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Multidimensional Opial inequalities for functions vanishing at an interior point

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Analisi matematica. — *Multidimensional Opial inequalities for functions vanishing at an interior point.* Nota (*) di GEORGE A. ANASTASSIOU, GISÈLE RUIZ GOLDSTEIN e JEROME A. GOLDSTEIN, presentata dal Socio G. Da Prato.

ABSTRACT. — In this paper we generalize Opial inequalities in the multidimensional case over balls. The inequalities carry weights and are proved to be sharp. The functions under consideration vanish at the center of the ball.

KEY WORDS: Integral inequalities; Opial-type multidimensional inequality; Sharp inequality.

RIASSUNTO. — *Disuguaglianze di Opial multidimensionali per funzioni che si annullano in un punto interno*. In questo lavoro si generalizzano alcune disuguaglianze di Opial su palle al caso multidimensionale. Si dimostra che tali disuguaglianze, che contengono pesi, sono ottimali. Le funzioni considerate si annullano al centro della palla.

1. INTRODUCTION

Opial [7] and Olech [6] in 1960 proved the following famous inequality.

THEOREM A. *Let* $c > 0$, *and* $y(x)$ *be real, continuously differentiable on* $[0, c]$, *with* $y(0) = y(c) = 0$. *Then*

$$\int_0^c |y(x)\, y'(x)|\, dx \leq \frac{c}{4} \int_0^c (y'(x))^2\, dx.$$

Equality holds for the function $y(x) = x$ *on* $\left[0, \frac{c}{2}\right]$, *and* $y(x) = c - x$ *on* $\left[\frac{c}{2}, c\right]$.

In 1962 Beesack [2] gave the following improvement.

THEOREM B. *Let* $b > 0$. *If* $y(x)$ *is real, continuously differentiable on* $[0, b]$, *and* $y(0) = 0$, *then*

$$\int_0^b |y(x)\, y'(x)|\, dx \leq \frac{b}{2} \int_0^b (y'(x))^2\, dx.$$

Equality holds only for $y(x) = mx$, *where* m *is a constant.*

Since then many people here worked on this type of inequalities in many directions; for an account see the important monograph of 1995 by Agarwal and Pang [1]. One inspiration for this paper is the interesting article of Troy [8] of 2001. His relevant result follows.

(*) Pervenuta in forma definitiva all'Accademia il 1 settembre 2003.

THEOREM C. *Let $p > -1$. Let $a, b \in \mathbb{R}$ with $0 \leqslant a < b$. If $y(x)$ is continuously differentiable on $[a, b]$, and $y(a) = 0$, then*

$$\int_a^b t^p \,|y(t)\, y'(t)|\, dt \leqslant \frac{1}{2\sqrt{p+1}} \int_a^b (b^{p+1} - at^p)(y'(t))^2\, dt.$$

Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^N$, $N \geqslant 1$. Let $B(0, R) := \{x \in \mathbb{R}^N\colon |x| < R\}$ be the open ball of radius R with center 0 in $\mathbb{R}^N$. Let $S^{N-1} := \{x \in \mathbb{R}^N\colon |x| = 1\}$ be the unit sphere in $\mathbb{R}^N$, centered at zero. Let $\omega_N = \dfrac{N\pi^{N/2}}{(N/2)\,\Gamma(N/2)}$ (see [3, p. 220]) be its surface area.

In this article we estimate the integral

$$I = \int_{B(0,R)} |x|^p\, |u(x)|\, |\nabla u(x)|\, dx,$$

for $p \in \mathbb{R}$ and $u \in C^1(\overline{B(0, R)})$. Using polar coordinates we get

$$I = \int_{S^{N-1}} \left(\int_0^R r^p\, |u(r\omega)|\, |\nabla u(r\omega)|\, r^{N-1}\, dr \right) d\omega,$$

where $x = r\omega$ with $r := |x|$ and $\omega := \dfrac{x}{r}$. For radial (spherically symmetric) functions u, this reduces to

$$I = \omega_N \int_0^R r^{p+N-1}\, |u(r)| \left| \frac{\partial u}{\partial r}(r) \right| dr,$$

where here $|\nabla u| = \left| \dfrac{\partial u}{\partial r} \right|$, with $\dfrac{\partial u(x)}{\partial r} = \nabla u(x) \cdot \dfrac{x}{|x|}$ the radial derivative of u. Spherically symmetric function means that

$$u(x) = u(r\omega) = u(r).$$

In general one has

$$\left| \frac{\partial u(x)}{\partial r} \right| \leqslant |\nabla u(x)|, \qquad \text{for any } u \in C^1(\overline{B(0, R)}).$$

