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Inner k-th Carathéodory-Reiffen completeness of Reinhardt domains

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Funzioni di variabili complesse. — *Inner k-th Carathéodory-Reiffen completeness of Reinhardt domains.* Nota (*) di PAWEŁ ZAPAŁOWSKI, presentata dal Socio E. Vesentini.

ABSTRACT. — A description of bounded pseudoconvex Reinhardt domains, which are complete with respect to the inner k-th Carathéodory-Reiffen distance, is given.

KEY WORDS: Completeness; Inner k-th Carathéodory-Reiffen distance; Pseudoconvex Reinhardt domain.

RIASSUNTO. — *Completezza di domini di Reinhardt per la distanza interna k-ta di Carathéodory-Reiffen.* Si presenta la descrizione di domini di Reinhardt limitati e pseudo-convessi, che sono completi per la distanza interna k-esima di Carathéodory-Reiffen.

In the class of bounded pseudoconvex Reinhardt domains the notion of completeness with respect to the Carathéodory and inner Carathéodory distances is completely understood. In 1994 S. Fu proved that all bounded pseudoconvex Reinhardt domains satisfying some geometric condition (see [1]) are Carathéodory complete. He also indicated that there are domains not satisfying this condition and not being Carathéodory complete. It was W. Zwonek who showed that each bounded pseudoconvex Reinhardt domain which is complete for the Carathéodory distance must fulfill a geometric condition from [1] (see [4]). In 2001 W. Zwonek proved that in the class of bounded pseudoconvex Reinhardt domains the notions of completenesses with respect to the Carathéodory and inner Carathéodory distances are equivalent (see [5]). We present a description of bounded pseudoconvex Reinhardt domains complete with respect to the inner k-th Carathéodory-Reiffen distance. It turns out that in the class of domains mentioned above the inner k-th Carathéodory-Reiffen completeness coincides with the Carathéodory completeness. Therefore this paper, where the same method as in [5] is used, sharpens the result from [5].

Before we state the result of the paper let us recall the notation and the definitions we shall need.

- $I_k^n := \{\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n : |\beta| = k\}$, $\iota_k^n := \# I_k^n$;
- if $X = (X_1, \dots, X_n) \in \mathbb{C}^n$, $\beta \in \mathbb{Z}^n$, then $X^\beta := \prod_{j=1}^n X_j^{\beta_j}$, $X_j \neq 0$ if $\beta_j < 0$;
- if $r, s \in \mathbb{R}$, $t \in \mathbb{R}_{>0}$ and $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}_{>0}^n$, then $\boldsymbol{t}^\gamma := (t^{\gamma_1}, \dots, t^{\gamma_n})$ and $\gamma^r \boldsymbol{t}^{\gamma - s} := (\gamma_1^r t^{\gamma_1 - s}, \dots, \gamma_n^r t^{\gamma_n - s})$.

Let E denote the unit disc in $\mathbb{C}$ and let $D \subset \mathbb{C}^n$ be an arbitrary domain. Let

$$c_D(z, w) := \sup\{p(f(z), f(w)) : f \in \mathcal{O}(D, E)\}, \quad z, w \in D,$$

(*) Pervenuta in forma definitiva all'Accademia il 27 ottobre 2003.

where p denotes the Poincaré distance on E. We call c_D the *Carathéodory pseudodistance*.

For $k \in \mathbb{N}$, $k \geq 1$, and $f \in \mathcal{O}(D, E)$ we define

$$f_{(k)}(z)\, X := \sum_{\beta \in I_k^n} \frac{1}{\beta!} \mathcal{D}^\beta f(z)\, X^\beta, \quad z \in D,\ X \in \mathbb{C}^n.$$

Let

$$\gamma_D^{(k)}(z; X) = \sup\{ |f_{(k)}(z)\, X|^{1/k} : f \in \mathcal{O}(D, E),\ \mathrm{ord}_z f \geq k\}, \quad z \in D,\ X \in \mathbb{C}^n,$$

where $\mathrm{ord}_z f$ denotes the order of the zero of f at z. We call $\gamma_D^{(k)}$ the *k-th Carathéodory-Reiffen pseudometric*.

