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Some properties of two-scale convergence


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Abstract. — We reformulate and extend G. Nguetseng’s notion of two-scale convergence by means of a variable transformation, and outline some of its properties. We approximate two-scale derivatives, and extend this convergence to spaces of differentiable functions. The two-scale limit of derivatives of bounded sequences in the Sobolev spaces $W^{1,p}(\mathbb{R}^N)$, $L^2_{div}(\mathbb{R}^3)$, $L^2_{rot}(\mathbb{R}^3)$ and $W^{2,p}(\mathbb{R}^N)$ is then characterized. The two-scale limit behaviour of the potentials of a two-scale convergent sequence of irrotational fields is finally studied.

Key words: Two-scale convergence; Two-scale decomposition; Sobolev spaces.

Riassunto. — Alcune proprietà della convergenza a due scale. Mediante una trasformazione di variabile, la nozione di convergenza a due scale di G. Nguetseng è qui riformulata ed estesa, ed alcune delle sue proprietà sono presentate. Tale convergenza è quindi estesa a spazi di funzioni differenziabili mediante l’approssimazione delle derivate a due scale. Inoltre si caratterizza il limite a due scale di derivate di successioni limitate negli spazi di Sobolev $W^{1,p}(\mathbb{R}^N)$, $L^2_{div}(\mathbb{R}^3)$, $L^2_{rot}(\mathbb{R}^3)$ e $W^{2,p}(\mathbb{R}^N)$. Infine si studia il limite a due scale dei potenziali di una successione convergente a due scale di campi irrotazionali.

Introduction

Let us fix any $N \geq 1$ and set $Y := [0, 1]^N$. The following concept was introduced by Nguetseng [15], and then studied in detail by Allaire [1] and others: a bounded sequence $\{u_\epsilon\}$ of $L^2(\mathbb{R}^N)$ is said (weakly) two-scale convergent to $u \in L^2(\mathbb{R}^N \times Y)$ iff

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} u_\epsilon(x) \varphi \left( x, \frac{x}{\epsilon} \right) dx = \int_{\mathbb{R}^N \times Y} u(x, y) \varphi(x, y) dxdy,
\]

for any smooth function $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ that is $Y$-periodic w.r.t. the second argument. Here is a canonic example: for any function $\varphi$ as above, $u_\epsilon(x) := \varphi(x, x/\epsilon)$ two-scale converges to $\varphi(x, y)$.

Two-scale convergence can account for occurrence of a fine-scale periodic structure, and indeed has been applied to a number of homogenization problems, see e.g. [1, 3, 5, 7, 8, 11-13, 15, 16, 18, 19]. For periodic homogenization problems, two-scale convergence can indeed represent an alternative to the classic energy method of Tartar, see e.g. [2, 4, 9, 10, 14, 17].

Along the lines of [3, 5, 8, 11, 12], in Sections 1-5 we reduce (1) to standard weak convergence in $L^2(\mathbb{R}^N \times Y)$, via a transformation of variable which can be interpreted as a two-scale decomposition. We characterize two-scale convergence, extend it to $L^p$ (for $p \in [1, +\infty)$) and $C^0$, and derive some basic properties. Some of these results are already known, cf. e.g. [1, 7, 8, 11, 12, 15]; here we organize these properties from the point of view of two-scale decomposition, in order to illustrate the potentialities of

that approach. We then study two-scale compactness, introduce approximate two-scale derivatives, and use them to extend two-scale convergence to spaces of differentiable functions. We thus show that several classic results (of Rellich-Kondrachov, Sobolev, Morrey, and so on) have a two-scale counterpart, that concerns sequences of functions instead of single functions.

In Sections 6-8 we characterize the two-scale limit of derivatives of bounded sequences in the Sobolev spaces $W^{1,p}(\mathbb{R}^N)$, $L^2_{\text{rot}}(\mathbb{R}^N)$, $L^2_{\text{div}}(\mathbb{R}^N)$ and $W^{2,p}(\mathbb{R}^N)$ ($p \in ]1, +\infty[)$. Theorem 6.1 may be compared with results of Nguetseng [15], of Allaire [1], and with one Cioranescu, Damlamian and Griso recently announced in [8]; the latter one is also based on two-scale decomposition, but uses a different approximation. Finally, in Section 9 we deal with the two-scale limit of the potential of a two-scale convergent vector field. Details, proofs and applications to homogenization problems will appear apart.

1. TWO-Scale CONVERGENCE

Two-Scale Decomposition. In this paper we denote by $\mathcal{Y}$ the set $Y = [0, 1]^N$, we equip with the topologic and differential structure of the $N$-dimensional torus, and identify any function on $\mathcal{Y}$ with its periodic extension to $\mathbb{R}^N$. For any $\varepsilon > 0$, we decompose real numbers and real vectors as follows:

$$
(1.1) \begin{cases}
\hat{n}(x) := \max \{ n \in \mathbb{Z} : n \leq x \}, & \hat{r}(x) := x - \hat{n}(x) (\in [0, 1[) \quad \forall x \in \mathbb{R}, \\
N(x) := (\hat{n}(x_1), \ldots, \hat{n}(x_N)) \in \mathbb{Z}^N, & R(x) := x - N(x) \in \mathcal{Y} \quad \forall x \in \mathbb{R}^N.
\end{cases}
$$

Thus $x = \varepsilon[N(x/\varepsilon) + R(x/\varepsilon)]$ for any $x \in \mathbb{R}^N$; $\varepsilon N(x/\varepsilon)$ and $R(x/\varepsilon)$ represent coarse-scale and fine-scale variables w.r.t. the scale $\varepsilon$, respectively. Besides this two-scale decomposition, we define the two-scale composition function:

$$
(1.2) S_\varepsilon(x, y) := \varepsilon N(x/\varepsilon) + \varepsilon y \quad \forall (x, y) \in \mathbb{R}^N \times \mathcal{Y}, \forall \varepsilon > 0.
$$

The next lemma can easily be proved via a variable transformation in the integral.

**Lemma 1.1.** Let $f : \mathbb{R}^N \times \mathcal{Y} \rightarrow \mathbb{R}$ be such that

$$
f \in L^1(\mathcal{Y}; (C^0 \cap L^\infty)(\mathbb{R}^N)) \cup L^1(\mathbb{R}^N; C^0(\mathcal{Y})),
$$

and extend it by periodicity to $\mathbb{R}^{2N}$. Then, for any $\varepsilon > 0$, the function $\mathbb{R}^N \times \mathcal{Y} \rightarrow \mathbb{R} : (x, y) \mapsto f(S_\varepsilon(x, y), y)$ is integrable, and

$$
(1.3) \int_{\mathbb{R}^N} f(x, x/\varepsilon) \, dx = \int_{\mathbb{R}^N \times \mathcal{Y}} f(S_\varepsilon(x, y), y) \, dxdy \quad \forall \varepsilon > 0.
$$

For any $p \in [1, +\infty]$, the operator $g \mapsto g \circ S_\varepsilon$ is then a linear isometry $L^p(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N \times \mathcal{Y})$.

