Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei Matematica e Applicazioni

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **15** (2004), n.1, p. 59–68.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2004_9_15_1_59_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2004.

Rend. Mat. Acc. Lincei
s. 9, v. 15:59-68 (2004)

Fisica matematica. — *The non-linear macroscopic model of Relativistic Extended Thermodynamics of an ultra-relativistic gas.* Nota (*) di FRANCESCO BORGHERO e SEBASTIANO PENNISI, presentata dal Socio T. Ruggeri.

ABSTRACT. — The model for an ultra-relativistic gas is here considered in the framework of Extended Thermodynamics. The closure, satisfying exactly the principles of relativity and of entropy, is obtained by following the approach «at a macroscopic level». Our results are compared with the ones of the kinetic approach.

KEY WORDS: Extended Thermodynamics; Fluid Models; Ultra-relativistic Gas; Entropy Principle.

RIASSUNTO. — *Il modello macroscopico non lineare della Termodinamica Estesa Relativistica di un gas ultra-relativistico.* Il modello di un gas-relativistico viene qui preso in considerazione, nell'ambito della Termodinamica Estesa. Seguendo l'approccio a «livello macroscopico», viene trovata la chiusura soddisfacente esattamente il principio d'entropia e quello di relatività. I nostri risultati vengono confrontati con quelli dell'approccio cinetico.

1. INTRODUCTION

In Relativistic Extended Thermodynamics the following balance equations are proposed by Liu, Müller and Ruggeri [1, 2]

$$\partial_\alpha V^\alpha = 0, \qquad \partial_\alpha T^{\alpha\beta} = 0, \qquad \partial_\alpha A^{\alpha\beta\gamma} = I^{\beta\gamma}. \tag{1}$$

V^α (particle number – particle flux vector) and $T^{\alpha\beta}$ (stress – energy – momentum tensor) are assumed as independent variables, while $I^{\beta\gamma}$ and $A^{\alpha\beta\gamma}$ are constitutive functions. In the kinetic approach to this theory there is only an unknown, *i.e.* the distribution function $f(x^\alpha, p^\alpha)$, after that V^α, $T^{\alpha\beta}$ and $A^{\alpha\beta\gamma}$ are defined as its moments

$$V^\alpha = \int f p^\alpha \, dP, \qquad T^{\alpha\beta} = \int f p^\alpha p^\beta \, dP, \qquad A^{\alpha\beta\gamma} = \int f p^\alpha p^\beta p^\gamma \, dP, \tag{2}$$

where p^α is the 4-momentum of the particle so that we have $p^\alpha p_\alpha = -(m_0)^2$ with m_0 the rest particle mass (the light velocity has been taken as unitary); moreover, $dP = \sqrt{-g}\,\dfrac{dp^1\,dp^2\,dp^3}{p^0}$ is the invariant element of momentum space.

The eqs. (2) show that $T^{\alpha\beta}$ and $A^{\alpha\beta\gamma}$ are symmetric with respect to all pairs of indices and that the following «trace condition» hold

$$A^{\alpha\beta}_{\beta} = -(m_0)^2 V^\alpha. \tag{3}$$

Consequently, we have that also $I^{\beta\gamma}$ is symmetric and traceless, as can be easily seen from eq. (1).

An important physical situation occurs when the quantity $\gamma = m_0/(kT)$ goes to zero, where k is the Boltzmann constant and T the absolute temperature.

(*) Pervenuta in forma definitiva all'Accademia il 23 ottobre 2003.

In this case we are dealing with an ultra-relativistic gas, even if this situation can be achieved in two ways

1. The absolute temperature T goes to infinity,
2. the rest particle mass m_0 is zero.

The first of these cases, can be obtained by taking the limit of the general one with $T \to \infty$, and interesting results to this regard are exposed in [2-4]. The second one needs an independent treatment. In fact, the eqs. (2) show that the following «trace conditions» substitute the eq. (3)

$$T^{\beta}_{\beta} = 0, \qquad A^{\alpha\beta}_{\beta} = 0. \tag{4}$$

In this way V^α and $T^{\alpha\beta}$ have only 13 independent components, despite the fact that the eqs. (1) have still 14 independent components. In [5] this problem has been overcame by taking as independent variables V^α, $T^{\alpha\beta}$ and one of the components of $A^{\alpha\beta\gamma}$ while $I^{\beta\gamma}$ and the remaining components of $A^{\alpha\beta\gamma}$ are constitutive functions; after that, the entropy principle and the relativity principle have been imposed approximately with respect to thermodynamical equilibrium. In [6] has been noted that the conditions, corresponding to a given order, are independent from those of the other orders; after that an exact solution has been found, but only up to second order. In the present paper these conditions are imposed up to whatever order.