We shall prove that

$$\int_{B(0,R)} |x|^p\, |u(x)|\, |\nabla u(x)|\, dx \leqslant C \int_{B(0,R)} |x|^p\, |\nabla u(x)|^2\, dx$$

for some constant C and functions vanishing at the origin. The idea is to use

$$\int_{B(0,R)} F(x)\, dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega)\, r^{N-1}\, dr \right) d\omega,$$

and to do first a one-dimensional analysis on the inner integral with ω fixed. The interior constraint ($u(0) = 0$) becomes a boundary condition ($F(0) = 0$). In a later paper we shall deal with the boundary condition $F(R) = 0$, which corresponds to the Dirichlet boundary condition $u(x)=0$ for $|x|=R$, *i.e.*, $u=0$ on $\partial B(0,R)$.

2. Results

We present our first basic result.

Theorem 1. *Let* $R>0$, $N\geqslant 1$, $B(0,R)$ *the ball centered at* 0 *of radius R in* $\mathbb{R}^N$. *Let* $p\in\mathbb{R}$ *such that* $p+N\in(0,2)$. *Consider* $u\in C^1(\overline{B(0,R)})$ *such that* $u(0)=0$. *Then one has*

$$\int_{B(0,R)} |x|^p\,|u(x)|\,|\nabla u(x)|\,dx \leqslant \left(\frac{R}{2\sqrt{2-p-N}}\right)\left(\int_{B(0,R)} |x|^p\,|\nabla u(x)|^2\,dx\right), \tag{1}$$

where ∇ *is the gradient operator.*

Remark 1. *Inequality* (1) *is sharp, more precisely it is attained by* $u(x)=|x|$, *for all* $x\in\overline{B(0,R)}$, *when* $p+N=1$.

Proof. Call $r:=|x|$, $0\leqslant r\leqslant R$. Clearly $u(0)=0$. Notice that

$$\frac{\partial u}{\partial r}=1 \quad \text{and} \quad |\nabla u(x)|=1\,.$$

We observe that

$$\text{L.H.S.}(1)=\omega_N\,\frac{R^{p+N+1}}{p+N+1}\,,$$

and

$$\text{R.H.S.}(1)=\omega_N\,\frac{R^{p+N+1}}{2(p+N)\sqrt{2-(p+N)}}\,.$$

Thus

$$\text{L.H.S.}(1)=\text{R.H.S.}(1) \quad \text{iff}$$

$$(p+N)+1=2(p+N)\sqrt{2-(p+N)} \quad \text{iff}$$

(calling $y:=p+N$, $y\in(0,2)$)

$$y+1=2y\sqrt{2-y} \quad \text{iff}$$

$$g(y):=4y^3-7y^2+2y+1=0, \qquad y\in(0,2).$$

See that $g(1)=0$, $g(0)=1$, $g(2)=9$, and

$$g'(y)=12(y-1)\left(y-\frac{1}{6}\right).$$

Thus g has critical numbers 1, $\frac{1}{6}$ with local maximum $g\left(\frac{1}{6}\right)=1.1574078$, and minimum $g(1)=0$. So $y=p+N=1$ is the only optimal value making inequality (1) attained. □

PROOF OF THEOREM 1. The integral in the R.H.S. (1) is finite since $2 > p + N > 0$. We can rewrite inequality (1) by the use of polar coordinates as follows, cf. [3, p. 217]:

$$(2)\quad \int_{S^{N-1}} \left(\int_0^R r^p \, |u(r\omega)| \, |\nabla u(r\omega)| \, r^{N-1} \, dr \right) d\omega \leqslant$$

$$\leqslant \frac{R}{2\sqrt{2-p-N}} \left(\int_{S^{N-1}} \left(\int_0^R r^p \, |\nabla u(r\omega)|^2 \, r^{N-1} \, dr \right) d\omega \right).$$

