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Augusto Visintin

Some properties of two-scale convergence

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Equazioni a derivate parziali. — *Some properties of two-scale convergence.* Nota di AUGUSTO VISINTIN, presentata (*) dal Socio E. Magenes.

ABSTRACT. — We reformulate and extend G. Nguetseng's notion of *two-scale convergence* by means of a variable transformation, and outline some of its properties. We approximate *two-scale derivatives*, and extend this convergence to spaces of differentiable functions. The two-scale limit of derivatives of bounded sequences in the Sobolev spaces $W^{1,p}(\boldsymbol{R}^N)$, $L^2_{\rm rot}(\boldsymbol{R}^3)^3$, $L^2_{\rm div}(\boldsymbol{R}^3)^3$ and $W^{2,p}(\boldsymbol{R}^N)$ is then characterized. The two-scale limit behaviour of the potentials of a two-scale convergent sequence of irrotational fields is finally studied.

KEY WORDS: Two-scale convergence; Two-scale decomposition; Sobolev spaces.

RIASSUNTO. — *Alcune proprietà della convergenza a due scale.* Mediante una trasformazione di variabile, la nozione di *convergenza a due scale* di G. Nguetseng è qui riformulata ed estesa, ed alcune delle sue proprietà sono presentate. Tale convergenza è quindi estesa a spazi di funzioni differenziabili mediante l'approssimazione delle *derivate a due scale.* Inoltre si caratterizza il limite a due scale di derivate di successioni limitate negli spazi di Sobolev $W^{1,p}(\boldsymbol{R}^N)$, $L^2_{\rm rot}(\boldsymbol{R}^3)^3$, $L^2_{\rm div}(\boldsymbol{R}^3)^3$ e $W^{2,p}(\boldsymbol{R}^N)$. Infine si studia il limite a due scale dei potenziali di una successione convergente a due scale di campi irrotazionali.

INTRODUCTION

Let us fix any $N \geqslant 1$ and set $Y := [0, 1[^N$. The following concept was introduced by Nguetseng [15], and then studied in detail by Allaire [1] and others: a bounded sequence $\{u_\varepsilon\}$ of $L^2(\boldsymbol{R}^N)$ is said (weakly) *two-scale convergent* to $u \in L^2(\boldsymbol{R}^N \times Y)$ iff

$$\lim_{\varepsilon \to 0} \int_{\boldsymbol{R}^N} u_\varepsilon(x)\, \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int\int_{\boldsymbol{R}^N \times Y} u(x, y)\, \psi(x, y)\, dxdy, \tag{1}$$

for any smooth function $\psi : \boldsymbol{R}^N \times \boldsymbol{R}^N \to \boldsymbol{R}$ that is Y-periodic w.r.t. the second argument. Here is a *canonic example:* for any function ψ as above, $u_\varepsilon(x) := \psi(x, x/\varepsilon)$ two-scale converges to $\psi(x, y)$.

Two-scale convergence can account for occurrence of a fine-scale periodic structure, and indeed has been applied to a number of homogenization problems, see *e.g.* [1, 3, 5, 7, 8, 11-13, 15, 16, 18, 19]. For periodic homogenization problems, two-scale convergence can indeed represent an alternative to the classic *energy method* of Tartar, see *e.g.* [2, 4, 9, 10, 14, 17].

Along the lines of [3, 5, 8, 11, 12], in Sections 1-5 we reduce (1) to standard weak convergence in $L^2(\boldsymbol{R}^N \times Y)$, via a transformation of variable which can be interpreted as a *two-scale decomposition*, We characterize two-scale convergence, extend it to L^p (for $p \in [1, +\infty])$ and C^0, and derive some basic properties. Some of these results are already known, cf. *e.g.* [1, 7, 8, 11, 12, 15]; here we organize these properties from the point of view of two-scale decomposition, in order to illustrate the potentialities of

(*) Nella seduta del 14 maggio 2004.

that approach. We then study two-scale compactness, introduce *approximate two-scale derivatives*, and use them to extend two-scale convergence to spaces of differentiable functions. We thus show that several classic results (of Rellich-Kondrachov, Sobolev, Morrey, and so on) have a two-scale counterpart, that concerns sequences of functions instead of single functions.

In Sections 6-8 we characterize the two-scale limit of derivatives of bounded sequences in the Sobolev spaces $W^{1,p}(\boldsymbol{R}^N)$, $L^2_{\rm rot}(\boldsymbol{R}^3)^3$, $L^2_{\rm div}(\boldsymbol{R}^3)^3$ and $W^{2,p}(\boldsymbol{R}^N)$ ($p \in]1, +\infty[$). Theorem 6.1 may be compared with results of Nguetseng [15], of Allaire [1], and with one Cioranescu, Damlamian and Griso recently announced in [8]; the latter one is also based on two-scale decomposition, but uses a different approximation. Finally, in Section 9 we deal with the two-scale limit of the potential of a two-scale convergent vector field. Details, proofs and applications to homogenization problems will appear apart.

1. Two-scale convergence

Two-Scale Decomposition. In this paper we denote by $\mathcal{Y}$ the set $Y = [0, 1[^N$, we equip with the topologic and differential structure of the N-dimensional torus, and identify any function on $\mathcal{Y}$ with its periodic extension to $\boldsymbol{R}^N$. For any $\varepsilon > 0$, we decompose real numbers and real vectors as follows:

$$(1.1)\qquad \begin{cases} \widehat{n}(x) := \max\{n \in \boldsymbol{Z} : n \leq x\}, & \widehat{r}(x) := x - \widehat{n}(x) (\in [0, 1[) \quad \forall x \in \boldsymbol{R}, \\ \mathcal{N}(x) := (\widehat{n}(x_1), \ldots, \widehat{n}(x_N)) \in \boldsymbol{Z}^N, & \mathcal{R}(x) := x - \mathcal{N}(x) \in \mathcal{Y} \quad \forall x \in \boldsymbol{R}^N. \end{cases}$$

Thus $x = \varepsilon[\mathcal{N}(x/\varepsilon) + \mathcal{R}(x/\varepsilon)]$ for any $x \in \boldsymbol{R}^N$; $\varepsilon\mathcal{N}(x/\varepsilon)$ and $\mathcal{R}(x/\varepsilon)$ represent coarse-scale and fine-scale variables w.r.t. the scale ε, respectively. Besides this *two-scale decomposition*, we define the *two-scale composition* function:

$$(1.2)\qquad S_\varepsilon(x, y) := \varepsilon\mathcal{N}(x/\varepsilon) + \varepsilon y \quad \forall (x, y) \in \boldsymbol{R}^N \times \mathcal{Y}, \forall \varepsilon > 0.$$

The next lemma can easily be proved via a variable transformation in the integral.

Lemma 1.1. *Let $f : \boldsymbol{R}^N \times \mathcal{Y} \to \boldsymbol{R}$ be such that*

$$f \in L^1(\mathcal{Y}; (C^0 \cap L^\infty)(\boldsymbol{R}^N)) \cup L^1(\boldsymbol{R}^N; C^0(\mathcal{Y})),$$

and extend it by periodicity to $\boldsymbol{R}^{2N}$. Then, for any $\varepsilon > 0$, the function $\boldsymbol{R}^N \times \mathcal{Y} \to \boldsymbol{R}$: $(x, y) \mapsto f(S_\varepsilon(x, y), y)$ is integrable, and

$$(1.3)\qquad \int_{\boldsymbol{R}^N} f(x, x/\varepsilon)\, dx = \int\!\!\int_{\boldsymbol{R}^N \times \mathcal{Y}} f(S_\varepsilon(x, y), y)\, dxdy \quad \forall \varepsilon > 0.$$

For any $p \in [1, +\infty]$, the operator $g \mapsto g \circ S_\varepsilon$ is then a linear isometry $L^p(\boldsymbol{R}^N) \to \to L^p(\boldsymbol{R}^N \times \mathcal{Y})$.

Two-Scale Convergence in $L^p(\boldsymbol{R}^N \times \mathcal{Y})$. In this *Note* by ε we represent the generic element of an arbitrary but prescribed, positive and vanishing sequence of real numbers; *e.g.*, $\varepsilon = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$. For any sequence of measurable functions,

$u_\varepsilon: \boldsymbol{R}^N \to \boldsymbol{R}$, and any measurable function, $u : \boldsymbol{R}^N \times \mathcal{Y} \to \boldsymbol{R}$, we say that u_ε *two-scale converges* to u (w.r.t. the prescribed sequence $\{\varepsilon_n\}$) in some specific sense, whenever $u_\varepsilon \circ S_\varepsilon \to u$ in the corresponding standard (*i.e.*, *one-scale*) sense. In this way, for any $p \in [1, +\infty]$ we define strong and weak (weak star for $p = \infty$) two-scale convergence in $L^p(\boldsymbol{R}^N \times \mathcal{Y})$; we then write $u_\varepsilon \underset{2}{\to} u$, $u_\varepsilon \underset{2}{\rightharpoonup} u$, $u_\varepsilon \underset{2}{\overset{*}{\rightharpoonup}} u$ (resp.). For any domain $\Omega \subset \boldsymbol{R}^N$, two-scale convergence in $L^p(\Omega \times \mathcal{Y})$ is then defined by extending functions to $\boldsymbol{R}^N \setminus \Omega$ with vanishing value.