Two-Scale Convergence in $L^p(\mathbb{R}^N \times \mathcal{Y})$. In this Note by $\varepsilon$ we represent the generic element of an arbitrary but prescribed, positive and vanishing sequence of real numbers; e.g., $\varepsilon = \{ 1, 1/2, 1/3, \ldots, 1/n, \ldots \}$. For any sequence of measurable functions,
$u_\epsilon : \mathbb{R}^N \to \mathbb{R}$, and any measurable function, $u : \mathbb{R}^N \times \mathcal{Y} \to \mathbb{R}$, we say that $u_\epsilon$ two-scale converges to $u$ (w.r.t. the prescribed sequence $\{\epsilon_n\}$) in some specific sense, whenever $u_\epsilon \circ S_\epsilon \to u$ in the corresponding standard (i.e., one-scale) sense. In this way, for any $p \in [1, + \infty]$ we define strong and weak (weak star for $p = \infty$) two-scale convergence in $L^p(\mathbb{R}^N \times \mathcal{Y})$; we then write $u_\epsilon \overset{2}{\rightharpoonup} u$, $u_\epsilon \overset{2}{\rightarrow} u$, $u_\epsilon \overset{2}{\ast} u$ (resp.). For any domain $\Omega \subset \mathbb{R}^N$, two-scale convergence in $L^p(\Omega \times \mathcal{Y})$ is then defined by extending functions to $\mathbb{R}^N \setminus \Omega$ with vanishing value.

Two-Scale Convergence in $C^0(\mathbb{R}^N \times \mathcal{Y})$. Because of the discontinuity of $S_\epsilon(\cdot, y)$, in general the function $u_\epsilon \circ S_\epsilon$ is discontinuous even if $u_\epsilon$ is continuous. A modification is thus needed, in order to extend the previous definitions to the space of continuous functions. For $i = 1, \ldots, N$, let us denote by $e_i$ the unit vector of the $x_i$-axis, set $x_i := x - x_i e_i$, for any $x \in \mathbb{R}$, and

$$(1.4) \quad \left\{ \begin{array}{l}
(I_{\epsilon,i}u)(x, y) := u(x_i + \epsilon \tilde{n}(x_i/e) e_i, y) + \\
\quad + R(x_i/e)(u(x_i + \epsilon \tilde{n}(x_i/e) e_i + \epsilon e_i, y) - u(x_i + \epsilon \tilde{n}(x_i/e) e_i, y))
\end{array} \right.
$$

Thus $L_{\epsilon,i} v$ is piecewise linear w.r.t. $x$, whereas $v \circ S_\epsilon$ is piecewise constant w.r.t. $x$. If $v \in C^0(\mathbb{R}^N)$, then $L_{\epsilon,i} v \in C^0(\mathbb{R}^N \times \mathcal{Y})$. For instance, for $N = 2$, let us set $r(x) := R(x, \epsilon)$ and $v_{ij}^m(y) := \nu(\epsilon(m + y) + \epsilon((i, j))$ for $i, j \in \{0, 1\}$ and for any $m \in \mathbb{Z}^N$; for any $x \in e_m \mathcal{Y}$ and any $y \in \mathcal{Y}$, then

$$(L_{\epsilon,i} v)(x, y) := (1 - r_1)(1 - r_2) v_{00}^m + r_1(1 - r_2) v_{10}^m + (1 - r_1) r_2 v_{01}^m + r_1 r_2 v_{11}^m.$$  

For any sequence $\{u_\epsilon\}$ in the $C^0(\mathbb{R}^N)$ and any $u \in C^0(\mathbb{R}^N \times \mathcal{Y})$, we say that $u_\epsilon$ strongly (weakly, resp.) two-scale converges to $u$ in $C^0(\mathbb{R}^N \times \mathcal{Y})$ iff $L_{\epsilon,i} u_\epsilon \rightharpoonup u$ (resp.) in $C^0(\mathbb{R}^N \times \mathcal{Y})$ w.r.t. to the usual topology of Fréchet space.

2. SOME PROPERTIES OF TWO-SCALE CONVERGENCE

It is easy to check that in $L^p$ weak/strong one-scale convergence and weak/strong two-scale convergence are related as follows. An analogous result holds in $C^0$.

**Proposition 2.1.** Let $p \in [1, + \infty[$ and $\{u_\epsilon\}$ be a sequence in $L^p(\mathbb{R}^N)$. Then:

$$(2.1) \quad u_\epsilon \rightharpoonup u \text{ in } L^p(\mathbb{R}^N) \iff \begin{cases} u_\epsilon \overset{2}{\rightharpoonup} u \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y}) \\ u \text{ is independent of } y, \end{cases}$$

$$(2.2) \quad u_\epsilon \overset{2}{\rightharpoonup} u \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y}) \implies u_\epsilon \overset{2}{\rightarrow} u \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y}),$$

$$(2.3) \quad u_\epsilon \overset{2}{\rightarrow} u \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y}) \implies u_\epsilon \rightarrow \int_y u(\cdot, y) \, dy \text{ in } L^p(\mathbb{R}^N).$$

**Limit Decomposition and Orthogonality.** Let $p \in [1, + \infty[$. If $u_\epsilon \overset{2}{\rightarrow} u$ in $L^p(\mathbb{R}^N \times \mathcal{Y})$,}
and \( u_e \to u_0 \) in \( L^p(\mathbb{R}^N) \), setting \( u_1 := u - u_0 \), by (2.3) we trivially get the two-scale decomposition

\[
\begin{aligned}
&\begin{cases}
  u(x, y) = u_0(x) + u_1(x, y) & \text{for a.a. } (x, y) \in \mathbb{R}^N \times \mathbb{Y},
  \\
  \int_{\mathbb{Y}} u_1(x, y) \, dy = 0 & \text{for a.a. } x \in \mathbb{R}^N.
\end{cases}
\end{aligned}
\]

(2.4)

Let us set \( p' := p/(p - 1) \) if \( p \neq 1 \), \( 1' := \infty \). If \( \varphi_{e} \to \varphi \) in \( L^p'(\mathbb{R}^N \times \mathbb{Y}) \) and \( \varphi_{e} \to \varphi_{0} \) in \( L^{p'}(\mathbb{R}^N) \), setting \( \varphi_{1} := \varphi - \varphi_{0} \) we then have

\[
\begin{aligned}
&\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} u_{\varepsilon}(x) \varphi_{\varepsilon}(x) \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{Y}} u(x, y) \varphi(x, y) \, dx \, dy = \\
&= \int_{\mathbb{R}^N} u_{0}(x) \varphi_{0}(x) \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{Y}} u_{1}(x, y) \varphi_{1}(x, y) \, dx \, dy.
\end{aligned}
\]

(2.5)

If \( p = 2 \), the decomposition (2.4) is orthogonal in \( L^2(\mathbb{R}^N \times \mathbb{Y}) \), and

\[
\|u\|_{L^2(\mathbb{R}^N \times \mathbb{Y})} = \|u_0\|_{L^2(\mathbb{R}^N)} + \|u_1\|_{L^2(\mathbb{R}^N \times \mathbb{Y})}.
\]

(2.6)

In Sections 6-8 we shall encounter examples of this two-scale decomposition of the limit.

The formula (2.7) below states the equivalence between the above definitions of weak and strong two-scale convergence and the original ones of Nguetseng [15] and Allaire [1]. The remainder is easily checked.