For a piecewise $\mathcal{C}^1$-curve $\alpha : [0, 1] \to D$ (we write $\alpha \in \mathcal{C}_p^1([0, 1], D)$) put

$$L_{\gamma_D^{(k)}}(\alpha) := \int_0^1 \gamma_D^{(k)}(\alpha(t); \alpha'(t))\, dt.$$

We call $L_{\gamma_D^{(k)}}(\alpha)$ the $\gamma_D^{(k)}$*-length of the curve* α. Define

$$\int \gamma_D^{(k)}(z, w) := \inf\{L_{\gamma_D^{(k)}}(\alpha) : \alpha \in \mathcal{C}_p^1([0, 1], D),\ \alpha(0) = z,\ \alpha(1) = w\}, \quad z, w \in D.$$

We call $\int \gamma_D^{(k)}$ the *integrated form of* $\gamma_D^{(k)}$ or the *inner k-th Carathéodory-Reiffen pseudodistance*.

In particular, we call the function $c_D^i := \int \gamma_D^{(1)}$ the *inner Carathéodory pseudodistance*.

For all the necessary information on these functions consult *e.g.* [2]. The following important inequalities hold on any domain D

$$c_D \leq c_D^i \leq \int \gamma_D^{(k)}, \quad k \geq 1. \tag{1}$$

Note that in the class of bounded domains all the pseudodistances defined above are distances. Recall that D is called d_D*-complete* if any d_D-Cauchy sequence is convergent in the natural topology of D, where $d = c$, c^i or $\int \gamma^{(k)}$.

A domain $D \subset \mathbb{C}^n$ is called *Reinhardt* if $(\lambda_1 z_1, \ldots, \lambda_n z_n) \in D$ for all points $z = (z_1, \ldots, z_n) \in D$ and $|\lambda_1| = \ldots = |\lambda_n| = 1$. From now we assume that D is always a bounded pseudoconvex Reinhardt domain.

Let us denote

$$V_j := \{z \in \mathbb{C}^n : z_j = 0\}, \quad j = 1, \ldots, n.$$

The description of complete bounded pseudoconvex Reinhardt domains is given in the following theorem.

THEOREM 1. *Let D be a bounded pseudoconvex Reinhardt domain.*

(a) (cf. [3, 1, 4]) *D is c_D-complete if and only if the following condition is satisfied*

$$\text{for any } j \in \{1, \ldots, n\}, \text{ if } \overline{D} \cap V_j \neq \varnothing, \text{ then } D \cap V_j \neq \varnothing. \tag{2}$$

(b) (cf. [5]) *D is c_D^i-complete if and only if D is c_D-complete.*

Our aim is to prove the following result.

THEOREM 2. *For every integer $k \geq 1$ a bounded pseudoconvex Reinhardt domain D is $\int \gamma_D^{(k)}$-complete if and only if D is c_D-complete.*

PROOF OF THEOREM 2. In view of the inequalities (1) and Theorem 1 we are done if we show that any domain D not satisfying (2) is not $\int \gamma_D^{(k)}$-complete. Proceeding exactly as in [4], the proof will be completed if we show that D is not $\int \gamma_D^{(k)}$-complete when $D \subset \mathbb{C}_*^n (\mathbb{C}_* := \mathbb{C} \setminus \{0\})$ is a pseudoconvex Reinhardt domain such that

$$\log D := \{0\} \times (\log \delta, -\log \delta)^{n-1} + \mathbb{R}_{<0} \gamma,$$

where $\delta \in (0, 1)$ and $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}_{>0}^n$, $\gamma_1 := 1$.

Fix such a D and $k \in \mathbb{N}$. Without loss of generality we may assume that $n \geq 2$. We show that the $\gamma_D^{(k)}$-length of the curve $(0, 1/2) \ni t \mapsto \boldsymbol{t}^\gamma \in D$ is finite, *i.e.*

$$\int_0^{1/2} \gamma_D^{(k)}(\boldsymbol{t}^\gamma; \gamma \boldsymbol{t}^{\gamma - 1})\, dt < \infty. \tag{3}$$

Reasoning exactly as in [5], for any function $f \in \mathcal{O}(D, E)$ we have

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n : \langle \alpha, \gamma \rangle \geq 0} a_\alpha z^\alpha, \quad z \in D,$$

and

$$|a_\alpha| \leq \delta^{|\alpha_2| + \dots + |\alpha_n|}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n : \langle \alpha, \gamma \rangle \geq 0, \tag{4}$$

where $|\alpha_j|$ denotes the modulus of the number α_j.