Two-Scale Convergence in $C^0(\boldsymbol{R}^N \times \mathcal{Y})$. Because of the discontinuity of $S_\varepsilon(\cdot, y)$, in general the function $u_\varepsilon \circ S_\varepsilon$ is discontinuous even if u_ε is continuous. A modification is thus needed, in order to extend the previous definitions to the space of continuous functions. For $i = 1, \dots, N$, let us denote by e_i the unit vector of the x_i-axis, set $x_{[i]} := = x - x_i e_i$ for any $x \in \boldsymbol{R}$, and

$$
(1.4)\qquad \begin{cases} (I_{\varepsilon, i} w)(x, y) := w(x_{[i]} + \varepsilon \widehat{n}(x_i/\varepsilon)\, e_i, y) + \\ \qquad + \mathcal{R}(x_i/\varepsilon)[w(x_{[i]} + \varepsilon \widehat{n}(x_i/\varepsilon)\, e_i + \varepsilon e_i, y) - w(x_{[i]} + \varepsilon \widehat{n}(x_i/\varepsilon)\, e_i, y)] \\ \forall (x, y) \in \boldsymbol{R}^N \times \mathcal{Y}, \forall w: \boldsymbol{R}^N \times \mathcal{Y} \to \boldsymbol{R}, \mathit{for}\, i = 1, \dots, N; \\ L_\varepsilon v := (I_{\varepsilon, 1} \circ \dots \circ I_{\varepsilon, N})(v \circ S_\varepsilon) \quad \forall v : \boldsymbol{R}^N \to \boldsymbol{R}. \end{cases}
$$

Thus $L_\varepsilon v$ is piecewise linear w.r.t. x, whereas $v \circ S_\varepsilon$ is piecewise constant w.r.t. x. If $v \in C^0(\boldsymbol{R}^N)$, then $L_\varepsilon v \in C^0(\boldsymbol{R}^N \times \mathcal{Y})$. For instance, for $N = 2$, let us set $r(x) := \mathcal{R}(x|\varepsilon)$ and $v_{ij}^m(y) := v(\varepsilon(m + y) + \varepsilon(i, j))$ for $i, j \in \{0, 1\}$ and for any $m \in \boldsymbol{Z}^N$; for any $x \in \varepsilon m \mathcal{Y}$ and any $y \in \mathcal{Y}$, then

$$
(L_\varepsilon v)(x, y) := (1 - r_1)(1 - r_2) v_{00}^m + r_1(1 - r_2) v_{10}^m + (1 - r_1) r_2 v_{01}^m + r_1 r_2 v_{11}^m .
$$

For any sequence $\{u_\varepsilon\}$ in the $C^0(\boldsymbol{R}^N)$ and any $u \in C^0(\boldsymbol{R}^N \times \mathcal{Y})$, we say that u_ε *strongly* (*weakly*, resp.) *two-scale converges* to u in $C^0(\boldsymbol{R}^N \times \mathcal{Y})$ iff $L_\varepsilon u_\varepsilon \to u$ ($L_\varepsilon u_\varepsilon \rightharpoonup u$ resp.) in $C^0(\boldsymbol{R}^N \times \mathcal{Y})$ w.r.t. to the usual topology of Fréchet space.

2. Some properties of two-scale convergence

It is easy to check that in L^p weak/strong one-scale convergence and weak/strong two-scale convergence are related as follows. An analogous result holds in C^0.

Proposition 2.1. *Let $p \in [1, +\infty[$ and $\{u_\varepsilon\}$ be a sequence in $L^p(\boldsymbol{R}^N)$. Then:*

$$
(2.1)\qquad u_\varepsilon \to u \ \text{in}\ L^p(\boldsymbol{R}^N) \Leftrightarrow \begin{cases} u_\varepsilon \underset{2}{\to} u \ \text{in}\ L^p(\boldsymbol{R}^N \times \mathcal{Y}) \\ u \ \text{is independent of}\ y, \end{cases}
$$

$$
(2.2)\qquad u_\varepsilon \underset{2}{\to} u \ \text{in}\ L^p(\boldsymbol{R}^N \times \mathcal{Y}) \Rightarrow u_\varepsilon \underset{2}{\rightharpoonup} u \ \text{in}\ L^p(\boldsymbol{R}^N \times \mathcal{Y}),
$$

$$
(2.3)\qquad u_\varepsilon \underset{2}{\rightharpoonup} u \ \text{in}\ L^p(\boldsymbol{R}^N \times \mathcal{Y}) \Rightarrow u_\varepsilon \rightharpoonup \int_{\mathcal{Y}} u(\cdot, y)\, dy \ \text{in}\ L^p(\boldsymbol{R}^N).
$$

Limit Decomposition and Orthogonality. Let $p \in [1, +\infty[$. If $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^p(\boldsymbol{R}^N \times \mathcal{Y})$, and $u_\varepsilon \rightharpoonup u_0$ in $L^p(\boldsymbol{R}^N)$, setting $u_1 := u - u_0$, by (2.3) we trivially get the *two-scale decomposition*

$$\text{(2.4)} \qquad \begin{cases} u(x, y) = u_0(x) + u_1(x, y) & \text{for a.a. } (x, y) \in \boldsymbol{R}^N \times \mathcal{Y}, \\ \int_{\mathcal{Y}} u_1(x, y)\, dy = 0 & \text{for a.a. } x \in \boldsymbol{R}^N. \end{cases}$$

Let us set $p' := p/(p-1)$ if $p \neq 1$, $1' := \infty$. If $\varphi_\varepsilon \underset{2}{\rightharpoonup} \varphi$ in $L^{p'}(\boldsymbol{R}^N \times \mathcal{Y})$ and $\varphi_\varepsilon \rightharpoonup \varphi_0$ in $L^{p'}(\boldsymbol{R}^N)$, setting $\varphi_1 := \varphi - \varphi_0$ we then have

$$\text{(2.5)} \qquad \lim_{\varepsilon \to 0} \int_{\boldsymbol{R}^N} u_\varepsilon(x)\, \varphi_\varepsilon(x)\, dx = \int\!\!\int_{\boldsymbol{R}^N \times \mathcal{Y}} u(x, y)\, \varphi(x, y)\, dxdy =$$
$$= \int_{\boldsymbol{R}^N} u_0(x)\, \varphi_0(x)\, dx + \int\!\!\int_{\boldsymbol{R}^N \times \mathcal{Y}} u_1(x, y)\, \varphi_1(x, y)\, dxdy.$$

If $p = 2$, the decomposition (2.4) is orthogonal in $L^2(\boldsymbol{R}^N \times \mathcal{Y})$, and

$$\text{(2.6)} \qquad \|u\|^2_{L^2(\boldsymbol{R}^N \times \mathcal{Y})} = \|u_0\|^2_{L^2(\boldsymbol{R}^N)} + \|u_1\|^2_{L^2(\boldsymbol{R}^N \times \mathcal{Y})}.$$

In Sections 6-8 we shall encounter examples of this two-scale decomposition of the limit.

The formula (2.7) below states the equivalence between the above definitions of weak and strong two-scale convergence and the original ones of Nguetseng [15] and Allaire [1]. The remainder is easily checked.

