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Abelian quasinormal subgroups of groups

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Teoria dei gruppi. — *Abelian quasinormal subgroups of groups.* Nota di STEWART E. STONEHEWER e GIOVANNI ZACHER, presentata (*) dal Socio G. Zacher.

ABSTRACT. — Let G be any group and let A be an abelian quasinormal subgroup of G. If n is any positive integer, either odd or divisible by 4, then we prove that the subgroup A^n is also quasinormal in G.

KEY WORDS: Quasinormal subgroup; Abelian groups.

RIASSUNTO. — *Sottogruppi abeliani quasi-normali dei gruppi.* Sia G un gruppo e sia A un sottogruppo abeliano e quasi-normale in G. Se n è un qualunque intero positivo dispari o divisibile per 4, allora si dimostra che il sottogruppo A^n è pure quasi-normale in G.

1. INTRODUCTION AND STATEMENT OF RESULTS

Among the most important concepts in group theory, arguably *the* most important, are those of composition and chief series, arising from normal subgroups. If *normal* is replaced by *quasinormal*, then, by analogy, little seems to be known. A subgroup A of a group G is said to be *quasinormal* (or *permutable*) in G if $AX = XA$ for all subgroups X of G. Obviously this is equivalent to the product AX being a subgroup. Of course normal subgroups are necessarily quasinormal, while the converse is not always true. But what is the structure of minimal quasinormal subgroups and what properties do maximal (*i.e.* unrefinable) chains of quasinormal subgroups possess? The answers to these and similar questions remain pitifully inadequate. In the present work, however, we show that when A is an abelian quasinormal subgroup of G, then certain canonical subgroups of A are also quasinormal in G. We denote the core of a subgroup A of a group G by A_G. When A is quasinormal and G is finite, the quotient A/A_G is always nilpotent [4], though there is no restriction on the class [2, 11]. Much of the published work on quasinormal subgroups has been about A/A_G, *i.e.* the core-free case. Here, however, our arguments apply equally to the core-free and the non-core-free situations. Our main result is the following.

THEOREM 1. *Let A be an abelian quasinormal subgroup of a group G and let n be a positive integer, either odd or divisible by* 4. *Then A^n is also quasinormal in G.*

We shall see in Section 3 that the restriction on n here is necessary, by constructing an example in which A^2 is not quasinormal. Theorem 1 is proved first for the case when G is a finite group. This in turn reduces easily to the case where G is a p-group, for some prime p. Then by straightforward induction arguments, we need to consider only the cases $n = p$, when p is odd, and $n = 4$ and 8, when $p = 2$. Thus we prove the following

(*) Nella seduta del 14 maggio 2004.

THEOREM 2. *Let A be an abelian quasinormal subgroup of a finite p-group G, where p is an odd prime. Then A^p is also quasinormal in G.*

THEOREM 3. *Let A be an abelian quasinormal subgroup of a finite 2-group G. Then* (i) A^4 *and* (ii) A^8 *are both quasinormal in G.*

We recall that every finite abelian group is self-dual, *i.e.* its subgroup lattice admits an inclusion-reversing bijection. Thus suppose that A is an abelian quasinormal subgroup of a finite p-group G, where p is an odd prime. Then in the light of Theorem 2, it is natural to ask if the subgroup $\Omega(A)$, generated by the elements of order p in A, is also quasinormal in G. However, this is not the case and we construct an example to show this in Section 3. To complete the picture, we give a further example of a finite 2-group G with an abelian quasinormal subgroup A such that $\Omega_2(A)$ is not quasinormal in G (here $\Omega_2(A)/\Omega(A) = \Omega(A/\Omega(A))$).

Section 4 deals with infinite groups G and deduces the full statement of Theorem 1 from the finite case. Also we include here two further examples which answer obvious questions.

Other notation is as follows. The centre of a group G is denoted by $Z(G)$ and a representative of the class of cyclic groups of order n by C_n. If H is a subgroup of a group G, then the lattice of subgroups between G and H is denoted by $[G : H]$. Also the normal closure of H in G is denoted by H^G.