On the other hand, for such a function and for any $j = 1, \dots, k$,

$$f^{(j)}(\boldsymbol{t}^\gamma) X \boldsymbol{t}^{\gamma - k/j} = \sum_{\alpha \in \mathbb{Z}^n : \langle \alpha, \gamma \rangle \geq 0} a_\alpha \sum_{\beta \in I_j^n} \frac{j!}{\beta!} p_\beta(\alpha) X^\beta t^{\langle \alpha, \gamma \rangle - k}, \quad X \in \mathbb{C}^n, \tag{5}$$

where $p_\beta(\alpha) := \prod_{j=1}^{n} p_{\beta_j}(\alpha_j)$ and $p_{\beta_j}(\alpha_j) := \prod_{l=0}^{\beta_j - 1} (\alpha_j - l)$, $j = 1, \dots, n$.

If $f \in \mathcal{O}(D, E)$ is such that $\mathrm{ord}_{t^\gamma} f \geq k$ then, in particular, $f^{(j)}(\boldsymbol{t}^\gamma) X = 0$ for any $X \in \mathbb{C}^n$ and $j = 0, \dots, k-1$. Thus, for such a function,

$$f^{(j)}(\boldsymbol{t}^\gamma) X \boldsymbol{t}^{\gamma - k/j} = 0, \quad X \in \mathbb{C}^n, \ j = 1, \dots, k-1. \tag{6}$$

Using (5) and (6) we will obtain

$$f^{(k)}(\boldsymbol{t}^\gamma) \gamma \boldsymbol{t}^{\gamma - 1} = \sum_{\alpha \in \mathbb{Z}^n : \langle \alpha, \gamma \rangle > 0} a_\alpha \langle \alpha, \gamma \rangle^k t^{\langle \alpha, \gamma \rangle - k} \tag{7}$$

for any function $f \in \mathcal{O}(D, E)$ such that $\mathrm{ord}_{t^\gamma} f \geq k$.

Assume for a while that (7) holds. Hence, by (7) and (4),

$$|f_{(k)}(\boldsymbol{t}^\gamma) \gamma \boldsymbol{t}^{\gamma - 1}|^{1/k} \leq \frac{1}{\sqrt[k]{k!}} \sum_{\alpha \in \mathbb{Z}^n : \langle \alpha, \gamma \rangle > 0} \delta^{(|\alpha_2| + \dots + |\alpha_n|)/k} \langle \alpha, \gamma \rangle t^{\langle \alpha, \gamma \rangle / k - 1}.$$

Hence, if we take the supremum over all $f \in \mathcal{O}(D, E)$ such that $\mathrm{ord}_{t^\gamma} f \geq k$, we obtain

$$\gamma_D^{(k)}(\boldsymbol{t}^\gamma;\gamma\boldsymbol{t}^{\gamma-1})\leqslant\frac{1}{\sqrt{k!}}\sum_{\alpha\in\mathbb{Z}^n:\langle\alpha,\gamma\rangle>0}\delta^{(|\alpha_2|+\ldots+|\alpha_n|)/k}\langle\alpha,\gamma\rangle t^{\langle\alpha,\gamma\rangle/k-1}.$$

Now, proceeding exactly as in [5], we get (3).

We are left with the proof of (7). Observe that, having (6), we obtain (7) if we show that there exist $M\in\mathbb{N}$, $X_1,\ldots,X_M\in\mathbb{C}^n$, $b_1,\ldots,b_M\in\mathbb{C}$ and $k_1,\ldots,k_M\in\{1,\ldots,k-1\}$ such that

$$\sum_{\beta\in I_k^n}\frac{k!}{\beta!}(p_\beta(\alpha)-\alpha^\beta)\gamma^\beta=\sum_{s=1}^M b_s\sum_{\beta\in I_{k_s}^n}\frac{k_s!}{\beta!}p_\beta(\alpha)X_s^\beta,\quad \alpha\in\mathbb{Z}^n.\tag{8}$$

Note that if $\beta\in I_r^n$, $1\leqslant r\leqslant k$, then $p_\beta(\alpha)-\alpha^\beta=\sum_{j=1}^{r-1}Q_j^\beta(\alpha)$, where Q_j^β is a homogeneous polynomial of the degree j. $Q_j^\beta(\alpha)$ is a sum of the monomials of the degree j which are of the form $q_\delta^\beta\alpha^{\beta-\delta}$, where $\delta\in I_{r-j}^n$ and $q_\delta^\beta\in\mathbb{Z}$ are such that $q_\delta^\beta=0$ if there exists an l such that $\beta_l<\delta_l$.