So, let us proceed in this direction; it is well known that the entropy principle for the balance equations (1) amounts to assuming the existence of the Lagrange multipliers ξ, λ_β, $\Sigma_{\beta\gamma}$ and of a 4-vectorial function h'^α (related to the entropy – entropy flux) such that [1, 2]:

$$V^\alpha = \frac{\partial h'^\alpha}{\partial \xi}, \qquad T^{\alpha\beta} = \frac{\partial h'^\alpha}{\partial \lambda_\beta}, \qquad A^{\alpha\beta\gamma} = \frac{\partial h'^\alpha}{\partial \Sigma_{\mu\nu}}\left(g^\beta_\mu g^\gamma_\nu - \frac{1}{4} g_{\mu\nu} g^{\beta\gamma}\right), \tag{5}$$

where the Lagrange multipliers have been taken as independent variables and it has been taken into account that $\Sigma_{\beta\gamma}$ is symmetric and traceless.

At this point we see that only the representation of h'^α is needed; in fact, from eqs. (5) one obtains those of V^α, $T^{\alpha\beta}$, $A^{\alpha\beta\gamma}$. The symmetry conditions on $T^{\alpha\beta}$ and $A^{\alpha\beta\gamma}$, and the trace condition (4), give restrictions on the expression of h'^α.

Let us start to solve these conditions; to this regard it is useful to remind that in [7, 8] it was shown that the symmetry condition on $T^{\alpha\beta}$ amounts to assuming the existence of the scalar function ϕ such that

$$h'^\alpha = \frac{\partial \phi}{\partial \lambda_\alpha}, \tag{6}$$

so that only the representation of ϕ, satisfying all the other conditions, is needed. At this stage there are three ways in which to proceed:

1) *Approach at a macroscopic level*: the kinetic approach is used only to obtain the above mentioned information on the form of eqs. (1) and (4); after that the most general expression ϕ is searched such that $A^{\alpha\beta\gamma}$ is symmetric and the trace conditions are satisfied.

2) *Approach at a kinetic level*: it is always kept into account that the fields in the equation (1) are defined from (2) in terms of the distribution function f, which is selected according to the entropy principle [3]. The eqs. (2) show that, with this approach, the symmetry conditions are automatically satisfied.

3) *Approach with the Maximum Entropy principle*: Another approach followed in literature imposes the Maximum Entropy principle [9]; it isn't necessary to dwell upon it because, in [3], Boillat and Ruggeri have proved that it is equivalent to their approach at a kinetic level.

Here we follow the first of these approaches; in this way in Section 2, by imposing only the symmetry condition on $A^{\alpha\beta\gamma}$ and the trace conditions, the following expression for ϕ will be obtained

$$\phi = \sum_{n=0}^{\infty} \frac{1}{n!} B_c^{\nu_1\sigma_1\dots\nu_n\sigma_n} \Sigma_{\nu_1\sigma_1}\dots\Sigma_{\nu_n\sigma_n}, \tag{7}$$

which is expressed in terms of the tensors

$$B_c^{\nu_1\sigma_1\dots\nu_n\sigma_n} = \sum_{S=0}^{n} \binom{2n}{2S} \frac{\gamma^{-(2n+2)}}{2S+1} A_{2n}(\xi)\, h^{(\nu_1\sigma_1}\dots h^{\nu_S\sigma_S} U^{\nu_{S+1}} U^{\sigma_{S+1}}\dots U^{\nu_n} U^{\sigma_n)}, \tag{8}$$

where round brackets enclosing some indices denote symmetrization over those indices and the following definitions are used

$$U^\alpha = (-\lambda^\mu \lambda_\mu)^{-1/2} \lambda^\alpha, \qquad h^{\alpha\beta} = g^{\alpha\beta} + U^\alpha U^\beta, \qquad \gamma = \sqrt{-\lambda^\alpha \lambda_\alpha}, \tag{9}$$

and $A_{2n}(\xi)$ are arbitrary single-variable functions.

In Section 4 it is shown that the exact particular solution, which has been found in [6], is a particular case of the present one, when $A_{2h}(\xi) = 0$ for every $h \geqslant 3$. In Section 5, the result of the present work is compared with the corresponding one of the approach at a kinetic level, investigated to a greater extent (see [3, 4, 10, 11]). There it is obtained that

$$\phi = \int F(\xi + \lambda_\alpha p^\alpha + \Sigma_{\alpha\beta} p^\alpha p^\beta)\, dP, \tag{10}$$

where the single-variable function $F(X)$ can be obtained from the distribution function at equilibrium, through

$$F''(X) = f_{eq}. \tag{11}$$

In particular, Jüttner's expression [12, 13] for f_{eq} can be used; it reads

$$f_{eq} = \frac{\omega}{h^3} \left(e^{X/k} \pm 1\right)^{-1}, \tag{12}$$

where the upper and the lower signs refer to Fermions and Bosons respectively, k is the Boltzmann constant, h is the Planck constant and $\omega = 2s + 1$ for particles with spin $s = h/2$.