2. THE FINITE CASE

We begin by showing how, when G is finite, Theorem 1 follows from Theorems 2 and 3. Thus let A be an abelian quasinormal subgroup of a finite group G and let n be a positive integer, odd or divisible by 4. Let p be a prime and let A_p be the p-component of A. Then A_p is quasinormal in G, by [8, Lemma 5.1.10]. Therefore we may assume that *A is a p-group.* Also we may assume that n is a p-power and indeed, by induction, either equal to p, if p is odd, or equal to 4 or 8, if $p = 2$. Clearly it suffices to show that $A^n X$ is a subgroup, for all cyclic subgroups X of G, and we can restrict ourselves to the case where X is a q-group, for some prime q. However, A is subnormal in AX, by [6], and so if $q \neq p$, then $A \lhd AX$. Thus $A^n \lhd AX$ in this case. Therefore we may suppose that $q = p$ and then Theorems 2 and 3 give the result.

In order to prove Theorems 2 and 3, we need two lemmas. The first is well known and the second is both intuitive and easily proved.

LEMMA 2.1 [7, 5.3.5]. *Let x and y be elements of a nilpotent group of class at most 2 and let m be an integer. Then $(xy)^m = x^m y^m [y, x]^{\binom{m}{2}}$.*

LEMMA 2.2. *Let A be a quasinormal subgroup of a group G and suppose that $G = AX$, where $X = \langle x \rangle$ is a cyclic subgroup. Then $A^G = AA^x$.*

PROOF. We may assume that A is not normal in G and $A_G = 1$. So $A \cap X = 1$, since $(A \cap X)^G = (A \cap X)^A \leqslant A$. Then by [10, Lemma 2.1], X is finite. Thus G is finite and we may argue by induction on $|G|$, assuming the usual induction hypothesis. By [5], A lies in the hypercentre of G, so G is nilpotent. Let p be a prime dividing $|AA^x \cap X|$ and let ax_1 be an element of order p in $Z(G)$, where $a \in A$, $x_1 \in X$. Then $[a, X] = 1$ and

$$\langle a \rangle^G = \langle a \rangle^A \leqslant A.$$

Hence $a = 1$ and $X_1 = \langle x_1 \rangle$ has order p and so lies in $AA^x \cap X \cap Z(G)$. By induction applied to G/X_1, $A^G = AA^x X_1 = AA^x$, as required. □

We are now in a position to be able to prove our main result for p-groups, when p is an odd prime.

PROOF OF THEOREM 2. We have a finite p-group G, where p is an odd prime, with an abelian quasinormal subgroup A and we must show that A^p is quasinormal in G. Thus suppose that the Theorem is false and let G be a counter-example of minimal order. Then there is a cyclic subgroup $X = \langle x \rangle$ of G such that $A^p X$ is not a subgroup, and so we have $G = AX$. Since $A \cap X \leqslant Z(G)$, we must have $A \cap X = 1$, otherwise $A^p X/(A \cap X)$ is a subgroup, by choice of G, whence so is $A^p X$. Similarly

$$X_G = (A^p)_G = 1.$$

Let M be a maximal subgroup of G containing X and let $B = A \cap M$. Then $|G : M| = |A : B| = p$. We distinguish two cases.

Case 1. *Suppose that B is not elementary.* Then $1 \neq B^p \leqslant A^p$. Since B is quasinormal in M, our choice of G implies that $B^p X$ is a subgroup. Thus

$$K = (B^p)^G = (B^p)^X \leqslant B^p X.$$

Again by choice of G, we know that $A^p XK/K$ is a subgroup and so $A^p XK$ is a subgroup. But

$$A^p X \subset A^p XK = A^p KX \subset A^p X,$$

and therefore $A^p X = A^p XK$ is a subgroup, a contradiction.

Case 2. *Suppose that B is elementary.* Clearly A is not elementary, so

$$A \cong C_{p^2} \times C_p \times \ldots \times C_p$$

and $B = \Omega(A)$. By Lemma 2.2, $A^G = AA^x = A(A^G \cap X)$. Thus with $L = A \cap A^x$, we have $[A^G : A] \cong [A^x : L]$ and this lattice is a chain. Therefore

$$|A^G : A| = p \text{ or } p^2$$

and we consider these cases separately (recall that A is not normal in G).