PROPOSITION 2.2. *Let $p \in [1, +\infty[$ and $\{u_\varepsilon\}$ be a sequence in $L^p(\boldsymbol{R}^N)$. Then*

$$\text{(2.7)} \qquad u_\varepsilon \underset{2}{\rightharpoonup} u \text{ in } L^p(\boldsymbol{R}^N \times \mathcal{Y}) \Leftrightarrow \{u_\varepsilon\} \text{ is bounded in } L^p(\boldsymbol{R}^N) \text{ and}$$
$$\int_{\boldsymbol{R}^N} u_\varepsilon(x)\, \psi(x, x/\varepsilon)\, dx \to \int\!\!\int_{\boldsymbol{R}^N \times \mathcal{Y}} u(x, y)\, \psi(x, y)\, dxdy \ \ \forall \psi \in \mathcal{D}(\boldsymbol{R}^N \times \mathcal{Y}),$$

$$\text{(2.8)} \qquad u_\varepsilon \underset{2}{\rightharpoonup} u \text{ in } L^p(\boldsymbol{R}^N \times \mathcal{Y}) \Rightarrow$$
$$\Rightarrow \liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\boldsymbol{R}^N)} \geq \|u\|_{L^p(\boldsymbol{R}^N \times \mathcal{Y})} \left(\geq \left\| \int_{\mathcal{Y}} u(\cdot, y)\, dy \right\|_{L^p(\boldsymbol{R}^N)} \right),$$

$$\text{(2.9)} \qquad \textit{if } p \in]1, +\infty[, \quad u_\varepsilon \underset{2}{\to} u \text{ in } L^p(\boldsymbol{R}^N \times \mathcal{Y}) \Leftrightarrow \begin{cases} u_\varepsilon \underset{2}{\rightharpoonup} u \text{ in } L^p(\boldsymbol{R}^N \times \mathcal{Y}) \\ \|u_\varepsilon\|_{L^p(\boldsymbol{R}^N)} \to \|u\|_{L^p(\boldsymbol{R}^N \times \mathcal{Y})}. \end{cases}$$

Two-Scale Convergence of Distributions. Let us denote the duality pairing between $\mathcal{D}'(\boldsymbol{R}^N)$ and $\mathcal{D}(\boldsymbol{R}^N)$ by $\langle \cdot, \cdot \rangle$, and that between $\mathcal{D}'(\boldsymbol{R}^N \times \mathcal{Y})$ and $\mathcal{D}(\boldsymbol{R}^N \times \mathcal{Y})$ by $\langle\langle \cdot, \cdot \rangle\rangle$. For any sequence $\{u_\varepsilon\}$ in $\mathcal{D}'(\boldsymbol{R}^N)$ and any $u \in \mathcal{D}'(\boldsymbol{R}^N \times \mathcal{Y})$, we say that u_ε *two-scale converges to u in $\mathcal{D}'(\boldsymbol{R}^N \times \mathcal{Y})$* iff

$$\text{(2.10)} \qquad \langle u_\varepsilon(x), \psi(x, x/\varepsilon) \rangle \to \langle\langle u(x, y), \psi(x, y) \rangle\rangle \qquad \forall \psi \in \mathcal{D}(\boldsymbol{R}^N \times \mathcal{Y}).$$

By (2.7), this extends the weak two-scale convergence in L^p. For instance, for $N = 1$, let $\{\varphi_\varepsilon\}$ be a sequence in $L^1(0, 1)$ such that $\varphi_\varepsilon(y) \rightharpoonup \delta_0(y - 1/2)$ (the Dirac mass at 1/2) in $\mathcal{D}'(0, 1)$, and extend φ_ε to $\boldsymbol{R}$ by periodicity. We have

$$x\varphi_\varepsilon(x/\varepsilon) \rightharpoonup x \quad \text{in } \mathcal{D}'(\boldsymbol{R}), \qquad x\varphi_\varepsilon(x/\varepsilon) \underset{2}{\rightharpoonup} x\delta_0(y - 1/2) \quad \text{in } \mathcal{D}'(\boldsymbol{R} \times \mathcal{Y}). \tag{2.11}$$

One can also define two-scale convergence in $\mathcal{D}'(\boldsymbol{R}^N \times Y^0)$ (Y^0 representing the interior of Y), by letting ψ range in $\mathcal{D}(\boldsymbol{R}^N \times Y^0)$ in (2.10). However this definition seems less convenient.

Two-scale convergence in the spaces of Radon measures, $C^0(\boldsymbol{R}^N \times \mathcal{Y})'$, is defined similarly.

3. Two-scale compactness

Let us say that a sequence $\{u_\varepsilon\}$ is compact iff it is possible to extract a convergent subsequence from any of its subsequences. Proposition 2.1 trivially entails the following statement.

Proposition 3.1. *Let $p \in [1, +\infty[$. For any sequence $\{u_\varepsilon\}$ in $L^p(\boldsymbol{R}^N)$,*

$$\begin{cases} \text{strong one-scale compactness entails strong two-scale compactness;} \\ \text{strong two-scale compactness entails weak two-scale compactness;} \\ \text{weak two-scale compactness entails weak one-scale compactness.} \end{cases} \tag{3.1}$$

The same holds for $C^0(\boldsymbol{R}^N)$, and (replacing weak compactness by weak star compactness) for $L^\infty(\boldsymbol{R}^N)$.

The next statement is also easily checked: parts (*i*) and (*ii*) follow from Lemma 1.1 and the Banach-Alaoglu theorem; part (*iii*) can be derived via the classic de la Vallée Poussin criterion.

Proposition 3.2 (Weak Two-Scale Compactness in L^p).

(*i*) *Let $p \in]1, +\infty]$. Any sequence $\{u_\varepsilon\}$ of $L^p(\boldsymbol{R}^N)$ is weakly star two-scale compact in $L^p(\boldsymbol{R}^N \times \mathcal{Y})$ iff it is bounded, hence iff it is weakly star one-scale compact in $L^p(\boldsymbol{R}^N)$.*

(*ii*) *Similarly, any sequence of $L^1(\boldsymbol{R}^N)$ is weakly star two-scale compact in $C_c^0(\boldsymbol{R}^N \times \mathcal{Y})'$ iff it is bounded, hence iff it is weakly star one-scale compact in $C_c^0(\boldsymbol{R}^N)'$.*

(*iii*) *Finally, any sequence of $L^1(\boldsymbol{R}^N)$ is weakly two-scale compact in $L^1(\boldsymbol{R}^N \times \mathcal{Y})$ iff it is weakly one-scale compact in $L^1(\boldsymbol{R}^N)$.*

We also have a two-scale version of Chacon's *biting lemma*, cf. [6].

Proposition 3.3 (Two-Scale Biting Lemma). *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^1(\boldsymbol{R}^N)$. Then there exist $u \in L^1(\boldsymbol{R}^N \times \mathcal{Y})$, a subsequence $\{u_{\varepsilon'}\}$, and a nondecreasing sequence $\{\Omega_k\}$ of measurable subsets of $\boldsymbol{R}^N$ such that the measure of $\boldsymbol{R}^N \setminus \Omega_k$ vanishes* *as* $k \to \infty$, *and*

$$u_{\varepsilon'|\Omega_k} \underset{2}{\to} u_{|\Omega_k \times \mathcal{Y}} \quad in\ L^1(\Omega_k \times \mathcal{Y}),\ as\ \varepsilon' \to 0,\ \forall k \in \mathbf{N}. \tag{3.2}$$

Strong one-scale compactness is not equivalent to strong two-scale compactness in L^p- and C^0-spaces. However, the classic Riesz and Ascoli-Arzelà compactness theorems entail the following results.

PROPOSITION 3.4 (Strong Two-Scale Compactness in L^p). *Let* $p \in [1, +\infty[$. *A sequence* $\{u_\varepsilon\}$ *of* $L^p(\boldsymbol{R}^N)$ *is strongly two-scale compact in* $L^p(\boldsymbol{R}^N \times \mathcal{Y})$ *iff it is bounded and*

$$\int_{\boldsymbol{R}^N} |u_\varepsilon(x + S_\varepsilon(h, k)) - u_\varepsilon(x)|^p\, dx \to 0 \quad as\ (h, k, \varepsilon) \to (0, 0, 0), \tag{3.3}$$

$$\sup_\varepsilon \int_{\boldsymbol{R}^N \setminus B(0, R)} |u_\varepsilon(x)|^p\, dx \to 0 \quad as\ R \to +\infty. \tag{3.4}$$

PROPOSITION 3.5 (Strong Two-Scale Compactness in C^0). *A sequence* $\{u_\varepsilon\}$ *of* $C^0(\boldsymbol{R}^N)$ *is strongly two-scale compact in the Fréchet space* $C^0(\boldsymbol{R}^N \times \mathcal{Y})$ *iff it is bounded and*

$$\sup_{x \in K} |u_\varepsilon(x + S_\varepsilon(h, k)) - u_\varepsilon(x)| \to 0 \quad as\ (h, k, \varepsilon) \to (0, 0, 0),\ \forall K \subset\subset \boldsymbol{R}^N. \tag{3.5}$$

In the two latter theorems $S_\varepsilon(h, k) := \varepsilon \mathcal{N}(h/\varepsilon) + \varepsilon k$ cannot be replaced by $h + \varepsilon k$: this more restrictive hypothesis would yield the strong one-scale compactness of $\{u_\varepsilon\}$ in $L^p(\boldsymbol{R}^N)$ (in $C^0(\boldsymbol{R}^N)$, resp.).

