega, \mu) \}
\]
is a self-adjoint and dissipative operator, provided $C_0^\infty (\Omega)$ is dense in $H^1(\Omega, \mu)$. We recall that $H^1(\Omega, \mu)$ is naturally defined as the set of all $u \in H^1_{loc}(\Omega)$ such that $u,$
\( D_i u \in L^2(\Omega, \mu) \), for \( i = 1, \ldots, N \). While it is easy to see that \( C_0^\infty(\Omega) \) is dense in \( L^2(\Omega, \mu) \), well-known counterexamples show that \( C_0^\infty(\Omega) \) is not dense in \( H^1(\Omega, \mu) \) in general. A sufficient condition in order that \( C_0^\infty(\Omega) \) be dense in \( H^1(\Omega, \mu) \) is

\[(1.4) \quad DU \in L^2(\Omega, \mu).\]

Once we know that \( C_0^\infty(\Omega) \) is dense in \( H^1(\Omega, \mu) \), it is not hard to show that for each \( u \in D(A) \) and \( \psi \in H^1(\Omega, \mu) \) we have

\[
\int_\Omega (\varOmega u)(x) \psi(x) \mu(dx) = -\frac{1}{2} \int_\Omega (Du(x), D\psi(x)) \mu(dx).
\]

This crucial integration formula implies that \( A \) is symmetric and dissipative. The next step is to prove that \( \lambda I - A \) is onto for \( \lambda > 0 \), so that \( A \) is \( m \)-dissipative. This is done by approximation, solving first, for each \( \lambda > 0 \) and \( f \in C_0^\infty(\Omega) \),

\[(1.5) \quad \lambda u_\alpha(x) - (\varOmega u_\alpha)(x) = f(x), \quad x \in \mathbb{R}^N,
\]

where \( A_\alpha \) is defined as \( \varOmega \), with \( U \) replaced by its Moreau-Yosida approximation \( U_\alpha \). To be more precise, first we extend \( f \) and \( U \) to the whole \( \mathbb{R}^N \) setting \( f(x) = 0 \) and \( U(x) = +\infty \) for \( x \) outside \( \Omega \); since the extension of \( U \) is lower semicontinuous and convex the Moreau-Yosida approximations \( U_\alpha \) are well defined and differentiable with Lipschitz continuous gradient in \( \mathbb{R}^N \). Then (1.5) has a unique solution \( u_\alpha \in H^2(\mathbb{R}^N, \mu_\alpha) \), with \( \mu_\alpha(dx) = \left( \int_{\mathbb{R}^N} e^{-2U_\alpha(x)} dx \right)^{-1} e^{-2U_\alpha(x)} dx \), and the norm of \( u_\alpha \) in

\( H^2(\mathbb{R}^N, \mu_\alpha) \) is bounded by \( C(\lambda)\|f\|_{L^2(\mathbb{R}^N, \mu_\alpha)} \), where the constant \( C(\lambda) \) is independent of \( \alpha \), due to the estimates for equations in the whole \( \mathbb{R}^N \) already proved in [5]. Using the convergence properties of \( U_\alpha \) and of \( DU_\alpha \) to \( U \) and to \( DU \) respectively, we arrive at a solution \( u \in H^2(\Omega, \mu) \) of

\[(1.6) \quad \lambda u(x) - (\varOmega u)(x) = f(x), \quad x \in \Omega,
\]

that belongs to \( D(A) \), satisfies \( \|u\|_{H^2(\Omega, \mu)} \leq C(\lambda)\|f\|_{L^2(\Omega, \mu)} \) and is the unique solution to the resolvent equation because \( A \) is dissipative. If \( f \) is just in \( L^2(\Omega, \mu) \), (1.6) is solved approaching \( f \) by a sequence of functions in \( C_0^\infty(\Omega) \).

A lot of nice consequences follow: \( A \) generates an analytic contraction semigroup \( T(t) \) in \( L^2(\Omega, \mu) \), which is a Markov semigroup and it may be extended in a standard way to a contraction semigroup in \( L^p(\Omega, \mu) \) for each \( p \geq 1 \). The measure \( \mu \) is invariant for \( T(t) \), i.e.

\[
\int_\Omega (T(t)f)(x) \mu(dx) = \int_\Omega f(x) \mu(dx), \quad f \in L^1(\Omega, \mu),
\]

and moreover \( T(t)f \) converges to the mean value \( \bar{f} = \int_\Omega f(x) \mu(dx) \) of \( f \) as \( t \to +\infty \), for each \( f \in L^2(\Omega, \mu) \).

