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THE PROBLEMS OF BLOW-UP FOR NONLINEAR HEAT EQUATIONS. COMPLETE BLOW-UP AND AVALANCHE FORMATION

Abstract. — We review the main mathematical questions posed in blow-up problems for reaction-diffusion equations and discuss results of the author and collaborators on the subjects of continuation of solutions after blow-up, existence of transient blow-up solutions (so-called peaking solutions) and avalanche formation as a mechanism of complete blow-up.

Key words: Blow-up; Semilinear heat equations; Thermal avalanche.

1. Introduction

Many processes in the applied sciences are modeled by means of evolution equations involving differential operators, or systems of such equations. When suitable additional conditions – usually, initial and boundary conditions – are supplied, we obtain well-posed problems with a well-defined solution. The classical mathematical theories involve linear operators in the governing equations. However, many of the most important models in science, like those of relativity theory or fluid flows, reacting or not, involve nonlinear partial differential equations. These systems exhibit a number of properties which are absent from the linear theories and make them quite difficult to analyze. Moreover, these nonlinear properties are often related to essential features of the real world phenomena which the mathematical model aims at describing; the linear approximation is only a first-step procedure to prepare more realistic nonlinear analysis.

The study of nonlinear processes has been a continuous source of new problems and it has motivated the introduction of new methods in the areas of mathematical analysis, partial differential equations and other disciplines, becoming a most active area of mathematical research in the last decades.

To be specific, one of the most remarkable properties that distinguish nonlinear evolution problems from the linear ones is the possibility of eventual occurrence of *singularities* starting from perfectly smooth data, or more accurately, from classes of data for which a theory of existence, uniqueness and continuous dependence can be established for small time intervals, so-called *well-posedness in the small*. While singularities can arise in linear problems, this happens through the singularities contained in the coefficients or data of the problem (*fixed singularities*). On the contrary, in nonlinear systems they may arise from the nonlinear mechanisms of the problem and their time and location are to be determined by the mathematical analysis (*moving singularities*).

Blow-up for ordinary differential equations. The simplest form of spontaneous singularities in nonlinear problems appears when the variable or variables tend to infinity as time approaches a certain finite limit $T > 0$. This is what we call a *blow-up* phenomenon. Blow-up happens in an elementary form in the theory of ordinary differential equations (ODE's). The simplest example appears in the equation of *quadratic growth*: we consider the following problem for a real scalar variable $u = u(t)$:

$$u_t = u^2, \quad t > 0; \qquad u(0) = a. \tag{1.1}$$

For data $a > 0$ it is immediate that a unique solution exists in the time interval $0 < t < T$ with $T = 1/a$, given by the formula

$$u(t) = \frac{1}{T - t}. \tag{1.2}$$

Therefore, the evolution is given by a smooth function for $t < T$. As $t \to T^-$ (limit from the left), we see that $u(t) \to \infty$, *i.e.*, the solution *blows up*. Note that we also know how quickly the solution blows up, namely $u(t) = O((T-t)^{-1})$. Blow-up is referred to in Latin languages as *explosion*, and in fact the mathematical problems involved aim in many cases at describing explosive phenomena.

Starting from this example, the concept of blow-up can be widely generalized as the phenomenon whereby solutions cease to exist *globally in time* because of infinite growth of one of several of the variables describing the evolution process. A first extension step is given by ODE's of the form $u_t = u^p$, with $p > 1$ and, more generally,

$$u_t = f(u), \tag{1.3}$$

where f is positive and, say, continuous, under the condition

$$\int_1^\infty ds/f(s) < \infty. \tag{1.4}$$

This *Osgood's condition* in the ODE theory established in 1898, [32], is *necessary and sufficient* for the occurrence of blow-up in finite time for any solution with positive initial data. More generally, we can think of systems $u_t = f(t, u)$ for a vector variable $u \in \mathbb{R}^n$. In this case we may have blow-up due to the same mechanism if f is superlinear with respect to u for $|u|$ large, and also blow-up due to the singular character of f with respect to t at certain given times. It is the generalization of the first form that will be of concern in these notes.

The study of ODE's supplies basic tools and intuitions for the whole theory of blow-up, and, more generally, the study of singularities. Not always we will find explicit formulas like the ones above, but we will find detailed information about when, where, and how blow-up happens, how to calculate it and what happens after.

Blow-up for PDES. Fluid flows and other problems. The study of blow-up is considerably more difficult (and interesting) when the equations involved are PDEs, and indeed, it has become both a kind of industry and an art. The most typical scenario deals with evolution processes, and then the problem has both a spatial and a time structure, so that the unknowns depend not only on time, like the ODE case, but also on space. Specifically, $u = u(x, t)$, with $x \in \Omega$, a domain in $\mathbb{R}^n$, while time stretches to an interval $0 \leq t < T$.

Those problems are interesting because of their mathematical difficulty, but mainly because of their application to different sciences. Maybe the most famous blow-up problems at this moment are those asking whether the solutions of the two basic equations of fluid motion, the Euler equations and the Navier-Stokes equations, develop singularities in finite time for cases where the initial data are smooth and conveniently decreasing as $|x| \to \infty$. Unfortunately, the work reported here does not offer a new clue to these problems.

Blow-up has been investigated for many other equations and systems: reaction-diffusion equations, wave equations, conservation laws, Prandtl's equations, and so on. A quite large number of different phenomena have been discovered and analyzed, having in any case important common trends. This recommends specializing the subject to gain in depth.

In the forthcoming sections we will introduce the main problems in the framework of reaction-diffusion equations, one of the fields in which the blow-up theory has witnessed more effort and progress. This paper is a kind of continuation of the survey paper [17], written in collaboration with Victor Galaktionov, with emphasis on progress done by the author and collaborators in recent years. Most of the contents of the next two sections is taken from that source. In particular, we review the *Question List* for blow-up problems proposed in [17], that is now reformulated as a 7-item list. We also point out that this reference contains a number of historical facts and ends with extensive discussion of other blow-up and extinction problems for reaction-diffusion and for other equations.