4. Two-scale derivatives

Let $w \in \mathcal{D}(\boldsymbol{R}^N \times \mathcal{Y})$. Although $u_\varepsilon(x) := w(x, x/\varepsilon) \underset{2}{\to} w(x, y)$ in $L^p(\boldsymbol{R}^N \times \mathcal{Y})$ for any $p \in [1, +\infty[$, in general $\nabla w(x, y)$ is not the (weak) two-scale limit of $\nabla u_\varepsilon(x)$; actually, this sequence is bounded in $L^p(\boldsymbol{R}^N)^N$ only if $w(x, y)$ does not depend from y. In this section we show that nevertheless it is possible to express the gradient of the two-scale limit without evaluating the limit itself, via what we name *approximate two-scale gradient.*

For $i = 1, \dots, N$, let us denote by $\nabla_i \varphi$ the partial derivative w.r.t. x_i of any function $\varphi(x)$, and by $\nabla_{x_i} \psi$ and $\nabla_{y_i} \psi$ the partial derivatives of any function $\psi(x, y)$. Let us also denote by e_i the unit vector of the x_i-axis, define the shift operator $(\tau_\xi v)(x) := = v(x + \xi)$ for any $x, \xi \in \boldsymbol{R}^N$, set

$$\nabla_{\varepsilon, i} := \frac{\tau_{\varepsilon e_i} - I}{\varepsilon}, \quad \nabla_\varepsilon^\alpha := \prod_{i=1}^N \nabla_{\varepsilon, i}^{\alpha_i}, \quad \nabla^\alpha = \prod_{i=1}^N \nabla_i^{\alpha_i} \quad \forall \alpha \in \mathbf{N}^N,\ \forall \varepsilon > 0, \tag{4.1}$$

and define ∇_x^α, ∇_y^α similarly. Finally, for any $\varepsilon > 0$ let us set $\boldsymbol{R}_\varepsilon^N := \bigcup_{m \in \boldsymbol{Z}^N} \varepsilon(m + Y^0)$, and denote by $\tilde{\nabla}$ the gradient in the sense of $\mathcal{D}'(\boldsymbol{R}_\varepsilon^N)$.

PROPOSITION 4.1. *Let* $m \in \mathbf{N}$, $p \in]1, +\infty[$, *and* $\alpha, \beta \in \mathbf{N}^N$.

(*i*) *If* $\{u_\varepsilon\}$ *is a sequence in* $W^{m,p}(\boldsymbol{R}^N)$, $|\alpha| + |\beta| \le m$, *and*

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad in \ L^p(\boldsymbol{R}^N \times \mathcal{Y}), \qquad \sup_\varepsilon \|\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta u_\varepsilon\|_{L^p(\boldsymbol{R}^N)} < +\infty, \tag{4.2}$$

then

$$\nabla_x^\alpha \nabla_y^\beta u \in L^p(\boldsymbol{R}^N \times \mathcal{Y}), \qquad \nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta u_\varepsilon \underset{2}{\rightharpoonup} \nabla_x^\alpha \nabla_y^\beta u \quad in \ L^p(\boldsymbol{R}^N \times \mathcal{Y}), \tag{4.3}$$

$$\begin{cases} \forall i \in \{1, \dots, N\}, \forall \gamma \le \beta \ \textit{such that} \ \gamma_i < \beta_i, \\ \nabla_x^\alpha \nabla_y^\gamma u(x, \cdot) \ \textit{is 1-periodic w.r.t.} \ y_i, \textit{for a.a.} \ x \in \boldsymbol{R}^N. \end{cases} \tag{4.4}$$

(*ii*) *If* $\{u_\varepsilon\}$ *is a sequence in* $L^p(\boldsymbol{R}^N) \cap W^{m,p}(\boldsymbol{R}_\varepsilon^N)$, $|\beta| \le m$, *and*

$$u_\varepsilon \underset{2}{\rightharpoonup} u \ \ in \ L^p(\boldsymbol{R}^N \times \mathcal{Y}), \qquad \sup_\varepsilon \|(\varepsilon\tilde{\nabla})^\beta u_\varepsilon\|_{L^p(\boldsymbol{R}_\varepsilon^N)} < +\infty, \tag{4.5}$$

then

$$\nabla_y^\beta u \in L^p(\boldsymbol{R}^N \times \mathcal{Y}), \qquad (\varepsilon\tilde{\nabla})^\beta u_\varepsilon \underset{2}{\rightharpoonup} \nabla_y^\beta u \quad in \ L^p(\boldsymbol{R}^N \times \mathcal{Y}). \tag{4.6}$$

The $\mathcal{Y}$-periodicity may fail in case (*ii*). This proposition has natural corollaries for more general linear differential operators with constant coefficients. For instance, if $\varepsilon\nabla \cdot u_\varepsilon$ is bounded in $L^p(\boldsymbol{R}^N)$ ($\nabla\cdot := div$), then $\nabla_y \cdot u \in L^p(\boldsymbol{R}^N \times \mathcal{Y})$, and the normal component of $u(x,\cdot)$ fulfils the periodicity condition on ∂Y, for a.a. $x \in \boldsymbol{R}^N$. A similar statement holds for the curl operator.

Two-Scale Boundedness in $W^{1,p}(\boldsymbol{R}^N \times \mathcal{Y})$. Let us define the *approximate two-scale gradient* $\Lambda_\varepsilon := (\nabla_\varepsilon, \varepsilon\nabla)$, and say that a sequence $\{u_\varepsilon\}$ is *two-scale bounded in* $W^{1,p}(\boldsymbol{R}^N \times \mathcal{Y})$ iff $\{u_\varepsilon\}$ and $\{\Lambda_\varepsilon u_\varepsilon\}$ are bounded in $L^p(\boldsymbol{R}^N)$ and $L^p(\boldsymbol{R}^N)^{2N}$, resp. The above canonic example shows that in $W^{1,p}$ *two-scale* boundedness is strictly weaker than *one-scale* boundedness, at variance with what we saw for L^p.

The next statement can be proved by means of Proposition 3.4.

THEOREM 4.2 (Two-Scale Rellich-Kondrachov-Type Result). *Let* $p \in [1, +\infty]$. *Any sequence* $\{u_\varepsilon\}$ *of* $W^{1,p}(\boldsymbol{R}^N)$ *that is two-scale bounded in* $W^{1,p}(\boldsymbol{R}^N \times \mathcal{Y})$ *is strongly two-scale compact in* $L^p_{\rm loc}(\boldsymbol{R}^N \times \mathcal{Y})$.

One might also define an alternative (weaker) concept: a sequence $\{u_\varepsilon\} \subset W^{1,p}(\boldsymbol{R}_\varepsilon^N)$ is two-scale bounded in $W^{1,p}(\boldsymbol{R}^N \times Y^0)$ whenever the sequences $\{\|u_\varepsilon\|_{L^p(\boldsymbol{R}^N)}\}$, $\{\|\nabla_\varepsilon u_\varepsilon\|_{L^p(\boldsymbol{R}^N)^N}\}$, $\{\varepsilon\|\nabla u_\varepsilon\|_{L^p(\boldsymbol{R}_\varepsilon^N)^N}\}$ are bounded. This entails strong two-scale compactness in $L^p_{\rm loc}(\boldsymbol{R}^N \times Y^0)$. Henceforth however we shall just refer to the former definition.

Defining $I_{\varepsilon,i}$ and L_ε as in (1.4), it is easy to check that for any $p \in [1, +\infty[$ and any $v \in W^{1,p}(\boldsymbol{R}^N)$

$$\begin{cases} \nabla_{x_i} I_{\varepsilon,i}(v \circ S_\varepsilon) = (\nabla_{\varepsilon,i} v) \circ S_\varepsilon \\ \nabla_{y_i} I_{\varepsilon,i}(v \circ S_\varepsilon) = I_{\varepsilon,i}[\varepsilon(\nabla_i v) \circ S_\varepsilon] \end{cases} \quad \text{in } \boldsymbol{R}^N \times \mathcal{Y}, \forall i, \tag{4.7}$$

$$\begin{cases} \nabla_{x_i}(I_{\varepsilon,j} \circ I_{\varepsilon,i})(v \circ S_\varepsilon) = I_{\varepsilon,j} \nabla_{x_i} I_{\varepsilon,i}(v \circ S_\varepsilon) \\ \nabla_{y_i}(I_{\varepsilon,j} \circ I_{\varepsilon,i})(v \circ S_\varepsilon) = I_{\varepsilon,j} \nabla_{y_i} I_{\varepsilon,i}(v \circ S_\varepsilon) \end{cases} \quad \forall i, j. \tag{4.8}$$

This yields the next statement.

PROPOSITION 4.3. *Let* $p \in [1, +\infty[$. *For any sequence* $\{u_\varepsilon\}$ *in* $W^{1,p}(\boldsymbol{R}^N)$,

$$\begin{cases} u_\varepsilon \text{ is two-scale bounded in } W^{1,p}(\boldsymbol{R}^N \times \mathcal{Y}) \\ \Leftrightarrow L_\varepsilon u_\varepsilon \text{ is one-scale bounded in the same space}, \end{cases} \tag{4.9}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \;\Leftrightarrow\; L_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{in } W^{1,p}(\boldsymbol{R}^N \times \mathcal{Y})^{2N}. \tag{4.10}$$

An equivalence analogous to (4.10) holds for strong convergence.