In Section 5, the expansion of eq. (10) is performed around the state with $\Sigma_{\alpha\beta} = 0$

and we find again the result (7) and (8), but $A_{2n}(\xi)$ are no more arbitrary function; on the contrary, they are given by

$$A_{2n}(\xi) = 4\pi \int_0^{\infty} F^{(n)}(\xi + \sigma)\, \sigma^{2n+1}\, d\sigma. \tag{13}$$

We note that from eq. (13), with two integrations by parts, it follows

$$\begin{aligned} A'_{2n}(\xi) &= 2n(2n+1)\, A_{2n-2}(\xi) + \\ &+4\pi \lim_{\sigma \to +\infty} \left[F^{(n)}(\xi+\sigma)\, \sigma^{2n+1} - (2n+1)\, F^{(n-1)}(\xi+\sigma)\, \sigma^{2n} \right] = 2n(2n+1)\, A_{2n-2}(\xi), \end{aligned} \tag{14}$$

where we have taken into account eqs. (11) and (12) when calculating the limit in eq. (14).

If we leave at this stage the kinetic approach, *i.e.*, if we don't consider any more the eqs. (11), (12) and (13) but we still keep the recurrence formula (14) (which is one of their consequences), we obtain the same result (24) of the approach at a macroscopic level, but only if it is constrained by the further condition which will be considered in Section 3.

2. The symmetry condition on $A^{\alpha\beta\gamma}$ and the trace conditions

Let us consider the tensor

$$B^{\alpha_1 \cdots \alpha_p} = \sum_{S=0}^{[p/2]} \binom{p}{2S} \frac{1}{2S+1}\, g_{p,2S}(\xi, \gamma)\, h^{(\alpha_1\alpha_2} \cdots h^{\alpha_{2S-1}\alpha_{2S}} U^{\alpha_{2S+1}} \cdots U^{\alpha_p)}, \tag{15}$$

where the definitions (9) are used. In [14, Appendix A], it has been proved that the tensor $C^{\alpha_1 \cdots \alpha_{p+1}} = \dfrac{\partial B^{\alpha_1 \cdots \alpha_p}}{\partial \lambda_{\alpha_{p+1}}}$ is symmetric iff $g_{p,2S}$ satisfies the following recurrence formula

$$g_{p,2S-2} = \frac{-1}{2S+1} \left[\gamma \frac{\partial g_{p,2S}}{\partial \gamma} + (p - 2S + 1)\, g_{p,2S} \right]. \tag{16}$$

Moreover, it has been obtained that

$$\begin{aligned} \frac{\partial B^{\alpha_1 \cdots \alpha_p}}{\partial \lambda_{\alpha_{p+1}}} &= \\ &= \sum_{S=0}^{[(p+1)/2]} \binom{p+1}{2S} \frac{1}{2S+1}\, G_{p+1,2S}(\xi, \gamma)\, h^{(\alpha_1\alpha_2} \cdots h^{\alpha_{2S-1}\alpha_{2S}} U^{\alpha_{2S+1}} \cdots U^{\alpha_{p+1})}, \end{aligned} \tag{17}$$

with $G_{p+1,2S}$ satisfying the eq. (16) and its value with the greatest value of S is

$$G_{p+1,p} = -\frac{\partial}{\partial \gamma} g_{p,p} \qquad \text{if } p \text{ is even}$$

$$G_{p+1,p+1} = \frac{p+2}{\gamma} g_{p,p-1} \qquad \text{if } p \text{ is odd}.$$

We have also that $D^{\alpha_1 \dots \alpha_{p+2}} = \dfrac{\partial C^{\alpha_1 \dots \alpha_{p+1}}}{\partial \lambda_{\alpha_{p+2}}}$ is symmetric because $G_{p+1,\,2S}$ satisfies the eq. (16).

Similarly, with the same passages of [14, Appendix B] we obtain that the relation

$$B^{\mu\nu\nu_1\sigma_1 \dots \nu_N\sigma_N} g_{\mu\nu} = 0\,, \tag{18}$$

holds iff $g_{p,\,2S}$ satisfies the equation

$$g_{p,\,2S} - g_{p,\,2S-2} = 0 \quad \text{for} \quad S = 1, \dots, [p/2]. \tag{19}$$

Substitution of $g_{p,\,2S-2}$ from eq. (19) allows us to rewrite eq. (16) as

$$\gamma \frac{\partial g_{p,\,2S}}{\partial \gamma} + (p+2)\, g_{p,\,2S} = 0\,,$$

which can be integrated and gives

$$g_{p,\,2S} = \gamma^{-(p+2)} A_p(\xi) \tag{20}$$

where $A_p(\xi)$ are constants, with respect to γ, arising from integration. They don't depend on S for eq. (19).