The last sections report on the subjects of complete blow-up and continuation after blow-up, and on the mechanism of avalanche formation.

Let us finally note that the formation of singularities in finite time is also important in the attack to geometric problems by means of the technique of evolving curvature, like the Ricci flow, proposed by Hamilton [21, 22], where curvature may blow-up in finite time at certain manifold locations. The Ricci flow is strongly related to the nonlinear heat problems treated below, [1].

2. Blow-up in reaction-diffusion equations

Reaction-diffusion equations have developed into a quite large subtopic in PDE research, cf. [37]. In the theories of thermal propagation and combustion it is natural to consider quasilinear equations of the form:

$$u_t = \nabla \cdot \mathcal{A}(u, \nabla u, x, t) + B(u, \nabla u, x, t) \tag{2.5}$$

with standard ellipticity conditions on the operator $\mathcal{A}$ and growth and regularity conditions on both $\mathcal{A}$ and B. To fix ideas, we typically think of (2.5) as a nonlinear heat propagation model in a reactive medium, and then u is a temperature.

Reaction-diffusion models have played a prominent role in the study of blow-up. The concept of blow-up is now formulated in its simplest form in the following framework.

(*i*) We start from the well-posedness of the mathematical problem in a certain framework and for small times; thus, assuming nice regularity conditions on $\mathcal{A}$ and B, we will have an existence and uniqueness theory for, say, the Cauchy problem or one of the initial-boundary value problems, in a certain class of bounded and nonnegative data, so that the solutions evolve being bounded for some time $0 < t < T$.

(*ii*) We have also a regularity and continuation theory in this framework which says that bounded solutions have the necessary smoothness so that they can be continued locally in time. For classical solutions of parabolic equations this theory is based on Schauder estimates. Corresponding estimates exist for weak solutions of divergence-form equations in Sobolev spaces, or even for fully nonlinear equations.

Blow-up occurs then if the solution becomes infinite at some (or many) points as t approaches a certain finite time T. Namely, there exists a time $T < \infty$, called the *blow-up time*, such that the solution is well defined for all $0 < t < T$, while

$$\sup_{x \in \Omega} |u(x,t)| \to \infty \qquad \text{as } t \to T^- . \tag{2.6}$$

The *mathematical theory* has been actively investigated by researchers in the 60's mainly after general approaches to blow-up by Kaplan [23], Fujita [10, 11], Friedman [9] and some others; there is as yet no complete theory developed in the generality presented above, but detailed studies have been performed on a hierarchy of models of increasing complexity and there is nowadays a very extensive literature on the subject. There are two classical scalar models. One of them is the *exponential reaction model*

$$u_t = \Delta u + \lambda e^u, \qquad \lambda > 0, \tag{2.7}$$

which is important in combustion theory [41] under the name of *solid-fuel model* (*Frank-Kamenetsky equation*), and also in other areas. It is also of interest in differential geometry, [24], and other applications. The occurrence and type of blow-up depends on the parameter $\lambda > 0$, the initial data and the domain. The other classical blow-up equation is

$$u_t = \Delta u + u^p. \tag{2.8}$$

For exponents $p > 1$ we have the property of blow-up in (2.8); depending on the value of p it may happen not only for some but for all the solutions in a given class. There are a number of other popular models of evolution problems involving nonlinear parabolic equations, possibly degenerate. Here are models with the *porous medium* and *p-Laplace* operators,

$$u_t = \Delta u^m + u^p \ \ (m > 0) \quad \text{and} \quad u_t = \nabla \cdot (|\nabla u|^\sigma \nabla u) + u^p \ \ (\sigma > -1). \tag{2.9}$$

All these models take the form $u_t = \boldsymbol{A}(u) + f(u)$, where $\boldsymbol{A}$ is a second-order elliptic operator, maybe nonlinear and degenerate, representing a *diffusion*, and $f(u)$ is a superlinear function of u representing *reaction*.

There exist some good texts which display many of the results which are known, like the books by Bebernes and Eberly [4] and Samarskii *et al.* [36]. However, this is a very active field where there are many new developments. More detailed references to recent literature are discussed for instance in Bandle-Bruner's [2] and in [17].

A more general framework. Extinction. As we have said, it is very useful to insert the study of blow-up in a more general framework by considering it as a special type of *singularity* that develops for a certain evolution process. To be specific, we may have an evolution process described by a law and an initial condition:

$$u_t = \mathcal{A}(u) \quad \text{for } t > 0, \quad u(0) = u_0, \tag{2.10}$$

and we want to study the existence of a solution $u = u(t)$ as a curve living in a certain functional space, $u(t) \in X$. Frequent instances of X are $C(\Omega) \cap L^\infty(\Omega)$, or $L^p(\Omega)$, or the Sobolev space $H^1(\Omega)$. Typically, we are able to prove that for initial data $u_0 \in X$ the problem is well-posed in the small (*i.e.*, the solution u with initial data u_0 is well-defined and lives in X for some time $0 < t < T = T(u_0) > 0$), and we face the problem that the solution leaves the space X as $t \to T^-$. As we see, the occurrence of a singularity becomes in this view contingent upon the type of ambient space and the concept of solution used. However, an *essential blow-up* will defy all standard choices of solution and functional framework. This will be apparent in the type of blow-up called *complete blow-up* which will be introduced in Section 3 and discussed in later sections.

The general view of treating blow-up problems as singularities gives a unified approach in which to address very interesting related problems like *extinction*. A typical example of the latter is the semilinear heat equation with absorption

$$u_t = \Delta u - u^p, \qquad p < 1, \tag{2.11}$$

where solutions with positive data are considered and the exponent p may be allowed to pass to the so-called singular range, $p \leqslant 0$, see the survey paper [29]. The difference with the blow-up problem lies in the fact that here the singularity that hinders the continuation of the solution past a given time is not a blow-up of the unknown u, but rather the blow-up of its derivative u_t and the absorption term $f(u)$. Blow-up of derivatives is therefore another reasonable way in which nonlinear evolution equations develop singularities which may or may not stop the evolution of their solutions.