**Proposition 2.2.** Let \( p \in [1, +\infty[ \) and \( \{u_{\varepsilon}\} \) be a sequence in \( L^p(\mathbb{R}^N) \). Then

\[
\begin{aligned}
&\begin{cases}
  u_{\varepsilon} \rightharpoonup u & \text{in } L^p(\mathbb{R}^N \times \mathbb{Y}) \iff \{u_{\varepsilon}\} \text{ is bounded in } L^p(\mathbb{R}^N),
  \\
  \left\| \int_{\mathbb{R}^N} u_{\varepsilon}(x) \psi(x, x/\varepsilon) \, dx \to \int_{\mathbb{R}^N} \int_{\mathbb{Y}} u(x, y) \psi(x, y) \, dx \, dy \right\|_{\mathcal{D}'(\mathbb{R}^N \times \mathbb{Y})} \forall \psi \in \mathcal{D}(\mathbb{R}^N \times \mathbb{Y}),
\end{cases}
\end{aligned}
\]

(2.7)

\[
\begin{aligned}
&u_{\varepsilon} \to u & \text{in } L^p(\mathbb{R}^N \times \mathbb{Y}) \implies \\
&\liminf_{\varepsilon \to 0} \|u_{\varepsilon}\|_{L^p(\mathbb{R}^N)} \geq \|u\|_{L^p(\mathbb{R}^N \times \mathbb{Y})} \left( = \left\| \int_{\mathbb{Y}} u(\cdot, y) \, dy \right\|_{L^p(\mathbb{R}^N)} \right),
\end{aligned}
\]

(2.8)

\[
\begin{aligned}
&\text{if } p \in ]1, +\infty[, \quad u_{\varepsilon} \to u & \text{in } L^p(\mathbb{R}^N \times \mathbb{Y}) \iff \\
&\left\| u_{\varepsilon}\right\|_{L^p(\mathbb{R}^N)} \to \|u\|_{L^p(\mathbb{R}^N \times \mathbb{Y})}.
\end{aligned}
\]

(2.9)

**Two-Scale Convergence of Distributions.** Let us denote the duality pairing between \( \mathcal{D}'(\mathbb{R}^N) \) and \( \mathcal{D}(\mathbb{R}^N) \) by \( \langle \cdot, \cdot \rangle \), and that between \( \mathcal{D}'(\mathbb{R}^N \times \mathbb{Y}) \) and \( \mathcal{D}(\mathbb{R}^N \times \mathbb{Y}) \) by \( \langle \langle \cdot, \cdot \rangle \rangle \). For any sequence \( \{u_{\varepsilon}\} \) in \( \mathcal{D}'(\mathbb{R}^N) \) and any \( u \in \mathcal{D}'(\mathbb{R}^N \times \mathbb{Y}) \), we say that \( u_{\varepsilon} \) two-scale converges to \( u \) in \( \mathcal{D}'(\mathbb{R}^N \times \mathbb{Y}) \) iff

\[
\langle u_{\varepsilon}(x), \psi(x, x/\varepsilon) \rangle \to \langle \langle u(x, y), \psi(x, y) \rangle \rangle \quad \forall \psi \in \mathcal{D}(\mathbb{R}^N \times \mathbb{Y}).
\]

(2.10)
By (2.7), this extends the weak two-scale convergence in $L^p$. For instance, for $N = 1$, let $\{v_\varepsilon\}$ be a sequence in $L^1(0, 1)$ such that $v_\varepsilon(y) \to \delta_0(y - 1/2)$ (the Dirac mass at 1/2) in $\mathcal{D}'(0, 1)$, and extend $v_\varepsilon$ to $\mathbb{R}$ by periodicity. We have
\[
(2.11) \quad x v_\varepsilon(x/\varepsilon) \to 1 \quad \text{in } \mathcal{D}'(\mathbb{R}), \quad x v_\varepsilon(x/\varepsilon) \to x \delta_0(y - 1/2) \quad \text{in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}).
\]

One can also define two-scale convergence in $\mathcal{D}'(\mathbb{R}^N)$, by letting $\psi$ range in $\mathcal{D}(\mathbb{R}^N \times \mathbb{R}^N)$ in (2.10). However this definition seems less convenient.

Two-scale convergence in the spaces of Radon measures, $C^0(\mathbb{R}^N \times \mathbb{R})'$, is defined similarly.

3. Two-Scale Compactness

Let us say that a sequence $\{u_n\}$ is compact iff it is possible to extract a convergent subsequence from any of its subsequences. Proposition 2.1 trivially entails the following statement.

**Proposition 3.1.** Let $p \in [1, +\infty]$. For any sequence $\{u_n\}$ in $L^p(\mathbb{R}^N)$,
\[
\begin{align*}
\text{strong one-scale compactness} &\implies \text{strong two-scale compactness}; \\
\text{strong two-scale compactness} &\implies \text{weak two-scale compactness}; \\
\text{weak two-scale compactness} &\implies \text{weak one-scale compactness}.
\end{align*}
\]
The same holds for $C^0(\mathbb{R}^N)$, and (replacing weak compactness by weak star compactness) for $L^\infty(\mathbb{R}^N)$.

The next statement is also easily checked: parts (i) and (ii) follow from Lemma 1.1 and the Banach-Alaoglu theorem; part (iii) can be derived via the classic de la Vallée Poussin criterion.

**Proposition 3.2 (Weak Two-Scale Compactness in $L^p$).**

(i) Let $p \in [1, +\infty]$. Any sequence $\{u_n\}$ of $L^p(\mathbb{R}^N)$ is weakly star two-scale compact in $L^p(\mathbb{R}^N \times \mathbb{R})$ iff it is bounded, hence iff it is weakly star one-scale compact in $L^p(\mathbb{R}^N)$.

(ii) Similarly, any sequence of $L^1(\mathbb{R}^N)$ is weakly star two-scale compact in $C^0(\mathbb{R}^N \times \mathbb{R})'$ iff it is bounded, hence iff it is weakly star one-scale compact in $C^0(\mathbb{R}^N)'$.

(iii) Finally, any sequence of $L^1(\mathbb{R}^N)$ is weakly two-scale compact in $L^1(\mathbb{R}^N \times \mathbb{R})$ iff it is weakly one-scale compact in $L^1(\mathbb{R}^N)$.

We also have a two-scale version of Chacon’s biting lemma, cf. [6].

**Proposition 3.3 (Two-Scale Biting Lemma).** Let $\{u_n\}$ be a bounded sequence in $L^1(\mathbb{R}^N)$. Then there exist $u \in L^1(\mathbb{R}^N \times \mathbb{R})$, a subsequence $\{u_{n_k}\}$, and a nondecreasing sequence $\{\Omega_k\}$ of measurable subsets of $\mathbb{R}^N$ such that the measure of $\mathbb{R}^N \setminus \Omega_k$ vanishes...
as \( k \to \infty \), and
\[
(3.2) \quad u_{\varepsilon}(x + S_{\varepsilon}(b, k)) - u_{\varepsilon}(x) \to 0 \quad \text{in} \quad L^1(\Omega \times \mathcal{Y}), \quad \text{as} \quad \varepsilon' \to 0, \quad \forall k \in \mathbb{N}.
\]

Strong one-scale compactness is not equivalent to strong two-scale compactness in \( L^p \)- and \( C^0 \)-spaces. However, the classic Riesz and Ascoli-Arzelà compactness theorems entail the following results.

**Proposition 3.4 (Strong Two-Scale Compactness in \( L^p \)).** Let \( p \in [1, + \infty[ \). A sequence \( \{u_{\varepsilon}\} \) of \( L^p(\mathbb{R}^N) \) is strongly two-scale compact in \( L^p(\mathbb{R}^N \times \mathcal{Y}) \) iff it is bounded and
\[
(3.3) \quad \int_{\mathbb{R}^N} |u_{\varepsilon}(x + S_{\varepsilon}(b, k)) - u_{\varepsilon}(x)|^p \, dx \to 0 \quad \text{as} \quad (b, k, \varepsilon) \to (0, 0, 0),
\]

\[
(3.4) \quad \sup_{R \to + \infty} \int_{\mathbb{R}^N \setminus B(0, R)} |u_{\varepsilon}(x)|^p \, dx \to 0 \quad \text{as} \quad R \to + \infty.
\]

**Proposition 3.5 (Strong Two-Scale Compactness in \( C^0 \)).** A sequence \( \{u_{\varepsilon}\} \) of \( C^0(\mathbb{R}^N) \) is strongly two-scale compact in the Fréchet space \( C^0(\mathbb{R}^N \times \mathcal{Y}) \) iff it is bounded and
\[
(3.5) \quad \sup_{x \in K} |u_{\varepsilon}(x + S_{\varepsilon}(b, k)) - u_{\varepsilon}(x)| \to 0 \quad \text{as} \quad (b, k, \varepsilon) \to (0, 0, 0), \quad \forall K \subset \mathbb{R}^N.
\]

In the two latter theorems \( S_{\varepsilon}(b, k) := \varepsilon N(b/\varepsilon) + \varepsilon k \) cannot be replaced by \( b + \varepsilon k \); this more restrictive hypothesis would yield the strong one-scale compactness of \( \{u_{\varepsilon}\} \) in \( L^p(\mathbb{R}^N) \) (in \( C^0(\mathbb{R}^N) \), resp.).