The derivative of eq. (18) with respect to λ_α is

$$C^{\alpha\mu\nu\nu_1\sigma_1 \dots \nu_N\sigma_N} g_{\mu\nu} = 0\,. \tag{21}$$

Similarly, the derivative of this equation with respect to λ_β is

$$D^{\alpha\beta\mu\nu\nu_1\sigma_1 \dots \nu_N\sigma_N} g_{\mu\nu} = 0\,. \tag{22}$$

If the above conditions are satisfied, by eqs. (6), $(5)_{2,3}$ and (7) we have that

$$h'^{\alpha} = \sum_{n=0}^{\infty} \frac{1}{n!} C^{\alpha\nu_1\sigma_1 \dots \nu_n\sigma_n} \Sigma_{\nu_1\sigma_1} \dots \Sigma_{\nu_n\sigma_n}\,,$$

$$T^{\alpha\beta} = \sum_{n=0}^{\infty} \frac{1}{n!} D^{\alpha\beta\nu_1\sigma_1 \dots \nu_n\sigma_n} \Sigma_{\nu_1\sigma_1} \dots \Sigma_{\nu_n\sigma_n}\,,$$

$$\begin{aligned} A^{\alpha\beta\gamma} = \frac{\partial h'^{\alpha}}{\partial \Sigma_{\mu\nu}} \left(g^{\beta}_{\mu} g^{\gamma}_{\nu} - \frac{1}{4} g_{\mu\nu} g^{\beta\gamma} \right) &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} C^{\alpha\beta\gamma\nu_1\sigma_1 \dots \nu_{n-1}\sigma_{n-1}} \Sigma_{\nu_1\sigma_1} \dots \Sigma_{\nu_{n-1}\sigma_{n-1}} + \\ &\quad - \frac{1}{4} g^{\beta\gamma} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} C^{\alpha\mu\nu\nu_1\sigma_1 \dots \nu_{n-1}\sigma_{n-1}} g_{\mu\nu} \Sigma_{\nu_1\sigma_1} \dots \Sigma_{\nu_{n-1}\sigma_{n-1}} = \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} C^{\alpha\beta\gamma\nu_1\sigma_1 \dots \nu_{n-1}\sigma_{n-1}} \Sigma_{\nu_1\sigma_1} \dots \Sigma_{\nu_{n-1}\sigma_{n-1}}\,, \end{aligned}$$

where eqs. $(4)_2$ and (20) have been used. From eq. (22) we see also that the above expression of $T^{\alpha\beta}$ satisfies the trace condition $(4)_1$. Moreover, the symmetry conditions on $T^{\alpha\beta}$ and $A^{\alpha\beta\gamma}$ hold. By using eqs. (15) and (20), it results the expression (8).

3. A further condition due to the presence of the electromagnetic field

For the sake of completeness, we recall that in literature [15, 16], a further condition has been found for the case of a charged gas and when the electromagnetic field acts as an external body force. For our case it reads

$$V_{[\alpha}\lambda_{\beta]} + 2T^{\gamma}_{[\alpha}\Sigma_{\beta]\gamma} = 0. \tag{23}$$

Regarding this condition, we have to say that it has been studied in [17] for the non ultra-relativistic case, leading to many identities and to some equations which determine the functions A_4, A_6, A_8 as polynomials in the variable ξ, except for a corresponding number of constants; here the same result is obtained, for every one of the functions $A_{2h}(\xi)$, *i.e.*

(24) $A_{2h}(\xi) = (2h+1)!\,B_{2h}(\xi)$, where the B_{2h} satisfy the recurrence formula

$$B_4 = C_4, \quad B_{2h+2}(\xi) = \int B_{2h}(\xi)\,d\xi + C_{2h+2},$$

for $h \geqslant 2$ and with C_4, C_{2h+2} constants arising from integration.