5. Two-scale convergence in spaces of differentiable functions

In this section we define two-scale convergence in spaces of either weakly or strongly differentiable functions, by means of the *approximate two-scale gradient*, $\Lambda_\varepsilon := (\nabla_\varepsilon, \varepsilon\nabla)$, cf. (4.1).

Two-Scale Convergence in $W^{m,p}(\boldsymbol{R}^N \times \mathcal{Y})$. Let $m \in \mathbf{N}$ and $p \in [1, +\infty[$. For any sequence $\{u_\varepsilon\}$ in $W^{m,p}(\boldsymbol{R}^N)$ and any $u \in W^{m,p}(\boldsymbol{R}^N \times \mathcal{Y})$, we say that u_ε *weakly two-scale converges* to u in $W^{m,p}(\boldsymbol{R}^N \times \mathcal{Y})$ iff

$$\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta u_\varepsilon \underset{2}{\rightharpoonup} \nabla_x^\alpha \nabla_y^\beta u \quad \text{in } L^p(\boldsymbol{R}^N \times \mathcal{Y}), \forall \alpha, \beta \in \mathbf{N}^N, |\alpha| + |\beta| \le m, \tag{5.1,2}$$

and define strong two-scale convergence similarly. We also say that a sequence $\{u_\varepsilon\}$ is two-scale bounded in $W^{m,p}(\boldsymbol{R}^N \times \mathcal{Y})$ iff the set $\{\nabla_\varepsilon^\alpha(\varepsilon\nabla)^\beta u_\varepsilon : \alpha, \beta \in \mathbf{N}^N, |\alpha| + |\beta| \le m\}$ is bounded in $L^p(\boldsymbol{R}^N)$.

The next statement follows from Propositions 3.2 and 4.1.

PROPOSITION 5.1. *For any* $m \in \mathbf{N}$ *and any* $p \in]1, +\infty[$, *any sequence of* $W^{m,p}(\boldsymbol{R}^N)$ *that is two-scale bounded in* $W^{m,p}(\boldsymbol{R}^N \times \mathcal{Y})$ *has a weakly two-scale convergent subsequence in the latter space.*

Weak Two-Scale Convergence in $W^{m,p}(\boldsymbol{R}^N \times \mathcal{Y})'$. Let us fix any $m \in \mathbf{N}$, any $p \in [1, +\infty[$, and denote by $\langle \cdot, \cdot \rangle$ ($\langle\langle \cdot, \cdot \rangle\rangle$, resp.) the duality pairing between $W^{m,p}(\boldsymbol{R}^N)$ ($W^{m,p}(\boldsymbol{R}^N \times \mathcal{Y})$, resp.) and the respective dual space. For any sequence $\{u_\varepsilon\}$ in $W^{m,p}(\boldsymbol{R}^N)'$ and any $u \in W^{m,p}(\boldsymbol{R}^N \times \mathcal{Y})'$, we say that u_ε *weakly two-scale converges* to u in $W^{m,p}(\boldsymbol{R}^N \times \mathcal{Y})'$ iff

$$\begin{cases} \langle u_\varepsilon(x), \psi_\varepsilon(x) \rangle \to \langle\langle u(x,y), \psi(x,y) \rangle\rangle \\ \forall \{\psi_\varepsilon\} \subset W^{m,p}(\boldsymbol{R}^N) \text{ such that } \psi_\varepsilon \underset{2}{\to} \psi \text{ in } W^{m,p}(\boldsymbol{R}^N \times \mathcal{Y}). \end{cases} \tag{5.3}$$

The next statement can be proved by transposing derivatives and applying the above definitions of two-scale convergence in the spaces $W^{m,p}(\boldsymbol{R}^N \times \mathcal{Y})$ and in $W^{m,p}(\boldsymbol{R}^N \times \mathcal{Y})'$.

PROPOSITION 5.2. *For any* $p \in [1, +\infty[$ *and any sequence* $\{u_\varepsilon\}$ *in* $L^{p'}(\boldsymbol{R}^N)$, *if* $u_\varepsilon \underset{2}{\rightharpoonup} u$ *in* $L^{p'}(\boldsymbol{R}^N \times \mathcal{Y})$ *then*

$$\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta u_\varepsilon \underset{2}{\rightharpoonup} \nabla_x^\alpha \nabla_y^\beta u \quad \text{in } W^{|\alpha|+|\beta|, p}(\boldsymbol{R}^N \times \mathcal{Y})', \ \forall \alpha, \beta \in \boldsymbol{N}^N. \tag{5.4}$$

Two-Scale Convergence in $C^{0,\lambda}(\boldsymbol{R}^N \times \mathcal{Y})$. For any $\lambda \in]0, 1]$, any sequence $\{u_\varepsilon\}$ in $C^{0,\lambda}(\boldsymbol{R}^N)$ and any $u \in C^{0,\lambda}(\boldsymbol{R}^N \times \mathcal{Y})$, we say that u_ε *weakly star two-scale converges* to u in $C^{0,\lambda}(\boldsymbol{R}^N \times \mathcal{Y})$ iff $L_\varepsilon u_\varepsilon \overset{*}{\rightharpoonup} u$ in the latter space. Strong two-scale convergence in $C^{0,\lambda}(\boldsymbol{R}^N \times \mathcal{Y})$ can be defined similarly.

A sequence $\{u_\varepsilon\}$ of $C^{0,\lambda}(\boldsymbol{R}^N)$ is said two-scale bounded in $C^{0,\lambda}(\boldsymbol{R}^N \times \mathcal{Y})$ whenever the sequence $\{L_\varepsilon u_\varepsilon\}$ is bounded in $C^{0,\lambda}(\boldsymbol{R}^N \times \mathcal{Y})$.

PROPOSITION 5.3. *For any* $\lambda \in]0, 1]$, *any sequence of* $C^{0,\lambda}(\boldsymbol{R}^N)$ *that is two-scale bounded in* $C^{0,\lambda}(\boldsymbol{R}^N \times \mathcal{Y})$ *has a weakly star two-scale convergent subsequence in the latter space.*

Two-Scale Convergence in $C^m(\boldsymbol{R}^N \times \mathcal{Y})$. For any integer $m > 0$, any sequence $\{u_\varepsilon\}$ in the Fréchet subspace $C^m(\boldsymbol{R}^N)$ and any $u \in C^m(\boldsymbol{R}^N \times \mathcal{Y})$, we say that u_ε *weakly two-scale converges* to u in $C^m(\boldsymbol{R}^N \times \mathcal{Y})$ iff

$$\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta L_\varepsilon u_\varepsilon \underset{2}{\rightharpoonup} \nabla_x^\alpha \nabla_y^\beta u \quad \text{in } C^0(\boldsymbol{R}^N \times \mathcal{Y}), \forall \alpha, \beta \in \boldsymbol{N}^N, \ |\alpha| + |\beta| \leqslant m, \tag{5.5}$$

and analogously for strong two-scale convergence.

One might also define two-scale convergence in $C^{m,\lambda}(\boldsymbol{R}^N \times \mathcal{Y})$, but here we omit that issue.

Two-Scale Convergence in $\mathcal{D}(\boldsymbol{R}^N \times \mathcal{Y})$. If $\{u_\varepsilon\}$ is a sequence in $\mathcal{D}(\boldsymbol{R}^N)$ and $u \in \mathcal{D}(\boldsymbol{R}^N \times \mathcal{Y})$, we say that u_ε two-scale converges to u in $\mathcal{D}(\boldsymbol{R}^N \times \mathcal{Y})$ iff

$$\begin{cases} \exists K \subset\subset \boldsymbol{R}^N : \forall \varepsilon, \ u_\varepsilon \equiv 0 \ \text{ in } \boldsymbol{R}^N \backslash K, \text{ and} \\ \nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta L_\varepsilon u_\varepsilon \underset{2}{\rightarrow} \nabla_x^\alpha \nabla_y^\beta u \quad \text{in } C^0(\boldsymbol{R}^N \times \mathcal{Y}), \forall \alpha, \beta \in \boldsymbol{N}^N. \end{cases} \tag{5.6}$$

One might similarly define two-scale convergence of a sequence in $\mathcal{D}(\boldsymbol{R}_\varepsilon^N)$ to an element of $\mathcal{D}(\boldsymbol{R}^N \times Y^0)$.

Imbedding-Type Results. By applying Proposition 4.3 and the classic Sobolev and Morrey theorems to the sequence $\{L_\varepsilon u_\varepsilon\}$ in $\boldsymbol{R}^N \times \mathcal{Y}$, one gets the following result.