Other examples of the more general framework will occur for instance when the reaction term depends on the spatial gradient and the latter blows-up, even if the solution stays bounded, or in free boundary problems when the free boundary develops a cusp while the solution is regular in its domain, like in some Stefan or Hele-Shaw flows.

Let us point out that there is a great amount of work on blow-up for elliptic and other stationary equations which shares with the above presentation the idea that a singularity develops at a certain point (or at some points). Its scope and techniques have a different flavor.

3. The basic questions

We concentrate next on the analysis of the main questions raised in the study of blow-up for reaction-diffusion equations. This list can be suitably adapted to other *singularity formation* problems. We have proposed in [17] a list of questions that reflect the different aspects of blow-up. Slightly improved, it reads as follows: it starts with three classical questions

(1) *Does blow-up occur?* (2) *When?* (3) *Where?*
followed by two questions on *How?*:

(4) *How fast?* (5) *Which pattern?*
it continues with the questions of

(6) *What happens later?*
and ends with

(7) *How to compute blow-up numerically.*

Naturally, the last questions have come to the field only later, and they are very actively pursued at this time. Let us describe what these questions mean in brief terms.

(**1**) The first question is: *Does blow-up occur?* The blow-up problem is properly formulated only when a suitable class of solutions is chosen. Usually, the existence and uniqueness of the solutions of the problem can be formulated in different functional settings, and blow-up is just the inability to continue the solutions in that framework up to or past a given time. By default, we deal with classical solutions, but weak, viscosity or other kinds of generalized solutions can be more natural to a given problem. We may consider cases where blow-up happens in a functional framework and not in another one, for instance for classical solutions but not for weak L^1 solutions.

The general question can be split into these two aspects:

1.*i*) Which equations and problems do exhibit blow-up in finite time? The answer is determined by the form of the equation (in terms of its coefficients, or more generally its structural conditions) and the form of the data. Recall that explosive phenomena can be caused by the boundary conditions.

1.*ii*) In case the previous question has a positive answer, we may ask which solutions do blow-up in finite time? The possibilities for the last question are two-fold: blow-up occurs for all solutions in the given class, or it only occurs for some solutions (which should be identified). A problem for which all solutions blow-up is called a *Fujita problem*. The classical example is the semilinear heat equation $u_t = \Delta u + u^p$ posed in $\mathbb{R}^n$ where all classical nonnegative solutions blow-up in finite time when the exponent lies in the range $p \in (1, (n+2)/n)$. The upper bound is the so-called *Fujita exponent*.

Note that if a particular solution does not blow-up, then it lives globally in time.

(**2**) The second question is: *When?* Granted that blow-up occurs in finite time, can we estimate the blow-up time? Indeed, the property of blow-up can also happen in a less striking form in infinite time, when the solution exists in the given functional framework for all $0 < t < \infty$ but becomes unbounded as $t \to \infty$. Thus, we have the alternative: finite versus infinite-time blow-up. Indeed, a four-option table occurs for the solutions of reaction-diffusion systems:

2.*i*) global solutions which remain uniformly bounded in time (*i.e.*, no blow-up),

2.*ii*) global solutions with blow-up at infinity, *infinite-time blow-up*,

2.*iii*) solutions with *finite-time blow-up* (the standard blow-up case), and

2.*iv*) *instantaneous blow-up*, *i.e.*, the solution blows up at $t = 0$ in a sense to be specified.

The latter is a very striking nonlinear phenomenon, but we have shown that it occurs for such a simple equation as the exponential reaction equation $u_t = \Delta u + \lambda e^u$, cf. [33, 38].

(**3**) Next comes the question of *Where?* Firstly, for a solution $u = u(x, t)$ in $Q_T = \Omega \times (0, T)$, which blows up at a time $T > 0$, we define the *blow-up set* as

$$B(u_0) = \{x \in \Omega : \exists \{x_n, t_n\} \subset Q_T,\ t_n \to T^-,\ x_n \to x,\ u(x_n, t_n) \to \infty\}. \tag{3.1}$$

This is a closed set. Its points are the *blow-up points*. A smaller blow-up set is

$$B_1(u_0) = \{x \in \Omega : \exists \{t_n\} \subset (0, T), \{t_n\} \to T^-,\ u(x, t_n) \to \infty\}. \tag{3.2}$$

Typical alternatives when $\Omega = \mathbb{R}^n$ are: *single-point blow-up*, where $B(u_0)$ consists of a single point (or of a finite number of points), *regional blow-up*, where the measure of $B(u_0)$ is finite and positive, and *global blow-up*, where $B(u_0) = \mathbb{R}^n$. These notions are naturally adapted when Ω is not the whole Euclidean space. In the Russian literature of the 1970-80s these types of blow-up are called LS-regime, S-regime and HS-regime of blow-up, respectively [36]. In the first two cases the blow-up solutions are called *localized*.

(**4**) Next question is: *How fast* does blow-up occur? In other words, we want to calculate the *rate* at which u diverges as t approaches the blow-up time and x approaches a blow-up point.

Usually, this information is replaced by some norm estimate of $u(\cdot, t)$ as t approaches blow-up, or better by an asymptotic expansion. In general, the question How? proceeds via a change of variables (*renormalization*) that rescales the evolution orbit to bounded size, followed by the study of the limits of these orbits, which are now restricted to the blow-up set instead of the non-blowing points.

For many equations like (2.8) there is scale-invariance which implies the existence of solutions which blow-up at a power rate [19, 12]. *Self-similar blow-up* becomes then the usual form of blow-up and fixes the blow-up rates. However, it has been found that for large values of the exponent p in (2.8) the actual rate of blow-up can be given by a larger function than the self-similar power, a phenomenon called sometimes *fast blow-up*, though there are good reasons to call it *slow*. Such a blow-up has been detected in a number of problems and its rates and profiles are difficult to obtain.