In order to prove this result, it is useful to note that from eq. (20) it follows

$$G_{p+1,\,p} = \gamma^{-(p+3)} A^1_{p+1}(\xi)$$

with $A^1_{p+1} = (p+2)\,A_p(\xi)$, and this is true in both cases, p even or p odd. Moreover, also $C^{\alpha_1\ldots\alpha_{p+1}}$ satisfies the conditions to have a symmetric derivative with respect to λ_α and a zero trace; it follows that $G_{p+1,\,2S}$ doesn't depend on S. Therefore,

$$G_{p+1,\,2S} = \gamma^{-(p+3)} A^1_{p+1}(\xi). \tag{25}$$

Similarly, we have

$$\frac{\partial^2 B^{\alpha_1\ldots\alpha_p}}{\partial\lambda_{\alpha_{p+1}}\partial\lambda_{\alpha_{p+2}}} = \tag{26}$$

$$= \sum_{S=0}^{[(p+2)/2]} \binom{p+2}{2S} \frac{1}{2S+1}\,\theta_{p+2,\,2S}\, h^{(\alpha_1\alpha_2}\ldots h^{\alpha_{2S-1}\alpha_{2S}} U^{\alpha_{2S+1}}\ldots U^{\alpha_p}U^{\alpha_{p+1}}U^{\alpha_{p+2})},$$

with

$$\theta_{p+2,2S} = \gamma^{-(p+4)}A^2_{p+2}(\xi) = \gamma^{-(p+4)}(p+3)\,A^1_{p+1}(\xi) = \gamma^{-(p+4)}(p+3)(p+2)\,A_p(\xi).$$

Let us consider now eq. (23). We see that by use of eqs. (6) and $(5)_{1,2}$, it can be written as

$$\frac{\partial^2\phi}{\partial\xi\partial\lambda_{[\alpha}}\lambda_{\beta]} + 2\,\frac{\partial^2\phi}{\partial\lambda_\gamma\partial\lambda_{[\alpha}}\Sigma_{\beta]\gamma} = 0. \tag{27}$$

Now the following identity holds, as a consequence of only the representation theorems,

$$\frac{\partial^2\phi}{\partial\xi\partial\lambda_{[\alpha}}\lambda_{\beta]} + 2\,\frac{\partial^2\phi}{\partial\xi\partial\Sigma_{\gamma[\alpha}}\Sigma_{\beta]\gamma} = 0. \tag{28}$$

It is simply a consequence of the fact that ϕ depends on ξ, λ_α, $\Sigma_{\gamma\alpha}$ through ξ, $G_0 = \lambda_\alpha\lambda^\alpha$, $G_1 = \lambda_\gamma\Sigma^{\gamma\alpha}\lambda_\alpha$, $G_2 = \lambda_\gamma\Sigma^{\gamma\beta}\Sigma_{\beta\alpha}\lambda^\alpha$, $G_3 = \lambda_\gamma\Sigma^{\gamma\beta}\Sigma_{\beta\delta}\Sigma^{\delta\alpha}\lambda_\alpha$, $Q_2 = \Sigma^{\gamma\beta}\Sigma_{\beta\gamma}$, $Q_3 = \Sigma^{\gamma\beta}\Sigma_{\beta\delta}\Sigma^{\delta}_{\gamma}$, $Q_4 = \Sigma^{\gamma\beta}\Sigma_{\beta\delta}\Sigma^{\delta\varepsilon}\Sigma_{\varepsilon\gamma}$.

The identity (28) allows us to write the condition (27) as

$$\left(\frac{\partial^2 \phi}{\partial \lambda_\gamma \partial \lambda_{[\alpha}} - \frac{\partial^2 \phi}{\partial \xi \partial \Sigma_{\gamma[\alpha}} \right) \Sigma_{\beta]\gamma} = 0, \tag{29}$$

and this is surely satisfied, if

$$\frac{\partial^2 \phi}{\partial \lambda_\gamma \partial \lambda_\alpha} = \frac{\partial^2 \phi}{\partial \xi \partial \Sigma_{\gamma\alpha}} + D g^{\gamma\alpha}, \tag{30}$$

holds, where the presence of the scalar D is due to the fact that the components of $\Sigma_{\gamma\alpha}$ are not independent because its trace is zero. By using eq. (7), we see that eq. (30) becomes

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^2}{\partial \lambda_\gamma \partial \lambda_\alpha} B_c^{\nu_1 \sigma_1 \dots \nu_n \sigma_n} \Sigma_{\nu_1 \sigma_1} \dots \Sigma_{\nu_n \sigma_n} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \frac{\partial}{\partial \xi} B^{\nu_1 \sigma_1 \dots \nu_{n-1} \sigma_{n-1} \gamma\alpha} \Sigma_{\nu_1 \sigma_1} \dots \Sigma_{\nu_{n-1} \sigma_{n-1}} + D^* g^{\gamma\alpha} =$$

$$= \sum_{N=0}^{\infty} \frac{1}{N!} \frac{\partial}{\partial \xi} B^{\nu_1 \sigma_1 \dots \nu_N \sigma_N \gamma\alpha} \Sigma_{\nu_1 \sigma_1} \dots \Sigma_{\nu_N \sigma_N} + D^* g^{\gamma\alpha},$$

which is surely satisfied if $D^* = 0$ and

$$\frac{\partial^2 B^{\nu_1 \sigma_1 \dots \nu_n \sigma_n}}{\partial \lambda_\gamma \partial \lambda_\alpha} = \frac{\partial B^{\nu_1 \sigma_1 \dots \nu_n \sigma_n \gamma\alpha}}{\partial \xi}. \tag{31}$$