THEOREM 5.4 (Two-Scale Sobolev- and Morrey-Type Results). *For any* $p \in [1, +\infty[$, *there exists a constant* $C = C_{N,p}$ *such that, for any sequence* $\{u_\varepsilon\}$ *in* $W^{1,p}(\boldsymbol{R}^N)$ *that is two-scale bounded in* $W^{1,p}(\boldsymbol{R}^N \times \mathcal{Y})$ *and any* ε, (cf. (1.4))

$$p < 2N \Rightarrow \|u_\varepsilon\|_{L^{\tilde{p}}(\boldsymbol{R}^N)} \leqslant C \|\nabla(L_\varepsilon u_\varepsilon)\|_{L^p(\boldsymbol{R}^N \times \mathcal{Y})^{2N}} \qquad \left(\tilde{p} := \frac{2Np}{2N - p}\right), \tag{5.7}$$

$$p = 2N \Rightarrow \|u_\varepsilon\|_{L^q(\boldsymbol{R}^N)} \leqslant C \|\nabla(L_\varepsilon u_\varepsilon)\|_{L^p(\boldsymbol{R}^N \times \mathcal{Y})^{2N}} \qquad \forall q \in [p, +\infty[, \tag{5.8}$$

$$(5.9)\qquad p > 2N \Rightarrow \|u_\varepsilon\|_{C^{0,\lambda}(\mathbf{R}^N)} \leqslant C\|\nabla(L_\varepsilon u_\varepsilon)\|_{L^p(\mathbf{R}^N\times\mathcal{Y})^{2N}} \qquad \left(\lambda := 1-\frac{2N}{p}\right).$$

(By (4.9), the right-hand side of each of these formulae is bounded).

By a standard argument Theorems 4.2 and 5.4 entail the next two-scale compactness result.

THEOREM 5.5. *For any sequence* $\{u_\varepsilon\}$ *in* $W^{1,p}(\mathbf{R}^N)$ *that is two-scale bounded in* $W^{1,p}(\mathbf{R}^N\times\mathcal{Y})$,

$$(5.10)\qquad p<2N \Rightarrow \{u_\varepsilon\} \text{ is two-scale strongly compact in } L^q_{\rm loc}(\mathbf{R}^N\times\mathcal{Y})\ \forall q<\frac{2Np}{2N-p},$$

$$(5.11)\qquad p=2N \Rightarrow \{u_\varepsilon\} \text{ is two-scale strongly compact in } L^q_{\rm loc}(\mathbf{R}^N\times\mathcal{Y})\ \forall q<+\infty,$$

$$(5.12)\qquad p>2N \Rightarrow \{u_\varepsilon\} \text{ is two-scale strongly compact in } C^{0,\lambda}_{\rm loc}(\mathbf{R}^N\times\mathcal{Y})\ \forall \lambda<1-\frac{2N}{p}.$$

6. TWO-SCALE CONVERGENCE OF GRADIENTS

Let us set $\mathbf{R}^N_\varepsilon := \bigcup_{m\in\mathbf{Z}^N} \varepsilon(m+]0,1[^N)\ (\neq \mathbf{R}^N)$, for any $\varepsilon>0$.

THEOREM 6.1. *Let* $p\in]1,+\infty[$, *and a sequence* $\{u_\varepsilon\}$ *be such that* $u_\varepsilon \rightharpoonup u$ *in* $W^{1,p}(\mathbf{R}^N)$. *For any* ε, *there exists a unique* $u_{1\varepsilon}\in W^{1,p}(\mathbf{R}^N_\varepsilon)$ *such that, for any* $m\in\mathbf{Z}^N$, *(omitting restrictions)*

$$(6.1)\qquad u_{1\varepsilon}\in W^{1,p}(\varepsilon(m+\mathcal{Y})), \qquad \int_{\varepsilon(m+\mathcal{Y})} u_{1\varepsilon}(x)\,dx = 0,$$

$$(6.2)\qquad \int_{\varepsilon(m+\mathcal{Y})} (\varepsilon\nabla u_{1\varepsilon}-\nabla u_\varepsilon)\cdot\nabla\psi\,dx = 0 \qquad \forall\psi\in W^{1,p'}(\varepsilon(m+\mathcal{Y})).$$

Then there exists $u_1\in L^p(\mathbf{R}^N; W^{1,p}(\mathcal{Y}))$ *such that* $\int_{\mathcal{Y}} u_1(x,y)\,dy=0$ *for a.a.* $x\in\mathbf{R}^N$, *and, as* $\varepsilon\to 0$ *along a suitable subsequence,*

$$(6.3)\qquad u_{1\varepsilon}\underset{2}{\rightharpoonup} u_1 \quad \text{in } L^p(\mathbf{R}^N\times\mathcal{Y}), \qquad \varepsilon\nabla u_{1\varepsilon}\underset{2}{\rightharpoonup}\nabla_y u_1 \quad \text{in } L^p(\mathbf{R}^N\times\mathcal{Y})^N,$$

$$(6.4)\qquad \nabla u_\varepsilon \underset{2}{\rightharpoonup} \nabla u + \nabla_y u_1 \qquad \text{in } L^p(\mathbf{R}^N\times\mathcal{Y})^N.$$

We remind the reader that in Section 1 we defined $\mathcal{Y}$ to be the N-dimensional torus; hence $W^{1,p}(\varepsilon(m+\mathcal{Y}))\neq W^{1,p}(\varepsilon(m+]0,1[^N))$ for any $m\in\mathbf{Z}^N$, although $L^p(\varepsilon(m+\mathcal{Y}))=L^p(\varepsilon(m+]0,1[^N))$. The $]0,1[^N$-periodic extension of any $v\in W^{1,p}(\varepsilon(m+\mathcal{Y}))$ to $\mathbf{R}^N$ is locally of class $W^{1,p}$; its gradient in the sense of $\mathcal{D}'(\varepsilon(m+\mathcal{Y}))^N$ then coincides with that in the sense of $\mathcal{D}'(\mathbf{R}^N)^N$. In (6.3) then $\nabla u_{1\varepsilon}\in L^p(\mathbf{R}^N)^N$. This type of remark will also apply to Sections 7, 8.

Here is a simple example. Let $N=1$ and $u_\varepsilon:\mathbf{R}\to\mathbf{R}$ be such that $u_\varepsilon(x):=x+$

$+\,\varepsilon \sin(2\pi x/\varepsilon)$ for any x in some neighbourhood of $[0, 1]$. After (6.1) and (6.2),

$$
\text{(6.5)}\qquad \begin{cases} u_{1\varepsilon}(x) = \sin(2\pi x/\varepsilon) \underset{2}{\rightharpoonup} \sin(2\pi y) = u_1(y), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } L^p(]0, 1[\times \mathcal{Y}),\ \forall p \in]1, +\infty[. \\ \varepsilon D u_{1\varepsilon}(x) = 2\pi\cos(2\pi x/\varepsilon) \underset{2}{\rightharpoonup} 2\pi\cos(2\pi y) = D_y u_1(y) \end{cases}
$$

Theorem 6.1 can be compared with Theorem 3 of [15] and Proposition 1.14 of [1], where (for $p=2$) existence of a function $u_1 \in L^p(\boldsymbol{R}^N; W^{1,p}(\mathcal{Y}))$ as in (6.4) is proved without exhibiting its relation with the sequence $\{u_\varepsilon\}$. That relation is however derived in Theorem 1 of [8], via a construction different from (6.1) and (6.2).

Denoting the weak one-scale (two-scale, resp.) limit by $\lim\limits_{\varepsilon\to 0}^{(1)}$ ($\lim\limits_{\varepsilon\to 0}^{(2)}$, resp.), (6.4) also reads

$$
\text{(6.6)}\qquad \lim_{\varepsilon\to 0}{}^{(2)} \nabla u_\varepsilon = \lim_{\varepsilon\to 0}{}^{(1)} \nabla u_\varepsilon + \nabla_y u_1 \Big(= \nabla \lim_{\varepsilon\to 0}{}^{(1)} u_\varepsilon + \nabla_y u_1 \Big) \qquad \text{a.e. in } \boldsymbol{R}^N \times \mathcal{Y};
$$

this may be compared with (2.4). For $p = 2$, this decomposition is orthogonal in $L^2(\boldsymbol{R}^N \times \mathcal{Y})^N$.

7. Two-scale convergence of curls and divergences

Two-Scale Convergence of Curls. In this section we assume that $N = 3$ and $p = 2$. We remind the reader that $L^2_{\text{rot}}(\boldsymbol{R}^3)^3 := \{v \in L^2(\boldsymbol{R}^3)^3 \colon \nabla \times v \in L^2(\boldsymbol{R}^3)^3\}$ ($\nabla \times :=$ $=$ curl) is a Hilbert space equipped with the graph norm.