(5) Next is the question of *Which pattern*, *i.e.*, the final-time blow-up profiles as limits of $u(x, t)$ when $t \to T^-$ at the blow-up points or at the non-blowing points.

The renormalized analysis usually leads to generic shape in the form of a *stable blow-up pattern*. The general classification of the singularity implies the further study of other unstable patterns. Typically, finite-time singularities, like blow-up, generate a countable discrete (not continuous!) spectra of structurally different patterns called *eigenfunctions of nonlinear media* in [36].

(6) A question that has received until recently less attention, but is of great importance for the practical application of mathematical models involving blow-up, is: *What happens after* a finite-time blow-up singularity occurs? This is the problem of *Continuation after blow-up*, also referred to as *Beyond blow-up*. A basic prerequisite is to find a suitable concept of continued solution. This is typically done, see below, by means of monotone approximation of the reaction term and the data so that the approximate problems have global solutions. Passing then to the limit we have to decide whether the solution becomes trivial (*i.e.*, identically infinite) after T or not [3]. This is the natural approach in the application to thermal propagation and combustion. With this method essentially three alternatives appear:

6.*i*) The solution cannot be continued. In the models we discuss below this happens because if continued it must be infinite everywhere in a natural sense. We call it *Complete Blow-up*.

6.*ii*) The solution can be continued in some region of space-time after T, but it is infinite in the complement, *Incomplete Blow-Up*.

6.*iii*) The solution becomes bounded again after T. This is a *Transient Blow-up*. We have found it in the form of *peaking blow-up*, where it becomes bounded immediately after T. In the models investigated so far, this is a very unstable phenomenon, a transition between more stable evolution patterns.

Alternative methods of continuation are not excluded and can be useful in suitable contexts, like *continuation in complex time*, cf. [30], who shows a way to continue the solution of $u_t = \Delta u + u^2$ past $t = T$ along a certain sector of times in the complex plane, avoiding the singularity. Unfortunately, this analytic continuation is not unique. Pointing out the interest in the study of continuation after blow-up is one of the main concerns of these notes.

(7) The final question refers to the *numerical methods* to detect the blow-up phenomenon and compute or approximate the blow-up solutions, times and profiles. The first problem is how to produce the *computational solution*. For instance, semi-discretization in space leads to an initial-value problem for a system of nonlinear ODEs. We can then use finite differences, collocation or finite elements to treat the spatial derivatives. Some of the relevant questions when we want the computational solution to display the properties of the blow-up phenomenon are the choice of spatial and temporal meshes, the choice of time integrator, the use of adaptive methods, and in a more theoretical direction the analysis of convergence. A good numerical approximation should be able to give an explanation of why a solution cannot be continued in complete blow-up. We refer to the survey paper [2] for more information.

4. Continuation after blow-up

We devote the rest of the paper to discuss concrete results, focusing on some problems where the author has been involved. An important question that we have studied in some detail is the possibility of natural continuation of the solutions of a blow-up problem *after* a blow-up occurs in finite time.

Concept of continuation. Proper solutions. A preliminary step consists of examining the most reasonable physical or mathematical options at our disposal when classical continuation fails. The experts in evolution problems aim at defining a continuous semigroup of maps in a certain space which will represent the time evolution of the physical problem, for all time if possible.

Baras and Cohen addressed this problem in 1987, [3], for the semilinear heat equations (2.8): $u_t = \Delta u + u^p$, posed for $\Omega \subset \mathbb{R}^N$ and $t > 0$ with exponent $p > 1$ in the framework of nonnegative solutions. The way favored by them and, more generally, by the Reaction-Diffusion community is to introduce a sequence of approximate problems that admit solutions globally in time. Namely, we assume that the reactive term $f(u)$, source of the explosive event, is replaced by a nicer term $f_n(u)$ where f_n is a smooth function with not more than the linear growth in $u \gg 1$ and the monotone convergence $f_n \nearrow f$ holds, so that the corresponding solution of the initial-value problem u_n is globally defined in time. If the Maximum Principle applies, as is the case, the sequence $\{u_n\}$ is monotone increasing in n. Therefore, we can pass to the limit by *monotonicity* and obtain a function

$$\tilde{u}(x, t) = \lim_{n \to \infty} u_n(x, t). \tag{4.1}$$

Function $\tilde{u}$ extends the classical solution u past the blow-up time, with finite or infinite values. In the particular case above, [3] proved that for $f(u) = u^p$ in the *subcritical Sobolev parameter range* $1 < p < p_s = (n+2)/(n-2)$, if $n \geqslant 3$ (or $1 < p < \infty$ if $n = 1, 2$) a continuation defined in a natural or physical way is not possible, because it leads to the conclusion that

$$\tilde{u}(x, t) = \infty \quad \text{for all} \quad x \in \Omega, \quad t > T. \tag{4.2}$$

They labeled the phenomenon *complete blow-up*. The case $p > p_s$ remained open. Brezis posed then the problem of finding equations where a nontrivial natural continuation exists, *i.e.*, the existence of reaction-diffusion equations with *incomplete blow-up*. This problem has been addressed by the authors in three works [14-16]. The results show that the continuation after blow-up is a relatively simple phenomenon in 1D.

This method of defining continuation by «approximations from below» works for quasilinear equations and also allows to treat unbounded initial data (by approximating them in a monotone increasing manner with bounded ones). It produces a «solution», finite or infinite, and can work in various settings where the maximum principle applies, like fully nonlinear equations. But, as always the case with so-called *limit sol-*

utions, it poses the problem of uniqueness of the limit object, *i.e.*, independence of the approximation choices. In [15] we have proved that this kind of definition does not depend on the approximations performed on the reaction term and/or the data. It is a continuation in the sense that it coincides with the classical or weak solution as long as it exists and is bounded, or with the *minimal* nonnegative solution in cases of non-uniqueness. We have given it the name of *proper solution* of the initial-value problem to distinguish it from other possible methods of constructing a limit solution. It is a kind of *viscosity solution* of the problem.