Suppose that $|A^G : A| = p$. Here A^G is the product of two normal abelian subgroups, therefore A^G has class at most 2. Also $(A^G)^p = (AX_1)^p$, where $X_1 = \Omega(X)$.

Thus by Lemma 2.1,

$$(A^G)^p = A^p[X_1, A]^{\binom{p}{2}} = A^p,$$

since $[X_1, A] \leqslant B$. Therefore $A^p \lhd G$, contradicting our choice of G.

Finally, *suppose that* $|A^G : A| = p^2$. Since $A/L \cong C_{p^2}$, it follows that $A = \langle a \rangle \times L$, where $\langle a \rangle \cong C_{p^2}$ and $L \leqslant B$. Also $L = A \cap A^x \leqslant Z(A^G) = Z$ (say) $\leqslant A$, since $[A, X_1] \neq 1$. Thus $Z \leqslant A_G \leqslant A \cap A^x \leqslant Z$ and so $L = Z \lhd G$. Now $|B/L| = p$ and B normalises X modulo L. Therefore $[B, X_2] \leqslant L$, where $X_2 = \Omega_2(X)$. Let

$$H = A^G \cap M = BX_2.$$

Thus $H \lhd G$ and $H' = [B, X_2] \leqslant L \leqslant Z(H)$. Then H has class at most 2, and so by Lemma 2.1, $H^p = X_2^p = X_1 \lhd G$, again contradicting our choice of G.

This completes the proof of Theorem 2. $\square$

Next we move on to consider the 2-group situation.

PROOF OF THEOREM 3(*i*). Again we suppose that the Theorem is false and let G be a 2-group which is a counter-example of minimal order. As in Theorem 2, $G = AX$, where A is an abelian quasinormal subgroup of G, $X = \langle x \rangle$ is a cyclic subgroup, and A^4X is *not* a subgroup. Also as before

$$A \cap X = 1 = X_G = (A^4)_G.$$

By Lemma 2.2, $A^G = AA^x$. Let $L = A \cap A^x$ and $Z = Z(A^G)$. Then $L \leqslant Z \lhd G$. Also $Z \leqslant A$, otherwise $\Omega(X)$ commutes with A and lies in $Z(G)$. Therefore $L \leqslant Z \leqslant A_G \leqslant L$ and

$$L = Z = A_G \lhd G,$$

as in Case 2 of Theorem 2. Thus L has exponent at most 4 and $A = \langle a \rangle L$, for some element a in A. Clearly $|a| \geqslant 8$.

Suppose that $|a| \geqslant 16$. Then $\langle a^2 \rangle L$ is quasinormal in the subgroup $\langle a^2 \rangle LX$, so by choice of G, $\langle a^8 \rangle X$ is a subgroup. Therefore

$$\langle A^8 \rangle^G = \langle a^8 \rangle^X \leqslant \langle a^8 \rangle X.$$

Again by choice of G (as in Case 1 of Theorem 2), we see that A^4X is a subgroup, which is not the case. Therefore

$$|a| = 8 \quad \text{and} \quad |A^G : A| = 2, 4 \text{ or } 8.$$

We claim that

(1) *G has a unique minimal normal subgroup.*

For, suppose that N_1 and N_2 are distinct minimal normal subgroups of G. Since $X_G = 1$, they both lie in A. Also by choice of G, A^4N_1X and A^4N_2X are subgroups and so we must have

$$A^4N_1X = A^4N_2X = \langle A^4, X \rangle.$$

Then intersecting with A, we obtain $A^4N_1 = A^4N_2$, contradicting $(A^4)_G = 1$. Therefore (1) is true.