### 4. Two-scale derivatives

Let \( w \in \mathcal{O}(\mathbb{R}^N \times \mathcal{Y}) \). Although \( u_{\varepsilon}(x) := w(x, x/\varepsilon) \to w(x, y) \) in \( L^p(\mathbb{R}^N \times \mathcal{Y}) \) for any \( p \in [1, + \infty[ \), in general \( \nabla w(x, y) \) is not the (weak) two-scale limit of \( \nabla u_{\varepsilon}(x) \); actually, this sequence is bounded in \( L^p(\mathbb{R}^N)^N \) only if \( w(x, y) \) does not depend from \( y \). In this section we show that nevertheless it is possible to express the gradient of the two-scale limit without evaluating the limit itself, via what we name *approximate two-scale gradient*.

For \( i = 1, \ldots, N \), let us denote by \( \nabla_i \varphi \) the partial derivative w.r.t. \( x_i \) of any function \( \varphi(x) \), and by \( \nabla_{\xi_i} \psi \) and \( \nabla_{\xi_j} \psi \) the partial derivatives of any function \( \psi(x, y) \). Let us also denote by \( e_i \) the unit vector of the \( x_i \)-axis, define the shift operator \( (\tau_{\xi} v)(x) := v(x + \xi) \) for any \( x, \xi \in \mathbb{R}^N \), set
\[
(4.1) \quad \nabla_{\varepsilon, i} := \frac{\tau_{\varepsilon e_i} - I}{\varepsilon}, \quad \nabla^a := \prod_{i=1}^N \nabla_{\varepsilon, i}^{a_i}, \quad \nabla^a := \prod_{i=1}^N \nabla_i^{a_i} \quad \forall a \in \mathbb{N}^N, \quad \forall \varepsilon > 0,
\]
and define \( \nabla_{\varepsilon, i}^a, \nabla_{\varepsilon}^a \) similarly. Finally, for any \( \varepsilon > 0 \) let us set \( \mathbb{R}^N_\varepsilon := \bigcup_{m \in \mathbb{Z}^N} \varepsilon(m + Y^0) \), and denote by \( \vec{\nabla} \) the gradient in the sense of \( \mathcal{O}'(\mathbb{R}^N_\varepsilon) \).

**Proposition 4.1.** Let \( m \in \mathbb{N}, \ p \in ]1, + \infty[, \ and \ a, \beta \in \mathbb{N}^N. \)
(i) If \( \{u_\varepsilon\} \) is a sequence in \( W^{m,p}(\mathbb{R}^N), \) \( |\alpha| + |\beta| \leq m, \) and
\[
\sup \|\nabla_\varepsilon (e \nabla)^\beta u_\varepsilon\|_{L^p(\mathbb{R}^N)} < +\infty,
\]
then
\[
\nabla_x u_\varepsilon \nabla_y u_\varepsilon \in L^p(\mathbb{R}^N \times \mathcal{Y}),
\]
\[
\nabla_x (e \nabla)^\beta u_\varepsilon \nabla_y u_\varepsilon \in L^p(\mathbb{R}^N \times \mathcal{Y}),
\]
and
\[
\nabla_\varepsilon (e \nabla)^\beta u_\varepsilon \nabla_\varepsilon u_\varepsilon \in L^p(\mathbb{R}^N \times \mathcal{Y}),
\]
then
\[
\nabla_x u_\varepsilon \nabla_y u_\varepsilon \in L^p(\mathbb{R}^N \times \mathcal{Y}),
\]
\[
\nabla_x (e \nabla)^\beta u_\varepsilon \nabla_y u_\varepsilon \in L^p(\mathbb{R}^N \times \mathcal{Y}),
\]
and
\[
\nabla_\varepsilon (e \nabla)^\beta u_\varepsilon \nabla_\varepsilon u_\varepsilon \in L^p(\mathbb{R}^N \times \mathcal{Y}),
\]

(ii) If \( \{u_\varepsilon\} \) is a sequence in \( L^p(\mathbb{R}^N) \cap W^{m,p}(\mathbb{R}^N), \) \( |\beta| \leq m, \) and
\[
\sup \|\nabla_\varepsilon (e \nabla)^\beta u_\varepsilon\|_{L^p(\mathbb{R}^N)} < +\infty,
\]
then
\[
\nabla_x u_\varepsilon \nabla_y u_\varepsilon \in L^p(\mathbb{R}^N \times \mathcal{Y}),
\]
\[
\nabla_x (e \nabla)^\beta u_\varepsilon \nabla_y u_\varepsilon \in L^p(\mathbb{R}^N \times \mathcal{Y}),
\]
\[
\nabla_\varepsilon (e \nabla)^\beta u_\varepsilon \nabla_\varepsilon u_\varepsilon \in L^p(\mathbb{R}^N \times \mathcal{Y}),
\]

The \( \mathcal{Y} \)-periodicity may fail in case (ii). This proposition has natural corollaries for more general linear differential operators with constant coefficients. For instance, if \( e \nabla \cdot u_\varepsilon \) is bounded in \( L^p(\mathbb{R}^N) \) (\( \nabla \cdot := div \)), then \( \nabla \cdot u_\varepsilon \in L^p(\mathbb{R}^N \times \mathcal{Y}), \) and the normal component of \( u(x, \cdot) \) fulfils the periodicity condition on \( \partial Y \), for a.a. \( x \in \mathbb{R}^N \). A similar statement holds for the curl operator.

**Two-Scale Boundedness in \( W^{1,p}(\mathbb{R}^N \times \mathcal{Y}) \).** Let us define the approximate two-scale gradient \( A_\varepsilon := (\nabla_\varepsilon, e \nabla) \), and say that a sequence \( \{u_\varepsilon\} \) is two-scale bounded in \( W^{1,p}(\mathbb{R}^N \times \mathcal{Y}) \) iff \( \{u_\varepsilon\} \) and \( \{A_\varepsilon u_\varepsilon\} \) are bounded in \( L^p(\mathbb{R}^N) \) and \( L^p(\mathbb{R}^N)2^N \), resp. The above canonic example shows that in \( W^{1,p} \) two-scale boundedness is strictly weaker than one-scale boundedness, at variance with what we saw for \( L^p \).

The next statement can be proved by means of Proposition 3.4.

**Theorem 4.2 (Two-Scale Rellich-Kondrachov-Type Result).** Let \( p \in [1, +\infty[ \). Any sequence \( \{u_\varepsilon\} \) of \( W^{1,p}(\mathbb{R}^N) \) that is two-scale bounded in \( W^{1,p}(\mathbb{R}^N \times \mathcal{Y}) \) is strongly two-scale compact in \( L^p_{loc}(\mathbb{R}^N \times \mathcal{Y}) \).