Therefore, it is easy to see that we get (8) if we show that for any $1\leqslant r\leqslant k$, $1\leqslant j\leqslant r-1$, $\delta\in I_{r-j}^n$ and $X\in\mathbb{C}^n$, there exist $N\in\mathbb{N}$, $Y_1,\ldots,Y_N\in\mathbb{C}^n$ and $c_1,\ldots,c_N\in\mathbb{C}$ such that

$$\sum_{\beta\in I_r^n}\frac{r!}{\beta!}q_\delta^\beta\alpha^{\beta-\delta}X^\beta=\sum_{\nu\in I_j^n}\frac{j!}{\nu!}\alpha^\nu\sum_{s=1}^N c_sY_s^\nu,\quad \alpha\in\mathbb{Z}^n.\tag{9}$$

Since

$$\sum_{\beta\in I_r^n}\frac{r!}{\beta!}q_\delta^\beta\alpha^{\beta-\delta}X^\beta=\sum_{\substack{\beta\in I_r^n:\\ \beta_l\geqslant\delta_l,\,1\leqslant l\leqslant n}}\frac{r!}{\beta!}q_\delta^\beta\alpha^{\beta-\delta}X^\beta=$$
$$=\sum_{\nu\in I_j^n}\frac{r!}{(\nu+\delta)!}q_\delta^{\nu+\delta}\alpha^\nu X^{\nu+\delta}=\sum_{\nu\in I_j^n}\frac{j!}{\nu!}\alpha^\nu\frac{r!\nu!}{j!(\nu+\delta)!}q_\delta^{\nu+\delta}X^{\nu+\delta},$$

we will obtain (9) if we apply the following lemma with the constants

$$C_\nu=C_\nu(r,j,\delta,X):=\frac{r!\nu!}{j!(\nu+\delta)!}q_\delta^{\nu+\delta}X^{\nu+\delta},\quad \nu\in I_j^n,$$

which ends the proof. □

Lemma 3. *For any $j\in\mathbb{N}$, $j\geqslant1$, and for any sequence $\{C_\nu\}_{\nu\in I_j^n}\subset\mathbb{C}$ there exist $N\in\mathbb{N}$, $Y_1,\ldots,Y_N\in\mathbb{C}^n$ and $c_1,\ldots,c_N\in\mathbb{C}$ such that*

$$\sum_{s=1}^N c_sY_s^\nu=C_\nu,\quad \nu\in I_j^n.\tag{10}$$

Proof of Lemma 3. For any $\nu,\mu\in I_j^n$, $\nu\neq\mu$, let $m=m(\nu,\mu):=\max\{l:\nu_l\neq\mu_l\}$. In the set I_j^n we introduce the following order: for any $\nu,\mu\in I_j^n$, $\nu\neq\mu$, we say that $\nu<\mu$ if and only if $\nu_m<\mu_m$. Thus we may write $I_j^n=\{\nu^1,\ldots,\nu^{\iota_j^n}\}$, where $\nu^l<\nu^{l+1}$ for any $l=1,\ldots,\iota_j^n-1$.

Let

$$\begin{cases} a_1 := 1 \\ a_l := ja_{l-1} - j + 2, \quad l = 2, \dots, n. \end{cases}$$