In the sequel continuation will be discussed in the framework of proper solutions, and will omit the tilde in the notation, $\tilde{u} = u$.

Characterization of complete vs incomplete blow-up in 1D. This problem has been solved in [14] in one space dimension for equations of the general form

$$u_t = \phi(u)_{xx} + f(u), \tag{4.3}$$

under quite general assumptions on ϕ and f. In particular, ϕ and f are positive for $u > 0$ and ϕ is increasing. More precisely, we assume that $\phi \in C([0, \infty)) \cap C^1(0, \infty)$, with $\phi'(u) > 0$ for $u > 0$ and $\phi(0) = 0$. The reaction term $f(u)$ is assumed to be continuous and positive for $u > 0$. We study the Cauchy problem $u(x, 0) = u_0(x)$, $x \in \mathbb{R}$, with bounded, continuous, nonnegative and nontrivial initial data.

We obtain a characterization of the possibility of nontrivial continuation of solutions of the Cauchy problem after blow-up, in terms of the properties of the constitutive nonlinearities ϕ and f. In principle, the alternative will also depend on the initial data u_0, but this will not be the case once the class of initial functions is fixed. In order to fix this setting we admit the assumption, typical in the blow-up literature, that the data are *bell-shaped*, *i.e.*, that u_0 has only one maximum point and goes to zero at infinity. The main results generalize to data with several humps.

Integral conditions and main results. Let us begin by stating the conditions which determine the form of continuation for bell-shaped data. The presence of complete or incomplete blow-up depends on the behaviour at infinity of three integrals.

We recall that finite-time blow-up occurs whenever $f(u)$ is super-linear for large u, as is well-known. More precisely, the first condition, (B1), is the convergence at infinity of the integral

$$I_1(u) = \int_1^u ds/f(s), \tag{4.4}$$

i.e., $I_1(\infty) < \infty$. Precisely, blow-up for positive constant initial data occurs if and only if the integral is finite. This follows from the ODE (1.3) satisfied by all spatially flat solutions $u(t)$. The problem of blow-up and continuation is thus completely solved for *flat* data and solutions (*i.e.*, not dependent on x) and the result does not depend on the data. For non-flat initial data the necessity of this condition for blow-up follows from the Maximum Principle.

Now, for bell-shaped data it is not always sufficient for blow-up, ϕ has a say! Indeed, the second integral that comes into play is

$$I_2(u) = \int_1^u f(s)\phi'(s)\,ds, \tag{4.5}$$

which allows to measure the relative influence of diffusion and reaction, and is reflected in our proofs in the existence of a certain type of blow-up traveling waves through the boundedness of the ratio

$$F(u) = I_2(u)/u^2 \tag{4.6}$$

as $u \to \infty$ (condition (B2) in [14]). Finally, we need to control

$$I_3(u) = \int_1^\infty [\phi'(s)/s]\,ds, \tag{4.7}$$

which affects only the strength of the diffusion nonlinearity. The boundedness of this integral (condition (B3)) is equivalent to the property of finite speed of propagation for large values of u, reflected in the existence of traveling waves which become infinite at a finite distance. In order to get an intuitive idea of what these conditions mean we may consider the case of power-like nonlinearities, equation

$$u\,t = (u^m)_{xx} + u^p, \tag{4.8}$$

$m > 0$ and $p > 0$. Then (B1) holds for $p > 1$, (B2) for $m + p \leqslant 2$ and (B3) for $m < 1$. In terms of these integrals we can formulate the following blow-up results.

THEOREM 4.1 (Global continuation). *Let u be a proper solution to the above problem under the stated general conditions. If* (B1), (B2) *and* (B3) *hold, then u can be continued in a non-trivial way for all times* $t > 0$ (i.e., $u(\cdot, t) \not\equiv \infty$ *for* $t > T$) *even if u blows up at a time* $T < \infty$.

In fact, the result says that the *burnt zone* $B[u](t)$, *i.e.*, the set of points where the continuation is infinite, is a bounded subset of $\mathbb{R}^n$ or the empty set. (B1)-(B3) are valid for equation (4.8) in 1D precisely when $p > 1$ and $p + m \leqslant 2$. This means that $m \leqslant 2 - p < 1$, a regime of fast diffusion. In particular, (B3) excludes linear diffusion, probably the main reason why the phenomenon of nontrivial continuation after blow-up in such a simple type of equation was unnoticed. The propagation of the burnt zone after blow-up, with the appearance of *singular interfaces*, is studied in [16].

The opposite situation happens when (B2) fails.

THEOREM 4.2 (Complete blow-up). *Let u be a solution of equation* (4.3) *under the above general conditions and assume that it blows up at time* $0 < T < \infty$. *If* (B2) *does not hold then* $u \equiv \infty$ *for* $t > T$.

Notice that we do not have still a complete characterization since the assumptions of the two results are not complementary of each other, due to the presence of condition (B3). They would combine into a complete results if it could be eliminated. This is almost true: under a number of light extra assumptions (B1) and (B2) imply (B3). It is for instance true for the 1D power equation (3.8). Besides, if condition (B2) is strengthened into (B2')

$$\phi'(u)f(u)/u \leqslant C \text{ for all large } u,$$

then we have $\phi'(u)/u \leqslant C/f(u)$, hence (B1) + (B2') imply (B3). The implication follows also when f is monotone. However, we have shown in [14] that there exist «pathological» choices of ϕ and f for which (B1) and (B2) hold but (B3) does not. Equations in the pathological class have curious properties.