One might also define an alternative (weaker) concept: a sequence \( \{u_\varepsilon\} \subset W^{1,p}(\mathbb{R}^N) \) is two-scale bounded in \( W^{1,p}(\mathbb{R}^N \times Y^0) \) whenever the sequences \( \{\|u_\varepsilon\|_{L^p(\mathbb{R}^N)}\}, \{\|\nabla_\varepsilon u_\varepsilon\|_{L^p(\mathbb{R}^N)}\}, \{\|e\nabla_\varepsilon u_\varepsilon\|_{L^p(\mathbb{R}^N)}\} \) are bounded. This entails strong two-scale compactness in \( L^p_{loc}(\mathbb{R}^N \times Y^0) \). Henceforth however we shall just refer to the former definition.

Defining \( I_{\varepsilon,i} \) and \( L_\varepsilon \) as in (1.4), it is easy to check that for any \( p \in [1, +\infty[ \) and any \( v \in W^{1,p}(\mathbb{R}^N) \),
\[
\nabla_x I_{\varepsilon,i}(v \circ S_\varepsilon) = (\nabla_{\varepsilon,i}v) \circ S_\varepsilon,
\]
\[
\nabla_y I_{\varepsilon,i}(v \circ S_\varepsilon) = I_{\varepsilon,i}[(\varepsilon \nabla v) \circ S_\varepsilon]
\]
in \( \mathbb{R}^N \times \mathcal{Y}, \forall i \).
This yields the next statement.

**Proposition 4.3.** Let \( p \in [1, + \infty[ \). For any sequence \( \{ u_\varepsilon \} \) in \( W^{1, p}(\mathbb{R}^N) \),

\[
\begin{aligned}
\{ u_\varepsilon \} \text{ is two-scale bounded in } W^{1, p}(\mathbb{R}^N) & \iff L_\varepsilon u_\varepsilon \text{ is one-scale bounded in the same space,} \\
\end{aligned}
\]

(4.9)

and define strong two-scale convergence similarly. We also say that a sequence \( \{ u_\varepsilon \} \) is two-scale bounded in \( W^{m, p}(\mathbb{R}^N \times \mathcal{Y}) \) iff the set \( \{ \nabla^\alpha_x (\varepsilon \nabla^\beta_y u_\varepsilon) : \alpha, \beta \in \mathbb{N}^N, |\alpha| + |\beta| \leq m \} \) is bounded in \( L^p(\mathbb{R}^N) \).

The next statement follows from Propositions 3.2 and 4.1.

**Proposition 5.1.** For any \( m, n \in \mathbb{N} \) and any \( p \in [1, + \infty[ \), any sequence of \( W^{m, p}(\mathbb{R}^N) \) that is two-scale bounded in \( W^{m, p}(\mathbb{R}^N \times \mathcal{Y}) \) has a weakly two-scale convergent subsequence in the latter space.

**Weak Two-Scale Convergence in** \( W^{m, p}(\mathbb{R}^N \times \mathcal{Y})' \). Let us fix any \( m, n \in \mathbb{N} \), any \( p \in [1, + \infty[ \), and denote by \( \langle \cdot, \cdot \rangle \) (\( \langle \langle \cdot, \cdot \rangle \rangle \), resp.) the duality pairing between \( W^{m, p}(\mathbb{R}^N) \) (\( W^{m, p}(\mathbb{R}^N \times \mathcal{Y}) \), resp.) and the respective dual space. For any sequence \( \{ u_\varepsilon \} \) in \( W^{m, p}(\mathbb{R}^N) \) and any \( u \in W^{m, p}(\mathbb{R}^N \times \mathcal{Y}) \), we say that \( u_\varepsilon \) weakly two-scale converges to \( u \) in \( W^{m, p}(\mathbb{R}^N \times \mathcal{Y})' \) iff

\[
\begin{aligned}
\{ u_\varepsilon(x) , \psi_\varepsilon(x) \} \rightharpoonup \langle \langle u(x, y) , \psi(x, y) \rangle \rangle & \quad \forall x \in \mathbb{R}^N, y \in \mathcal{Y}, \\
\forall \{ \psi_\varepsilon \} \in W^{m, p}(\mathbb{R}^N) \text{ such that } \psi_\varepsilon \rightharpoonup \psi \text{ in } W^{m, p}(\mathbb{R}^N \times \mathcal{Y}).
\end{aligned}
\]

(5.3)

The next statement can be proved by transposing derivatives and applying the above definitions of two-scale convergence in the spaces \( W^{m, p}(\mathbb{R}^N \times \mathcal{Y}) \) and in \( W^{m, p}(\mathbb{R}^N \times \mathcal{Y})' \).
**Proposition 5.2.** For any $p \in [1, +\infty[$ and any sequence $\{u_\varepsilon\}$ in $L^p(R^N)$, if $u_\varepsilon \rightharpoonup u$ in $L^{p'}(R^N \times Y)$ then

$$
\nabla_x^a(\varepsilon\nabla)\beta u_\varepsilon \rightharpoonup \nabla_x^a \nabla_y^\beta u \quad \text{in} \quad W^{[\alpha]+[\beta],p}(R^N \times Y)', \forall \alpha, \beta \in \mathbb{N}.
$$

Two-Scale Convergence in $C^{0,\lambda}(R^N \times Y)$. For any $\lambda \in ]0, 1]$, any sequence $\{u_\varepsilon\}$ in $C^{0,\lambda}(R^N)$ and any $u \in C^{0,\lambda}(R^N \times Y)$, we say that $u_\varepsilon$ weakly star two-scale converges to $u$ in $C^{0,\lambda}(R^N \times Y)$ iff $L_\varepsilon u_\varepsilon \rightharpoonup u$ in the latter space. Strong two-scale convergence in $C^{0,\lambda}(R^N \times Y)$ can be defined similarly.

A sequence $\{u_\varepsilon\}$ of $C^{0,\lambda}(R^N)$ is said two-scale bounded in $C^{0,\lambda}(R^N \times Y)$ whenever the sequence $\{L_\varepsilon u_\varepsilon\}$ is bounded in $C^{0,\lambda}(R^N \times Y)$.

**Proposition 5.3.** For any $\lambda \in ]0, 1]$, any sequence of $C^{0,\lambda}(R^N)$ that is two-scale bounded in $C^{0,\lambda}(R^N \times Y)$ has a weakly star two-scale convergent subsequence in the latter space.

Two-Scale Convergence in $C^m(R^N \times Y)$. For any integer $m > 0$, any sequence $\{u_\varepsilon\}$ in the Fréchet subspace $C^m(R^N)$ and any $u \in C^m(R^N \times Y)$, we say that $u_\varepsilon$ weakly two-scale converges to $u$ in $C^m(R^N \times Y)$ iff

$$
\nabla_x^a(\varepsilon\nabla)\beta L_\varepsilon u_\varepsilon \rightharpoonup \nabla_x^a \nabla_y^\beta u \quad \text{in} \quad C^0(R^N \times Y), \forall \alpha, \beta \in \mathbb{N}, |\alpha| + |\beta| \leq m,
$$

and analogously for strong two-scale convergence.

One might also define two-scale convergence in $C^{m,\lambda}(R^N \times Y)$, but here we omit that issue.

Two-Scale Convergence in $O(R^N \times Y)$. If $\{u_\varepsilon\}$ is a sequence in $O(R^N)$ and $u \in O(R^N \times Y)$, we say that $u_\varepsilon$ two-scale converges to $u$ in $O(R^N \times Y)$ iff

$$
\exists K \subset R^N: \forall \varepsilon, \ u_\varepsilon \equiv 0 \ \text{in} \ R^N \setminus K, \text{ and } \nabla_x^a(\varepsilon\nabla)\beta L_\varepsilon u_\varepsilon \rightharpoonup \nabla_x^a \nabla_y^\beta u \quad \text{in} \quad C^0(R^N \times Y), \forall \alpha, \beta \in \mathbb{N}.
$$

One might similarly define two-scale convergence of a sequence in $O(R^N)$ to an element of $O(R^N \times Y^0)$.