If, in addition, \( U - \omega |x|^2 / 2 \) is still convex for some \( \omega > 0 \), \( T(t) \) enjoys further properties. 0 comes out to be a simple isolated eigenvalue in \( \sigma(A) \), the rest of the spectrum is contained in \((-\infty, -\omega)\), and \( T(t)f \) converges to \( \bar{f} \) at an exponential rate as \( t \to +\infty \). Moreover, \( T(t) \) is a bounded operator (with norm not exceeding 1) from \( L^p(\Omega, \mu) \) to \( L^q(t)(\Omega, \mu) \), with \( q(t) = 1 + (p - 1)e^{2\omega t} \). This hypercontractivity prop-
ty is the best we can expect in weighted Lebesgue spaces with general weight, and there is no hope that $T(t)$ maps, say, $L^2(\Omega, \mu)$ into $L^\infty(\Omega)$. Similarly, Sobolev embeddings are not available in general. The best we can prove is a logarithmic Sobolev inequality,

$$\int_\Omega f^2(x) \log(f^2(x)) \mu(dx) \leq \frac{1}{\omega} \int_\Omega |Df(x)|^2 \mu(dx) + f^4 \log(f^4), \quad f \in H^1(\Omega, \mu).$$

2. Preliminaries: Operators in the whole $\mathbb{R}^N$

Let $U: \mathbb{R}^N \to \mathbb{R}$ be a convex $C^1$ function, satisfying

$$\lim_{|x| \to +\infty} U(x) = +\infty. \tag{2.1}$$

Then there are $a \in \mathbb{R}, b > 0$ such that $U(x) \geq a + b|x|$, for each $x \in \mathbb{R}^N$. It follows that the probability measure $\nu(dx) = e^{-2U(x)} dx / \int e^{-2U(x)} dx$ is well defined.

The spaces $H^1(\mathbb{R}^N, \nu)$ and $H^2(\mathbb{R}^N, \nu)$, consist of the functions $u \in H^1_{loc}(\mathbb{R}^N)$ (respectively, $u \in H^2_{loc}(\mathbb{R}^N)$) such that $u$ and its first (resp., first and second) order derivatives are in $L^2(\mathbb{R}^N, \nu)$.

We recall some results proved in [5] on the realization $A$ of $\mathcal{A}$ in $L^2(\mathbb{R}^N, \nu)$. It is defined by

$$D(A) = \{u \in H^2(\mathbb{R}^N, \nu): \mathcal{A}u \in L^2(\mathbb{R}^N, \nu)\}$$

$$= \{u \in H^2(\mathbb{R}^N, \nu): (DU, Du) \in L^2(\mathbb{R}^N, \nu)\}, \quad (Au)(x) = \mathcal{A}u(x), \quad x \in \mathbb{R}^N. \tag{2.2}$$

**Theorem 2.1.** Let $U : \mathbb{R}^N \to \mathbb{R}$ be a convex function satisfying assumption (2.1). Then the resolvent set of $A$ contains $(0, +\infty)$ and

$$\begin{cases}
(i) \quad \|R(\lambda, A) f\|_{L^2(\mathbb{R}^N, \nu)} \leq \frac{1}{\lambda} \|f\|_{L^2(\mathbb{R}^N, \nu)}, \\
(ii) \quad \|DR(\lambda, A) f\|_{L^2(\mathbb{R}^N, \nu)} \leq \frac{2}{\sqrt{\lambda}} \|f\|_{L^2(\mathbb{R}^N, \nu)}, \\
(iii) \quad \|D^2 R(\lambda, A) f\|_{L^2(\mathbb{R}^N, \nu)} \leq 4 \|f\|_{L^2(\mathbb{R}^N, \nu)}. \end{cases} \tag{2.3}$$

**Theorem 2.2.** Let $U : \mathbb{R}^N \to \mathbb{R}$ satisfy (2.1), and be such that $x \mapsto U(x) - \omega|x|^2 / 2$ is convex, for some $\omega > 0$. Then, setting $\overline{u} = \int \mathcal{A}u(x) \nu(dx)$, we have

$$\int_{\mathbb{R}^N} |u(x) - \overline{u}|^2 \nu(dx) \leq \frac{1}{2\omega} \int_{\mathbb{R}^N} |Du(x)|^2 \nu(dx),$$

$$\int_{\mathbb{R}^N} u^2(x) \log(u^2(x)) \nu(dx) \leq \frac{1}{\omega} \int_{\mathbb{R}^N} |Du(x)|^2 \nu(dx) + \overline{u}^2 \log(\overline{u}^2),$$

for each $u \in H^1(\mathbb{R}^N, \nu)$ (we adopt the convention $0 \log 0 = 0$).
3. The Operator $A$