THEOREM 4.3. *If* (B1) *and* (B2) *hold and* (B3) *does not, then all the solutions with flat initial data* $u_0 \equiv \text{const} > 0$ *blow-up in finite time, while no solution with bell-shaped compactly supported data does.*

Our results are true not only for solutions to the Cauchy problem, but can also be directly applied to initial-boundary value problems in bounded spatial domains with Dirichlet or Neumann boundary conditions. In fact, the analysis of complete/incomplete blow-up is local in the sense that the behaviour for $t > T$ depends only on the behaviour of the solution in a small neighbourhood of a given blow-up point, thus being independent of the boundary conditions.

The proof of the results is based on the Method of Traveling Waves, that replaces in the analysis the more usual method of investigation of blow-up phenomena, the *Method of Stationary States*, see [36], Chapter 7. A complete classification of existing traveling waves is done and then careful comparison arguments allow to pass the information to the blow-up solutions.

It is to be noted that only possibilities (6.*i*) and (6.*ii*) of item (6) of the Question list occur in one dimension. We will see that the situation in several space dimensions is much more complicated and admits option (6.*iii*).

Finally, we recall that the same techniques allowed us to treat the problem of Continuation after Extinction for equations of the form

$$u_t = \phi(u)_{xx} - f(u), \tag{4.9}$$

which was also open, cf. [14]. There are again three conditions that control the occurrence of complete or incomplete extinction.

5. TRANSIENT BLOW-UP IN SEVERAL DIMENSIONS

In contrast to the results of previous section, the situation in several dimensions is much more complex, and we have found in [15] new forms of blow-up described by a kind of blow-up solutions that we have called *peaking solutions*: they blow-up only at one point and one instant of time, and have a classical continuation afterwards. They represent a transient form of blow-up, and a minimal one at that.

Let us present this contribution of paper [15] in some detail. We consider equation

$$u_t = \Delta u^m + u^p, \tag{5.1}$$

for $x \in \mathbb{R}^n$ and $t > 0$. We assume that $n \geqslant 3$ and $m > m_c = (n-2)/n$. While for $1 < p \leqslant p_s$ the alternative between complete and incomplete blow-up reminds us of the 1D case, the novelty of the peaking solutions appears in the *supercritical Sobolev range*,

$$p > p_s = m(n+2)/(n-2). \tag{5.2}$$

Our main result can be stated as follows.

THEOREM 5.1. *There exists a number $p_p > p_s$ depending on m, n such that for $p \in (p_s, p_p)$ there exist proper solutions globally defined in time, which blow-up at finite $T > 0$ and are bounded at all other times. For $n \leqslant 10$ we can take $p_p = \infty$. Our solutions satisfy $u(x, t) \leqslant S(x)$ for all $t \geqslant T$, where S is the singular stationary solution.*

The singular solution is given by

$$S(x) = c_s\, |x|^{-2/(p-m)},$$

for some $c_s > 0$ that depends on m, p and n. $S(x)$ is defined if $n \geqslant 3$ and $p > p_{st} = mn/(n-2)$ (otherwise, no suitable $c_s \in \mathbb{R}$ exists). Observe that $S(x)$ is locally integrable and moreover $S^p \in L^1_{\text{loc}}(\mathbb{R}^n)$. Note that, even if S has a point singularity that may qualify as incomplete blow-up, it does not fit our blow-up framework since it does not start from bounded and smooth data at $t = 0$.

As for the range of allowed p's, in dimensions $n \geqslant 11$, and denoting $N = n - 10 > 0$, we have a finite value for p_p given by the formula

$$p_p = 1 + \frac{3m + [(m-1)^2 N^2 + 2(m-1)(5m-4)N + 9m^2]^{1/2}}{N}. \tag{5.3}$$

This new (and impressive) exponent was first introduced in [28] for $m = 1$, where it takes the simpler form $p_p = 1 + 6/(n-10)$. The subindex in the exponent p_p refers to «peaking». Observe that p_p is larger than another *critical exponent* $p_u = m[1 + 4/(n - 4 - 2\sqrt{n-1})]$, $n \geqslant 11$; p_u appears in the theory as responsible for the uniqueness of a solution with the singular initial data, cf. [15].

The peaking solution is constructed with a self-similar form, both for $t < T$ and $t > T$. This means that

$$u(x, t) = (T-t)^{\alpha}\theta_1(|x|(T-t)^{-\beta}), \qquad u(x, t) = (t-T)^{\alpha}\theta_2(|x|(t-T)^{-\beta}), \tag{5.4}$$

resp. for $t < T$ and $t > T$, with $\alpha = -1/(p-1)$ and $\beta = (p-m)/2(p-1)$ and θ_1 and θ_2 suitable one-dimensional profile functions such that

$$\lim_{s \to \infty} \theta_1(s) s^{2/(p-m)} = \lim_{s \to \infty} \theta_2(s) s^{2/(p-m)} > 0. \tag{5.5}$$

This is the matching condition, that allows them to be parts of one global solution. The construction relies on an ODE stability study for the singular stationary solution S, which happens to be completely different for $p < p_u$ and $p \geqslant p_u$. Moreover, for $p < p_u$ we can construct infinitely many different peaking solutions, while for $p \geqslant p_u$ only a finite number can be shown to exist by our method. We claim that, at least for self-similar solutions, the above value p_p is optimal for the theorem.

These peaking solutions have the weakest possible form of blow-up. In fact, it is easy a posteriori to check that $u \in C([0, \infty): L^r_{loc}(\mathbb{R}^n))$ for every $1 < r < n(p-m)/2$, a number that for $p > p_s$ is larger than $2mn/(n-2)$. They are also *weak solutions* of the equation in the standard sense of integration by parts.

Let us mention as a precedent the work of Lacey and Tzanetis [27] for the exponential equation (2.7), but there the matching of both developments is formal. Note also that *Incomplete* blow-up patterns are known to be structurally *unstable*. Moreover, stable self-similar patterns exhibit *complete* blow-up which is expected to be a *stable* mode, see [15, Section 14]. Construction of peaking solutions under conditions that do not allow for self-similarity is not easy, and recent results are due to Fila and Matano, cf. [8]. The existence of solutions which peak up several times is a quite difficult problem [added in proof: studied recently by Mizoguchi, 2004].