Thus let N be the unique minimal normal subgroup of G. Note that $A^4 = \langle a^4 \rangle$, so again by choice of G, $\langle a^4 \rangle NX$ is a subgroup. Also modulo N, $\langle a^4 \rangle$ is quasinormal in this subgroup. Therefore for some integer i,

$$[x, a^4] \equiv x^{i2^{n-1}} \bmod N,$$

where $|x| = 2^n$. Thus $[x^2, a^4] = [x, a^4]^2 = 1$ and so $a^4 \in Z = L$. Since $A = \langle a \rangle L$, it follows that $|A^G : A| = 2$ or 4. Suppose that $|A^G : A| = 2$, so $A \lhd A^G$, $|A : L| = 2$ and A^G has class 2. Then putting $X_1 = \Omega(X)$ and using Lemma 2.1, we have

$$(A^G)^4 = (AX_1)^4 = \langle a^4 \rangle [\langle a \rangle L, X_1]^2 = \langle a^4 \rangle,$$

since $[a, X_1]^2 = [a^2, X_1] \leqslant [L, X_1] = 1$. Thus $\langle a^4 \rangle = A^4 \lhd G$ and we have a contradiction. Therefore we must have

$$|A^G : A| = 4.$$

We see now that A^G/L is the product of two cyclic quasinormal subgroups of order 4. Then it is easy to deduce that A^G/L is abelian and so again A^G has class 2. Modulo L, G is the product of cyclic subgroups $\langle a \rangle$ of order 4 and X of order 2^n. Thus by [3, Satz 2], $G^4 \equiv X^4 \bmod \mathrm{L}$ and hence $LX^4 \lhd G$. Let $K = LX^4$. Then A^2K/K has order 2 and is quasinormal in A^2XK/K and $|XK/K| = 4$. Thus A^2XK/K is abelian and so $A^2K = H$ (say) is normal in G. Similarly AH/H has order 2 and is quasinormal in $G/H = AX/H$ and $|XH/H| = 4$. Thus G/H is abelian and so $AK \lhd G$. To complete the proof, we distinguish two cases.

Case 1. *Suppose that G/K is not abelian.* Thus in the quotient G/K, x must invert the element a, *i.e.* $a^x = a^{-1}\ell x^{4j}$, where $\ell \in L$ and j is an integer. Since $[L, X] = [L, G] \lhd G$, it follows that

$$a^{x^2} = (a^{-1}\ell x^{4j})^x = x^{-4j}\ell^{-1}a\ell^x x^{4j} \equiv a^{x^{4j}} \bmod [L, X].$$

Hence $[a, x^2] = v$ (say) lies in $[L, X]$. We have $n \geqslant 3$ and by (1), $\Omega(L)$ is an indecomposable X-module. Therefore it has rank at most 2^{n-2}, since $x^{2^{n-2}}$ centralises L. However, modulo $\Omega(L)$ $(=\Omega(A))$, A is the direct product of $\langle a \rangle$ (of order 4) and L, and so

$$\operatorname{rank}(L/\Omega(L)) < \operatorname{rank}(A/\Omega(A)) \leqslant \operatorname{rank}(A) \leqslant 2^{n-2}.$$

Therefore $L/\Omega(L)$ has rank at most $2^{n-2} - 1$. Now viewing $L/\Omega(L)$ as additive X-module, we have

$$[a, x^2] = v \equiv \ell_1(x-1) \bmod \Omega(L),$$

for some $\ell_1 \in L$. Thus $[a, x^{2^{n-2}}] \equiv \ell_1(x-1)^{2^{n-2}-1} \equiv 0 \bmod \Omega(L)$. Therefore $[a, x^{2^{n-2}}]$ has order at most 2 and then, by Lemma 2.1,

$$(A^G)^4 = (A\Omega_2(X))^4 = \langle a^4 \rangle [\langle a \rangle, \Omega_2(X)]^2 = \langle a^4 \rangle \lhd G,$$

a contradiction.

Case 2. *Suppose that G/K is abelian.* Here we must have $n \geqslant 4$, otherwise if $|X| = 8$, then $A^G = A[A, X] \leqslant A\Omega(X) < A^G$. Thus $[a, x] = \ell_2 x^{k2^{n-2}}$, where $\ell_2 \in L$ and k is an integer. Therefore

$$[a, x^2] = \ell_2^2[\ell_2, x]\, x^{k2^{n-1}} \equiv [\ell_2, x] x^{k2^{n-1}} \bmod \Omega(L).$$

Then $[a, x^4] \equiv [\ell_2, x, x^2] \bmod \Omega(L)$, *i.e.* again viewing $L/\Omega(L)$ as X-module, $[a, x^4] \equiv \ell_2(x-1)^3$ and hence $[a, x^{2^{n-2}}] \equiv \ell_2(x-1)^{2^{n-2}-1} \equiv 0$, as in Case 1. Thus we obtain the same contradiction as before and the proof of Theorem 3(*i*) is complete. □

To prove the second part of Theorem 3, we can now use the first part and a somewhat shorter argument suffices.