Let $U : \Omega \mapsto \mathbb{R}$ be a convex function satisfying assumption (1.2), and let us extend it to the whole $\mathbb{R}^N$ setting

\[ U(x) = + \infty, \quad x \notin \Omega. \tag{3.1} \]

The extension, that we shall still call $U$, is lower semicontinuous and convex. For each $x \in \mathbb{R}^N$, the subdifferential $\partial U(x)$ of $U$ at $x$ is the set $\{ y \in \mathbb{R}^N : U(y) \geq U(x) + (y, \xi - x), \forall \xi \in \mathbb{R}^N \}$. At each $x \in \Omega$, since $U$ is real valued and continuous, $\partial U(x)$ is not empty and it has a unique element with minimal norm, that we denote by $DU(x)$. Of course if $U$ is differentiable at $x$, $DU(x)$ is the usual gradient. At each $x \notin \Omega$, $\partial U(x)$ is empty and $DU(x)$ is not defined.

**Lemma 3.1.** There are $a \in \mathbb{R}$, $b > 0$ such that

\[ U(x) \geq a + b|x| \quad \text{for each} \quad x \in \Omega. \]

**Proof.** The statement is obvious if $\Omega$ is bounded. If $\Omega$ is unbounded, we may assume without loss of generality that $0 \notin \Omega$. Assume by contradiction that there is a sequence $x_n$ with $|x_n| \to + \infty$ such that $\lim_{n \to \infty} U(x_n)/|x_n| = 0$. Let $R$ be so large that

\[ \min_{x \in \Omega} U(x) \geq \frac{R}{|x_n|} \]

so that

\[ \limsup_{n \to \infty} U\left( \frac{R}{|x_n|} x_n \right) - U(0) \leq \lim_{n \to \infty} \frac{R}{|x_n|} U(x_n) - \frac{R}{|x_n|} U(0) = 0, \]

a contradiction. \qed

We set as usual $e^{-\infty} = 0$. The function

\[ x \mapsto e^{-2U(x)}, \quad x \in \mathbb{R}^N, \]

is continuous, it is positive in $\Omega$, and it vanishes outside $\Omega$. Lemma 3.1 implies that it is in $L^1(\Omega)$. Therefore, the probability measure (1.3) is well defined, and it has $\Omega$ as support.

**Lemma 3.2.** $C_0^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N, \mu)$, in $H^1(\mathbb{R}^N, \mu)$ and in $H^2(\mathbb{R}^N, \mu)$. Moreover,

(i) $C_0^\infty(\Omega)$ is dense in $L^2(\Omega, \mu)$;

(ii) If (1.4) holds, then $C_0^\infty(\Omega)$ is dense in $H^1(\Omega, \mu)$.

**Proof.** The proof of the first statement is identical to the proof of [5, Lemma 2.1], and we omit it.

Let $\theta_n : \mathbb{R} \mapsto \mathbb{R}$ be a sequence of smooth functions such that $0 \leq \theta_n(y) \leq 1$ for
each \( y, \theta_n \equiv 1 \) for \( y \leq n \), \( \theta_n \equiv 0 \) for \( y \geq 2n \), and such that
\[
|\theta_n'(y)| \leq \frac{C}{n}, \quad y \in \mathbb{R}.
\]
For each \( u \in L^2(\Omega, \mu) \) set
\[
(3.2) \quad u_n(x) = u(x) \theta_n(U(x)), \quad x \in \Omega, \quad u_n(x) = 0, \quad x \notin \Omega.
\]
Then \( u_n \) has compact support, and \( u_n \to u \) in \( L^2(\mathbb{R}^N, \mu) \). Indeed,
\[
\int_{\mathbb{R}^N} |u_n - u|^2 \mu(dx) \leq \int_{\{x \in \Omega: U(x) \equiv n\}} |u|^2 \mu(dx)
\]
which goes to 0 as \( n \to \infty \). In its turn, \( u_n \) may be approximated in \( L^2(\Omega) \) by a sequence of \( C^0_0(\Omega) \) functions obtained by convolution with smooth mollifiers. Since \( u_n \) has compact support, such a sequence approximates \( u_n \) also in \( L^2(\Omega, \mu) \), and statement (i) follows.

Statement (ii) is proved in three steps. First, we note that any \( u \in H^1(\Omega, \mu) \) may be approached by functions in \( H^1(\Omega, \mu) \cap L^\infty(\Omega) \). Then we approach any function in \( H^1(\Omega, \mu) \cap L^\infty(\Omega) \) by functions in \( H^1(\Omega, \mu) \) with compact support. Third, any function in \( H^1(\Omega, \mu) \) with compact support is approximated by a sequence of \( C^0_0(\Omega) \) functions obtained as above by convolution with smooth mollifiers.