Imbedding-Type Results. By applying Proposition 4.3 and the classic Sobolev and Morrey theorems to the sequence $\{L_\varepsilon u_\varepsilon\}$ in $R^N \times Y$, one gets the following result.

**Theorem 5.4 (Two-Scale Sobolev- and Morrey-Type Results).** For any $p \in [1, +\infty [$, there exists a constant $C = C_{N, p}$ such that, for any sequence $\{u_\varepsilon\}$ in $W^{1,p}(R^N)$ that is two-scale bounded in $W^{1,\tilde{p}}(R^N \times Y)$ and any $\varepsilon$, (cf. (1.4))

$$
p < 2N \Rightarrow \|u_\varepsilon\|_{L^{\tilde{p}}(R^N)} \leq C\|\nabla(L_\varepsilon u_\varepsilon)\|_{L^{\tilde{p}}(R^N \times Y)^{2N}} \left( \tilde{p} := \frac{2Np}{2N - p} \right),$$

$$
p = 2N \Rightarrow \|u_\varepsilon\|_{L^\infty(R^N)} \leq C\|\nabla(L_\varepsilon u_\varepsilon)\|_{L^{\tilde{p}}(R^N \times Y)^{2N}} \quad \forall q \in [p, +\infty [,$$
(5.9) \[ p > 2N \Rightarrow \| u_\varepsilon \|_{C^0(\mathbb{R}^N)} \leq C \| \nabla (L \varepsilon u_\varepsilon) \|_{L^p(\mathbb{R}^N \times \mathcal{Y})^2N} \quad (\lambda := 1 - \frac{2N}{p}). \]

(By (4.9), the right-hand side of each of these formulae is bounded).

By a standard argument Theorems 4.2 and 5.4 entail the next two-scale compactness result.

**Theorem 5.5.** For any sequence \( \{ u_\varepsilon \} \) in \( W^{1,p}(\mathbb{R}^N) \) that is two-scale bounded in \( W^{1,p}(\mathbb{R}^N \times \mathcal{Y}) \),

(5.10) \[ p < 2N \Rightarrow \{ u_\varepsilon \} \text{ is two-scale strongly compact in } L^{q \lambda}_{\text{loc}}(\mathbb{R}^N \times \mathcal{Y}) \forall q < \frac{2Np}{2N-p}, \]

(5.11) \[ p = 2N \Rightarrow \{ u_\varepsilon \} \text{ is two-scale strongly compact in } L^{q \lambda}_{\text{loc}}(\mathbb{R}^N \times \mathcal{Y}) \forall q < + \infty, \]

(5.12) \[ p > 2N \Rightarrow \{ u_\varepsilon \} \text{ is two-scale strongly compact in } C^{0,\lambda}_{\text{loc}}(\mathbb{R}^N \times \mathcal{Y}) \forall \lambda < 1 - \frac{2N}{p}. \]

6. **Two-scale Convergence of Gradients**

Let us set \( R^N_\varepsilon := \bigcup_{m \in \mathbb{Z}^N} \varepsilon(m + ]0, 1[)^N \) (\( \neq \mathbb{R}^N \)), for any \( \varepsilon > 0 \).

**Theorem 6.1.** Let \( p \in ]1, + \infty[ \), and a sequence \( \{ u_\varepsilon \} \) be such that \( u_\varepsilon \rightharpoonup u \) in \( W^{1,p}(\mathbb{R}^N) \). For any \( \varepsilon \), there exists a unique \( u_{1\varepsilon} \in W^{1,p}(R^N_\varepsilon) \) such that, for any \( m \in \mathbb{Z}^N \),

(6.1) \[ u_{1\varepsilon} \in W^{1,p}(\varepsilon(m + ]0, 1[)^N), \quad \int_{\varepsilon(m + ]0, 1[)} u_{1\varepsilon}(x) \, dx = 0, \]

(6.2) \[ \int_{\varepsilon(m + ]0, 1[)} (\varepsilon \nabla u_{1\varepsilon} - \nabla u_\varepsilon) \cdot \nabla \psi \, dx = 0 \quad \forall \psi \in W^{1,p}(\varepsilon(m + ]0, 1[)). \]

Then there exists \( u_1 \in L^p(\mathbb{R}^N; W^{1,p}(\mathcal{Y})) \) such that \( \int_{\mathcal{Y}} u_1(x, y) \, dy = 0 \) for a.a. \( x \in \mathbb{R}^N \), and, as \( \varepsilon \to 0 \) along a suitable subsequence,

(6.3) \[ u_{1\varepsilon} \rightharpoonup^* u_1 \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y}), \quad \varepsilon \nabla u_{1\varepsilon} \rightharpoonup \nabla u_1 \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y}), \]

(6.4) \[ \nabla u_{1\varepsilon} \rightharpoonup \nabla u + \nabla_y u_1 \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y}). \]

We remind the reader that in Section 1 we defined \( \mathcal{Y} \) to be the \( N \)-dimensional torus; hence \( W^{1,p}(\varepsilon(m + ]0, 1[)) \neq W^{1,p}(\varepsilon(m + ]0, 1[)^N) \) for any \( m \in \mathbb{Z}^N \), although \( L^p(\varepsilon(m + ]0, 1[)) = L^p(\varepsilon(m + ]0, 1[)^N) \). The \( ]0, 1[ \)-periodic extension of any \( \psi \in W^{1,p}(\varepsilon(m + ]0, 1[)) \) to \( \mathbb{R}^N \) is locally of class \( W^{1,p}; \) its gradient in the sense of \( \partial^\varepsilon(\varepsilon(m + ]0, 1[)^N) \) then coincides with that in the sense of \( \partial^\varepsilon(\mathbb{R}^N)^N \). In (6.3) then \( \nabla u_{1\varepsilon} \in L^p(\mathbb{R}^N)^N \). This type of remark will also apply to Sections 7, 8.

Here is a simple example. Let \( N = 1 \) and \( u_\varepsilon : \mathbb{R} \to \mathbb{R} \) be such that \( u_\varepsilon(x) := x + \)
+ ε sin (2πx/ε) for any x in some neighbourhood of [0, 1]. After (6.1) and (6.2),
\[
\begin{align*}
\epsilon Du_{1\epsilon}(x) &= 2\pi\cos(2\pi x/\epsilon) \quad \text{in } L^p(0, 1 \times \mathcal{Y}), \forall \rho \in ]1, + \infty[. \\
\epsilon Du_{1\epsilon}(x) &= 2\pi\cos(2\pi y/\epsilon) = D_y u_1(y)
\end{align*}
\]

(6.5)

Theorem 6.1 can be compared with Theorem 3 of [15] and Proposition 1.14 of [1], where (for \( p = 2 \)) existence of a function \( u \in L^p(R^N; W^{1,p}(\mathcal{Y})) \) as in (6.4) is proved without exhibiting its relation with the sequence \( \{u_\epsilon\} \). That relation is however derived in Theorem 1 of [8], via a construction different from (6.1) and (6.2). Denoting the weak one-scale (two-scale, resp.) limit by \( \lim_{\epsilon \to 0} (\lim_{\epsilon \to 0}) \), (6.4) also reads
\[
\begin{align*}
\lim_{\epsilon \to 0} (\nabla u_\epsilon &= \nabla u_1 + \nabla y u_1 \quad \text{a.e. in } R^N \times \mathcal{Y}; \\
\text{this may be compared with (2.4). For } p = 2, \text{ this decomposition is orthogonal in } L^2(R^N \times \mathcal{Y})^N.
\end{align*}
\]

7. **Two-scale convergence of curls and divergences**

**Two-Scale Convergence of Curls.** In this section we assume that \( N = 3 \) and \( p = 2 \). We remind the reader that \( L^2_{\text{curl}}(R^3)^3 := \{v \in L^2(R^3)^3; \nabla \times \psi \in L^2(R^3)^3\} \) (\( \nabla \times := \text{curl} \)) is a Hilbert space equipped with the graph norm.