Here $0 \neq x \in \overline{B(0,R)}$ is written as $x := r\omega$ with $r := |x|$, $0 < r \leqslant R$, and $\omega = \frac{x}{|x|} \in S^{N-1}$. So it is enough to prove that

$$(3)\quad \int_0^R r^{p+N-1} \, |u(r\omega)| \, |\nabla u(r\omega)| \, dr \leqslant \left(\frac{R}{2\sqrt{2-p-N}} \right) \left(\int_0^R r^{p+N-1} \, |\nabla u(r\omega)|^2 \, dr \right).$$

We call

$$z(r) := \int_0^r s^{p+N-1} \, |\nabla u(s\omega)|^2 \, ds, \qquad 0 \leqslant r \leqslant R.$$

Here $z(r) \geqslant 0$ and $z(0) = 0$. Thus

$$z'(r) = r^{p+N-1} \, |\nabla u(r\omega)|^2 \geqslant 0, \qquad 0 < r \leqslant R.$$

Whence

$$(4)\quad r^{\frac{p+N-1}{2}} \, |\nabla u(r\omega)| = (z'(r))^{1/2}, \qquad 0 < r \leqslant R.$$

By the fundamental theorem of calculus we have

$$u(r\omega) = \int_0^r \frac{\partial u(s\omega)}{\partial s} \, ds.$$

Then

$$|u(r\omega)| \leqslant \int_0^r \left| \frac{\partial u}{\partial s}(s\omega) \right| ds = \int_0^r s^{-\left(\frac{p+N-1}{2}\right)} s^{\left(\frac{p+N-1}{2}\right)} \left| \frac{\partial u}{\partial s}(s\omega) \right| ds \leqslant$$

(by the Cauchy-Schwarz inequality)

$$\leqslant \left(\int_0^r s^{-(p+N-1)} \, ds \right)^{1/2} \left(\int_0^r s^{(p+N-1)} \left| \frac{\partial u}{\partial s}(s\omega) \right|^2 ds \right)^{1/2} \leqslant$$

$$\leqslant \left(\frac{r^{2-p-N}}{2-p-N} \right)^{1/2} \left(\int_0^r s^{(p+N-1)} \, |\nabla u(s\omega)|^2 \, ds \right)^{1/2} = \left(\frac{r^{2-p-N}}{2-p-N} \right)^{1/2} (z(r))^{1/2}.$$

So we have proved that

$$|u(r\omega)| \leqslant \left(\frac{r^{2-p-N}}{2-p-N}\right)^{1/2} (z(r))^{1/2}, \qquad \text{all } 0 \leqslant r \leqslant R.$$

And

$$r^{\left(\frac{p+N-1}{2}\right)} |u(r\omega)| \leqslant \frac{\sqrt{r}}{\sqrt{2-p-N}} (z(r))^{1/2}, \qquad \text{all } 0 \leqslant r \leqslant R. \tag{5}$$

Consequently by (4) and (5) we obtain

$$r^{(p+N-1)} |u(r\omega)| |\nabla u(r\omega)| \leqslant \frac{\sqrt{r}}{\sqrt{2-p-N}} (z(r))^{1/2} (z'(r))^{1/2}, \quad \text{all } 0 < r \leqslant R. \tag{6}$$

Next we integrate (6) and use the Cauchy-Schwarz inequality to get

$$\int_0^R r^{p+N-1} |u(r\omega)| |\nabla u(r\omega)| dr \leqslant \frac{1}{\sqrt{2-p-N}} \int_0^R \sqrt{r} (z(r))^{1/2} (z'(r))^{1/2} dr \leqslant$$

$$\leqslant \frac{R}{\sqrt{2}\sqrt{2-p-N}} \left(\int_0^R z(r)\, dz(r) \right)^{1/2} =$$

$$= \frac{R}{2\sqrt{2-p-N}} z(R) = \frac{R}{2\sqrt{2-p-N}} \left(\int_0^R r^{p+N-1} |\nabla u(r\omega)|^2 dr \right),$$

establishing (3). □

A similar result follows.