6. Avalanche formation at complete blow-up

The last subject we will treat in this paper is the occurrence of complete blow-up. This is a quite intriguing phenomenon that needs explanation. We will review the conclusions of two papers coauthored with J. Rossi and F. Quirós [34, 35] where complete blow-up is explained as the limit of a process of *avalanche formation*.

To be specific, let us consider as in [35] the semilinear heat equation

$$u_t = \Delta u + u^p, \tag{6.6}$$

posed in $\mathbb{R}^N$ or in a bounded domain with homogeneous Dirichlet boundary conditions, with $1 < p < p_s$, p_s is the Sobolev exponent ($p_s = (n+2)/(n-2) < \infty$ if $n \geqslant 3$). This problem has solutions with finite-time blow-up, *i.e.*, for large enough initial data there exists $T < \infty$ such that u is a classical solution for $0 < t < T$, while it becomes unbounded as $t \nearrow T$. In order to understand the situation for $t > T$ we consider a natural approximation by reaction problems of the form $u_t = \Delta u + f_n(u)$, with $f_n(u)$ a Lipschitz continuous approximation of the power u^p, so that these equation have global solutions u_n. We then pass to the limit $n \to \infty$. As has been said, the limit solution undergoes complete blow-up: after it blows up at $t = T$, the continuation is identically infinite for all $t > T$. Therefore, a quite strong discontinuity takes place at $t = T$, in the form of a jump from the finite profile formed at $t = T-$ to the infinite values taken at $t = T+$. This is what we want to explain.

Actually, we contend that the singularity set of a solution that blows up as $t \nearrow T$ propagates instantaneously at time $t = T$ to cover the whole space, producing the catastrophic discontinuity between $t = T-$ and $t = T+$. This is called the avalanche. We visualize it by looking at what happens to the approximations u_n on which the construction of the proper solution is based: as t proceeds past T, the solutions of the approximate problems, u_n, approach a certain asymptotic size and shape in the inner core, near the place and time where blow-up takes place for u. We perform a suitable renormalization or scaling that blows up time and space near the first blow-up point with a rate depending on the approximation parameter n. Then, the renormalized solutions v_n tend to a structure, the inner layer, which is the solution of a simple limit problem in terms of an ordinary differential equation. Moreover, this inner build-up produces in the outer region a traveling wave whose speed we can compute. The speed scales like $n^{(p-1)/2}$, which confirms in a quantitative way the fact that in the limit the proper solution propagates instantaneously at time $t = T$ to cover the whole space. In case the spatial domain is bounded, the traveling waves are modified by the effect of the boundary conditions, that become dominant at a later stage.

Outline of results. We start the analysis of paper [35] by collecting information on what happens immediately before blow-up. We are in the situation of point-wise blow-up. For definiteness, we assume that u blows up at $x = 0 \in \Omega$. Actually, we know that the blow-up rate is given by

$$c(T-t)^{-1/(p-1)} \leq \|u(\cdot, t)\|_\infty \leq C(T-t)^{-1/(p-1)} \tag{6.7}$$

if p is subcritical, *i.e.*, for $1 < p < (N+2)/(N-2)_+$, see [19]. Moreover, in this case the asymptotic behaviour close to the blow-up point $x = 0$ and the blow-up time T is given by

$$\lim_{t \nearrow T} (T-t)^{1/(p-1)} u(\xi(T-t)^{1/2}, t) = (p-1)^{-1/(p-1)},$$

uniformly on sets $|\xi| \leq R$, see [13, 18-20, 40]. (*Remark.* This blow-up rate may fail for large p in high dimensions, hence the restriction on p.)

We then construct a sequence of approximate solutions with suitable truncations f_n of the reaction term u^p. There are two typical approximations: the flat truncation, where $f_n(u)$ is defined for all $u \leq 0$ as

$$f_n(u) = \min\{u^p, n\},$$

and the linear truncation, where f_n is defined for $u > n$ as the linear function $f_n(u) = n^p + \alpha(u-n)$, $\alpha > 0$. When $\alpha = pn^{p-1}$ we get the tangent truncation.

The whole project aims at showing that for any sequence of truncations f_n of u^p which grow at most linearly at infinity, the global solutions u_n undergo at times $t \approx T$ the transition from an approximation of the blow-up profile of u just outlined (first approximate blow-up stage), through an intermediate «parabolic stage» towards the resolution to different asymptotics which depend on the boundary conditions and have partly parabolic, partly hyperbolic character.

If Ω is the whole space, the third stage consists of two main regions

(*i*) An *outer region* of the form $|x| \geq c_n(t - \check{T}_n)$, $\check{T}_n \approx T$, where the solution u_n is below the level n.

(*ii*) An *inner region* in which the solution is larger than n so that the reaction term takes on the truncated value.

It is proved that there exists a traveling wave which propagates at the precise speed

$$v(n) = c_* n^{(p-1)/2}.$$

The normalized wave speed c_* is calculated in [35] as *the only value of the parameter c that produces the correct connection in a phase plane*. Note that c_* depends on the type of truncations used. This traveling wave gives the asymptotic behaviour of general solutions u_n in the following sense: the level sets $S(t; k, n)$ of points x where $u_n(x, t) = k$ tend to travel with constant speed $c_* n^{(p-1)/2}$ for large t. This had been predicted in [15], based on the ODE analysis of the different types of traveling waves performed in [14]. As $n \to \infty$ we have $v(n) \to \infty$, and this is a precise quantitative explanation of why complete blow-up occurs.