For any \( u \in H^1(\Omega, \mu) \) we set
\[
u(x) = \frac{u(x)}{1 + \varepsilon u(x)^2}.
\]
Then
\[
\int_{\Omega} |u - \nu|^2 \mu(dx) = \int_{\Omega} u^2 \left(1 - \frac{1}{1 + \varepsilon u^2}\right)^2 \mu(dx)
\]
goesto 0 as \( \varepsilon \to 0 \), and
\[
D\nu = \frac{D\nu}{1 + \varepsilon u^2} - \frac{2 \varepsilon u^2 D\nu}{(1 + \varepsilon u^2)^2}
\]
so that \( |D\nu| \) goes to 0 in \( L^2(\Omega, \mu) \) as well. So, \( \nu \) is approximated by a sequence of bounded \( H^1 \) functions.

Now, let \( u \in H^1(\Omega, \mu) \cap L^\infty(\Omega) \), and define \( u_n \) by (3.2).

Since \( U \) is convex, it is locally Lipschitz continuous, so that it is differentiable almost everywhere with locally \( L^\infty \) gradient. It follows that \( u_n \) is differentiable a.e. and for almost each \( x \) in \( \Omega \) we have
\[
D\nu(x) = D\nu(x) \theta_n(U(x)) + u(x) \theta'_n(U(x)) DU(x).
\]
Here \( D\nu \theta_n(U) \) goes to \( D\nu \) in \( L^2(\Omega, \mu) \), and \( u \theta'_n(U) DU \) goes to 0 in \( L^2(\Omega, \mu) \) as \( n \to \infty \) because \( u \in L^\infty \), \( DU \in L^2(\Omega, \mu) \) and \( |\theta'_n| \leq C/n \). Statement (ii) follows. \( \Box \)

We remark that in general \( C^0_0(\Omega) \) is not dense in \( H^1(\mathbb{R}^N, \mu) \). See next Example 4.1.

We introduce now the main tool in our study, i.e. the Moreau-Yosida approximation.
tions of $U$,
\[ U_a(x) = \inf \left\{ U(y) + \frac{1}{2\alpha} |x - y|^2 : y \in \mathbb{R}^N \right\}, \quad x \in \mathbb{R}^N, \quad \alpha > 0, \]
that are real valued on the whole $\mathbb{R}^N$ and enjoy good regularity properties: they are convex, differentiable, and for each $x \in \mathbb{R}^N$ we have (see e.g. [2, Prop. 2.6, Prop. 2.11])
\[ U_a(x) \leq U(x), |DU_a(x)| \leq |DU(x)|, \quad \lim_{\alpha \to 0} U_a(x) = U(x), \quad x \in \mathbb{R}^N, \]
\[ \lim_{\alpha \to 0} DU_a(x) = DU(x), \quad x \in \Omega; \quad \lim_{\alpha \to 0} |DU_a(x)| = +\infty, \quad x \notin \Omega. \]
Moreover $DU_a$ is Lipschitz continuous for each $\alpha$, with Lipschitz constant $1/\alpha$.

Let us define now the realization $A$ of $\mathcal{A}$ in $L^2(\Omega, \mu)$ by
\[
\begin{cases}
D(A) = \{ u \in H^2(\Omega, \mu) : \langle DU, Du \rangle \in L^2(\Omega, \mu) \}, \\
(Au)(x) = \mathcal{A} u(x), \quad x \in \Omega.
\end{cases}
\]

We shall show that $A$ is a self-adjoint dissipative operator, provided $C_0^\infty(\Omega)$ is dense in $H^1(\mathbb{R}^N, \mu)$. The fact that $A$ is symmetric is a consequence of the next lemma.

**Lemma 3.3.** If $C_0^\infty(\Omega)$ is dense in $H^1(\mathbb{R}^N, \mu)$, then for each $u \in D(A)$,
\[ \psi \in H^1(\mathbb{R}^N, \mu) \] we have
\[
\int_{\Omega} (\mathcal{A} u)(x) \psi(x) \nu(dx) = -\frac{1}{2} \int_{\Omega} \langle Du(x), D\psi(x) \rangle \mu(dx).
\]

**Proof.** Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N, \mu)$ it is sufficient to show that (3.4) hold for each $\psi \in C_0^\infty(\mathbb{R}^N)$.