Proof of Theorem 3(*ii*). We have a finite 2-group G with an abelian quasinormal subgroup A and we have to show that A^8 is quasinormal in G. As before, we suppose that this is not the case and let G be a counter-example of minimal order. So $G = AX$, where $X = \langle x \rangle$ is cyclic, A^8X is not a subgroup and

$$A \cap X = 1 = X_G = (A^8)_G.$$

But A^4X is a subgroup, by part (*i*). Also $A^G = AA^x$, by Lemma 2.2; and exactly as in part (*i*), we must have

$$L = A \cap A^x = Z = Z(A^G) = A_G \lhd G.$$

Then L has exponent at most 8 and $A = \langle a \rangle L$, for some element a in A. By analogy with part (*i*), we have $|a| = 16$. Thus

$$2 \leqslant |A^G : A| \leqslant 16.$$

If $L^4 = 1$, then $A^4 = \langle a^4 \rangle$ and $\langle a^4 \rangle X$ is a subgroup, by (*i*). Therefore $A^8X = \langle a^8 \rangle X$ is also a subgroup, contradicting our assumption. Thus

$$L^4 \neq 1$$

and L has exponent exactly 8. Let N be a minimal normal subgroup of G contained in L^4 and consider A^4X modulo L^4. This quotient is the product of $\langle a^4 \rangle$ (of order 2 or 4) and X. Modulo N, A^8 is quasinormal in G and so $[a^8, x] \in NX^{2^{n-1}}$, where $|X| = 2^n$. Therefore $[a^8, x^2] = 1$ and hence $a^8 \in Z = L$. Thus $|A : L| = |A^G : A| \leqslant 8$. It follows that

$$[a^8, x] \in NX^{2^{n-1}} \cap Z = N \leqslant L^4.$$

Therefore $A^8 = \langle a^8 \rangle$ is central in G modulo L^4 and so $[a^4, x]$ lies in $A^8L^4X^{2^{n-1}}$ (recalling that A^4 is quasinormal in G). Thus

$$[a^4, x] = a^{8i}\ell^4 x^{2^{n-1}},$$

for some integer i and element $\ell \in L$ (observe that the factor $x^{2^{n-1}}$ is required here, otherwise $[a^8, x] = 1$). Then

$$(2) \qquad [a^8, x] = (a^{8i}\ell^4 x^{2^{n-1}})^{a^4} a^{8i}\ell^4 x^{2^{n-1}} = [a^4, x^{2^{n-1}}],$$

since a^8 and ℓ both commute with $x^{2^{n-1}}$. Also $[a^4, x^2] = [a^4, x, x] = [a^{8i}, x][\ell^4, x]$, and so $[a^4, x^4] = [\ell^4, x][\ell^4, x]^{x^2} = \ell^4(x-1)^3$, viewing L^4 as additive X-module.

Continuing, we see that

$$(3) \qquad [a^4, x^{2^{n-1}}] = \ell^4(x-1)^{2^{n-1}-1} = 0.$$

For, if $|A^G : A| = 2$, then $a^2 \in L$, *i.e.* $[a^2, x^{2^{n-1}}] = 1$ and (3) holds. On the other hand, if $|A^G : A| \geqslant 4$, then L^4 is an $X/X^{2^{n-2}}$-module, and so $\ell^4(x-1)^{2^{n-2}} = 0$. Therefore $\ell^4(x-1)^{2^{n-1}-1} = 0$, and again (3) holds. Finally we see from (2) that (3) implies $[a^8, x] = 1$, a contradiction.

This completes the proof of Theorem 3(*ii*). □

3. Examples

We begin by showing why n in Theorem 1 cannot be twice an odd integer and why p in Theorem 2 has to be odd.