**Theorem 7.1.** Let \( \{u_\epsilon\} \) be a bounded sequence in \( L^2_{\text{curl}}(R^3)^3 \) such that \( u_\epsilon \rightharpoonup u \) in \( L^2(R^3 \times \mathcal{Y})^3 \). For any \( \epsilon \), there exists a unique \( u_{1\epsilon} \in H^1(R^3)^3 \) such that, for any \( m \in \mathbb{Z}^3 \),

(omitting restrictions)
\[
\begin{align*}
u_{1\epsilon} &\in H^1(\epsilon(m + \mathcal{Y}))^3, & \int_{\epsilon(m + \mathcal{Y})} u_{1\epsilon}(x) \, dx = 0,
\end{align*}
\]

(7.1)

\[
\begin{align*}
\nabla \cdot u_{1\epsilon} &= 0 \quad \text{a.e. in } \epsilon(m + \mathcal{Y}), \\
\int_{\epsilon(m + \mathcal{Y})} (\epsilon \nabla \times u_{1\epsilon} - \nabla \times u_\epsilon) \cdot \nabla \times \psi \, dx = 0 \quad \forall \psi \in H^1(\epsilon(m + \mathcal{Y}))^3.
\end{align*}
\]

(7.2)

Then there exists \( u_1 \in L^2(R^3; H^1(\mathcal{Y}))^3 \) such that \( \int_{\mathcal{Y}} u_1(x, y) \, dy = 0 \) for a.a. \( x \in R^3 \), \( \nabla_y u_1 = 0 \) a.e. in \( R^3 \times \mathcal{Y} \), and, as \( \epsilon \to 0 \) along a suitable subsequence,
\[
\begin{align*}
\nabla \times u_{1\epsilon} &\rightharpoonup \nabla \times u_1, & \epsilon \nabla \times u_{1\epsilon} &\rightharpoonup \nabla \times u_1 \quad \text{in } L^2(R^3 \times \mathcal{Y})^3.
\end{align*}
\]

Moreover, setting \( \bar{u}(x) = \int_{\mathcal{Y}} u(x, y) \, dy \) for a.a. \( x \in R^3 \),
\[
\begin{align*}
\nabla \times u_\epsilon &\rightharpoonup \nabla \times \bar{u} + \nabla \times u_1, & \epsilon \nabla \times u_\epsilon &\rightharpoonup \nabla \times u_1 \quad \text{in } L^2(R^3 \times \mathcal{Y})^3.
\end{align*}
\]

(7.3)

Finally, \( \epsilon \nabla \times u_\epsilon \rightharpoonup \nabla \times \bar{u} \quad \text{in } L^2(R^3 \times \mathcal{Y})^3 \).

(7.4)
\[ \nabla_x \times \bar{u} \text{ and } \nabla_y \times u_1 \text{ are orthogonal in } L^2(\mathbb{R}^3 \times \mathfrak{Y})^3, \text{ and (7.4) also reads} \]

\[ \lim_{\varepsilon \to 0}^{(2)} \nabla \times u_\varepsilon = \nabla \times \lim_{\varepsilon \to 0}^{(1)} u_\varepsilon + \nabla_y \times u_1 = \lim_{\varepsilon \to 0}^{(1)} \nabla \times u_\varepsilon + \nabla_y \times u_1 \quad \text{a.e. in } \mathbb{R}^3 \times \mathfrak{Y}, \]

which may be compared with (2.4) and (6.6).

**Two-Scale Convergence of Divergences.** A result similar to Theorems 7.1 holds if the curl is replaced by the divergence, and \( L^2_{\text{rot}} \) by \( L^2_{\text{div}} \).

**Theorem 7.2.** Let \( \{u_\varepsilon\} \) be a bounded sequence in \( L^2_{\text{div}}(\mathbb{R}^3)^3 \) such that \( u_\varepsilon \rightharpoonup u \) in \( L^2(\mathbb{R}^3 \times \mathfrak{Y})^3 \). For any \( \varepsilon \), there exists a unique \( u_{1\varepsilon} \in H^1(\mathbb{R}^3)^3 \) such that, for any \( m \in \mathbb{Z}^3 \),

\( \text{(omitting restrictions)} \)

\[ u_{1\varepsilon} \in H^1(\varepsilon(m + \mathfrak{Y}))^3, \quad \int_{\varepsilon(m + \mathfrak{Y})} u_{1\varepsilon}(x) \, dx = 0, \]

\[ \left\{ \begin{array}{c} \nabla \times u_{1\varepsilon} = 0 \quad \text{a.e. in } \varepsilon(m + \mathfrak{Y}), \\ \int_{\varepsilon(m + \mathfrak{Y})} (\varepsilon \nabla \cdot u_{1\varepsilon} - \nabla \cdot u_\varepsilon) \nabla \cdot \psi \, dx = 0 \quad \forall \psi \in H^1(\varepsilon(m + \mathfrak{Y}))^3. \end{array} \right. \]

Then there exists \( u_1 \in L^2(\mathbb{R}^3; H^1(\mathfrak{Y}))^3 \) such that \( \int_{\mathfrak{Y}} u_1(x, y) \, dy = 0 \) for a.a. \( x \in \mathbb{R}^3 \), \( \nabla_y \times u_1 = 0 \) a.e. in \( \mathbb{R}^3 \times \mathfrak{Y} \), and, as \( \varepsilon \to 0 \) along a suitable subsequence,

\[ u_{1\varepsilon} \rightharpoonup u_1 \text{ in } L^2(\mathbb{R}^3 \times \mathfrak{Y})^3, \quad \varepsilon \nabla \cdot u_{1\varepsilon} \rightharpoonup \nabla_y \cdot u_1 \text{ in } L^2(\mathbb{R}^3 \times \mathfrak{Y}). \]

Moreover, setting \( \bar{u}(x) = \int_{\mathfrak{Y}} u(x, y) \, dy \) for a.a. \( x \in \mathbb{R}^3 \),

\[ \nabla \cdot u_\varepsilon \rightharpoonup \nabla_x \cdot \bar{u} + \nabla_y \cdot u_1 \text{ in } L^2(\mathbb{R}^3 \times \mathfrak{Y}). \]

Finally, \( \varepsilon \nabla \cdot u_\varepsilon \rightharpoonup \nabla_y \cdot u = 0 \) in \( L^2(\mathbb{R}^3 \times \mathfrak{Y}) \).

\( \nabla_x \cdot \bar{u} \) and \( \nabla_y \cdot u_1 \) are orthogonal in \( L^2(\mathbb{R}^3 \times \mathfrak{Y}) \), and a formula like (7.5) holds for divergences.