From eqs. (26) and (10) we see that eq. (31) is equivalent to

$$\theta_{2n+2, 2S} = \frac{\partial g_{2n+2, 2S}}{\partial \xi}, \quad i.e., \quad A'_{2n+2} = (2n+2)(2n+3) A_{2n}. \tag{32}$$

Eq. (32) can be rewritten by use of the definition $(24)_1$ as

$$B'_{2n+2} = B_{2n}. \tag{33}$$

This equation can be integrated to give eq. $(24)_2$, which we wanted to prove.

4. A comparison with the second order approximation

In [6] the same problem has been considered of the present work, but restricting to search only one of the exact solutions, and up to second order with respect to equilibrium. We want now to verify that those results are a particular case of the corresponding ones in the present paper. In [6], the following expression has been found for ϕ (the apex «2» indicates that it is an expression up to second order with respect to $\Sigma_{\mu\nu}$)

$$\overset{2}{\phi} = G_0^{-1} k_0(\xi) + k_1(\xi) + 8 G_0^{-3} q_0(\xi) \Sigma_{\mu\nu} \lambda^\mu \lambda^\nu + \tag{34}$$

$$+ g_0(\xi) [24 G_0^{-5} (\Sigma_{\mu\nu} \lambda^\mu \lambda^\nu)^2 - 12 G_0^{-4} \lambda_\mu \Sigma^{\mu\nu} \Sigma_{\nu\delta} \lambda^\delta + G_0^{-3} \Sigma_{\mu\nu} \Sigma^{\mu\nu}] + g_1(\xi) \Sigma_{\mu\nu} \Sigma^{\mu\nu},$$

with $G_0 = \lambda_\mu \lambda^\mu$ so that, in the notation of the present work, we have $G_0 = -\gamma^2$.

Instead of this, the corresponding expression for ϕ, which has been found in the present paper, is

$$\overset{2}{\phi} = \gamma^{-2} A_0(\xi) + \gamma^{-4} A_2(\xi) \left(U^{\nu_1} U^{\sigma_1} + \frac{1}{3} h^{\nu_1 \sigma_1} \right) \Sigma_{\nu_1 \sigma_1} +$$

$$+ \frac{1}{2} \gamma^{-6} A_4(\xi) \left(U^{\nu_1} U^{\sigma_1} U^{\nu_2} U^{\sigma_2} + 2 h^{(\nu_1 \sigma_1} U^{\nu_2} U^{\sigma_2)} + \frac{1}{5} h^{(\nu_1 \sigma_1} h^{\nu_2 \sigma_2)} \right) \Sigma_{\nu_1 \sigma_1} \Sigma_{\nu_2 \sigma_2} =$$

$$= \gamma^{-2} A_0(\xi) + \frac{4}{3} \gamma^{-4} A_2(\xi)\, U^{\nu_1} U^{\sigma_1} \Sigma_{\nu_1 \sigma_1} +$$

$$+ \frac{1}{2} \gamma^{-6} A_4(\xi) \left(U^{\nu_1} U^{\sigma_1} U^{\nu_2} U^{\sigma_2} + \frac{2}{3} h^{\nu_1 \sigma_1} U^{\nu_2} U^{\sigma_2} + \frac{4}{3} h^{\nu_1 \nu_2} U^{\sigma_1} U^{\sigma_2} + \right.$$

$$\left. + \frac{1}{15} h^{\nu_1 \sigma_1} h^{\nu_2 \sigma_2} + \frac{2}{15} h^{\nu_1 \nu_2} h^{\sigma_1 \sigma_2} \right) \Sigma_{\nu_1 \sigma_1} \Sigma_{\nu_2 \sigma_2} =$$

$$= \gamma^{-2} A_0(\xi) + \frac{4}{3} \gamma^{-4} A_2(\xi)\, U^{\nu_1} U^{\sigma_1} \Sigma_{\nu_1 \sigma_1} +$$

$$+ \frac{1}{2} \gamma^{-6} A_4(\xi) \left(\frac{16}{5} U^{\nu_1} U^{\sigma_1} U^{\nu_2} U^{\sigma_2} + \frac{8}{5} g^{\nu_1 \nu_2} U^{\sigma_1} U^{\sigma_2} + \frac{2}{15} g^{\nu_1 \nu_2} g^{\sigma_1 \sigma_2} \right) \Sigma_{\nu_1 \sigma_1} \Sigma_{\nu_2 \sigma_2}.$$