THEOREM 2. *Let $R > 0$, $N \geqslant 1$, $B(0, R)$ the ball centered at 0 of radius R in $\mathbb{R}^N$. Let $p \in \mathbb{R}$ such that $p + N > 0$. Consider $u \in C^1(\overline{B(0, R)})$ such that $u(0) = 0$. Then one has*

$$\int_{B(0,R)} |x|^p |u(x)| |\nabla u(x)| dx \leqslant \frac{R^{p+N}}{2\sqrt{p+N}} \int_{B(0,R)} |x|^{1-N} |\nabla u(x)|^2 dx. \tag{7}$$

REMARK 2. *Inequality (7) is sharp, namely it is attained by $u(x) = |x|$, for all $x \in \overline{B(0, R)}$, when $p + N = 1$.*

PROOF. Call $r := |x|$, $0 \leqslant r \leqslant R$. Clearly $u(0) = 0$. Note that $\frac{\partial u}{\partial r} = 1$ and $|\nabla u(x)| = 1$. We have that

$$\text{L.H.S.}(7) = \omega_N \frac{R^{p+N+1}}{p+N+1},$$

and

$$\text{R.H.S.}(7) = \omega_N \frac{R^{p+N+1}}{2\sqrt{p+N}}.$$

Thus

$$\begin{aligned} \text{L.H.S.}(7) &= \text{R.H.S.}(7) \quad \text{iff} \\ (p+N)+1 &= 2\sqrt{p+N} \quad \text{iff} \\ y+1 &= 2\sqrt{y}, \quad \text{where } y := p+N > 0, \quad \text{iff} \\ y &= 1. \quad \square \end{aligned}$$

PROOF OF THEOREM 2. Clearly the R.H.S.(7) is finite. As in the proof of Theorem 1 it is enough to prove

$$\int_0^R r^p |u(r\omega)| |\nabla u(r\omega)| r^{N-1} dr \leqslant \frac{R^{p+N}}{2\sqrt{p+N}} \left(\int_0^R |\nabla u(r\omega)|^2 dr \right). \tag{8}$$

Therefore we begin with

$$\int_0^R r^p |u(r\omega)| |\nabla u(r\omega)| r^{N-1} dr = \int_0^R r^{p+N-1} |u(r\omega)| |\nabla u(r\omega)| dr =$$

$$= \int_0^R \left(r^{\left(\frac{p+N}{2}\right)} |\nabla u(r\omega)| \right) \left(r^{\left(\frac{p+N-2}{2}\right)} |u(r\omega)| dr \right) \leqslant$$

(by the Cauchy-Schwarz inequality)

$$\leqslant \left(\int_0^R r^{p+N} |\nabla u(r\omega)|^2 dr \right)^{1/2} \left(\int_0^R r^{(p+N-1)} r^{-1} (u(r\omega))^2 dr \right)^{1/2} =: (*).$$

We have again

$$u(r\omega) = \int_0^r \frac{\partial u}{\partial s}(s\omega)\, ds, \qquad 0 \leqslant r \leqslant R.$$

Therefore

$$|u(r\omega)| \leqslant \int_0^r 1 \cdot \left| \frac{\partial u}{\partial s}(s\omega) \right| ds \leqslant$$

(by the Cauchy-Schwarz inequality)

$$\leqslant \sqrt{r} \left(\int_0^r \left| \frac{\partial u}{\partial s}(s\omega) \right|^2 ds \right)^{1/2} \leqslant \sqrt{r} \left(\int_0^r |\nabla u(s\omega)|^2 ds \right)^{1/2}.$$

That is

$$(u(r\omega))^2 \leqslant r \int_0^r |\nabla u(s\omega)|^2\, ds. \tag{9}$$

Hence by (9) we get

$$(*) \leqslant \left(\int_0^R r^{p+N} |\nabla u(r\omega)|^2\, dr\right)^{1/2} \cdot \left(\int_0^R r^{(p+N-1)} \left(\int_0^r |\nabla u(s\omega)|^2\, ds\right) dr\right)^{1/2} =$$

$$= \left(\int_0^R r^{p+N} |\nabla u(r\omega)|^2\, dr\right)^{1/2} \cdot$$

$$\cdot \left[\frac{R^{p+N}}{p+N}\left(\int_0^R |\nabla u(r\omega)|^2\, dr\right) - \frac{1}{p+N}\int_0^R r^{p+N} |\nabla u(r\omega)|^2\, dr\right]^{1/2} =: (*\ *).$$