THEOREM 7.1. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2_{\text{rot}}(\boldsymbol{R}^3)^3$ such that $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^2(\boldsymbol{R}^3 \times \mathcal{Y})^3$. For any ε, there exists a unique $u_{1\varepsilon} \in H^1(\boldsymbol{R}^3_\varepsilon)^3$ such that, for any $m \in \boldsymbol{Z}^3$, (omitting restrictions)*

$$
\text{(7.1)}\qquad u_{1\varepsilon} \in H^1(\varepsilon(m+\mathcal{Y}))^3, \qquad \int_{\varepsilon(m+\mathcal{Y})} u_{1\varepsilon}(x)\,dx = 0,
$$

$$
\text{(7.2)}\qquad \begin{cases} \nabla\cdot u_{1\varepsilon} = 0 \quad \text{a.e. in } \varepsilon(m+\mathcal{Y}), \\ \displaystyle\int_{\varepsilon(m+\mathcal{Y})} (\varepsilon\nabla \times u_{1\varepsilon} - \nabla \times u_\varepsilon)\cdot\nabla\times\psi\,dx = 0 \qquad \forall \psi \in H^1(\varepsilon(m+\mathcal{Y}))^3. \end{cases}
$$

Then there exists $u_1 \in L^2(\boldsymbol{R}^3; H^1(\mathcal{Y})^3)$ such that $\int_{\mathcal{Y}} u_1(x, y)\,dy = 0$ for a.a. $x \in \boldsymbol{R}^3$, $\nabla_y\cdot u_1 = 0$ a.e. in $\boldsymbol{R}^3 \times \mathcal{Y}$, and, as $\varepsilon \to 0$ along a suitable subsequence,

$$
\text{(7.3)}\qquad u_{1\varepsilon} \underset{2}{\rightharpoonup} u_1, \qquad \varepsilon\nabla\times u_{1\varepsilon} \underset{2}{\rightharpoonup} \nabla_y \times u_1 \qquad \text{in } L^2(\boldsymbol{R}^3 \times \mathcal{Y})^3.
$$

Moreover, setting $\overline{u}(x) = \int_{\mathcal{Y}} u(x, y)\,dy$ for a.a. $x \in \boldsymbol{R}^3$,

$$
\text{(7.4)}\qquad \nabla \times u_\varepsilon \underset{2}{\rightharpoonup} \nabla_x \times \overline{u} + \nabla_y \times u_1 \qquad \text{in } L^2(\boldsymbol{R}^3 \times \mathcal{Y})^3.
$$

Finally, $\varepsilon\nabla \times u_\varepsilon \underset{2}{\rightharpoonup} \nabla_y \times u = 0$ in $L^2(\boldsymbol{R}^3 \times \mathcal{Y})^3$.

$\nabla_x \times \bar{u}$ and $\nabla_y \times u_1$ are orthogonal in $L^2(\boldsymbol{R}^3 \times \mathcal{Y})^3$, and (7.4) also reads

$$\lim_{\varepsilon \to 0}{}^{(2)} \nabla \times u_\varepsilon = \nabla \times \lim_{\varepsilon \to 0}{}^{(1)} u_\varepsilon + \nabla_y \times u_1 = \lim_{\varepsilon \to 0}{}^{(1)} \nabla \times u_\varepsilon + \nabla_y \times u_1 \quad \text{a.e. in } \boldsymbol{R}^3 \times \mathcal{Y}, \tag{7.5}$$

which may be compared with (2.4) and (6.6).

Two-Scale Convergence of Divergences. A result similar to Theorems 7.1 holds if the curl is replaced by the divergence, and $L^2_{\rm rot}$ by $L^2_{\rm div}$.

THEOREM 7.2. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2_{\rm div}(\boldsymbol{R}^3)^3$ such that $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^2(\boldsymbol{R}^3 \times \mathcal{Y})^3$. For any ε, there exists a unique $u_{1\varepsilon} \in H^1(\boldsymbol{R}^3_\varepsilon)^3$ such that, for any $m \in \boldsymbol{Z}^3$, (omitting restrictions)*

$$u_{1\varepsilon} \in H^1(\varepsilon(m + \mathcal{Y}))^3, \qquad \int_{\varepsilon(m+\mathcal{Y})} u_{1\varepsilon}(x)\, dx = 0, \tag{7.6}$$

$$\begin{cases} \nabla \times u_{1\varepsilon} = 0 \quad \text{a.e. in } \varepsilon(m + \mathcal{Y}), \\ \displaystyle\int_{\varepsilon(m+\mathcal{Y})} (\varepsilon \nabla \cdot u_{1\varepsilon} - \nabla \cdot u_\varepsilon) \nabla \cdot \psi \, dx = 0 \qquad \forall \psi \in H^1(\varepsilon(m+\mathcal{Y}))^3. \end{cases} \tag{7.7}$$

Then there exists $u_1 \in L^2(\boldsymbol{R}^3; H^1(\mathcal{Y})^3)$ such that $\int_{\mathcal{Y}} u_1(x, y)\, dy = 0$ for a.a. $x \in \boldsymbol{R}^3$, $\nabla_y \times \times u_1 = 0$ a.e. in $\boldsymbol{R}^3 \times \mathcal{Y}$, and, as $\varepsilon \to 0$ along a suitable subsequence,

$$u_{1\varepsilon} \underset{2}{\rightharpoonup} u_1 \quad \text{in } L^2(\boldsymbol{R}^3 \times \mathcal{Y})^3, \qquad \varepsilon \nabla \cdot u_{1\varepsilon} \underset{2}{\rightharpoonup} \nabla_y \cdot u_1 \quad \text{in } L^2(\boldsymbol{R}^3 \times \mathcal{Y}). \tag{7.8}$$

Moreover, setting $\bar{u}(x) = \int_{\mathcal{Y}} u(x, y)\, dy$ for a.a. $x \in \boldsymbol{R}^3$,

$$\nabla \cdot u_\varepsilon \underset{2}{\rightharpoonup} \nabla_x \cdot \bar{u} + \nabla_y \cdot u_1 \qquad \text{in } L^2(\boldsymbol{R}^3 \times \mathcal{Y}). \tag{7.9}$$

Finally, $\varepsilon \nabla \cdot u_\varepsilon \underset{2}{\rightharpoonup} \nabla_y \cdot u = 0$ in $L^2(\boldsymbol{R}^3 \times \mathcal{Y})$.

$\nabla_x \cdot \bar{u}$ and $\nabla_y \cdot u_1$ are orthogonal in $L^2(\boldsymbol{R}^3 \times \mathcal{Y})$, and a formula like (7.5) holds for divergences.

8. TWO-SCALE CONVERGENCE OF THE LAPLACE OPERATOR

THEOREM 8.1. *Let $p \in]1, +\infty[$, and a sequence $\{u_\varepsilon\}$ be such that $u_\varepsilon \rightharpoonup u$ in $W^{2,p}(\boldsymbol{R}^N)$. For any ε, there exists a unique $u_{2\varepsilon} \in W^{2,p}(\boldsymbol{R}^N_\varepsilon)$ such that, for any $m \in \boldsymbol{Z}^N$, (omitting restrictions)*

$$u_{2\varepsilon} \in W^{2,p}(\varepsilon(m + \mathcal{Y})), \qquad \int_{\varepsilon(m+\mathcal{Y})} u_{2\varepsilon}(x)\, dx = 0, \tag{8.1}$$

$$\int_{\varepsilon(m+\mathcal{Y})} (\varepsilon^2 \Delta u_{2\varepsilon} - \Delta u_\varepsilon) \Delta \psi \, dx = 0 \qquad \forall \psi \in W^{2,p'}(\varepsilon(m + \mathcal{Y})). \tag{8.2}$$

Then there exists $u_2 \in L^p(\boldsymbol{R}^N; W^{2,p}(\mathcal{Y}))$ such that $\int_{\mathcal{Y}} u_2(x, y)\, dy = 0$ for a.a. $x \in \boldsymbol{R}^N$, *and, as* $\varepsilon \to 0$ *along a suitable subsequence,*

(8.3) $$u_{2\varepsilon} \underset{2}{\rightharpoonup} u_2 \quad in\ L^p(\mathbf{R}^N \times \mathcal{Y}), \qquad \varepsilon^2 \Delta u_{2\varepsilon} \underset{2}{\rightharpoonup} \Delta_y u_2 \quad in\ L^p(\mathbf{R}^N \times \mathcal{Y}),$$

(8.4) $$\Delta u_\varepsilon \underset{2}{\rightharpoonup} \Delta u + \Delta_y u_2 \qquad in\ L^p(\mathbf{R}^N \times \mathcal{Y}).$$

The latter formula also reads

(8.5) $$\lim_{\varepsilon \to 0}{}^{(2)} \Delta u_\varepsilon = \Delta \lim_{\varepsilon \to 0}{}^{(1)} u_\varepsilon + \Delta_y u_2 = \lim_{\varepsilon \to 0}{}^{(1)} \Delta u_\varepsilon + \Delta_y u_2 \qquad \text{a.e. in } \mathbf{R}^N \times \mathcal{Y},$$

this may be compared with (2.4), (6.6), (7.5). For $p = 2$ this decomposition is orthogonal in $L^2(\mathbf{R}^N \times \mathcal{Y})$. This theorem can be extended to more general linear elliptic operators.