Example 3.1. *There is a finite 2-group G with an abelian quasinormal subgroup A such that A^2 is not quasinormal in G.*

To see this, let B be an elementary abelian 2-group of rank 4 with basis $\{a_1, a_2, a_3, a_4\}$. Let $X = \langle x \rangle$ be a cyclic group of order 8 and form a split extension of B by X as follows:

$$(4) \qquad [a_i, x] = a_{i+1}, \quad i = 1, 2, 3; \quad [a_4, x] = 1.$$

Thus B is an indecomposable X/X^4-module of dimension 4 and we put $M = B \rtimes X$. We claim that M admits an automorphism α defined by

$$a_i^\alpha = a_i, \quad i = 1, 2, 3, 4; \quad x^\alpha = a_1 x^5.$$

For, $(a_1 x^5)^8 = (a_1 x)^8 =$ (in additive notation) $a_1(x-1)^7 = 0$. Also the relations of B and (4) are all preserved by α. Since α is clearly a surjective map, it follows that α is indeed an automorphism of M.

Now $x^{\alpha^2} = (a_1 x^5)^\alpha = a_1(a_1 x^5)^5 = a_1(a_1 x)^5 x^4 = x(a_1 x)^4 x^4 = a_4 x = x^{a_3}$. Thus the action of α^2 on M coincides with conjugation by a_3. Therefore by [9, Theorem 9.7.1(*ii*)], there is an extension G of M by a group of order 2, defined by $G = M\langle a \rangle$, where $a^2 = a_3$ and conjugation by a on M agrees with α.

Let $A = B\langle a \rangle = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle a \rangle \times \langle a_4 \rangle \cong C_2 \times C_2 \times C_4 \times C_2$. We claim that A is quasinormal in G. For, $B \lhd G$, and modulo B, all subgroups of G are quasinormal, since G/B ($\cong C_8 \rtimes C_2$) has a modular subgroup lattice. Therefore A is quasinormal in G. But $A^2 = \langle a_3 \rangle$ *is not quasinormal in* G, because $\langle a_3, X \rangle = (\langle a_3 \rangle \times \times \langle a_4 \rangle) \rtimes X$. □

Our next example shows that the self-duality of the subgroup lattice of a finite abelian group does not lead to a result dual to Theorem 2.

Example 3.2. *For each odd prime p, there is a finite p-group G with an abelian quasinormal subgroup A such that $\Omega(A)$ is not quasinormal in G.*

Thus, consider the abelian group H of order p^4 defined by

$$H = \langle a, b, y \mid a^p = b^{p^2} = 1, \ y^p = b^p, \ [a, b] = [a, y] = [b, y] = 1 \rangle.$$

It is easy to see that H has an automorphism θ of order p defined by

$$\theta : a \mapsto aby^{-1}, \quad b \mapsto b, \quad y \mapsto y.$$

Therefore by [9, Theorem 9.7.1(*ii*)], there is an extension G of H by a group of order p defined by

$$G = \langle a, b, x \mid a^p = b^{p^2} = 1, \ [a, b] = 1, \ x^{p^2} = b^p, \ a^x = abx^{-p}, \ b^x = b \rangle.$$

This group G has order p^5. Let $A = \langle a, b \rangle$. We claim that

(5) *A is quasinormal in G.*

For, certainly $b \in Z(G)$. Therefore to prove (5), we may factor G by $\langle b \rangle$. But the quotient is isomorphic to $C_p \ltimes C_{p^2}$ and so has all its subgroups quasinormal. Thus (5) is true.

However, $\Omega(A) = \langle a \rangle \times \langle b^p \rangle$ and, with $X = \langle x \rangle$, $\Omega(A)X = \langle a \rangle X$ is not a subgroup, otherwise a would normalise X, which is not the case. Therefore $\Omega(A)$ is *not* quasinormal in G and our example is established. □

The above construction can easily be modified to include the case $p = 2$. But by Example 3.1, A^2 is not always quasinormal and so, in the present context, the modified example has no interest. More relevant is the question of whether $\Omega_2(A)$ is always quasinormal in a finite 2-group having A as an abelian quasinormal subgroup. But again this is answered negatively by the following modification of Example 3.2.