If $A \geqslant 0$, $B \geqslant 0$ and $\varepsilon > 0$ we have

$$(AB)^{1/2} \leqslant \frac{\varepsilon}{2} A + \frac{1}{2\varepsilon} B. \tag{10}$$

Hence

$$(*\ *) \leqslant \frac{\varepsilon}{2}\left(\int_0^R r^{p+N} |\nabla u(r\omega)|^2\, dr\right) +$$

$$+ \frac{1}{2\varepsilon}\left[\frac{R^{p+N}}{p+N}\left(\int_0^R |\nabla u(r\omega)|^2\, dr\right) - \frac{1}{p+N}\left(\int_0^R r^{p+N} |\nabla u(r\omega)|^2\, dr\right)\right] =$$

$$= \frac{1}{2}\left(\varepsilon - \frac{1}{\varepsilon(p+N)}\right)\left(\int_0^R r^{p+N} |\nabla u(r\omega)|^2\, dr\right) + \frac{R^{p+N}}{2\varepsilon(p+N)}\left(\int_0^R |\nabla u(r\omega)|^2\, dr\right) =$$

$$\left(\text{choosing } \varepsilon = \frac{1}{\sqrt{p+N}}\right)$$

$$= \frac{R^{p+N}}{2\sqrt{p+N}}\left(\int_0^R |\nabla u(r\omega)|^2\, dr\right).$$

We have established (8). □

Finally we present a generalization and extension of Theorem 1.

Theorem 3. *Let* $R > 0$, $N \geqslant 1$, $\varrho > 1$, $\alpha, \beta > 0$, $\varrho \geqslant \alpha + \beta$ *and* $p \in \mathbb{R}$ *such that* $0 < p + N < \varrho$.

Consider $u \in C^1(\overline{B(0,R)})$ *such that* $u(0)=0$. *Then one has*

$$\int_{B(0,R)} |x|^{\left[(p+N-1)\left(\frac{\alpha+\beta}{\varrho}\right)+(1-N)\right]} |u(x)|^{\beta} |\nabla u(x)|^{\alpha} dx \leqslant \tag{11}$$

$$\leqslant L\omega_N^{\left(\frac{\varrho-\alpha-\beta}{\varrho}\right)} \left(\int_{B(0,R)} |x|^p |\nabla u(x)|^{\varrho} dx \right)^{\left(\frac{\alpha+\beta}{\varrho}\right)},$$

where

$$L := \left(\frac{\varrho-1}{\varrho-p-N}\right)^{\beta\left(\frac{\varrho-1}{\varrho}\right)} \left(\frac{\varrho-a}{\beta(\varrho-1)+(\varrho-\alpha)}\right)^{\left(\frac{\varrho-\alpha}{\varrho}\right)} \left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{\varrho}} R^{\left(\frac{\beta(\varrho-1)+(\varrho-\alpha)}{\varrho}\right)}. \tag{12}$$

REMARK 3. *Inequality* (11) *is sharp, namely it is attained by* $u(x)=|x|$, *for all* $x \in \overline{B(0,R)}$, *when* $p+N=1$, $\varrho=\alpha+\beta$ *and* $\alpha^{\left(\frac{\alpha}{\alpha+\beta}\right)} = \dfrac{\alpha+\beta}{1+\beta}$.

PROOF. Call $r := |x|$, $0 \leqslant r \leqslant R$. Clearly $u(0)=0$. Note that $\dfrac{\partial u}{\partial r}=1$ and $|\nabla u(x)|=1$. We have

$$\text{L.H.S.}(11) = \omega_N \frac{R^{\beta+1}}{\beta+1},$$

and

$$\text{R.H.S.}(11) = \omega_N \frac{\alpha^{\alpha/\varrho}}{\varrho} R^{\beta+1}.$$

It is obvious now that (11) holds as equality. □

PROOF OF THEOREM 3. The integral in the R.H.S.(11) is finite by $p+N>0$. Here for $0 \neq x \in \overline{B(0,R)}$ we set $\omega := \dfrac{x}{|x|} = \dfrac{x}{r}$, where $r := |x|$, $0 < r \leqslant R$. That is $x = r\omega$. By the fundamental theorem of calculus we have

$$u(r\omega) = \int_0^r \frac{\partial u}{\partial s}(s\omega)\, ds, \qquad 0 \leqslant r \leqslant R.$$