By using also the relation $\lambda^\alpha = \gamma U^\alpha$, we see that the two results agree exactly if and only if

$$k_1(\xi) = 0, \; k_0(\xi) = -A_0(\xi), \; q_0(\xi) = -\frac{1}{6} A_2(\xi), \qquad (35)$$

$$g_1(\xi) = 0, \; g_0(\xi) = \frac{-1}{360} A_4(\xi).$$

Now some of these relations, in particular eqs. $(35)_{2,3,5}$ give simply the correspondence between the expressions $k_0(\xi)$, $q_0(\xi)$, $g_0(\xi)$ of [6] and the functions $A_0(\xi)$, $A_2(\xi)$, $A_4(\xi)$ in the present paper. The eqs. $(35)_{1,4}$ may induce us to think that the present work is more restrictive than [6]; however, from eq. (34) we see that the terms with $k_1(\xi)$ and $g_1(\xi)$ don't depend on λ^α, so that they give a zero contribution to $h'^\alpha = \frac{\partial \phi}{\partial \lambda_\alpha}$. But we know that ϕ contributes to the model only by means of h'^α; then the present result isn't more restrictive than the corresponding one in [6].

5. A comparison with the kinetic approach

We have already reported, in the introduction, the solution (10) to our problem, which has been found by Boillat and Ruggeri in the framework of the kinetic approach; we have also seen that it is a particular solution of the present approach at a macroscopic level. We have postponed to prove in this section that the expansion of eq. (10) around $\Sigma_{\mu\nu} = 0$ coincides with eqs. (7) and (8), but with A_{2n} given by eq. (13). Now it is evident that from eq. (10) it follows eq. (7) with

$$B_c^{\nu_1 \sigma_1 \dots \nu_n \sigma_n} = \int F^{(n)}(\xi + \lambda_\mu p^\mu)\, p^{\nu_1} p^{\sigma_1} \dots p^{\nu_n} p^{\sigma_n} \, dP. \qquad (36)$$

To evaluate this tensor, it is useful to calculate the integral

(37) $$B^{\alpha_1 \dots \alpha_m} = \int F(\xi + \lambda_\mu p^\mu)\, p^{\alpha_1} \dots p^{\alpha_m} dP.$$

Let us start by calculating, in the reference frame with $U_\mu \equiv (1, 0, 0, 0)$, the following tensor

(38) $$h^{\beta_1}_{\alpha_1} \dots h^{\beta_r}_{\alpha_r} U_{\alpha_{r+1}} \dots U_{\alpha_m} B^{\alpha_1 \dots \alpha_r \alpha_{r+1} \dots \alpha_m} = \int F(\xi + \gamma p^0)(p^0)^{m-r}(p^0)^r q^{\beta_1} \dots q^{\beta_r} dP,$$

where we have used $q^{\beta_j} = h^{\beta_j}_{\alpha_j} p^{\alpha_j} \dfrac{1}{p^0}$ with $j = 1, 2, \dots r$. By using the following change of variables

(39) $$p^0 = \varrho,\ q^1 = \sin\theta\cos\phi,\ q^2 = \sin\theta\sin\phi,\ q^3 = \cos\theta,$$

the eq. (38) assumes the form

$$h^{\beta_1}_{\alpha_1} \dots h^{\beta_r}_{\alpha_r} U_{\alpha_{r+1}} \dots U_{\alpha_m} B^{\alpha_1 \dots \alpha_r \alpha_{r+1} \dots \alpha_m} = G_{m+1}(\xi, \gamma)\, I^{\beta_1 \dots \beta_r},$$

with

$$G_{m+1}(\xi, \gamma) = \int_0^\infty F(\xi + \gamma\varrho)\, \varrho^{m+1} d\varrho$$

$$I^{\beta_1 \dots \beta_r} = \int_0^{2\pi} d\phi \int_0^\pi q^{\beta_1} \dots q^{\beta_r} \sin\theta d\theta = \begin{cases} \dfrac{4\pi}{r+1} h^{(\beta_1\beta_2} \dots h^{\beta_{r-1}\beta_r)} & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd}, \end{cases}$$

where, in the last passage, we have used a well known result (a proof can be found in [18, Appendix A]).