EXAMPLE 3.3. *There is a group G of order 2^7 with an abelian quasinormal subgroup A such that $\Omega_2(A)$ is not quasinormal in G.*

To see this, let

$$H = \langle a, b, y \mid a^2 = b^8 = 1, \ y^4 = b^2, \ [a, b] = [a, y] = [b, y] = 1 \rangle.$$

Then H is abelian of order 2^6 and has an automorphism of order 2 defined by

$$a \mapsto aby^{-2}, \quad b \mapsto b, \quad y \mapsto y.$$

By analogy with Example 3.2, we see that H can be extended by a group of order 2 to give the group

$$G = \langle a, b, x \mid a^2 = b^8 = 1, \ x^8 = b^2, \ a^x = abx^{-4}, \ b^x = b \rangle.$$

Let $A = \langle a, b \rangle$. Then we easily see that A is quasinormal in G. But $\Omega_2(A) = \langle a \rangle \times \langle b^2 \rangle$ and $\Omega_2(A)X$ is not a subgroup. □

4. THE INFINITE CASE

Extending our result to include abelian quasinormal subgroups of infinite groups is fairly straightforward. We already know that Theorem 1 is true for finite groups G. Thus suppose that G is any group with an abelian quasinormal subgroup A and n is a positive integer, either odd or divisible by 4. In order to complete the proof of Theorem 1, clearly we may assume that

$$G = AX,$$

where $X = \langle x \rangle$ is cyclic. We must show that $A^n X$ is a subgroup.

If X is infinite and $A \cap X = 1$, then by [10, Lemma 2.1], X normalises A and so X also normalises A^n. Therefore we may assume that

$$|G : A| \text{ is finite}.$$

Let H be any finitely generated subgroup of A and let $K = \langle H, X \rangle$. Then $B = A \cap K$ is quasinormal in K and has finite index in K. Thus B is finitely generated. Also if $B^n X$ is a subgroup for all H, then it follows that $A^n X$ is also a subgroup. In other words, we may assume that A is finitely generated. Thus $|A : A^n|$ is finite and so $|G : A^n|$ is finite. Let $N = (A^n)_G$. So G/N is finite. Now by the finite version of Theorem 1, $A^n XN$ is a subgroup. But $A^n XN = A^n NX = A^n X$ and this establishes the infinite case of Theorem 1. $\square$

Originally we conjectured that when A is a *torsion-free* abelian quasinormal subgroup of a group G, then A^n is also quasinormal in G, for *all* positive integers n. At one point we even had a fallacious proof of this statement. Also it is true when A has very small rank, but it fails in general, as the following example (in which A has rank 5) shows.

EXAMPLE 4.1. *There is a group G with a torsion-free abelian quasinormal subgroup A such that A^2 is not quasinormal in G.*

We begin with an abelian group K presented as follows:

$$K = \langle a_1, \dots, a_5, w \,|\, [a_i, a_j] = [a_i, w] = 1, \quad \text{all } i, j;\ a_5^2 = w^4 \rangle.$$

Thus K is the direct product of a free abelian group of rank 5 and a group of order 2. Then K has an automorphism of order 2 defined by

$$a_i \mapsto a_i,\ i \neq 2;\ a_2 \mapsto a_2 a_5 w^{-2},\ w \mapsto w.$$

Therefore by [9, Theorem 9.7.1(*ii*)], there is an extension H of K by a group of order 2, where H is generated by elements $a_1, \dots, a_5, y$ subject to the relations

$$[a_i, a_j] = 1, \quad \text{all } i, j, \quad [a_i, y] = 1, \quad i \neq 2;\ [a_2, y] = a_5 y^{-4}, \quad a_5^2 = y^8.$$

This group H is nilpotent of class 2, with derived subgroup $\langle a_5 y^{-4} \rangle$ of order 2. We wish to extend H by a cyclic group of order 4 generated by x, with $x^4 = y$. This requires first an extension by a group of order 2, and we considered the most general of these, consistent with obvious restrictions, using the theory of integral representations of cyclic groups of prime order, described for example in [1, §74]. Then we made a second extension by a group of order 2, satisfying the constraints necessary to produce our example. We obtained many solutions, of which the following is one of the simplest.