We observe that

$$|u(r\omega)| \leqslant \int_0^r \left| \frac{\partial u}{\partial s}(s\omega) \right| ds = \int_0^r s^{-\left(\frac{p+N-1}{\varrho}\right)} s^{\left(\frac{p+N-1}{\varrho}\right)} \left| \frac{\partial u}{\partial s}(s\omega) \right| ds \leqslant$$

$$\left(\text{using Hölder's inequality with indices } \varrho \text{ and } \frac{\varrho}{\varrho-1}\right)$$

$$\leqslant \left(\int_0^r s^{-\frac{(p+N-1)}{\varrho-1}} ds \right)^{\left(\frac{\varrho-1}{\varrho}\right)} \left(\int_0^r s^{(p+N-1)} \left| \frac{\partial u}{\partial s}(s\omega) \right|^{\varrho} ds \right)^{1/\varrho} \leqslant$$

$$\leqslant \left(\frac{\varrho-1}{\varrho-p-N}\right)^{\left(\frac{\varrho-1}{\varrho}\right)} r^{\frac{(\varrho-p-N)}{\varrho}} \left(\int_0^r s^{(p+N-1)} |\nabla u(s\omega)|^{\varrho} ds \right)^{1/\varrho}.$$

I.e.

$$|u(r\omega)| \leqslant \left(\frac{\varrho - 1}{\varrho - p - N}\right)^{\left(\frac{\varrho-1}{\varrho}\right)} r^{\left(\frac{\varrho-p-N}{\varrho}\right)} z(r)^{1/\varrho}, \qquad 0 \leqslant r \leqslant R.$$

Here

$$+\infty > z(r) := \int_0^r s^{(p+N-1)} |\nabla u(s\omega)|^{\varrho} \, ds \geqslant 0,$$

with $z(0) = 0$. We have

$$z'(r) = r^{p+N-1} |\nabla u(r\omega)|^{\varrho} \geqslant 0, \qquad 0 < r \leqslant R,$$

and

$$r^{\left(\frac{p+N-1}{\varrho}\right)\alpha} |\nabla u(r\omega)|^{\alpha} = (z'(r))^{\alpha/\varrho}, \qquad 0 < r \leqslant R. \tag{13}$$

We see that

$$|u(r\omega)|^{\beta} \leqslant \left(\frac{\varrho - 1}{\varrho - p - N}\right)^{\beta\left(\frac{\varrho-1}{\varrho}\right)} r^{\beta\frac{(\varrho-p-N)}{\varrho}} (z(r))^{\beta/\varrho},$$

and

$$r^{\left(\frac{p+N-1}{\varrho}\right)\beta} |u(r\omega)|^{\beta} \leqslant \left(\frac{\varrho - 1}{\varrho - p - N}\right)^{\beta\left(\frac{\varrho-1}{\varrho}\right)} \cdot r^{\beta\left(\frac{\varrho-1}{\varrho}\right)} z(r)^{\beta/\varrho}, \qquad 0 \leqslant r \leqslant R. \tag{14}$$

By multiplying (13) and (14) we set

$$\begin{aligned} r^{(p+N-1)\left(\frac{\alpha+\beta}{\varrho}\right)} |u(r\omega)|^{\beta} |\nabla u(r\omega)|^{\alpha} \leqslant \\ \leqslant \left(\frac{\varrho - 1}{\varrho - p - N}\right)^{\beta\left(\frac{\varrho-1}{\varrho}\right)} r^{\beta\left(\frac{\varrho-1}{\varrho}\right)} (z(r))^{\beta/\varrho} (z'(r))^{\alpha/\varrho}, \qquad 0 < r \leqslant R. \end{aligned} \tag{15}$$