If $\psi \in C_0^\infty(\Omega)$, then the function $\psi \exp(-2U)$ is continuously differentiable and it has compact support in $\Omega$. Integrating by parts $(Au)(x) \psi(x) \exp(-2U(x))$ we get
\[
\frac{1}{2} \int_{\Omega} (\Delta u)(x) \psi(x) e^{-2U(x)} dx = -\frac{1}{2} \int_{\Omega} \langle Du(x), D(\psi(x) e^{-2U(x)}) \rangle dx =
\]
\[
- \int_{\Omega} \langle Du(x), D\psi(x) \rangle e^{-2U(x)} dx + \frac{1}{2} \int_{\Omega} \langle Du(x), 2DU(x) \psi(x) e^{-2U(x)} dx
\]so that (3.4) holds. \[\Box\]

Taking $\psi = u$ in (3.4) shows that $A$ is symmetric.

Once we have the integration formula (3.4) and the powerful tool of the Moreau-Yosida approximations at our disposal, the proof of the dissipativity of $A$ is similar to the proof of Theorem 2.4 of [5]. However we write down all the details for the reader’s convenience.
THEOREM 3.4. Let $U : \Omega \mapsto \mathbb{R}$ be a convex function satisfying assumption (1.2), and be such that $C_0^\infty (\Omega)$ is dense in $H^1 (\Omega, \mu)$. Then the resolvent set of $A$ contains $(0, + \infty)$ and

$$
\begin{cases}
(i) & \| R(\lambda, A) f \|_{L^2 (\Omega, \mu)} \leq \frac{1}{\lambda} \| f \|_{L^2 (\Omega, \mu)}, \\
(ii) & \| DR(\lambda, A) f \|_{L^2 (\Omega, \mu)} \leq \frac{2}{\sqrt{\lambda}} \| f \|_{L^2 (\Omega, \mu)}, \\
(iii) & \| D^2 R(\lambda, A) f \|_{L^2 (\Omega, \mu)} \leq 4 \| f \|_{L^2 (\Omega, \mu)}.
\end{cases}
$$

(3.5)

Moreover the resolvent $R(\lambda, A)$ is positivity preserving, and $R(\lambda, A) 1 = 1/\lambda$.

PROOF. For $\lambda > 0$ and $f \in L^2 (\Omega, \mu)$ consider the resolvent equation

$$
\lambda u - Au = f.
$$

(3.6)

It has at most a solution, because if $u \in D(A)$ satisfies $\lambda u = Au$ then by (3.4) we have

$$
\int_\Omega \lambda (u(x))^2 \mu(dx) = \int_\Omega (Au(x)) u(x) \mu(dx) = -\frac{1}{2} \int_\Omega |Du(x)|^2 \mu(dx) \leq 0,
$$

so that $u = 0$.

To find a solution to (3.6), we approximate $U$ by the Moreau-Yosida approximations $U_a$ defined above, we consider the measures $v_a(dx) = e^{-2U_a(x)} dx/ \int e^{-2U_a(x)} dx$ in $\mathbb{R}^N$ and the operators $\mathcal{A}_a$ defined by $\mathcal{A}_a u = \Delta u - (Du_a, Du_a)$. Since the functions $U_a$ are convex and satisfy (2.1), the results of Theorem 2.1 hold for the operators $A_a : D(A_a) = H^2 (\mathbb{R}^N, v_a) \mapsto L^2 (\mathbb{R}^N, v_a)$. In particular, for each $f \in C_0^\infty (\mathbb{R}^N)$ with support contained in $\Omega$, the equation

$$
\lambda u_a - A_a u_a = f,
$$

(3.7)

has a unique solution $u_a \in D(A_a)$. Moreover, each $u_a$ is bounded with bounded and H"older continuous second order derivatives, thanks to the Schauder estimates and the maximum principle that hold for operators with Lipschitz continuous coefficients, see [10].

Estimates (2.3) imply that

$$
\begin{cases}
\| u_a \|_{L^2 (\mathbb{R}^N, v_a)} \leq \frac{1}{\lambda} \| f \|_{L^2 (\mathbb{R}^N, v_a)}, \\
\| Du_a \|_{L^2 (\mathbb{R}^N, v_a)} \leq \frac{2}{\sqrt{\lambda}} \| f \|_{L^2 (\mathbb{R}^N, v_a)}, \\
\| D^2 u_a \|_{L^2 (\mathbb{R}^N, v_a)} \leq 4 \| f \|_{L^2 (\mathbb{R}^N, v_a)},
\end{cases}
$$