9. Two-scale convergence of potentials

Finally, we deal with the two-scale limit of a sequence of solutions φ_ε of the equation $(\Lambda_\varepsilon \varphi_\varepsilon :=)\ (\nabla_\varepsilon, \varepsilon\nabla)\varphi_\varepsilon = u_\varepsilon$, as $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $C^1(\mathbf{R}^N \times \mathcal{Y})^{2N}$. For the sake of simplicity we confine ourselves to $N = 3$.

Theorem 9.1. *Let two sequences* $\{u_{1\varepsilon}\}, \{u_{2\varepsilon}\}$ *of* $C^1(\mathbf{R}^3)^3$ *and* $u_1, u_2 \in C^1(\mathbf{R}^3 \times \mathcal{Y})^3$ *be such that*

(9.1) $$u_{1\varepsilon} \underset{2}{\rightharpoonup} u_1, \quad u_{2\varepsilon} \underset{2}{\rightharpoonup} u_2 \qquad in\ C^1(\mathbf{R}^3 \times \mathcal{Y})^3,$$

(9.2) $$\nabla_\varepsilon \times u_{1\varepsilon} = \nabla \times u_{2\varepsilon} = 0, \quad \varepsilon \nabla u_{1\varepsilon} = \nabla_\varepsilon u_{2\varepsilon} \qquad in\ \mathbf{R}^3, \forall \varepsilon,$$

(9.3) $$\varepsilon u_{1\varepsilon}(\varepsilon m)\cdot e_i = \int_0^1 u_{2\varepsilon}(\varepsilon m + \varepsilon t e_i)\cdot e_i\, dt \qquad \forall m \in \mathbf{Z}^3, \forall \varepsilon, for\ i = 1, 2, 3.$$

For any $(x, y) \in \mathbf{R}^3 \times \mathcal{Y}$, *let* ξ_x, η_y *and the sequences* $\{\xi^\varepsilon_{\varepsilon\mathcal{N}(x/\varepsilon)}\}, \{\eta^\varepsilon_y\}$ *in* $C^1([0,1])^3$ *be such that*

(9.4) $$\begin{cases} \xi^\varepsilon_{\varepsilon\mathcal{N}(x/\varepsilon)}(0) = \eta^\varepsilon_y(0) = 0, \quad \xi^\varepsilon_{\varepsilon\mathcal{N}(x/\varepsilon)}(1) = \varepsilon\mathcal{N}(x/\varepsilon), \quad \eta^\varepsilon_y(1) = y \quad \forall \varepsilon, \\ \xi^\varepsilon_{\varepsilon\mathcal{N}(x/\varepsilon)} \to \xi_x, \qquad \eta^\varepsilon_y \to \eta_y \quad in\ C^1([0,1])^3. \end{cases}$$

[*This determines the sequence* $\{\eta^\varepsilon_{\mathcal{R}(x/\varepsilon)}\}$ *via diagonalization*]. *Finally, let us set*

(9.5) $$\begin{cases} \varphi_\varepsilon(x) := \displaystyle\int_0^1 u_{1\varepsilon}(\xi^\varepsilon_{\varepsilon\mathcal{N}(x/\varepsilon)}(t))\cdot(\xi^\varepsilon_{\varepsilon\mathcal{N}(x/\varepsilon)})'(t)\, dt + \\ \qquad + \displaystyle\int_0^1 u_{2\varepsilon}(\varepsilon\mathcal{N}(x/\varepsilon) + \varepsilon\eta^\varepsilon_{\mathcal{R}(x/\varepsilon)}(t))\cdot(\eta^\varepsilon_{\mathcal{R}(x/\varepsilon)})'(t)\, dt \quad \forall x \in \mathbf{R}^3, \forall \varepsilon, \\ \varphi(x, y) := \displaystyle\int_0^1 u_1(\xi_x(t), 0)\cdot \xi'_x(t)\, dt + \\ \qquad + \displaystyle\int_0^1 u_2(x, \eta_y(t))\cdot \eta'_y(t)\, dt \quad \forall (x, y) \in \mathbf{R}^3 \times \mathcal{Y}. \end{cases}$$

Then

$$(9.6) \qquad \varphi_\varepsilon \in C^2(\boldsymbol{R}^3), \quad \nabla_\varepsilon \varphi_\varepsilon = u_{1\varepsilon}, \quad \varepsilon\nabla\varphi_\varepsilon = u_{2\varepsilon} \quad in \ \boldsymbol{R}^3,$$

$$(9.7) \qquad \varphi \in C^2(\boldsymbol{R}^3 \times \mathcal{Y}), \quad \varphi_\varepsilon(x) \underset{2}{\rightharpoonup} \varphi(x, y) \quad in \ C^2(\boldsymbol{R}^3 \times \mathcal{Y}).$$

Moreover, the φ_ε's and φ are path-independent (in $\boldsymbol{R}^3$ and in $\boldsymbol{R}^3 \times \mathcal{Y}$, resp.); that is, they do not depend on the specific choice of the sequences $\{\xi^\varepsilon_{\varepsilon\mathcal{N}(x/\varepsilon)}\}, \{\eta^\varepsilon_y\}$.

(Although $\mathcal{N}(x/\varepsilon)$ and $\mathcal{R}(x/\varepsilon)$ are discontinuous at any x such that $\mathcal{R}(x/\varepsilon)_i = 0$ for some i, by (9.3) the φ_ε's are continuous everywhere in $\boldsymbol{R}^3$). An analogous result holds for *strong* two-scale convergence.

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References

[1] G. Allaire, *Homogenization and two-scale convergence.* S.I.A.M. J. Math. Anal., 23, 1992, 1482-1518.

[2] G. Allaire, *Shape Optimization by the Homogenization Method.* Springer, New York 2002.

[3] T. Arbogast - J. Douglas - U. Hornung, *Derivation of the double porosity model of single phase flow via homogenization theory.* S.I.A.M. J. Math. Anal., 21, 1990, 823-836.

[4] A. Bensoussan - J.L. Lions - G. Papanicolaou, *Asymptotic Analysis for Periodic Structures.* North-Holland, Amsterdam 1978.

[5] A. Bourgeat - S. Luckhaus - A. Mikelic, *Convergence of the homogenization process for a double-porosity model of immiscible two-phase flow.* S.I.A.M. J. Math. Anal., 27, 1996, 1520-1543.

[6] J.K. Brooks - R.V. Chacon, *Continuity and compactness of measures.* Adv. in Math., 37, 1980, 16-26.

[7] J. Casado-Diaz - I. Gayte, *A general compactness result and its application to two-scale convergence of almost periodic functions.* C.R. Acad. Sci. Paris, Ser. I, 323, 1996, 329-334.

[8] D. Cioranescu - A. Damlamian - G. Griso, *Periodic unfolding and homogenization.* C.R. Acad. Sci. Paris, Ser. I, 335, 2002, 99-104.

[9] D. Cioranescu - P. Donato, *An Introduction to Homogenization.* Oxford Univ. Press, New York 1999.

[10] V.V. Jikov - S.M. Kozlov - O.A. Oleinik, *Homogenization of Differential Operators and Integral Functionals.* Springer, Berlin 1994.

[11] M. Lenczner, *Homogénéisation d'un circuit électrique.* C.R. Acad. Sci. Paris, Ser. II, 324, 1997, 537-542.

[12] M. Lenczner - G. Senouci, *Homogenization of electrical networks including voltage-to-voltage amplifiers.* Math. Models Meth. Appl. Sci., 9, 1999, 899-932.

[13] D. Lukkassen - G. Nguetseng - P. Wall, *Two-scale convergence.* Int. J. Pure Appl. Math., 2, 2002, 35-86.

[14] F. Murat - L. Tartar, *H-convergence.* In: A. Cherkaev - R. Kohn (eds.), *Topics in the Mathematical Modelling of Composite Materials*. Birkhäuser, Boston 1997, 21-44.

[15] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization.* S.I.A.M. J. Math. Anal., 20, 1989, 608-623.

[16] G. NGUETSENG, *Asymptotic analysis for a stiff variational problem arising in mechanics.* S.I.A.M. J. Math. Anal., 21, 1990, 1394-1414.
[17] L. TARTAR, *Mathematical tools for studying oscillations and concentrations: from Young measures to H-measures and their variants.* In: N. ANTONIĆ - C.J. VAN DUIJN - W. JÄGER - A. MIKELIĆ (eds.), *Multiscale Problems in Science and Technology*. Springer, Berlin 2002, 1-84.
[18] E. WEINAN, *Homogenization of linear and nonlinear transport equations.* Comm. Pure Appl. Math., 45, 1992, 301-326.
[19] V.V. ZHIKOV, *On an extension of the method of two-scale convergence and its applications.* Sb. Math., 191, 2000, 973-1014.

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Dipartimento di Matematica
Università degli Studi di Trento
Via Sommarive, 14 - 38050 POVO DI TRENTO (TN)
visintin@science.unitn.it