**8. Two-scale convergence of the Laplace operator**

**Theorem 8.1.** Let \( p \in ]1, + \infty[, \) and a sequence \( \{u_\varepsilon\} \) be such that \( u_\varepsilon \rightharpoonup u \) in \( \mathcal{W}^{2, p}(\mathbb{R}^N) \). For any \( \varepsilon \), there exists a unique \( u_{2\varepsilon} \in \mathcal{W}^{2, p}(\mathbb{R}^N) \) such that, for any \( m \in \mathbb{Z}^N \),

\( \text{(omitting restrictions)} \)

\[ u_{2\varepsilon} \in \mathcal{W}^{2, p}(\varepsilon(m + \mathfrak{Y})), \quad \int_{\varepsilon(m + \mathfrak{Y})} u_{2\varepsilon}(x) \, dx = 0, \]

\[ \int_{\varepsilon(m + \mathfrak{Y})} (\varepsilon^2 \Delta u_{2\varepsilon} - \Delta u_\varepsilon) \Delta \psi \, dx = 0 \quad \forall \psi \in \mathcal{W}^{2, p'}(\varepsilon(m + \mathfrak{Y})). \]

Then there exists \( u_2 \in L^p(\mathbb{R}^N; \mathcal{W}^{2, p}(\mathfrak{Y})) \) such that \( \int_{\mathfrak{Y}} u_2(x, y) \, dy = 0 \) for a.a. \( x \in \mathbb{R}^N \),
and, as $\varepsilon \to 0$ along a suitable subsequence,

\begin{equation}
(8.3) \quad u_{2\varepsilon} \xrightarrow{2} u_2 \quad \text{in} \quad L^p(\mathbb{R}^N \times \mathbb{Y}), \quad \varepsilon^2 \Delta u_{2\varepsilon} \xrightarrow{2} \Delta_y u_2 \quad \text{in} \quad L^p(\mathbb{R}^N \times \mathbb{Y}),
\end{equation}

\begin{equation}
(8.4) \quad \Delta u_{\varepsilon} \xrightarrow{2} \Delta u + \Delta_y u_2 \quad \text{in} \quad L^p(\mathbb{R}^N \times \mathbb{Y}).
\end{equation}

The latter formula also reads

\begin{equation}
(8.5) \quad \lim_{\varepsilon \to 0} (2) \Delta u_{\varepsilon} = \Delta \lim_{\varepsilon \to 0} (1) u_{\varepsilon} + \Delta_y u_2 = \lim_{\varepsilon \to 0} (1) \Delta u_{\varepsilon} + \Delta_y u_2 \quad \text{a.e. in} \quad \mathbb{R}^N \times \mathbb{Y},
\end{equation}

this may be compared with (2.4), (6.6), (7.5). For $p = 2$ this decomposition is orthogonal in $L^2(\mathbb{R}^N \times \mathbb{Y})$. This theorem can be extended to more general linear elliptic operators.

9. Two-scale convergence of potentials

Finally, we deal with the two-scale limit of a sequence of solutions $\varphi_{\varepsilon}$ of the equation $(\varepsilon \varphi_{\varepsilon} := (\nabla_{\varepsilon}, \varepsilon \nabla)) \varphi_{\varepsilon} = u_{\varepsilon}$, as $u_{\varepsilon} \xrightarrow{2} u \in C^1(\mathbb{R}^N \times \mathbb{Y})^3$.$^2$. For the sake of simplicity we confine ourselves to $N = 3$.

**Theorem 9.1.** Let two sequences $\{u_{1\varepsilon}\}, \{u_{2\varepsilon}\}$ of $C^1(\mathbb{R}^3)^3$ and $u_1, u_2 \in C^1(\mathbb{R}^1 \times \mathbb{Y})^3$ be such that

\begin{equation}
(9.1) \quad u_{1\varepsilon} \xrightarrow{2} u_1, \quad u_{2\varepsilon} \xrightarrow{2} u_2 \quad \text{in} \quad C^1(\mathbb{R}^3 \times \mathbb{Y})^3,
\end{equation}

\begin{equation}
(9.2) \quad \nabla_{\varepsilon} \times u_{1\varepsilon} = \nabla \times u_{2\varepsilon} = 0, \quad \varepsilon \nabla u_{1\varepsilon} = \nabla_{\varepsilon} u_{2\varepsilon} \quad \text{in} \quad \mathbb{R}^3, \forall \varepsilon,
\end{equation}

\begin{equation}
(9.3) \quad \varepsilon u_{1\varepsilon}(\varepsilon m) \cdot e_i = \int_0^1 u_{2\varepsilon}(\varepsilon m + \varepsilon te_i) \cdot e_i dt \quad \forall m \in \mathbb{Z}^3, \forall \varepsilon, \text{fori} = 1, 2, 3.
\end{equation}

For any $(x, y) \in \mathbb{R}^3 \times \mathbb{Y}$, let $\xi_x, \eta_y$ and the sequences $\{\xi_{\varepsilon N(x/\varepsilon)}\}, \{\eta_{\varepsilon y}\}$ in $C^1([0, 1])^3$ be such that

\begin{equation}
(9.4) \quad \left\{ \begin{array}{ll}
\xi_{\varepsilon N(x/\varepsilon)}(0) = \eta_{\varepsilon y}(0) = 0, & \xi_{\varepsilon N(x/\varepsilon)}(1) = \varepsilon N(x/\varepsilon), & \eta_{\varepsilon y}(1) = y & \forall \varepsilon, \\
\xi_{\varepsilon N(x/\varepsilon)} \to \xi_x, & \eta_{\varepsilon y} \to \eta_y & \text{in} \quad C^1([0, 1])^3.
\end{array} \right.
\end{equation}

[This determines the sequence $\{\eta_{0(\varepsilon x/\varepsilon)}\}$ via diagonalization]. Finally, let us set

\begin{equation}
(9.5) \quad \varphi_{\varepsilon}(x) := \int_0^1 u_{1\varepsilon}(\xi_{\varepsilon N(x/\varepsilon)}(t)) \cdot (\xi_{\varepsilon N(x/\varepsilon)}(t))' dt +
\end{equation}

\begin{equation}
\begin{aligned}
&+ \int_0^1 u_{2\varepsilon}(\varepsilon N(x/\varepsilon) + \varepsilon \eta_{\varepsilon y}(t)) \cdot (\eta_{\varepsilon y}(t))' dt \quad \forall x \in \mathbb{R}^3, \forall \varepsilon,
\end{aligned}
\end{equation}

\begin{equation}
\varphi(x, y) := \int_0^1 u_1(\xi_{x}(t), 0) \cdot \xi_{x}'(t) dt +
\end{equation}

\begin{equation}
\begin{aligned}
&+ \int_0^1 u_2(x, \eta_{y}(t)) \cdot \eta_{y}'(t) dt \quad \forall (x, y) \in \mathbb{R}^3 \times \mathbb{Y}.
\end{aligned}
\end{equation}
Then
\begin{align}
\varphi_\epsilon & \in C^2(\mathbb{R}^3), \quad \nabla_\epsilon \varphi_\epsilon = u_{1\epsilon}, \quad \epsilon \nabla \varphi_\epsilon = u_{2\epsilon} \quad \text{in} \quad \mathbb{R}^3, \\
\varphi & \in C^2(\mathbb{R}^3 \times \mathbb{Y}), \quad \varphi_\epsilon(x) \rightharpoonup \varphi(x, y) \quad \text{in} \quad C^2(\mathbb{R}^3 \times \mathbb{Y}).
\end{align}
Moreover, the \( \varphi_\epsilon \)'s and \( \varphi \) are path-independent (in \( \mathbb{R}^3 \) and in \( \mathbb{R}^3 \times \mathbb{Y} \), resp.; that is, they do not depend on the specific choice of the sequences \( \{ \tilde{\xi}_\epsilon, N(x/\epsilon) \} \), \( \{ \eta_\epsilon \} \).

(Although \( N(x/\epsilon) \) and \( \partial R(x/\epsilon) \) are discontinuous at any \( x \) such that \( \partial R(x/\epsilon) = 0 \) for some \( i \), by (9.3) the \( \varphi_\epsilon \)'s are continuous everywhere in \( \mathbb{R}^3 \)). An analogous result holds for \textit{strong} two-scale convergence.

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