(3.8)

so that

$$
\| u_a \|_{H^2 (\mathbb{R}^N, v_a)} \leq C \| f \|_{L^2 (\mathbb{R}^N, v_a)}
$$

with $C = C(\lambda)$ independent of $\alpha$. Since $U_a(x)$ goes to $U(x)$ monotonically as $\alpha \to 0$, then $\exp (-2U_a(x))$ goes to $\exp (-2U(x))$ monotonically, and $\left( \int_{\mathbb{R}^N} e^{-2U_a(x)} dx \right)^{-1}$
goes to \( \left( \int_{\mathbb{R}^N} e^{-2U(x)} \, dx \right)^{-1} \), \( \| f \|_{L^2(\mathbb{R}^N, \nu \alpha)} \) goes to \( \| f \|_{L^2(\mathbb{R}^N, \mu)} \) as \( \alpha \to 0 \). It follows that the norm \( \| u_\alpha \|_{H^2(\mathbb{R}^N, \nu \alpha)} \) is bounded by a constant independent of \( \alpha \), and consequently also the norm \( \| u_\alpha \|_{H^2(\mathbb{R}^N, \mu)} \) is bounded by a constant independent of \( \alpha \). Therefore there is a sequence \( u_{\alpha_n} \) that converges weakly in \( H^2(\mathbb{R}^N, \mu) \) to a function \( u \in H^2(\mathbb{R}^N, \mu) \), and converges to \( u \) in \( H^1(K) \) for each compact subset \( K \subset \Omega \). This implies easily that \( u \) solves (3.6). Indeed, let \( \phi \in C_0^\infty(\Omega) \). For each \( n \in \mathbb{N} \) we have

\[
\int_{\mathbb{R}^N} \left( \lambda u_{\alpha_n} \right) - \frac{1}{2} \Delta u_{\alpha_n} + \langle Du_{\alpha_n}, D\phi \rangle - f \mu e^{-2U(x)} \, dx = 0.
\]

Letting \( n \to \infty \), we get immediately that \( \int_{\mathbb{R}^N} \left( \lambda u - \frac{1}{2} \Delta u \right) \phi e^{-2U(x)} \, dx \) goes to

\[
\int_{\mathbb{R}^N} \left( \lambda u - \frac{1}{2} \Delta u \right) \phi e^{-2U(x)} \, dx.
\]

Moreover \( \int_{\mathbb{R}^N} \langle Du_{\alpha_n}, D\phi \rangle \mu e^{-2U(x)} \, dx \) goes to \( \int_{\mathbb{R}^N} \langle Du, D\phi \rangle \mu e^{-2U(x)} \, dx \) because \( Du_{\alpha_n} \) goes to \( Du \) in \( L^2(\text{supp} \phi) \). Therefore letting \( n \to \infty \) we get

\[
\int_{\mathbb{R}^N} \left( \lambda u - \Delta u - f \right) \phi e^{-2U(x)} \, dx = 0
\]

for each \( \phi \in C_0^\infty(\mathbb{R}^N) \), and hence \( \lambda u - \Delta u = f \) almost everywhere in \( \Omega \). So, \( u_{|\Omega} \in D(A) \) is the solution of the resolvent equation, and letting \( \alpha \to 0 \) in (3.8) we get

\[
\left\{ \begin{align*}
\| u \|_{L^2(\Omega, \mu)} & \leq \frac{1}{\lambda} \| f \|_{L^2(\Omega, \mu)}, \\
\| Du \|_{L^2(\Omega, \mu)} & \leq \frac{2}{\sqrt{\lambda}} \| f \|_{L^2(\Omega, \nu)}, \\
\| D^2 u \|_{L^2(\Omega, \mu)} & \leq 4 \| f \|_{L^2(\Omega, \nu)}.
\end{align*} \right.
\]

(3.9)

Let now \( f \in L^2(\Omega, \mu) \) and let \( f_n \) be a sequence of \( C_0^\infty(\Omega) \) functions going to \( f \) in \( L^2(\Omega, \mu) \) as \( n \to \infty \). Thanks to estimates (3.9), the solutions \( u_n \) of

\[
\lambda u_n - \Delta u_n = f_n
\]

are a Cauchy sequence in \( H^2(\Omega, \mu) \), and converge to a solution \( u \in H^2(\Omega, \mu) \) of (3.6). Due again to estimates (3.9), \( u \) satisfies (3.5).

If in addition \( f(x) \geq 0 \) a.e. in \( \Omega \), we may take \( f_n(x) \geq 0 \) in \( \Omega \), see the proof of Lemma 3.2. Each \( u_n \), solution to (3.7) with \( f \) replaced by \( f_n \), has nonnegative values thanks to the maximum principle for elliptic operators with Lipschitz continuous coefficients proved in [10]. Our limiting procedure gives \( R(\lambda, A) f_n(x) \geq 0 \) for each \( x \), and \( R(\lambda, A) f(x) \geq 0 \) for each \( x \). So, \( R(\lambda, A) \) is a positivity preserving operator.  

\[\square\]