Moreover, the following identity holds (see, for example, [18, Appendix A])

$$B^{\alpha_1\alpha_2 \dots \alpha_m} = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} h^{(\alpha_1}_{\beta_1} \dots h^{\alpha_r}_{\beta_r} U^{\alpha_{r+1}} \dots U^{\alpha_m)} U_{\beta_{r+1}} \dots U_{\beta_m} B^{\beta_1 \dots \beta_r \beta_{r+1} \dots \beta_m},$$

which, for the previous result, becomes (we can put $r = 2s$ because we obtain zero when r is odd)

$$B^{\alpha_1\alpha_2 \dots \alpha_m} = \sum_{S=0}^{[m/2]} (-1)^{m-2S} \binom{m}{2S} G_{m+1}(\xi, \gamma) \frac{4\pi}{2S+1} h^{(\alpha_1\alpha_2} \dots h^{\alpha_{2S-1}\alpha_{2S}} U^{\alpha_{2S+1}} \dots U^{\alpha_m)}.$$

Moreover, with the change of variable $\varrho = \dfrac{\sigma}{\gamma}$, we obtain

$$G_{m+1}(\xi, \gamma) = \gamma^{-(m+2)} \widetilde{G}_{m+1}(\xi) \quad \text{with} \quad \widetilde{G}_{m+1}(\xi) = \int_0^\infty F(\xi + \sigma)\, \sigma^{m+1} d\sigma.$$

Now from eqs. (36) and (37), it follows

$$B_c^{\nu_1\sigma_1 \dots \nu_n\sigma_n} = \frac{\partial^n}{\partial\xi^n} B^{\nu_1\sigma_1 \dots \nu_n\sigma_n};$$

then we find eq. (8), provided that

$$A_{2n}(\xi) = 4\pi \widetilde{G}^{(n)}_{2n+1}(\xi),$$

from which eq. (13) follows.

Acknowledgements

This work has been partially supported by GNFM - INDAM and Ministero dell'Università e della Ricerca Scientifica e Tecnologica.

References

[1] I.-S. Liu - I. Müller - T. Ruggeri, *Relativistic thermodynamics of gases*. Ann. of Phys., 169, 1986, 191-219.

[2] I. Müller - T. Ruggeri, *Rational Extended Thermodynamics*. 2nd ed., Springer Tracts in Natural Philosophy, 37, Springer-Verlag, New York, 1998, 396 pp.

[3] G. Boillat - T. Ruggeri, *Moment equations in the kinetic theory of gases and wave velocities*. Continuum Mech. Thermodyn., 9, 1997, 205-212.

[4] G. Boillat - T. Ruggeri, *Relativistic gas: Moment equations and maximum wave velocity*. J. Math. Phys., 40, 1999, 6399-6404.

[5] S. Pennisi, *Extended Thermodynamics of non degenerate ultrarelativistic gases*. Il Nuovo Cimento, 104B, 1989, 273-290.

[6] S. Pennisi, *Soluzioni esatte per la termodinamica estesa di un gas ultrarelativistico*. Rend. Sem. Fac. Scienze Univ. Cagliari, 70, 2000, 273-286.

[7] S. Pennisi, *Some considerations on a non-linear approach to extended thermodynamics*. Pitagora Editrice Bologna, 1987, 259-264.

[8] R. Geroch - L. Lindblom, *Dissipative relativistic fluid theorie of divergence type*. Physical Review D., 41, 1990, 1855-1861.

[9] W. Dreyer, *Maximisation of the entropy in non-equilibrium*. J. Phys. A: Math. Gen., 20, 1987, 6505-6517.

[10] G. Boillat - T. Ruggeri, *Maximum wave velocity in the moments system of a relativistic gas*. Continuum Mech. Thermodyn., 11, 1999, 107-111.

[11] F. Brini - T. Ruggeri, *Maximum velocity for wave propagation in a relativistic rarefied gas*. Continuum Mech. Thermodyn., 11, 1999, 331-338.

[12] F. Jüttner, *Das Maxwellsche Gesetz der Geshwindigkeitsverteilung in der Relativtheorie*. Annalen der Physik, 34, 1911, 856-882.

[13] F. Jüttner, *Die Relativistische Quantentheorie des Idealen Gases*. Zeitschrift f. Physik, 47, 1928, 542-566.

[14] S. Pennisi, *A family of exact solutions in Relativistic Extended Thermodynamics*. Continuum Mech. Thermodyn., 44, 2002, 377-387.

[15] P. Amendt - H. Weitzner, *Relativistically covariant warm charged fluid beam modeling*. Phys. Fluids, 28, 1985, 949-957.

[16] A.M. Anile - S. Pennisi, *Fluid models for relativistic electron beams*. Continuum Mech. Thermodyn., 1, 1989, 267-282.

[17] S. Pennisi, *A fourth order approach to relativistic extended thermodynamics*. Continuum Mech. Thermodyn., 11, 1999, 51-71.

[18] S. Pennisi, *Relativistic Extended Thermodynamics with 30 independent fields*. Atti Sem. Mat. Fis. Univ. Modena, L, 2002, 429-449.

Pervenuta il 16 giugno 2003,
in forma definitiva il 23 ottobre 2003.

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