As for the inner region, we perform the detailed analysis in the case of flat truncation, $f_n(u) = \min\{u^p, n^p\}$, and obtain a *conical pattern* for the solution of the form

$$\lim_{t \to \infty} \frac{u_n(\theta n^{(p-1)/2} t, t)}{n^p t} = \left(1 - \frac{|\theta|}{c_*}\right)_+, \tag{6.8}$$

uniformly for $\theta \in \mathbb{R}^N$, where $(v)_+ = \max\{v, 0\}$. This looks like a *sandpile* growing linearly in time. It is to be noted that this behaviour is independent of the space dimension. The result can be explained in simple terms by looking at the simplified problem consisting of the heat equation

$$v_t = \Delta v + 1,$$

posed in the cone $\mathcal{C} = \{(x, t): t > 0, |x| \leqslant c_* t\}$ with zero boundary conditions on the side of the cone. The solution of this problem has the asymptotic shape (6.8), as can be shown by manipulating Gaussian kernels. Note also that the effect of the Laplacian is confined to an even smaller inner region of the form $|x| = o(t)$, while in the rest of the inner region we have $u_t \sim n^p$.

While the appearance of a traveling wave behaviour is a general phenomenon, the formation of a conical pattern in the inner region depends on the flat truncation. In the case of the tangent truncation, $f_n(u) = \min\{u^p, n^p + pn^{p-1}(u-n)_+\}$, it is shown that there is an exponential increase in time and a formation of a diffusive spatial pattern.

An important step in the study of the third stage is to show that u_n stays large once it becomes big. This is done via some technical, though crucial, estimates, that make the whole analysis rather long and difficult.

Problem with boundary blow-up. Actually, the structure just described is so complicated that we needed a preliminary study of a related but simpler model with complete blow-up in order to prepare the machinery.

Thus, in [34] we study the continuation after blow-up of solutions $u(x, t)$ of the heat equation, $u_t = u_{xx}$, with a nonlinear flux condition at the boundary, $-u_x(0, t) = f(u(0, t))$ and some related problems. The main problem we address is

$$\begin{cases} u_t = u_{xx} & (x, t) \in (0, 1) \times (0, T), \\ -u_x(0, t) = f(u(0, t)) & t \in (0, T), \\ u_x(1, t) = 0 & t \in (0, T), \\ u(x, 0) = u_0(x) & x \in (0, 1), \end{cases} \tag{6.9}$$

where f is positive and continuous. We assume that u_0 is continuous, nonnegative and nontrivial ($u_0 \not\equiv 0$). Hence, the solution of problem (6.9) exists for a certain time interval $0 < t < T$ and is C^1 up to the boundary for all $t > 0$ as long as it is bounded. The problem can be thought of as a model to describe heat propagation with constant thermal conductivity (also referred to as linear diffusion) in a medium with a nonlinear radiation law at the left boundary, the right one being thermally insulated.

Our first result states that blow-up for problem (6.9) is complete for any radiation function f as above for which there is blow-up, in the sense that the proper solution is identically infinite for $t > T$.

We are interested in the avalanche formation at blow-up. We describe it in the case $f(u) = u^p$, $p > 1$, as a boundary layer which appears in the limit of the approximate problems by choosing a suitable scaling and passing to self-similar variables. We then show that the layer is described by the solution of a limit problem. We also describe the asymptotic behaviour for the approximate problems as t goes to infinity.

More precisely, we show that for a sequence of truncations f_n of f which are linear at infinity with increasing slope $f_n'(s) \sim cn^{p-1}$, the global solutions u_n *undergo at times* $t \approx T$ *the transition from an approximation of the blow-up profile of* u *towards an approximate traveling wave which propagates at speed* $v = v(n)$. *As* $n \to \infty$ *we have* $v(n) \to \infty$ *and complete blow-up occurs*. This intermediate profile evolves later towards different asymptotics of parabolic type which we also describe. Let us remark that the study of the approximate problems for large n is in many cases (combustion, chemistry) more realistic than the blow-up problem, which is a mathematical idealization.

Finally, we consider the effect of a nonlinear diffusion in the equation. For simplicity we take the usual power form $u_t = (u^m)_{xx}$ with diffusion exponent $m > 0$. For $m > 1$ this equation is called the porous medium equation, while for $m < 1$ it is known as the fast-diffusion equation. In any case, we arrive to

$$\text{(6.10)} \qquad \begin{cases} u_t = (u^m)_{xx} & (x, t) \in (0, 1) \times (0, T), \\ -(u^m)_x(0, t) = f(u(0, t)) & t \in (0, T), \\ (u^m)_x(1, t) = 0 & t \in (0, T), \\ u(x, 0) = u_0(x) & x \in (0, 1). \end{cases}$$

We find the same phenomenon of complete blow-up for all f provoking blow-up if and only if $m \geqslant 1$, *i.e.*, for porous medium equations. The thermal avalanche is also described in this case. On the contrary, for the so-called fast diffusion equations, $0 < m < 1$, continuation after blow-up is always possible. Moreover, for $m < 1$ the only blow-up point is the origin $x = 0$ for all $t \geqslant T$ and we are able to describe the form of the isolated singularity after blow-up.

The last results show the fundamental role played by diffusion in the propagation of the ∞-level set. In case $m > 1$ the infinite level propagates with infinite speed and there is no possible continuation, no matter which is the nonlinearity f that we are considering at the boundary. However, when $m < 1$ the ∞-level set does not propagate at all and there is an extension beyond $t = T$ that remains finite everywhere except at the boundary, $x = 0$. Actually, as explained in [7], the label fast-diffusion is misleading for blow-up problems since the diffusivity $D(u) = mu^{m-1}$ goes to zero as $u \to \infty$ for $m < 1$.

Avalanche is therefore related to the appearance of traveling waves that transport the blow-up information from the core. There are many aspects that are not well understood in this connection. The author has been inspired by Bebernes' work, cf. [5]. On the other hand, the formation and properties of travelling waves are a main topic in the mathematical theory of combustion, cf. for instance the study of plane deflagration waves in [6].

Finally, let us mention that numerical results illustrate these stages and are reported in the papers.

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