Integrate (15) and use Hölder's inequality with indices $\dfrac{\varrho}{\alpha}$, $\dfrac{\varrho}{\varrho - \alpha}$ to find

$$\int_0^R r^{(p+N-1)\left(\frac{\alpha+\beta}{\varrho}\right)} |u(r\omega)|^{\beta} |\nabla u(r\omega)|^{\alpha} \, dr \leqslant$$

$$\leqslant \left(\frac{\varrho - 1}{\varrho - p - N}\right)^{\beta\left(\frac{\varrho-1}{\varrho}\right)} \int_0^R r^{\beta\left(\frac{\varrho-1}{\varrho}\right)} (z(r))^{\beta/\varrho} (z'(r))^{\alpha/\varrho} \, dr \leqslant$$

$$\leqslant \left(\frac{\varrho - 1}{\varrho - p - N}\right)^{\beta\left(\frac{\varrho-1}{\varrho}\right)} \left(\int_0^R r^{\beta\left(\frac{\varrho-1}{\varrho-\alpha}\right)} dr\right)^{\left(\frac{\varrho-\alpha}{\varrho}\right)} \cdot \left(\int_0^R (z(r))^{\beta/\alpha} z'(r) \, dr\right)^{\alpha/\varrho} \overset{(12)}{=}$$

$$\overset{(12)}{=} L \cdot z(R)^{\frac{\alpha+\beta}{\varrho}} = L \left(\int_0^R r^{p+N-1} |\nabla u(r\omega)|^{\varrho} \, dr\right)^{\left(\frac{\alpha+\beta}{\varrho}\right)}.$$

We have established that

$$\text{(16)}\qquad \left(\int_0^R r^{\left[(p+N-1)\left(\frac{\alpha+\beta}{\varrho}\right)+(1-N)\right]}|u(r\omega)|^{\beta}|\nabla u(r\omega)|^{\alpha} r^{N-1}\,dr\right)\leqslant$$

$$\leqslant L\left(\int_0^R r^p|\nabla u(r\omega)|^{\varrho} r^{N-1}\,dr\right)^{\left(\frac{\alpha+\beta}{\varrho}\right)}.$$

Integrating (16) over s^{N-1} we obtain

$$\text{(17)}\qquad \int_{S^{N-1}}\left(\int_0^R r^{\left[(p+N-1)\left(\frac{\alpha+\beta}{\varrho}\right)+(1-N)\right]}|u(r\omega)|^{\beta}|\nabla u(r\omega)|^{\alpha} r^{N-1}\,dr\right)d\omega\leqslant$$

$$\leqslant L\cdot\int_{S^{N-1}}\left(\int_0^R r^p|\nabla u(r\omega)|^{\varrho} r^{N-1}\,dr\right)^{\left(\frac{\alpha+\beta}{\varrho}\right)} d\omega=:(***).$$

If $\varrho>\alpha+\beta$ then apply again Hölder's inequality with indices $\left(\dfrac{\varrho}{\varrho-\alpha-\beta}\right)$ and $\left(\dfrac{\varrho}{\alpha+\beta}\right)$ to find

$$\text{(18)}\qquad (***)\leqslant L\left(\int_{S^{N-1}} 1^{\frac{\varrho}{\varrho-\alpha-\beta}}\,d\omega\right)^{\left(\frac{\varrho-\alpha-\beta}{\varrho}\right)}\cdot\left(\int_{S^{N-1}}\left(\int_0^R r^p|\nabla u(r\omega)|^{\varrho} r^{N-1}\,dr\right)d\omega\right)^{\left(\frac{\alpha+\beta}{\varrho}\right)}=$$

$$=L\omega_N^{\left(\frac{\varrho-\alpha-\beta}{\varrho}\right)}\cdot\left(\int_{S^{N-1}}\left(\int_0^R r^p|\nabla u(r\omega)|^{\varrho} r^{N-1}\,dr\right)d\omega\right)^{\left(\frac{\alpha+\beta}{\varrho}\right)}.$$

From (17) and (18) we conclude (11). □

The work of Nečaev [5] in 1973 is related to our work; see also [1, p. 275].

THEOREM [5]. *Let* $u\in C^1(\overline{B(0,R)})$ *be such that* $u(0)=0$, *and* $N+p<2$, $N\geqslant 1$. *Then one has*

$$\int_{B(0,R)}|x|^{1-N}|u(x)||\nabla u(x)|\,dx\leqslant\frac{R^{2-N-p}}{2(2-N-p)}\int_{B(0,R)}|x|^p|\nabla u(x)|^2\,dx,$$

with equality holding when

$$u(x)=c|x|^{2-N-p}$$

for a real constant c.

This result can be compared to our Theorem 2 if and only if $p=1-N$, in which case the conclusions are identical, with the common constant being $R/2$.

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