4. Examples and consequences

Example 4.1. Let \( \Omega \) be the unit open ball in \( \mathbb{R}^N \), and let \( U(x) = -\frac{\alpha}{2} \log (1 - |x|) \) for \( x \in \Omega \), with \( \alpha > 0 \). Then

\[
\exp \left( -2 U(x) \right) = (1 - |x|) \alpha, \quad DU(x) = \frac{\alpha x}{2 |x|(1 - |x|)} , \quad 0 < |x| < 1,
\]

\[
\text{for } x/\text{EMV}, \text{with } a = 0. \text{ Then}
\]

\[
\exp \left( -2 U(x) \right) = (1 - |x|) \alpha, \quad DU(x) = \frac{\alpha x}{2 |x|(1 - |x|)} , \quad 0 < |x| < 1,
\]

\[
\text{for } x/\text{EMV}, \text{with } a = 0. \text{ Then}
\]
and it is known that $C_0^\alpha(\Omega)$ is dense in $H^1(\Omega, \mu)$ iff $\alpha \geq 1$. See e.g. [14, Theorem 3.6.1]. In this case the result of Theorem 3.4 holds, and $A$ is a self-adjoint dissipative operator in $L^2(\Omega, \mu)$.

Note that assumption (1.4) is satisfied only for $\alpha > 1$. This shows that assumption (1.4) is not equivalent to the fact that $C_0^\alpha(\Omega)$ is dense in $H^1(\mathbb{R}^N, \mu)$, however it is not very far. □

Under the assumptions of Theorem 3.4, $A$ is the infinitesimal generator of an analytic contraction semigroup $T(t)$ in $L^2(\Omega, \mu)$.

Since the resolvent $R(\lambda, A)$ is positivity preserving for $\lambda > 0$, also $T(t)$ is positivity preserving. Since $R(\lambda, A) 1 = 1/\lambda$, then $T(t) 1 = 1$ for each $t > 0$. Therefore, $T(t)$ is a Markov semigroup and it may be extended in a standard way to a contraction semigroup (that we shall still call $T(t)$) in $L^p(\Omega, \mu)$, $1 \leq p \leq \infty$. $T(t)$ is strongly continuous in $L^p(\Omega, \mu)$ for $1 \leq p < \infty$, and it is analytic for $1 < p < \infty$. See e.g. [4, Chapter 1].

The infinitesimal generator of $T(t)$ in $L^p(\Omega, \mu)$ is denoted by $A_p$. The characterization of the domain of $A_p$ in $L^p(\Omega, \mu)$ for $p \neq 2$ is an interesting open problem.

An important optimal regularity result for evolution equations follows, see [9].

**Corollary 4.2.** Let $1 < p < \infty$, $T > 0$. For each $f \in L^p((0, T); L^p(\Omega, \mu))$ (i.e. $(t, x) \mapsto f(t)(x) \in L^p((0, T) \times \Omega; dt \times d\mu)$) the problem

\[
\begin{cases}
  u'(t) = A_p u(t) + f(t), & 0 < t < T, \\
  u(0) = 0,
\end{cases}
\]

has a unique solution $u \in L^p((0, T); D(A_p)) \cap W^{1,p}((0, T); L^p(\Omega, \mu))$.

From Lemma 3.3 we get, taking $\psi \equiv 1$,

\[
\int_\Omega Au(x)d\mu = 0, \quad u \in D(A),
\]

and hence,

\[
\int_\Omega T(t)f d\mu = \int_\Omega f d\mu, \quad t > 0,
\]

for each $f \in L^2(\Omega, \mu)$. Since $L^2(\Omega, \mu)$ is dense in $L^1(\Omega, \mu)$, the above equality holds for each $f \in L^1(\Omega, \mu)$. In other words, $\mu$ is an invariant measure for the semigroup $T(t)$.

From Lemma 3.3 we get also

\[
u \in D(A), \quad Au = 0 \Rightarrow Du = 0,
\]

and hence the kernel of $A$ consists of the constant functions. Let us prove now that

\[
\lim_{t \to +\infty} T(t)f = \int_\Omega f(y)d\mu = L^2(\Omega, \mu),
\]

for all $f \in L^2(\Omega, \mu)$.
Indeed, since the function \( t \to q(t) = \int (T(t)f)^2 \mu(dx) \) is nonincreasing and bounded, there exists the limit \( \lim_{t \to +\infty} q(t) = \lim_{t \to +\infty} \langle T(2t)f, f \rangle_{L^2(\Omega, \mu)} \). By a standard argument it follows that there exists a symmetric nonnegative operator \( Q \in \mathcal{L}(L^2(\Omega, \mu)) \) such that
\[
\lim_{t \to +\infty} T(t)f = Qf, \quad f \in L^2(H, \mu).
\]
On the other hand, using the Mean Ergodic Theorem in Hilbert space (see e.g. [11, p. 24]) we get easily
\[
\lim_{t \to +\infty} T(t)f = \mathcal{P} \left( \int_0^1 T(s)f ds \right),
\]
where \( \mathcal{P} \) is the orthogonal projection on the kernel of \( A \). Since the kernel of \( A \) consists of the constant functions, (4.1) follows.