We claim that H has an automorphism ϕ defined as follows:

$$\phi : a_1 \mapsto a_1 a_3^{-1},\ a_2 \mapsto a_2 a_3 a_4 y,\ a_3 \mapsto a_1^2 a_3^{-1} a_5^{-1},$$
$$a_4 \mapsto a_1^{-1} a_4^{-1},\quad a_5 \mapsto a_5,\quad y \mapsto y.$$

It is easy to see that ϕ is surjective and preserves all the relations of H. Therefore ϕ is an automorphism, as claimed.

Next we claim that

(6) *ϕ^4 coincides with conjugation in H by y.*

For, one checks easily the following:

$$\phi^2 : a_1 \mapsto a_1^{-1} a_5,\quad a_2 \mapsto a_1 a_2 a_5^{-1} y^2,\quad a_3 \mapsto a_3^{-1},$$
$$a_4 \mapsto a_3 a_4,\quad a_5 \mapsto a_5,\quad y \mapsto y.$$

Then we find

$$\phi^4 : a_1 \mapsto a_1,\quad a_2 \mapsto a_2 a_5^{-1} y^4,\quad a_3 \mapsto a_3,$$
$$a_4 \mapsto a_4,\quad a_5 \mapsto a_5,\quad y \mapsto y.$$

Therefore (6) is true and by the now familiar result in [9], we may extend H by a cyclic group of order 4 generated by x, to get

$$G = H\langle x\rangle,$$

where $x^4 = y$ and x acts on H according to the automorphism ϕ.

Let $A = \langle a_i \mid i = 1, \dots, 5\rangle$. Then A is a torsion-free abelian group of rank 5. Also $H = AY$, where $Y = \langle y\rangle \leqslant \langle x\rangle = X$, say. So $G = AX$. We claim that

(7) *A is a quasinormal subgroup of G.*

For, $\langle a_1, a_3, a_4, a_5\rangle^G = \langle a_1, a_3, a_4, a_5\rangle^X \leqslant A$. Therefore we may assume that $a_1 = a_3 = = a_4 = a_5 = 1$ and then G becomes $\langle a_2\rangle X$, where $\langle a_2\rangle = A \cong C_\infty$ and $X \cong C_{32}$. Now $[a_2, x] = y = x^4$ and $[a_2^8, x] = 1$. So we may assume that $a_2^8 = 1$ and then G becomes a modular group of order 2^8 (see Iwasawa's Structure Theorem, for example in [8, Theorem 2.3.1]). Thus (7) holds.

However,

(8) *$A^2 = \langle a_1^2, a_2^2, a_3^2, a_4^2, a_5^2\rangle$ is not quasinormal in G.*

To see this, we observe that A^2X is *not* a subgroup. For,

$$A^2X = \langle a_1^2, a_2^2, a_3^2, a_4^2\rangle X$$

and $\langle a_1^2, a_2^2, a_3^2, a_4^2\rangle \cap X = 1$. But

$$\langle a_2^2\rangle^x = a_2^2 a_3^2 a_4^2 a_5 y^{-2}$$

and $a_5 \notin A^2X$. Therefore $A^2X \neq \langle A^2, X\rangle$, and so (8) follows. This verifies that our example has the required properties. □

To conclude, we construct an example which answers another natural question concerning infinite abelian quasinormal subgroups.

EXAMPLE 4.2. *There is a group G with an infinite abelian quasinormal subgroup A such that the torsion subgroup of A is not quasinormal in G.*

For, let p be an odd prime, $\langle a \rangle \cong C_p$ and $\langle x \rangle \cong C_{p^2}$. We form the split extension $H = \langle a \rangle \ltimes \langle x \rangle$ according to $x^a = x^{1+p}$. Then $\langle a \rangle$ is quasinormal in H. Now form $G = H \times \langle b \rangle$, where $\langle b \rangle \cong C_\infty$. So $A = \langle a, b \rangle = \langle a \rangle \times \langle b \rangle \cong C_p \times C_\infty$ and A is quasinormal in G. But the torsion subgroup of A is $\langle a \rangle$, which is not quasinormal in G. For, the element xb has infinite order and therefore normalises any quasinormal subgroup from which it is disjoint, by [10, Lemma 2.1]. However, xb does *not* normalises $\langle a \rangle$.

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