We define $Y_1 := (2^{a_1}, \dots, 2^{a_n})$. Thus we obtain

$$Y_1^{\nu^l} = 2^{t_l}, \quad l = 1, \dots, \iota_j^n, \tag{11}$$

where $t_l \in \mathbb{N}$ and $j = t_1 < \dots < t_{\iota_j^n}$. Indeed, according to the order introduced in the set I_j^n and the definition of the numbers a_l only the following two cases are possible:

a) $\nu^l = (\nu_1^l, \nu_2^l, \nu_3^l, \dots, \nu_n^l)$, $\nu^{l+1} = (\nu_1^l - 1, \nu_2^l + 1, \nu_3^l, \dots, \nu_n^l)$, where $\nu_1^l > 0$; then

$$t_{l+1} - t_l = \nu_1^l - 1 + 2(\nu_2^l + 1) - \nu_1^l - 2\nu_2^l = 1 > 0.$$

b) $\nu^l = (\underbrace{0, \dots, 0, \nu_s^l}_{s}, \nu_{s+1}^l, \dots, \nu_n^l)$, $\nu^{l+1} = (\underbrace{j-R-1, 0, \dots, 0}_{s}, \nu_{s+1}^l + 1, \dots, \nu_n^l)$, where $R := \sum_{p=s+1}^{n} \nu_p^l$; then

$$t_{l+1} - t_l = j - R - 1 + a_{s+1}(\nu_{s+1}^l + 1) - a_s \nu_s^l - a_{s+1}\nu_{s+1}^l =$$
$$= j - R - 1 + a_{s+1} - a_s \nu_s^l = ja_s - a_s \nu_s^l - R + 1 \geqslant a_s(j - \nu_s^l - R) + 1 = 1,$$

since $j - \nu_s^l - R = \sum_{p=1}^{s-1} \nu_p^l = 0$.

Therefore, there are only two possibilities.

1. $t_l = j + l - 1$ for all $l = 1, \dots, \iota_j^n$; then we put $N := \iota_j^n$ and define $Y_s := := (2^{sa_1}, \dots, 2^{sa_n})$ for $s = 2, \dots, N$.

Now we construct the numbers $c_1, \dots, c_N$. Note that the system of the equations (10) is equal to

$$\sum_{s=1}^{N} 2^{s(j+l-1)} c_s = C_{\nu^l}, \quad l = 1, \dots, N. \tag{12}$$

It is clear that (12) has a unique solution $c_1, \dots, c_N$ if only its main determinant is not zero. Fortunately,

$$\det [2^{s(j+l-1)}]_{s,\,l=1}^N = 2^{j(1+\dots+M)} \det [2^{s(l-1)}]_{s,\,l=1}^N = 2^{jM(M+1)/2} \prod_{N \geqslant s > l \geqslant 1} (2^s - 2^l)$$

is not zero and we are done.

2. There exists a number l such that $t_{l+1} > t_l + 1$; then we fill in the gaps in the sequence t_l with the missing natural numbers and, after renaming it, we obtain a new sequence $j = u_1, \dots, u_N := t_{\iota_j^n}$ such that $u_l = j + l - 1$ for any $l = 1, \dots, N$. Note that it is also a definition of N. For newly introduced numbers u_l, *i.e.* for these u_l which do not have a counterpart amongst the numbers t_l, we define $\nu^l := (u_l, 0, \dots, 0) \in \mathbb{Z}_+^n$ and $C_{\nu^l} := 0$. Thus we obtain the property (11) for all numbers u_l. Then, proceeding exactly as in the case 1, we construct vectors $Y_2, \dots, Y_N$ and prove the existence of numbers $c_1, \dots, c_N$. In this case, however, in order to verify (10) we only need these equations from (12) which come from the original indices $\nu^l \in I_j^n$, *i.e.* which existed before filling in the gaps in the sequence t_l. $\square$

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References

[1] S. Fu, *On completeness of invariant metrics of Reinhardt domains*. Arch. Math., 63, 1994, 166-172.

[2] M. Jarnicki - P. Pflug, *Invariant Distances and Metrics in Complex Analysis*. de Gruyter Expositions Math. 9, Berlin 1993.

[3] P. Pflug, *About the Carathéodory completeness of all Reinhardt domains*. In: G. Zapata (ed.), *Functional Analysis, Holomorphy and Approximation Theory II*. North-Holland, Amsterdam 1984, 331-337.

[4] W. Zwonek, *On Carathéodory completeness of pseudoconvex Reinhardt domains*. Proc. Amer. Math. Soc., 128(3), 2000, 857-864.

[5] W. Zwonek, *Inner Carathéodory completeness of Reinhardt domains*. Rend. Mat. Acc. Lincei, s. 9, v. 12, 2001, 153-157.

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