From now on we make a strict convexity assumption on \( U \):
\[
\exists \omega > 0 \text{ such that } x \mapsto U(x) - \omega |x|^2/2 \text{ is convex.}
\]
This will allow us to prove further properties for \( T(t) \), through Poincaré and Log-Sobolev inequalities.

If \((\Omega, m)\) is any measure space and \( u \in L^1(\Omega, m) \) we set
\[
\overline{u}_m = \int \limits_{\Omega} u(x) m(dx).
\]

**Proposition 4.3.** Let the assumptions of Theorem 3.4 and (4.2) hold. Then
\[
\int \limits_{\Omega} |u(x) - \overline{u}_m|^2 \mu(dx) \leq \frac{1}{2\omega} \int \limits_{\Omega} |Du(x)|^2 \, d\mu(dx), \quad u \in H^1(\Omega, \mu),
\]
and
\[
\int \limits_{\Omega} u^2(x) \log(u^2(x)) \mu(dx) \leq \frac{1}{\omega} \int \limits_{\Omega} |Du(x)|^2 \mu(dx) + \overline{u}_m^2 \log(\overline{u}_m^2), \quad u \in H^1(\Omega, \mu).
\]

**Proof.** Let \( u \in C_0^\infty(\mathbb{R}^N) \) have support in \( \Omega \). Let \( U_\alpha \) be the Moreau-Yosida approximations of \( U \), and set as usual \( \nu_\alpha(dx) = \left( \int_{\mathbb{R}^N} e^{-2U_\alpha(x)} \, dx \right)^{-1} e^{-2U_\alpha(x)} \, dx \). Since \( x \mapsto U_\alpha(x) - \omega(1 - \alpha) |x|^2 \) is convex in the whole \( \mathbb{R}^N \), by Theorem 2.2 we have, for \( \alpha \in (0, 1) \),
\[
\int \limits_{\mathbb{R}^N} |u(x) - \overline{u}_\alpha|^2 \nu_\alpha(dx) \leq \frac{1}{2\omega(1 - \alpha)} \int \limits_{\mathbb{R}^N} |Du(x)|^2 \nu_\alpha(dx),
\]
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(where $\bar{u}_a$ stands for $\bar{u}_{w_a}$) and

$$\int_{\mathbb{R}^N} u^2(x) \log (u^2(x)) v_a (dx) \leq \frac{1}{\alpha (1 - \alpha)} \int_{\mathbb{R}^N} |Du(x)|^2 v_a (dx) + \bar{u}_a^2 \log (\bar{u}_a^2).$$

Since

$$\lim_{a \to 0} U_a(x) = \left\{ \begin{array}{ll} U(x) & \text{if } x \in \Omega \\ + \infty & \text{if } x \notin \Omega, \end{array} \right.$$ 

then $\bar{u}_a$ goes to $\bar{u}_\mu = \int_{\Omega} u(x) \mu(dx)$, $\bar{u}_a^2$ goes to $\bar{u}_\mu^2$ as $\alpha$ goes to 0, and letting $\alpha$ go to 0 in (4.6), (4.7) we obtain that $u$ satisfies (4.4) and (4.5). Since $C_0^\infty (\Omega)$ is dense in $H^1 (\Omega, \mu)$, the statement follows. □

Proposition 4.3 yields other properties of $T(t)$, listed in the next corollary. The proof is identical to the proof of [5, Corollary 4.3], and we omit it.

**Corollary 4.4.** Let the assumptions of Theorem 3.4 and (4.2) hold. Then 0 is a simple isolated eigenvalue of $A$. The rest of the spectrum, $\sigma (A) \setminus \{ 0 \}$ is contained in $(- \infty , - \omega ]$, and

$$\| T(t) u - \bar{u}_\mu \|_{L^2 (\Omega, \mu)} \leq e^{-\omega t} \| u - \bar{u}_\mu \|_{L^2 (\Omega, \mu)}, \quad u \in L^2 (\Omega, \mu), \quad t \geq 0.$$ 

Moreover we have

$$\| T(t) \varphi \|_{L^{p(t)} (\Omega, \mu)} \leq \| \varphi \|_{L^p (\Omega, \mu)}, \quad p \geq 2, \quad \varphi \in L^p (\Omega, \mu),$$

where

$$q(t) = 1 + (p - 1) e^{2\omega t}, \quad t > 0.$$ 

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